EUCLID'S ELEMENTS OF GEOMETRY

The Greek text of J.L. Heiberg (1883–1885)

from *Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus B.G. Teubneri, 1883–1885*

edited, and provided with a modern English translation, by

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Introduction

Euclid's Elements is by far the most famous mathematical work of classical antiquity, and also has the distinction of being the world's oldest continuously used mathematical textbook. Little is known about the author, beyond the fact that he lived in Alexandria around 300 BCE. The main subjects of the work are geometry, proportion, and number theory.

Most of the theorems appearing in the Elements were not discovered by Euclid himself, but were the work of earlier Greek mathematicians such as Pythagoras (and his school), Hippocrates of Chios, Theaetetus of Athens, and Eudoxus of Cnidos. However, Euclid is generally credited with arranging these theorems in a logical manner, so as to demonstrate (admittedly, not always with the rigour demanded by modern mathematics) that they necessarily follow from five simple axioms. Euclid is also credited with devising a number of particularly ingenious proofs of previously discovered theorems: *e.g.*, Theorem 48 in Book 1.

The geometrical constructions employed in the Elements are restricted to those which can be achieved using a straight-rule and a compass. Furthermore, empirical proofs by means of measurement are strictly forbidden: *i.e.*, any comparison of two magnitudes is restricted to saying that the magnitudes are either equal, or that one is greater than the other.

The Elements consists of thirteen books. Book 1 outlines the fundamental propositions of plane geometry, including the three cases in which triangles are congruent, various theorems involving parallel lines, the theorem regarding the sum of the angles in a triangle, and the Pythagorean theorem. Book 2 is commonly said to deal with "geometric algebra", since most of the theorems contained within it have simple algebraic interpretations. Book 3 investigates circles and their properties, and includes theorems on tangents and inscribed angles. Book 4 is concerned with regular polygons inscribed in, and circumscribed around, circles. Book 5 develops the arithmetic theory of proportion. Book 6 applies the theory of proportion to plane geometry, and contains theorems on similar figures. Book 7 deals with elementary number theory: *e.g.*, prime numbers, greatest common denominators, *etc.* Book 8 is concerned with geometric series. Book 9 contains various applications of results in the previous two books, and includes theorems on the infinitude of prime numbers, as well as the sum of a geometric series. Book 10 attempts to classify incommensurable (*i.e.*, irrational) magnitudes using the so-called "method of exhaustion", an ancient precursor to integration. Book 11 deals with the fundamental propositions of three-dimensional geometry. Book 12 calculates the relative volumes of cones, pyramids, cylinders, and spheres using the method of exhaustion. Finally, Book 13 investigates the five so-called Platonic solids.

This edition of Euclid's Elements presents the definitive Greek text—*i.e.*, that edited by J.L. Heiberg (1883–1885)—accompanied by a modern English translation, as well as a Greek-English lexicon. Neither the spurious books 14 and 15, nor the extensive scholia which have been added to the Elements over the centuries, are included. The aim of the translation is to make the mathematical argument as clear and unambiguous as possible, whilst still adhering closely to the meaning of the original Greek. Text within square parenthesis (in both Greek and English) indicates material identified by Heiberg as being later interpolations to the original text (some particularly obvious or unhelpful interpolations have been omitted altogether). Text within round parenthesis (in English) indicates material which is implied, but not actually present, in the Greek text.

My thanks to Mariusz Wodzicki (Berkeley) for typesetting advice, and to Sam Watson & Jonathan Fenno (U. Mississippi), and Gregory Wong (UCSD) for pointing out a number of errors in Book 1.

ELEMENTS BOOK 1

Fundamentals of Plane Geometry Involving Straight-Lines

Definitions

1. A point is that of which there is no part.

2. And a line is a length without breadth.

3. And the extremities of a line are points.

4. A straight-line is (any) one which lies evenly with points on itself.

5. And a surface is that which has length and breadth only.

6. And the extremities of a surface are lines.

7. A plane surface is (any) one which lies evenly with the straight-lines on itself.

8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.

11. An obtuse angle is one greater than a right-angle.

12. And an acute angle (is) one less than a right-angle.

13. A boundary is that which is the extremity of something.

14. A figure is that which is contained by some boundary or boundaries.

15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

16. And the point is called the center of the circle.

17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.†

18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.

19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions). † This should really be counted as a postulate, rather than as part of a definition.

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡ † The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative could be translated as "let it be postulated", in the sense "let it stand as postulated", but not "let the postulate be now brought forward". The literal translation "let it have been postulated" sounds awkward in English, but more accurately captures the meaning of the Greek.

‡ This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

Common Notions

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things then the wholes are equal.

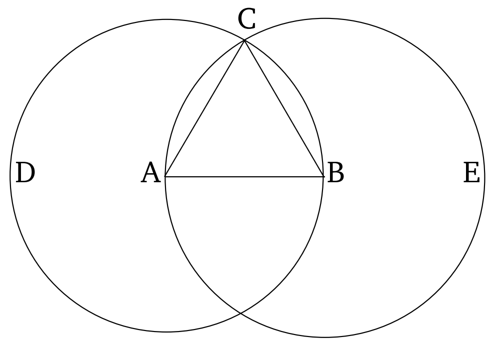
3. And if equal things are subtracted from equal things then the remainders are equal.†

4. And things coinciding with one another are equal to one another.

5. And the whole [is] greater than the part. † As an obvious extension of C.N.s 2 & 3—if equal things are added or subtracted from the two sides of an inequality then the inequality remains an inequality of the same type.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

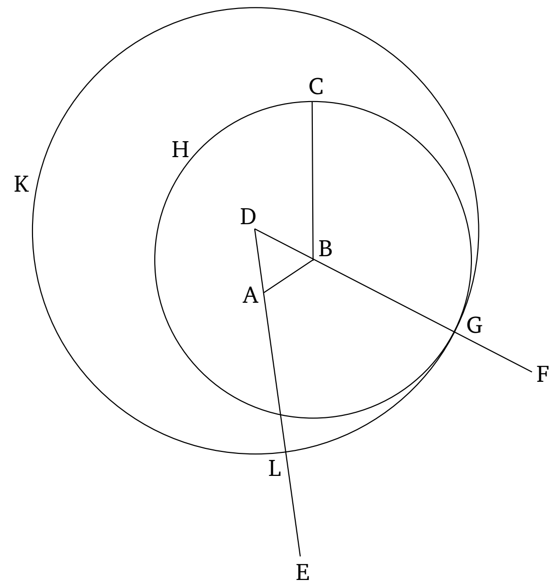
Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do. † The assumption that the circles do indeed cut one another should be counted as an additional postulate. There is also an implicit assumption that two straight-lines cannot share a common segment.

Proposition 2†

To place a straight-line equal to a given straight-line at a given point (as an extremity).

Let A be the given point, and BC the given straight-line. So it is required to place a straight-line at point A equal to the given straight-line BC.

For let the straight-line AB have been joined from point A to point B [Post. 1], and let the equilateral triangle DAB have been been constructed upon it [Prop. 1.1]. And let the straight-lines AE and BF have been produced in a straight-line with DA and DB (respectively) [Post. 2]. And let the circle CGH with center B and radius BC have been drawn [Post. 3], and again let the circle GKL with center D and radius DG have been drawn [Post. 3].



Therefore, since the point B is the center of (the circle) CGH, BC is equal to BG [Def. 1.15]. Again, since the point D is the center of the circle GKL, DL is equal to DG [Def. 1.15]. And within these, DA is equal to DB. Thus, the remainder AL is equal to the remainder BG [C.N. 3]. But BC was also shown (to be) equal to BG. Thus, AL and BC are each equal to BG. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, AL is also equal to BC.

Thus, the straight-line AL, equal to the given straight-line BC, has been placed at the given point A. (Which is) the very thing it was required to do. † This proposition admits of a number of different cases, depending on the relative positions of the point A and the line BC. In such situations, Euclid invariably only considers one particular case—usually, the most difficult—and leaves the remaining cases as exercises for the reader.

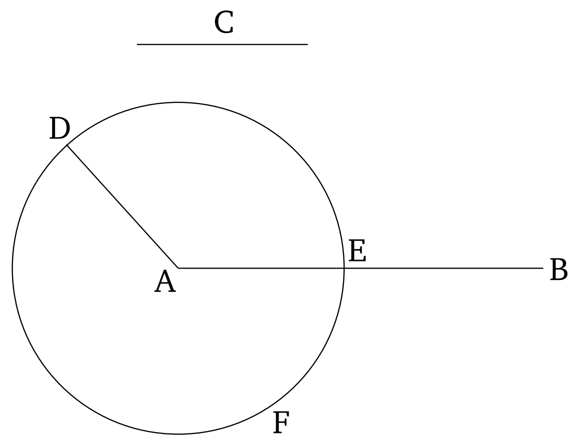
Proposition 3

For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.

Let AB and C be the two given unequal straight-lines, of which let the greater be AB. So it is required to cut off a straight-line equal to the lesser C from the greater AB.

Let the line AD, equal to the straight-line C, have been placed at point A [Prop. 1.2]. And let the circle DEF have been drawn with center A and radius AD [Post. 3].

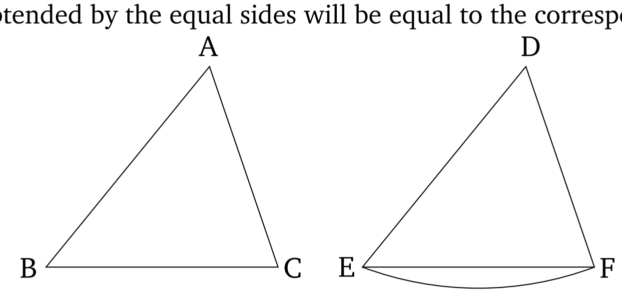
And since point A is the center of circle DEF, AE is equal to AD [Def. 1.15]. But, C is also equal to AD. Thus, AE and C are each equal to AD. So AE is also equal to C [C.N. 1].



Thus, for two given unequal straight-lines, AB and C, the (straight-line) AE, equal to the lesser C, has been cut off from the greater AB. (Which is) the very thing it was required to do.

Proposition 4

If two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is) AB to DE, and AC to DF. And (let) the angle BAC (be) equal to the angle EDF. I say that the base BC is also equal to the base EF, and triangle ABC will be equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (That is) ABC to DEF, and ACB to DFE.

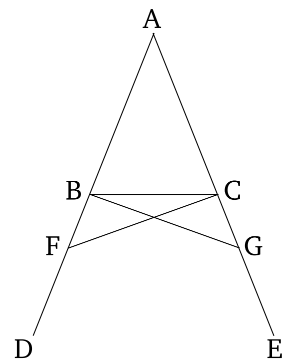
For if triangle ABC is applied to triangle DEF,[†] the point A being placed on the point D, and the straight-line AB on DE, then the point B will also coincide with E, on account of AB being equal to DE. So (because of) AB coinciding with DE, the straight-line AC will also coincide with DF, on account of the angle BAC being equal to EDF. So the point C will also coincide with the point F, again on account of AC being equal to DF. But, point B certainly also coincided with point E, so that the base BC will coincide with the base EF. For if B coincides with E, and C with F, and the base BC does not coincide with EF, then two straight-lines will encompass an area. The very thing is impossible [Post. 1].[‡] Thus, the base BC will coincide with EF, and will be equal to it [C.N. 4]. So the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it [C.N. 4]. And the remaining angles will coincide with the remaining angles, and will be equal to them [C.N. 4]. (That is) ABC to DEF, and ACB to DFE [C.N. 4].

Thus, if two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-line equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (Which is) the very thing it was required to show. † The application of one figure to another should be counted as an additional postulate.

‡ Since Post. 1 implicitly assumes that the straight-line joining two given points is unique.

Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.



Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight-lines BD and CE have been produced in a straight-line with AB and AC (respectively) [Post. 2]. I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

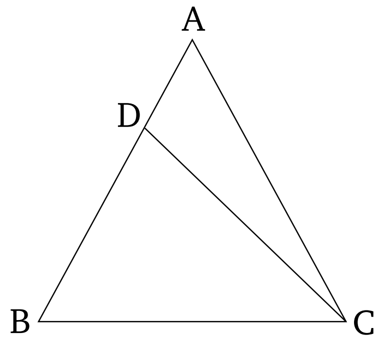
For let the point F have been taken at random on BD, and let AG have been cut off from the greater AE, equal to the lesser AF [Prop. 1.3]. Also, let the straight-lines FC and GB have been joined [Post. 1].

In fact, since AF is equal to AG, and AB to AC, the two (straight-lines) FA, AC are equal to the two (straight-lines) GA, AB, respectively. They also encompass a common angle, FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtendend by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG, and AFC to AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [C.N. 3]. But FC was also shown (to be) equal to GB. So the two (straight-lines) BF, FC are equal to the two (straight-lines) CG, GB, respectively, and the angle BFC (is) equal to the angle CGB, and the base BC is common to them. Thus, the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, FBC is equal to GCB, and BCF to CBG. Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB [C.N. 3]. And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

Proposition 6

If a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another.



Let ABC be a triangle having the angle ABC equal to the angle ACB. I say that side AB is also equal to side AC.

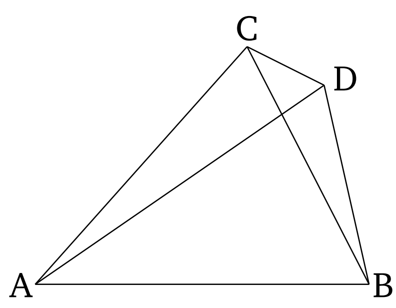
For if AB is unequal to AC then one of them is greater. Let AB be greater. And let DB, equal to the lesser AC, have been cut off from the greater AB [Prop. 1.3]. And let DC have been joined [Post. 1].

Therefore, since DB is equal to AC, and BC (is) common, the two sides DB, BC are equal to the two sides AC, CB, respectively, and the angle DBC is equal to the angle ACB. Thus, the base DC is equal to the base AB, and the triangle DBC will be equal to the triangle ACB [Prop. 1.4], the lesser to the greater. The very notion (is) absurd [C.N. 5]. Thus, AB is not unequal to AC. Thus, (it is) equal.†

Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show. † Here, use is made of the previously unmentioned common notion that if two quantities are not unequal then they must be equal. Later on, use is made of the closely related common notion that if two quantities are not greater than or less than one another, respectively, then they must be equal to one another.

Proposition 7

On the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines.



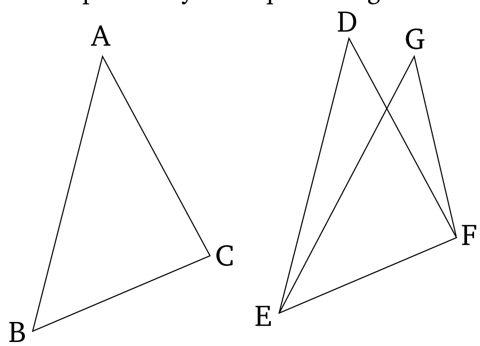
For, if possible, let the two straight-lines AC, CB, equal to two other straight-lines AD, DB, respectively, have been constructed on the same straight-line AB, meeting at different points, C and D, on the same side (of AB), and having the same ends (on AB). So CA is equal to DA, having the same end A as it, and CB is equal to DB, having the same end B as it. And let CD have been joined [Post. 1].

Therefore, since AC is equal to AD, the angle ACD is also equal to angle ADC [Prop. 1.5]. Thus, ADC (is) greater than DCB [C.N. 5]. Thus, CDB is much greater than DCB [C.N. 5]. Again, since CB is equal to DB, the angle CDB is also equal to angle DCB [Prop. 1.5]. But it was shown that the former (angle) is also much greater (than the latter). The very thing is impossible.

Thus, on the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines. (Which is) the very thing it was required to show.

Proposition 8

If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, then they will also have equal the angles encompassed by the equal straight-lines.



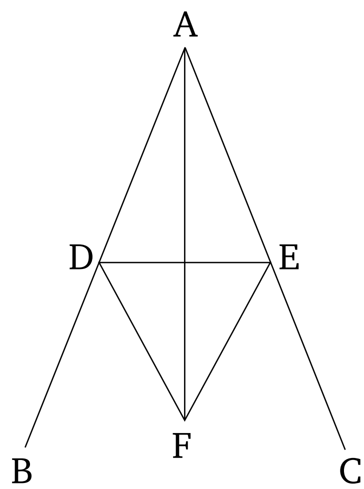
Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is) AB to DE, and AC to DF. Let them also have the base BC equal to the base EF. I say that the angle BAC is also equal to the angle EDF.

For if triangle ABC is applied to triangle DEF, the point B being placed on point E, and the straight-line BC on EF, then point C will also coincide with F, on account of BC being equal to EF. So (because of) BC coinciding with EF, (the sides) BA and CA will also coincide with ED and DF (respectively). For if base BC coincides with base EF, but the sides AB and AC do not coincide with ED and DF (respectively), but miss like EG and GF (in the above figure), then we will have constructed upon the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines, and (meeting) at a different point on the same side (of the straight-line), but having the same ends. But (such straight-lines) cannot be constructed [Prop. 1.7]. Thus, the base BC being applied to the base EF, the sides BA and AC cannot not coincide with ED and DF (respectively). Thus, they will coincide. So the angle BAC will also coincide with angle EDF, and will be equal to it [C.N. 4].

Thus, if two triangles have two sides equal to two side, respectively, and have the base equal to the base, then they will also have equal the angles encompassed by the equal straight-lines. (Which is) the very thing it was required to show.

Proposition 9

To cut a given rectilinear angle in half.



Let BAC be the given rectilinear angle. So it is required to cut it in half.

Let the point D have been taken at random on AB, and let AE, equal to AD, have been cut off from AC [Prop. 1.3], and let DE have been joined. And let the equilateral triangle DEF have been constructed upon DE [Prop. 1.1], and let AF have been joined. I say that the angle BAC has been cut in half by the straight-line AF.

For since AD is equal to AE, and AF is common, the two (straight-lines) DA, AF are equal to the two (straight-lines) EA, AF, respectively. And the base DF is equal to the base EF. Thus, angle DAF is equal to angle EAF [Prop. 1.8].

Thus, the given rectilinear angle BAC has been cut in half by the straight-line AF. (Which is) the very thing it was required to do.

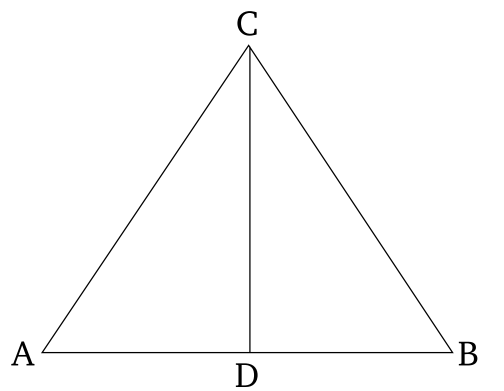
Proposition 10

To cut a given finite straight-line in half.

Let AB be the given finite straight-line. So it is required to cut the finite straight-line AB in half.

Let the equilateral triangle ABC have been constructed upon (AB) [Prop. 1.1], and let the angle ACB have been cut in half by the straight-line CD [Prop. 1.9]. I say that the straight-line AB has been cut in half at point D.

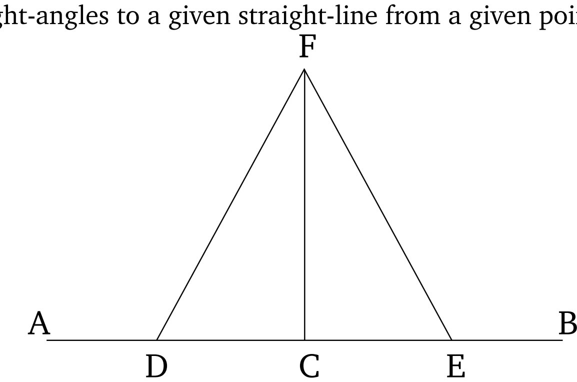
For since AC is equal to CB, and CD (is) common, the two (straight-lines) AC, CD are equal to the two (straight-lines) BC, CD, respectively. And the angle ACD is equal to the angle BCD. Thus, the base AD is equal to the base BD [Prop. 1.4].



Thus, the given finite straight-line AB has been cut in half at (point) D. (Which is) the very thing it was required to do.

Proposition 11

To draw a straight-line at right-angles to a given straight-line from a given point on it.



Let AB be the given straight-line, and C the given point on it. So it is required to draw a straight-line from the point C at right-angles to the straight-line AB.

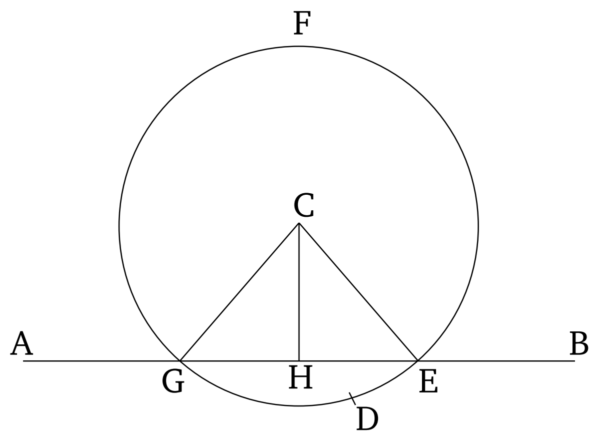
Let the point D be have been taken at random on AC, and let CE be made equal to CD [Prop. 1.3], and let the equilateral triangle FDE have been constructed on DE [Prop. 1.1], and let FC have been joined. I say that the straight-line FC has been drawn at right-angles to the given straight-line AB from the given point C on it.

For since DC is equal to CE, and CF is common, the two (straight-lines) DC, CF are equal to the two (straight-lines), EC, CF, respectively. And the base DF is equal to the base FE. Thus, the angle DCF is equal to the angle ECF [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, each of the (angles) DCF and FCE is a right-angle.

Thus, the straight-line CF has been drawn at right-angles to the given straight-line AB from the given point C on it. (Which is) the very thing it was required to do.

Proposition 12

To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.



Let AB be the given infinite straight-line and C the given point, which is not on (AB). So it is required to draw a straight-line perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB).

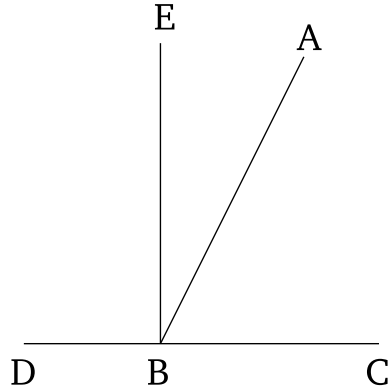
For let point D have been taken at random on the other side (to C) of the straight-line AB, and let the circle EFG have been drawn with center C and radius CD [Post. 3], and let the straight-line EG have been cut in half at (point) H [Prop. 1.10], and let the straight-lines CG, CH, and CE have been joined. I say that the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB).

For since GH is equal to HE, and HC (is) common, the two (straight-lines) GH, HC are equal to the two (straight-lines) EH, HC, respectively, and the base CG is equal to the base CE. Thus, the angle CHG is equal to the angle EHC [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands [Def. 1.10].

Thus, the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB). (Which is) the very thing it was required to do.

Proposition 13

If a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles.



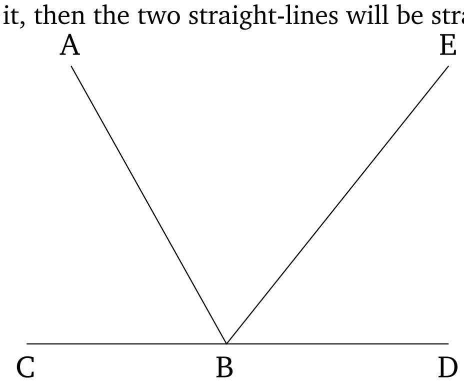
For let some straight-line AB stood on the straight-line CD make the angles CBA and ABD. I say that the angles CBA and ABD are certainly either two right-angles, or (have a sum) equal to two right-angles.

In fact, if CBA is equal to ABD then they are two right-angles [Def. 1.10]. But, if not, let BE have been drawn from the point B at right-angles to [the straight-line] CD [Prop. 1.11]. Thus, CBE and EBD are two right-angles. And since CBE is equal to the two (angles) CBA and ABE, let EBD have been added to both. Thus, the (sum of the angles) CBE and EBD is equal to the (sum of the) three (angles) CBA, ABE, and EBD [C.N. 2]. Again, since DBA is equal to the two (angles) DBE and EBA, let ABC have been added to both. Thus, the (sum of the angles) DBA and ABC is equal to the (sum of the) three (angles) DBE, EBA, and ABC [C.N. 2]. But (the sum of) CBE and EBD was also shown (to be) equal to the (sum of the) same three (angles). And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, (the sum of) CBE and EBD is also equal to (the sum of) DBA and ABC. But, (the sum of) CBE and EBD is two right-angles. Thus, (the sum of) ABD and ABC is also equal to two right-angles.

Thus, if a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 14

If two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another.



For let two straight-lines BC and BD, not lying on the same side, make adjacent angles ABC and ABD (whose sum is) equal to two right-angles with some straight-line AB, at the point B on it. I say that BD is straight-on with respect to CB.

For if BD is not straight-on to BC then let BE be straight-on to CB.

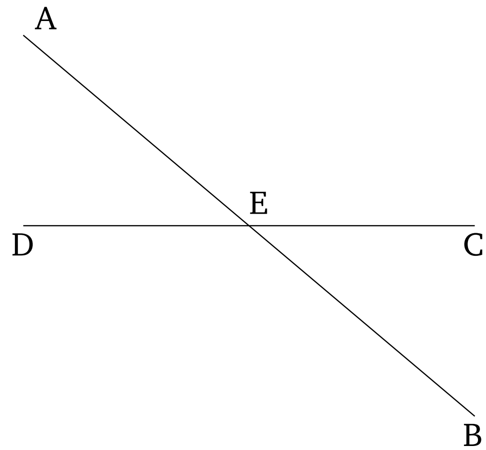
Therefore, since the straight-line AB stands on the straight-line CBE, the (sum of the) angles ABC and ABE is thus equal to two right-angles [Prop. 1.13]. But (the sum of) ABC and ABD is also equal to two right-angles. Thus, (the sum of angles) CBA and ABE is equal to (the sum of angles) CBA and ABD [C.N. 1]. Let (angle) CBA have been subtracted from both. Thus, the remainder ABE is equal to the remainder ABD [C.N. 3], the lesser to the greater. The very thing is impossible. Thus, BE is not straight-on with respect to CB. Similarly, we can show that neither (is) any other (straight-line) than BD. Thus, CB is straight-on with respect to BD.

Thus, if two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another. (Which is) the very thing it was required to show.

Proposition 15

If two straight-lines cut one another then they make the vertically opposite angles equal to one another.

For let the two straight-lines AB and CD cut one another at the point E. I say that angle AEC is equal to (angle) DEB, and (angle) CEB to (angle) AED.



For since the straight-line AE stands on the straight-line CD, making the angles CEA and AED, the (sum of the) angles CEA and AED is thus equal to two right-angles [Prop. 1.13]. Again, since the straight-line DE stands on the straight-line AB, making the angles AED and DEB, the (sum of the) angles AED and DEB is thus equal to two right-angles [Prop. 1.13]. But (the sum of) CEA and AED was also shown (to be) equal to two right-angles. Thus, (the sum of) CEA and AED is equal to (the sum of) AED and DEB [C.N. 1]. Let AED have been subtracted from both. Thus, the remainder CEA is equal to the remainder BED [C.N. 3]. Similarly, it can be shown that CEB and DEA are also equal.

Thus, if two straight-lines cut one another then they make the vertically opposite angles equal to one another. (Which is) the very thing it was required to show.

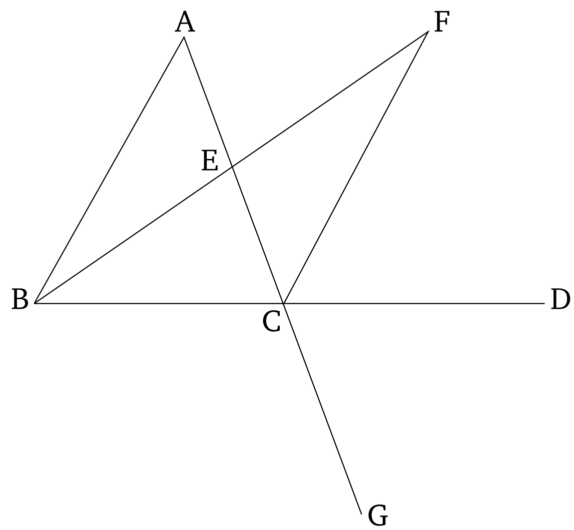
Proposition 16

For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

Let ABC be a triangle, and let one of its sides BC have been produced to D. I say that the external angle ACD is greater than each of the internal and opposite angles, CBA and BAC.

Let the (straight-line) AC have been cut in half at (point) E [Prop. 1.10]. And BE being joined, let it have been produced in a straight-line to (point) F.[†] And let EF be made equal to BE [Prop. 1.3], and let FC have been joined, and let AC have been drawn through to (point) G.

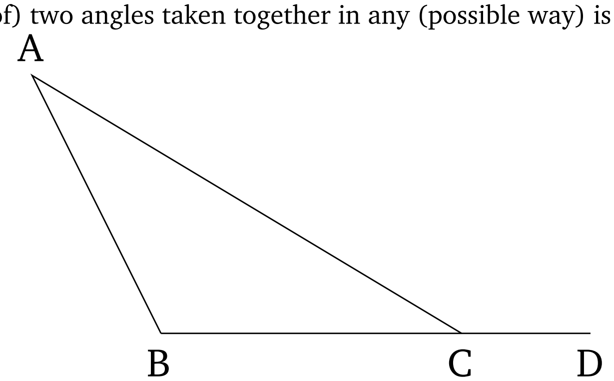
Therefore, since AE is equal to EC, and BE to EF, the two (straight-lines) AE, EB are equal to the two (straight-lines) CE, EF, respectively. Also, angle AEB is equal to angle FEC, for (they are) vertically opposite [Prop. 1.15]. Thus, the base AB is equal to the base FC, and the triangle ABE is equal to the triangle FEC, and the remaining angles subtended by the equal sides are equal to the corresponding remaining angles [Prop. 1.4]. Thus, BAE is equal to ECF. But ECD is greater than ECF. Thus, ACD is greater than BAE. Similarly, by having cut BC in half, it can be shown (that) BCG—that is to say, ACD—(is) also greater than ABC.



Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles. (Which is) the very thing it was required to show. † The implicit assumption that the point F lies in the interior of the angle ABC should be counted as an additional postulate.

Proposition 17

For any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles.



Let ABC be a triangle. I say that (the sum of) two angles of triangle ABC taken together in any (possible way) is less than two right-angles.

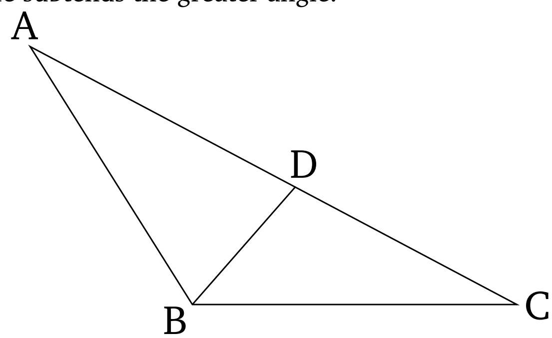
For let BC have been produced to D.

And since the angle ACD is external to triangle ABC, it is greater than the internal and opposite angle ABC [Prop. 1.16]. Let ACB have been added to both. Thus, the (sum of the angles) ACD and ACB is greater than the (sum of the angles) ABC and BCA. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ABC and BCA is less than two right-angles. Similarly, we can show that (the sum of) BAC and ACB is also less than two right-angles, and further (that the sum of) CAB and ABC (is less than two right-angles).

Thus, for any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles. (Which is) the very thing it was required to show.

Proposition 18

In any triangle, the greater side subtends the greater angle.



For let ABC be a triangle having side AC greater than AB. I say that angle ABC is also greater than BCA.

For since AC is greater than AB, let AD be made equal to AB [Prop. 1.3], and let BD have been joined.

And since angle ADB is external to triangle BCD, it is greater than the internal and opposite (angle) DCB [Prop. 1.16]. But ADB (is) equal to ABD, since side AB is also equal to side AD [Prop. 1.5]. Thus, ABD is also greater than ACB. Thus, ABC is much greater than ACB.

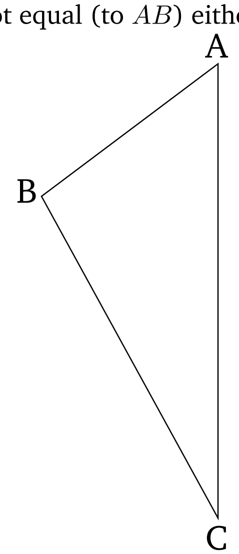
Thus, in any triangle, the greater side subtends the greater angle. (Which is) the very thing it was required to show.

Proposition 19

In any triangle, the greater angle is subtended by the greater side.

Let ABC be a triangle having the angle ABC greater than BCA. I say that side AC is also greater than side AB.

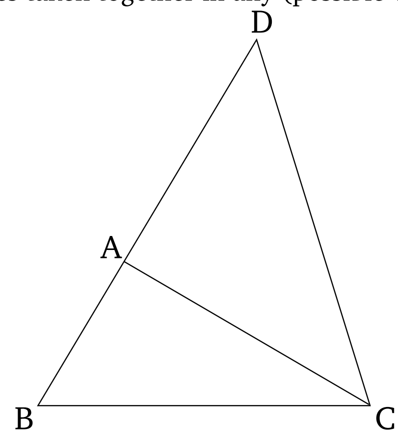
For if not, AC is certainly either equal to, or less than, AB. In fact, AC is not equal to AB. For then angle ABC would also have been equal to ACB [Prop. 1.5]. But it is not. Thus, AC is not equal to AB. Neither, indeed, is AC less than AB. For then angle ABC would also have been less than ACB [Prop. 1.18]. But it is not. Thus, AC is not less than AB. But it was shown that (AC) is not equal (to AB) either. Thus, AC is greater than AB.



Thus, in any triangle, the greater angle is subtended by the greater side. (Which is) the very thing it was required to show.

Proposition 20

In any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side).



For let ABC be a triangle. I say that in triangle ABC (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (So), (the sum of) BA and AC (is greater) than BC, (the sum of) AB and BC than AC, and (the sum of) BC and CA than AB.

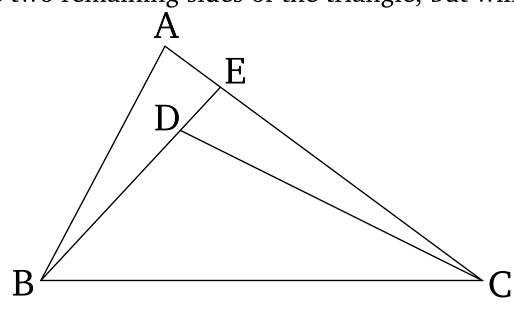
For let BA have been drawn through to point D, and let AD be made equal to CA [Prop. 1.3], and let DC have been joined.

Therefore, since DA is equal to AC, the angle ADC is also equal to ACD [Prop. 1.5]. Thus, BCD is greater than ADC. And since DCB is a triangle having the angle BCD greater than BDC, and the greater angle subtends the greater side [Prop. 1.19], DB is thus greater than BC. But DA is equal to AC. Thus, (the sum of) BA and AC is greater than BC. Similarly, we can show that (the sum of) AB and BC is also greater than CA, and (the sum of) BC and CA than AB.

Thus, in any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (Which is) the very thing it was required to show.

Proposition 21

If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle.



For let the two internal straight-lines BD and DC have been constructed on one of the sides BC of the triangle ABC, from its ends B and C (respectively). I say that BD and DC are less than the (sum of the) two remaining sides of the triangle BA and AC, but encompass an angle BDC greater than BAC.

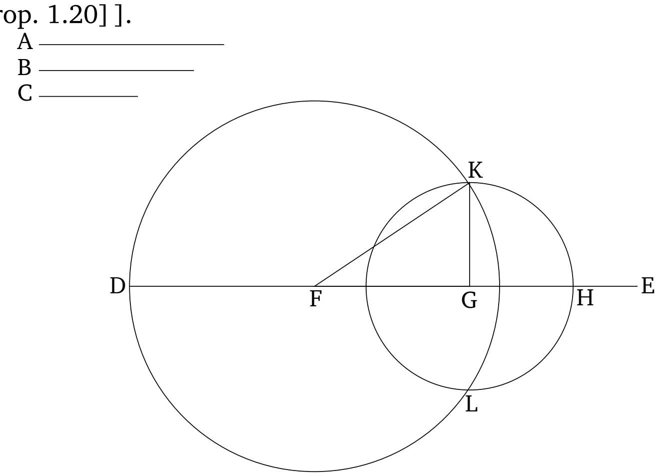
For let BD have been drawn through to E. And since in any triangle (the sum of any) two sides is greater than the remaining (side) [Prop. 1.20], in triangle ABE the (sum of the) two sides AB and AE is thus greater than BE. Let EC have been added to both. Thus, (the sum of) BA and AC is greater than (the sum of) BE and EC. Again, since in triangle CED the (sum of the) two sides CE and ED is greater than CD, let DB have been added to both. Thus, (the sum of) CE and EB is greater than (the sum of) CD and DB. But, (the sum of) BA and AC was shown (to be) greater than (the sum of) BE and EC. Thus, (the sum of) BA and AC is much greater than (the sum of) BD and DC.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED. Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC. But, BDC was shown (to be) greater than CEB. Thus, BDC is much greater than BAC.

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].



Let A, B, and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C, (the sum of) A and C than B, and also (the sum of) B and C than A. So it is required to construct a triangle from (straight-lines) equal to A, B, and C.

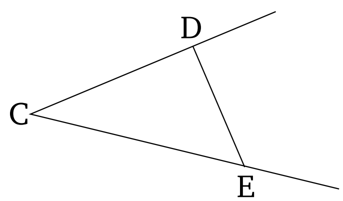
Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 1.3]. And let the circle DKL have been drawn with center F and radius FD. Again, let the circle KLH have been drawn with center G and radius GH. And let KF and KG have been joined. I say that the triangle KFG has been constructed from three straight-lines equal to A, B, and C.

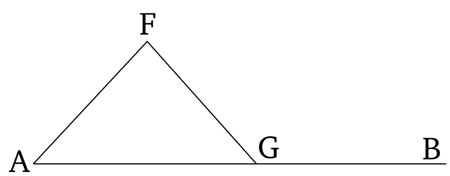
For since point F is the center of the circle DKL, FD is equal to FK. But, FD is equal to A. Thus, KF is also equal to A. Again, since point G is the center of the circle LKH, GH is equal to GK. But, GH is equal to C. Thus, KG is also equal to C. And FG is also equal to B. Thus, the three straight-lines KF, FG, and GK are equal to A, B, and C (respectively).

Thus, the triangle KFG has been constructed from the three straight-lines KF, FG, and GK, which are equal to the three given straight-lines A, B, and C (respectively). (Which is) the very thing it was required to do.

Proposition 23

To construct a rectilinear angle equal to a given rectilinear angle at a (given) point on a given straight-line.





Let AB be the given straight-line, A the (given) point on it, and DCE the given rectilinear angle. So it is required to construct a rectilinear angle equal to the given rectilinear angle DCE at the (given) point A on the given straight-line AB.

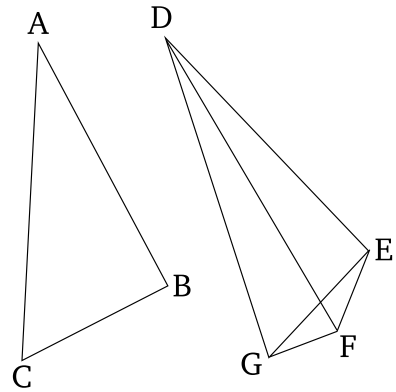
Let the points D and E have been taken at random on each of the (straight-lines) CD and CE (respectively), and let DE have been joined. And let the triangle AFG have been constructed from three straight-lines which are equal to CD, DE, and CE, such that CD is equal to AF, CE to AG, and further DE to FG [Prop. 1.22].

Therefore, since the two (straight-lines) DC, CE are equal to the two (straight-lines) FA, AG, respectively, and the base DE is equal to the base FG, the angle DCE is thus equal to the angle FAG [Prop. 1.8].

Thus, the rectilinear angle FAG, equal to the given rectilinear angle DCE, has been constructed at the (given) point A on the given straight-line AB. (Which is) the very thing it was required to do.

Proposition 24

If two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter).



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is), AB (equal) to DE, and AC to DF. Let them also have the angle at A greater than the angle at D. I say that the base BC is also greater than the base EF.

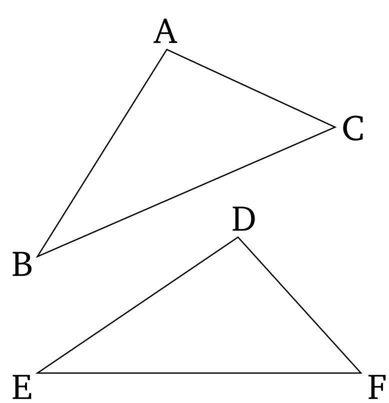
For since angle BAC is greater than angle EDF, let (angle) EDG, equal to angle BAC, have been constructed at the point D on the straight-line DE [Prop. 1.23]. And let DG be made equal to either of AC or DF [Prop. 1.3], and let EG and FG have been joined.

Therefore, since AB is equal to DE and AC to DG, the two (straight-lines) BA, AC are equal to the two (straight-lines) ED, DG, respectively. Also the angle BAC is equal to the angle EDG. Thus, the base BC is equal to the base EG [Prop. 1.4]. Again, since DF is equal to DG, angle DGF is also equal to angle DFG [Prop. 1.5]. Thus, DFG (is) greater than EGF. Thus, EFG is much greater than EGF. And since triangle EFG has angle EFG greater than EGF, and the greater angle is subtended by the greater side [Prop. 1.19], side EG (is) thus also greater than EF. But EG (is) equal to BC. Thus, BC (is) also greater than EF.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter). (Which is) the very thing it was required to show.

Proposition 25

If two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter).



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively (That is), AB (equal) to DE, and AC to DF. And let the base BC be greater than the base EF. I say that angle BAC is also greater than EDF.

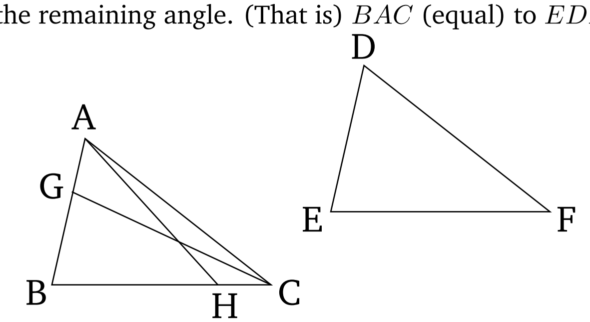
For if not, (BAC) is certainly either equal to, or less than, (EDF). In fact, BAC is not equal to EDF. For then the base BC would also have been equal to the base EF [Prop. 1.4]. But it is not. Thus, angle BAC is not equal to EDF. Neither, indeed, is BAC less than EDF. For then the base BC would also have been less than the base EF [Prop. 1.24]. But it is not. Thus, angle BAC is not less than EDF. But it was shown that (BAC is) not equal (to EDF) either. Thus, BAC is greater than EDF.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter). (Which is) the very thing it was required to show.

Proposition 26

If two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the [corresponding] remaining sides, and the remaining angle (equal) to the remaining angle.

Let ABC and DEF be two triangles having the two angles ABC and BCA equal to the two (angles) DEF and EFD, respectively. (That is) ABC (equal) to DEF, and BCA to EFD. And let them also have one side equal to one side. First of all, the (side) by the equal angles. (That is) BC (equal) to EF. I say that they will have the remaining sides equal to the corresponding remaining sides. (That is) AB (equal) to DE, and AC to DF. And (they will have) the remaining angle (equal) to the remaining angle. (That is) BAC (equal) to EDF.



For if AB is unequal to DE then one of them is greater. Let AB be greater, and let BG be made equal to DE [Prop. 1.3], and let GC have been joined.

Therefore, since BG is equal to DE, and BC to EF, the two (straight-lines) GB, $BC^{\dagger}$ are equal to the two (straight-lines) DE, EF, respectively. And angle GBC is equal to angle DEF. Thus, the base GC is equal to the base DF, and triangle GBC is equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, GCB (is equal) to DFE. But, DFE was assumed (to be) equal to BCA. Thus, BCG is also equal to BCA, the lesser to the greater. The very thing (is) impossible. Thus, AB is not unequal to DE. Thus, (it is) equal. And BC is also equal to EF. So the two (straight-lines) AB, BC are equal to the two (straight-lines) DE, EF, respectively. And angle ABC is equal to angle DEF. Thus, the base AC is equal to the base DF, and the remaining angle BAC is equal to the remaining angle EDF [Prop. 1.4].

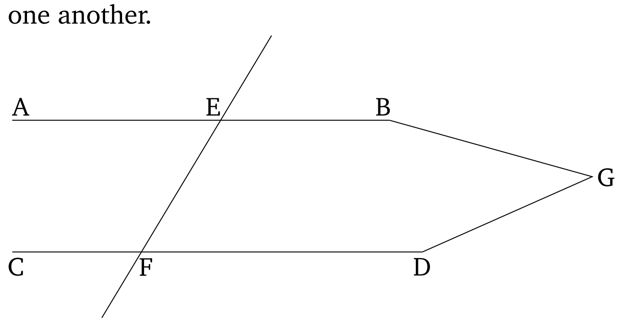
But, again, let the sides subtending the equal angles be equal: for instance, (let) AB (be equal) to DE. Again, I say that the remaining sides will be equal to the remaining sides. (That is) AC (equal) to DF, and BC to EF. Furthermore, the remaining angle BAC is equal to the remaining angle EDF.

For if BC is unequal to EF then one of them is greater. If possible, let BC be greater. And let BH be made equal to EF [Prop. 1.3], and let AH have been joined. And since BH is equal to EF, and AB to DE, the two (straight-lines) AB, BH are equal to the two (straight-lines) DE, EF, respectively. And the angles they encompass (are also equal). Thus, the base AH is equal to the base DF, and the triangle ABH is equal to the triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, angle BHA is equal to EFD. But, EFD is equal to BCA. So, in triangle AHC, the external angle BHA is equal to the internal and opposite angle BCA. The very thing (is) impossible [Prop. 1.16]. Thus, BC is not unequal to EF. Thus, (it is) equal. And AB is also equal to DE. So the two (straight-lines) AB, BC are equal to the two (straight-lines) DE, EF, respectively. And they encompass equal angles. Thus, the base AC is equal to the base DF, and triangle ABC (is) equal to triangle DEF, and the remaining angle BAC (is) equal to the remaining angle EDF [Prop. 1.4].

Thus, if two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle (equal) to the remaining angle. (Which is) the very thing it was required to show. † The Greek text has “BG, BC”, which is obviously a mistake.

Proposition 27

If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.



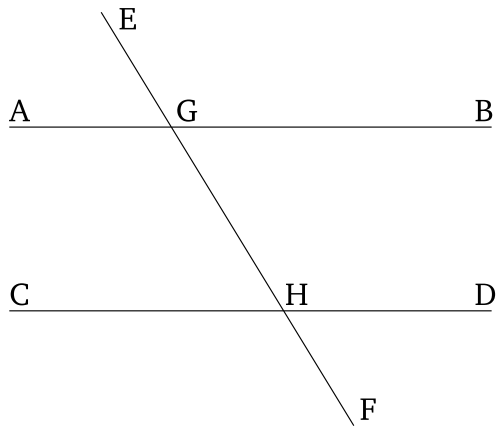
For let the straight-line EF, falling across the two straight-lines AB and CD, make the alternate angles AEF and EFD equal to one another. I say that AB and CD are parallel.

For if not, being produced, AB and CD will certainly meet together: either in the direction of B and D, or (in the direction) of A and C [Def. 1.23]. Let them have been produced, and let them meet together in the direction of B and D at (point) G. So, for the triangle GEF, the external angle AEF is equal to the interior and opposite (angle) EFG. The very thing is impossible [Prop. 1.16]. Thus, being produced, AB and CD will not meet together in the direction of B and D. Similarly, it can be shown that neither (will they meet together) in (the direction of) A and C. But (straight-lines) meeting in neither direction are parallel [Def. 1.23]. Thus, AB and CD are parallel.

Thus, if a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Proposition 28

If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, then the (two) straight-lines will be parallel to one another.



For let EF, falling across the two straight-lines AB and CD, make the external angle EGB equal to the internal and opposite angle GHD, or the (sum of the) internal (angles) on the same side, BGH and GHD, equal to two right-angles. I say that AB is parallel to CD.

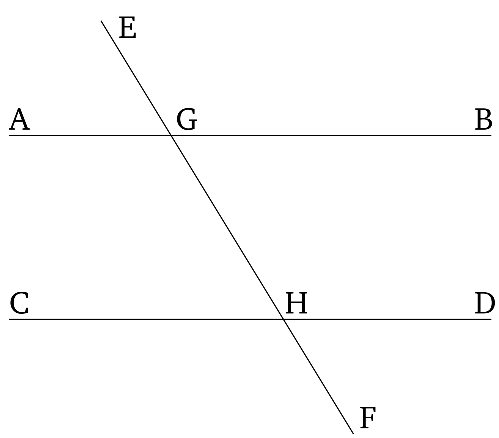
For since (in the first case) EGB is equal to GHD, but EGB is equal to AGH [Prop. 1.15], AGH is thus also equal to GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

Again, since (in the second case, the sum of) BGH and GHD is equal to two right-angles, and (the sum of) AGH and BGH is also equal to two right-angles [Prop. 1.13], (the sum of) AGH and BGH is thus equal to (the sum of) BGH and GHD. Let BGH have been subtracted from both. Thus, the remainder AGH is equal to the remainder GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

Thus, if a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Proposition 29

A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.



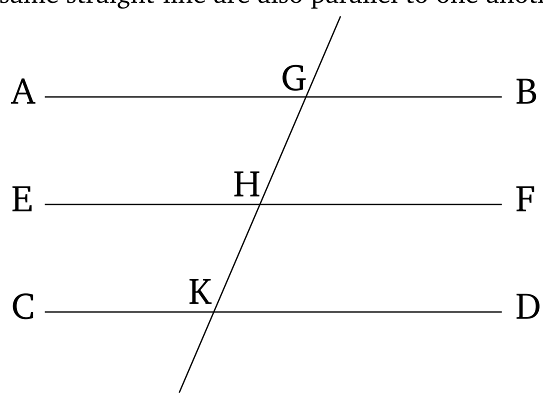
For let the straight-line EF fall across the parallel straight-lines AB and CD. I say that it makes the alternate angles, AGH and GHD, equal, the external angle EGB equal to the internal and opposite (angle) GHD, and the (sum of the) internal (angles) on the same side, BGH and GHD, equal to two right-angles.

For if AGH is unequal to GHD then one of them is greater. Let AGH be greater. Let BGH have been added to both. Thus, (the sum of) AGH and BGH is greater than (the sum of) BGH and GHD. But, (the sum of) AGH and BGH is equal to two right-angles [Prop 1.13]. Thus, (the sum of) BGH and GHD is [also] less than two right-angles. But (straight-lines) being produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, AB and CD, being produced to infinity, will meet together. But they do not meet, on account of them (initially) being assumed parallel (to one another) [Def. 1.23]. Thus, AGH is not unequal to GHD. Thus, (it is) equal. But, AGH is equal to EGB [Prop. 1.15]. And EGB is thus also equal to GHD. Let BGH be added to both. Thus, (the sum of) EGB and BGH is equal to (the sum of) BGH and GHD. But, (the sum of) EGB and BGH is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) BGH and GHD is also equal to two right-angles.

Thus, a straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 30

(Straight-lines) parallel to the same straight-line are also parallel to one another.



Let each of the (straight-lines) AB and CD be parallel to EF. I say that AB is also parallel to CD.

For let the straight-line GK fall across (AB, CD, and EF).

And since the straight-line GK has fallen across the parallel straight-lines AB and EF, (angle) AGK (is) thus equal to GHF [Prop. 1.29]. Again, since the straight-line GK has fallen across the parallel straight-lines EF and CD, (angle) GHF is equal to GKD [Prop. 1.29]. But AGK was also shown (to be) equal to GHF. Thus, AGK is also equal to GKD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

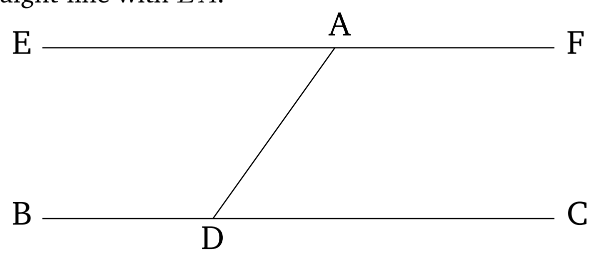
[Thus, (straight-lines) parallel to the same straight-line are also parallel to one another.] (Which is) the very thing it was required to show.

Proposition 31

To draw a straight-line parallel to a given straight-line, through a given point.

Let A be the given point, and BC the given straight-line. So it is required to draw a straight-line parallel to the straight-line BC, through the point A.

Let the point D have been taken a random on BC, and let AD have been joined. And let (angle) DAE, equal to angle ADC, have been constructed on the straight-line DA at the point A on it [Prop. 1.23]. And let the straight-line AF have been produced in a straight-line with EA.

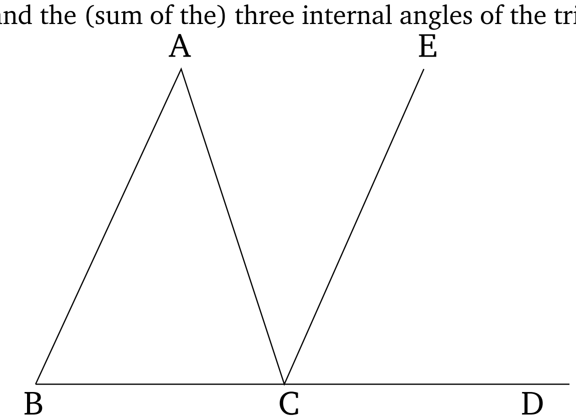


And since the straight-line AD, (in) falling across the two straight-lines BC and EF, has made the alternate angles EAD and ADC equal to one another, EAF is thus parallel to BC [Prop. 1.27].

Thus, the straight-line EAF has been drawn parallel to the given straight-line BC, through the given point A. (Which is) the very thing it was required to do.

Proposition 32

In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.



Let ABC be a triangle, and let one of its sides BC have been produced to D. I say that the external angle ACD is equal to the (sum of the) two internal and opposite angles CAB and ABC, and the (sum of the) three internal angles of the triangle—ABC, BCA, and CAB—is equal to two right-angles.

For let CE have been drawn through point C parallel to the straight-line AB [Prop. 1.31].

And since AB is parallel to CE, and AC has fallen across them, the alternate angles BAC and ACE are equal to one another [Prop. 1.29]. Again, since AB is parallel to CE, and the straight-line BD has fallen across them, the external angle ECD is equal to the internal and opposite (angle) ABC [Prop. 1.29]. But ACE was also shown (to be) equal to BAC. Thus, the whole angle ACD is equal to the (sum of the) two internal and opposite (angles) BAC and ABC.

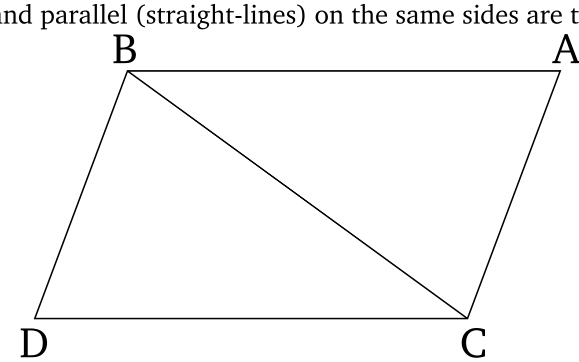
Let ACB have been added to both. Thus, (the sum of) ACD and ACB is equal to the (sum of the) three (angles)

ABC, BCA, and CAB. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ACB, CBA, and CAB is also equal to two right-angles.

Thus, in any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 33

Straight-lines joining equal and parallel (straight-lines) on the same sides are themselves also equal and parallel.



Let AB and CD be equal and parallel (straight-lines), and let the straight-lines AC and BD join them on the same sides. I say that AC and BD are also equal and parallel.

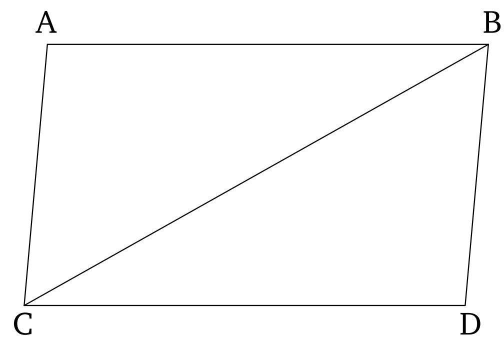
Let BC have been joined. And since AB is parallel to CD, and BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. And since AB is equal to CD, and BC is common, the two (straight-lines) AB, BC are equal to the two (straight-lines) DC, CB.[†]And the angle ABC is equal to the angle BCD. Thus, the base AC is equal to the base BD, and triangle ABC is equal to triangle DCB[‡], and the remaining angles will be equal to the corresponding remaining angles subtended by the equal sides [Prop. 1.4]. Thus, angle ACB is equal to CBD. Also, since the straight-line BC, (in) falling across the two straight-lines AC and BD, has made the alternate angles (ACB and CBD) equal to one another, AC is thus parallel to BD [Prop. 1.27]. And (AC) was also shown (to be) equal to (BD).

Thus, straight-lines joining equal and parallel (straight-lines) on the same sides are themselves also equal and parallel. (Which is) the very thing it was required to show. [†] The Greek text has "BC, CD", which is obviously a mistake.

[‡] The Greek text has "DCB", which is obviously a mistake.

Proposition 34

In parallelogrammic figures the opposite sides and angles are equal to one another, and a diagonal cuts them in half.



Let $ACDB$ be a parallelogrammic figure, and BC its diagonal. I say that for parallelogram $ACDB$, the opposite sides and angles are equal to one another, and the diagonal BC cuts it in half.

For since AB is parallel to CD, and the straight-line BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. Again, since AC is parallel to BD, and BC has fallen across them, the alternate angles ACB and CBD are equal to one another [Prop. 1.29]. So ABC and BCD are two triangles having the two angles ABC and BCA equal to the two (angles) BCD and CBD, respectively, and one side equal to one side—the (one) by the equal angles and common to them, (namely) BC. Thus, they will also have the remaining sides equal to the corresponding remaining (sides), and the remaining angle (equal) to the remaining angle [Prop. 1.26]. Thus, side AB is equal to CD, and AC to BD. Furthermore, angle BAC is equal to CDB. And since angle ABC is equal to BCD, and CBD to ACB, the whole (angle) ABD is thus equal to the whole (angle) ACD. And BAC was also shown (to be) equal to CDB.

Thus, in parallelogrammic figures the opposite sides and angles are equal to one another.

And, I also say that a diagonal cuts them in half. For since AB is equal to CD, and BC (is) common, the two (straight-lines) AB, BC are equal to the two (straight-lines) DC, CB†, respectively. And angle ABC is equal to angle BCD. Thus, the base AC (is) also equal to DB, and triangle ABC is equal to triangle BCD [Prop. 1.4].

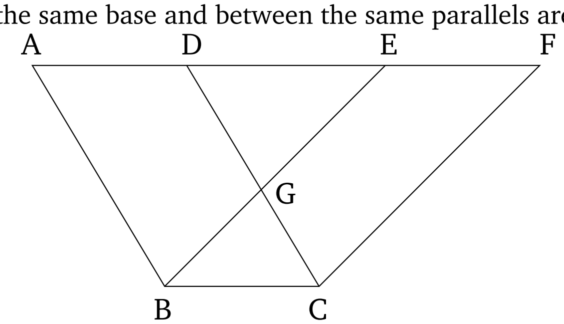
Thus, the diagonal BC cuts the parallelogram $ACDB$‡ in half. (Which is) the very thing it was required to show.

† The Greek text has "CD, BC", which is obviously a mistake.

‡ The Greek text has "$ABCD$", which is obviously a mistake.

Proposition 35

Parallelograms which are on the same base and between the same parallels are equal† to one another.



Let $ABCD$ and $EBCF$ be parallelograms on the same base BC, and between the same parallels AF and BC. I say that $ABCD$ is equal to parallelogram $EBCF$.

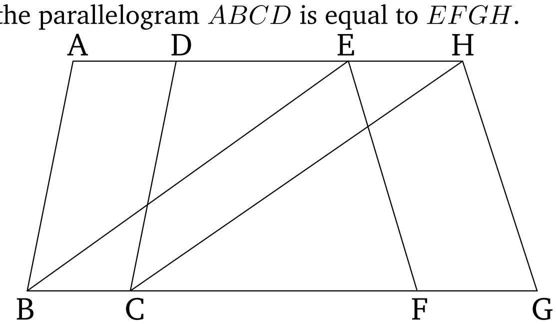
For since $ABCD$ is a parallelogram, AD is equal to BC [Prop. 1.34]. So, for the same (reasons), EF is also equal to BC. So AD is also equal to EF. And DE is common. Thus, the whole (straight-line) AE is equal to the whole (straight-line) DF. And AB is also equal to DC. So the two (straight-lines) EA, AB are equal to the two (straight-lines) FD, DC, respectively. And angle FDC is equal to angle EAB, the external to the internal [Prop. 1.29]. Thus, the base EB is equal to the base FC, and triangle EAB will be equal to triangle DFC [Prop. 1.4]. Let DGE have been taken away from both. Thus, the remaining trapezium $ABGD$ is equal to the remaining trapezium $EGCF$. Let triangle GBC have been added to both. Thus, the whole parallelogram $ABCD$ is equal to the whole parallelogram $EBCF$.

Thus, parallelograms which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show. † Here, for the first time, "equal" means "equal in area", rather than "congruent".

Proposition 36

Parallelograms which are on equal bases and between the same parallels are equal to one another.

Let $ABCD$ and $EFGH$ be parallelograms which are on the equal bases BC and FG, and (are) between the same parallels AH and BG. I say that the parallelogram $ABCD$ is equal to $EFGH$.

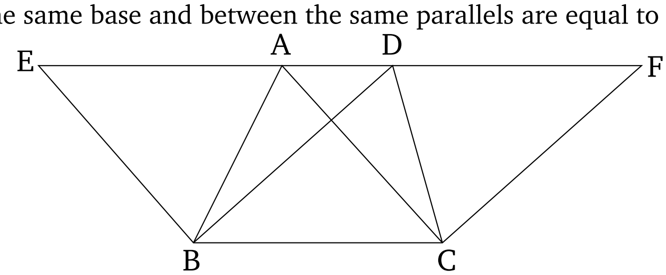


For let BE and CH have been joined. And since BC is equal to FG, but FG is equal to EH [Prop. 1.34], BC is thus equal to EH. And they are also parallel, and EB and HC join them. But (straight-lines) joining equal and parallel (straight-lines) on the same sides are (themselves) equal and parallel [Prop. 1.33] [thus, EB and HC are also equal and parallel]. Thus, $EBCH$ is a parallelogram [Prop. 1.34], and is equal to $ABCD$. For it has the same base, BC, as ($ABCD$), and is between the same parallels, BC and AH, as ($ABCD$) [Prop. 1.35]. So, for the same (reasons), $EFGH$ is also equal to the same (parallelogram) $EBCH$ [Prop. 1.34]. So that the parallelogram $ABCD$ is also equal to $EFGH$.

Thus, parallelograms which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Proposition 37

Triangles which are on the same base and between the same parallels are equal to one another.



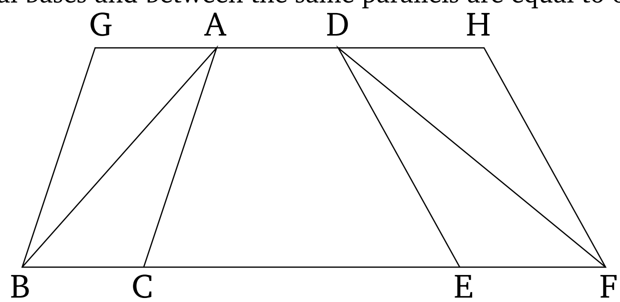
Let ABC and DBC be triangles on the same base BC, and between the same parallels AD and BC. I say that triangle ABC is equal to triangle DBC.

Let AD have been produced in both directions to E and F, and let the (straight-line) BE have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) CF have been drawn through C parallel to BD [Prop. 1.31]. Thus, $EBCA$ and $DBCF$ are both parallelograms, and are equal. For they are on the same base BC, and between the same parallels BC and EF [Prop. 1.35]. And the triangle ABC is half of the parallelogram $EBCA$. For the diagonal AB cuts the latter in half [Prop. 1.34]. And the triangle DBC (is) half of the parallelogram $DBCF$. For the diagonal DC cuts the latter in half [Prop. 1.34]. [And the halves of equal things are equal to one another.]† Thus, triangle ABC is equal to triangle DBC.

Thus, triangles which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show. † This is an additional common notion.

Proposition 38

Triangles which are on equal bases and between the same parallels are equal to one another.



Let ABC and DEF be triangles on the equal bases BC and EF, and between the same parallels BF and AD. I say that triangle ABC is equal to triangle DEF.

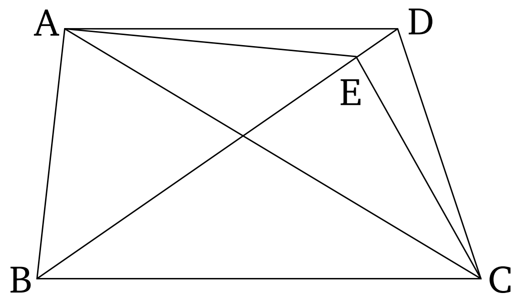
For let AD have been produced in both directions to G and H, and let the (straight-line) BG have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) FH have been drawn through F parallel to DE [Prop. 1.31]. Thus, $GBCA$ and $DEFH$ are each parallelograms. And $GBCA$ is equal to $DEFH$. For they are on the equal bases BC and EF, and between the same parallels BF and GH [Prop. 1.36]. And triangle ABC is half of the parallelogram $GBCA$. For the diagonal AB cuts the latter in half [Prop. 1.34]. And triangle FED (is) half of parallelogram $DEFH$. For the diagonal DF cuts the latter in half. [And the halves of equal things are equal to one another.] Thus, triangle ABC is equal to triangle DEF.

Thus, triangles which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Proposition 39

Equal triangles which are on the same base, and on the same side, are also between the same parallels.

Let ABC and DBC be equal triangles which are on the same base BC, and on the same side (of it). I say that they are also between the same parallels.



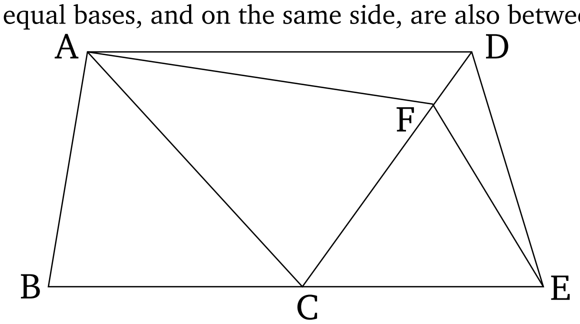
For let AD have been joined. I say that AD and BC are parallel.

For, if not, let AE have been drawn through point A parallel to the straight-line BC [Prop. 1.31], and let EC have been joined. Thus, triangle ABC is equal to triangle EBC. For it is on the same base as it, BC, and between the same parallels [Prop. 1.37]. But ABC is equal to DBC. Thus, DBC is also equal to EBC, the greater to the lesser. The very thing is impossible. Thus, AE is not parallel to BC. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BC.

Thus, equal triangles which are on the same base, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

Proposition 40†

Equal triangles which are on equal bases, and on the same side, are also between the same parallels.



Let ABC and CDE be equal triangles on the equal bases BC and CE (respectively), and on the same side (of BE). I say that they are also between the same parallels.

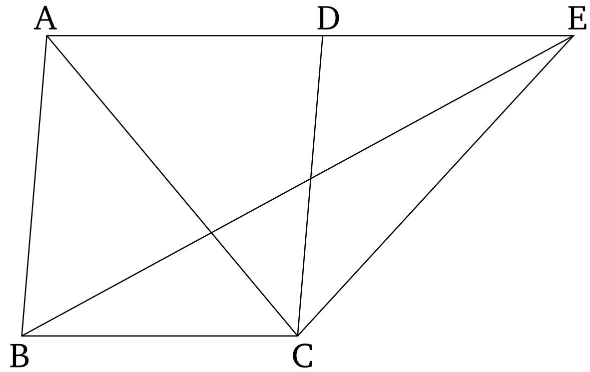
For let AD have been joined. I say that AD is parallel to BE.

For if not, let AF have been drawn through A parallel to BE [Prop. 1.31], and let FE have been joined. Thus, triangle ABC is equal to triangle FCE. For they are on equal bases, BC and CE, and between the same parallels, BE and AF [Prop. 1.38]. But, triangle ABC is equal to [triangle] DCE. Thus, [triangle] DCE is also equal to triangle FCE, the greater to the lesser. The very thing is impossible. Thus, AF is not parallel to BE. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BE.

Thus, equal triangles which are on equal bases, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show. † This whole proposition is regarded by Heiberg as a relatively early interpolation to the original text.

Proposition 41

If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.



For let parallelogram $ABCD$ have the same base BC as triangle EBC, and let it be between the same parallels, BC and AE. I say that parallelogram $ABCD$ is double (the area) of triangle BEC.

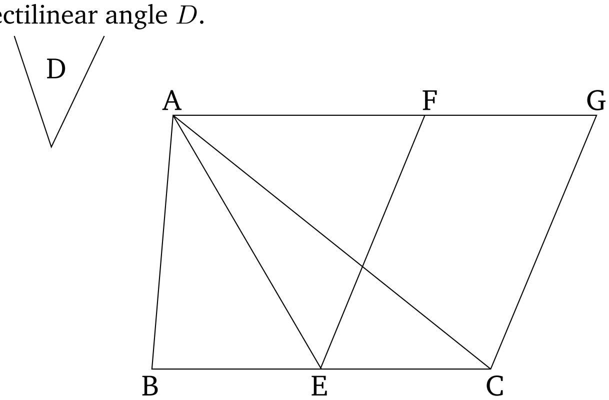
For let AC have been joined. So triangle ABC is equal to triangle EBC. For it is on the same base, BC, as (EBC), and between the same parallels, BC and AE [Prop. 1.37]. But, parallelogram $ABCD$ is double (the area) of triangle ABC. For the diagonal AC cuts the former in half [Prop. 1.34]. So parallelogram $ABCD$ is also double (the area) of triangle EBC.

Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

Proposition 42

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Let ABC be the given triangle, and D the given rectilinear angle. So it is required to construct a parallelogram equal to triangle ABC in the rectilinear angle D.



Let BC have been cut in half at E [Prop. 1.10], and let AE have been joined. And let (angle) CEF, equal to angle D, have been constructed at the point E on the straight-line EC [Prop. 1.23]. And let AG have been drawn through A parallel to EC [Prop. 1.31], and let CG have been drawn through C parallel to EF [Prop. 1.31]. Thus, $FECG$ is a parallelogram. And since BE is equal to EC, triangle ABE is also equal to triangle AEC. For they are on the equal bases, BE and EC, and between the same parallels, BC and AG [Prop. 1.38]. Thus, triangle ABC is double (the area) of triangle AEC. And parallelogram $FECG$ is also double (the area) of triangle AEC. For it has the same base as (AEC), and is between the same parallels as (AEC) [Prop. 1.41]. Thus, parallelogram $FECG$ is equal to triangle ABC. ($FECG$) also has the angle CEF equal to the given (angle) D.

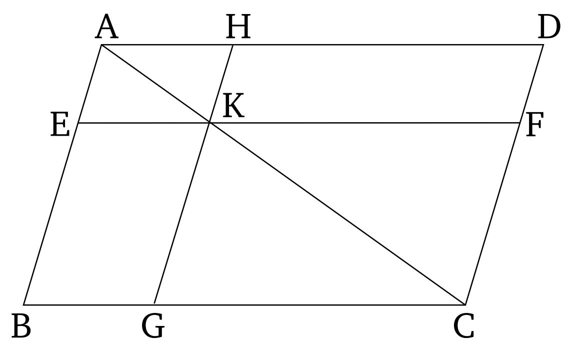
Thus, parallelogram $FECG$, equal to the given triangle ABC, has been constructed in the angle CEF, which is equal to D. (Which is) the very thing it was required to do.

Proposition 43

For any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.

Let $ABCD$ be a parallelogram, and AC its diagonal. And let EH and FG be the parallelograms about AC, and BK and KD the so-called complements (about AC). I say that the complement BK is equal to the complement KD.

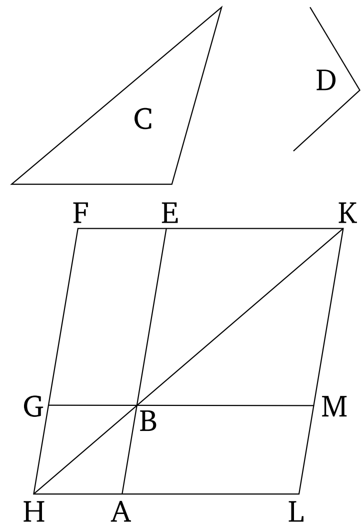
For since $ABCD$ is a parallelogram, and AC its diagonal, triangle ABC is equal to triangle ACD [Prop. 1.34]. Again, since EH is a parallelogram, and AK is its diagonal, triangle AEK is equal to triangle AHK [Prop. 1.34]. So, for the same (reasons), triangle KFC is also equal to (triangle) KGC. Therefore, since triangle AEK is equal to triangle AHK, and KFC to KGC, triangle AEK plus KGC is equal to triangle AHK plus KFC. And the whole triangle ABC is also equal to the whole (triangle) ADC. Thus, the remaining complement BK is equal to the remaining complement KD.



Thus, for any parallelogramic figure, the complements of the parallelograms about the diagonal are equal to one another. (Which is) the very thing it was required to show.

Proposition 44

To apply a parallelogram equal to a given triangle to a given straight-line in a given rectilinear angle.



Let AB be the given straight-line, C the given triangle, and D the given rectilinear angle. So it is required to apply a parallelogram equal to the given triangle C to the given straight-line AB in an angle equal to (angle) D.

Let the parallelogram $BEFG$, equal to the triangle C, have been constructed in the angle EBG, which is equal to D [Prop. 1.42]. And let it have been placed so that BE is straight-on to AB.† And let FG have been drawn through to H, and let AH have been drawn through A parallel to either of BG or EF [Prop. 1.31], and let HB have been joined. And since the straight-line HF falls across the parallels AH and EF, the (sum of the) angles AHF and HFE is thus equal to two right-angles [Prop. 1.29]. Thus, (the sum of) BHG and GFE is less than two right-angles. And (straight-lines) produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced, HB and FE will meet together. Let them have been produced, and let them meet together at K. And let KL have been drawn through point K parallel to either of EA or FH [Prop. 1.31]. And let HA and GB have been produced to points L and M (respectively). Thus, $HLKF$ is a parallelogram, and HK its diagonal. And AG and ME (are) parallelograms, and LB and BF the so-called complements, about HK. Thus, LB is equal to BF [Prop. 1.43]. But, BF is equal to triangle C. Thus, LB is also equal to C. Also, since angle GBE is equal to ABM [Prop. 1.15], but GBE is equal to D, ABM is thus also equal to angle D.

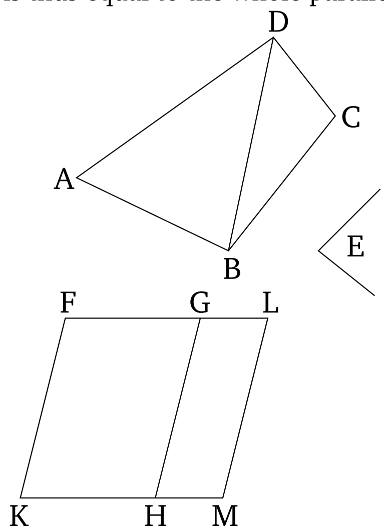
Thus, the parallelogram LB, equal to the given triangle C, has been applied to the given straight-line AB in the angle ABM, which is equal to D. (Which is) the very thing it was required to do. † This can be achieved using Props. 1.3, 1.23, and 1.31.

Proposition 45

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Let $ABCD$ be the given rectilinear figure,† and E the given rectilinear angle. So it is required to construct a parallelogram equal to the rectilinear figure $ABCD$ in the given angle E.

Let DB have been joined, and let the parallelogram FH, equal to the triangle ABD, have been constructed in the angle HKF, which is equal to E [Prop. 1.42]. And let the parallelogram GM, equal to the triangle DBC, have been applied to the straight-line GH in the angle GHM, which is equal to E [Prop. 1.44]. And since angle E is equal to each of (angles) HKF and GHM, (angle) HKF is thus also equal to GHM. Let KHG have been added to both. Thus, (the sum of) FKH and KHG is equal to (the sum of) KHG and GHM. But, (the sum of) FKH and KHG is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) KHG and GHM is also equal to two right-angles. So two straight-lines, KH and HM, not lying on the same side, make adjacent angles with some straight-line GH, at the point H on it, (whose sum is) equal to two right-angles. Thus, KH is straight-on to HM [Prop. 1.14]. And since the straight-line HG falls across the parallels KM and FG, the alternate angles MHG and HGF are equal to one another [Prop. 1.29]. Let HGL have been added to both. Thus, (the sum of) MHG and HGL is equal to (the sum of) HGF and HGL. But, (the sum of) MHG and HGL is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) HGF and HGL is also equal to two right-angles. Thus, FG is straight-on to GL [Prop. 1.14]. And since FK is equal and parallel to HG [Prop. 1.34], but also HG to ML [Prop. 1.34], KF is thus also equal and parallel to ML [Prop. 1.30]. And the straight-lines KM and FL join them. Thus, KM and FL are equal and parallel as well [Prop. 1.33]. Thus, $KFLM$ is a parallelogram. And since triangle ABD is equal to parallelogram FH, and DBC to GM, the whole rectilinear figure $ABCD$ is thus equal to the whole parallelogram $KFLM$.



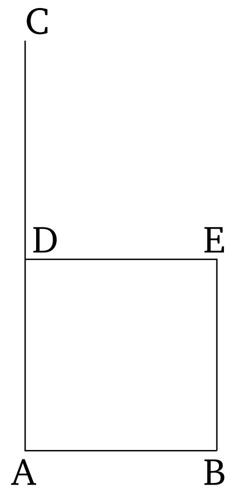
Thus, the parallelogram $KFLM$, equal to the given rectilinear figure $ABCD$, has been constructed in the angle FKM, which is equal to the given (angle) E. (Which is) the very thing it was required to do. † The proof is only given for a four-sided figure. However, the extension to many-sided figures is trivial.

Proposition 46

To describe a square on a given straight-line.

Let AB be the given straight-line. So it is required to describe a square on the straight-line AB.

Let AC have been drawn at right-angles to the straight-line AB from the point A on it [Prop. 1.11], and let AD have been made equal to AB [Prop. 1.3]. And let DE have been drawn through point D parallel to AB [Prop. 1.31], and let BE have been drawn through point B parallel to AD [Prop. 1.31]. Thus, $ADEB$ is a parallelogram. Therefore, AB is equal to DE, and AD to BE [Prop. 1.34]. But, AB is equal to AD. Thus, the four (sides) BA, AD, DE, and EB are equal to one another. Thus, the parallelogram $ADEB$ is equilateral. So I say that (it is) also right-angled. For since the straight-line AD falls across the parallels AB and DE, the (sum of the) angles BAD and ADE is equal to two right-angles [Prop. 1.29]. But BAD (is a) right-angle. Thus, ADE (is) also a right-angle. And for parallelogrammic figures, the opposite sides and angles are equal to one another [Prop. 1.34]. Thus, each of the opposite angles ABE and BED (are) also right-angles. Thus, $ADEB$ is right-angled. And it was also shown (to be) equilateral.



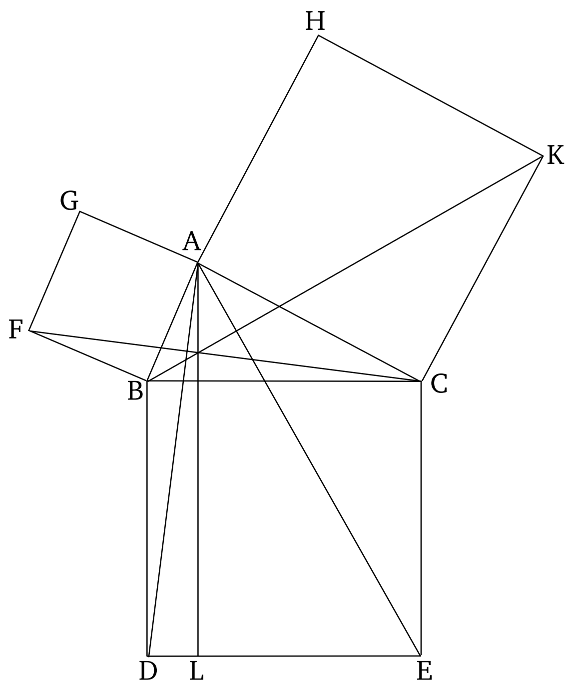
Thus, ($ADEB$) is a square [Def. 1.22]. And it is described on the straight-line AB. (Which is) the very thing it was required to do.

Proposition 47

In right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides containing the right-angle.

Let ABC be a right-angled triangle having the angle BACa right-angle. I say that the square on BC is equal to the (sum of the) squares on BA and AC.

For let the square $BDEC$ have been described on BC, and (the squares) GB and HC on AB and AC (respectively) [Prop. 1.46]. And let AL have been drawn through point A parallel to either of BD or CE [Prop. 1.31]. And let AD and FC have been joined. And since angles BAC and BAG are each right-angles, then two straight-lines AC and AG, not lying on the same side, make the adjacent angles with some straight-line BA, at the point A on it, (whose sum is) equal to two right-angles. Thus, CA is straight-on to AG [Prop. 1.14]. So, for the same (reasons), BA is also straight-on to AH. And since angle DBC is equal to FBA, for (they are) both right-angles, let ABC have been added to both. Thus, the whole (angle) DBA is equal to the whole (angle) FBC. And since DB is equal to BC, and FB to BA, the two (straight-lines) DB, BA are equal to the two (straight-lines) CB, BF,† respectively. And angle DBA (is) equal to angle FBC. Thus, the base AD [is] equal to the base FC, and the triangle ABD is equal to the triangle FBC [Prop. 1.4]. And parallelogram BL [is] double (the area) of triangle ABD. For they have the same base, BD, and are between the same parallels, BD and AL [Prop. 1.41]. And square GB is double (the area) of triangle FBC. For again they have the same base, FB, and are between the same parallels, FB and GC [Prop. 1.41]. [And the doubles of equal things are equal to one another.]‡ Thus, the parallelogram BL is also equal to the square GB. So, similarly, AE and BK being joined, the parallelogram CL can be shown (to be) equal to the square HC. Thus, the whole square $BDEC$ is equal to the (sum of the) two squares GB and HC. And the square $BDEC$ is described on BC, and the (squares) GB and HC on BA and AC (respectively). Thus, the square on the side BC is equal to the (sum of the) squares on the sides BA and AC.

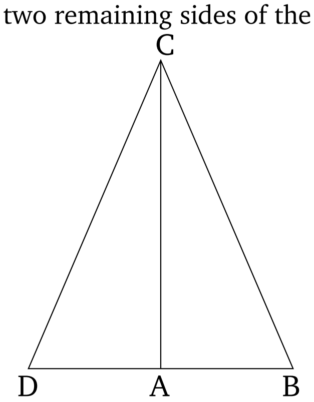


Thus, in right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-[angle]. (Which is) the very thing it was required to show. † The Greek text has "FB, BC", which is obviously a mistake.

‡ This is an additional common notion.

Proposition 48

If the square on one of the sides of a triangle is equal to the (sum of the) squares on the two remaining sides of the triangle then the angle contained by the two remaining sides of the triangle is a right-angle.



For let the square on one of the sides, BC, of triangle ABC be equal to the (sum of the) squares on the sides BA and AC. I say that angle BAC is a right-angle.

For let AD have been drawn from point A at right-angles to the straight-line AC [Prop. 1.11], and let AD have been made equal to BA [Prop. 1.3], and let DC have been joined. Since DA is equal to AB, the square on DA is thus also equal to the square on AB.† Let the square on AC have been added to both. Thus, the (sum of the) squares on DA and AC is equal to the (sum of the) squares on BA and AC. But, the (square) on DC is equal to the (sum of the squares) on DA and AC. For angle DAC is a right-angle [Prop. 1.47]. But, the (square) on BC is equal to (sum of the squares) on BA and AC. For (that) was assumed. Thus, the square on DC is equal to the square on BC. So side DC is also equal to (side) BC. And since DA is equal to AB, and AC (is) common, the two (straight-lines) DA, AC are equal to the two (straight-lines) BA, AC. And the base DC is equal to the base BC. Thus, angle DAC [is] equal to angle BAC [Prop. 1.8]. But DAC is a right-angle. Thus, BAC is also a right-angle.

Thus, if the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining two sides of the triangle then the angle contained by the remaining two sides of the triangle is a right-angle. (Which is) the very thing it was required to show. † Here, use is made of the additional common notion that the squares of equal things are themselves equal. Later on, the inverse notion is used.

ELEMENTS BOOK 2

Fundamentals of Geometric Algebra

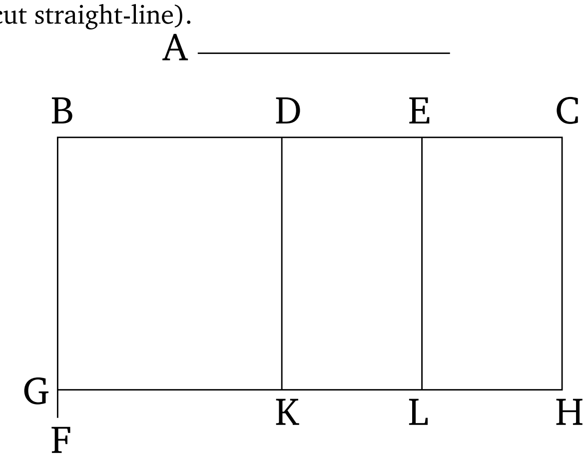
Definitions

1. Any rectangular parallelogram is said to be contained by the two straight-lines containing the right-angle.

2. And in any parallelogrammic figure, let any one whatsoever of the parallelograms about its diagonal, (taken) with its two complements, be called a gnomon.

Proposition 1†

If there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line).



Let A and BC be the two straight-lines, and let BC be cut, at random, at points D and E. I say that the rectangle contained by A and BC is equal to the rectangle(s) contained by A and BD, by A and DE, and, finally, by A and EC.

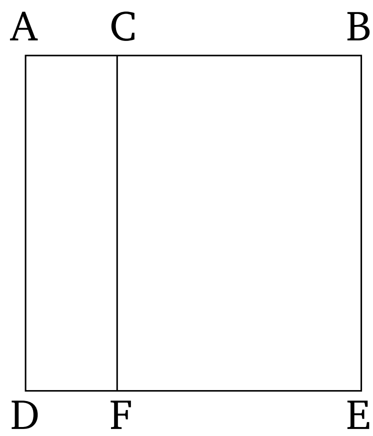
For let BF have been drawn from point B, at right-angles to BC [Prop. 1.11], and let BG be made equal to A [Prop. 1.3], and let GH have been drawn through (point) G, parallel to BC [Prop. 1.31], and let DK, EL, and CH have been drawn through (points) D, E, and C (respectively), parallel to BG [Prop. 1.31].

So the (rectangle) BH is equal to the (rectangles) BK, DL, and EH. And BH is the (rectangle contained) by A and BC. For it is contained by GB and BC, and BG (is) equal to A. And BK (is) the (rectangle contained) by A and BD. For it is contained by GB and BD, and BG (is) equal to A. And DL (is) the (rectangle contained) by A and DE. For DK, that is to say BG [Prop. 1.34], (is) equal to A. Similarly, EH (is) also the (rectangle contained) by A and EC. Thus, the (rectangle contained) by A and BC is equal to the (rectangles contained) by A and BD, by A and DE, and, finally, by A and EC.

Thus, if there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line). (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $a\,(b + c + d + \cdots) = a\,b + a\,c + a\,d + \cdots$.

Proposition 2†

If a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole.



For let the straight-line AB have been cut, at random, at point C. I say that the rectangle contained by AB and BC, plus the rectangle contained by BA and AC, is equal to the square on AB.

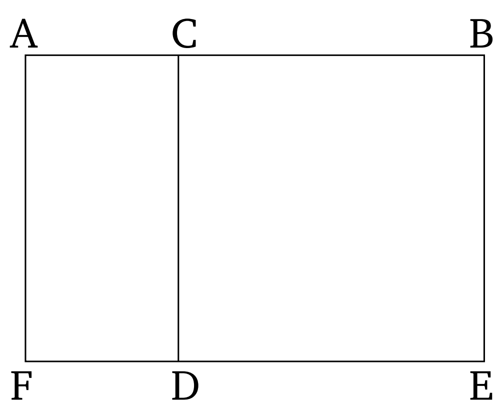
For let the square $ADEB$ have been described on AB [Prop. 1.46], and let CF have been drawn through C, parallel to either of AD or BE [Prop. 1.31].

So the (square) AE is equal to the (rectangles) AF and CE. And AE is the square on AB. And AF (is) the rectangle contained by the (straight-lines) BA and AC. For it is contained by DA and AC, and AD (is) equal to AB. And CE (is) the (rectangle contained) by AB and BC. For BE (is) equal to AB. Thus, the (rectangle contained) by BA and AC, plus the (rectangle contained) by AB and BC, is equal to the square on AB.

Thus, if a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole. (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $a\,b + a\,c = a^2$ if $a = b + c$.

Proposition 3†

If a straight-line is cut at random then the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both of) the pieces, and the square on the aforementioned piece.



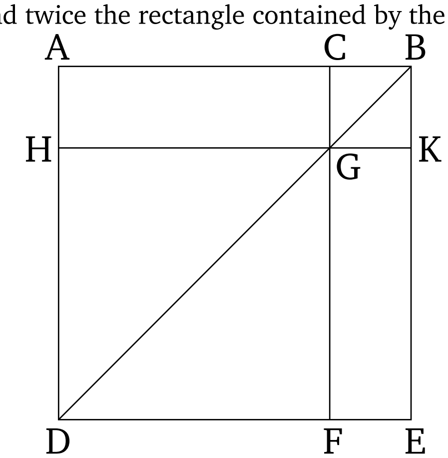
For let the straight-line AB have been cut, at random, at (point) C. I say that the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

For let the square $CDEB$ have been described on CB [Prop. 1.46], and let ED have been drawn through to F, and let AF have been drawn through A, parallel to either of CD or BE [Prop. 1.31]. So the (rectangle) AE is equal to the (rectangle) AD and the (square) CE. And AE is the rectangle contained by AB and BC. For it is contained by AB and BE, and BE (is) equal to BC. And AD (is) the (rectangle contained) by AC and CB. For DC (is) equal to CB. And DB (is) the square on CB. Thus, the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

Thus, if a straight-line is cut at random then the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both of) the pieces, and the square on the aforementioned piece. (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $(a+b)\,a = a\,b + a^2$.

Proposition 4†

If a straight-line is cut at random then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces.



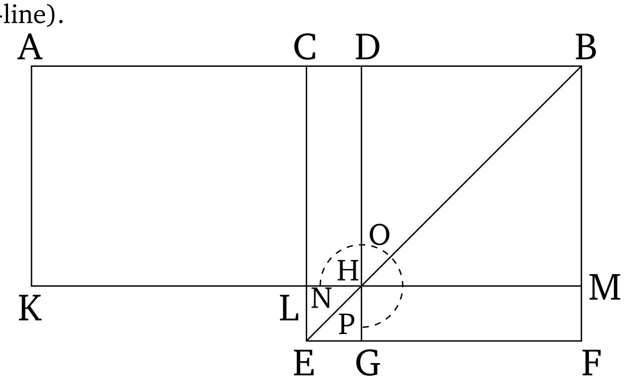
For let the straight-line AB have been cut, at random, at (point) C. I say that the square on AB is equal to the (sum of the) squares on AC and CB, and twice the rectangle contained by AC and CB.

For let the square $ADEB$ have been described on AB [Prop. 1.46], and let BD have been joined, and let CF have been drawn through C, parallel to either of AD or EB [Prop. 1.31], and let HK have been drawn through G, parallel to either of AB or DE [Prop. 1.31]. And since CF is parallel to AD, and BD has fallen across them, the external angle CGB is equal to the internal and opposite (angle) ADB [Prop. 1.29]. But, ADB is equal to ABD, since the side BA is also equal to AD [Prop. 1.5]. Thus, angle CGB is also equal to GBC. So the side BC is equal to the side CG [Prop. 1.6]. But, CB is equal to GK, and CG to KB [Prop. 1.34]. Thus, GK is also equal to KB. Thus, $CGKB$ is equilateral. So I say that (it is) also right-angled. For since CG is parallel to BK [and the straight-line CB has fallen across them], the angles KBC and GCB are thus equal to two right-angles [Prop. 1.29]. But KBC (is) a right-angle. Thus, BCG (is) also a right-angle. So the opposite (angles) CGK and GKB are also right-angles [Prop. 1.34]. Thus, $CGKB$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square. And it is on CB. So, for the same (reasons), HF is also a square. And it is on HG, that is to say [on] AC [Prop. 1.34]. Thus, the squares HF and KC are on AC and CB (respectively). And the (rectangle) AG is equal to the (rectangle) GE [Prop. 1.43]. And AG is the (rectangle contained) by AC and CB. For GC (is) equal to CB. Thus, GE is also equal to the (rectangle contained) by AC and CB. Thus, the (rectangles) AG and GE are equal to twice the (rectangle contained) by AC and CB. And HF and CK are the squares on AC and CB (respectively). Thus, the four (figures) HF, CK, AG, and GE are equal to the (sum of the) squares on AC and BC, and twice the rectangle contained by AC and CB. But, the (figures) HF, CK, AG, and GE are (equivalent to) the whole of $ADEB$, which is the square on AB. Thus, the square on AB is equal to the (sum of the) squares on AC and CB, and twice the rectangle contained by AC and CB.

Thus, if a straight-line is cut at random then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces. (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $(a+b)^2 = a^2 + b^2 + 2\,a\,b$.

Proposition 5‡

If a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line).



For let any straight-line AB have been cut—equally at C, and unequally at D. I say that the rectangle contained by AD and DB, plus the square on CD, is equal to the square on CB.

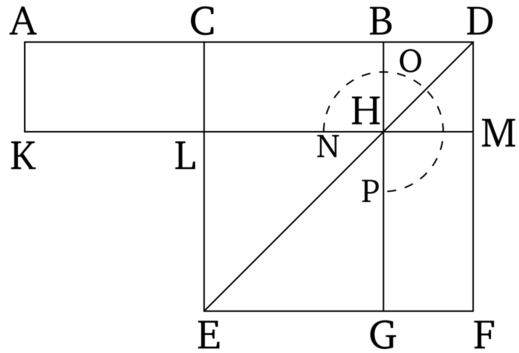
For let the square $CEFB$ have been described on CB [Prop. 1.46], and let BE have been joined, and let DG have been drawn through D, parallel to either of CE or BF [Prop. 1.31], and again let KM have been drawn through H, parallel to either of AB or EF [Prop. 1.31], and again let AK have been drawn through A, parallel to either of CL or BM [Prop. 1.31]. And since the complement CH is equal to the complement HF [Prop. 1.43], let the (square) DM have been added to both. Thus, the whole (rectangle) CM is equal to the whole (rectangle) DF. But, (rectangle) CM is equal to (rectangle) AL, since AC is also equal to CB [Prop. 1.36]. Thus, (rectangle) AL is also equal to (rectangle) DF. Let (rectangle) CH have been added to both. Thus, the whole (rectangle) AH is equal to the gnomon NOP. But, AH is the (rectangle contained) by AD and DB. For DH (is) equal to DB. Thus, the gnomon NOP is also equal to the (rectangle contained) by AD and DB. Let LG, which is equal to the (square) on CD, have been added to both. Thus, the gnomon NOP and the (square) LG are equal to the rectangle contained by AD and DB, and the square on CD. But, the gnomon NOP and the (square) LG is (equivalent to) the whole square $CEFB$, which is on CB. Thus, the rectangle contained by AD and DB, plus the square on CD, is equal to the square on CB.

Thus, if a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line). (Which is) the very thing it was required to show. † Note the (presumably mistaken) double use of the label M in the Greek text.

‡ This proposition is a geometric version of the algebraic identity: $a\,b + [(a+b)/2 - b]^2 = [(a+b)/2]^2$.

Proposition 6†

If a straight-line is cut in half, and any straight-line added to it straight-on, then the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added.



For let any straight-line AB have been cut in half at point C, and let any straight-line BD have been added to it straight-on. I say that the rectangle contained by AD and DB, plus the square on CB, is equal to the square on CD.

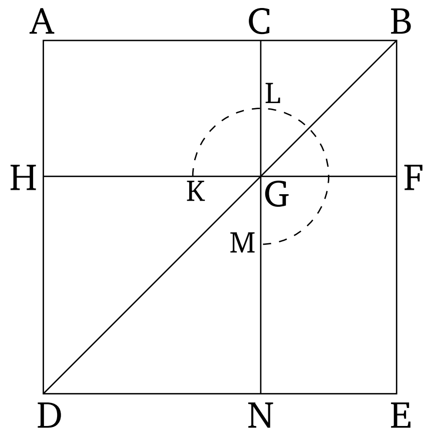
For let the square $CEFD$ have been described on CD [Prop. 1.46], and let DE have been joined, and let BG have been drawn through point B, parallel to either of EC or DF [Prop. 1.31], and let KM have been drawn through point H, parallel to either of AB or EF [Prop. 1.31], and finally let AK have been drawn through A, parallel to either of CL or DM [Prop. 1.31].

Therefore, since AC is equal to CB, (rectangle) AL is also equal to (rectangle) CH [Prop. 1.36]. But, (rectangle) CH is equal to (rectangle) HF [Prop. 1.43]. Thus, (rectangle) AL is also equal to (rectangle) HF. Let (rectangle) CM have been added to both. Thus, the whole (rectangle) AM is equal to the gnomon NOP. But, AM is the (rectangle contained) by AD and DB. For DM is equal to DB. Thus, gnomon NOP is also equal to the [rectangle contained] by AD and DB. Let LG, which is equal to the square on BC, have been added to both. Thus, the rectangle contained by AD and DB, plus the square on CB, is equal to the gnomon NOP and the (square) LG. But the gnomon NOP and the (square) LG is (equivalent to) the whole square $CEFD$, which is on CD. Thus, the rectangle contained by AD and DB, plus the square on CB, is equal to the square on CD.

Thus, if a straight-line is cut in half, and any straight-line added to it straight-on, then the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added. (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $(2\,a+b)\,b+a^2=(a+b)^2$.

Proposition 7†

If a straight-line is cut at random then the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece.



For let any straight-line AB have been cut, at random, at point C. I say that the (sum of the) squares on AB and BC is equal to twice the rectangle contained by AB and BC, and the square on CA.

For let the square $ADEB$ have been described on AB [Prop. 1.46], and let the (rest of) the figure have been drawn.

Therefore, since (rectangle) AG is equal to (rectangle) GE [Prop. 1.43], let the (square) CF have been added to both. Thus, the whole (rectangle) AF is equal to the whole (rectangle) CE. Thus, (rectangle) AF plus (rectangle) CE is double (rectangle) AF. But, (rectangle) AF plus (rectangle) CE is the gnomon KLM, and the square CF. Thus, the gnomon KLM, and the square CF, is double the (rectangle) AF. But double the (rectangle) AF is also twice the (rectangle contained) by AB and BC. For BF (is) equal to BC. Thus, the gnomon KLM, and the square CF, are equal to twice the (rectangle contained) by AB and BC. Let DG, which is the square on AC, have been added to both. Thus, the gnomon KLM, and the squares BG and GD, are equal to twice the rectangle contained by AB and BC, and the square on AC. But, the gnomon KLM and the squares BG and GD is (equivalent to) the whole of $ADEB$ and CF, which are the squares on AB and BC (respectively). Thus, the (sum of the) squares on AB and BC is equal to twice the rectangle contained by AB and BC, and the square on AC.

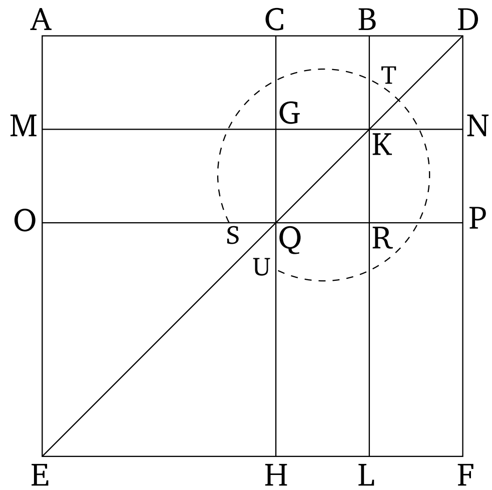
Thus, if a straight-line is cut at random then the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece. (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $(a+b)^2 + a^2 = 2\,(a+b)\,a + b^2$.

Proposition 8†

If a straight-line is cut at random then four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line).

For let any straight-line AB have been cut, at random, at point C. I say that four times the rectangle contained by AB and BC, plus the square on AC, is equal to the square described on AB and BC, as on one (complete straight-line).

For let BD have been produced in a straight-line [with the straight-line AB], and let BD be made equal to CB [Prop. 1.3], and let the square $AEFD$ have been described on AD [Prop. 1.46], and let the (rest of the) figure have been drawn double.



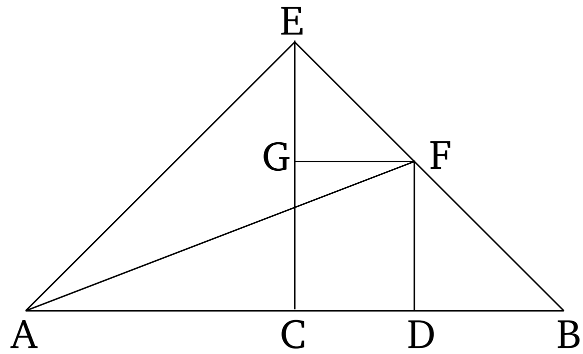
Therefore, since CB is equal to BD, but CB is equal to GK [Prop. 1.34], and BD to KN [Prop. 1.34], GK is thus also equal to KN. So, for the same (reasons), QR is equal to RP. And since BC is equal to BD, and GK to KN, (square) CK is thus also equal to (square) KD, and (square) GR to (square) RN [Prop. 1.36]. But, (square) CK is equal to (square) RN. For (they are) complements in the parallelogram CP [Prop. 1.43]. Thus, (square) KD is also equal to (square) GR. Thus, the four (squares) DK, CK, GR, and RN are equal to one another. Thus, the four (taken together) are quadruple (square) CK. Again, since CB is equal to BD, but BD (is) equal to BK—that is to say, CG—and CB is equal to GK—that is to say, GQ—CG is thus also equal to GQ. And since CG is equal to GQ, and QR to RP, (rectangle) AG is also equal to (rectangle) MQ, and (rectangle) QL to (rectangle) RF [Prop. 1.36]. But, (rectangle) MQ is equal to (rectangle) QL. For (they are) complements in the parallelogram ML [Prop. 1.43]. Thus, (rectangle) AG is also equal to (rectangle) RF. Thus, the four (rectangles) AG, MQ, QL, and RF are equal to one another. Thus, the four (taken together) are quadruple (rectangle) AG. And it was also shown that the four (squares) CK, KD, GR, and RN (taken together are) quadruple (square) CK. Thus, the eight (figures taken together), which comprise the gnomon STU, are quadruple (rectangle) AK. And since AK is the (rectangle contained) by AB and BD, for BK (is) equal to BD, four times the (rectangle contained) by AB and BD is quadruple (rectangle) AK. But the gnomon STU was also shown (to be equal to) quadruple (rectangle) AK. Thus, four times the (rectangle contained) by AB and BD is equal to the gnomon STU. Let OH, which is equal to the square on AC, have been added to both. Thus, four times the rectangle contained by AB and BD, plus the square on AC, is equal to the gnomon STU, and the (square) OH. But, the gnomon STU and the (square) OH is (equivalent to) the whole square $AEFD$, which is on AD. Thus, four times the (rectangle contained) by AB and BD, plus the (square) on AC, is equal to the square on AD. And BD (is) equal to BC. Thus, four times the rectangle contained by AB and BC, plus the square on AC, is equal to the (square) on AD, that is to say the square described on AB and BC, as on one (complete straight-line).

Thus, if a straight-line is cut at random then four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line). (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $4\,(a+b)\,a+b^2=[(a+b)+a]^2$.

Proposition 9†

If a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces.



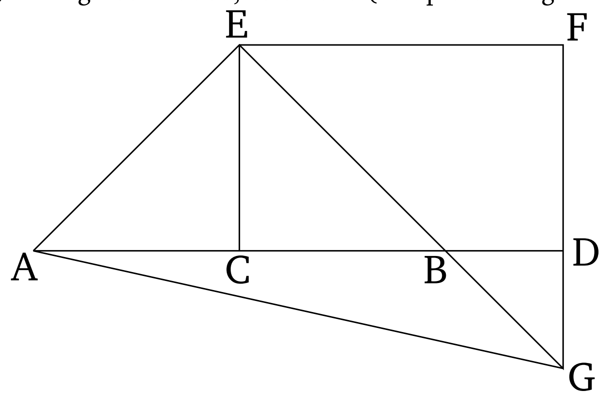
For let any straight-line AB have been cut—equally at C, and unequally at D. I say that the (sum of the) squares on AD and DB is double the (sum of the squares) on AC and CD.

For let CE have been drawn from (point) C, at right-angles to AB [Prop. 1.11], and let it be made equal to each of AC and CB [Prop. 1.3], and let EA and EB have been joined. And let DF have been drawn through (point) D, parallel to EC [Prop. 1.31], and (let) FG (have been drawn) through (point) F, (parallel) to AB [Prop. 1.31]. And let AF have been joined. And since AC is equal to CE, the angle EAC is also equal to the (angle) AEC [Prop. 1.5]. And since the (angle) at C is a right-angle, the (sum of the) remaining angles (of triangle AEC), EAC and AEC, is thus equal to one right-angle [Prop. 1.32]. And they are equal. Thus, (angles) CEA and CAE are each half a right-angle. So, for the same (reasons), (angles) CEB and EBC are also each half a right-angle. Thus, the whole (angle) AEB is a right-angle. And since GEF is half a right-angle, and EGF (is) a right-angle—for it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) EFG is thus half a right-angle [Prop. 1.32]. Thus, angle GEF [is] equal to EFG. So the side EG is also equal to the (side) GF [Prop. 1.6]. Again, since the angle at B is half a right-angle, and (angle) FDB (is) a right-angle—for again it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) BFD is half a right-angle [Prop. 1.32]. Thus, the angle at B (is) equal to DFB. So the side FD is also equal to the side DB [Prop. 1.6]. And since AC is equal to CE, the (square) on AC (is) also equal to the (square) on CE. Thus, the (sum of the) squares on AC and CE is double the (square) on AC. And the square on EA is equal to the (sum of the) squares on AC and CE. For angle ACE (is) a right-angle [Prop. 1.47]. Thus, the (square) on EA is double the (square) on AC. Again, since EG is equal to GF, the (square) on EG (is) also equal to the (square) on GF. Thus, the (sum of the squares) on EG and GF is double the square on GF. And the square on EF is equal to the (sum of the) squares on EG and GF [Prop. 1.47]. Thus, the square on EF is double the (square) on GF. And GF (is) equal to CD [Prop. 1.34]. Thus, the (square) on EF is double the (square) on CD. And the (square) on EA is also double the (square) on AC. Thus, the (sum of the) squares on AE and EF is double the (sum of the) squares on AC and CD. And the square on AF is equal to the (sum of the squares) on AE and EF. For the angle AEF is a right-angle [Prop. 1.47]. Thus, the square on AF is double the (sum of the squares) on AC and CD. And the (sum of the squares) on AD and DF (is) equal to the (square) on AF. For the angle at D is a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AD and DF is double the (sum of the) squares on AC and CD. And DF (is) equal to DB. Thus, the (sum of the) squares on AD and DB is double the (sum of the) squares on AC and CD.

Thus, if a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces. (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $a^2 + b^2 = 2[([a+b]/2)^2 + ([a+b]/2 - b)^2]$.

Proposition 10†

If a straight-line is cut in half, and any straight-line added to it straight-on, then the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line).



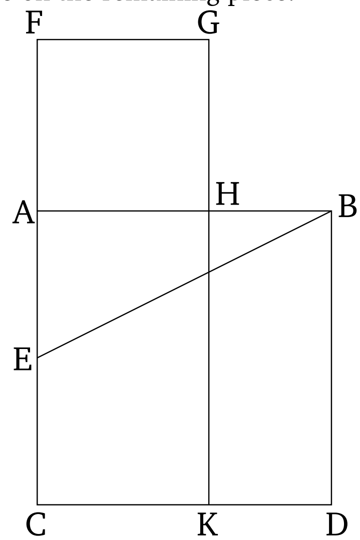
For let any straight-line AB have been cut in half at (point) C, and let any straight-line BD have been added to it straight-on. I say that the (sum of the) squares on AD and DB is double the (sum of the) squares on AC and CD.

For let CE have been drawn from point C, at right-angles to AB [Prop. 1.11], and let it be made equal to each of AC and CB [Prop. 1.3], and let EA and EB have been joined. And let EF have been drawn through E, parallel to AD [Prop. 1.31], and let FD have been drawn through D, parallel to CE [Prop. 1.31]. And since some straight-line EF falls across the parallel straight-lines EC and FD, the (internal angles) CEF and EFD are thus equal to two right-angles [Prop. 1.29]. Thus, FEB and EFD are less than two right-angles. And (straight-lines) produced from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced in the direction of B and D, the (straight-lines) EB and FD will meet. Let them have been produced, and let them meet together at G, and let AG have been joined. And since AC is equal to CE, angle EAC is also equal to (angle) AEC [Prop. 1.5]. And the (angle) at C (is) a right-angle. Thus, EAC and AEC [are] each half a right-angle [Prop. 1.32]. So, for the same (reasons), CEB and EBC are also each half a right-angle. Thus, (angle) AEB is a right-angle. And since EBC is half a right-angle, DBG (is) thus also half a right-angle [Prop. 1.15]. And BDG is also a right-angle. For it is equal to DCE. For (they are) alternate (angles) [Prop. 1.29]. Thus, the remaining (angle) DGB is half a right-angle. Thus, DGB is equal to DBG. So side BD is also equal to side GD [Prop. 1.6]. Again, since EGF is half a right-angle, and the (angle) at F (is) a right-angle, for it is equal to the opposite (angle) at C [Prop. 1.34], the remaining (angle) FEG is thus half a right-angle. Thus, angle EGF (is) equal to FEG. So the side GF is also equal to the side EF [Prop. 1.6]. And since [EC is equal to CA] the square on EC is [also] equal to the square on CA. Thus, the (sum of the) squares on EC and CA is double the square on CA. And the (square) on EA is equal to the (sum of the squares) on EC and CA [Prop. 1.47]. Thus, the square on EA is double the square on AC. Again, since FG is equal to EF, the (square) on FG is also equal to the (square) on FE. Thus, the (sum of the squares) on GF and FE is double the (square) on EF. And the (square) on EG is equal to the (sum of the squares) on GF and FE [Prop. 1.47]. Thus, the (square) on EG is double the (square) on EF. And EF (is) equal to CD [Prop. 1.34]. Thus, the square on EG is double the (square) on CD. But it was also shown that the (square) on EA (is) double the (square) on AC. Thus, the (sum of the) squares on AE and EG is double the (sum of the) squares on AC and CD. And the square on AG is equal to the (sum of the) squares on AE and EG [Prop. 1.47]. Thus, the (square) on AG is double the (sum of the squares) on AC and CD. And the (sum of the squares) on AD and DG is equal to the (square) on AG [Prop. 1.47]. Thus, the (sum of the) [squares] on AD and DG is double the (sum of the) [squares] on AC and CD. And DG (is) equal to DB. Thus, the (sum of the) [squares] on AD and DB is double the (sum of the) squares on AC and CD.

Thus, if a straight-line is cut in half, and any straight-line added to it straight-on, then the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line). (Which is) the very thing it was required to show. † This proposition is a geometric version of the algebraic identity: $(2\,a+b)^2+b^2=2\,[a^2+(a+b)^2]$.

Proposition 11†

To cut a given straight-line such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the square on the remaining piece.



Let AB be the given straight-line. So it is required to cut AB such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the square on the remaining piece.

For let the square $ABDC$ have been described on AB [Prop. 1.46], and let AC have been cut in half at point E [Prop. 1.10], and let BE have been joined. And let CA have been drawn through to (point) F, and let EF be made equal to BE [Prop. 1.3]. And let the square FH have been described on AF [Prop. 1.46], and let GH have been drawn through to (point) K. I say that AB has been cut at H such as to make the rectangle contained by AB and BH equal to the square on AH.

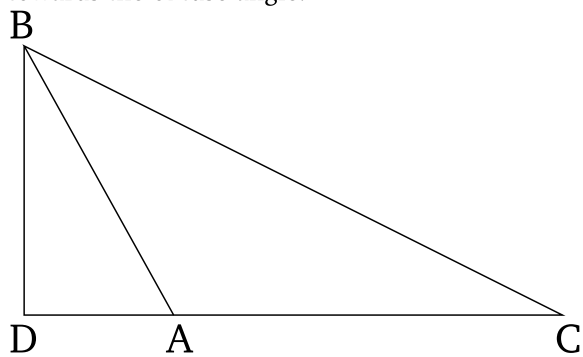
For since the straight-line AC has been cut in half at E, and FA has been added to it, the rectangle contained by CF and FA, plus the square on AE, is thus equal to the square on EF [Prop. 2.6]. And EF (is) equal to EB. Thus, the (rectangle contained) by CF and FA, plus the (square) on AE, is equal to the (square) on EB. But, the (sum of the squares) on BA and AE is equal to the (square) on EB. For the angle at A (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by CF and FA, plus the (square) on AE, is equal to the (sum of the squares) on BA and AE. Let the square on AE have been subtracted from both. Thus, the remaining rectangle contained by CF and FA is equal to the square on AB. And FK is the (rectangle contained) by CF and FA. For AF (is) equal to FG. And AD (is) the (square) on AB. Thus, the (rectangle) FK is equal to the (square) AD. Let (rectangle) AK have been subtracted from both. Thus, the remaining (square) FH is equal to the (rectangle) HD. And HD is the (rectangle contained) by AB and BH. For AB (is) equal to BD. And FH (is) the (square) on AH. Thus, the rectangle contained by AB and BH is equal to the square on HA.

Thus, the given straight-line AB has been cut at (point) H such as to make the rectangle contained by AB and BH equal to the square on HA. (Which is) the very thing it was required to do.

† This manner of cutting a straight-line—so that the ratio of the whole to the larger piece is equal to the ratio of the larger to the smaller piece—is sometimes called the “Golden Section”.

Proposition 12[†]

In obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle.



Let ABC be an obtuse-angled triangle, having the angle BAC obtuse. And let BD be drawn from point B, perpendicular to CA produced [Prop. 1.12]. I say that the square on BC is greater than the (sum of the) squares on BA and AC, by twice the rectangle contained by CA and AD.

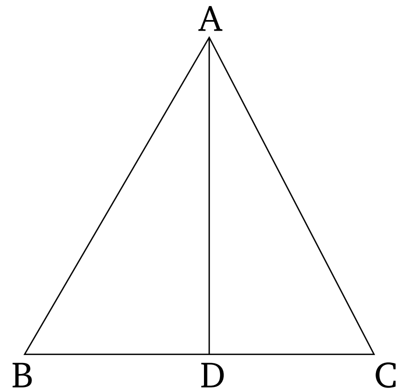
For since the straight-line CD has been cut, at random, at point A, the (square) on DC is thus equal to the (sum of the) squares on CA and AD, and twice the rectangle contained by CA and AD [Prop. 2.4]. Let the (square) on DB have been added to both. Thus, the (sum of the squares) on CD and DB is equal to the (sum of the) squares on CA, AD, and DB, and twice the [rectangle contained] by CA and AD. But, the (square) on CB is equal to the (sum of the squares) on CD and DB. For the angle at D (is) a right-angle [Prop. 1.47]. And the (square) on AB (is) equal to the (sum of the squares) on AD and DB [Prop. 1.47]. Thus, the square on CB is equal to the (sum of the) squares on CA and AB, and twice the rectangle contained by CA and AD. So the square on CB is greater than the (sum of the) squares on CA and AB by twice the rectangle contained by CA and AD.

Thus, in obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle. (Which is) the very thing it was required to show.

[†] This proposition is equivalent to the well-known cosine formula: $BC^2 = AB^2 + AC^2 - 2\,AB\,AC\,\cos BAC$, since $\cos BAC = -AD/AB$.

Proposition 13[†]

In acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle.



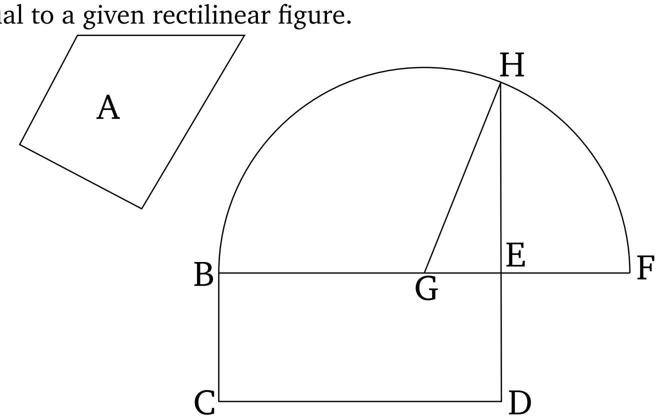
Let ABC be an acute-angled triangle, having the angle at (point) B acute. And let AD have been drawn from point A, perpendicular to BC [Prop. 1.12]. I say that the square on AC is less than the (sum of the) squares on CB and BA, by twice the rectangle contained by CB and BD.

For since the straight-line CB has been cut, at random, at (point) D, the (sum of the) squares on CB and BD is thus equal to twice the rectangle contained by CB and BD, and the square on DC [Prop. 2.7]. Let the square on DA have been added to both. Thus, the (sum of the) squares on CB, BD, and DA is equal to twice the rectangle contained by CB and BD, and the (sum of the) squares on AD and DC. But, the (square) on AB (is) equal to the (sum of the squares) on BD and DA. For the angle at (point) D is a right-angle [Prop. 1.47]. And the (square) on AC (is) equal to the (sum of the squares) on AD and DC [Prop. 1.47]. Thus, the (sum of the squares) on CB and BA is equal to the (square) on AC, and twice the (rectangle contained) by CB and BD. So the (square) on AC alone is less than the (sum of the) squares on CB and BA by twice the rectangle contained by CB and BD.

Thus, in acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle. (Which is) the very thing it was required to show. † This proposition is equivalent to the well-known cosine formula: $AC^2 = AB^2 + BC^2 - 2\,AB\,BC\,\cos ABC$, since $\cos ABC = BD/AB$.

Proposition 14

To construct a square equal to a given rectilinear figure.



Let A be the given rectilinear figure. So it is required to construct a square equal to the rectilinear figure A.

For let the right-angled parallelogram BD, equal to the rectilinear figure A, have been constructed [Prop. 1.45]. Therefore, if BE is equal to ED then that (which) was prescribed has taken place. For the square BD, equal to the rectilinear figure A, has been constructed. And if not, then one of the (straight-lines) BE or ED is greater (than the other). Let BE be greater, and let it have been produced to F, and let EF be made equal to ED [Prop. 1.3]. And let BF have been cut in half at (point) G [Prop. 1.10]. And, with center G, and radius one of the (straight-lines) GB or GF, let the semi-circle BHF have been drawn. And let DE have been produced to H, and let GH have been joined.

Therefore, since the straight-line BF has been cut—equally at G, and unequally at E—the rectangle contained by BE and EF, plus the square on EG, is thus equal to the square on GF [Prop. 2.5]. And GF (is) equal to GH. Thus, the (rectangle contained) by BE and EF, plus the (square) on GE, is equal to the (square) on GH. And the (sum of the) squares on HE and EG is equal to the (square) on GH [Prop. 1.47]. Thus, the (rectangle contained) by BE and EF, plus the (square) on GE, is equal to the (sum of the squares) on HE and EG. Let the square on GE have been taken from both. Thus, the remaining rectangle contained by BE and EF is equal to the square on EH. But, BD is the (rectangle contained) by BE and EF. For EF (is) equal to ED. Thus, the parallelogram BD is equal to the square on HE. And BD (is) equal to the rectilinear figure A. Thus, the rectilinear figure A is also equal to the square (which) can be described on EH.

Thus, a square—(namely), that (which) can be described on EH—has been constructed, equal to the given rectilinear figure A. (Which is) the very thing it was required to do.

ELEMENTS BOOK 3

Fundamentals of Plane Geometry Involving Circles

Definitions

1. Equal circles are (circles) whose diameters are equal, or whose (distances) from the centers (to the circumferences) are equal (i.e., whose radii are equal).

2. A straight-line said to touch a circle is any (straight-line) which, meeting the circle and being produced, does not cut the circle.

3. Circles said to touch one another are any (circles) which, meeting one another, do not cut one another.

4. In a circle, straight-lines are said to be equally far from the center when the perpendiculars drawn to them from the center are equal.

5. And (that straight-line) is said to be further (from the center) on which the greater perpendicular falls (from the center).

6. A segment of a circle is the figure contained by a straight-line and a circumference of a circle.

7. And the angle of a segment is that contained by a straight-line and a circumference of a circle.

8. And the angle in a segment is the angle contained by the joined straight-lines, when any point is taken on the circumference of a segment, and straight-lines are joined from it to the ends of the straight-line which is the base of the segment.

9. And when the straight-lines containing an angle cut off some circumference, the angle is said to stand upon that (circumference).

10. And a sector of a circle is the figure contained by the straight-lines surrounding an angle, and the circumference cut off by them, when the angle is constructed at the center of a circle.

11. Similar segments of circles are those accepting equal angles, or in which the angles are equal to one another.

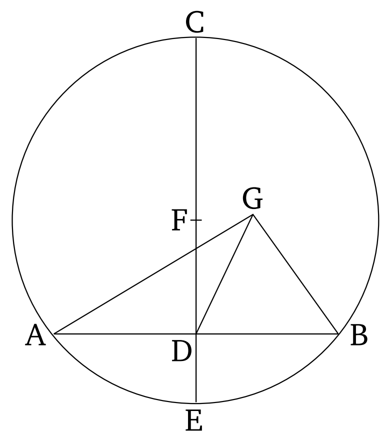
Proposition 1

To find the center of a given circle.

Let ABC be the given circle. So it is required to find the center of circle ABC.

Let some straight-line AB have been drawn through (ABC), at random, and let (AB) have been cut in half at point D [Prop. 1.9]. And let DC have been drawn from D, at right-angles to AB [Prop. 1.11]. And let (CD) have been drawn through to E. And let CE have been cut in half at F [Prop. 1.9]. I say that (point) F is the center of the [circle] ABC.

For (if) not then, if possible, let G (be the center of the circle), and let GA, GD, and GB have been joined. And since AD is equal to DB, and DG (is) common, the two (straight-lines) AD, DG are equal to the two (straight-lines) BD, DG,† respectively. And the base GA is equal to the base GB. For (they are both) radii. Thus, angle ADG is equal to angle GDB [Prop. 1.8]. And when a straight-line stood upon (another) straight-line make adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, GDB is a right-angle. And FDB is also a right-angle. Thus, FDB (is) equal to GDB, the greater to the lesser. The very thing is impossible. Thus, (point) G is not the center of the circle ABC. So, similarly, we can show that neither is any other (point) except F.



Thus, point F is the center of the [circle] ABC.

Corollary

So, from this, (it is) manifest that if any straight-line in a circle cuts any (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line). — (Which is) the very thing it was required to do. † The Greek text has "GD, DB", which is obviously a mistake.

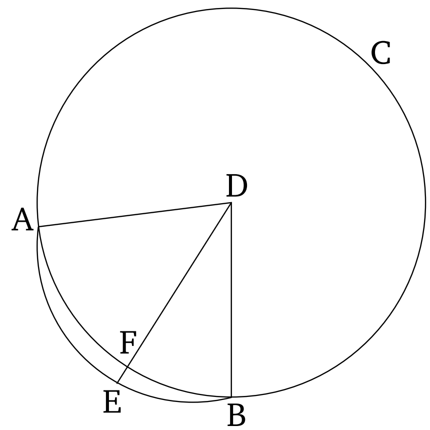
Proposition 2

If two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle.

Let ABC be a circle, and let two points A and B have been taken at random on its circumference. I say that the straight-line joining A to B will fall inside the circle.

For (if) not then, if possible, let it fall outside (the circle), like AEB (in the figure). And let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) D. And let DA and DB have been joined, and let DFE have been drawn through.

Therefore, since DA is equal to DB, the angle DAE (is) thus also equal to DBE [Prop. 1.5]. And since in triangle DAE the one side, AEB, has been produced, angle DEB (is) thus greater than DAE [Prop. 1.16]. And DAE (is) equal to DBE [Prop. 1.5]. Thus, DEB (is) greater than DBE. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, DB (is) greater than DE. And DB (is) equal to DF. Thus, DF (is) greater than DE, the lesser than the greater. The very thing is impossible. Thus, the straight-line joining A to B will not fall outside the circle. So, similarly, we can show that neither (will it fall) on the circumference itself. Thus, (it will fall) inside (the circle).



Thus, if two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle. (Which is) the very thing it was required to show.

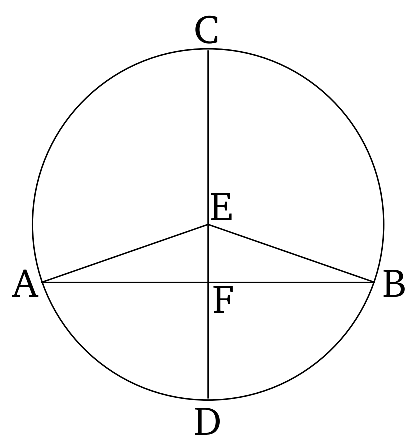
Proposition 3

In a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half.

Let ABC be a circle, and, within it, let some straight-line through the center, CD, cut in half some straight-line not through the center, AB, at the point F. I say that (CD) also cuts (AB) at right-angles.

For let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) E, and let EA and EB have been joined.

And since AF is equal to FB, and FE (is) common, two (sides of triangle AFE) [are] equal to two (sides of triangle BFE). And the base EA (is) equal to the base EB. Thus, angle AFE is equal to angle BFE [Prop. 1.8]. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, AFE and BFE are each right-angles. Thus, the (straight-line) CD, which is through the center and cuts in half the (straight-line) AB, which is not through the center, also cuts (AB) at right-angles.



And so let CD cut AB at right-angles. I say that it also cuts (AB) in half. That is to say, that AF is equal to FB.

For, with the same construction, since EA is equal to EB, angle EAF is also equal to EBF [Prop. 1.5]. And the right-angle AFE is also equal to the right-angle BFE. Thus, EAF and EFB are two triangles having two angles equal to two angles, and one side equal to one side—(namely), their common (side) EF, subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, AF (is) equal to FB.

Thus, in a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half. (Which is) the very thing it was required to show.

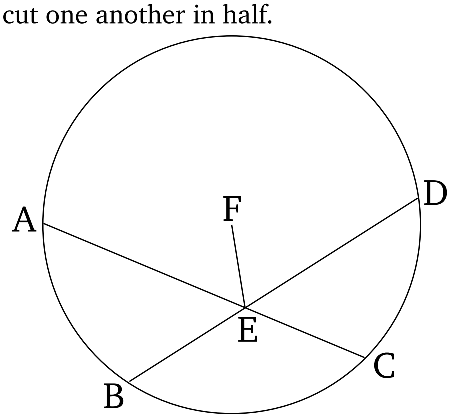
Proposition 4

In a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half.

Let $ABCD$ be a circle, and within it, let two straight-lines, AC and BD, which are not through the center, cut one another at (point) E. I say that they do not cut one another in half.

For, if possible, let them cut one another in half, such that AE is equal to EC, and BE to ED. And let the center of the circle $ABCD$ have been found [Prop. 3.1], and let it be (at point) F, and let FE have been joined.

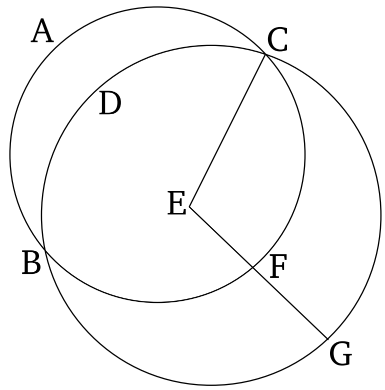
Therefore, since some straight-line through the center, FE, cuts in half some straight-line not through the center, AC, it also cuts it at right-angles [Prop. 3.3]. Thus, FEA is a right-angle. Again, since some straight-line FE cuts in half some straight-line BD, it also cuts it at right-angles [Prop. 3.3]. Thus, FEB (is) a right-angle. But FEA was also shown (to be) a right-angle. Thus, FEA (is) equal to FEB, the lesser to the greater. The very thing is impossible. Thus, AC and BD do not cut one another in half.



Thus, in a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half. (Which is) the very thing it was required to show.

Proposition 5

If two circles cut one another then they will not have the same center.



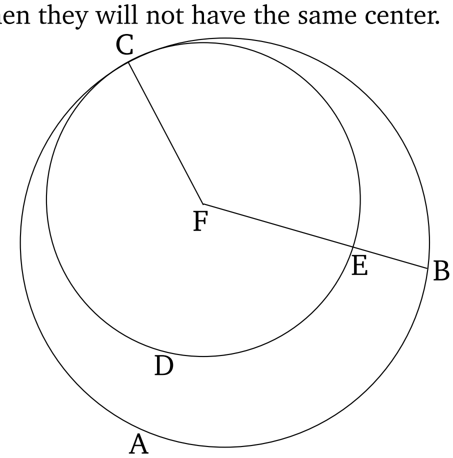
For let the two circles ABC and CDG cut one another at points B and C. I say that they will not have the same center.

For, if possible, let E be (the common center), and let EC have been joined, and let EFG have been drawn through (the two circles), at random. And since point E is the center of the circle ABC, EC is equal to EF. Again, since point E is the center of the circle CDG, EC is equal to EG. But EC was also shown (to be) equal to EF. Thus, EF is also equal to EG, the lesser to the greater. The very thing is impossible. Thus, point E is not the (common) center of the circles ABC and CDG.

Thus, if two circles cut one another then they will not have the same center. (Which is) the very thing it was required to show.

Proposition 6

If two circles touch one another then they will not have the same center.



For let the two circles ABC and CDE touch one another at point C. I say that they will not have the same center.

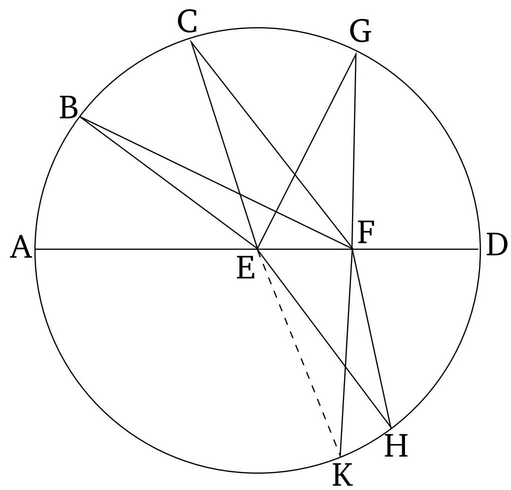
For, if possible, let F be (the common center), and let FC have been joined, and let FEB have been drawn through (the two circles), at random.

Therefore, since point F is the center of the circle ABC, FC is equal to FB. Again, since point F is the center of the circle CDE, FC is equal to FE. But FC was shown (to be) equal to FB. Thus, FE is also equal to FB, the lesser to the greater. The very thing is impossible. Thus, point F is not the (common) center of the circles ABC and CDE.

Thus, if two circles touch one another then they will not have the same center. (Which is) the very thing it was required to show.

Proposition 7

If some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer† to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).



Let $ABCD$ be a circle, and let AD be its diameter, and let some point F, which is not the center of the circle, have been taken on AD. Let E be the center of the circle. And let some straight-lines, FB, FC, and FG, radiate from F towards (the circumference of) circle $ABCD$. I say that FA is the greatest (straight-line), FD the least, and of the others, FB (is) greater than FC, and FC than FG.

For let BE, CE, and GE have been joined. And since for every triangle (any) two sides are greater than the remaining (side) [Prop. 1.20], EB and EF is thus greater than BF. And AE (is) equal to BE [thus, BE and EF is equal to AF]. Thus, AF (is) greater than BF. Again, since BE is equal to CE, and FE (is) common, the two (straight-lines) BE, EF are equal to the two (straight-lines) CE, EF (respectively). But, angle BEF (is) also greater than angle CEF.‡ Thus, the base BF is greater than the base CF. Thus, the base BF is greater than the base CF [Prop. 1.24]. So, for the same (reasons), CF is also greater than FG.

Again, since GF and FE are greater than EG [Prop. 1.20], and EG (is) equal to ED, GF and FE are thus greater than ED. Let EF have been taken from both. Thus, the remainder GF is greater than the remainder FD. Thus, FA (is) the greatest (straight-line), FD the least, and FB (is) greater than FC, and FC than FG.

I also say that from point F only two equal (straight-lines) will radiate towards (the circumference of) circle $ABCD$, (one) on each (side) of the least (straight-line) FD. For let the (angle) FEH, equal to angle GEF, have been constructed on the straight-line EF, at the point E on it [Prop. 1.23], and let FH have been joined. Therefore, since GE is equal to EH, and EF (is) common, the two (straight-lines) GE, EF are equal to the two (straight-lines) HE, EF (respectively). And angle GEF (is) equal to angle HEF. Thus, the base FG is equal to the base FH [Prop. 1.4]. So I say that another (straight-line) equal to FG will not radiate towards (the circumference of) the circle from point F. For, if possible, let FK (so) radiate. And since FK is equal to FG, but FH [is equal] to FG,

FK is thus also equal to FH, the nearer to the (straight-line) through the center equal to the further away. The very thing (is) impossible. Thus, another (straight-line) equal to GF will not radiate from the point F towards (the circumference of) the circle. Thus, (there is) only one (such straight-line).

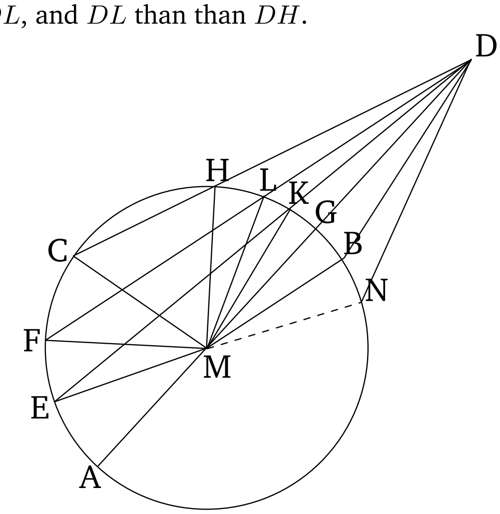
Thus, if some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the same point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show. † Presumably, in an angular sense.

‡ This is not proved, except by reference to the figure.

Proposition 8

If some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer† to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).

Let ABC be a circle, and let some point D have been taken outside ABC, and from it let some straight-lines, DA, DE, DF, and DC, have been drawn through (the circle), and let DA be through the center. I say that for the straight-lines radiating towards the concave (part of the) circumference, $AEFC$, the greatest is the one (passing) through the center, (namely) AD, and (that) DE (is) greater than DF, and DF than DC. For the straight-lines radiating towards the convex (part of the) circumference, $HLKG$, the least is the one between the point and the diameter AG, (namely) DG, and a (straight-line) nearer to the least (straight-line) DG is always less than one farther away, (so that) DK (is less) than DL, and DL than than DH.



For let the center of the circle have been found [Prop. 3.1], and let it be (at point) M [Prop. 3.1]. And let ME, MF, MC, MK, ML, and MH have been joined.

And since AM is equal to EM, let MD have been added to both. Thus, AD is equal to EM and MD. But, EM and MD is greater than ED [Prop. 1.20]. Thus, AD is also greater than ED. Again, since ME is equal to MF, and

MD (is) common, the (straight-lines) EM, MD are thus equal to FM, MD. And angle EMD is greater than angle FMD.‡ Thus, the base ED is greater than the base FD [Prop. 1.24]. So, similarly, we can show that FD is also greater than CD. Thus, AD (is) the greatest (straight-line), and DE (is) greater than DF, and DF than DC.

And since MK and KD is greater than MD [Prop. 1.20], and MG (is) equal to MK, the remainder KD is thus greater than the remainder GD. So GD is less than KD. And since in triangle MLD, the two internal straight-lines MK and KD were constructed on one of the sides, MD, then MK and KD are thus less than ML and LD [Prop. 1.21]. And MK (is) equal to ML. Thus, the remainder DK is less than the remainder DL. So, similarly, we can show that DL is also less than DH. Thus, DG (is) the least (straight-line), and DK (is) less than DL, and DL than DH.

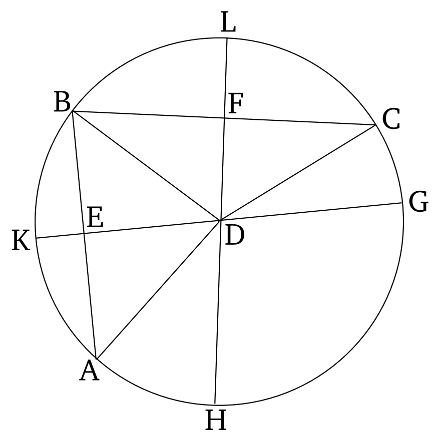
I also say that only two equal (straight-lines) will radiate from point D towards (the circumference of) the circle, (one) on each (side) on the least (straight-line), DG. Let the angle DMB, equal to angle KMD, have been constructed on the straight-line MD, at the point M on it [Prop. 1.23], and let DB have been joined. And since MK is equal to MB, and MD (is) common, the two (straight-lines) KM, MD are equal to the two (straight-lines) BM, MD, respectively. And angle KMD (is) equal to angle BMD. Thus, the base DK is equal to the base DB [Prop. 1.4]. [So] I say that another (straight-line) equal to DK will not radiate towards the (circumference of the) circle from point D. For, if possible, let (such a straight-line) radiate, and let it be DN. Therefore, since DK is equal to DN, but DK is equal to DB, then DB is thus also equal to DN, (so that) a (straight-line) nearer to the least (straight-line) DG [is] equal to one further away. The very thing was shown (to be) impossible. Thus, not more than two equal (straight-lines) will radiate towards (the circumference of) circle ABC from point D, (one) on each side of the least (straight-line) DG.

Thus, if some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show. † Presumably, in an angular sense. ‡ This is not proved, except by reference to the figure.

Proposition 9

If some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle.

Let ABC be a circle, and D a point inside it, and let more than two equal straight-lines, DA, DB, and DC, radiate from D towards (the circumference of) circle ABC. I say that point D is the center of circle ABC.



For let AB and BC have been joined, and (then) have been cut in half at points E and F (respectively) [Prop. 1.10]. And ED and FD being joined, let them have been drawn through to points G, K, H, and L.

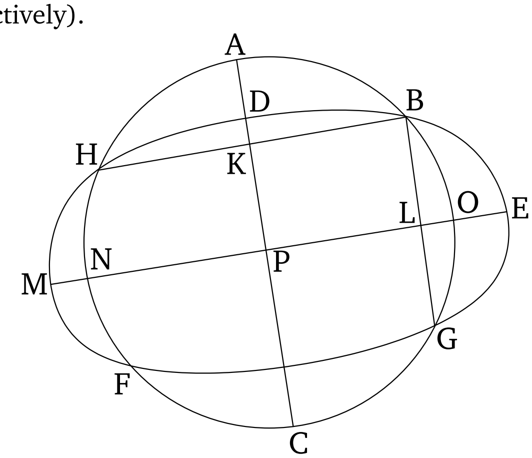
Therefore, since AE is equal to EB, and ED (is) common, the two (straight-lines) AE, ED are equal to the two (straight-lines) BE, ED (respectively). And the base DA (is) equal to the base DB. Thus, angle AED is equal to angle BED [Prop. 1.8]. Thus, angles AED and BED (are) each right-angles [Def. 1.10]. Thus, GK cuts AB in half, and at right-angles. And since, if some straight-line in a circle cuts some (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line) [Prop. 3.1 corr.], the center of the circle is thus on GK. So, for the same (reasons), the center of circle ABC is also on HL. And the straight-lines GK and HL have no common (point) other than point D. Thus, point D is the center of circle ABC.

Thus, if some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle. (Which is) the very thing it was required to show.

Proposition 10

A circle does not cut a(nother) circle at more than two points.

For, if possible, let the circle ABC cut the circle DEF at more than two points, B, G, F, and H. And BH and BG being joined, let them (then) have been cut in half at points K and L (respectively). And KC and LM being drawn at right-angles to BH and BG from K and L (respectively) [Prop. 1.11], let them (then) have been drawn through to points A and E (respectively).



Therefore, since in circle ABC some straight-line AC cuts some (other) straight-line BH in half, and at right-angles, the center of circle ABC is thus on AC [Prop. 3.1 corr.]. Again, since in the same circle ABC some straight-line NO cuts some (other straight-line) BG in half, and at right-angles, the center of circle ABC is thus on NO [Prop. 3.1 corr.]. And it was also shown (to be) on AC. And the straight-lines AC and NO meet at no other (point) than P. Thus, point P is the center of circle ABC. So, similarly, we can show that P is also the center of circle DEF. Thus, two circles cutting one another, ABC and DEF, have the same center P. The very thing is impossible [Prop. 3.5].

Thus, a circle does not cut a(nother) circle at more than two points. (Which is) the very thing it was required to show.

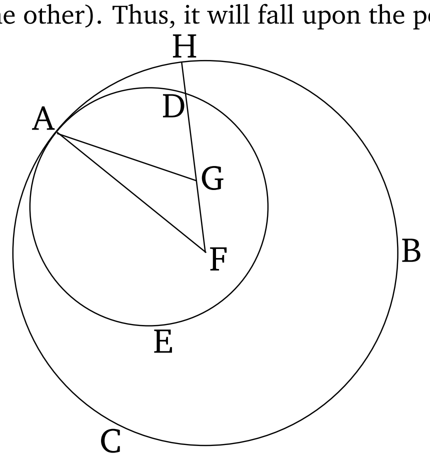
Proposition 11

If two circles touch one another internally, and their centers are found, then the straight-line joining their centers, being produced, will fall upon the point of union of the circles.

For let two circles, ABC and ADE, touch one another internally at point A, and let the center F of circle ABC have been found [Prop. 3.1], and (the center) G of (circle) ADE [Prop. 3.1]. I say that the straight-line joining G to F, being produced, will fall on A.

For (if) not then, if possible, let it fall like FGH (in the figure), and let AF and AG have been joined.

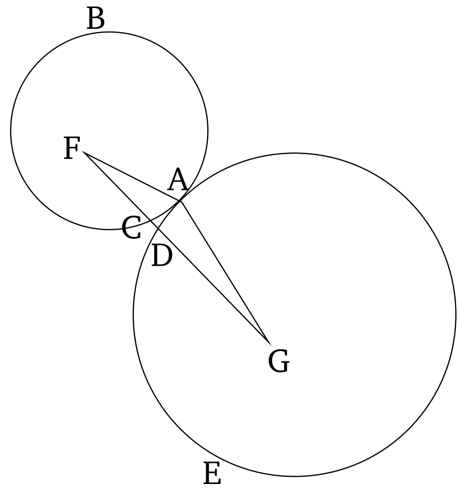
Therefore, since AG and GF is greater than FA, that is to say FH [Prop. 1.20], let FG have been taken from both. Thus, the remainder AG is greater than the remainder GH. And AG (is) equal to GD. Thus, GD is also greater than GH, the lesser than the greater. The very thing is impossible. Thus, the straight-line joining F to G will not fall outside (one circle but inside the other). Thus, it will fall upon the point of union (of the circles) at point A.



Thus, if two circles touch one another internally, [and their centers are found], then the straight-line joining their centers, [being produced], will fall upon the point of union of the circles. (Which is) the very thing it was required to show.

Proposition 12

If two circles touch one another externally then the (straight-line) joining their centers will go through the point of union.



For let two circles, ABC and ADE, touch one another externally at point A, and let the center F of ABC have been found [Prop. 3.1], and (the center) G of ADE [Prop. 3.1]. I say that the straight-line joining F to G will go through the point of union at A.

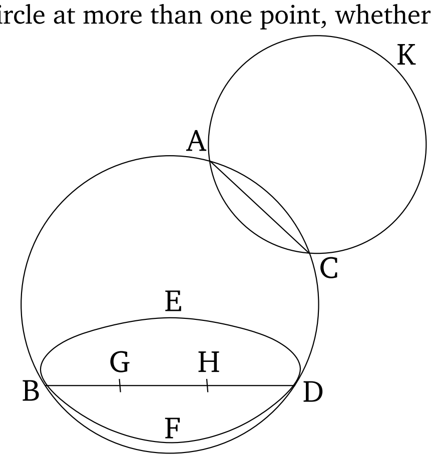
For (if) not then, if possible, let it go like $FCDG$ (in the figure), and let AF and AG have been joined.

Therefore, since point F is the center of circle ABC, FA is equal to FC. Again, since point G is the center of circle ADE, GA is equal to GD. And FA was also shown (to be) equal to FC. Thus, the (straight-lines) FA and AG are equal to the (straight-lines) FC and GD. So the whole of FG is greater than FA and AG. But, (it is) also less [Prop. 1.20]. The very thing is impossible. Thus, the straight-line joining F to G cannot not go through the point of union at A. Thus, (it will go) through it.

Thus, if two circles touch one another externally then the [straight-line] joining their centers will go through the point of union. (Which is) the very thing it was required to show.

Proposition 13

A circle does not touch a(nother) circle at more than one point, whether they touch internally or externally.



For, if possible, let circle $ABDC^{\dagger}$ touch circle $EBFD$—first of all, internally—at more than one point, D and B.

And let the center G of circle $ABDC$ have been found [Prop. 3.1], and (the center) H of $EBFD$ [Prop. 3.1].

Thus, the (straight-line) joining G and H will fall on B and D [Prop. 3.11]. Let it fall like $BGHD$ (in the figure). And since point G is the center of circle $ABDC$, BG is equal to GD. Thus, BG (is) greater than HD. Thus, BH (is) much greater than HD. Again, since point H is the center of circle $EBFD$, BH is equal to HD. But it was also shown (to be) much greater than it. The very thing (is) impossible. Thus, a circle does not touch a(nother) circle internally at more than one point.

So, I say that neither (does it touch) externally (at more than one point).

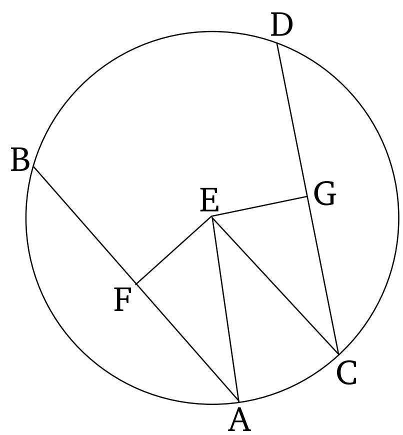
For, if possible, let circle ACK touch circle $ABDC$ externally at more than one point, A and C. And let AC have been joined.

Therefore, since two points, A and C, have been taken at random on the circumference of each of the circles $ABDC$ and ACK, the straight-line joining the points will fall inside each (circle) [Prop. 3.2]. But, it fell inside $ABDC$, and outside ACK [Def. 3.3]. The very thing (is) absurd. Thus, a circle does not touch a(nother) circle externally at more than one point. And it was shown that neither (does it) internally.

Thus, a circle does not touch a(nother) circle at more than one point, whether they touch internally or externally. (Which is) the very thing it was required to show. † The Greek text has "$ABCD$", which is obviously a mistake.

Proposition 14

In a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another.



Let $ABDC$† be a circle, and let AB and CD be equal straight-lines within it. I say that AB and CD are equally far from the center.

For let the center of circle $ABDC$ have been found [Prop. 3.1], and let it be (at) E. And let EF and EG have been drawn from (point) E, perpendicular to AB and CD (respectively) [Prop. 1.12]. And let AE and EC have been joined.

Therefore, since some straight-line, EF, through the center (of the circle), cuts some (other) straight-line, AB, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, AF (is) equal to FB. Thus, AB (is) double AF. So, for the same (reasons), CD is also double CG. And AB is equal to CD. Thus, AF (is) also equal to CG. And since AE is equal to EC, the (square) on AE (is) also equal to the (square) on EC. But, the (sum of the squares) on AF and EF (is) equal to the (square) on AE. For the angle at F (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on EG and GC (is) equal to the (square) on EC. For the angle at G (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AF and FE is equal to the (sum of the squares) on CG and GE, of which the (square) on AF is equal to the (square) on CG. For AF is equal to CG. Thus, the remaining (square) on FE is equal to the (remaining square) on EG. Thus, EF (is) equal to EG. And straight-lines in a circle are said to be equally far from the center when perpendicular (straight-lines) which are drawn to them from the center are equal [Def. 3.4]. Thus, AB and CD are equally far from the center.

So, let the straight-lines AB and CD be equally far from the center. That is to say, let EF be equal to EG. I say that AB is also equal to CD.

For, with the same construction, we can, similarly, show that AB is double AF, and CD (double) CG. And since AE is equal to CE, the (square) on AE is equal to the (square) on CE. But, the (sum of the squares) on EF and FA is equal to the (square) on AE [Prop. 1.47]. And the (sum of the squares) on EG and GC (is) equal to the (square) on CE [Prop. 1.47]. Thus, the (sum of the squares) on EF and FA is equal to the (sum of the squares) on EG and GC, of which the (square) on EF is equal to the (square) on EG. For EF (is) equal to EG. Thus, the remaining (square) on AF is equal to the (remaining square) on CG. Thus, AF (is) equal to CG. And AB is double AF, and CD double CG. Thus, AB (is) equal to CD.

Thus, in a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another. (Which is) the very thing it was required to show. † The Greek text has "$ABCD$", which is obviously a mistake.

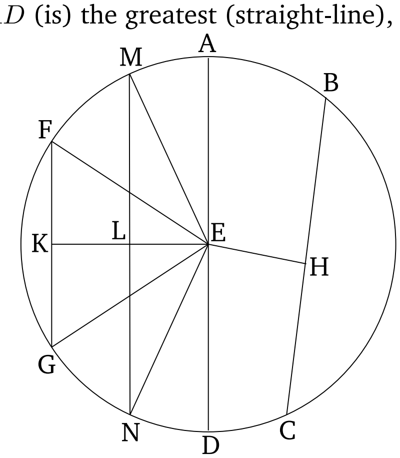
Proposition 15

In a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away.

Let $ABCD$ be a circle, and let AD be its diameter, and E (its) center. And let BC be nearer to the diameter AD,† and FG further away. I say that AD is the greatest (straight-line), and BC (is) greater than FG.

For let EH and EK have been drawn from the center E, at right-angles to BC and FG (respectively) [Prop. 1.12]. And since BC is nearer to the center, and FG further away, EK (is) thus greater than EH [Def. 3.5]. Let EL be made equal to EH [Prop. 1.3]. And LM being drawn through L, at right-angles to EK [Prop. 1.11], let it have been drawn through to N. And let ME, EN, FE, and EG have been joined.

And since EH is equal to EL, BC is also equal to MN [Prop. 3.14]. Again, since AE is equal to EM, and ED to EN, AD is thus equal to ME and EN. But, ME and EN is greater than MN [Prop. 1.20] [also AD is greater than MN], and MN (is) equal to BC. Thus, AD is greater than BC. And since the two (straight-lines) ME, EN are equal to the two (straight-lines) FE, EG (respectively), and angle MEN [is] greater than angle FEG,‡ the base MN is thus greater than the base FG [Prop. 1.24]. But, MN was shown (to be) equal to BC [(so) BC is also greater than FG]. Thus, the diameter AD (is) the greatest (straight-line), and BC (is) greater than FG.



Thus, in a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away. (Which is) the very thing it was required to show. † Euclid should have said "to the center", rather than "to the diameter AD", since BC, AD and FG are not necessarily parallel.

‡ This is not proved, except by reference to the figure.

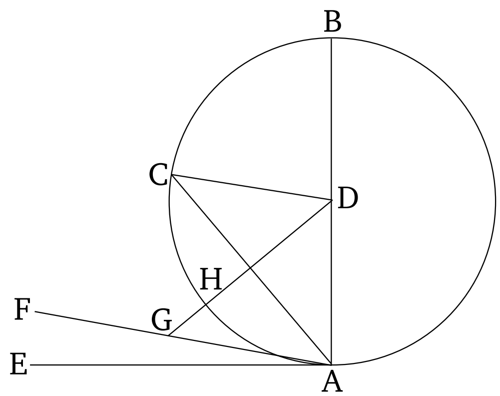
Proposition 16

A (straight-line) drawn at right-angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the (aforementioned) straight-line and the circumference. And the angle of the semi-circle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle).

Let ABC be a circle around the center D and the diameter AB. I say that the (straight-line) drawn from A, at right-angles to AB [Prop 1.11], from its end, will fall outside the circle.

For (if) not then, if possible, let it fall inside, like CA (in the figure), and let DC have been joined.

Since DA is equal to DC, angle DAC is also equal to angle ACD [Prop. 1.5]. And DAC (is) a right-angle. Thus, ACD (is) also a right-angle. So, in triangle ACD, the two angles DAC and ACD are equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, the (straight-line) drawn from point A, at right-angles to BA, will not fall inside the circle. So, similarly, we can show that neither (will it fall) on the circumference. Thus, (it will fall) outside (the circle).



Let it fall like AE (in the figure). So, I say that another straight-line cannot be inserted into the space between the straight-line AE and the circumference CHA.

For, if possible, let it be inserted like FA (in the figure), and let DG have been drawn from point D, perpendicular to FA [Prop. 1.12]. And since AGD is a right-angle, and DAG (is) less than a right-angle, AD (is) thus greater than DG [Prop. 1.19]. And DA (is) equal to DH. Thus, DH (is) greater than DG, the lesser than the greater. The very thing is impossible. Thus, another straight-line cannot be inserted into the space between the straight-line (AE) and the circumference.

And I also say that the semi-circular angle contained by the straight-line BA and the circumference CHA is greater than any acute rectilinear angle whatsoever, and the remaining (angle) contained by the circumference CHA and the straight-line AE is less than any acute rectilinear angle whatsoever.

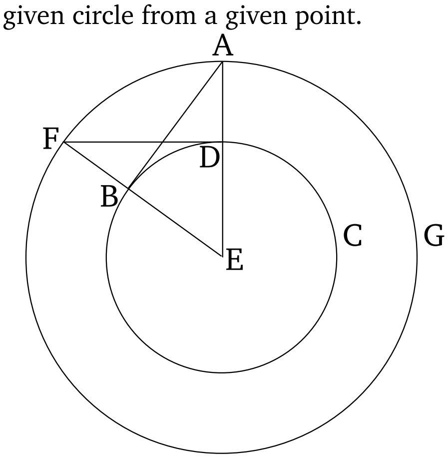
For if any rectilinear angle is greater than the (angle) contained by the straight-line BA and the circumference CHA, or less than the (angle) contained by the circumference CHA and the straight-line AE, then a straight-line can be inserted into the space between the circumference CHA and the straight-line AE—anything which will make (an angle) contained by straight-lines greater than the angle contained by the straight-line BA and the circumference CHA, or less than the (angle) contained by the circumference CHA and the straight-line AE. But (such a straight-line) cannot be inserted. Thus, an acute (angle) contained by straight-lines cannot be greater than the angle contained by the straight-line BA and the circumference CHA, neither (can it be) less than the (angle) contained by the circumference CHA and the straight-line AE.

Corollary

So, from this, (it is) manifest that a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [and that the straight-line touches the circle at a single point, inasmuch as it was also shown that a (straight-line) meeting (the circle) at two (points) falls inside it [Prop. 3.2]]. (Which is) the very thing it was required to show.

Proposition 17

To draw a straight-line touching a given circle from a given point.



Let A be the given point, and BCD the given circle. So it is required to draw a straight-line touching circle BCD from point A.

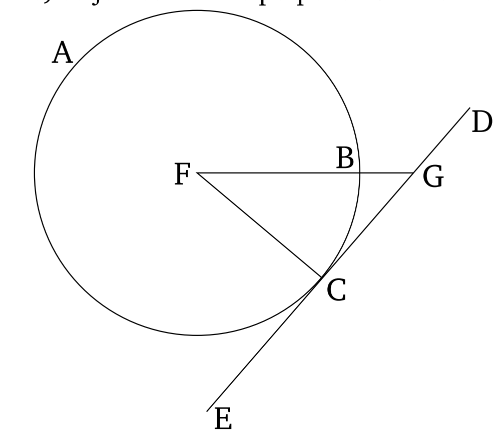
For let the center E of the circle have been found [Prop. 3.1], and let AE have been joined. And let (the circle) AFG have been drawn with center E and radius EA. And let DF have been drawn from from (point) D, at right-angles to EA [Prop. 1.11]. And let EF and AB have been joined. I say that the (straight-line) AB has been drawn from point A touching circle BCD.

For since E is the center of circles BCD and AFG, EA is thus equal to EF, and ED to EB. So the two (straight-lines) AE, EB are equal to the two (straight-lines) FE, ED (respectively). And they contain a common angle at E. Thus, the base DF is equal to the base AB, and triangle DEF is equal to triangle EBA, and the remaining angles (are equal) to the (corresponding) remaining angles [Prop. 1.4]. Thus, (angle) EDF (is) equal to EBA. And EDF (is) a right-angle. Thus, EBA (is) also a right-angle. And EB is a radius. And a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [Prop. 3.16 corr.]. Thus, AB touches circle BCD.

Thus, the straight-line AB has been drawn touching the given circle BCD from the given point A. (Which is) the very thing it was required to do.

Proposition 18

If some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent.



For let some straight-line DE touch the circle ABC at point C, and let the center F of circle ABC have been found [Prop. 3.1], and let FC have been joined from F to C. I say that FC is perpendicular to DE.

For if not, let FG have been drawn from F, perpendicular to DE [Prop. 1.12].

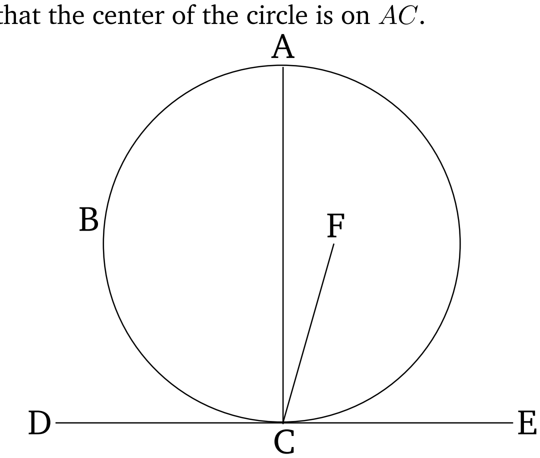
Therefore, since angle FGC is a right-angle, (angle) FCG is thus acute [Prop. 1.17]. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, FC (is) greater than FG. And FC (is) equal to FB. Thus, FB (is) also greater than FG, the lesser than the greater. The very thing is impossible. Thus, FG is not perpendicular to DE. So, similarly, we can show that neither (is) any other (straight-line) except FC. Thus, FC is perpendicular to DE.

Thus, if some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent. (Which is) the very thing it was required to show.

Proposition 19

If some straight-line touches a circle, and a straight-line is drawn from the point of contact, at right-[angles] to the tangent, then the center (of the circle) will be on the (straight-line) so drawn.

For let some straight-line DE touch the circle ABC at point C. And let CA have been drawn from C, at right-angles to DE [Prop. 1.11]. I say that the center of the circle is on AC.



For (if) not, if possible, let F be (the center of the circle), and let CF have been joined.

[Therefore], since some straight-line DE touches the circle ABC, and FC has been joined from the center to the point of contact, FC is thus perpendicular to DE [Prop. 3.18]. Thus, FCE is a right-angle. And ACE is also a right-angle. Thus, FCE is equal to ACE, the lesser to the greater. The very thing is impossible. Thus, F is not the center of circle ABC. So, similarly, we can show that neither is any (point) other (than one) on AC.

Thus, if some straight-line touches a circle, and a straight-line is drawn from the point of contact, at right-angles to the tangent, then the center (of the circle) will be on the (straight-line) so drawn. (Which is) the very thing it was required to show.

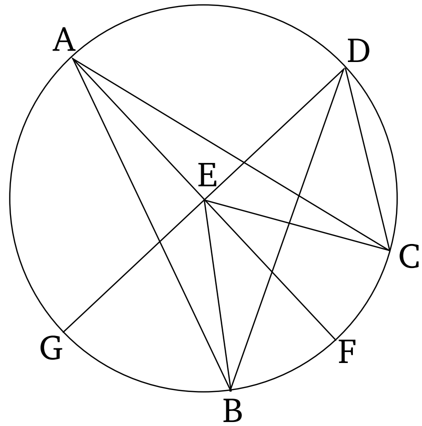
Proposition 20

In a circle, the angle at the center is double that at the circumference, when the angles have the same circumference base.

Let ABC be a circle, and let BEC be an angle at its center, and BAC (one) at (its) circumference. And let them have the same circumference base BC. I say that angle BEC is double (angle) BAC.

For being joined, let AE have been drawn through to F.

Therefore, since EA is equal to EB, angle EAB (is) also equal to EBA [Prop. 1.5]. Thus, angle EAB and EBA is double (angle) EAB. And BEF (is) equal to EAB and EBA [Prop. 1.32]. Thus, BEF is also double EAB. So, for the same (reasons), FEC is also double EAC. Thus, the whole (angle) BEC is double the whole (angle) BAC.

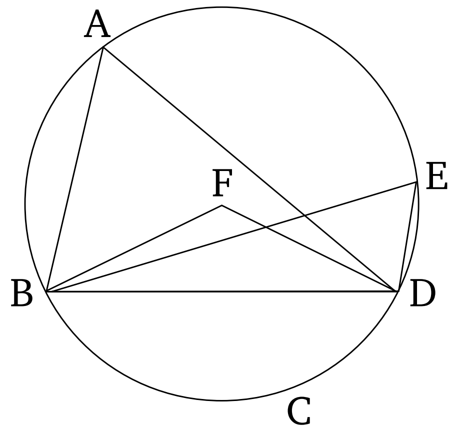


So let another (straight-line) have been inflected, and let there be another angle, BDC. And DE being joined, let it have been produced to G. So, similarly, we can show that angle GEC is double EDC, of which GEB is double EDB. Thus, the remaining (angle) BEC is double the (remaining angle) BDC.

Thus, in a circle, the angle at the center is double that at the circumference, when [the angles] have the same circumference base. (Which is) the very thing it was required to show.

Proposition 21

In a circle, angles in the same segment are equal to one another.



Let $ABCD$ be a circle, and let BAD and BED be angles in the same segment $BAED$. I say that angles BAD and BED are equal to one another.

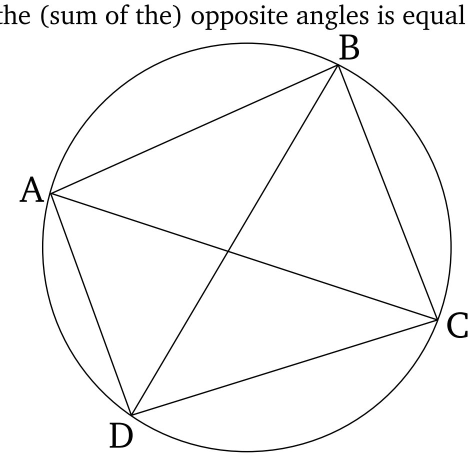
For let the center of circle $ABCD$ have been found [Prop. 3.1], and let it be (at point) F. And let BF and FD have been joined.

And since angle BFD is at the center, and BAD at the circumference, and they have the same circumference base BCD, angle BFD is thus double BAD [Prop. 3.20]. So, for the same (reasons), BFD is also double BED. Thus, BAD (is) equal to BED.

Thus, in a circle, angles in the same segment are equal to one another. (Which is) the very thing it was required to show.

Proposition 22

For quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles.



Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral within it. I say that the (sum of the) opposite angles is equal to two right-angles.

Let AC and BD have been joined.

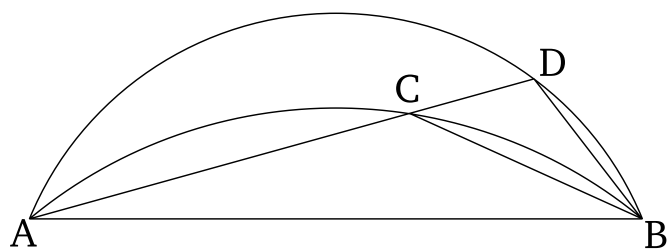
Therefore, since the three angles of any triangle are equal to two right-angles [Prop. 1.32], the three angles CAB, ABC, and BCA of triangle ABC are thus equal to two right-angles. And CAB (is) equal to BDC. For they are in the same segment $BADC$ [Prop. 3.21]. And ACB (is equal) to ADB. For they are in the same segment $ADCB$ [Prop. 3.21]. Thus, the whole of ADC is equal to BAC and ACB. Let ABC have been added to both. Thus, ABC, BAC, and ACB are equal to ABC and ADC. But, ABC, BAC, and ACB are equal to two right-angles. Thus, ABC and ADC are also equal to two right-angles. Similarly, we can show that angles BAD and DCB are also equal to two right-angles.

Thus, for quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 23

Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

For, if possible, let the two similar and unequal segments of circles, ACB and ADB, have been constructed on the same side of the same straight-line AB. And let ACD have been drawn through (the segments), and let CB and DB have been joined.

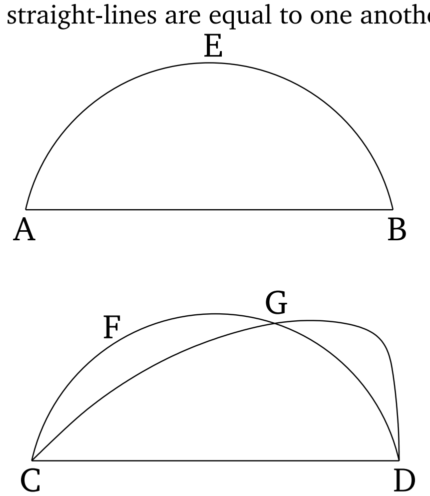


Therefore, since segment ACB is similar to segment ADB, and similar segments of circles are those accepting equal angles [Def. 3.11], angle ACB is thus equal to ADB, the external to the internal. The very thing is impossible [Prop. 1.16].

Thus, two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

Proposition 24

Similar segments of circles on equal straight-lines are equal to one another.



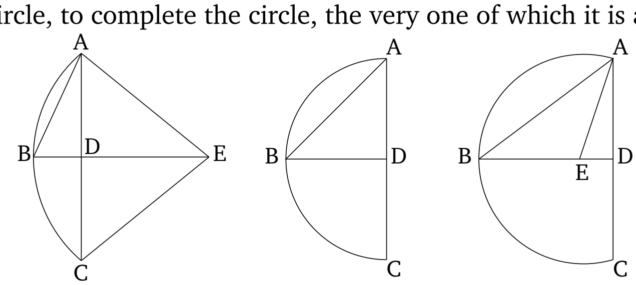
For let AEB and CFD be similar segments of circles on the equal straight-lines AB and CD (respectively). I say that segment AEB is equal to segment CFD.

For if the segment AEB is applied to the segment CFD, and point A is placed on (point) C, and the straight-line AB on CD, then point B will also coincide with point D, on account of AB being equal to CD. And if AB coincides with CD then the segment AEB will also coincide with CFD. For if the straight-line AB coincides with CD, and the segment AEB does not coincide with CFD, then it will surely either fall inside it, outside (it),† or it will miss like CGD (in the figure), and a circle (will) cut (another) circle at more than two points. The very thing is impossible [Prop. 3.10]. Thus, if the straight-line AB is applied to CD, the segment AEB cannot not also coincide with CFD. Thus, it will coincide, and will be equal to it [C.N. 4].

Thus, similar segments of circles on equal straight-lines are equal to one another. (Which is) the very thing it was required to show. † Both this possibility, and the previous one, are precluded by Prop. 3.23.

Proposition 25

For a given segment of a circle, to complete the circle, the very one of which it is a segment.



Let ABC be the given segment of a circle. So it is required to complete the circle for segment ABC, the very one of which it is a segment.

For let AC have been cut in half at (point) D [Prop. 1.10], and let DB have been drawn from point D, at right-angles to AC [Prop. 1.11]. And let AB have been joined. Thus, angle ABD is surely either greater than, equal to, or less than (angle) BAD.

First of all, let it be greater. And let (angle) BAE, equal to angle ABD, have been constructed on the straight-line BA, at the point A on it [Prop. 1.23]. And let DB have been drawn through to E, and let EC have been joined. Therefore, since angle ABE is equal to BAE, the straight-line EB is thus also equal to EA [Prop. 1.6]. And since AD is equal to DC, and DE (is) common, the two (straight-lines) AD, DE are equal to the two (straight-lines) CD, DE, respectively. And angle ADE is equal to angle CDE. For each (is) a right-angle. Thus, the base AE is equal to the base CE [Prop. 1.4]. But, AE was shown (to be) equal to BE. Thus, BE is also equal to CE. Thus, the three (straight-lines) AE, EB, and EC are equal to one another. Thus, if a circle is drawn with center E, and radius one of AE, EB, or EC, it will also go through the remaining points (of the segment), and the (associated circle) will have been completed [Prop. 3.9]. Thus, a circle has been completed from the given segment of a circle. And (it is) clear that the segment ABC is less than a semi-circle, because the center E happens to lie outside it.

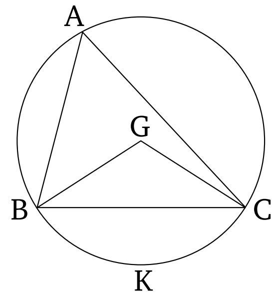
[And], similarly, even if angle ABD is equal to BAD, (since) AD becomes equal to each of BD [Prop. 1.6] and DC, the three (straight-lines) DA, DB, and DC will be equal to one another. And point D will be the center of the completed circle. And ABC will manifestly be a semi-circle.

And if ABD is less than BAD, and we construct (angle BAE), equal to angle ABD, on the straight-line BA, at the point A on it [Prop. 1.23], then the center will fall on DB, inside the segment ABC. And segment ABC will manifestly be greater than a semi-circle.

Thus, a circle has been completed from the given segment of a circle. (Which is) the very thing it was required to do.

Proposition 26

In equal circles, equal angles stand upon equal circumferences whether they are standing at the center or at the circumference.



Let ABC and DEF be equal circles, and within them let BGC and EHF be equal angles at the center, and BAC and EDF (equal angles) at the circumference. I say that circumference BKC is equal to circumference ELF.

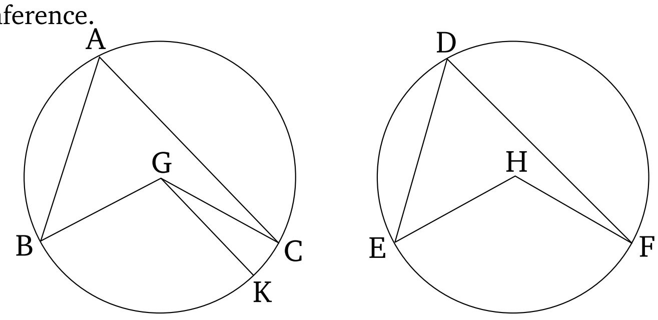
For let BC and EF have been joined.

And since circles ABC and DEF are equal, their radii are equal. So the two (straight-lines) BG, GC (are) equal to the two (straight-lines) EH, HF (respectively). And the angle at G (is) equal to the angle at H. Thus, the base BC is equal to the base EF [Prop. 1.4]. And since the angle at A is equal to the (angle) at D, the segment BAC is thus similar to the segment EDF [Def. 3.11]. And they are on equal straight-lines [BC and EF]. And similar segments of circles on equal straight-lines are equal to one another [Prop. 3.24]. Thus, segment BAC is equal to (segment) EDF. And the whole circle ABC is also equal to the whole circle DEF. Thus, the remaining circumference BKC is equal to the (remaining) circumference ELF.

Thus, in equal circles, equal angles stand upon equal circumferences, whether they are standing at the center or at the circumference. (Which is) the very thing which it was required to show.

Proposition 27

In equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference.



For let the angles BGC and EHF at the centers G and H, and the (angles) BAC and EDF at the circumferences, stand upon the equal circumferences BC and EF, in the equal circles ABC and DEF (respectively). I say that angle BGC is equal to (angle) EHF, and BAC is equal to EDF.

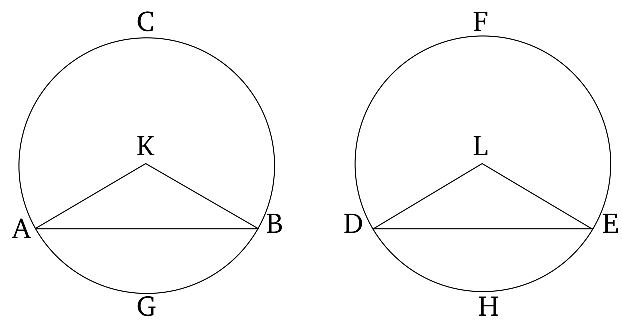
For if BGC is unequal to EHF, one of them is greater. Let BGC be greater, and let the (angle) BGK, equal to angle EHF, have been constructed on the straight-line BG, at the point G on it [Prop. 1.23]. But equal angles (in equal circles) stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference BK (is) equal to circumference EF. But, EF is equal to BC. Thus, BK is also equal to BC, the lesser to the greater. The very thing is impossible. Thus, angle BGC is not unequal to EHF. Thus, (it is) equal. And the (angle) at A is half BGC, and the (angle) at D half EHF [Prop. 3.20]. Thus, the angle at A (is) also equal to the (angle) at D.

Thus, in equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference. (Which is) the very thing it was required to show.

Proposition 28

In equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser.

Let ABC and DEF be equal circles, and let AB and DE be equal straight-lines in these circles, cutting off the greater circumferences ACB and DFE, and the lesser (circumferences) AGB and DHE (respectively). I say that the greater circumference ACB is equal to the greater circumference DFE, and the lesser circumference AGB to (the lesser) DHE.



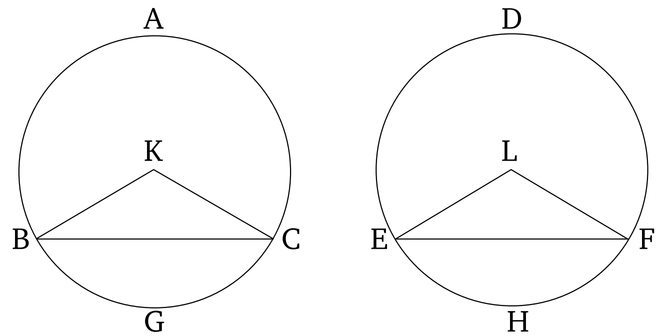
For let the centers of the circles, K and L, have been found [Prop. 3.1], and let AK, KB, DL, and LE have been joined.

And since (ABC and DEF) are equal circles, their radii are also equal [Def. 3.1]. So the two (straight-lines) AK, KB are equal to the two (straight-lines) DL, LE (respectively). And the base AB (is) equal to the base DE. Thus, angle AKB is equal to angle DLE [Prop. 1.8]. And equal angles stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference AGB (is) equal to DHE. And the whole circle ABC is also equal to the whole circle DEF. Thus, the remaining circumference ACB is also equal to the remaining circumference DFE.

Thus, in equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser. (Which is) the very thing it was required to show.

Proposition 29

In equal circles, equal straight-lines subtend equal circumferences.



Let ABC and DEF be equal circles, and within them let the equal circumferences BGC and EHF have been cut off. And let the straight-lines BC and EF have been joined. I say that BC is equal to EF.

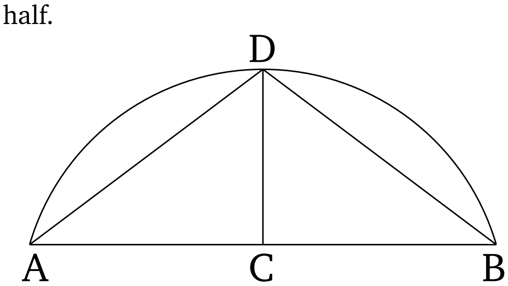
For let the centers of the circles have been found [Prop. 3.1], and let them be (at) K and L. And let BK, KC, EL, and LF have been joined.

And since the circumference BGC is equal to the circumference EHF, the angle BKC is also equal to (angle) ELF [Prop. 3.27]. And since the circles ABC and DEF are equal, their radii are also equal [Def. 3.1]. So the two (straight-lines) BK, KC are equal to the two (straight-lines) EL, LF (respectively). And they contain equal angles. Thus, the base BC is equal to the base EF [Prop. 1.4].

Thus, in equal circles, equal straight-lines subtend equal circumferences. (Which is) the very thing it was required to show.

Proposition 30

To cut a given circumference in half.



Let ADB be the given circumference. So it is required to cut circumference ADB in half.

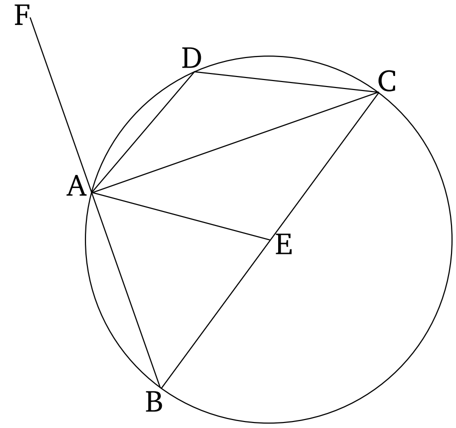
Let AB have been joined, and let it have been cut in half at (point) C [Prop. 1.10]. And let CD have been drawn from point C, at right-angles to AB [Prop. 1.11]. And let AD, and DB have been joined.

And since AC is equal to CB, and CD (is) common, the two (straight-lines) AC, CD are equal to the two (straight-lines) BC, CD (respectively). And angle ACD (is) equal to angle BCD. For (they are) each right-angles. Thus, the base AD is equal to the base DB [Prop. 1.4]. And equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser [Prop. 1.28]. And the circumferences AD and DB are each less than a semi-circle. Thus, circumference AD (is) equal to circumference DB.

Thus, the given circumference has been cut in half at point D. (Which is) the very thing it was required to do.

Proposition 31

In a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser segment (is) greater than a right-angle. And, further, the angle of a segment greater (than a semi-circle) is greater than a right-angle, and the angle of a segment less (than a semi-circle) is less than a right-angle.



Let $ABCD$ be a circle, and let BC be its diameter, and E its center. And let BA, AC, AD, and DC have been joined. I say that the angle BAC in the semi-circle BAC is a right-angle, and the angle ABC in the segment ABC, (which is) greater than a semi-circle, is less than a right-angle, and the angle ADC in the segment ADC, (which is) less than a semi-circle, is greater than a right-angle.

Let AE have been joined, and let BA have been drawn through to F.

And since BE is equal to EA, angle ABE is also equal to BAE [Prop. 1.5]. Again, since CE is equal to EA, ACE is also equal to CAE [Prop. 1.5]. Thus, the whole (angle) BAC is equal to the two (angles) ABC and ACB. And FAC, (which is) external to triangle ABC, is also equal to the two angles ABC and ACB [Prop. 1.32]. Thus, angle BAC (is) also equal to FAC. Thus, (they are) each right-angles. [Def. 1.10]. Thus, the angle BAC in the semi-circle BAC is a right-angle.

And since the two angles ABC and BAC of triangle ABC are less than two right-angles [Prop. 1.17], and BAC is a right-angle, angle ABC is thus less than a right-angle. And it is in segment ABC, (which is) greater than a semi-circle.

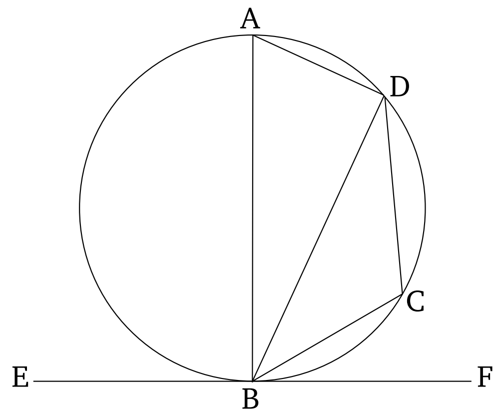
And since $ABCD$ is a quadrilateral within a circle, and for quadrilaterals within circles the (sum of the) opposite angles is equal to two right-angles [Prop. 3.22] [angles ABC and ADC are thus equal to two right-angles], and (angle) ABC is less than a right-angle. The remaining angle ADC is thus greater than a right-angle. And it is in segment ADC, (which is) less than a semi-circle.

I also say that the angle of the greater segment, (namely) that contained by the circumference ABC and the straight-line AC, is greater than a right-angle. And the angle of the lesser segment, (namely) that contained by the circumference $AD[C]$ and the straight-line AC, is less than a right-angle. And this is immediately apparent. For since the (angle contained by) the two straight-lines BA and AC is a right-angle, the (angle) contained by the circumference ABC and the straight-line AC is thus greater than a right-angle. Again, since the (angle contained by) the straight-lines AC and AF is a right-angle, the (angle) contained by the circumference $AD[C]$ and the straight-line CA is thus less than a right-angle.

Thus, in a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser [segment] (is) greater than a right-angle. And, further, the [angle] of a segment greater (than a semi-circle) [is] greater than a right-angle, and the [angle] of a segment less (than a semi-circle) is less than a right-angle. (Which is) the very thing it was required to show.

Proposition 32

If some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle.



For let some straight-line EF touch the circle $ABCD$ at the point B, and let some (other) straight-line BD have been drawn from point B into the circle $ABCD$, cutting it (in two). I say that the angles BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle. That is to say, that angle FBD is equal to the angle constructed in segment BAD, and angle EBD is equal to the angle constructed in segment DCB.

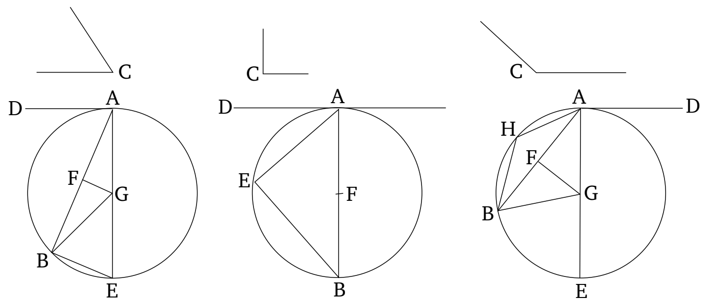
For let BA have been drawn from B, at right-angles to EF [Prop. 1.11]. And let the point C have been taken at random on the circumference BD. And let AD, DC, and CB have been joined.

And since some straight-line EF touches the circle $ABCD$ at point B, and BA has been drawn from the point of contact, at right-angles to the tangent, the center of circle $ABCD$ is thus on BA [Prop. 3.19]. Thus, BA is a diameter of circle $ABCD$. Thus, angle ADB, being in a semi-circle, is a right-angle [Prop. 3.31]. Thus, the remaining angles (of triangle ADB) BAD and ABD are equal to one right-angle [Prop. 1.32]. And ABF is also a right-angle. Thus, ABF is equal to BAD and ABD. Let ABD have been subtracted from both. Thus, the remaining angle DBF is equal to the angle BAD in the alternate segment of the circle. And since $ABCD$ is a quadrilateral in a circle, (the sum of) its opposite angles is equal to two right-angles [Prop. 3.22]. And DBF and DBE is also equal to two right-angles [Prop. 1.13]. Thus, DBF and DBE is equal to BAD and BCD, of which BAD was shown (to be) equal to DBF. Thus, the remaining (angle) DBE is equal to the angle DCB in the alternate segment DCB of the circle.

Thus, if some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle. (Which is) the very thing it was required to show.

Proposition 33

To draw a segment of a circle, accepting an angle equal to a given rectilinear angle, on a given straight-line.



Let AB be the given straight-line, and C the given rectilinear angle. So it is required to draw a segment of a circle, accepting an angle equal to C, on the given straight-line AB.

So the [angle] C is surely either acute, a right-angle, or obtuse. First of all, let it be acute. And, as in the first diagram (from the left), let (angle) BAD, equal to angle C, have been constructed on the straight-line AB, at the point A (on it) [Prop. 1.23]. Thus, BAD is also acute. Let AE have been drawn, at right-angles to DA [Prop. 1.11]. And let AB have been cut in half at F [Prop. 1.10]. And let FG have been drawn from point F, at right-angles to AB [Prop. 1.11]. And let GB have been joined.

And since AF is equal to FB, and FG (is) common, the two (straight-lines) AF, FG are equal to the two (straight-lines) BF, FG (respectively). And angle AFG (is) equal to [angle] BFG. Thus, the base AG is equal to the base BG [Prop. 1.4]. Thus, the circle drawn with center G, and radius GA, will also go through B (as well as A). Let it have been drawn, and let it be (denoted) ABE. And let EB have been joined. Therefore, since AD is at the extremity of diameter AE, (namely, point) A, at right-angles to AE, the (straight-line) AD thus touches the circle ABE [Prop. 3.16 corr.]. Therefore, since some straight-line AD touches the circle ABE, and some (other) straight-line AB has been drawn across from the point of contact A into circle ABE, angle DAB is thus equal to the angle AEB in the alternate segment of the circle [Prop. 3.32]. But, DAB is equal to C. Thus, angle C is also equal to AEB.

Thus, a segment AEB of a circle, accepting the angle AEB (which is) equal to the given (angle) C, has been drawn on the given straight-line AB.

And so let C be a right-angle. And let it again be necessary to draw a segment of a circle on AB, accepting an angle equal to the right-[angle] C. Let the (angle) BAD [again] have been constructed, equal to the right-angle C [Prop. 1.23], as in the second diagram (from the left). And let AB have been cut in half at F [Prop. 1.10]. And let the circle AEB have been drawn with center F, and radius either FA or FB.

Thus, the straight-line AD touches the circle ABE, on account of the angle at A being a right-angle [Prop. 3.16 corr.]. And angle BAD is equal to the angle in segment AEB. For (the latter angle), being in a semi-circle, is also a right-angle [Prop. 3.31]. But, BAD is also equal to C. Thus, the (angle) in (segment) AEB is also equal to C.

Thus, a segment AEB of a circle, accepting an angle equal to C, has again been drawn on AB.

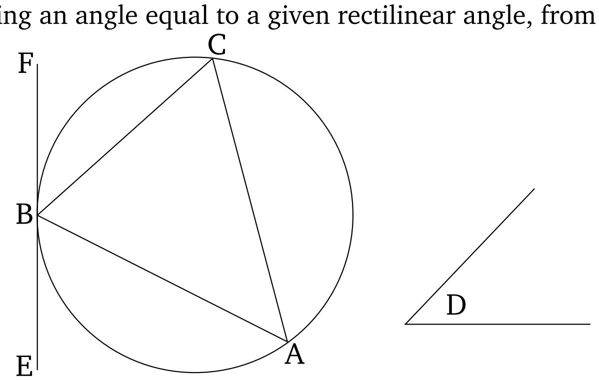
And so let (angle) C be obtuse. And let (angle) BAD, equal to (C), have been constructed on the straight-line AB, at the point A (on it) [Prop. 1.23], as in the third diagram (from the left). And let AE have been drawn, at right-angles to AD [Prop. 1.11]. And let AB have again been cut in half at F [Prop. 1.10]. And let FG have been drawn, at right-angles to AB [Prop. 1.10]. And let GB have been joined.

And again, since AF is equal to FB, and FG (is) common, the two (straight-lines) AF, FG are equal to the two (straight-lines) BF, FG (respectively). And angle AFG (is) equal to angle BFG. Thus, the base AG is equal to the base BG [Prop. 1.4]. Thus, a circle of center G, and radius GA, being drawn, will also go through B (as well as A). Let it go like AEB (in the third diagram from the left). And since AD is at right-angles to the diameter AE, at its extremity, AD thus touches circle AEB [Prop. 3.16 corr.]. And AB has been drawn across (the circle) from the point of contact A. Thus, angle BAD is equal to the angle constructed in the alternate segment AHB of the circle [Prop. 3.32]. But, angle BAD is equal to C. Thus, the angle in segment AHB is also equal to C.

Thus, a segment AHB of a circle, accepting an angle equal to C, has been drawn on the given straight-line AB. (Which is) the very thing it was required to do.

Proposition 34

To cut off a segment, accepting an angle equal to a given rectilinear angle, from a given circle.



Let ABC be the given circle, and D the given rectilinear angle. So it is required to cut off a segment, accepting an angle equal to the given rectilinear angle D, from the given circle ABC.

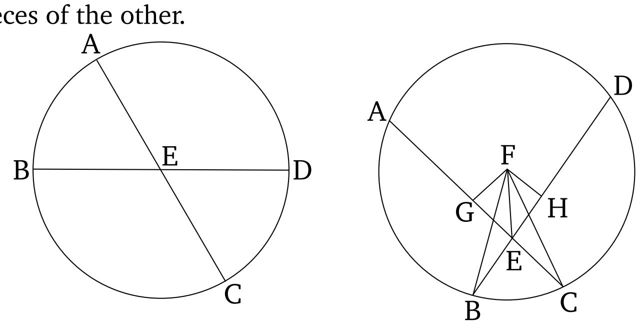
Let EF have been drawn touching ABC at point B.† And let (angle) FBC, equal to angle D, have been constructed on the straight-line FB, at the point B on it [Prop. 1.23].

Therefore, since some straight-line EF touches the circle ABC, and BC has been drawn across (the circle) from the point of contact B, angle FBC is thus equal to the angle constructed in the alternate segment BAC [Prop. 1.32]. But, FBC is equal to D. Thus, the (angle) in the segment BAC is also equal to [angle] D.

Thus, the segment BAC, accepting an angle equal to the given rectilinear angle D, has been cut off from the given circle ABC. (Which is) the very thing it was required to do. † Presumably, by finding the center of ABC [Prop. 3.1], drawing a straight-line between the center and point B, and then drawing EF through point B, at right-angles to the aforementioned straight-line [Prop. 1.11].

Proposition 35

If two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other.



For let the two straight-lines AC and BD, in the circle $ABCD$, cut one another at point E. I say that the rectangle contained by AE and EC is equal to the rectangle contained by DE and EB.

In fact, if AC and BD are through the center (as in the first diagram from the left), so that E is the center of circle $ABCD$, then (it is) clear that, AE, EC, DE, and EB being equal, the rectangle contained by AE and EC is also equal to the rectangle contained by DE and EB.

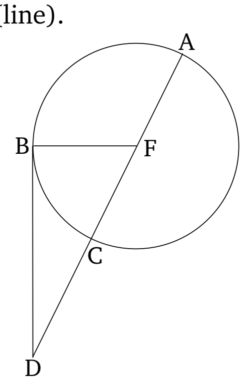
So let AC and DB not be though the center (as in the second diagram from the left), and let the center of $ABCD$ have been found [Prop. 3.1], and let it be (at) F. And let FG and FH have been drawn from F, perpendicular to the straight-lines AC and DB (respectively) [Prop. 1.12]. And let FB, FC, and FE have been joined.

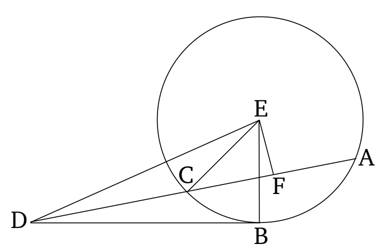
And since some straight-line, GF, through the center, cuts at right-angles some (other) straight-line, AC, not through the center, then it also cuts it in half [Prop. 3.3]. Thus, AG (is) equal to GC. Therefore, since the straight-line AC is cut equally at G, and unequally at E, the rectangle contained by AE and EC plus the square on EG is thus equal to the (square) on GC [Prop. 2.5]. Let the (square) on GF have been added [to both]. Thus, the (rectangle contained) by AE and EC plus the (sum of the squares) on GE and GF is equal to the (sum of the squares) on CG and GF. But, the (square) on FE is equal to the (sum of the squares) on EG and GF [Prop. 1.47], and the (square) on FC is equal to the (sum of the squares) on CG and GF [Prop. 1.47]. Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (square) on FC. And FC (is) equal to FB. Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (square) on FB. So, for the same (reasons), the (rectangle contained) by DE and EB plus the (square) on FE is equal to the (square) on FB. And the (rectangle contained) by AE and EC plus the (square) on FE was also shown (to be) equal to the (square) on FB. Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (rectangle contained) by DE and EB plus the (square) on FE. Let the (square) on FE have been taken from both. Thus, the remaining rectangle contained by AE and EC is equal to the rectangle contained by DE and EB.

Thus, if two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other. (Which is) the very thing it was required to show.

Proposition 36

If some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and the (other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line).





For let some point D have been taken outside circle ABC, and let two straight-lines, $DC[A]$ and DB, radiate from D towards circle ABC. And let DCA cut circle ABC, and let BD touch (it). I say that the rectangle contained by AD and DC is equal to the square on DB.

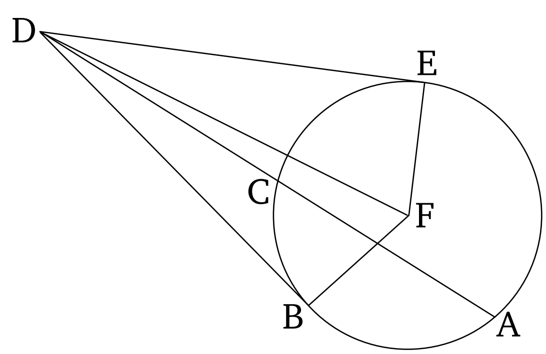
$[D]CA$ is surely either through the center, or not. Let it first of all be through the center, and let F be the center of circle ABC, and let FB have been joined. Thus, (angle) FBD is a right-angle [Prop. 3.18]. And since straight-line AC is cut in half at F, let CD have been added to it. Thus, the (rectangle contained) by AD and DC plus the (square) on FC is equal to the (square) on FD [Prop. 2.6]. And FC (is) equal to FB. Thus, the (rectangle contained) by AD and DC plus the (square) on FB is equal to the (square) on FD. And the (square) on FD is equal to the (sum of the squares) on FB and BD [Prop. 1.47]. Thus, the (rectangle contained) by AD and DC plus the (square) on FB is equal to the (sum of the squares) on FB and BD. Let the (square) on FB have been subtracted from both. Thus, the remaining (rectangle contained) by AD and DC is equal to the (square) on the tangent DB.

And so let DCA not be through the center of circle ABC, and let the center E have been found, and let EF have been drawn from E, perpendicular to AC [Prop. 1.12]. And let EB, EC, and ED have been joined. (Angle) EBD (is) thus a right-angle [Prop. 3.18]. And since some straight-line, EF, through the center, cuts some (other) straight-line, AC, not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, AF is equal to FC. And since the straight-line AC is cut in half at point F, let CD have been added to it. Thus, the (rectangle contained) by AD and DC plus the (square) on FC is equal to the (square) on FD [Prop. 2.6]. Let the (square) on FE have been added to both. Thus, the (rectangle contained) by AD and DC plus the (sum of the squares) on CF and FE is equal to the (sum of the squares) on FD and FE. But the (square) on EC is equal to the (sum of the squares) on CF and FE. For [angle] EFC [is] a right-angle [Prop. 1.47]. And the (square) on ED is equal to the (sum of the squares) on DF and FE [Prop. 1.47]. Thus, the (rectangle contained) by AD and DC plus the (square) on EC is equal to the (square) on ED. And EC (is) equal to EB. Thus, the (rectangle contained) by AD and DC plus the (square) on EB is equal to the (square) on ED. And the (sum of the squares) on EB and BD is equal to the (square) on ED. For EBD (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by AD and DC plus the (square) on EB is equal to the (sum of the squares) on EB and BD. Let the (square) on EB have been subtracted from both. Thus, the remaining (rectangle contained) by AD and DC is equal to the (square) on BD.

Thus, if some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and (the other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line). (Which is) the very thing it was required to show.

Proposition 37

If some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle.



For let some point D have been taken outside circle ABC, and let two straight-lines, DCA and DB, radiate from D towards circle ABC, and let DCA cut the circle, and let DB meet (the circle). And let the (rectangle contained) by AD and DC be equal to the (square) on DB. I say that DB touches circle ABC.

For let DE have been drawn touching ABC [Prop. 3.17], and let the center of the circle ABC have been found, and let it be (at) F. And let FE, FB, and FD have been joined. (Angle) FED is thus a right-angle [Prop. 3.18]. And since DE touches circle ABC, and DCA cuts (it), the (rectangle contained) by AD and DC is thus equal to the (square) on DE [Prop. 3.36]. And the (rectangle contained) by AD and DC was also equal to the (square) on DB. Thus, the (square) on DE is equal to the (square) on DB. Thus, DE (is) equal to DB. And FE is also equal to FB. So the two (straight-lines) DE, EF are equal to the two (straight-lines) DB, BF (respectively). And their base, FD, is common. Thus, angle DEF is equal to angle DBF [Prop. 1.8]. And DEF (is) a right-angle. Thus, DBF (is) also a right-angle. And FB produced is a diameter, And a (straight-line) drawn at right-angles to a diameter of a circle, at its extremity, touches the circle [Prop. 3.16 corr.]. Thus, DB touches circle ABC. Similarly, (the same thing) can be shown, even if the center happens to be on AC.

Thus, if some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle. (Which is) the very thing it was required to show.

ELEMENTS BOOK 4

Construction of Rectilinear Figures In and Around Circles

Definitions

1. A rectilinear figure is said to be inscribed in a(nother) rectilinear figure when the respective angles of the inscribed figure touch the respective sides of the (figure) in which it is inscribed.

2. And, similarly, a (rectilinear) figure is said to be circumscribed about a(nother rectilinear) figure when the respective sides of the circumscribed (figure) touch the respective angles of the (figure) about which it is circumscribed.

3. A rectilinear figure is said to be inscribed in a circle when each angle of the inscribed (figure) touches the circumference of the circle.

4. And a rectilinear figure is said to be circumscribed about a circle when each side of the circumscribed (figure) touches the circumference of the circle.

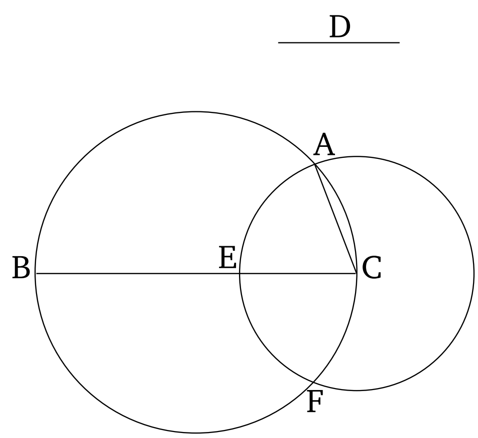
5. And, similarly, a circle is said to be inscribed in a (rectilinear) figure when the circumference of the circle touches each side of the (figure) in which it is inscribed.

6. And a circle is said to be circumscribed about a rectilinear (figure) when the circumference of the circle touches each angle of the (figure) about which it is circumscribed.

7. A straight-line is said to be inserted into a circle when its extemities are on the circumference of the circle.

Proposition 1

To insert a straight-line equal to a given straight-line into a circle, (the latter straight-line) not being greater than the diameter of the circle.



Let ABC be the given circle, and D the given straight-line (which is) not greater than the diameter of the circle. So it is required to insert a straight-line, equal to the straight-line D, into the circle ABC.

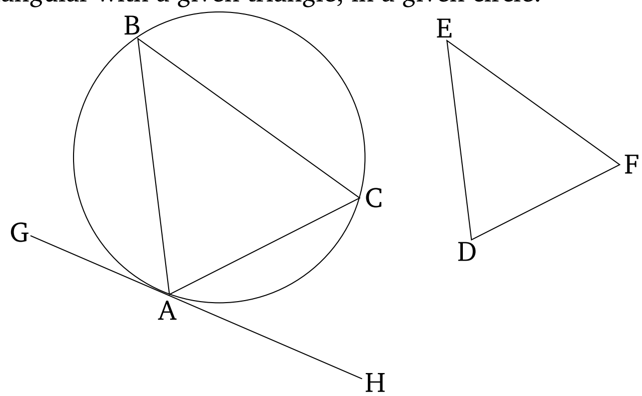
Let a diameter BC of circle ABC have been drawn.† Therefore, if BC is equal to D then that (which) was prescribed has taken place. For the (straight-line) BC, equal to the straight-line D, has been inserted into the circle ABC. And if BC is greater than D then let CE be made equal to D [Prop. 1.3], and let the circle EAF have been drawn with center C and radius CE. And let CA have been joined.

Therefore, since the point C is the center of circle EAF, CA is equal to CE. But, CE is equal to D. Thus, D is also equal to CA.

Thus, CA, equal to the given straight-line D, has been inserted into the given circle ABC. (Which is) the very thing it was required to do. † Presumably, by finding the center of the circle [Prop. 3.1], and then drawing a line through it.

Proposition 2

To inscribe a triangle, equiangular with a given triangle, in a given circle.



Let ABC be the given circle, and DEF the given triangle. So it is required to inscribe a triangle, equiangular with triangle DEF, in circle ABC.

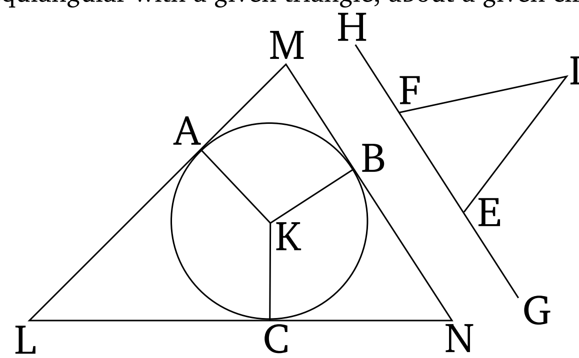
Let GH have been drawn touching circle ABC at A.† And let (angle) HAC, equal to angle DEF, have been constructed on the straight-line AH at the point A on it, and (angle) GAB, equal to [angle] DFE, on the straight-line AG at the point A on it [Prop. 1.23]. And let BC have been joined.

Therefore, since some straight-line AH touches the circle ABC, and the straight-line AC has been drawn across (the circle) from the point of contact A, (angle) HAC is thus equal to the angle ABC in the alternate segment of the circle [Prop. 3.32]. But, HAC is equal to DEF. Thus, angle ABC is also equal to DEF. So, for the same (reasons), ACB is also equal to DFE. Thus, the remaining (angle) BAC is equal to the remaining (angle) EDF [Prop. 1.32]. [Thus, triangle ABC is equiangular with triangle DEF, and has been inscribed in circle ABC].

Thus, a triangle, equiangular with the given triangle, has been inscribed in the given circle. (Which is) the very thing it was required to do. † See the footnote to Prop. 3.34.

Proposition 3

To circumscribe a triangle, equiangular with a given triangle, about a given circle.



Let ABC be the given circle, and DEF the given triangle. So it is required to circumscribe a triangle, equiangular with triangle DEF, about circle ABC.

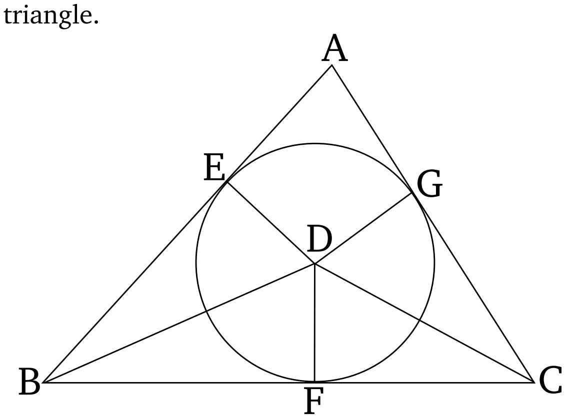
Let EF have been produced in each direction to points G and H. And let the center K of circle ABC have been found [Prop. 3.1]. And let the straight-line KB have been drawn, at random, across (ABC). And let (angle) BKA, equal to angle DEG, have been constructed on the straight-line KB at the point K on it, and (angle) BKC, equal to DFH [Prop. 1.23]. And let the (straight-lines) LAM, MBN, and NCL have been drawn through the points A, B, and C (respectively), touching the circle ABC.†

And since LM, MN, and NL touch circle ABC at points A, B, and C (respectively), and KA, KB, and KC are joined from the center K to points A, B, and C (respectively), the angles at points A, B, and C are thus right-angles [Prop. 3.18]. And since the (sum of the) four angles of quadrilateral $AMBK$ is equal to four right-angles, inasmuch as $AMBK$ (can) also (be) divided into two triangles [Prop. 1.32], and angles KAM and KBM are (both) right-angles, the (sum of the) remaining (angles), AKB and AMB, is thus equal to two right-angles. And DEG and DEF is also equal to two right-angles [Prop. 1.13]. Thus, AKB and AMB is equal to DEG and DEF, of which AKB is equal to DEG. Thus, the remainder AMB is equal to the remainder DEF. So, similarly, it can be shown that LNB is also equal to DFE. Thus, the remaining (angle) MLN is also equal to the [remaining] (angle) EDF [Prop. 1.32]. Thus, triangle LMN is equiangular with triangle DEF. And it has been drawn around circle ABC.

Thus, a triangle, equiangular with the given triangle, has been circumscribed about the given circle. (Which is) the very thing it was required to do. † See the footnote to Prop. 3.34.

Proposition 4

To inscribe a circle in a given triangle.



Let ABC be the given triangle. So it is required to inscribe a circle in triangle ABC.

Let the angles ABC and ACB have been cut in half by the straight-lines BD and CD (respectively) [Prop. 1.9], and let them meet one another at point D, and let DE, DF, and DG have been drawn from point D, perpendicular to the straight-lines AB, BC, and CA (respectively) [Prop. 1.12].

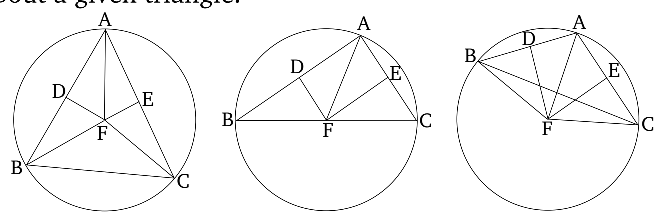
And since angle ABD is equal to CBD, and the right-angle BED is also equal to the right-angle BFD, EBD and FBD are thus two triangles having two angles equal to two angles, and one side equal to one side—the (one) subtending one of the equal angles (which is) common to the (triangles)—(namely), BD. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, DE (is) equal to DF. So, for the same (reasons), DG is also equal to DF. Thus, the three straight-lines DE, DF, and DG are equal to one another. Thus, the circle drawn with center D, and radius one of E, F, or G,† will also go through the remaining points, and will touch the straight-lines AB, BC, and CA, on account of the angles at E, F, and G being right-angles.

For if it cuts (one of) them then it will be a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, falling inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center D, and radius one of E, F, or G, does not cut the straight-lines AB, BC, and CA. Thus, it will touch them and will be the circle inscribed in triangle ABC. Let it have been (so) inscribed, like FGE (in the figure).

Thus, the circle EFG has been inscribed in the given triangle ABC. (Which is) the very thing it was required to do. † Here, and in the following propositions, it is understood that the radius is actually one of DE, DF, or DG.

Proposition 5

To circumscribe a circle about a given triangle.



Let ABC be the given triangle. So it is required to circumscribe a circle about the given triangle ABC.

Let the straight-lines AB and AC have been cut in half at points D and E (respectively) [Prop. 1.10]. And let DF and EF have been drawn from points D and E, at right-angles to AB and AC (respectively) [Prop. 1.11]. So (DF and EF) will surely either meet inside triangle ABC, on the straight-line BC, or beyond BC.

Let them, first of all, meet inside (triangle ABC) at (point) F, and let FB, FC, and FA have been joined. And since AD is equal to DB, and DF is common and at right-angles, the base AF is thus equal to the base FB [Prop. 1.4]. So, similarly, we can show that CF is also equal to AF. So that FB is also equal to FC. Thus, the three (straight-lines) FA, FB, and FC are equal to one another. Thus, the circle drawn with center F, and radius one of A, B, or C, will also go through the remaining points. And the circle will have been circumscribed about triangle ABC. Let it have been (so) circumscribed, like ABC (in the first diagram from the left).

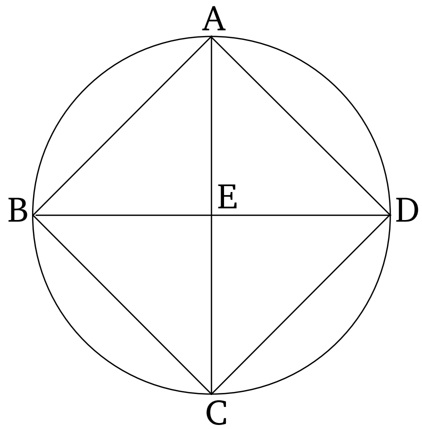
And so, let DF and EF meet on the straight-line BC at (point) F, like in the second diagram (from the left). And let AF have been joined. So, similarly, we can show that point F is the center of the circle circumscribed about triangle ABC.

And so, let DF and EF meet outside triangle ABC, again at (point) F, like in the third diagram (from the left). And let AF, BF, and CF have been joined. And, again, since AD is equal to DB, and DF is common and at right-angles, the base AF is thus equal to the base BF [Prop. 1.4]. So, similarly, we can show that CF is also equal to AF. So that BF is also equal to FC. Thus, [again] the circle drawn with center F, and radius one of FA, FB, and FC, will also go through the remaining points. And it will have been circumscribed about triangle ABC.

Thus, a circle has been circumscribed about the given triangle. (Which is) the very thing it was required to do.

Proposition 6

To inscribe a square in a given circle.



Let $ABCD$ be the given circle. So it is required to inscribe a square in circle $ABCD$.

Let two diameters of circle $ABCD$, AC and BD, have been drawn at right-angles to one another.† And let AB, BC, CD, and DA have been joined.

And since BE is equal to ED, for E (is) the center (of the circle), and EA is common and at right-angles, the base AB is thus equal to the base AD [Prop. 1.4]. So, for the same (reasons), each of BC and CD is equal to each of AB and AD. Thus, the quadrilateral $ABCD$ is equilateral. So I say that (it is) also right-angled. For since the straight-line BD is a diameter of circle $ABCD$, BAD is thus a semi-circle. Thus, angle BAD (is) a right-angle [Prop. 3.31]. So, for the same (reasons), (angles) ABC, BCD, and CDA are also each right-angles. Thus, the quadrilateral $ABCD$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been inscribed in circle $ABCD$.

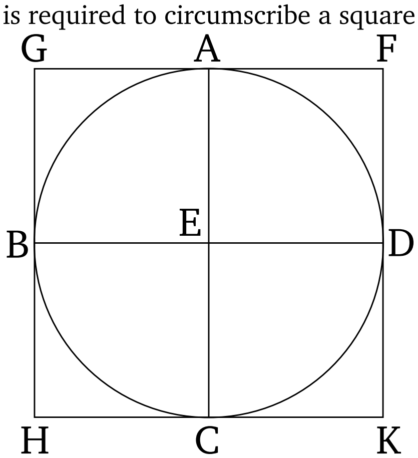
Thus, the square $ABCD$ has been inscribed in the given circle. (Which is) the very thing it was required to do.

† Presumably, by finding the center of the circle [Prop. 3.1], drawing a line through it, and then drawing a second line through it, at right-angles to the first [Prop. 1.11].

Proposition 7

To circumscribe a square about a given circle.

Let $ABCD$ be the given circle. So it is required to circumscribe a square about circle $ABCD$.



Let two diameters of circle $ABCD$, AC and BD, have been drawn at right-angles to one another.† And let FG, GH, HK, and KF have been drawn through points A, B, C, and D (respectively), touching circle $ABCD$.‡

Therefore, since FG touches circle $ABCD$, and EA has been joined from the center E to the point of contact A, the angles at A are thus right-angles [Prop. 3.18]. So, for the same (reasons), the angles at points B, C, and D are also right-angles. And since angle AEB is a right-angle, and EBG is also a right-angle, GH is thus parallel to AC [Prop. 1.29]. So, for the same (reasons), AC is also parallel to FK. So that GH is also parallel to FK [Prop. 1.30]. So, similarly, we can show that GF and HK are each parallel to BED. Thus, GK, GC, AK, FB, and BK are (all) parallelograms. Thus, GF is equal to HK, and GH to FK [Prop. 1.34]. And since AC is equal to BD, but AC (is) also (equal) to each of GH and FK, and BD is equal to each of GF and HK [Prop. 1.34] [and each of GH and FK is thus equal to each of GF and HK], the quadrilateral $FGHK$ is thus equilateral. So I say that (it is) also right-angled. For since $GBEA$ is a parallelogram, and AEB is a right-angle, AGB is thus also a right-angle [Prop. 1.34]. So, similarly, we can show that the angles at H, K, and F are also right-angles. Thus, $FGHK$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been circumscribed about circle $ABCD$.

Thus, a square has been circumscribed about the given circle. (Which is) the very thing it was required to do.

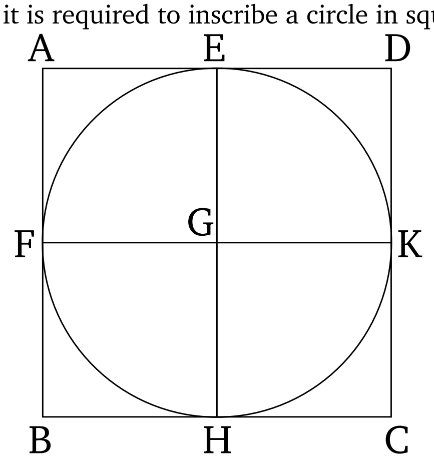
† See the footnote to the previous proposition.

‡ See the footnote to Prop. 3.34.

Proposition 8

To inscribe a circle in a given square.

Let the given square be $ABCD$. So it is required to inscribe a circle in square $ABCD$.



Let AD and AB each have been cut in half at points E and F (respectively) [Prop. 1.10]. And let EH have been drawn through E, parallel to either of AB or CD, and let FK have been drawn through F, parallel to either of AD or BC [Prop. 1.31]. Thus, AK, KB, AH, HD, AG, GC, BG, and GD are each parallelograms, and their opposite sides [are] manifestly equal [Prop. 1.34]. And since AD is equal to AB, and AE is half of AD, and AF half of AB, AE (is) thus also equal to AF. So that the opposite (sides are) also (equal). Thus, FG (is) also equal to GE. So, similarly, we can also show that each of GH and GK is equal to each of FG and GE. Thus, the four (straight-lines) GE, GF, GH, and GK [are] equal to one another. Thus, the circle drawn with center G, and radius one of E, F, H, or K, will also go through the remaining points. And it will touch the straight-lines AB, BC, CD, and DA, on account of the angles at E, F, H, and K being right-angles. For if the circle cuts AB, BC, CD, or DA, then a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, will fall inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center G, and radius one of E, F, H, or K, does not cut the straight-lines AB, BC, CD, or DA. Thus, it will touch them, and will have been inscribed in the square $ABCD$.

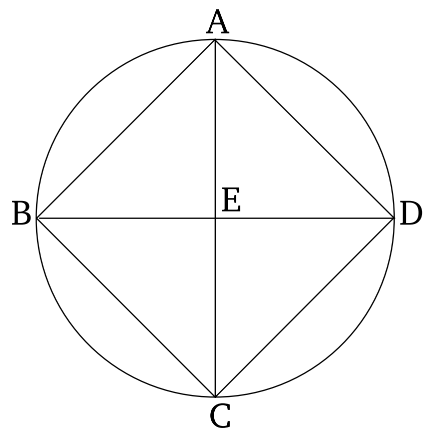
Thus, a circle has been inscribed in the given square. (Which is) the very thing it was required to do.

Proposition 9

To circumscribe a circle about a given square.

Let $ABCD$ be the given square. So it is required to circumscribe a circle about square $ABCD$.

AC and BD being joined, let them cut one another at E.



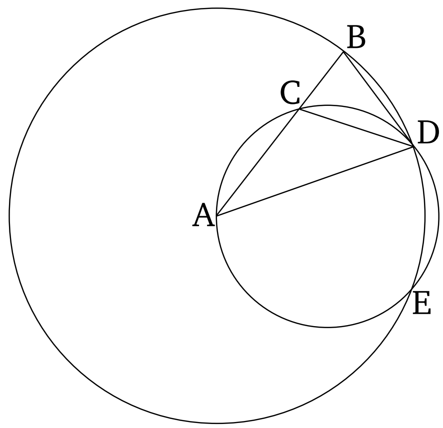
And since DA is equal to AB, and AC (is) common, the two (straight-lines) DA, AC are thus equal to the two (straight-lines) BA, AC. And the base DC (is) equal to the base BC. Thus, angle DAC is equal to angle BAC [Prop. 1.8]. Thus, the angle DAB has been cut in half by AC. So, similarly, we can show that ABC, BCD, and CDA have each been cut in half by the straight-lines AC and DB. And since angle DAB is equal to ABC, and EAB is half of DAB, and EBA half of ABC, EAB is thus also equal to EBA. So that side EA is also equal to EB [Prop. 1.6]. So, similarly, we can show that each of the [straight-lines] EA and EB are also equal to each of EC and ED. Thus, the four (straight-lines) EA, EB, EC, and ED are equal to one another. Thus, the circle drawn with center E, and radius one of A, B, C, or D, will also go through the remaining points, and will have been circumscribed about the square $ABCD$. Let it have been (so) circumscribed, like $ABCD$ (in the figure).

Thus, a circle has been circumscribed about the given square. (Which is) the very thing it was required to do.

Proposition 10

To construct an isosceles triangle having each of the angles at the base double the remaining (angle).

Let some straight-line AB be taken, and let it have been cut at point C so that the rectangle contained by AB and BC is equal to the square on CA [Prop. 2.11]. And let the circle BDE have been drawn with center A, and radius AB. And let the straight-line BD, equal to the straight-line AC, being not greater than the diameter of circle BDE, have been inserted into circle BDE [Prop. 4.1]. And let AD and DC have been joined. And let the circle ACD have been circumscribed about triangle ACD [Prop. 4.5].

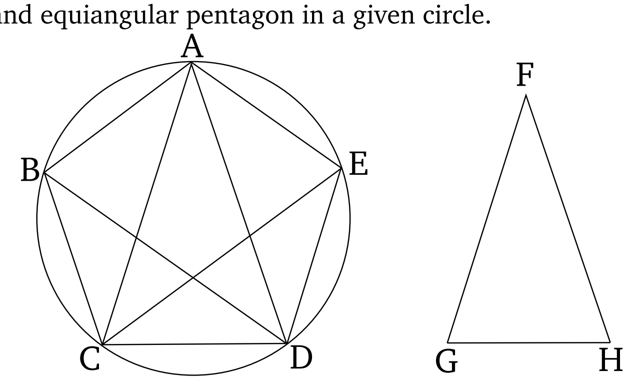


And since the (rectangle contained) by AB and BC is equal to the (square) on AC, and AC (is) equal to BD, the (rectangle contained) by AB and BC is thus equal to the (square) on BD. And since some point B has been taken outside of circle ACD, and two straight-lines BA and BD have radiated from B towards the circle ACD, and (one) of them cuts (the circle), and (the other) meets (the circle), and the (rectangle contained) by AB and BC is equal to the (square) on BD, BD thus touches circle ACD [Prop. 3.37]. Therefore, since BD touches (the circle), and DC has been drawn across (the circle) from the point of contact D, the angle BDC is thus equal to the angle DAC in the alternate segment of the circle [Prop. 3.32]. Therefore, since BDC is equal to DAC, let CDA have been added to both. Thus, the whole of BDA is equal to the two (angles) CDA and DAC. But, the external (angle) BCD is equal to CDA and DAC [Prop. 1.32]. Thus, BDA is also equal to BCD. But, BDA is equal to CBD, since the side AD is also equal to AB [Prop. 1.5]. So that DBA is also equal to BCD. Thus, the three (angles) BDA, DBA, and BCD are equal to one another. And since angle DBC is equal to BCD, side BD is also equal to side DC [Prop. 1.6]. But, BD was assumed (to be) equal to CA. Thus, CA is also equal to CD. So that angle CDA is also equal to angle DAC [Prop. 1.5]. Thus, CDA and DAC is double DAC. But BCD (is) equal to CDA and DAC. Thus, BCD is also double CAD. And BCD (is) equal to to each of BDA and DBA. Thus, BDA and DBA are each double DAB.

Thus, the isosceles triangle ABD has been constructed having each of the angles at the base BD double the remaining (angle). (Which is) the very thing it was required to do.

Proposition 11

To inscribe an equilateral and equiangular pentagon in a given circle.



Let $ABCDE$ be the given circle. So it is required to inscribed an equilateral and equiangular pentagon in circle $ABCDE$.

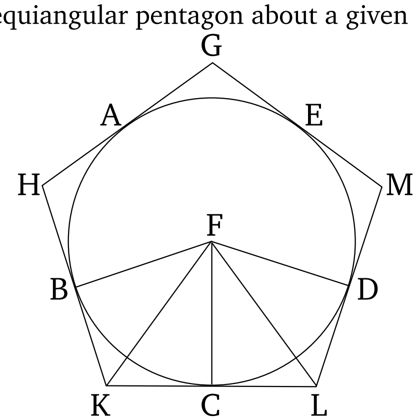
Let the the isosceles triangle FGH be set up having each of the angles at G and H double the (angle) at F [Prop. 4.10]. And let triangle ACD, equiangular to FGH, have been inscribed in circle $ABCDE$, such that CAD is equal to the angle at F, and the (angles) at G and H (are) equal to ACD and CDA, respectively [Prop. 4.2]. Thus, ACD and CDA are each double CAD. So let ACD and CDA have been cut in half by the straight-lines CE and DB, respectively [Prop. 1.9]. And let AB, BC, DE and EA have been joined.

Therefore, since angles ACD and CDA are each double CAD, and are cut in half by the straight-lines CE and DB, the five angles DAC, ACE, ECD, CDB, and BDA are thus equal to one another. And equal angles stand upon equal circumferences [Prop. 3.26]. Thus, the five circumferences AB, BC, CD, DE, and EA are equal to one another [Prop. 3.29]. Thus, the pentagon $ABCDE$ is equilateral. So I say that (it is) also equiangular. For since the circumference AB is equal to the circumference DE, let BCD have been added to both. Thus, the whole circumference $ABCD$ is equal to the whole circumference $EDCB$. And the angle AED stands upon circumference $ABCD$, and angle BAE upon circumference $EDCB$. Thus, angle BAE is also equal to AED [Prop. 3.27]. So, for the same (reasons), each of the angles ABC, BCD, and CDE is also equal to each of BAE and AED. Thus, pentagon $ABCDE$ is equiangular. And it was also shown (to be) equilateral.

Thus, an equilateral and equiangular pentagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

Proposition 12

To circumscribe an equilateral and equiangular pentagon about a given circle.



Let $ABCDE$ be the given circle. So it is required to circumscribe an equilateral and equiangular pentagon about circle $ABCDE$.

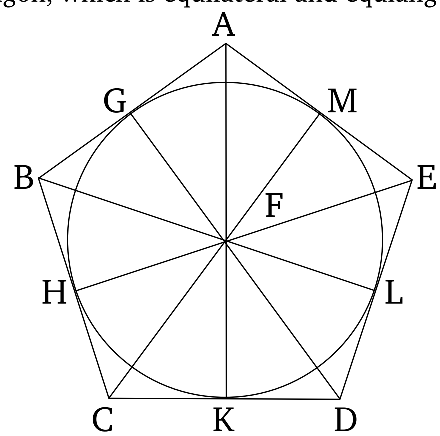
Let A, B, C, D, and E have been conceived as the angular points of a pentagon having been inscribed (in circle $ABCDE$) [Prop. 3.11], such that the circumferences AB, BC, CD, DE, and EA are equal. And let GH, HK, KL, LM, and MG have been drawn through (points) A, B, C, D, and E (respectively), touching the circle.† And let the center F of the circle $ABCDE$ have been found [Prop. 3.1]. And let FB, FK, FC, FL, and FD have been joined.

And since the straight-line KL touches (circle) $ABCDE$ at C, and FC has been joined from the center F to the point of contact C, FC is thus perpendicular to KL [Prop. 3.18]. Thus, each of the angles at C is a right-angle. So, for the same (reasons), the angles at B and D are also right-angles. And since angle FCK is a right-angle, the (square) on FK is thus equal to the (sum of the squares) on FC and CK [Prop. 1.47]. So, for the same (reasons), the (square) on FK is also equal to the (sum of the squares) on FB and BK. So that the (sum of the squares) on FC and CK is equal to the (sum of the squares) on FB and BK, of which the (square) on FC is equal to the (square) on FB. Thus, the remaining (square) on CK is equal to the remaining (square) on BK. Thus, BK (is) equal to CK. And since FB is equal to FC, and FK (is) common, the two (straight-lines) BF, FK are equal to the two (straight-lines) CF, FK. And the base BK [is] equal to the base CK. Thus, angle BFK is equal to [angle] KFC [Prop. 1.8]. And BKF (is equal) to FKC [Prop. 1.8]. Thus, BFC (is) double KFC, and BKC (is double) FKC. So, for the same (reasons), CFD is also double CFL, and DLC (is also double) FLC. And since circumference BC is equal to CD, angle BFC is also equal to CFD [Prop. 3.27]. And BFC is double KFC, and DFC (is double) LFC. Thus, KFC is also equal to LFC. And angle FCK is also equal to FCL. So, FKC and FLC are two triangles having two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle to the remaining angle [Prop. 1.26]. Thus, the straight-line KC (is) equal to CL, and the angle FKC to FLC. And since KC is equal to CL, KL (is) thus double KC. So, for the same (reasons), it can be shown that HK (is) also double BK. And BK is equal to KC. Thus, HK is also equal to KL. So, similarly, each of HG, GM, and ML can also be shown (to be) equal to each of HK and KL. Thus, pentagon $GHKLM$ is equilateral. So I say that (it is) also equiangular. For since angle FKC is equal to FLC, and HKL was shown (to be) double FKC, and KLM double FLC, HKL is thus also equal to KLM. So, similarly, each of KHG, HGM, and GML can also be shown (to be) equal to each of HKL and KLM. Thus, the five angles GHK, HKL, KLM, LMG, and MGH are equal to one another. Thus, the pentagon $GHKLM$ is equiangular. And it was also shown (to be) equilateral, and has been circumscribed about circle $ABCDE$.

[Thus, an equilateral and equiangular pentagon has been circumscribed about the given circle]. (Which is) the very thing it was required to do. † See the footnote to Prop. 3.34.

Proposition 13

To inscribe a circle in a given pentagon, which is equilateral and equiangular.



Let $ABCDE$ be the given equilateral and equiangular pentagon. So it is required to inscribe a circle in pentagon $ABCDE$.

For let angles BCD and CDE have each been cut in half by each of the straight-lines CF and DF (respectively) [Prop. 1.9]. And from the point F, at which the straight-lines CF and DF meet one another, let the straight-lines FB, FA, and FE have been joined. And since BC is equal to CD, and CF (is) common, the two (straight-lines) BC, CF are equal to the two (straight-lines) DC, CF. And angle BCF [is] equal to angle DCF. Thus, the base BF is equal to the base DF, and triangle BCF is equal to triangle DCF, and the remaining angles will be equal to the (corresponding) remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle CBF (is) equal to CDF. And since CDE is double CDF, and CDE (is) equal to ABC, and CDF to CBF, CBA is thus also double CBF. Thus, angle ABF is equal to FBC. Thus, angle ABC has been cut in half by the straight-line BF. So, similarly, it can be shown that BAE and AED have been cut in half by the straight-lines FA and FE, respectively. So let FG, FH, FK, FL, and FM have been drawn from point F, perpendicular to the straight-lines AB, BC, CD, DE, and

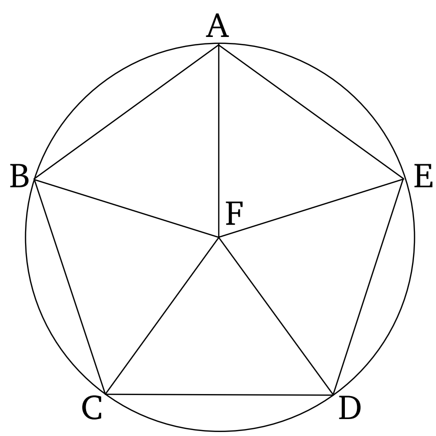
EA (respectively) [Prop. 1.12]. And since angle HCF is equal to KCF, and the right-angle FHC is also equal to the [right-angle] FKC, FHC and FKC are two triangles having two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC, subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, the perpendicular FH (is) equal to the perpendicular FK. So, similarly, it can be shown that FL, FM, and FG are each equal to each of FH and FK. Thus, the five straight-lines FG, FH, FK, FL, and FM are equal to one another. Thus, the circle drawn with center F, and radius one of G, H, K, L, or M, will also go through the remaining points, and will touch the straight-lines AB, BC, CD, DE, and EA, on account of the angles at points G, H, K, L, and M being right-angles. For if it does not touch them, but cuts them, it follows that a (straight-line) drawn at right-angles to the diameter of the circle, from its extremity, falls inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center F, and radius one of G, H, K, L, or M, does not cut the straight-lines AB, BC, CD, DE, or EA. Thus, it will touch them. Let it have been drawn, like $GHKLM$ (in the figure).

Thus, a circle has been inscribed in the given pentagon which is equilateral and equiangular. (Which is) the very thing it was required to do.

Proposition 14

To circumscribe a circle about a given pentagon which is equilateral and equiangular.

Let $ABCDE$ be the given pentagon which is equilateral and equiangular. So it is required to circumscribe a circle about the pentagon $ABCDE$.



So let angles BCD and CDE have been cut in half by the (straight-lines) CF and DF, respectively [Prop. 1.9]. And let the straight-lines FB, FA, and FE have been joined from point F, at which the straight-lines meet, to the points B, A, and E (respectively). So, similarly, to the (proposition) before this (one), it can be shown that angles CBA, BAE, and AED have also been cut in half by the straight-lines FB, FA, and FE, respectively. And since angle BCD is equal to CDE, and FCD is half of BCD, and CDF half of CDE, FCD is thus also equal to FDC. So that side FC is also equal to side FD [Prop. 1.6]. So, similarly, it can be shown that FB, FA, and FE are also each equal to each of FC and FD. Thus, the five straight-lines FA, FB, FC, FD, and FE are equal to one another. Thus, the circle drawn with center F, and radius one of FA, FB, FC, FD, or FE, will also go through the remaining points, and will have been circumscribed. Let it have been (so) circumscribed, and let it be $ABCDE$.

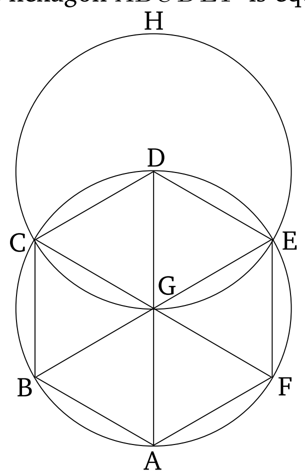
Thus, a circle has been circumscribed about the given pentagon, which is equilateral and equiangular. (Which is) the very thing it was required to do.

Proposition 15

To inscribe an equilateral and equiangular hexagon in a given circle.

Let $ABCDEF$ be the given circle. So it is required to inscribe an equilateral and equiangular hexagon in circle $ABCDEF$.

Let the diameter AD of circle $ABCDEF$ have been drawn,† and let the center G of the circle have been found [Prop. 3.1]. And let the circle $EGCH$ have been drawn, with center D, and radius DG. And EG and CG being joined, let them have been drawn across (the circle) to points B and F (respectively). And let AB, BC, CD, DE, EF, and FA have been joined. I say that the hexagon $ABCDEF$ is equilateral and equiangular.



For since point G is the center of circle $ABCDEF$, GE is equal to GD. Again, since point D is the center of circle GCH, DE is equal to DG. But, GE was shown (to be) equal to GD. Thus, GE is also equal to ED. Thus, triangle EGD is equilateral. Thus, its three angles EGD, GDE, and DEG are also equal to one another, inasmuch as the angles at the base of isosceles triangles are equal to one another [Prop. 1.5]. And the three angles of the triangle are equal to two right-angles [Prop. 1.32]. Thus, angle EGD is one third of two right-angles. So, similarly, DGC can also be shown (to be) one third of two right-angles. And since the straight-line CG, standing on EB, makes adjacent angles EGC and CGB equal to two right-angles [Prop. 1.13], the remaining angle CGB is thus also one third of two right-angles. Thus, angles EGD, DGC, and CGB are equal to one another. And hence the (angles) opposite to them BGA, AGF, and FGE are also equal [to EGD, DGC, and CGB (respectively)] [Prop. 1.15]. Thus, the six angles EGD, DGC, CGB, BGA, AGF, and FGE are equal to one another. And equal angles stand on equal circumferences [Prop. 3.26]. Thus, the six circumferences AB, BC, CD, DE, EF, and FA are equal to one another. And equal circumferences are subtended by equal straight-lines [Prop. 3.29]. Thus, the six straight-lines (AB, BC, CD, DE, EF, and FA) are equal to one another. Thus, hexagon $ABCDEF$ is equilateral. So, I say that (it is) also equiangular. For since circumference FA is equal to circumference ED, let circumference $ABCD$ have been added to both. Thus, the whole of $FABCD$ is equal to the whole of $EDCBA$. And angle FED stands on circumference $FABCD$, and angle AFE on circumference $EDCBA$. Thus, angle AFE is equal to DEF [Prop. 3.27]. Similarly, it can also be shown that the remaining angles of hexagon $ABCDEF$ are individually equal to each of the angles AFE and FED. Thus, hexagon $ABCDEF$ is equiangular. And it was also shown (to be) equilateral. And it has been inscribed in circle $ABCDE$.

Thus, an equilateral and equiangular hexagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

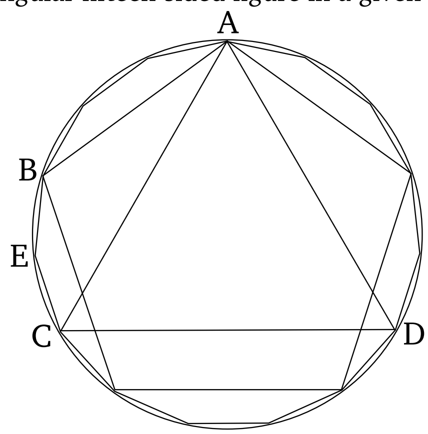
Corollary

So, from this, (it is) manifest that a side of the hexagon is equal to the radius of the circle.

And similarly to a pentagon, if we draw tangents to the circle through the (sixfold) divisions of the (circumference of the) circle, an equilateral and equiangular hexagon can be circumscribed about the circle, analogously to the aforementioned pentagon. And, further, by (means) similar to the aforementioned pentagon, we can inscribe and circumscribe a circle in (and about) a given hexagon. (Which is) the very thing it was required to do. † See the footnote to Prop. 4.6.

Proposition 16

To inscribe an equilateral and equiangular fifteen-sided figure in a given circle.



Let $ABCD$ be the given circle. So it is required to inscribe an equilateral and equiangular fifteen-sided figure in circle $ABCD$.

Let the side AC of an equilateral triangle inscribed in (the circle) [Prop. 4.2], and (the side) AB of an (inscribed) equilateral pentagon [Prop. 4.11], have been inscribed in circle $ABCD$. Thus, just as the circle $ABCD$ is (made up) of fifteen equal pieces, the circumference ABC, being a third of the circle, will be (made up) of five such (pieces), and the circumference AB, being a fifth of the circle, will be (made up) of three. Thus, the remainder BC (will be made up) of two equal (pieces). Let (circumference) BC have been cut in half at E [Prop. 3.30]. Thus, each of the circumferences BE and EC is one fifteenth of the circle $ABCDE$.

Thus, if, joining BE and EC, we continuously insert straight-lines equal to them into circle $ABCD[E]$ [Prop. 4.1], then an equilateral and equiangular fifteen-sided figure will have been inserted into (the circle). (Which is) the very thing it was required to do.

And similarly to the pentagon, if we draw tangents to the circle through the (fifteenfold) divisions of the (circumference of the) circle, we can circumscribe an equilateral and equiangular fifteen-sided figure about the circle. And, further, through similar proofs to the pentagon, we can also inscribe and circumscribe a circle in (and about) a given fifteen-sided figure. (Which is) the very thing it was required to do.

ELEMENTS BOOK 5

Proportion†

†The theory of proportion set out in this book is generally attributed to Eudoxus of Cnidus. The novel feature of this theory is its ability to deal with irrational magnitudes, which had hitherto been a major stumbling block for Greek mathematicians. Throughout the footnotes in this book, α, β, γ, *etc.*, denote general (possibly irrational) magnitudes, whereas m, n, l, *etc.*, denote positive integers.

Definitions

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.[†]

2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.

3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.[‡]

4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.[§]

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.[¶]

6. And let magnitudes having the same ratio be called proportional.[*]

7. And when for equal multiples (as in Def. 5), the multiple of the first (magnitude) exceeds the multiple of the second, and the multiple of the third (magnitude) does not exceed the multiple of the fourth, then the first (magnitude) is said to have a greater ratio to the second than the third (magnitude has) to the fourth.

8. And a proportion in three terms is the smallest (possible).[$]

9. And when three magnitudes are proportional, the first is said to have to the third the squared[‖] ratio of that (it has) to the second.[††]

10. And when four magnitudes are (continuously) proportional, the first is said to have to the fourth the cubed[‡‡] ratio of that (it has) to the second.[§§] And so on, similarly, in successive order, whatever the (continuous) proportion might be.

11. These magnitudes are said to be corresponding (magnitudes): the leading to the leading (of two ratios), and the following to the following.

12. An alternate ratio is a taking of the (ratio of the) leading (magnitude) to the leading (of two equal ratios), and (setting it equal to) the (ratio of the) following (magnitude) to the following.[¶¶]

13. An inverse ratio is a taking of the (ratio of the) following (magnitude) as the leading and the leading (magnitude) as the following.[**]

14. A composition of a ratio is a taking of the (ratio of the) leading plus the following (magnitudes), as one, to the following (magnitude) by itself.[$$]

15. A separation of a ratio is a taking of the (ratio of the) excess by which the leading (magnitude) exceeds the following to the following (magnitude) by itself.[‖‖]

16. A conversion of a ratio is a taking of the (ratio of the) leading (magnitude) to the excess by which the leading (magnitude) exceeds the following.[†††]

17. There being several magnitudes, and other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, a ratio via equality (or *ex aequali*) occurs when as the first is to the last in the first (set of) magnitudes, so the first (is) to the last in the second (set of) magnitudes. Or alternately, (it is) a taking of the (ratio of the) outer (magnitudes) by the removal of the inner (magnitudes).[‡‡‡]

18. There being three magnitudes, and other (magnitudes) of equal number to them, a perturbed proportion occurs when as the leading is to the following in the first (set of) magnitudes, so the leading (is) to the following in the second (set of) magnitudes, and as the following (is) to some other (*i.e.*, the remaining magnitude) in the first (set of) magnitudes, so some other (is) to the leading in the second (set of) magnitudes.[§§§]

† In other words, α is said to be a part of β if $\beta = m\,\alpha$.

‡ In modern notation, the ratio of two magnitudes, α and β, is denoted $\alpha : \beta$.

§ In other words, α has a ratio with respect to β if $m\,\alpha > \beta$ and $n\,\beta > \alpha$, for some m and n.

¶ In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\,\alpha > n\,\beta$ whenever $m\,\gamma > n\,\delta$, and $m\,\alpha = n\,\beta$ whenever $m\,\gamma = n\,\delta$, and $m\,\alpha < n\,\beta$ whenever $m\,\gamma < n\,\delta$, for all m and n. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α, β, *etc.*, are irrational.

* Thus if α and β have the same ratio as γ and δ then they are proportional. In modern notation, $\alpha : \beta :: \gamma : \delta$.

$ In modern notation, a proportion in three terms—α, β, and γ—is written: $\alpha : \beta :: \beta : \gamma$.

‖ Literally, "double".

†† In other words, if $\alpha : \beta :: \beta : \gamma$ then $\alpha : \gamma :: \alpha^2 : \beta^2$.

‡‡ Literally, "triple".

§§ In other words, if $\alpha : \beta :: \beta : \gamma :: \gamma : \delta$ then $\alpha : \delta :: \alpha^3 : \beta^3$.

¶¶ In other words, if $\alpha : \beta :: \gamma : \delta$ then the alternate ratio corresponds to $\alpha : \gamma :: \beta : \delta$.

** In other words, if $\alpha : \beta$ then the inverse ratio corresponds to $\beta : \alpha$.

$$ In other words, if $\alpha : \beta$ then the composed ratio corresponds to $\alpha + \beta : \beta$.

‖‖ In other words, if $\alpha : \beta$ then the separated ratio corresponds to $\alpha - \beta : \beta$.

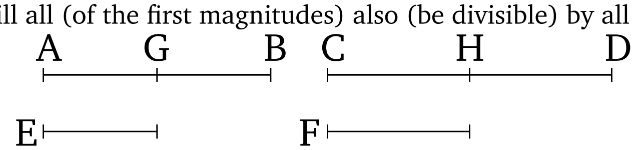
††† In other words, if $\alpha : \beta$ then the converted ratio corresponds to $\alpha : \alpha - \beta$.

‡‡‡ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha : \beta : \gamma :: \delta : \epsilon : \zeta$, then the ratio via equality (or *ex aequali*) corresponds to $\alpha : \gamma :: \delta : \zeta$.

§§§ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha : \beta :: \delta : \epsilon$ as well as $\beta : \gamma :: \zeta : \delta$, then the proportion is said to be perturbed.

Proposition 1[†]

If there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second).



Let there be any number of magnitudes whatsoever, AB, CD, (which are) equal multiples, respectively, of some (other) magnitudes, E, F, of equal number (to them). I say that as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.

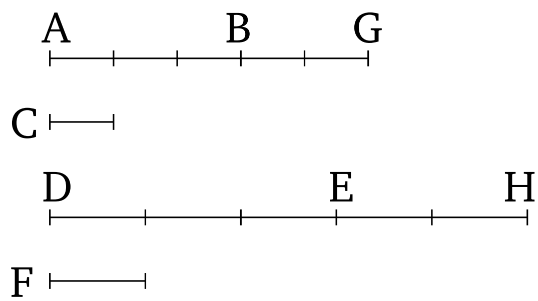
For since AB, CD are equal multiples of E, F, thus as many magnitudes as (there) are in AB equal to E, so many (are there) also in CD equal to F. Let AB have been divided into magnitudes AG, GB, equal to E, and CD into (magnitudes) CH, HD, equal to F. So, the number of (divisions) AG, GB will be equal to the number of (divisions) CH, HD. And since AG is equal to E, and CH to F, AG (is) thus equal to E, and AG, CH to E, F. So, for the same (reasons), GB is equal to E, and GB, HD to E, F. Thus, as many (magnitudes) as (there) are in AB equal to E, so many (are there) also in AB, CD equal to E, F. Thus, as many times as AB is (divisible) by E, so many times will AB, CD also be (divisible) by E, F.

Thus, if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second). (Which is) the very thing it was required to show. † In modern notation, this proposition reads $m\,\alpha + m\,\beta + \cdots = m\,(\alpha + \beta + \cdots)$.

Proposition 2†

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and the sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively).

For let a first (magnitude) AB and a third DE be equal multiples of a second C and a fourth F (respectively). And let a fifth (magnitude) BG and a sixth EH also be (other) equal multiples of the second C and the fourth F (respectively). I say that the first (magnitude) and the fifth, being added together, (to give) AG, and the third (magnitude) and the sixth, (being added together, to give) DH, will also be equal multiples of the second (magnitude) C and the fourth F (respectively).



For since AB and DE are equal multiples of C and F (respectively), thus as many (magnitudes) as (there) are in AB equal to C, so many (are there) also in DE equal to F. And so, for the same (reasons), as many (magnitudes) as (there) are in BG equal to C, so many (are there) also in EH equal to F. Thus, as many (magnitudes) as (there) are in the whole of AG equal to C, so many (are there) also in the whole of DH equal to F. Thus, as many times as AG is (divisible) by C, so many times will DH also be divisible by F. Thus, the first (magnitude) and the fifth, being added together, (to give) AG, and the third (magnitude) and the sixth, (being added together, to give) DH, will also be equal multiples of the second (magnitude) C and the fourth F (respectively).

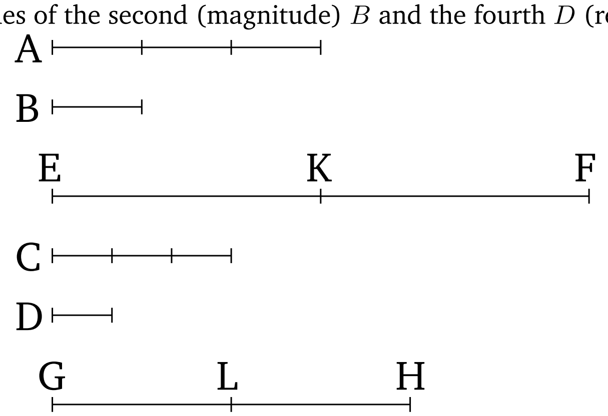
Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively). (Which is) the very thing it was required to show. † In modern notation, this propostion reads $m\,\alpha + n\,\alpha = (m + n)\,\alpha$.

Proposition 3†

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively.

For let a first (magnitude) A and a third C be equal multiples of a second B and a fourth D (respectively), and let the equal multiples EF and GH have been taken of A and C (respectively). I say that EF and GH are equal multiples of B and D (respectively).

For since EF and GH are equal multiples of A and C (respectively), thus as many (magnitudes) as (there) are in EF equal to A, so many (are there) also in GH equal to C. Let EF have been divided into magnitudes EK, KF equal to A, and GH into (magnitudes) GL, LH equal to C. So, the number of (magnitudes) EK, KF will be equal to the number of (magnitudes) GL, LH. And since A and C are equal multiples of B and D (respectively), and EK (is) equal to A, and GL to C, EK and GL are thus equal multiples of B and D (respectively). So, for the same (reasons), KF and LH are equal multiples of B and D (respectively). Therefore, since the first (magnitude) EK and the third GL are equal multiples of the second B and the fourth D (respectively), and the fifth (magnitude) KF and the sixth LH are also equal multiples of the second B and the fourth D (respectively), then the first (magnitude) and fifth, being added together, (to give) EF, and the third (magnitude) and sixth, (being added together, to give) GH, are thus also equal multiples of the second (magnitude) B and the fourth D (respectively) [Prop. 5.2].

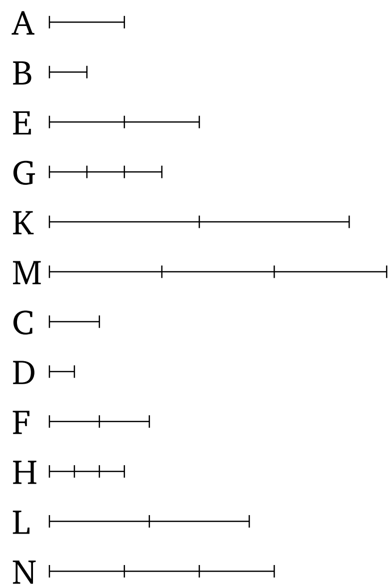


Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively. (Which is) the very thing it was required to show. † In modern notation, this proposition reads $m(n\,\alpha) = (m\,n)\,\alpha$.

Proposition 4†

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever.

For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D. And let equal multiples E and F have been taken of A and C (respectively), and other random equal multiples G and H of B and D (respectively). I say that as E (is) to G, so F (is) to H.



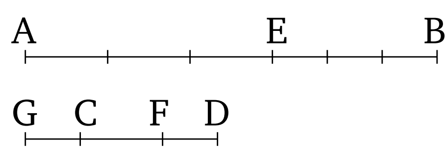
For let equal multiples K and L have been taken of E and F (respectively), and other random equal multiples M and N of G and H (respectively).

[And] since E and F are equal multiples of A and C (respectively), and the equal multiples K and L have been taken of E and F (respectively), K and L are thus equal multiples of A and C (respectively) [Prop. 5.3]. So, for the same (reasons), M and N are equal multiples of B and D (respectively). And since as A is to B, so C (is) to D, and the equal multiples K and L have been taken of A and C (respectively), and the other random equal multiples M and N of B and D (respectively), then if K exceeds M then L also exceeds N, and if (K is) equal (to M then L is also) equal (to N), and if (K is) less (than M then L is also) less (than N) [Def. 5.5]. And K and L are equal multiples of E and F (respectively), and M and N other random equal multiples of G and H (respectively). Thus, as E (is) to G, so F (is) to H [Def. 5.5].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $m\,\alpha : n\,\beta :: m\,\gamma : n\,\delta$, for all m and n.

Proposition 5†

If a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively).



For let the magnitude AB be the same multiple of the magnitude CD that the (part) taken away AE (is) of the (part) taken away CF (respectively). I say that the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

For as many times as AE is (divisible) by CF, so many times let EB also have been made (divisible) by CG.

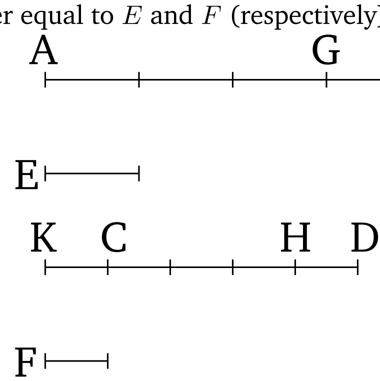
And since AE and EB are equal multiples of CF and GC (respectively), AE and AB are thus equal multiples of CF and GF (respectively) [Prop. 5.1]. And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, AB is an equal multiple of each of GF and CD. Thus, GF (is) equal to CD. Let CF have been subtracted from both. Thus, the remainder GC is equal to the remainder FD. And since AE and EB are equal multiples of CF and GC (respectively), and GC (is) equal to DF, AE and EB are thus equal multiples of CF and FD (respectively). And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, EB and AB are equal multiples of FD and CD (respectively). Thus, the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

Thus, if a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively). (Which is) the very thing it was required to show. † In modern notation, this proposition reads $m\,\alpha - m\,\beta = m\,(\alpha - \beta)$.

Proposition 6†

If two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples of them (respectively).

For let two magnitudes AB and CD be equal multiples of two magnitudes E and F (respectively). And let the (parts) taken away (from the former) AG and CH be equal multiples of E and F (respectively). I say that the remainders GB and HD are also either equal to E and F (respectively), or (are) equal multiples of them.



For let GB be, first of all, equal to E. I say that HD is also equal to F.

For let CK be made equal to F. Since AG and CH are equal multiples of E and F (respectively), and GB (is) equal to E, and KC to F, AB and KH are thus equal multiples of E and F (respectively) [Prop. 5.2]. And AB and CD are assumed (to be) equal multiples of E and F (respectively). Thus, KH and CD are equal multiples of F and F (respectively). Therefore, KH and CD are each equal multiples of F. Thus, KH is equal to CD. Let CH have be taken away from both. Thus, the remainder KC is equal to the remainder HD. But, F is equal to KC. Thus, HD is also equal to F. Hence, if GB is equal to E then HD will also be equal to F.

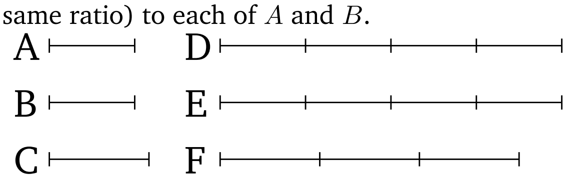
So, similarly, we can show that even if GB is a multiple of E then HD will also be the same multiple of F.

Thus, if two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples of them (respectively). (Which is) the very thing it was required to show. † In modern notation, this proposition reads $m\,\alpha - n\,\alpha = (m-n)\,\alpha$.

Proposition 7

Equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Let A and B be equal magnitudes, and C some other random magnitude. I say that A and B each have the same ratio to C, and (that) C (has the same ratio) to each of A and B.



For let the equal multiples D and E have been taken of A and B (respectively), and the other random multiple F of C.

Therefore, since D and E are equal multiples of A and B (respectively), and A (is) equal to B, D (is) thus also equal to E. And F (is) different, at random. Thus, if D exceeds F then E also exceeds F, and if (D is) equal (to F then E is also) equal (to F), and if (D is) less (than F then E is also) less (than F). And D and E are equal multiples of A and B (respectively), and F another random multiple of C. Thus, as A (is) to C, so B (is) to C [Def. 5.5].

[So] I say that C† also has the same ratio to each of A and B.

For, similarly, we can show, by the same construction, that D is equal to E. And F (has) some other (value). Thus, if F exceeds D then it also exceeds E, and if (F is) equal (to D then it is also) equal (to E), and if (F is) less (than D then it is also) less (than E). And F is a multiple of C, and D and E other random equal multiples of A and B. Thus, as C (is) to A, so C (is) to B [Def. 5.5].

Thus, equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Corollary‡

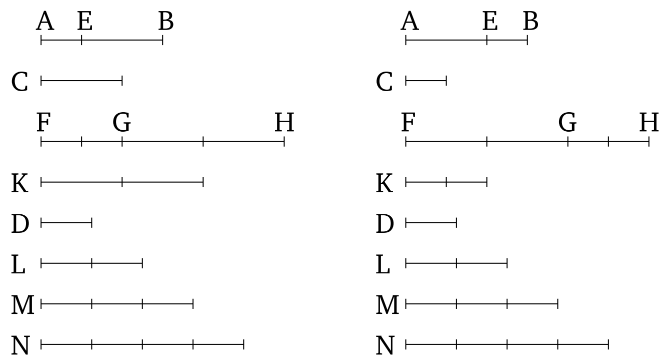
So (it is) clear, from this, that if some magnitudes are proportional then they will also be proportional inversely. (Which is) the very thing it was required to show. † The Greek text has "E", which is obviously a mistake.

‡ In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\beta : \alpha :: \delta : \gamma$.

Proposition 8

For unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater.

Let AB and C be unequal magnitudes, and let AB be the greater (of the two), and D another random magnitude. I say that AB has a greater ratio to D than C (has) to D, and (that) D has a greater ratio to C than (it has) to AB.



For since AB is greater than C, let BE be made equal to C. So, the lesser of AE and EB, being multiplied, will sometimes be greater than D [Def. 5.4]. First of all, let AE be less than EB, and let AE have been multiplied, and let FG be a multiple of it which (is) greater than D. And as many times as FG is (divisible) by AE, so many times let GH also have become (divisible) by EB, and K by C. And let the double multiple L of D have been taken, and the triple multiple M, and several more, (each increasing) in order by one, until the (multiple) taken becomes the first multiple of D (which is) greater than K. Let it have been taken, and let it also be the quadruple multiple N of D—the first (multiple) greater than K.

Therefore, since K is less than N first, K is thus not less than M. And since FG and GH are equal multiples of AE and EB (respectively), FG and FH are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. And FG and K are equal multiples of AE and C (respectively). Thus, FH and K are equal multiples of AB and C (respectively). Thus, FH, K are equal multiples of AB, C. Again, since GH and K are equal multiples of EB and C, and EB (is) equal to C, GH (is) thus also equal to K. And K is not less than M. Thus, GH not less than M either. And FG (is) greater than D. Thus, the whole of FH is greater than D and M (added) together. But, D and M (added) together is equal to N, inasmuch as M is three times D, and M and D (added) together is four times D, and N is also four times D. Thus, M and D (added) together is equal to N. But, FH is greater than M and D. Thus, FH exceeds N. And K does not exceed N. And FH, K are equal multiples of AB, C, and N another random multiple of D. Thus, AB has a greater ratio to D than C (has) to D [Def. 5.7].

So, I say that D also has a greater ratio to C than D (has) to AB.

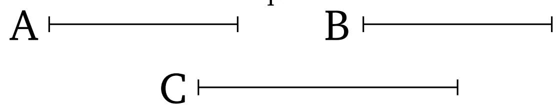
For, similarly, by the same construction, we can show that N exceeds K, and N does not exceed FH. And N is a multiple of D, and FH, K other random equal multiples of AB, C (respectively). Thus, D has a greater ratio to C than D (has) to AB [Def. 5.5].

And so let AE be greater than EB. So, the lesser, EB, being multiplied, will sometimes be greater than D. Let it have been multiplied, and let GH be a multiple of EB (which is) greater than D. And as many times as GH is (divisible) by EB, so many times let FG also have become (divisible) by AE, and K by C. So, similarly (to the above), we can show that FH and K are equal multiples of AB and C (respectively). And, similarly (to the above), let the multiple N of D, (which is) the first (multiple) greater than FG, have been taken. So, FG is again not less than M. And GH (is) greater than D. Thus, the whole of FH exceeds D and M, that is to say N. And K does not exceed N, inasmuch as FG, which (is) greater than GH—that is to say, K—also does not exceed N. And, following the above (arguments), we (can) complete the proof in the same manner.

Thus, for unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater. (Which is) the very thing it was required to show.

Proposition 9

(Magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal.



For let A and B each have the same ratio to C. I say that A is equal to B.

For if not, A and B would not each have the same ratio to C [Prop. 5.8]. But they do. Thus, A is equal to B.

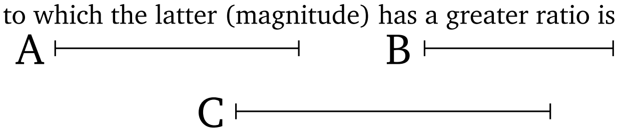
So, again, let C have the same ratio to each of A and B. I say that A is equal to B.

For if not, C would not have the same ratio to each of A and B [Prop. 5.8]. But it does. Thus, A is equal to B.

Thus, (magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal. (Which is) the very thing it was required to show.

Proposition 10

For (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser.



For let A have a greater ratio to C than B (has) to C. I say that A is greater than B.

For if not, A is surely either equal to or less than B. In fact, A is not equal to B. For (then) A and B would each have the same ratio to C [Prop. 5.7]. But they do not. Thus, A is not equal to B. Neither, indeed, is A less than B. For (then) A would have a lesser ratio to C than B (has) to C [Prop. 5.8]. But it does not. Thus, A is not less than B. And it was shown not (to be) equal either. Thus, A is greater than B.

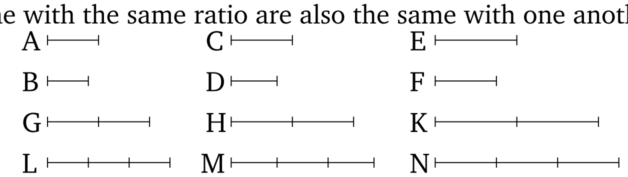
So, again, let C have a greater ratio to B than C (has) to A. I say that B is less than A.

For if not, (it is) surely either equal or greater. In fact, B is not equal to A. For (then) C would have the same ratio to each of A and B [Prop. 5.7]. But it does not. Thus, A is not equal to B. Neither, indeed, is B greater than A. For (then) C would have a lesser ratio to B than (it has) to A [Prop. 5.8]. But it does not. Thus, B is not greater than A. And it was shown that (it is) not equal (to A) either. Thus, B is less than A.

Thus, for (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser. (Which is) the very thing it was required to show.

Proposition 11†

(Ratios which are) the same with the same ratio are also the same with one another.



For let it be that as A (is) to B, so C (is) to D, and as C (is) to D, so E (is) to F. I say that as A is to B, so E (is) to F.

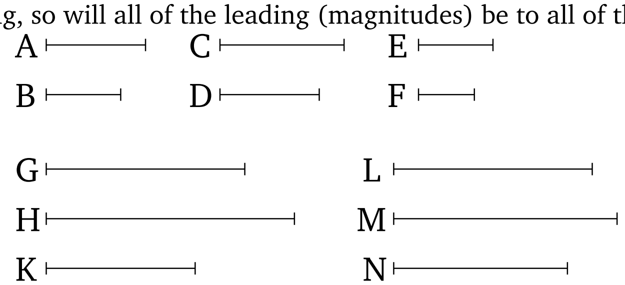
For let the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively).

And since as A is to B, so C (is) to D, and the equal multiples G and H have been taken of A and C (respectively), and the other random equal multiples L and M of B and D (respectively), thus if G exceeds L then H also exceeds M, and if (G is) equal (to L then H is also) equal (to M), and if (G is) less (than L then H is also) less (than M) [Def. 5.5]. Again, since as C is to D, so E (is) to F, and the equal multiples H and K have been taken of C and E (respectively), and the other random equal multiples M and N of D and F (respectively), thus if H exceeds M then K also exceeds N, and if (H is) equal (to M then K is also) equal (to N), and if (H is) less (than M then K is also) less (than N) [Def. 5.5]. But (we saw that) if H was exceeding M then G was also exceeding L, and if (H was) equal (to M then G was also) equal (to L), and if (H was) less (than M then G was also) less (than L). And, hence, if G exceeds L then K also exceeds N, and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N). And G and K are equal multiples of A and E (respectively), and L and N other random equal multiples of B and F (respectively). Thus, as A is to B, so E (is) to F [Def. 5.5].

Thus, (ratios which are) the same with the same ratio are also the same with one another. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta :: \epsilon : \zeta$ then $\alpha : \beta :: \epsilon : \zeta$.

Proposition 12†

If there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following.



Let there be any number of magnitudes whatsoever, A, B, C, D, E, F, (which are) proportional, (so that) as A (is) to B, so C (is) to D, and E to F. I say that as A is to B, so A, C, E (are) to B, D, F.

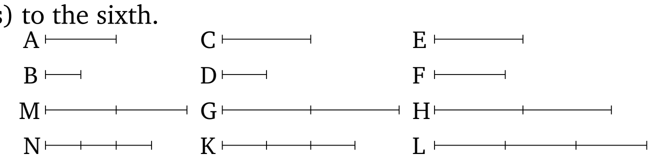
For let the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively).

And since as A is to B, so C (is) to D, and E to F, and the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively), thus if G exceeds L then H also exceeds M, and K (exceeds) N, and if (G is) equal (to L then H is also) equal (to M, and K to N), and if (G is) less (than L then H is also) less (than M, and K than N) [Def. 5.5]. And, hence, if G exceeds L then G, H, K also exceed L, M, N, and if (G is) equal (to L then G, H, K are also) equal (to L, M, N) and if (G is) less (than L then G, H, K are also) less (than L, M, N). And G and G, H, K are equal multiples of A and A, C, E (respectively), inasmuch as if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second) [Prop. 5.1]. So, for the same (reasons), L and L, M, N are also equal multiples of B and B, D, F (respectively). Thus, as A is to B, so A, C, E (are) to B, D, F (respectively).

Thus, if there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \alpha' :: \beta : \beta' :: \gamma : \gamma'$ *etc.* then $\alpha : \alpha' :: (\alpha + \beta + \gamma + \cdots) : (\alpha' + \beta' + \gamma' + \cdots)$.

Proposition 13†

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the third (magnitude) has a greater ratio to the fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth.



For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D, and let the third (magnitude) C have a greater ratio to the fourth D than a fifth E (has) to a sixth F. I say that the first (magnitude) A will also have a greater ratio to the second B than the fifth E (has) to the sixth F.

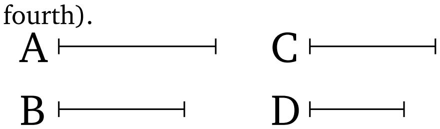
For since there are some equal multiples of C and E, and other random equal multiples of D and F, (for which) the multiple of C exceeds the (multiple) of D, and the multiple of E does not exceed the multiple of F [Def. 5.7], let them have been taken. And let G and H be equal multiples of C and E (respectively), and K and L other random equal multiples of D and F (respectively), such that G exceeds K, but H does not exceed L. And as many times as G is (divisible) by C, so many times let M be (divisible) by A. And as many times as K (is divisible) by D, so many times let N be (divisible) by B.

And since as A is to B, so C (is) to D, and the equal multiples M and G have been taken of A and C (respectively), and the other random equal multiples N and K of B and D (respectively), thus if M exceeds N then G exceeds K, and if (M is) equal (to N then G is also) equal (to K), and if (M is) less (than N then G is also) less (than K) [Def. 5.5]. And G exceeds K. Thus, M also exceeds N. And H does not exceeds L. And M and H are equal multiples of A and E (respectively), and N and L other random equal multiples of B and F (respectively). Thus, A has a greater ratio to B than E (has) to F [Def. 5.7].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and a third (magnitude) has a greater ratio to a fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta > \epsilon : \zeta$ then $\alpha : \beta > \epsilon : \zeta$.

Proposition 14†

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is) equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth).



For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D. And let A be greater than C. I say that B is also greater than D.

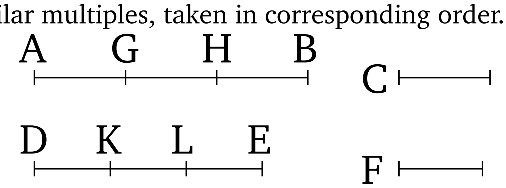
For since A is greater than C, and B (is) another random [magnitude], A thus has a greater ratio to B than C (has) to B [Prop. 5.8]. And as A (is) to B, so C (is) to D. Thus, C also has a greater ratio to D than C (has) to B. And that (magnitude) to which the same (magnitude) has a greater ratio is the lesser [Prop. 5.10]. Thus, D (is) less than B. Hence, B is greater than D.

So, similarly, we can show that even if A is equal to C then B will also be equal to D, and even if A is less than C then B will also be less than D.

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is) equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth). (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha \gtreqless \gamma$ as $\beta \gtreqless \delta$.

Proposition 15†

Parts have the same ratio as similar multiples, taken in corresponding order.



For let AB and DE be equal multiples of C and F (respectively). I say that as C is to F, so AB (is) to DE.

For since AB and DE are equal multiples of C and F (respectively), thus as many magnitudes as there are in AB equal to C, so many (are there) also in DE equal to F. Let AB have been divided into (magnitudes) AG, GH, HB, equal to C, and DE into (magnitudes) DK, KL, LE, equal to F. So, the number of (magnitudes) AG, GH, HB will equal the number of (magnitudes) DK, KL, LE. And since AG, GH, HB are equal to one another, and DK, KL, LE are also equal to one another, thus as AG is to DK, so GH (is) to KL, and HB to LE [Prop. 5.7]. And, thus (for proportional magnitudes), as one of the leading (magnitudes) will be to one of the following, so all of the leading (magnitudes will be) to all of the following [Prop. 5.12]. Thus, as AG is to DK, so AB (is) to DE. And AG is equal to C, and DK to F. Thus, as C is to F, so AB (is) to DE.

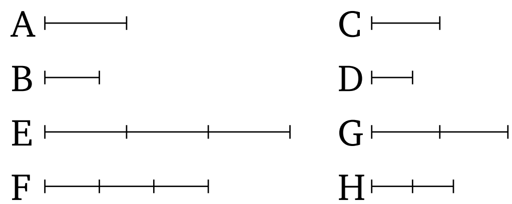
Thus, parts have the same ratio as similar multiples, taken in corresponding order. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that $\alpha : \beta :: m\,\alpha : m\,\beta$.

Proposition 16†

If four magnitudes are proportional then they will also be proportional alternately.

Let A, B, C and D be four proportional magnitudes, (such that) as A (is) to B, so C (is) to D. I say that they will also be [proportional] alternately, (so that) as A (is) to C, so B (is) to D.

For let the equal multiples E and F have been taken of A and B (respectively), and the other random equal multiples G and H of C and D (respectively).

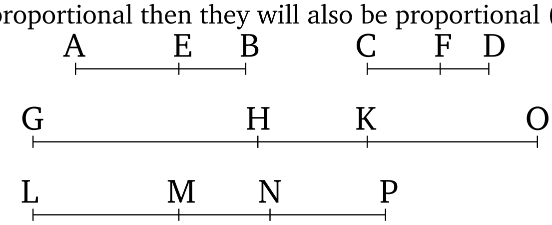


And since E and F are equal multiples of A and B (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as A is to B, so E (is) to F. But as A (is) to B, so C (is) to D. And, thus, as C (is) to D, so E (is) to F [Prop. 5.11]. Again, since G and H are equal multiples of C and D (respectively), thus as C is to D, so G (is) to H [Prop. 5.15]. But as C (is) to D, [so] E (is) to F. And, thus, as E (is) to F, so G (is) to H [Prop. 5.11]. And if four magnitudes are proportional, and the first is greater than the third then the second will also be greater than the fourth, and if (the first is) equal (to the third then the second will also be) equal (to the fourth), and if (the first is) less (than the third then the second will also be) less (than the fourth) [Prop. 5.14]. Thus, if E exceeds G then F also exceeds H, and if (E is) equal (to G then F is also) equal (to H), and if (E is) less (than G then F is also) less (than H). And E and F are equal multiples of A and B (respectively), and G and H other random equal multiples of C and D (respectively). Thus, as A is to C, so B (is) to D [Def. 5.5].

Thus, if four magnitudes are proportional then they will also be proportional alternately. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \gamma :: \beta : \delta$.

Proposition 17†

If composed magnitudes are proportional then they will also be proportional (when) separarted.



Let AB, BE, CD, and DF be composed magnitudes (which are) proportional, (so that) as AB (is) to BE, so CD (is) to DF. I say that they will also be proportional (when) separated, (so that) as AE (is) to EB, so CF (is) to DF.

For let the equal multiples GH, HK, LM, and MN have been taken of AE, EB, CF, and FD (respectively), and the other random equal multiples KO and NP of EB and FD (respectively).

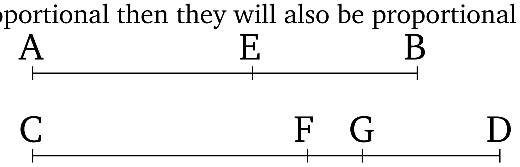
And since GH and HK are equal multiples of AE and EB (respectively), GH and GK are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. But GH and LM are equal multiples of AE and CF (respectively). Thus, GK and LM are equal multiples of AB and CF (respectively). Again, since LM and MN are equal multiples of CF and FD (respectively), LM and LN are thus equal multiples of CF and CD (respectively) [Prop. 5.1]. And LM and GK were equal multiples of CF and AB (respectively). Thus, GK and LN are equal multiples of AB and CD (respectively). Thus, GK, LN are equal multiples of AB, CD. Again, since HK and MN are equal multiples of EB and FD (respectively), and KO and NP are also equal multiples of EB and FD (respectively), then, added together, HO and MP are also equal multiples of EB and FD (respectively) [Prop. 5.2]. And since as AB (is) to BE, so CD (is) to DF, and the equal multiples GK, LN have been taken of AB, CD, and the equal multiples HO, MP of EB, FD, thus if GK exceeds HO then LN also exceeds MP, and if (GK is) equal (to HO then LN is also) equal (to MP), and if (GK is) less (than HO then LN is also) less (than MP) [Def. 5.5]. So let GK exceed HO, and thus, HK being taken away from both, GH exceeds KO. But (we saw that) if GK was exceeding HO then LN

was also exceeding MP. Thus, LN also exceeds MP, and, MN being taken away from both, LM also exceeds NP. Hence, if GH exceeds KO then LM also exceeds NP. So, similarly, we can show that even if GH is equal to KO then LM will also be equal to NP, and even if (GH is) less (than KO then LM will also be) less (than NP). And GH, LM are equal multiples of AE, CF, and KO, NP other random equal multiples of EB, FD. Thus, as AE is to EB, so CF (is) to FD [Def. 5.5].

Thus, if composed magnitudes are proportional then they will also be proportional (when) separarted. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha+\beta : \beta :: \gamma+\delta : \delta$ then $\alpha : \beta :: \gamma : \delta$.

Proposition 18†

If separated magnitudes are proportional then they will also be proportional (when) composed.



Let AE, EB, CF, and FD be separated magnitudes (which are) proportional, (so that) as AE (is) to EB, so CF (is) to FD. I say that they will also be proportional (when) composed, (so that) as AB (is) to BE, so CD (is) to FD.

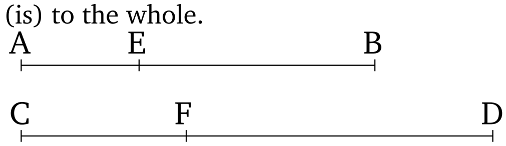
For if (it is) not (the case that) as AB is to BE, so CD (is) to FD, then it will surely be (the case that) as AB (is) to BE, so CD is either to some (magnitude) less than DF, or (some magnitude) greater (than DF).‡

Let it, first of all, be to (some magnitude) less (than DF), (namely) DG. And since composed magnitudes are proportional, (so that) as AB is to BE, so CD (is) to DG, they will thus also be proportional (when) separated [Prop. 5.17]. Thus, as AE is to EB, so CG (is) to GD. But it was also assumed that as AE (is) to EB, so CF (is) to FD. Thus, (it is) also (the case that) as CG (is) to GD, so CF (is) to FD [Prop. 5.11]. And the first (magnitude) CG (is) greater than the third CF. Thus, the second (magnitude) GD (is) also greater than the fourth FD [Prop. 5.14]. But (it is) also less. The very thing is impossible. Thus, (it is) not (the case that) as AB is to BE, so CD (is) to less than FD. Similarly, we can show that neither (is it the case) to greater (than FD). Thus, (it is the case) to the same (as FD).

Thus, if separated magnitudes are proportional then they will also be proportional (when) composed. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha+\beta : \beta :: \gamma+\delta : \delta$.
‡ Here, Euclid assumes, without proof, that a fourth magnitude proportional to three given magnitudes can always be found.

Proposition 19†

If as the whole is to the whole so the (part) taken away is to the (part) taken away then the remainder to the remainder will also be as the whole (is) to the whole.



For let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF. I say that the remainder EB to the remainder FD will also be as the whole AB (is) to the whole CD.

For since as AB is to CD, so AE (is) to CF, (it is) also (the case), alternately, (that) as BA (is) to AE, so DC (is) to CF [Prop. 5.16]. And since composed magnitudes are proportional then they will also be proportional (when) separated, (so that) as BE (is) to EA, so DF (is) to CF [Prop. 5.17]. Also, alternately, as BE (is) to DF, so EA (is) to FC [Prop. 5.16]. And it was assumed that as AE (is) to CF, so the whole AB (is) to the whole CD. And, thus, as the remainder EB (is) to the remainder FD, so the whole AB will be to the whole CD.

Thus, if as the whole is to the whole so the (part) taken away is to the (part) taken away then the remainder to the remainder will also be as the whole (is) to the whole. [(Which is) the very thing it was required to show.]

[And since it was shown (that) as AB (is) to CD, so EB (is) to FD, (it is) also (the case), alternately, (that) as AB (is) to BE, so CD (is) to FD. Thus, composed magnitudes are proportional. And it was shown (that) as BA (is) to AE, so DC (is) to CF. And (the latter) is converted (from the former).]

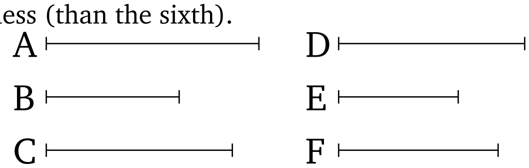
Corollary‡

So (it is) clear, from this, that if composed magnitudes are proportional then they will also be proportional (when) converted. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \beta :: \alpha - \gamma : \beta - \delta$.

‡ In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \alpha - \beta :: \gamma : \gamma - \delta$.

Proposition 20†

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



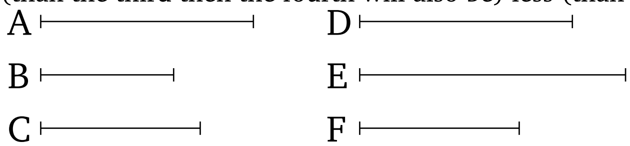
Let A, B, and C be three magnitudes, and D, E, F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, (so that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. And let A be greater than C, via equality. I say that D will also be greater than F. And if (A is) equal (to C then D will also be) equal (to F). And if (A is) less (than C then D will also be) less (than F).

For since A is greater than C, and B some other (magnitude), and the greater (magnitude) has a greater ratio than the lesser to the same (magnitude) [Prop. 5.8], A thus has a greater ratio to B than C (has) to B. But as A (is) to B, [so] D (is) to E. And, inversely, as C (is) to B, so F (is) to E [Prop. 5.7 corr.]. Thus, D also has a greater ratio to E than F (has) to E [Prop. 5.13]. And for (magnitudes) having a ratio to the same (magnitude), that having the greater ratio is greater [Prop. 5.10]. Thus, D (is) greater than F. Similarly, we can show that even if A is equal to C then D will also be equal to F, and even if (A is) less (than C then D will also be) less (than F).

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third, then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And (if the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \delta : \epsilon$ and $\beta : \gamma :: \epsilon : \zeta$ then $\alpha \gtreqless \gamma$ as $\delta \gtreqless \zeta$.

Proposition 21†

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



Let A, B, and C be three magnitudes, and D, E, F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B, so E (is) to F, and as B (is) to C, so D (is) to E. And let A be greater than C, via equality. I say that D will also be greater than F. And if (A is) equal (to C then D will also be) equal (to F). And if (A is) less (than C then D will also be) less (than F).

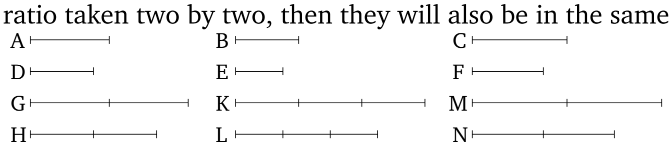
For since A is greater than C, and B some other (magnitude), A thus has a greater ratio to B than C (has) to B [Prop. 5.8]. But as A (is) to B, so E (is) to F. And, inversely, as C (is) to B, so E (is) to D [Prop. 5.7 corr.]. Thus, E also has a greater ratio to F than E (has) to D [Prop. 5.13]. And that (magnitude) to which the same (magnitude) has a greater ratio is (the) lesser (magnitude) [Prop. 5.10]. Thus, F is less than D. Thus, D is greater than F. Similarly, we can show that even if A is equal to C then D will also be equal to F, and even if (A is) less (than C then D will also be) less (than F).

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha \gtreqless \gamma$ as $\delta \gtreqless \zeta$.

Proposition 22†

If there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.



Let there be any number of magnitudes whatsoever, A, B, C, and (some) other (magnitudes), D, E, F, of equal number to them, (which are) in the same ratio taken two by two, (so that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that they will also be in the same ratio via equality. (That is, as A is to C, so D is to F.)

For let the equal multiples G and H have been taken of A and D (respectively), and the other random equal multiples K and L of B and E (respectively), and the yet other random equal multiples M and N of C and F (respectively).

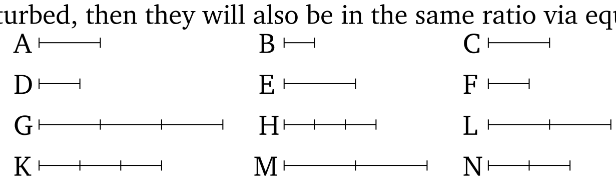
And since as A is to B, so D (is) to E, and the equal multiples G and H have been taken of A and D (respectively), and the other random equal multiples K and L of B and E (respectively), thus as G is to K, so H (is) to L [Prop. 5.4]. And, so, for the same (reasons), as K (is) to M, so L (is) to N. Therefore, since G, K, and M are three magnitudes, and H, L, and N other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, thus, via equality, if G exceeds M then H also exceeds N, and if (G is) equal (to M then H is also) equal (to N), and if (G is) less (than M then H is also) less (than N) [Prop. 5.20]. And G and H are equal multiples of A and D

(respectively), and M and N other random equal multiples of C and F (respectively). Thus, as A is to C, so D (is) to F [Def. 5.5].

Thus, if there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \zeta : \eta$ and $\gamma : \delta :: \eta : \theta$ then $\alpha : \delta :: \epsilon : \theta$.

Proposition 23†

If there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality.



Let A, B, and C be three magnitudes, and D, E and F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B, so E (is) to F, and as B (is) to C, so D (is) to E. I say that as A is to C, so D (is) to F.

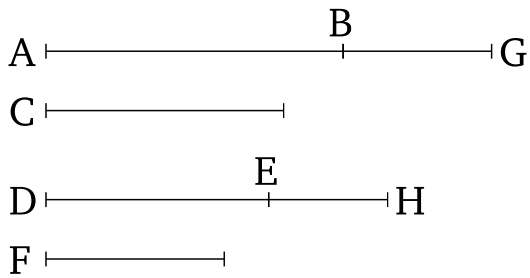
Let the equal multiples G, H, and K have been taken of A, B, and D (respectively), and the other random equal multiples L, M, and N of C, E, and F (respectively).

And since G and H are equal multiples of A and B (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as A (is) to B, so G (is) to H. And, so, for the same (reasons), as E (is) to F, so M (is) to N. And as A is to B, so E (is) to F. And, thus, as G (is) to H, so M (is) to N [Prop. 5.11]. And since as B is to C, so D (is) to E, also, alternately, as B (is) to D, so C (is) to E [Prop. 5.16]. And since H and K are equal multiples of B and D (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as B is to D, so H (is) to K. But, as B (is) to D, so C (is) to E. And, thus, as H (is) to K, so C (is) to E [Prop. 5.11]. Again, since L and M are equal multiples of C and E (respectively), thus as C is to E, so L (is) to M [Prop. 5.15]. But, as C (is) to E, so H (is) to K. And, thus, as H (is) to K, so L (is) to M [Prop. 5.11]. Also, alternately, as H (is) to L, so K (is) to M [Prop. 5.16]. And it was also shown (that) as G (is) to H, so M (is) to N. Therefore, since G, H, and L are three magnitudes, and K, M, and N other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, and their proportion is perturbed, thus, via equality, if G exceeds L then K also exceeds N, and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N) [Prop. 5.21]. And G and K are equal multiples of A and D (respectively), and L and N of C and F (respectively). Thus, as A (is) to C, so D (is) to F [Def. 5.5].

Thus, if there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha : \gamma :: \delta : \zeta$.

Proposition 24†

If a first (magnitude) has to a second the same ratio that third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, then the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and sixth (added together, have) to the fourth.



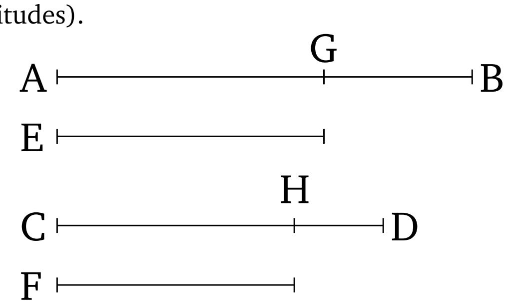
For let a first (magnitude) AB have the same ratio to a second C that a third DE (has) to a fourth F. And let a fifth (magnitude) BG also have the same ratio to the second C that a sixth EH (has) to the fourth F. I say that the first (magnitude) and the fifth, added together, AG, will also have the same ratio to the second C that the third (magnitude) and the sixth, (added together), DH, (has) to the fourth F.

For since as BG is to C, so EH (is) to F, thus, inversely, as C (is) to BG, so F (is) to EH [Prop. 5.7 corr.]. Therefore, since as AB is to C, so DE (is) to F, and as C (is) to BG, so F (is) to EH, thus, via equality, as AB is to BG, so DE (is) to EH [Prop. 5.22]. And since separated magnitudes are proportional then they will also be proportional (when) composed [Prop. 5.18]. Thus, as AG is to GB, so DH (is) to HE. And, also, as BG is to C, so EH (is) to F. Thus, via equality, as AG is to C, so DH (is) to F [Prop. 5.22].

Thus, if a first (magnitude) has to a second the same ratio that a third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, then the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and the sixth (added together, have) to the fourth. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\epsilon : \beta :: \zeta : \delta$ then $\alpha + \epsilon : \beta :: \gamma + \zeta : \delta$.

Proposition 25†

If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes).



Let AB, CD, E, and F be four proportional magnitudes, (such that) as AB (is) to CD, so E (is) to F. And let AB be the greatest of them, and F the least. I say that AB and F is greater than CD and E.

For let AG be made equal to E, and CH equal to F.

[In fact,] since as AB is to CD, so E (is) to F, and E (is) equal to AG, and F to CH, thus as AB is to CD, so AG (is) to CH. And since the whole AB is to the whole CD as the (part) taken away AG (is) to the (part) taken away CH, thus the remainder GB will also be to the remainder HD as the whole AB (is) to the whole CD [Prop. 5.19]. And AB (is) greater than CD. Thus, GB (is) also greater than HD. And since AG is equal to E, and CH to F, thus AG and F is equal to CH and E. And [since] if [equal (magnitudes) are added to unequal (magnitudes) then the wholes are unequal, thus if] AG and F are added to GB, and CH and E to HD—GB and HD being unequal, and GB greater—it is inferred that AB and F (is) greater than CD and E.

Thus, if four magnitudes are proportional then the (sum of the) largest and the smallest of them is greater than the (sum of the) remaining two (magnitudes). (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$, and α is the greatest and δ the least, then $\alpha + \delta > \beta + \gamma$.

ELEMENTS BOOK 6

Similar Figures

Definitions

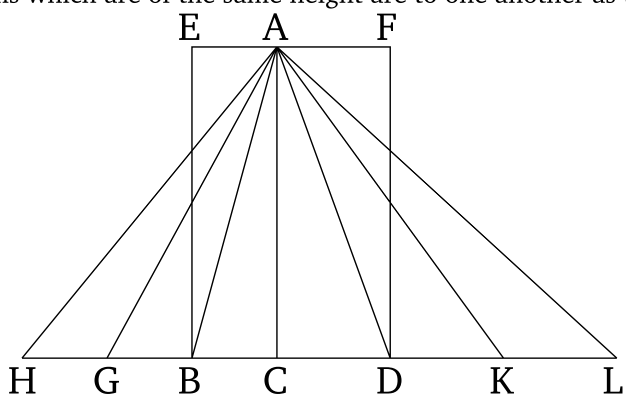
1. Similar rectilinear figures are those (which) have (their) angles separately equal and the (corresponding) sides about the equal angles proportional.

2. A straight-line is said to have been cut in extreme and mean ratio when as the whole is to the greater segment so the greater (segment is) to the lesser.

3. The height of any figure is the (straight-line) drawn from the vertex perpendicular to the base.

Proposition 1†

Triangles and parallelograms which are of the same height are to one another as their bases.



Let ABC and ACD be triangles, and EC and CF parallelograms, of the same height AC. I say that as base BC is to base CD, so triangle ABC (is) to triangle ACD, and parallelogram EC to parallelogram CF.

For let the (straight-line) BD have been produced in each direction to points H and L, and let [any number] (of straight-lines) BG and GH be made equal to base BC, and any number (of straight-lines) DK and KL equal to base CD. And let AG, AH, AK, and AL have been joined.

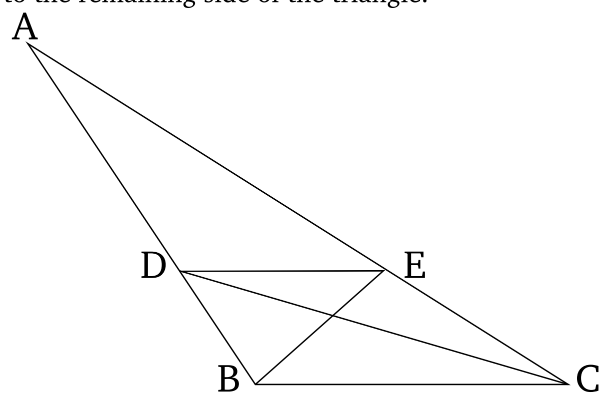
And since CB, BG, and GH are equal to one another, triangles AHG, AGB, and ABC are also equal to one another [Prop. 1.38]. Thus, as many times as base HC is (divisible by) base BC, so many times is triangle AHC also (divisible) by triangle ABC. So, for the same (reasons), as many times as base LC is (divisible) by base CD, so many times is triangle ALC also (divisible) by triangle ACD. And if base HC is equal to base CL then triangle AHC is also equal to triangle ACL [Prop. 1.38]. And if base HC exceeds base CL then triangle AHC also exceeds triangle ACL.‡ And if (HC is) less (than CL then AHC is also) less (than ACL). So, their being four magnitudes, two bases, BC and CD, and two triangles, ABC and ACD, equal multiples have been taken of base BC and triangle ABC—(namely), base HC and triangle AHC—and other random equal multiples of base CD and triangle ADC—(namely), base LC and triangle ALC. And it has been shown that if base HC exceeds base CL then triangle AHC also exceeds triangle ALC, and if (HC is) equal (to CL then AHC is also) equal (to ALC), and if (HC is) less (than CL then AHC is also) less (than ALC). Thus, as base BC is to base CD, so triangle ABC (is) to triangle ACD [Def. 5.5]. And since parallelogram EC is double triangle ABC, and parallelogram FC is double triangle ACD [Prop. 1.34], and parts have the same ratio as similar multiples [Prop. 5.15], thus as triangle ABC is to triangle ACD, so parallelogram EC (is) to parallelogram FC. In fact, since it was shown that as base BC (is) to CD, so triangle ABC (is) to triangle ACD, and as triangle ABC (is) to triangle ACD, so parallelogram EC (is) to parallelogram CF, thus, also, as base BC (is) to base CD, so parallelogram EC (is) to parallelogram FC [Prop. 5.11].

Thus, triangles and parallelograms which are of the same height are to one another as their bases. (Which is) the very thing it was required to show. † As is easily demonstrated, this proposition holds even when the triangles, or parallelograms, do not share a common side, and/or are not right-angled.

‡ This is a straight-forward generalization of Prop. 1.38.

Proposition 2

If some straight-line is drawn parallel to one of the sides of a triangle then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle.



For let DE have been drawn parallel to one of the sides BC of triangle ABC. I say that as BD is to DA, so CE (is) to EA.

For let BE and CD have been joined.

Thus, triangle BDE is equal to triangle CDE. For they are on the same base DE and between the same parallels DE and BC [Prop. 1.38]. And ADE is some other triangle. And equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7]. Thus, as triangle BDE is to [triangle] ADE, so triangle CDE (is) to triangle ADE. But, as triangle BDE (is) to triangle ADE, so (is) BD to DA. For, having the same height—(namely), the (straight-line) drawn from E perpendicular to AB—they are to one another as their bases [Prop. 6.1]. So, for the same (reasons), as triangle CDE (is) to ADE, so CE (is) to EA. And, thus, as BD (is) to DA, so CE (is) to EA [Prop. 5.11].

And so, let the sides AB and AC of triangle ABC have been cut proportionally (such that) as BD (is) to DA, so CE (is) to EA. And let DE have been joined. I say that DE is parallel to BC.

For, by the same construction, since as BD is to DA, so CE (is) to EA, but as BD (is) to DA, so triangle BDE (is) to triangle ADE, and as CE (is) to EA, so triangle CDE (is) to triangle ADE [Prop. 6.1], thus, also, as triangle BDE (is) to triangle ADE, so triangle CDE (is) to triangle ADE [Prop. 5.11]. Thus, triangles BDE and CDE each have the same ratio to ADE. Thus, triangle BDE is equal to triangle CDE [Prop. 5.9]. And they are on the same base DE. And equal triangles, which are also on the same base, are also between the same parallels [Prop. 1.39]. Thus, DE is parallel to BC.

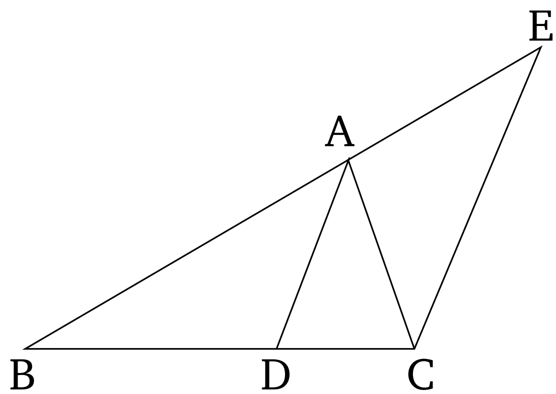
Thus, if some straight-line is drawn parallel to one of the sides of a triangle, then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally, then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle. (Which is) the very thing it was required to show.

Proposition 3

If an angle of a triangle is cut in half, and the straight-line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half.

Let ABC be a triangle. And let the angle BAC have been cut in half by the straight-line AD. I say that as BD is to CD, so BA (is) to AC.

For let CE have been drawn through (point) C parallel to DA. And, BA being drawn through, let it meet (CE) at (point) E.†



And since the straight-line AC falls across the parallel (straight-lines) AD and EC, angle ACE is thus equal to CAD [Prop. 1.29]. But, (angle) CAD is assumed (to be) equal to BAD. Thus, (angle) BAD is also equal to ACE. Again, since the straight-line BAE falls across the parallel (straight-lines) AD and EC, the external angle BAD is equal to the internal (angle) AEC [Prop. 1.29]. And (angle) ACE was also shown (to be) equal to BAD. Thus, angle ACE is also equal to AEC. And, hence, side AE is equal to side AC [Prop. 1.6]. And since AD has been drawn parallel to one of the sides EC of triangle BCE, thus, proportionally, as BD is to DC, so BA (is) to AE [Prop. 6.2]. And AE (is) equal to AC. Thus, as BD (is) to DC, so BA (is) to AC.

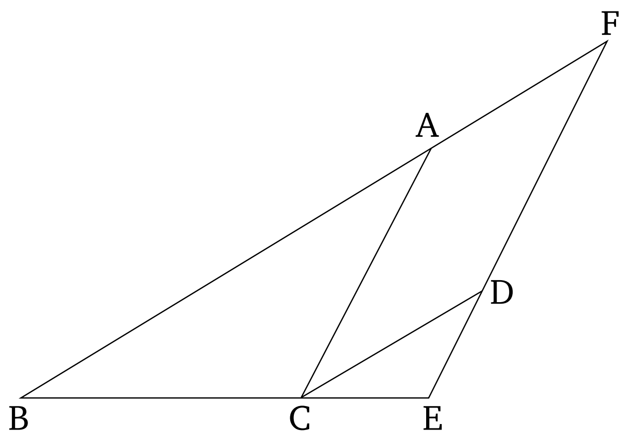
And so, let BD be to DC, as BA (is) to AC. And let AD have been joined. I say that angle BAC has been cut in half by the straight-line AD.

For, by the same construction, since as BD is to DC, so BA (is) to AC, then also as BD (is) to DC, so BA is to AE. For AD has been drawn parallel to one (of the sides) EC of triangle BCE [Prop. 6.2]. Thus, also, as BA (is) to AC, so BA (is) to AE [Prop. 5.11]. Thus, AC (is) equal to AE [Prop. 5.9]. And, hence, angle AEC is equal to ACE [Prop. 1.5]. But, AEC [is] equal to the external (angle) BAD, and ACE is equal to the alternate (angle) CAD [Prop. 1.29]. Thus, (angle) BAD is also equal to CAD. Thus, angle BAC has been cut in half by the straight-line AD.

Thus, if an angle of a triangle is cut in half, and the straight-line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half. (Which is) the very thing it was required to show. † The fact that the two straight-lines meet follows because the sum of ACE and CAE is less than two right-angles, as can easily be demonstrated. See Post. 5.

Proposition 4

In equiangular triangles the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.



Let ABC and DCE be equiangular triangles, having angle ABC equal to DCE, and (angle) BAC to CDE, and, further, (angle) ACB to CED. I say that in triangles ABC and DCE the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.

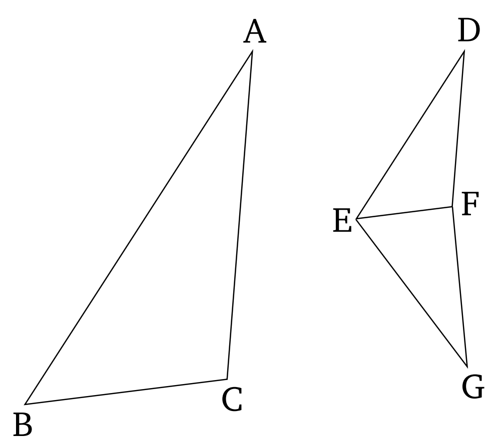
Let BC be placed straight-on to CE. And since angles ABC and ACB are less than two right-angles [Prop 1.17], and ACB (is) equal to DEC, thus ABC and DEC are less than two right-angles. Thus, BA and ED, being produced, will meet [C.N. 5]. Let them have been produced, and let them meet at (point) F.

And since angle DCE is equal to ABC, BF is parallel to CD [Prop. 1.28]. Again, since (angle) ACB is equal to DEC, AC is parallel to FE [Prop. 1.28]. Thus, $FACD$ is a parallelogram. Thus, FA is equal to DC, and AC to FD [Prop. 1.34]. And since AC has been drawn parallel to one (of the sides) FE of triangle FBE, thus as BA is to AF, so BC (is) to CE [Prop. 6.2]. And AF (is) equal to CD. Thus, as BA (is) to CD, so BC (is) to CE, and, alternately, as AB (is) to BC, so DC (is) to CE [Prop. 5.16]. Again, since CD is parallel to BF, thus as BC (is) to CE, so FD (is) to DE [Prop. 6.2]. And FD (is) equal to AC. Thus, as BC is to CE, so AC (is) to DE, and, alternately, as BC (is) to CA, so CE (is) to ED [Prop. 6.2]. Therefore, since it was shown that as AB (is) to BC, so DC (is) to CE, and as BC (is) to CA, so CE (is) to ED, thus, via equality, as BA (is) to AC, so CD (is) to DE [Prop. 5.22].

Thus, in equiangular triangles the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond. (Which is) the very thing it was required to show.

Proposition 5

If two triangles have proportional sides then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.



Let ABC and DEF be two triangles having proportional sides, (so that) as AB (is) to BC, so DE (is) to EF, and as BC (is) to CA, so EF (is) to FD, and, further, as BA (is) to AC, so ED (is) to DF. I say that triangle ABC is equiangular to triangle DEF, and (that the triangles) will have the angles which corresponding sides subtend equal. (That is), (angle) ABC (equal) to DEF, BCA to EFD, and, further, BAC to EDF.

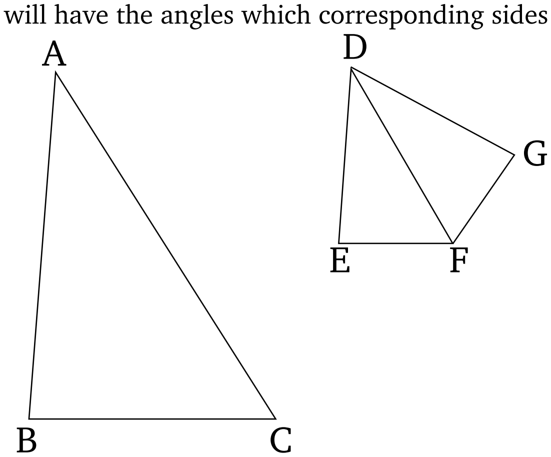
For let (angle) FEG, equal to angle ABC, and (angle) EFG, equal to ACB, have been constructed on the straight-line EF at the points E and F on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at A is equal to the remaining (angle) at G [Prop. 1.32].

Thus, triangle ABC is equiangular to [triangle] EGF. Thus, for triangles ABC and EGF, the sides about the equal angles are proportional, and (those) sides subtending equal angles correspond [Prop. 6.4]. Thus, as AB is to BC, [so] GE (is) to EF. But, as AB (is) to BC, so, it was assumed, (is) DE to EF. Thus, as DE (is) to EF, so GE (is) to EF [Prop. 5.11]. Thus, DE and GE each have the same ratio to EF. Thus, DE is equal to GE [Prop. 5.9]. So, for the same (reasons), DF is also equal to GF. Therefore, since DE is equal to EG, and EF (is) common, the two (sides) DE, EF are equal to the two (sides) GE, EF (respectively). And base DF [is] equal to base FG. Thus, angle DEF is equal to angle GEF [Prop. 1.8], and triangle DEF (is) equal to triangle GEF, and the remaining angles (are) equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle DFE is also equal to GFE, and (angle) EDF to EGF. And since (angle) FED is equal to GEF, and (angle) GEF to ABC, angle ABC is thus also equal to DEF. So, for the same (reasons), (angle) ACB is also equal to DFE, and, further, the (angle) at A to the (angle) at D. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have proportional sides then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

Proposition 6

If two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.



Let ABC and DEF be two triangles having one angle, BAC, equal to one angle, EDF (respectively), and the sides about the equal angles proportional, (so that) as BA (is) to AC, so ED (is) to DF. I say that triangle ABC is equiangular to triangle DEF, and will have angle ABC equal to DEF, and (angle) ACB to DFE.

For let (angle) FDG, equal to each of BAC and EDF, and (angle) DFG, equal to ACB, have been constructed on the straight-line AF at the points D and F on it (respectively) [Prop. 1.23]. Thus, the remaining angle at B is equal to the remaining angle at G [Prop. 1.32].

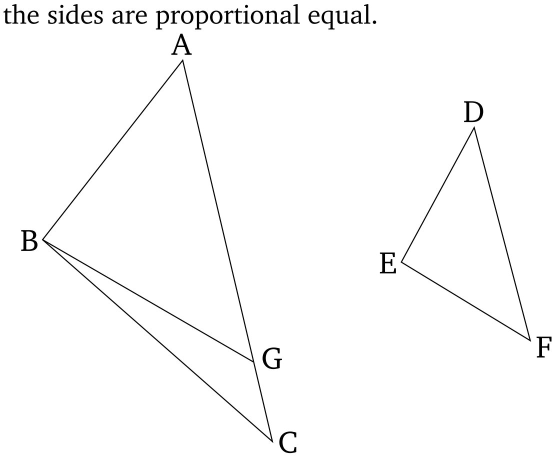
Thus, triangle ABC is equiangular to triangle DGF. Thus, proportionally, as BA (is) to AC, so GD (is) to DF [Prop. 6.4]. And it was also assumed that as BA is) to AC, so ED (is) to DF. And, thus, as ED (is) to DF, so

GD (is) to DF [Prop. 5.11]. Thus, ED (is) equal to DG [Prop. 5.9]. And DF (is) common. So, the two (sides) ED, DF are equal to the two (sides) GD, DF (respectively). And angle EDF [is] equal to angle GDF. Thus, base EF is equal to base GF, and triangle DEF is equal to triangle GDF, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, (angle) DFG is equal to DFE, and (angle) DGF to DEF. But, (angle) DFG is equal to ACB. Thus, (angle) ACB is also equal to DFE. And (angle) BAC was also assumed (to be) equal to EDF. Thus, the remaining (angle) at B is equal to the remaining (angle) at E [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

Proposition 7

If two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles either both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides are proportional equal.



Let ABC and DEF be two triangles having one angle, BAC, equal to one angle, EDF (respectively), and the sides about (some) other angles, ABC and DEF (respectively), proportional, (so that) as AB (is) to BC, so DE (is) to EF, and the remaining (angles) at C and F, first of all, both less than right-angles. I say that triangle ABC is equiangular to triangle DEF, and (that) angle ABC will be equal to DEF, and (that) the remaining (angle) at C (will be) manifestly equal to the remaining (angle) at F.

For if angle ABC is not equal to (angle) DEF then one of them is greater. Let ABC be greater. And let (angle) ABG, equal to (angle) DEF, have been constructed on the straight-line AB at the point B on it [Prop. 1.23].

And since angle A is equal to (angle) D, and (angle) ABG to DEF, the remaining (angle) AGB is thus equal to the remaining (angle) DFE [Prop. 1.32]. Thus, triangle ABG is equiangular to triangle DEF. Thus, as AB is to BG, so DE (is) to EF [Prop. 6.4]. And as DE (is) to EF, [so] it was assumed (is) AB to BC. Thus, AB has the same ratio to each of BC and BG [Prop. 5.11]. Thus, BC (is) equal to BG [Prop. 5.9]. And, hence, the angle at C is equal to angle BGC [Prop. 1.5]. And the angle at C was assumed (to be) less than a right-angle. Thus, (angle) BGC is also less than a right-angle. Hence, the adjacent angle to it, AGB, is greater than a right-angle [Prop. 1.13]. And (AGB) was shown to be equal to the (angle) at F. Thus, the (angle) at F is also greater than a right-angle. But it was assumed (to be) less than a right-angle. The very thing is absurd. Thus, angle ABC is not unequal to (angle) DEF. Thus, (it is) equal. And the (angle) at A is also equal to the (angle) at D. And thus the remaining (angle) at C is equal to the remaining (angle) at F [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF.

But, again, let each of the (angles) at C and F be assumed (to be) not less than a right-angle. I say, again, that triangle ABC is equiangular to triangle DEF in this case also.

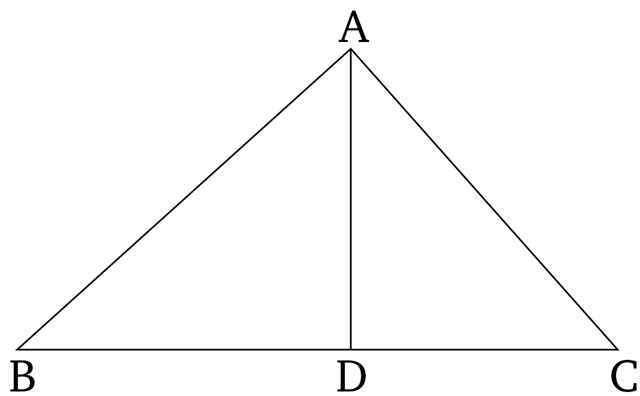
For, with the same construction, we can similarly show that BC is equal to BG. Hence, also, the angle at C is equal to (angle) BGC. And the (angle) at C (is) not less than a right-angle. Thus, BGC (is) not less than a right-angle either. So, in triangle BGC the (sum of) two angles is not less than two right-angles. The very thing is impossible [Prop. 1.17]. Thus, again, angle ABC is not unequal to DEF. Thus, (it is) equal. And the (angle) at A is also equal to the (angle) at D. Thus, the remaining (angle) at C is equal to the remaining (angle) at F [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF.

Thus, if two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides (are) proportional equal. (Which is) the very thing it was required to show.

Proposition 8

If, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the triangles around the perpendicular are similar to the whole (triangle), and to one another.

Let ABC be a right-angled triangle having the angle BAC a right-angle, and let AD have been drawn from A, perpendicular to BC [Prop. 1.12]. I say that triangles ABD and ADC are each similar to the whole (triangle) ABC and, further, to one another.



For since (angle) BAC is equal to ADB—for each (are) right-angles—and the (angle) at B (is) common to the two triangles ABC and ABD, the remaining (angle) ACB is thus equal to the remaining (angle) BAD [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle ABD. Thus, as BC, subtending the right-angle in triangle ABC, is to BA, subtending the right-angle in triangle ABD, so the same AB, subtending the angle at C in triangle ABC, (is) to BD, subtending the equal (angle) BAD in triangle ABD, and, further, (so is) AC to AD, (both) subtending the angle at B common to the two triangles [Prop. 6.4]. Thus, triangle ABC is equiangular to triangle ABD, and has the sides about the equal angles proportional. Thus, triangle ABC [is] similar to triangle ABD [Def. 6.1]. So, similarly, we can show that triangle ABC is also similar to triangle ADC. Thus, [triangles] ABD and ADC are each similar to the whole (triangle) ABC.

So I say that triangles ABD and ADC are also similar to one another.

For since the right-angle BDA is equal to the right-angle ADC, and, indeed, (angle) BAD was also shown (to be) equal to the (angle) at C, thus the remaining (angle) at B is also equal to the remaining (angle) DAC [Prop. 1.32]. Thus, triangle ABD is equiangular to triangle ADC. Thus, as BD, subtending (angle) BAD in triangle ABD, is to DA, subtending the (angle) at C in triangle ADC, (which is) equal to (angle) BAD, so (is) the same AD, subtending the angle at B in triangle ABD, to DC, subtending (angle) DAC in triangle ADC, (which is) equal to the (angle) at B, and, further, (so is) BA to AC, (each) subtending right-angles [Prop. 6.4]. Thus, triangle ABD is similar to triangle ADC [Def. 6.1].

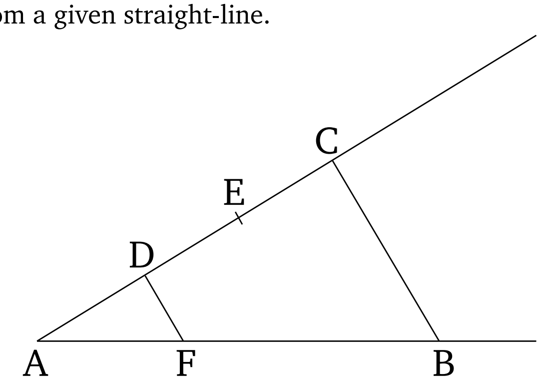
Thus, if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the triangles around the perpendicular are similar to the whole (triangle), and to one another. [(Which is) the very thing it was required to show.]

Corollary

So (it is) clear, from this, that if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the (straight-line so) drawn is in mean proportion to the pieces of the base.† (Which is) the very thing it was required to show. † In other words, the perpendicular is the geometric mean of the pieces.

Proposition 9

To cut off a prescribed part from a given straight-line.



Let AB be the given straight-line. So it is required to cut off a prescribed part from AB.

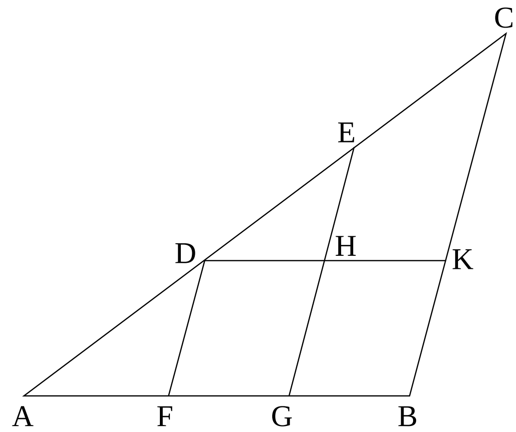
So let a third (part) have been prescribed. [And] let some straight-line AC have been drawn from (point) A, encompassing a random angle with AB. And let a random point D have been taken on AC. And let DE and EC be made equal to AD [Prop. 1.3]. And let BC have been joined. And let DF have been drawn through D parallel to it [Prop. 1.31].

Therefore, since FD has been drawn parallel to one of the sides, BC, of triangle ABC, then, proportionally, as CD is to DA, so BF (is) to FA [Prop. 6.2]. And CD (is) double DA. Thus, BF (is) also double FA. Thus, BA (is) triple AF.

Thus, the prescribed third part, AF, has been cut off from the given straight-line, AB. (Which is) the very thing it was required to do.

Proposition 10

To cut a given uncut straight-line similarly to a given cut (straight-line).



Let AB be the given uncut straight-line, and AC a (straight-line) cut at points D and E, and let (AC) be laid down so as to encompass a random angle (with AB). And let CB have been joined. And let DF and EG have been drawn through (points) D and E (respectively), parallel to BC, and let DHK have been drawn through (point) D, parallel to AB [Prop. 1.31].

Thus, FH and HB are each parallelograms. Thus, DH (is) equal to FG, and HK to GB [Prop. 1.34]. And since the straight-line HE has been drawn parallel to one of the sides, KC, of triangle DKC, thus, proportionally, as CE is to ED, so KH (is) to HD [Prop. 6.2]. And KH (is) equal to BG, and HD to GF. Thus, as CE is to ED, so BG (is) to GF. Again, since FD has been drawn parallel to one of the sides, GE, of triangle AGE, thus, proportionally, as ED is to DA, so GF (is) to FA [Prop. 6.2]. And it was also shown that as CE (is) to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF, and as ED (is) to DA, so GF (is) to FA.

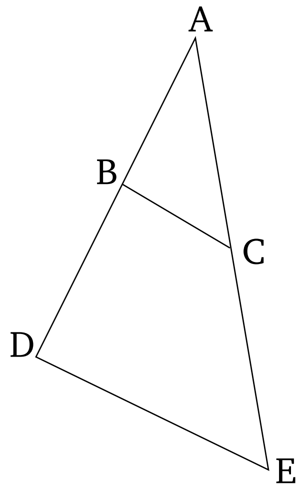
Thus, the given uncut straight-line, AB, has been cut similarly to the given cut straight-line, AC. (Which is) the very thing it was required to do.

Proposition 11

To find a third (straight-line) proportional to two given straight-lines.

Let BA and AC be the [two] given [straight-lines], and let them be laid down encompassing a random angle. So it is required to find a third (straight-line) proportional to BA and AC. For let (BA and AC) have been produced to points D and E (respectively), and let BD be made equal to AC [Prop. 1.3]. And let BC have been joined. And let DE have been drawn through (point) D parallel to it [Prop. 1.31].

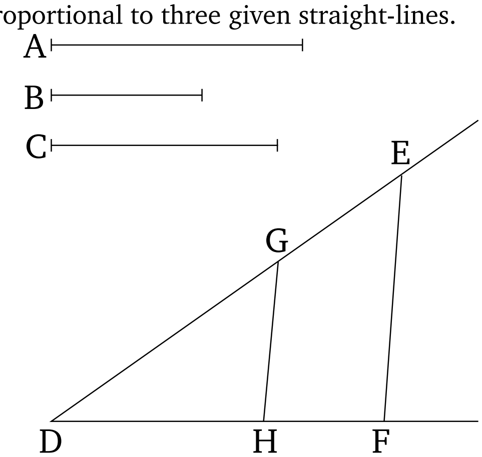
Therefore, since BC has been drawn parallel to one of the sides DE of triangle ADE, proportionally, as AB is to BD, so AC (is) to CE [Prop. 6.2]. And BD (is) equal to AC. Thus, as AB is to AC, so AC (is) to CE.



Thus, a third (straight-line), CE, has been found (which is) proportional to the two given straight-lines, AB and AC. (Which is) the very thing it was required to do.

Proposition 12

To find a fourth (straight-line) proportional to three given straight-lines.



Let A, B, and C be the three given straight-lines. So it is required to find a fourth (straight-line) proportional to A, B, and C.

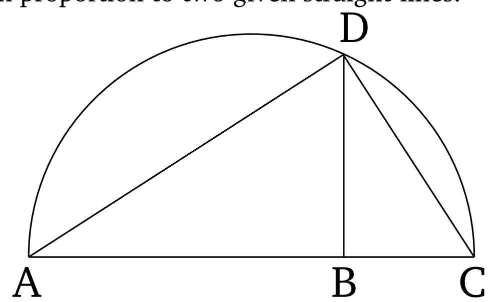
Let the two straight-lines DE and DF be set out encompassing the [random] angle EDF. And let DG be made equal to A, and GE to B, and, further, DH to C [Prop. 1.3]. And GH being joined, let EF have been drawn through (point) E parallel to it [Prop. 1.31].

Therefore, since GH has been drawn parallel to one of the sides EF of triangle DEF, thus as DG is to GE, so DH (is) to HF [Prop. 6.2]. And DG (is) equal to A, and GE to B, and DH to C. Thus, as A is to B, so C (is) to HF.

Thus, a fourth (straight-line), HF, has been found (which is) proportional to the three given straight-lines, A, B, and C. (Which is) the very thing it was required to do.

Proposition 13

To find the (straight-line) in mean proportion to two given straight-lines.†



Let AB and BC be the two given straight-lines. So it is required to find the (straight-line) in mean proportion to AB and BC.

Let (AB and BC) be laid down straight-on (with respect to one another), and let the semi-circle ADC have been drawn on AC [Prop. 1.10]. And let BD have been drawn from (point) B, at right-angles to AC [Prop. 1.11]. And let AD and DC have been joined.

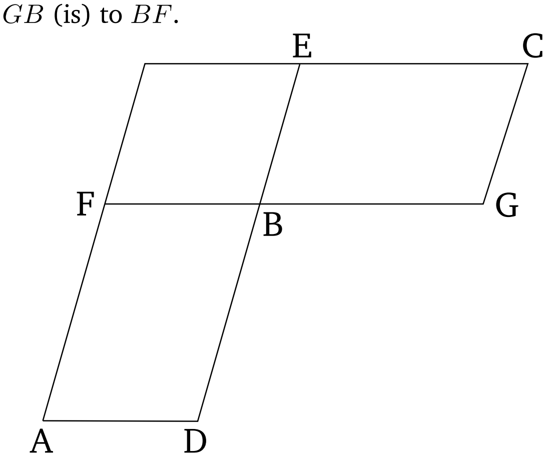
And since ADC is an angle in a semi-circle, it is a right-angle [Prop. 3.31]. And since, in the right-angled triangle ADC, the (straight-line) DB has been drawn from the right-angle perpendicular to the base, DB is thus the mean proportional to the pieces of the base, AB and BC [Prop. 6.8 corr.].

Thus, DB has been found (which is) in mean proportion to the two given straight-lines, AB and BC. (Which is) the very thing it was required to do. † In other words, to find the geometric mean of two given straight-lines.

Proposition 14

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let AB and BC be equal and equiangular parallelograms having the angles at B equal. And let DB and BE be laid down straight-on (with respect to one another). Thus, FB and BG are also straight-on (with respect to one another) [Prop. 1.14]. I say that the sides of AB and BC about the equal angles are reciprocally proportional, that is to say, that as DB is to BE, so GB (is) to BF.



For let the parallelogram FE have been completed. Therefore, since parallelogram AB is equal to parallelogram BC, and FE (is) some other (parallelogram), thus as (parallelogram) AB is to FE, so (parallelogram) BC (is) to

FE [Prop. 5.7]. But, as (parallelogram) AB (is) to FE, so DB (is) to BE, and as (parallelogram) BC (is) to FE, so GB (is) to BF [Prop. 6.1]. Thus, also, as DB (is) to BE, so GB (is) to BF. Thus, in parallelograms AB and BC the sides about the equal angles are reciprocally proportional.

And so, let DB be to BE, as GB (is) to BF. I say that parallelogram AB is equal to parallelogram BC.

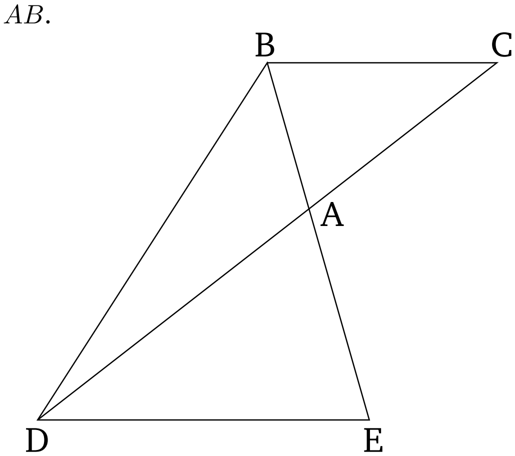
For since as DB is to BE, so GB (is) to BF, but as DB (is) to BE, so parallelogram AB (is) to parallelogram FE, and as GB (is) to BF, so parallelogram BC (is) to parallelogram FE [Prop. 6.1], thus, also, as (parallelogram) AB (is) to FE, so (parallelogram) BC (is) to FE [Prop. 5.11]. Thus, parallelogram AB is equal to parallelogram BC [Prop. 5.9].

Thus, in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal. (Which is) the very thing it was required to show.

Proposition 15

In equal triangles also having one angle equal to one (angle) the sides about the equal angles are reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal.

Let ABC and ADE be equal triangles having one angle equal to one (angle), (namely) BAC (equal) to DAE. I say that, in triangles ABC and ADE, the sides about the equal angles are reciprocally proportional, that is to say, that as CA is to AD, so EA (is) to AB.



For let CA be laid down so as to be straight-on (with respect) to AD. Thus, EA is also straight-on (with respect) to AB [Prop. 1.14]. And let BD have been joined.

Therefore, since triangle ABC is equal to triangle ADE, and BAD (is) some other (triangle), thus as triangle CAB is to triangle BAD, so triangle EAD (is) to triangle BAD [Prop. 5.7]. But, as (triangle) CAB (is) to BAD, so CA (is) to AD, and as (triangle) EAD (is) to BAD, so EA (is) to AB [Prop. 6.1]. And thus, as CA (is) to AD, so EA (is) to AB. Thus, in triangles ABC and ADE the sides about the equal angles (are) reciprocally proportional.

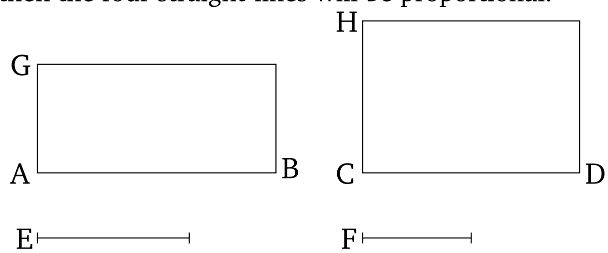
And so, let the sides of triangles ABC and ADE be reciprocally proportional, and (thus) let CA be to AD, as EA (is) to AB. I say that triangle ABC is equal to triangle ADE.

For, BD again being joined, since as CA is to AD, so EA (is) to AB, but as CA (is) to AD, so triangle ABC (is) to triangle BAD, and as EA (is) to AB, so triangle EAD (is) to triangle BAD [Prop. 6.1], thus as triangle ABC (is) to triangle BAD, so triangle EAD (is) to triangle BAD. Thus, (triangles) ABC and EAD each have the same ratio to BAD. Thus, [triangle] ABC is equal to triangle EAD [Prop. 5.9].

Thus, in equal triangles also having one angle equal to one (angle) the sides about the equal angles (are) reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal. (Which is) the very thing it was required to show.

Proposition 16

If four straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two) then the four straight-lines will be proportional.



Let AB, CD, E, and F be four proportional straight-lines, (such that) as AB (is) to CD, so E (is) to F. I say that the rectangle contained by AB and F is equal to the rectangle contained by CD and E.

[For] let AG and CH have been drawn from points A and C at right-angles to the straight-lines AB and CD (respectively) [Prop. 1.11]. And let AG be made equal to F, and CH to E [Prop. 1.3]. And let the parallelograms BG and DH have been completed.

And since as AB is to CD, so E (is) to F, and E (is) equal CH, and F to AG, thus as AB is to CD, so CH (is) to AG. Thus, in the parallelograms BG and DH the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.14]. Thus, parallelogram BG is equal to parallelogram DH. And BG is the (rectangle contained) by AB and F. For AG (is) equal to F. And DH (is) the (rectangle contained) by CD and E. For E (is) equal to CH. Thus, the rectangle contained by AB and F is equal to the rectangle contained by CD and E.

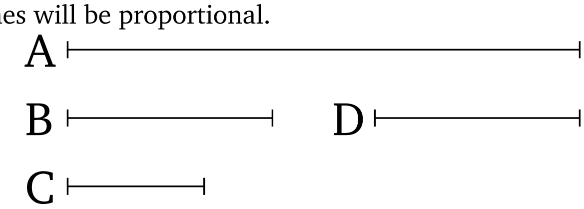
And so, let the rectangle contained by AB and F be equal to the rectangle contained by CD and E. I say that the four straight-lines will be proportional, (so that) as AB (is) to CD, so E (is) to F.

For, with the same construction, since the (rectangle contained) by AB and F is equal to the (rectangle contained) by CD and E. And BG is the (rectangle contained) by AB and F. For AG is equal to F. And DH (is) the (rectangle contained) by CD and E. For CH (is) equal to E. BG is thus equal to DH. And they are equiangular. And in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as AB is to CD, so CH (is) to AG. And CH (is) equal to E, and AG to F. Thus, as AB is to CD, so E (is) to F.

Thus, if four straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two) then the four straight-lines will be proportional. (Which is) the very thing it was required to show.

Proposition 17

If three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional.



Let A, B and C be three proportional straight-lines, (such that) as A (is) to B, so B (is) to C. I say that the rectangle contained by A and C is equal to the square on B.

Let D be made equal to B [Prop. 1.3].

And since as A is to B, so B (is) to C, and B (is) equal to D, thus as A is to B, (so) D (is) to C. And if four straight-lines are proportional then the [rectangle] contained by the (two) outermost is equal to the rectangle contained by the middle (two) [Prop. 6.16]. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by B and D. But, the (rectangle contained) by B and D is the (square) on B. For B (is) equal to D. Thus, the rectangle contained by A and C is equal to the square on B.

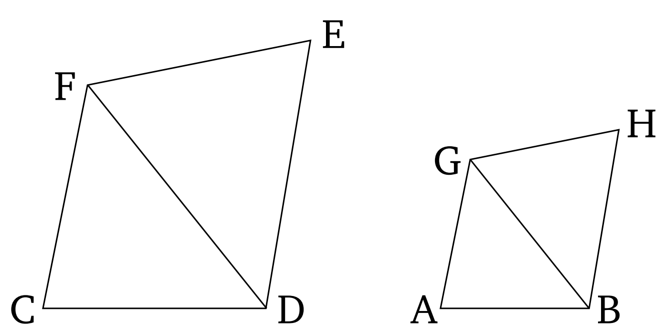
And so, let the (rectangle contained) by A and C be equal to the (square) on B. I say that as A is to B, so B (is) to C.

For, with the same construction, since the (rectangle contained) by A and C is equal to the (square) on B. But, the (square) on B is the (rectangle contained) by B and D. For B (is) equal to D. The (rectangle contained) by A and C is thus equal to the (rectangle contained) by B and D. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four straight-lines are proportional [Prop. 6.16]. Thus, as A is to B, so D (is) to C. And B (is) equal to D. Thus, as A (is) to B, so B (is) to C.

Thus, if three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional. (Which is) the very thing it was required to show.

Proposition 18

To describe a rectilinear figure similar, and similarly laid down, to a given rectilinear figure on a given straight-line.



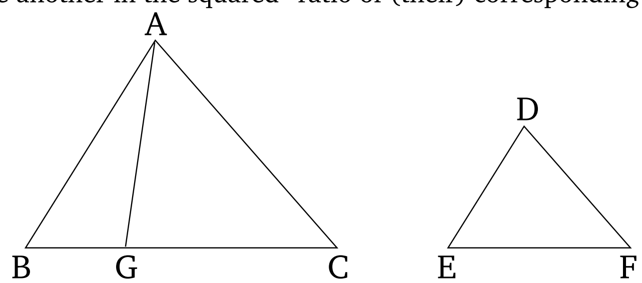
Let AB be the given straight-line, and CE the given rectilinear figure. So it is required to describe a rectilinear figure similar, and similarly laid down, to the rectilinear figure CE on the straight-line AB.

Let DF have been joined, and let GAB, equal to the angle at C, and ABG, equal to (angle) CDF, have been constructed on the straight-line AB at the points A and B on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) CFD is equal to AGB [Prop. 1.32]. Thus, triangle FCD is equiangular to triangle GAB. Thus, proportionally, as FD is to GB, so FC (is) to GA, and CD to AB [Prop. 6.4]. Again, let BGH, equal to angle DFE, and GBH equal to (angle) FDE, have been constructed on the straight-line BG at the points G and B on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at E is equal to the remaining (angle) at H [Prop. 1.32]. Thus, triangle FDE is equiangular to triangle GHB. Thus, proportionally, as FD is to GB, so FE (is) to GH, and ED to HB [Prop. 6.4]. And it was also shown (that) as FD (is) to GB, so FC (is) to GA, and CD to AB. Thus, also, as FC (is) to AG, so CD (is) to AB, and FE to GH, and, further, ED to HB. And since angle CFD is equal to AGB, and DFE to BGH, thus the whole (angle) CFE is equal to the whole (angle) AGH. So, for the same (reasons), (angle) CDE is also equal to ABH. And the (angle) at C is also equal to the (angle) at A, and the (angle) at E to the (angle) at H. Thus, (figure) AH is equiangular to CE. And (the two figures) have the sides about their equal angles proportional. Thus, the rectilinear figure AH is similar to the rectilinear figure CE [Def. 6.1].

Thus, the rectilinear figure AH, similar, and similarly laid down, to the given rectilinear figure CE has been constructed on the given straight-line AB. (Which is) the very thing it was required to do.

Proposition 19

Similar triangles are to one another in the squared† ratio of (their) corresponding sides.



Let ABC and DEF be similar triangles having the angle at B equal to the (angle) at E, and AB to BC, as DE (is) to EF, such that BC corresponds to EF. I say that triangle ABC has a squared ratio to triangle DEF with respect to (that side) BC (has) to EF.

For let a third (straight-line), BG, have been taken (which is) proportional to BC and EF, so that as BC (is) to EF, so EF (is) to BG [Prop. 6.11]. And let AG have been joined.

Therefore, since as AB is to BC, so DE (is) to EF, thus, alternately, as AB is to DE, so BC (is) to EF [Prop. 5.16]. But, as BC (is) to EF, so EF is to BG. And, thus, as AB (is) to DE, so EF (is) to BG. Thus, for triangles ABG and DEF, the sides about the equal angles are reciprocally proportional. And those triangles having one (angle) equal to one (angle) for which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.15]. Thus, triangle ABG is equal to triangle DEF. And since as BC (is) to EF, so EF (is) to BG, and if three straight-lines are proportional then the first has a squared ratio to the third with respect to the second [Def. 5.9], BC thus has a squared ratio to BG with respect to (that) CB (has) to EF. And as CB (is) to BG, so triangle ABC (is) to triangle ABG [Prop. 6.1]. Thus, triangle ABC also has a squared ratio to (triangle) ABG with respect to (that side) BC (has) to EF. And triangle ABG (is) equal to triangle DEF. Thus, triangle ABC also has a squared ratio to triangle DEF with respect to (that side) BC (has) to EF.

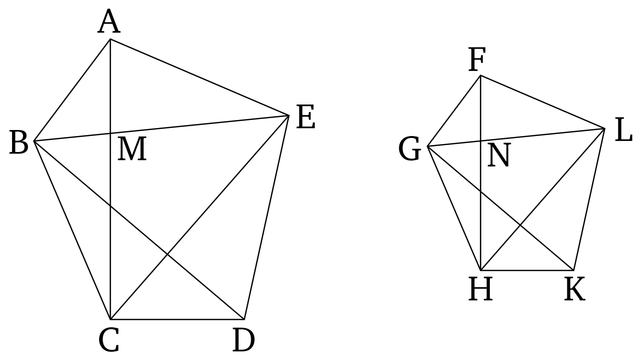
Thus, similar triangles are to one another in the squared ratio of (their) corresponding sides. [(Which is) the very thing it was required to show].

Corollary

So it is clear, from this, that if three straight-lines are proportional, then as the first is to the third, so the figure (described) on the first (is) to the similar, and similarly described, (figure) on the second. (Which is) the very thing it was required to show. † Literally, "double".

Proposition 20

Similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let $ABCDE$ and $FGHKL$ be similar polygons, and let AB correspond to FG. I say that polygons $ABCDE$ and $FGHKL$ can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and (that) polygon $ABCDE$ has a squared ratio to polygon $FGHKL$ with respect to that AB (has) to FG.

Let BE, EC, GL, and LH have been joined.

And since polygon $ABCDE$ is similar to polygon $FGHKL$, angle BAE is equal to angle GFL, and as BA is to AE, so GF (is) to FL [Def. 6.1]. Therefore, since ABE and FGL are two triangles having one angle equal to one angle and the sides about the equal angles proportional, triangle ABE is thus equiangular to triangle FGL [Prop. 6.6]. Hence, (they are) also similar [Prop. 6.4, Def. 6.1]. Thus, angle ABE is equal to (angle) FGL. And the whole (angle) ABC is equal to the whole (angle) FGH, on account of the similarity of the polygons. Thus, the remaining angle EBC is equal to LGH. And since, on account of the similarity of triangles ABE and FGL, as EB is to BA, so LG (is) to GF, but also, on account of the similarity of the polygons, as AB is to BC, so FG (is) to GH, thus, via equality, as EB is to BC, so LG (is) to GH [Prop. 5.22], and the sides about the equal angles, EBC and LGH, are proportional. Thus, triangle EBC is equiangular to triangle LGH [Prop. 6.6]. Hence, triangle EBC is also similar to triangle LGH [Prop. 6.4, Def. 6.1]. So, for the same (reasons), triangle ECD is also similar to triangle LHK. Thus, the similar polygons $ABCDE$ and $FGHKL$ have been divided into equal numbers of similar triangles.

I also say that (the triangles) correspond (in proportion) to the wholes. That is to say, the triangles are proportional: ABE, EBC, and ECD are the leading (magnitudes), and their (associated) following (magnitudes are) FGL, LGH, and LHK (respectively). (I) also (say) that polygon $ABCDE$ has a squared ratio to polygon $FGHKL$ with respect to (that) a corresponding side (has) to a corresponding side—that is to say, (side) AB to FG.

For let AC and FH have been joined. And since angle ABC is equal to FGH, and as AB is to BC, so FG (is) to GH, on account of the similarity of the polygons, triangle ABC is equiangular to triangle FGH [Prop. 6.6]. Thus, angle BAC is equal to GFH, and (angle) BCA to GHF. And since angle BAM is equal to GFN, and (angle) ABM is also equal to FGN (see earlier), the remaining (angle) AMB is thus also equal to the remaining (angle) FNG [Prop. 1.32]. Thus, triangle ABM is equiangular to triangle FGN. So, similarly, we can show that triangle BMC is also equiangular to triangle GNH. Thus, proportionally, as AM is to MB, so FN (is) to NG, and as BM (is) to MC, so GN (is) to NH [Prop. 6.4]. Hence, also, via equality, as AM (is) to MC, so FN (is) to NH [Prop. 5.22]. But, as AM (is) to MC, so [triangle] ABM is to MBC, and AME to EMC. For they are to one another as their bases [Prop. 6.1]. And as one of the leading (magnitudes) is to one of the following (magnitudes), so (the sum of) all the leading (magnitudes) is to (the sum of) all the following (magnitudes) [Prop. 5.12]. Thus, as triangle AMB (is) to BMC, so (triangle) ABE (is) to CBE. But, as (triangle) AMB (is) to BMC, so AM (is) to MC. Thus, also, as AM (is) to MC, so triangle ABE (is) to triangle EBC. And so, for the same (reasons), as FN (is) to NH, so triangle FGL (is) to triangle GLH. And as AM is to MC, so FN (is) to NH. Thus, also, as triangle ABE (is) to triangle BEC, so triangle FGL (is) to triangle GLH, and, alternately, as triangle ABE (is) to triangle FGL, so triangle BEC (is) to triangle GLH [Prop. 5.16]. So, similarly, we can also show, by joining BD and GK, that as triangle BEC (is) to triangle LGH, so triangle ECD (is) to triangle LHK. And since as triangle ABE is to triangle FGL, so (triangle) EBC (is) to LGH, and, further, (triangle) ECD to LHK, and also as one of the leading (magnitudes is) to one of the following, so (the sum of) all the leading (magnitudes is) to (the sum of) all the following [Prop. 5.12], thus as triangle ABE is to triangle FGL, so polygon $ABCDE$ (is) to polygon $FGHKL$. But, triangle ABE has a squared ratio to triangle FGL with respect to (that) the corresponding side AB (has) to the corresponding side FG. For, similar triangles are in the squared ratio of corresponding sides [Prop. 6.14]. Thus, polygon $ABCDE$ also has a squared ratio to polygon $FGHKL$ with respect to (that) the corresponding side AB (has) to the corresponding side FG.

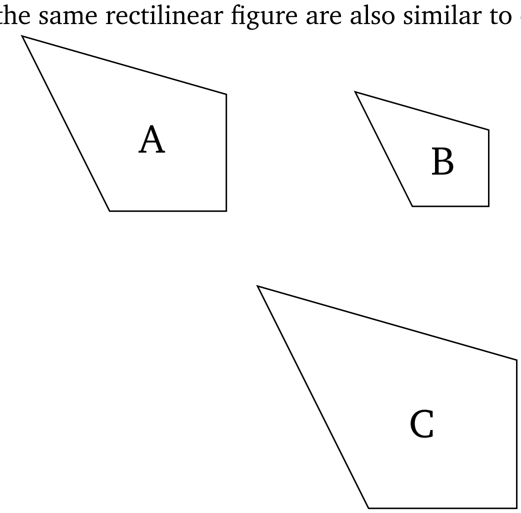
Thus, similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side. [(Which is) the very thing it was required to show].

Corollary

And, in the same manner, it can also be shown for [similar] quadrilaterals that they are in the squared ratio of (their) corresponding sides. And it was also shown for triangles. Hence, in general, similar rectilinear figures are also to one another in the squared ratio of (their) corresponding sides. (Which is) the very thing it was required to show.

Proposition 21

(Rectilinear figures) similar to the same rectilinear figure are also similar to one another.

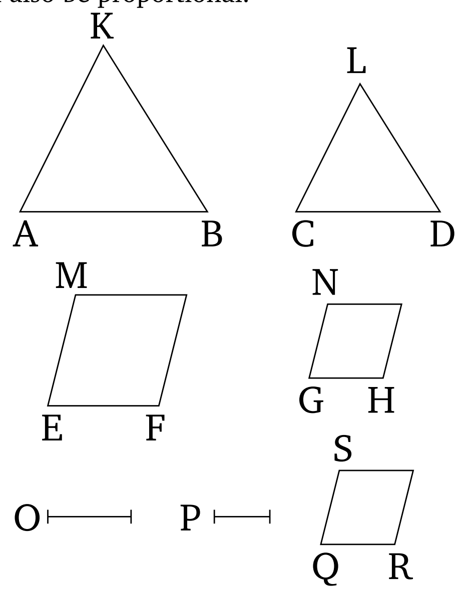


Let each of the rectilinear figures A and B be similar to (the rectilinear figure) C. I say that A is also similar to B.

For since A is similar to C, (A) is equiangular to (C), and has the sides about the equal angles proportional [Def. 6.1]. Again, since B is similar to C, (B) is equiangular to (C), and has the sides about the equal angles proportional [Def. 6.1]. Thus, A and B are each equiangular to C, and have the sides about the equal angles proportional [hence, A is also equiangular to B, and has the sides about the equal angles proportional]. Thus, A is similar to B [Def. 6.1]. (Which is) the very thing it was required to show.

Proposition 22

If four straight-lines are proportional then similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional then the straight-lines themselves will also be proportional.



Let AB, CD, EF, and GH be four proportional straight-lines, (such that) as AB (is) to CD, so EF (is) to GH. And let the similar, and similarly laid out, rectilinear figures KAB and LCD have been described on AB and CD (respectively), and the similar, and similarly laid out, rectilinear figures MF and NH on EF and GH (respectively). I say that as KAB is to LCD, so MF (is) to NH.

For let a third (straight-line) O have been taken (which is) proportional to AB and CD, and a third (straight-line) P proportional to EF and GH [Prop. 6.11]. And since as AB is to CD, so EF (is) to GH, and as CD (is) to O, so GH (is) to P, thus, via equality, as AB is to O, so EF (is) to P [Prop. 5.22]. But, as AB (is) to O, so [also] KAB (is) to LCD, and as EF (is) to P, so MF (is) to NH [Prop. 5.19 corr.]. And, thus, as KAB (is) to LCD, so MF (is) to NH.

And so let KAB be to LCD, as MF (is) to NH. I say also that as AB is to CD, so EF (is) to GH. For if as AB is to CD, so EF (is) not to GH, let AB be to CD, as EF (is) to QR [Prop. 6.12]. And let the rectilinear figure SR, similar, and similarly laid down, to either of MF or NH, have been described on QR [Props. 6.18, 6.21].

Therefore, since as AB is to CD, so EF (is) to QR, and the similar, and similarly laid out, (rectilinear figures) KAB and LCD have been described on AB and CD (respectively), and the similar, and similarly laid out, (rectilinear figures) MF and SR on EF and QR (resespectively), thus as KAB is to LCD, so MF (is) to SR (see above). And it was also assumed that as KAB (is) to LCD, so MF (is) to NH. Thus, also, as MF (is) to SR, so MF (is) to NH

[Prop. 5.11]. Thus, MF has the same ratio to each of NH and SR. Thus, NH is equal to SR [Prop. 5.9]. And it is also similar, and similarly laid out, to it. Thus, GH (is) equal to QR.† And since AB is to CD, as EF (is) to QR, and QR (is) equal to GH, thus as AB is to CD, so EF (is) to GH.

Thus, if four straight-lines are proportional, then similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional then the straight-lines themselves will also be proportional. (Which is) the very thing it was required to show. † Here, Euclid assumes, without proof, that if two similar figures are equal then any pair of corresponding sides is also equal.

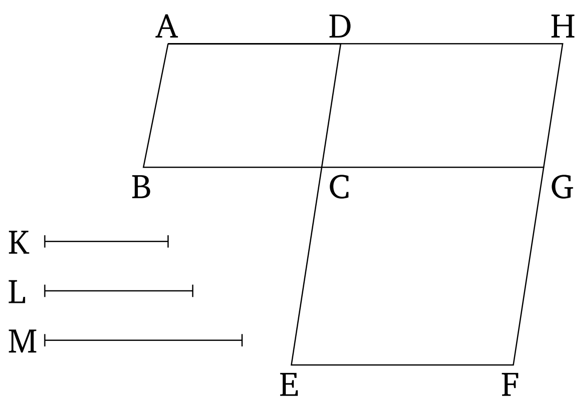
Proposition 23

Equiangular parallelograms have to one another the ratio compounded† out of (the ratios of) their sides.

Let AC and CF be equiangular parallelograms having angle BCD equal to ECG. I say that parallelogram AC has to parallelogram CF the ratio compounded out of (the ratios of) their sides.

For let BC be laid down so as to be straight-on to CG. Thus, DC is also straight-on to CE [Prop. 1.14]. And let the parallelogram DG have been completed. And let some straight-line K have been laid down. And let it be contrived that as BC (is) to CG, so K (is) to L, and as DC (is) to CE, so L (is) to M [Prop. 6.12].

Thus, the ratios of K to L and of L to M are the same as the ratios of the sides, (namely), BC to CG and DC to CE (respectively). But, the ratio of K to M is compounded out of the ratio of K to L and (the ratio) of L to M. Hence, K also has to M the ratio compounded out of (the ratios of) the sides (of the parallelograms). And since as BC is to CG, so parallelogram AC (is) to CH [Prop. 6.1], but as BC (is) to CG, so K (is) to L, thus, also, as K (is) to L, so (parallelogram) AC (is) to CH. Again, since as DC (is) to CE, so parallelogram CH (is) to CF [Prop. 6.1], but as DC (is) to CE, so L (is) to M, thus, also, as L (is) to M, so parallelogram CH (is) to parallelogram CF. Therefore, since it was shown that as K (is) to L, so parallelogram AC (is) to parallelogram CH, and as L (is) to M, so parallelogram CH (is) to parallelogram CF, thus, via equality, as K is to M, so (parallelogram) AC (is) to parallelogram CF [Prop. 5.22]. And K has to M the ratio compounded out of (the ratios of) the sides (of the parallelograms). Thus, (parallelogram) AC also has to (parallelogram) CF the ratio compounded out of (the ratio of) their sides.



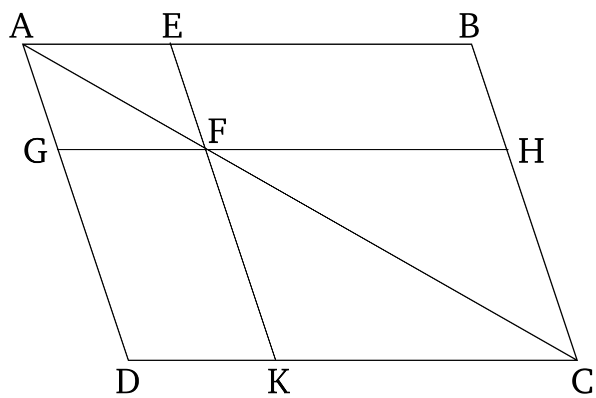
Thus, equiangular parallelograms have to one another the ratio compounded out of (the ratio of) their sides. (Which is) the very thing it was required to show. † In modern terminology, if two ratios are "compounded" then they are multiplied together.

Proposition 24

In any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another.

Let $ABCD$ be a parallelogram, and AC its diagonal. And let EG and HK be parallelograms about AC. I say that the parallelograms EG and HK are each similar to the whole (parallelogram) $ABCD$, and to one another.

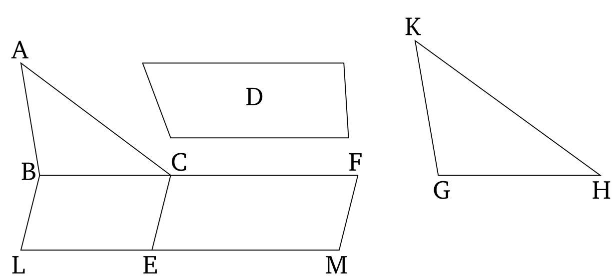
For since EF has been drawn parallel to one of the sides BC of triangle ABC, proportionally, as BE is to EA, so CF (is) to FA [Prop. 6.2]. Again, since FG has been drawn parallel to one (of the sides) CD of triangle ACD, proportionally, as CF is to FA, so DG (is) to GA [Prop. 6.2]. But, as CF (is) to FA, so it was also shown (is) BE to EA. And thus as BE (is) to EA, so DG (is) to GA. And, thus, compounding, as BA (is) to AE, so DA (is) to AG [Prop. 5.18]. And, alternately, as BA (is) to AD, so EA (is) to AG [Prop. 5.16]. Thus, in parallelograms $ABCD$ and EG the sides about the common angle BAD are proportional. And since GF is parallel to DC, angle AFG is equal to DCA [Prop. 1.29]. And angle DAC (is) common to the two triangles ADC and AGF. Thus, triangle ADC is equiangular to triangle AGF [Prop. 1.32]. So, for the same (reasons), triangle ACB is equiangular to triangle AFE, and the whole parallelogram $ABCD$ is equiangular to parallelogram EG. Thus, proportionally, as AD (is) to DC, so AG (is) to GF, and as DC (is) to CA, so GF (is) to FA, and as AC (is) to CB, so AF (is) to FE, and, further, as CB (is) to BA, so FE (is) to EA [Prop. 6.4]. And since it was shown that as DC is to CA, so GF (is) to FA, and as AC (is) to CB, so AF (is) to FE, thus, via equality, as DC is to CB, so GF (is) to FE [Prop. 5.22]. Thus, in parallelograms $ABCD$ and EG the sides about the equal angles are proportional. Thus, parallelogram $ABCD$ is similar to parallelogram EG [Def. 6.1]. So, for the same (reasons), parallelogram $ABCD$ is also similar to parallelogram KH. Thus, parallelograms EG and HK are each similar to [parallelogram] $ABCD$. And (rectilinear figures) similar to the same rectilinear figure are also similar to one another [Prop. 6.21]. Thus, parallelogram EG is also similar to parallelogram HK.



Thus, in any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another. (Which is) the very thing it was required to show.

Proposition 25

To construct a single (rectilinear figure) similar to a given rectilinear figure, and equal to a different given rectilinear figure.



Let ABC be the given rectilinear figure to which it is required to construct a similar (rectilinear figure), and D the (rectilinear figure) to which (the constructed figure) is required (to be) equal. So it is required to construct a single (rectilinear figure) similar to ABC, and equal to D.

For let the parallelogram BE, equal to triangle ABC, have been applied to (the straight-line) BC [Prop. 1.44], and the parallelogram CM, equal to D, (have been applied) to (the straight-line) CE, in the angle FCE, which is equal to CBL [Prop. 1.45]. Thus, BC is straight-on to CF, and LE to EM [Prop. 1.14]. And let the mean proportion GH have been taken of BC and CF [Prop. 6.13]. And let KGH, similar, and similarly laid out, to ABC have been described on GH [Prop. 6.18].

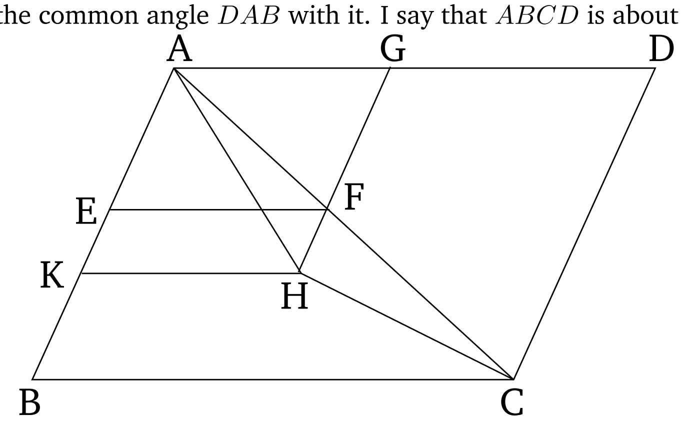
And since as BC is to GH, so GH (is) to CF, and if three straight-lines are proportional then as the first is to the third, so the figure (described) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.], thus as BC is to CF, so triangle ABC (is) to triangle KGH. But, also, as BC (is) to CF, so parallelogram BE (is) to parallelogram EF [Prop. 6.1]. And, thus, as triangle ABC (is) to triangle KGH, so parallelogram BE (is) to parallelogram EF. Thus, alternately, as triangle ABC (is) to parallelogram BE, so triangle KGH (is) to parallelogram EF [Prop. 5.16]. And triangle ABC (is) equal to parallelogram BE. Thus, triangle KGH (is) also equal to parallelogram EF. But, parallelogram EF is equal to D. Thus, KGH is also equal to D. And KGH is also similar to ABC.

Thus, a single (rectilinear figure) KGH has been constructed (which is) similar to the given rectilinear figure ABC, and equal to a different given (rectilinear figure) D. (Which is) the very thing it was required to do.

Proposition 26

If from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole.

For, from parallelogram $ABCD$, let (parallelogram) AF have been subtracted (which is) similar, and similarly laid out, to $ABCD$, having the common angle DAB with it. I say that $ABCD$ is about the same diagonal as AF.



For (if) not, then, if possible, let AHC be [$ABCD$'s] diagonal. And producing GF, let it have been drawn through to (point) H. And let HK have been drawn through (point) H, parallel to either of AD or BC [Prop. 1.31].

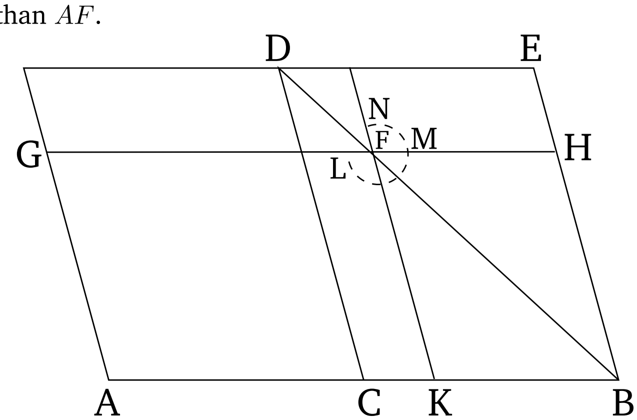
Therefore, since $ABCD$ is about the same diagonal as KG, thus as DA is to AB, so GA (is) to AK [Prop. 6.24]. And, on account of the similarity of $ABCD$ and EG, also, as DA (is) to AB, so GA (is) to AE. Thus, also, as GA (is) to AK, so GA (is) to AE. Thus, GA has the same ratio to each of AK and AE. Thus, AE is equal to AK [Prop. 5.9], the lesser to the greater. The very thing is impossible. Thus, $ABCD$ is not not about the same diagonal as AF. Thus, parallelogram $ABCD$ is about the same diagonal as parallelogram AF.

Thus, if from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole. (Which is) the very thing it was required to show.

Proposition 27

Of all the parallelograms applied to the same straight-line, and falling short by parallelogrammic figures similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line) which (is) similar to (that parallelogram) by which it falls short.

Let AB be a straight-line, and let it have been cut in half at (point) C [Prop. 1.10]. And let the parallelogram AD have been applied to the straight-line AB, falling short by the parallelogrammic figure DB (which is) applied to half of AB—that is to say, CB. I say that of all the parallelograms applied to AB, and falling short by [parallelogrammic] figures similar, and similarly laid out, to DB, the greatest is AD. For let the parallelogram AF have been applied to the straight-line AB, falling short by the parallelogrammic figure FB (which is) similar, and similarly laid out, to DB. I say that AD is greater than AF.



For since parallelogram DB is similar to parallelogram FB, they are about the same diagonal [Prop. 6.26]. Let their (common) diagonal DB have been drawn, and let the (rest of the) figure have been described.

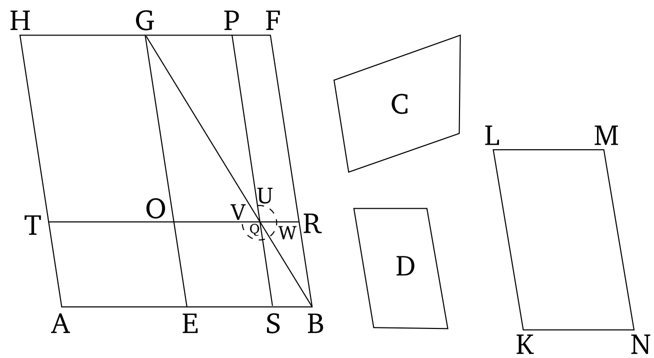
Therefore, since (complement) CF is equal to (complement) FE [Prop. 1.43], and (parallelogram) FB is common, the whole (parallelogram) CH is thus equal to the whole (parallelogram) KE. But, (parallelogram) CH is equal to CG, since AC (is) also (equal) to CB [Prop. 6.1]. Thus, (parallelogram) GC is also equal to EK. Let (parallelogram) CF have been added to both. Thus, the whole (parallelogram) AF is equal to the gnomon LMN. Hence, parallelogram DB—that is to say, AD—is greater than parallelogram AF.

Thus, for all parallelograms applied to the same straight-line, and falling short by a parallelogrammic figure similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line). (Which is) the very thing it was required to show.

Proposition 28†

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) falling short by a parallelogrammic figure similar to a given (parallelogram). It is necessary for the given rectilinear figure [to which it is required to apply an equal (parallelogram)] not to be greater than the (parallelogram) described on half (of the straight-line) and similar to the deficit.

Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to AB is required (to be) equal, [being] not greater than the (parallelogram) described on half of AB and similar to the deficit, and D the (parallelogram) to which the deficit is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C, to the straight-line AB, falling short by a parallelogrammic figure which is similar to D.



Let AB have been cut in half at point E [Prop. 1.10], and let (parallelogram) $EBFG$, (which is) similar, and similarly laid out, to (parallelogram) D, have been described on EB [Prop. 6.18]. And let parallelogram AG have been completed.

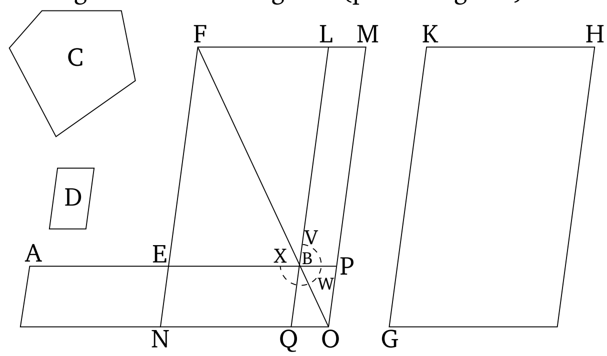
Therefore, if AG is equal to C then the thing prescribed has happened. For a parallelogram AG, equal to the given rectilinear figure C, has been applied to the given straight-line AB, falling short by a parallelogrammic figure GB which is similar to D. And if not, let HE be greater than C. And HE (is) equal to GB [Prop. 6.1]. Thus, GB (is) also greater than C. So, let (parallelogram) $KLMN$ have been constructed (so as to be) both similar, and similarly laid out, to D, and equal to the excess by which GB is greater than C [Prop. 6.25]. But, GB [is] similar to D. Thus, KM is also similar to GB [Prop. 6.21]. Therefore, let KL correspond to GE, and LM to GF. And since (parallelogram) GB is equal to (figure) C and (parallelogram) KM, GB is thus greater than KM. Thus, GE is also greater than KL, and GF than LM. Let GO be made equal to KL, and GP to LM [Prop. 1.3]. And let the parallelogram $OGPQ$ have been completed. Thus, [GQ] is equal and similar to KM [but, KM is similar to GB]. Thus, GQ is also similar to GB [Prop. 6.21]. Thus, GQ and GB are about the same diagonal [Prop. 6.26]. Let GQB be their (common) diagonal, and let the (remainder of the) figure have been described.

Therefore, since BG is equal to C and KM, of which GQ is equal to KM, the remaining gnomon UWV is thus equal to the remainder C. And since (the complement) PR is equal to (the complement) OS [Prop. 1.43], let (parallelogram) QB have been added to both. Thus, the whole (parallelogram) PB is equal to the whole (parallelogram) OB. But, OB is equal to TE, since side AE is equal to side EB [Prop. 6.1]. Thus, TE is also equal to PB. Let (parallelogram) OS have been added to both. Thus, the whole (parallelogram) TS is equal to the gnomon VWU. But, gnomon VWU was shown (to be) equal to C. Therefore, (parallelogram) TS is also equal to (figure) C.

Thus, the parallelogram ST, equal to the given rectilinear figure C, has been applied to the given straight-line AB, falling short by the parallelogrammic figure QB, which is similar to D [inasmuch as QB is similar to GQ [Prop. 6.24]]. (Which is) the very thing it was required to do. † This proposition is a geometric solution of the quadratic equation $x^2 - \alpha\,x + \beta = 0$. Here, x is the ratio of a side of the deficit to the corresponding side of figure D, α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the deficit running along AB, and β is the ratio of the areas of figures C and D. The constraint corresponds to the condition $\beta < \alpha^2/4$ for the equation to have real roots. Only the smaller root of the equation is found. The larger root can be found by a similar method.

Proposition 29†

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) overshooting by a parallelogrammic figure similar to a given (parallelogram).



Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to AB is required (to be) equal, and D the (parallelogram) to which the excess is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C, to the given straight-line AB, overshooting by a parallelogrammic figure similar to D.

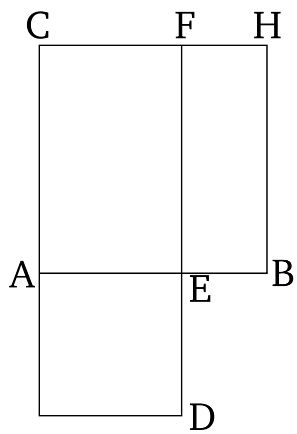
Let AB have been cut in half at (point) E [Prop. 1.10], and let the parallelogram BF, (which is) similar, and similarly laid out, to D, have been described on EB [Prop. 6.18]. And let (parallelogram) GH have been constructed (so as to be) both similar, and similarly laid out, to D, and equal to the sum of BF and C [Prop. 6.25]. And let KH correspond to FL, and KG to FE. And since (parallelogram) GH is greater than (parallelogram) FB, KH is thus also greater than FL, and KG than FE. Let FL and FE have been produced, and let FLM be (made) equal to KH, and FEN to KG [Prop. 1.3]. And let (parallelogram) MN have been completed. Thus, MN is equal and similar to GH. But, GH is similar to EL. Thus, MN is also similar to EL [Prop. 6.21]. EL is thus about the same diagonal as MN [Prop. 6.26]. Let their (common) diagonal FO have been drawn, and let the (remainder of the) figure have been described.

And since (parallelogram) GH is equal to (parallelogram) EL and (figure) C, but GH is equal to (parallelogram) MN, MN is thus also equal to EL and C. Let EL have been subtracted from both. Thus, the remaining gnomon XWV is equal to (figure) C. And since AE is equal to EB, (parallelogram) AN is also equal to (parallelogram) NB [Prop. 6.1], that is to say, (parallelogram) LP [Prop. 1.43]. Let (parallelogram) EO have been added to both. Thus, the whole (parallelogram) AO is equal to the gnomon VWX. But, the gnomon VWX is equal to (figure) C. Thus, (parallelogram) AO is also equal to (figure) C.

Thus, the parallelogram AO, equal to the given rectilinear figure C, has been applied to the given straight-line AB, overshooting by the parallelogrammic figure QP which is similar to D, since PQ is also similar to EL [Prop. 6.24]. (Which is) the very thing it was required to do. † This proposition is a geometric solution of the quadratic equation $x^2 + \alpha\,x - \beta = 0$. Here, x is the ratio of a side of the excess to the corresponding side of figure D, α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the excess running along AB, and β is the ratio of the areas of figures C and D. Only the positive root of the equation is found.

Proposition 30†

To cut a given finite straight-line in extreme and mean ratio.



Let AB be the given finite straight-line. So it is required to cut the straight-line AB in extreme and mean ratio.

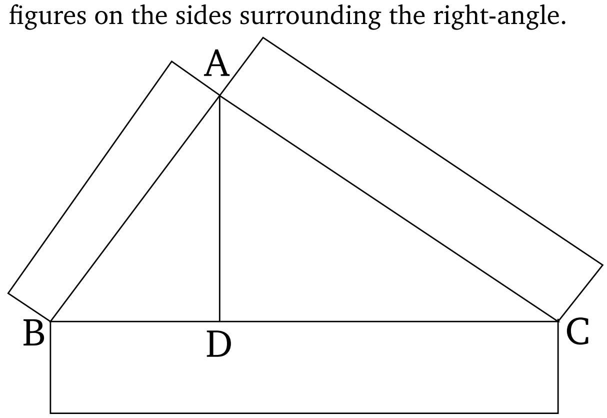
Let the square BC have been described on AB [Prop. 1.46], and let the parallelogram CD, equal to BC, have been applied to AC, overshooting by the figure AD (which is) similar to BC [Prop. 6.29].

And BC is a square. Thus, AD is also a square. And since BC is equal to CD, let (rectangle) CE have been subtracted from both. Thus, the remaining (rectangle) BF is equal to the remaining (square) AD. And it is also equiangular to it. Thus, the sides of BF and AD about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as FE is to ED, so AE (is) to EB. And FE (is) equal to AB, and ED to AE. Thus, as BA is to AE, so AE (is) to EB. And AB (is) greater than AE. Thus, AE (is) also greater than EB [Prop. 5.14].

Thus, the straight-line AB has been cut in extreme and mean ratio at E, and AE is its greater piece. (Which is) the very thing it was required to do. † This method of cutting a straight-line is sometimes called the "Golden Section"—see Prop. 2.11.

Proposition 31

In right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle.



Let ABC be a right-angled triangle having the angle BAC a right-angle. I say that the figure (drawn) on BC is equal to the (sum of the) similar, and similarly described, figures on BA and AC.

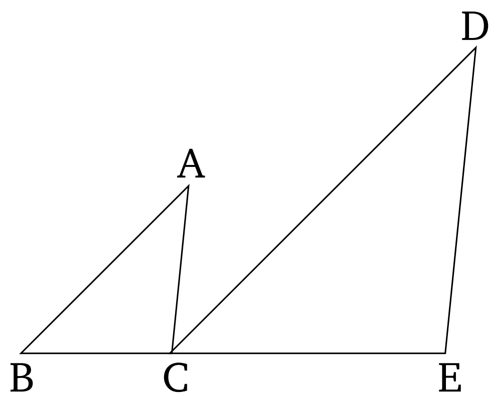
Let the perpendicular AD have been drawn [Prop. 1.12].

Therefore, since, in the right-angled triangle ABC, the (straight-line) AD has been drawn from the right-angle at A perpendicular to the base BC, the triangles ABD and ADC about the perpendicular are similar to the whole (triangle) ABC, and to one another [Prop. 6.8]. And since ABC is similar to ABD, thus as CB is to BA, so AB (is) to BD [Def. 6.1]. And since three straight-lines are proportional, as the first is to the third, so the figure (drawn) on the first is to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. Thus, as CB (is) to BD, so the figure (drawn) on CB (is) to the similar, and similarly described, (figure) on BA. And so, for the same (reasons), as BC (is) to CD, so the figure (drawn) on BC (is) to the (figure) on CA. Hence, also, as BC (is) to BD and DC, so the figure (drawn) on BC (is) to the (sum of the) similar, and similarly described, (figures) on BA and AC [Prop. 5.24]. And BC is equal to BD and DC. Thus, the figure (drawn) on BC (is) also equal to the (sum of the) similar, and similarly described, figures on BA and AC [Prop. 5.9].

Thus, in right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle. (Which is) the very thing it was required to show.

Proposition 32

If two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another).



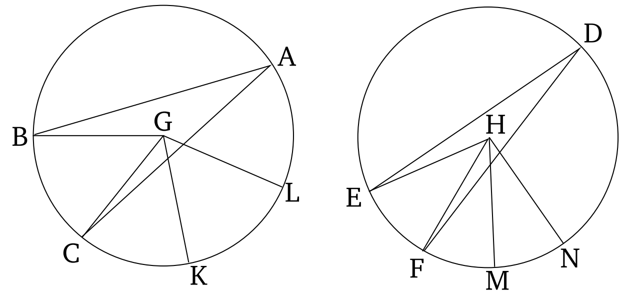
Let ABC and DCE be two triangles having the two sides BA and AC proportional to the two sides DC and DE—so that as AB (is) to AC, so DC (is) to DE—and (having side) AB parallel to DC, and AC to DE. I say that (side) BC is straight-on to CE.

For since AB is parallel to DC, and the straight-line AC has fallen across them, the alternate angles BAC and ACD are equal to one another [Prop. 1.29]. So, for the same (reasons), CDE is also equal to ACD. And, hence, BAC is equal to CDE. And since ABC and DCE are two triangles having the one angle at A equal to the one angle at D, and the sides about the equal angles proportional, (so that) as BA (is) to AC, so CD (is) to DE, triangle ABC is thus equiangular to triangle DCE [Prop. 6.6]. Thus, angle ABC is equal to DCE. And (angle) ACD was also shown (to be) equal to BAC. Thus, the whole (angle) ACE is equal to the two (angles) ABC and BAC. Let ACB have been added to both. Thus, ACE and ACB are equal to BAC, ACB, and CBA. But, BAC, ABC, and ACB are equal to two right-angles [Prop. 1.32]. Thus, ACE and ACB are also equal to two right-angles. Thus, the two straight-lines BC and CE, not lying on the same side, make adjacent angles ACE and ACB (whose sum is) equal to two right-angles with some straight-line AC, at the point C on it. Thus, BC is straight-on to CE [Prop. 1.14].

Thus, if two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another). (Which is) the very thing it was required to show.

Proposition 33

In equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences.



Let ABC and DEF be equal circles, and let BGC and EHF be angles at their centers, G and H (respectively), and BAC and EDF (angles) at their circumferences. I say that as circumference BC is to circumference EF, so angle BGC (is) to EHF, and (angle) BAC to EDF.

For let any number whatsoever of consecutive (circumferences), CK and KL, be made equal to circumference BC, and any number whatsoever, FM and MN, to circumference EF. And let GK, GL, HM, and HN have been joined.

Therefore, since circumferences BC, CK, and KL are equal to one another, angles BGC, CGK, and KGL are also equal to one another [Prop. 3.27]. Thus, as many times as circumference BL is (divisible) by BC, so many times is angle BGL also (divisible) by BGC. And so, for the same (reasons), as many times as circumference NE is (divisible) by EF, so many times is angle NHE also (divisible) by EHF. Thus, if circumference BL is equal to circumference EN then angle BGL is also equal to EHN [Prop. 3.27], and if circumference BL is greater than circumference EN then angle BGL is also greater than EHN,† and if (BL is) less (than EN then BGL is also) less (than EHN). So there are four magnitudes, two circumferences BC and EF, and two angles BGC and EHF. And equal multiples have been taken of circumference BC and angle BGC, (namely) circumference BL and angle BGL, and of circumference EF and angle EHF, (namely) circumference EN and angle EHN. And it has been shown that if circumference BL exceeds circumference EN then angle BGL also exceeds angle EHN, and if (BL is) equal (to EN then BGL is also) equal (to EHN), and if (BL is) less (than EN then BGL is also) less (than EHN). Thus, as circumference BC (is) to EF, so angle BGC (is) to EHF [Def. 5.5]. But as angle BGC (is) to EHF, so (angle) BAC (is) to EDF [Prop. 5.15]. For the former (are) double the latter (respectively) [Prop. 3.20]. Thus, also, as circumference BC (is) to circumference EF, so angle BGC (is) to EHF, and BAC to EDF.

Thus, in equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences. (Which is) the very thing it was required to show. † This is a straight-forward generalization of Prop. 3.27

ELEMENTS BOOK 7

Elementary Number Theory†

† The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

Definitions

1. A unit is (that) according to which each existing (thing) is said (to be) one.

2. And a number (is) a multitude composed of units.†

3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.‡

4. But (the lesser is) parts (of the greater) when it does not measure it.§

5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.

6. An even number is one (which can be) divided in half.

7. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.

8. An even-times-even number is one (which is) measured by an even number according to an even number.¶

9. And an even-times-odd number is one (which is) measured by an even number according to an odd number.*

10. And an odd-times-odd number is one (which is) measured by an odd number according to an odd number.$

11. A prime‖ number is one (which is) measured by a unit alone.

12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.

13. A composite number is one (which is) measured by some number.

14. And numbers composite to one another are those (which are) measured by some number as a common measure.

15. A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.

16. And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another.

17. And when three numbers multiplying one another make some (other number) then the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.

18. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.

19. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.

20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.

21. Similar plane and solid numbers are those having proportional sides.

22. A perfect number is that which is equal to its own parts.††

† In other words, a "number" is a positive integer greater than unity.

‡ In other words, a number a is part of another number b if there exists some number n such that $n\,a = b$.

§ In other words, a number a is parts of another number b (where $a < b$) if there exist distinct numbers, m and n, such that $n\,a = m\,b$.

¶ In other words. an even-times-even number is the product of two even numbers.

* In other words, an even-times-odd number is the product of an even and an odd number.

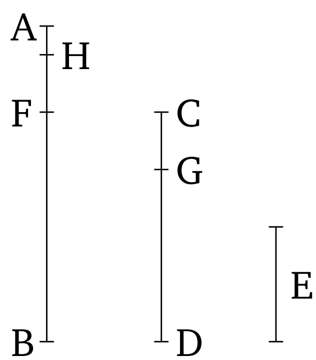
$ In other words, an odd-times-odd number is the product of two odd numbers.

‖ Literally, "first".

†† In other words, a perfect number is equal to the sum of its own factors.

Proposition 1

Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.



For two [unequal] numbers, AB and CD, the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains. I say that AB and CD are prime to one another—that is to say, that a unit alone measures (both) AB and CD.

For if AB and CD are not prime to one another then some number will measure them. Let (some number) measure them, and let it be E. And let CD measuring BF leave FA less than itself, and let AF measuring DG leave GC less than itself, and let GC measuring FH leave a unit, HA.

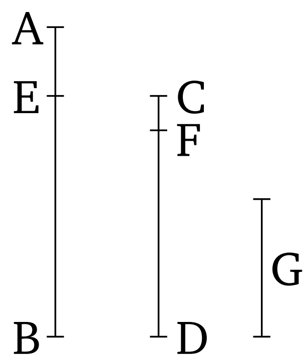
In fact, since E measures CD, and CD measures BF, E thus also measures BF.† And (E) also measures the whole of BA. Thus, (E) will also measure the remainder AF.‡ And AF measures DG. Thus, E also measures DG. And (E) also measures the whole of DC. Thus, (E) will also measure the remainder CG. And CG measures FH. Thus, E also measures FH. And (E) also measures the whole of FA. Thus, (E) will also measure the remaining unit AH, (despite) being a number. The very thing is impossible. Thus, some number does not measure (both) the numbers AB and CD. Thus, AB and CD are prime to one another. (Which is) the very thing it was required to show.

† Here, use is made of the unstated common notion that if a measures b, and b measures c, then a also measures c, where all symbols denote numbers.

‡ Here, use is made of the unstated common notion that if a measures b, and a measures part of b, then a also measures the remainder of b, where all symbols denote numbers.

Proposition 2

To find the greatest common measure of two given numbers (which are) not prime to one another.



Let AB and CD be the two given numbers (which are) not prime to one another. So it is required to find the greatest common measure of AB and CD.

In fact, if CD measures AB, CD is thus a common measure of CD and AB, (since CD) also measures itself. And (it is) manifest that (it is) also the greatest (common measure). For nothing greater than CD can measure CD.

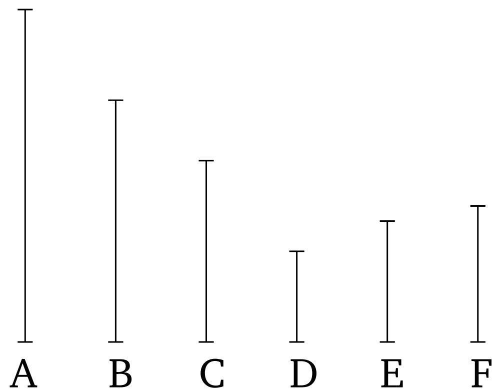
But if CD does not measure AB then some number will remain from AB and CD, the lesser being continually subtracted, in turn, from the greater, which will measure the (number) preceding it. For a unit will not be left. But if not, AB and CD will be prime to one another [Prop. 7.1]. The very opposite thing was assumed. Thus, some number will remain which will measure the (number) preceding it. And let CD measuring BE leave EA less than itself, and let EA measuring DF leave FC less than itself, and let CF measure AE. Therefore, since CF measures AE, and AE measures DF, CF will thus also measure DF. And it also measures itself. Thus, it will also measure the whole of CD. And CD measures BE. Thus, CF also measures BE. And it also measures EA. Thus, it will also measure the whole of BA. And it also measures CD. Thus, CF measures (both) AB and CD. Thus, CF is a common measure of AB and CD. So I say that (it is) also the greatest (common measure). For if CF is not the greatest common measure of AB and CD then some number which is greater than CF will measure the numbers AB and CD. Let it (so) measure (AB and CD), and let it be G. And since G measures CD, and CD measures BE, G thus also measures BE. And it also measures the whole of BA. Thus, it will also measure the remainder AE. And AE measures DF. Thus, G will also measure DF. And it also measures the whole of DC. Thus, it will also measure the remainder CF, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than CF cannot measure the numbers AB and CD. Thus, CF is the greatest common measure of AB and CD. [(Which is) the very thing it was required to show].

Corollary

So it is manifest, from this, that if a number measures two numbers then it will also measure their greatest common measure. (Which is) the very thing it was required to show.

Proposition 3

To find the greatest common measure of three given numbers (which are) not prime to one another.



Let A, B, and C be the three given numbers (which are) not prime to one another. So it is required to find the greatest common measure of A, B, and C.

For let the greatest common measure, D, of the two (numbers) A and B have been taken [Prop. 7.2]. So D either measures, or does not measure, C. First of all, let it measure (C). And it also measures A and B. Thus, D measures A, B, and C. Thus, D is a common measure of A, B, and C. So I say that (it is) also the greatest (common measure). For if D is not the greatest common measure of A, B, and C then some number greater than D will measure the numbers A, B, and C. Let it (so) measure (A, B, and C), and let it be E. Therefore, since E measures A, B, and C, it will thus also measure A and B. Thus, it will also measure the greatest common measure of A and B [Prop. 7.2 corr.]. And D is the greatest common measure of A and B. Thus, E measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than D cannot measure the numbers A, B, and C. Thus, D is the greatest common measure of A, B, and C.

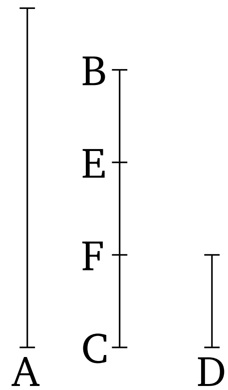
So let D not measure C. I say, first of all, that C and D are not prime to one another. For since A, B, C are not prime to one another, some number will measure them. So the (number) measuring A, B, and C will also measure A and B, and it will also measure the greatest common measure, D, of A and B [Prop. 7.2 corr.]. And it also measures C. Thus, some number will measure the numbers D and C. Thus, D and C are not prime to one another. Therefore, let their greatest common measure, E, have been taken [Prop. 7.2]. And since E measures D, and D measures A and B, E thus also measures A and B. And it also measures C. Thus, E measures A, B, and C. Thus, E is a common measure of A, B, and C. So I say that (it is) also the greatest (common measure). For if E is not the greatest common measure of A, B, and C then some number greater than E will measure the numbers A, B, and C. Let it (so) measure (A, B, and C), and let it be F. And since F measures A, B, and C, it also measures A and B. Thus, it will also measure the greatest common measure of A and B [Prop. 7.2 corr.]. And D is the greatest common measure of A and B. Thus, F measures D. And it also measures C. Thus, F measures D and C. Thus, it will also measure the greatest common measure of D and C [Prop. 7.2 corr.]. And E is the greatest common measure of D and C. Thus, F measures E, the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than E does not measure the numbers A, B, and C. Thus, E is the greatest common measure of A, B, and C. (Which is) the very thing it was required to show.

Proposition 4

Any number is either part or parts of any (other) number, the lesser of the greater.

Let A and BC be two numbers, and let BC be the lesser. I say that BC is either part or parts of A.

For A and BC are either prime to one another, or not. Let A and BC, first of all, be prime to one another. So separating BC into its constituent units, each of the units in BC will be some part of A. Hence, BC is parts of A.

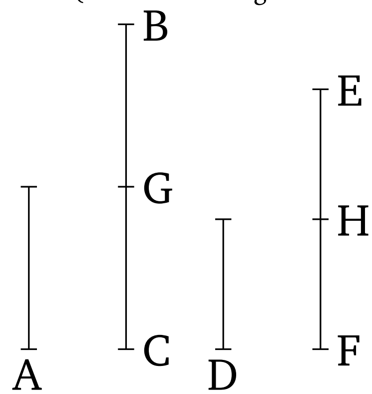


So let A and BC be not prime to one another. So BC either measures, or does not measure, A. Therefore, if BC measures A then BC is part of A. And if not, let the greatest common measure, D, of A and BC have been taken [Prop. 7.2], and let BC have been divided into BE, EF, and FC, equal to D. And since D measures A, D is a part of A. And D is equal to each of BE, EF, and FC. Thus, BE, EF, and FC are also each part of A. Hence, BC is parts of A.

Thus, any number is either part or parts of any (other) number, the lesser of the greater. (Which is) the very thing it was required to show.

Proposition 5†

If a number is part of a number, and another (number) is the same part of another, then the sum (of the leading numbers) will also be the same part of the sum (of the following numbers) that one (number) is of another.



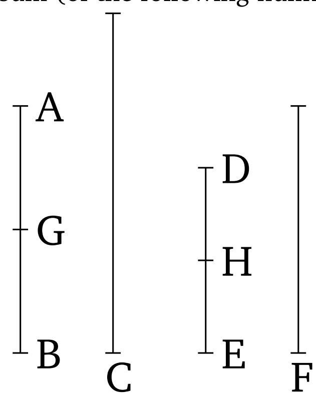
For let a number A be part of a [number] BC, and another (number) D (be) the same part of another (number) EF that A (is) of BC. I say that the sum A, D is also the same part of the sum BC, EF that A (is) of BC.

For since which(ever) part A is of BC, D is the same part of EF, thus as many numbers as are in BC equal to A, so many numbers are also in EF equal to D. Let BC have been divided into BG and GC, equal to A, and EF into EH and HF, equal to D. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF. And since BG is equal to A, and EH to D, thus BG, EH (is) also equal to A, D. So, for the same (reasons), GC, HF (is) also (equal) to A, D. Thus, as many numbers as [are] in BC equal to A, so many are also in BC, EF equal to A, D. Thus, as many times as BC is (divisible) by A, so many times is the sum BC, EF also (divisible) by the sum A, D. Thus, which(ever) part A is of BC, the sum A, D is also the same part of the sum BC, EF. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (1/n)\,b$ and $c = (1/n)\,d$ then $(a + c) = (1/n)\,(b + d)$, where all symbols denote numbers.

Proposition 6†

If a number is parts of a number, and another (number) is the same parts of another, then the sum (of the leading numbers) will also be the same parts of the sum (of the following numbers) that one (number) is of another.

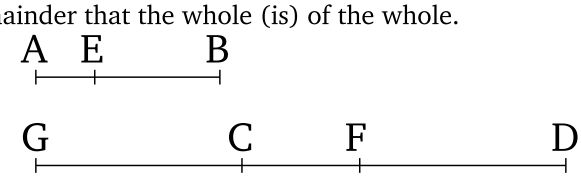


For let a number AB be parts of a number C, and another (number) DE (be) the same parts of another (number) F that AB (is) of C. I say that the sum AB, DE is also the same parts of the sum C, F that AB (is) of C.

For since which(ever) parts AB is of C, DE (is) also the same parts of F, thus as many parts of C as are in AB, so many parts of F are also in DE. Let AB have been divided into the parts of C, AG and GB, and DE into the parts of F, DH and HE. So the multitude of (divisions) AG, GB will be equal to the multitude of (divisions) DH, HE. And since which(ever) part AG is of C, DH is also the same part of F, thus which(ever) part AG is of C, the sum AG, DH is also the same part of the sum C, F [Prop. 7.5]. And so, for the same (reasons), which(ever) part GB is of C, the sum GB, HE is also the same part of the sum C, F. Thus, which(ever) parts AB is of C, the sum AB, DE is also the same parts of the sum C, F. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if $a = (m/n)\,b$ and $c = (m/n)\,d$ then $(a + c) = (m/n)\,(b + d)$, where all symbols denote numbers.

Proposition 7†

If a number is that part of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same part of the remainder that the whole (is) of the whole.

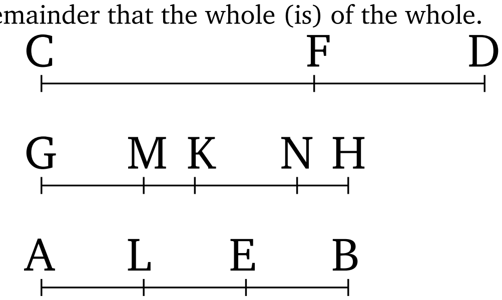


For let a number AB be that part of a number CD that a (part) taken away AE (is) of a part taken away CF. I say that the remainder EB is also the same part of the remainder FD that the whole AB (is) of the whole CD.

For which(ever) part AE is of CF, let EB also be the same part of CG. And since which(ever) part AE is of CF, EB is also the same part of CG, thus which(ever) part AE is of CF, AB is also the same part of GF [Prop. 7.5]. And which(ever) part AE is of CF, AB is also assumed (to be) the same part of CD. Thus, also, which(ever) part AB is of GF, (AB) is also the same part of CD. Thus, GF is equal to CD. Let CF have been subtracted from both. Thus, the remainder GC is equal to the remainder FD. And since which(ever) part AE is of CF, EB [is] also the same part of GC, and GC (is) equal to FD, thus which(ever) part AE is of CF, EB is also the same part of FD. But, which(ever) part AE is of CF, AB is also the same part of CD. Thus, the remainder EB is also the same part of the remainder FD that the whole AB (is) of the whole CD. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if $a = (1/n)\,b$ and $c = (1/n)\,d$ then $(a - c) = (1/n)\,(b - d)$, where all symbols denote numbers.

Proposition 8†

If a number is those parts of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same parts of the remainder that the whole (is) of the whole.

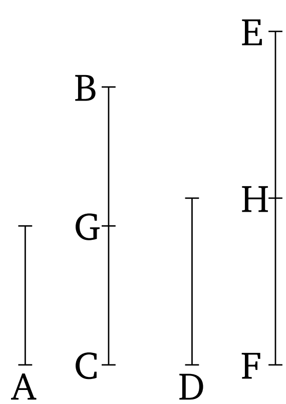


For let a number AB be those parts of a number CD that a (part) taken away AE (is) of a (part) taken away CF. I say that the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD.

For let GH be laid down equal to AB. Thus, which(ever) parts GH is of CD, AE is also the same parts of CF. Let GH have been divided into the parts of CD, GK and KH, and AE into the part of CF, AL and LE. So the multitude of (divisions) GK, KH will be equal to the multitude of (divisions) AL, LE. And since which(ever) part GK is of CD, AL is also the same part of CF, and CD (is) greater than CF, GK (is) thus also greater than AL. Let GM be made equal to AL. Thus, which(ever) part GK is of CD, GM is also the same part of CF. Thus, the remainder MK is also the same part of the remainder FD that the whole GK (is) of the whole CD [Prop. 7.5]. Again, since which(ever) part KH is of CD, EL is also the same part of CF, and CD (is) greater than CF, HK (is) thus also greater than EL. Let KN be made equal to EL. Thus, which(ever) part KH (is) of CD, KN is also the same part of CF. Thus, the remainder NH is also the same part of the remainder FD that the whole KH (is) of the whole CD [Prop. 7.5]. And the remainder MK was also shown to be the same part of the remainder FD that the whole GK (is) of the whole CD. Thus, the sum MK, NH is the same parts of DF that the whole HG (is) of the whole CD. And the sum MK, NH (is) equal to EB, and HG to BA. Thus, the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if $a = (m/n)\,b$ and $c = (m/n)\,d$ then $(a - c) = (m/n)\,(b - d)$, where all symbols denote numbers.

Proposition 9†

If a number is part of a number, and another (number) is the same part of another, also, alternately, which(ever) part, or parts, the first (number) is of the third, the second (number) will also be the same part, or the same parts, of the fourth.



For let a number A be part of a number BC, and another (number) D (be) the same part of another EF that A (is) of BC. I say that, also, alternately, which(ever) part, or parts, A is of D, BC is also the same part, or parts, of EF.

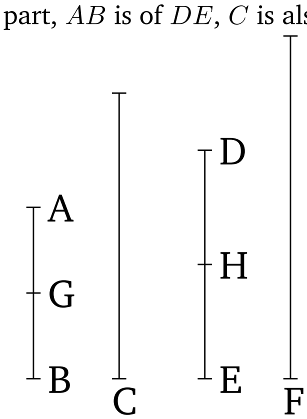
For since which(ever) part A is of BC, D is also the same part of EF, thus as many numbers as are in BC equal to A, so many are also in EF equal to D. Let BC have been divided into BG and GC, equal to A, and EF into EH and HF, equal to D. So the multitude of (divisions) BG, GC will be equal to the multitude of (divisions) EH, HF.

And since the numbers BG and GC are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) BG, GC is equal to the multitude of (divisions) EH, HC, thus which(ever) part, or parts, BG is of EH, GC is also the same part, or the same parts, of HF. And hence, which(ever) part, or parts, BG is of EH, the sum BC is also the same part, or the same parts, of the sum EF [Props. 7.5, 7.6]. And BG (is) equal to A, and EH to D. Thus, which(ever) part, or parts, A is of D, BC is also the same part, or the same parts, of EF. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if $a = (1/n)\, b$ and $c = (1/n)\, d$ then if $a = (k/l)\, c$ then $b = (k/l)\, d$, where all symbols denote numbers.

Proposition 10†

If a number is parts of a number, and another (number) is the same parts of another, also, alternately, which(ever) parts, or part, the first (number) is of the third, the second will also be the same parts, or the same part, of the fourth.

For let a number AB be parts of a number C, and another (number) DE (be) the same parts of another F. I say that, also, alternately, which(ever) parts, or part, AB is of DE, C is also the same parts, or the same part, of F.

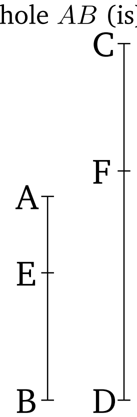


For since which(ever) parts AB is of C, DE is also the same parts of F, thus as many parts of C as are in AB, so many parts of F (are) also in DE. Let AB have been divided into the parts of C, AG and GB, and DE into the parts of F, DH and HE. So the multitude of (divisions) AG, GB will be equal to the multitude of (divisions) DH, HE. And since which(ever) part AG is of C, DH is also the same part of F, also, alternately, which(ever) part, or parts, AG is of DH, C is also the same part, or the same parts, of F [Prop. 7.9]. And so, for the same (reasons), which(ever) part, or parts, GB is of HE, C is also the same part, or the same parts, of F [Prop. 7.9]. And so [which(ever) part, or parts, AG is of DH, GB is also the same part, or the same parts, of HE. And thus, which(ever) part, or parts, AG is of DH, AB is also the same part, or the same parts, of DE [Props. 7.5, 7.6]. But, which(ever) part, or parts, AG is of DH, C was also shown (to be) the same part, or the same parts, of F. And, thus] which(ever) parts, or part, AB is of DE, C is also the same parts, or the same part, of F. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if $a = (m/n)\, b$ and $c = (m/n)\, d$ then if $a = (k/l)\, c$ then $b = (k/l)\, d$, where all symbols denote numbers.

Proposition 11

If as the whole (of a number) is to the whole (of another), so a (part) taken away (is) to a (part) taken away, then the remainder will also be to the remainder as the whole (is) to the whole.

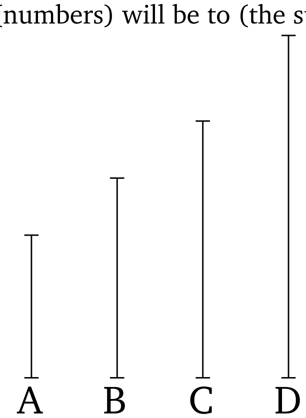
Let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF. I say that the remainder EB is to the remainder FD as the whole AB (is) to the whole CD.



(For) since as AB is to CD, so AE (is) to CF, thus which(ever) part, or parts, AB is of CD, AE is also the same part, or the same parts, of CF [Def. 7.20]. Thus, the remainder EB is also the same part, or parts, of the remainder FD that AB (is) of CD [Props. 7.7, 7.8]. Thus, as EB is to FD, so AB (is) to CD [Def. 7.20]. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if $a : b :: c : d$ then $a : b :: a - c : b - d$, where all symbols denote numbers.

Proposition 12†

If any multitude whatsoever of numbers are proportional then as one of the leading (numbers is) to one of the following so (the sum of) all of the leading (numbers) will be to (the sum of) all of the following.

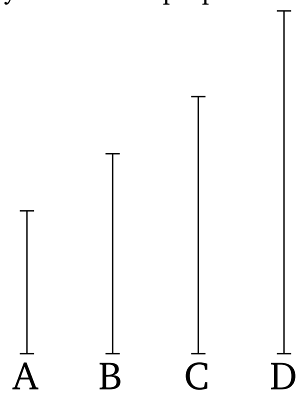


Let any multitude whatsoever of numbers, A, B, C, D, be proportional, (such that) as A (is) to B, so C (is) to D. I say that as A is to B, so A, C (is) to B, D.

For since as A is to B, so C (is) to D, thus which(ever) part, or parts, A is of B, C is also the same part, or parts, of D [Def. 7.20]. Thus, the sum A, C is also the same part, or the same parts, of the sum B, D that A (is) of B [Props. 7.5, 7.6]. Thus, as A is to B, so A, C (is) to B, D [Def. 7.20]. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if $a : b :: c : d$ then $a : b :: a + c : b + d$, where all symbols denote numbers.

Proposition 13†

If four numbers are proportional then they will also be proportional alternately.



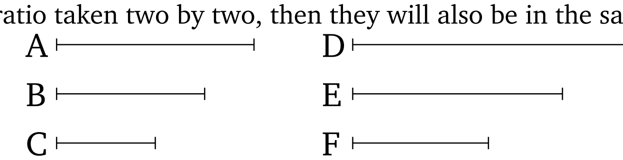
Let the four numbers A, B, C, and D be proportional, (such that) as A (is) to B, so C (is) to D. I say that they will also be proportional alternately, (such that) as A (is) to C, so B (is) to D.

For since as A is to B, so C (is) to D, thus which(ever) part, or parts, A is of B, C is also the same part, or the same parts, of D [Def. 7.20]. Thus, alterately, which(ever) part, or parts, A is of C, B is also the same part, or the same parts, of D [Props. 7.9, 7.10]. Thus, as A is to C, so B (is) to D [Def. 7.20]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a : b :: c : d$ then $a : c :: b : d$, where all symbols denote numbers.

Proposition 14†

If there are any multitude of numbers whatsoever, and (some) other (numbers) of equal multitude to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.



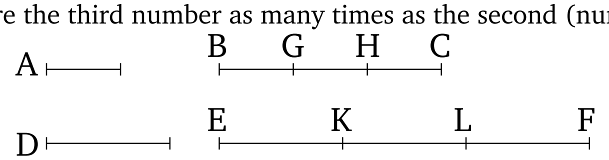
Let there be any multitude of numbers whatsoever, A, B, C, and (some) other (numbers), D, E, F, of equal multitude to them, (which are) in the same ratio taken two by two, (such that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that also, via equality, as A is to C, so D (is) to F.

For since as A is to B, so D (is) to E, thus, alternately, as A is to D, so B (is) to E [Prop. 7.13]. Again, since as B is to C, so E (is) to F, thus, alternately, as B is to E, so C (is) to F [Prop. 7.13]. And as B (is) to E, so A (is) to D. Thus, also, as A (is) to D, so C (is) to F. Thus, alternately, as A is to C, so D (is) to F [Prop. 7.13]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a : b :: d : e$ and $b : c :: e : f$ then $a : c :: d : f$, where all symbols denote numbers.

Proposition 15

If a unit measures some number, and another number measures some other number as many times, then, also, alternately, the unit will measure the third number as many times as the second (number measures) the fourth.

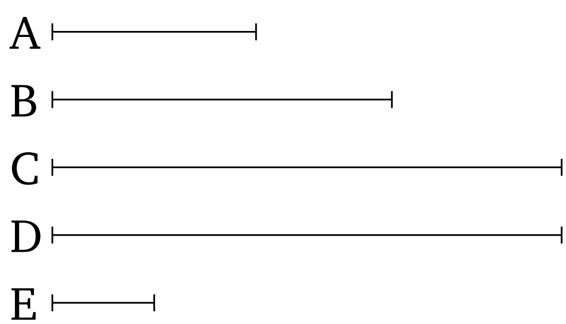


For let a unit A measure some number BC, and let another number D measure some other number EF as many times. I say that, also, alternately, the unit A also measures the number D as many times as BC (measures) EF.

For since the unit A measures the number BC as many times as D (measures) EF, thus as many units as are in BC, so many numbers are also in EF equal to D. Let BC have been divided into its constituent units, BG, GH, and HC, and EF into the (divisions) EK, KL, and LF, equal to D. So the multitude of (units) BG, GH, HC will be equal to the multitude of (divisions) EK, KL, LF. And since the units BG, GH, and HC are equal to one another, and the numbers EK, KL, and LF are also equal to one another, and the multitude of the (units) BG, GH, HC is equal to the multitude of the numbers EK, KL, LF, thus as the unit BG (is) to the number EK, so the unit GH will be to the number KL, and the unit HC to the number LF. And thus, as one of the leading (numbers is) to one of the following, so (the sum of) all of the leading will be to (the sum of) all of the following [Prop. 7.12]. Thus, as the unit BG (is) to the number EK, so BC (is) to EF. And the unit BG (is) equal to the unit A, and the number EK to the number D. Thus, as the unit A is to the number D, so BC (is) to EF. Thus, the unit A measures the number D as many times as BC (measures) EF [Def. 7.20]. (Which is) the very thing it was required to show. † This proposition is a special case of Prop. 7.9.

Proposition 16†

If two numbers multiplying one another make some (numbers) then the (numbers) generated from them will be equal to one another.

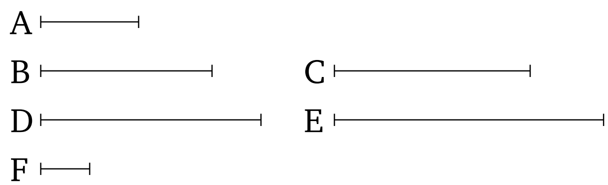


Let A and B be two numbers. And let A make C (by) multiplying B, and let B make D (by) multiplying A. I say that C is equal to D.

For since A has made C (by) multiplying B, B thus measures C according to the units in A [Def. 7.15]. And the unit E also measures the number A according to the units in it. Thus, the unit E measures the number A as many times as B (measures) C. Thus, alternately, the unit E measures the number B as many times as A (measures) C [Prop. 7.15]. Again, since B has made D (by) multiplying A, A thus measures D according to the units in B [Def. 7.15]. And the unit E also measures B according to the units in it. Thus, the unit E measures the number B as many times as A (measures) D. And the unit E was measuring the number B as many times as A (measures) C. Thus, A measures each of C and D an equal number of times. Thus, C is equal to D. (Which is) the very thing it was required to show. † In modern notation, this proposition states that $a\,b = b\,a$, where all symbols denote numbers.

Proposition 17†

If a number multiplying two numbers makes some (numbers) then the (numbers) generated from them will have the same ratio as the multiplied (numbers).

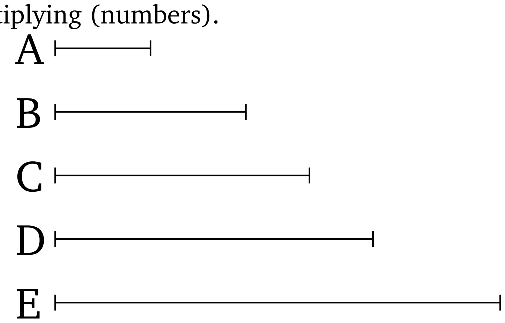


For let the number A make (the numbers) D and E (by) multiplying the two numbers B and C (respectively). I say that as B is to C, so D (is) to E.

For since A has made D (by) multiplying B, B thus measures D according to the units in A [Def. 7.15]. And the unit F also measures the number A according to the units in it. Thus, the unit F measures the number A as many times as B (measures) D. Thus, as the unit F is to the number A, so B (is) to D [Def. 7.20]. And so, for the same (reasons), as the unit F (is) to the number A, so C (is) to E. And thus, as B (is) to D, so C (is) to E. Thus, alternately, as B is to C, so D (is) to E [Prop. 7.13]. (Which is) the very thing it was required to show.† In modern notation, this proposition states that if $d = a\,b$ and $e = a\,c$ then $d : e :: b : c$, where all symbols denote numbers.

Proposition 18†

If two numbers multiplying some number make some (other numbers) then the (numbers) generated from them will have the same ratio as the multiplying (numbers).



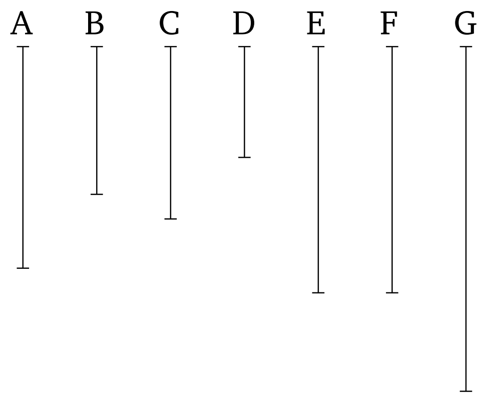
For let the two numbers A and B make (the numbers) D and E (respectively, by) multiplying some number C. I say that as A is to B, so D (is) to E.

For since A has made D (by) multiplying C, C has thus also made D (by) multiplying A [Prop. 7.16]. So, for the same (reasons), C has also made E (by) multiplying B. So the number C has made D and E (by) multiplying the two numbers A and B (respectively). Thus, as A is to B, so D (is) to E [Prop. 7.17]. (Which is) the very thing it was required to show. † In modern notation, this propositions states that if $a\,c = d$ and $b\,c = e$ then $a : b :: d : e$, where all symbols denote numbers.

Proposition 19†

If four number are proportional then the number created from (multiplying) the first and fourth will be equal to the number created from (multiplying) the second and third. And if the number created from (multiplying) the first and fourth is equal to the (number created) from (multiplying) the second and third then the four numbers will be proportional.

Let A, B, C, and D be four proportional numbers, (such that) as A (is) to B, so C (is) to D. And let A make E (by) multiplying D, and let B make F (by) multiplying C. I say that E is equal to F.



For let A make G (by) multiplying C. Therefore, since A has made G (by) multiplying C, and has made E (by) multiplying D, the number A has made G and E by multiplying the two numbers C and D (respectively). Thus, as C is to D, so G (is) to E [Prop. 7.17]. But, as C (is) to D, so A (is) to B. Thus, also, as A (is) to B, so G (is) to E. Again, since A has made G (by) multiplying C, but, in fact, B has also made F (by) multiplying C, the two numbers A and B have made G and F (respectively, by) multiplying some number C. Thus, as A is to B, so G (is) to F [Prop. 7.18]. But, also, as A (is) to B, so G (is) to E. And thus, as G (is) to E, so G (is) to F. Thus, G has the same ratio to each of E and F. Thus, E is equal to F [Prop. 5.9].

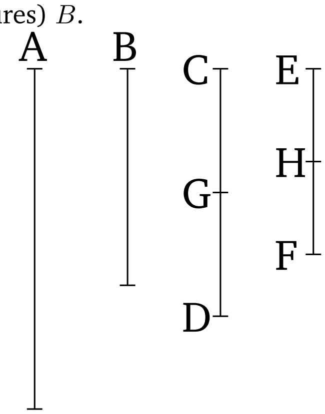
So, again, let E be equal to F. I say that as A is to B, so C (is) to D.

For, with the same construction, since E is equal to F, thus as G is to E, so G (is) to F [Prop. 5.7]. But, as G (is) to E, so C (is) to D [Prop. 7.17]. And as G (is) to F, so A (is) to B [Prop. 7.18]. And, thus, as A (is) to B, so C (is) to D. (Which is) the very thing it was required to show. † In modern notation, this proposition reads that if $a : b :: c : d$ then $a\,d = b\,c$, and *vice versa*, where all symbols denote numbers.

Proposition 20

The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

For let CD and EF be the least numbers having the same ratio as A and B (respectively). I say that CD measures A the same number of times as EF (measures) B.



For CD is not parts of A. For, if possible, let it be (parts of A). Thus, EF is also the same parts of B that CD (is) of A [Def. 7.20, Prop. 7.13]. Thus, as many parts of A as are in CD, so many parts of B are also in EF. Let CD

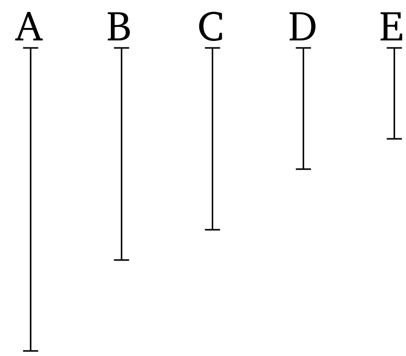
have been divided into the parts of A, CG and GD, and EF into the parts of B, EH and HF. So the multitude of (divisions) CG, GD will be equal to the multitude of (divisions) EH, HF. And since the numbers CG and GD are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) CG, GD is equal to the multitude of (divisions) EH, HF, thus as CG is to EH, so GD (is) to HF. Thus, as one of the leading (numbers is) to one of the following, so will (the sum of) all of the leading (numbers) be to (the sum of) all of the following [Prop. 7.12]. Thus, as CG is to EH, so CD (is) to EF. Thus, CG and EH are in the same ratio as CD and EF, being less than them. The very thing is impossible. For CD and EF were assumed (to be) the least of those (numbers) having the same ratio as them. Thus, CD is not parts of A. Thus, (it is) a part (of A) [Prop. 7.4]. And EF is the same part of B that CD (is) of A [Def. 7.20, Prop 7.13]. Thus, CD measures A the same number of times that EF (measures) B. (Which is) the very thing it was required to show.

Proposition 21

Numbers prime to one another are the least of those (numbers) having the same ratio as them.

Let A and B be numbers prime to one another. I say that A and B are the least of those (numbers) having the same ratio as them.

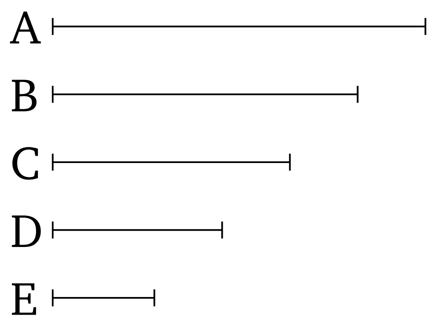
For if not then there will be some numbers less than A and B which are in the same ratio as A and B. Let them be C and D.



Therefore, since the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following—C thus measures A the same number of times that D (measures) B [Prop. 7.20]. So as many times as C measures A, so many units let there be in E. Thus, D also measures B according to the units in E. And since C measures A according to the units in E, E thus also measures A according to the units in C [Prop. 7.16]. So, for the same (reasons), E also measures B according to the units in D [Prop. 7.16]. Thus, E measures A and B, which are prime to one another. The very thing is impossible. Thus, there cannot be any numbers less than A and B which are in the same ratio as A and B. Thus, A and B are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

Proposition 22

The least numbers of those (numbers) having the same ratio as them are prime to one another.



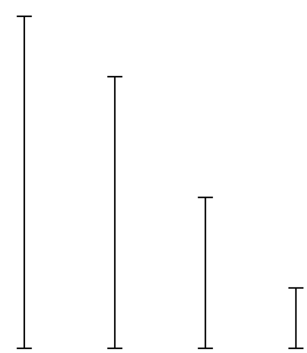
Let A and B be the least numbers of those (numbers) having the same ratio as them. I say that A and B are prime to one another.

For if they are not prime to one another then some number will measure them. Let it (so measure them), and let it be C. And as many times as C measures A, so many units let there be in D. And as many times as C measures B, so many units let there be in E.

Since C measures A according to the units in D, C has thus made A (by) multiplying D [Def. 7.15]. So, for the same (reasons), C has also made B (by) multiplying E. So the number C has made A and B (by) multiplying the two numbers D and E (respectively). Thus, as D is to E, so A (is) to B [Prop. 7.17]. Thus, D and E are in the same ratio as A and B, being less than them. The very thing is impossible. Thus, some number does not measure the numbers A and B. Thus, A and B are prime to one another. (Which is) the very thing it was required to show.

Proposition 23

If two numbers are prime to one another then a number measuring one of them will be prime to the remaining (one).

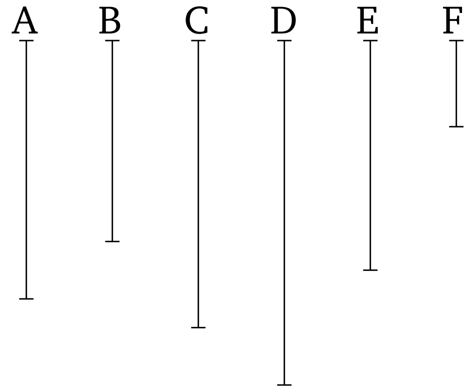


Let A and B be two numbers (which are) prime to one another, and let some number C measure A. I say that C and B are also prime to one another.

For if C and B are not prime to one another then [some] number will measure C and B. Let it (so) measure (them), and let it be D. Since D measures C, and C measures A, D thus also measures A. And (D) also measures B. Thus, D measures A and B, which are prime to one another. The very thing is impossible. Thus, some number does not measure the numbers C and B. Thus, C and B are prime to one another. (Which is) the very thing it was required to show.

Proposition 24

If two numbers are prime to some number then the number created from (multiplying) the former (two numbers) will also be prime to the latter (number).



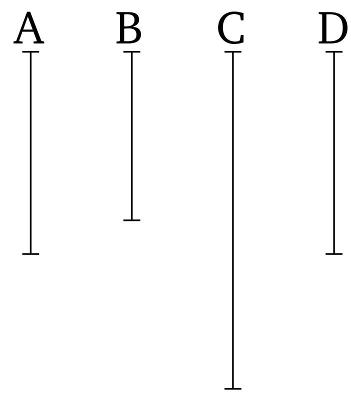
For let A and B be two numbers (which are both) prime to some number C. And let A make D (by) multiplying B. I say that C and D are prime to one another.

For if C and D are not prime to one another then [some] number will measure C and D. Let it (so) measure them, and let it be E. And since C and A are prime to one another, and some number E measures C, A and E are thus prime to one another [Prop. 7.23]. So as many times as E measures D, so many units let there be in F. Thus, F also measures D according to the units in E [Prop. 7.16]. Thus, E has made D (by) multiplying F [Def. 7.15]. But, in fact, A has also made D (by) multiplying B. Thus, the (number created) from (multiplying) E and F is equal to the (number created) from (multiplying) A and B. And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four numbers are proportional [Prop. 6.15]. Thus, as E is to A, so B (is) to F. And A and E (are) prime (to one another). And (numbers) prime (to one another) are also the least (of those numbers having the same ratio) [Prop. 7.21]. And the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures B. And it also measures C. Thus, E measures B and C, which are prime to one another. The very thing is impossible. Thus, some number cannot measure the numbers C and D. Thus, C and D are prime to one another. (Which is) the very thing it was required to show.

Proposition 25

If two numbers are prime to one another then the number created from (squaring) one of them will be prime to the remaining (number).

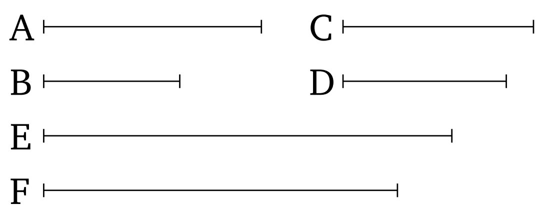
Let A and B be two numbers (which are) prime to one another. And let A make C (by) multiplying itself. I say that B and C are prime to one another.



For let D be made equal to A. Since A and B are prime to one another, and A (is) equal to D, D and B are thus also prime to one another. Thus, D and A are each prime to B. Thus, the (number) created from (multilying) D and A will also be prime to B [Prop. 7.24]. And C is the number created from (multiplying) D and A. Thus, C and B are prime to one another. (Which is) the very thing it was required to show.

Proposition 26

If two numbers are both prime to each of two numbers then the (numbers) created from (multiplying) them will also be prime to one another.

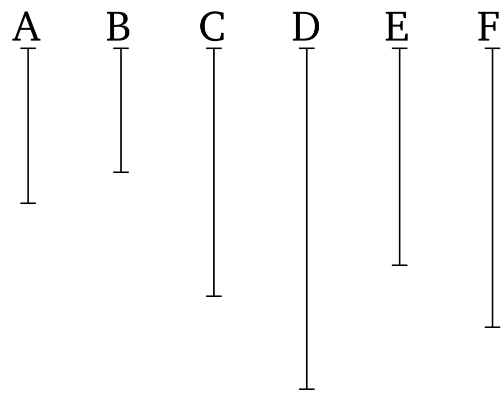


For let two numbers, A and B, both be prime to each of two numbers, C and D. And let A make E (by) multiplying B, and let C make F (by) multiplying D. I say that E and F are prime to one another.

For since A and B are each prime to C, the (number) created from (multiplying) A and B will thus also be prime to C [Prop. 7.24]. And E is the (number) created from (multiplying) A and B. Thus, E and C are prime to one another. So, for the same (reasons), E and D are also prime to one another. Thus, C and D are each prime to E. Thus, the (number) created from (multiplying) C and D will also be prime to E [Prop. 7.24]. And F is the (number) created from (multiplying) C and D. Thus, E and F are prime to one another. (Which is) the very thing it was required to show.

Proposition 27†

If two numbers are prime to one another and each makes some (number by) multiplying itself then the numbers created from them will be prime to one another, and if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be prime to one another [and this always happens with the extremes].

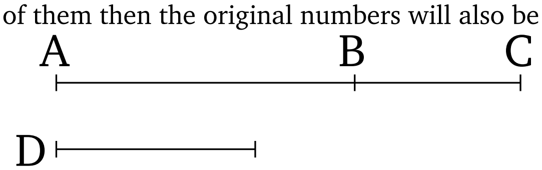


Let A and B be two numbers prime to one another, and let A make C (by) multiplying itself, and let it make D (by) multiplying C. And let B make E (by) multiplying itself, and let it make F by multiplying E. I say that C and E, and D and F, are prime to one another.

For since A and B are prime to one another, and A has made C (by) multiplying itself, C and B are thus prime to one another [Prop. 7.25]. Therefore, since C and B are prime to one another, and B has made E (by) multiplying itself, C and E are thus prime to one another [Prop. 7.25]. Again, since A and B are prime to one another, and B has made E (by) multiplying itself, A and E are thus prime to one another [Prop. 7.25]. Therefore, since the two numbers A and C are both prime to each of the two numbers B and E, the (number) created from (multiplying) A and C is thus prime to the (number created) from (multiplying) B and E [Prop. 7.26]. And D is the (number created) from (multiplying) A and C, and F the (number created) from (multiplying) B and E. Thus, D and F are prime to one another. (Which is) the very thing it was required to show. † In modern notation, this proposition states that if a is prime to b, then a^2 is also prime to b^2, as well as a^3 to b^3, *etc.*, where all symbols denote numbers.

Proposition 28

If two numbers are prime to one another then their sum will also be prime to each of them. And if the sum (of two numbers) is prime to any one of them then the original numbers will also be prime to one another.



For let the two numbers, AB and BC, (which are) prime to one another, be laid down together. I say that their sum AC is also prime to each of AB and BC.

For if CA and AB are not prime to one another then some number will measure CA and AB. Let it (so) measure (them), and let it be D. Therefore, since D measures CA and AB, it will thus also measure the remainder BC. And it also measures BA. Thus, D measures AB and BC, which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers CA and AB. Thus, CA and AB are prime to one another. So, for the same (reasons), AC and CB are also prime to one another. Thus, CA is prime to each of AB and BC.

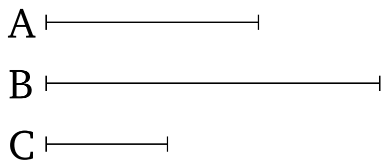
So, again, let CA and AB be prime to one another. I say that AB and BC are also prime to one another.

For if AB and BC are not prime to one another then some number will measure AB and BC. Let it (so) measure (them), and let it be D. And since D measures each of AB and BC, it will thus also measure the whole of CA. And it also measures AB. Thus, D measures CA and AB, which are prime to one another. The very thing is impossible.

Thus, some number cannot measure (both) the numbers AB and BC. Thus, AB and BC are prime to one another. (Which is) the very thing it was required to show.

Proposition 29

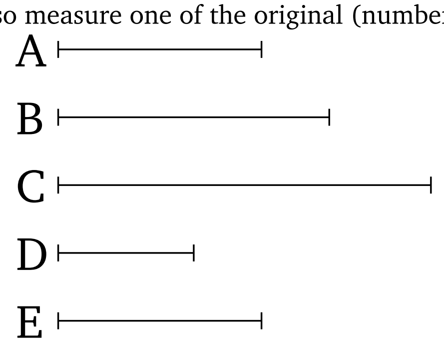
Every prime number is prime to every number which it does not measure.



Let A be a prime number, and let it not measure B. I say that B and A are prime to one another. For if B and A are not prime to one another then some number will measure them. Let C measure (them). Since C measures B, and A does not measure B, C is thus not the same as A. And since C measures B and A, it thus also measures A, which is prime, (despite) not being the same as it. The very thing is impossible. Thus, some number cannot measure (both) B and A. Thus, A and B are prime to one another. (Which is) the very thing it was required to show.

Proposition 30

If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, then it will also measure one of the original (numbers).



For let two numbers A and B make C (by) multiplying one another, and let some prime number D measure C. I say that D measures one of A and B.

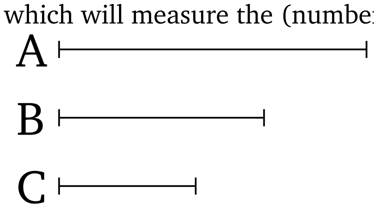
For let it not measure A. And since D is prime, A and D are thus prime to one another [Prop. 7.29]. And as many times as D measures C, so many units let there be in E. Therefore, since D measures C according to the units E, D has thus made C (by) multiplying E [Def. 7.15]. But, in fact, A has also made C (by) multiplying B. Thus, the (number created) from (multiplying) D and E is equal to the (number created) from (multiplying) A and B. Thus, as D is to A, so B (is) to E [Prop. 7.19]. And D and A (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, D measures B. So, similarly, we can also show that if (D) does not measure B then it will measure A. Thus, D measures one of A and B. (Which is) the very thing it was required to show.

Proposition 31

Every composite number is measured by some prime number.

Let A be a composite number. I say that A is measured by some prime number.

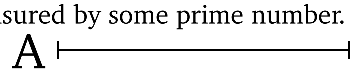
For since A is composite, some number will measure it. Let it (so) measure (A), and let it be B. And if B is prime then that which was prescribed has happened. And if (B is) composite then some number will measure it. Let it (so) measure (B), and let it be C. And since C measures B, and B measures A, C thus also measures A. And if C is prime then that which was prescribed has happened. And if (C is) composite then some number will measure it. So, in this manner of continued investigation, some prime number will be found which will measure (the number preceding it, which will also measure A). And if (such a number) cannot be found then an infinite (series of) numbers, each of which is less than the preceding, will measure the number A. The very thing is impossible for numbers. Thus, some prime number will (eventually) be found which will measure the (number) preceding it, which will also measure A.



Thus, every composite number is measured by some prime number. (Which is) the very thing it was required to show.

Proposition 32

Every number is either prime or is measured by some prime number.



Let A be a number. I say that A is either prime or is measured by some prime number.

In fact, if A is prime then that which was prescribed has happened. And if (it is) composite then some prime number will measure it [Prop. 7.31].

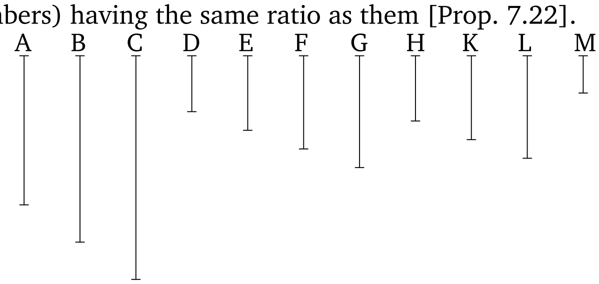
Thus, every number is either prime or is measured by some prime number. (Which is) the very thing it was required to show.

Proposition 33

To find the least of those (numbers) having the same ratio as any given multitude of numbers.

Let A, B, and C be any given multitude of numbers. So it is required to find the least of those (numbers) having the same ratio as A, B, and C.

For A, B, and C are either prime to one another, or not. In fact, if A, B, and C are prime to one another then they are the least of those (numbers) having the same ratio as them [Prop. 7.22].

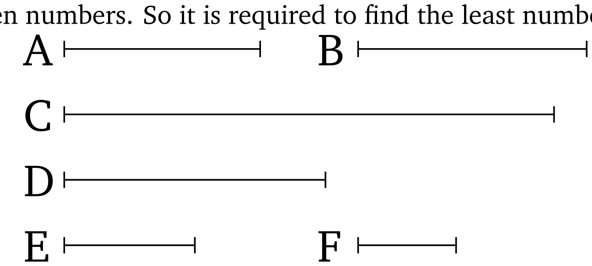


And if not, let the greatest common measure, D, of A, B, and C have be taken [Prop. 7.3]. And as many times as D measures A, B, C, so many units let there be in E, F, G, respectively. And thus E, F, G measure A, B, C, respectively, according to the units in D [Prop. 7.15]. Thus, E, F, G measure A, B, C (respectively) an equal number of times. Thus, E, F, G are in the same ratio as A, B, C (respectively) [Def. 7.20]. So I say that (they are) also the least (of those numbers having the same ratio as A, B, C). For if E, F, G are not the least of those (numbers) having the same ratio as A, B, C (respectively), then there will be [some] numbers less than E, F, G which are in the same ratio as A, B, C (respectively). Let them be H, K, L. Thus, H measures A the same number of times that K, L also measure B, C, respectively. And as many times as H measures A, so many units let there be in M. Thus, K, L measure B, C, respectively, according to the units in M. And since H measures A according to the units in M, M thus also measures A according to the units in H [Prop. 7.15]. So, for the same (reasons), M also measures B, C according to the units in K, L, respectively. Thus, M measures A, B, and C. And since H measures A according to the units in M, H has thus made A (by) multiplying M. So, for the same (reasons), E has also made A (by) multiplying D. Thus, the (number created) from (multiplying) E and D is equal to the (number created) from (multiplying) H and M. Thus, as E (is) to H, so M (is) to D [Prop. 7.19]. And E (is) greater than H. Thus, M (is) also greater than D [Prop. 5.13]. And (M) measures A, B, and C. The very thing is impossible. For D was assumed (to be) the greatest common measure of A, B, and C. Thus, there cannot be any numbers less than E, F, G which are in the same ratio as A, B, C (respectively). Thus, E, F, G are the least of (those numbers) having the same ratio as A, B, C (respectively). (Which is) the very thing it was required to show.

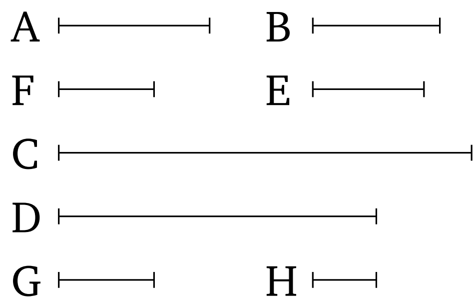
Proposition 34

To find the least number which two given numbers (both) measure.

Let A and B be the two given numbers. So it is required to find the least number which they (both) measure.



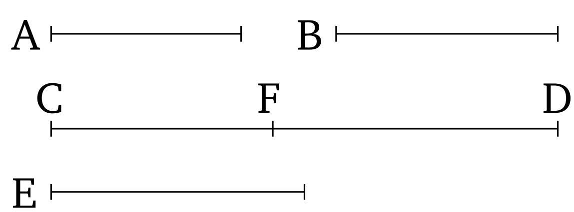
For A and B are either prime to one another, or not. Let them, first of all, be prime to one another. And let A make C (by) multiplying B. Thus, B has also made C (by) multiplying A [Prop. 7.16]. Thus, A and B (both) measure C. So I say that (C) is also the least (number which they both measure). For if not, A and B will (both) measure some (other) number which is less than C. Let them (both) measure D (which is less than C). And as many times as A measures D, so many units let there be in E. And as many times as B measures D, so many units let there be in F. Thus, A has made D (by) multiplying E, and B has made D (by) multiplying F. Thus, the (number created) from (multiplying) A and E is equal to the (number created) from (multiplying) B and F. Thus, as A (is) to B, so F (is) to E [Prop. 7.19]. And A and B are prime (to one another), and prime (numbers) are the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, B measures E, as the following (number measuring) the following. And since A has made C and D (by) multiplying B and E (respectively), thus as B is to E, so C (is) to D [Prop. 7.17]. And B measures E. Thus, C also measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, A and B do not (both) measure some number which is less than C. Thus, C is the least (number) which is measured by (both) A and B.



So let A and B be not prime to one another. And let the least numbers, F and E, have been taken having the same ratio as A and B (respectively) [Prop. 7.33]. Thus, the (number created) from (multiplying) A and E is equal to the (number created) from (multiplying) B and F [Prop. 7.19]. And let A make C (by) multiplying E. Thus, B has also made C (by) multiplying F. Thus, A and B (both) measure C. So I say that (C) is also the least (number which they both measure). For if not, A and B will (both) measure some number which is less than C. Let them (both) measure D (which is less than C). And as many times as A measures D, so many units let there be in G. And as many times as B measures D, so many units let there be in H. Thus, A has made D (by) multiplying G, and B has made D (by) multiplying H. Thus, the (number created) from (multiplying) A and G is equal to the (number created) from (multiplying) B and H. Thus, as A is to B, so H (is) to G [Prop. 7.19]. And as A (is) to B, so F (is) to E. Thus, also, as F (is) to E, so H (is) to G. And F and E are the least (numbers having the same ratio as A and B), and the least (numbers) measure those (numbers) having the same ratio an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, E measures G. And since A has made C and D (by) multiplying E and G (respectively), thus as E is to G, so C (is) to D [Prop. 7.17]. And E measures G. Thus, C also measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, A and B do not (both) measure some (number) which is less than C. Thus, C (is) the least (number) which is measured by (both) A and B. (Which is) the very thing it was required to show.

Proposition 35

If two numbers (both) measure some number then the least (number) measured by them will also measure the same (number).



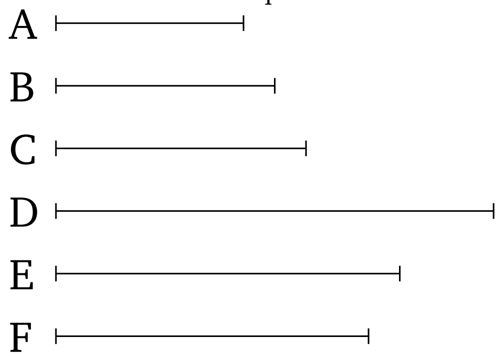
For let two numbers, A and B, (both) measure some number CD, and (let) E (be the) least (number measured by both A and B). I say that E also measures CD.

For if E does not measure CD then let E leave CF less than itself (in) measuring DF. And since A and B (both) measure E, and E measures DF, A and B will thus also measure DF. And (A and B) also measure the whole of CD. Thus, they will also measure the remainder CF, which is less than E. The very thing is impossible. Thus, E cannot not measure CD. Thus, (E) measures (CD). (Which is) the very thing it was required to show.

Proposition 36

To find the least number which three given numbers (all) measure.

Let A, B, and C be the three given numbers. So it is required to find the least number which they (all) measure.

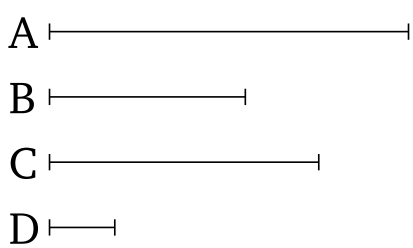


For let the least (number), D, measured by the two (numbers) A and B have been taken [Prop. 7.34]. So C either measures, or does not measure, D. Let it, first of all, measure (D). And A and B also measure D. Thus, A, B, and C (all) measure D. So I say that (D is) also the least (number measured by A, B, and C). For if not, A, B, and C will (all) measure [some] number which is less than D. Let them measure E (which is less than D). Since A, B, and C (all) measure E then A and B thus also measure E. Thus, the least (number) measured by A and B will also measure [E] [Prop. 7.35]. And D is the least (number) measured by A and B. Thus, D will measure E, the greater (measuring) the lesser. The very thing is impossible. Thus, A, B, and C cannot (all) measure some number which is less than D. Thus, A, B, and C (all) measure the least (number) D.

So, again, let C not measure D. And let the least number, E, measured by C and D have been taken [Prop. 7.34]. Since A and B measure D, and D measures E, A and B thus also measure E. And C also measures [E]. Thus, A, B, and C [also] measure E. So I say that (E is) also the least (number measured by A, B, and C). For if not, A, B, and C will (all) measure some (number) which is less than E. Let them measure F (which is less than E). Since A, B, and C (all) measure F, A and B thus also measure F. Thus, the least (number) measured by A and B will also measure F [Prop. 7.35]. And D is the least (number) measured by A and B. Thus, D measures F. And C also measures F. Thus, D and C (both) measure F. Hence, the least (number) measured by D and C will also measure F [Prop. 7.35]. And E is the least (number) measured by C and D. Thus, E measures F, the greater (measuring) the lesser. The very thing is impossible. Thus, A, B, and C cannot measure some number which is less than E. Thus, E (is) the least (number) which is measured by A, B, and C. (Which is) the very thing it was required to show.

Proposition 37

If a number is measured by some number then the (number) measured will have a part called the same as the measuring (number).



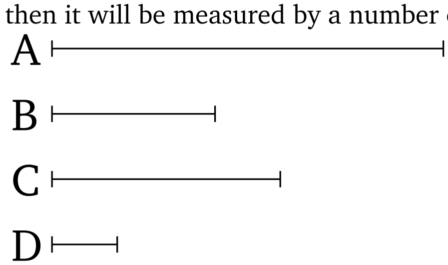
For let the number A be measured by some number B. I say that A has a part called the same as B.

For as many times as B measures A, so many units let there be in C. Since B measures A according to the units in C, and the unit D also measures C according to the units in it, the unit D thus measures the number C as many times as B (measures) A. Thus, alternately, the unit D measures the number B as many times as C (measures) A

[Prop. 7.15]. Thus, which(ever) part the unit D is of the number B, C is also the same part of A. And the unit D is a part of the number B called the same as it (*i.e.*, a Bth part). Thus, C is also a part of A called the same as B (*i.e.*, C is the Bth part of A). Hence, A has a part C which is called the same as B (*i.e.*, A has a Bth part). (Which is) the very thing it was required to show.

Proposition 38

If a number has any part whatever then it will be measured by a number called the same as the part.

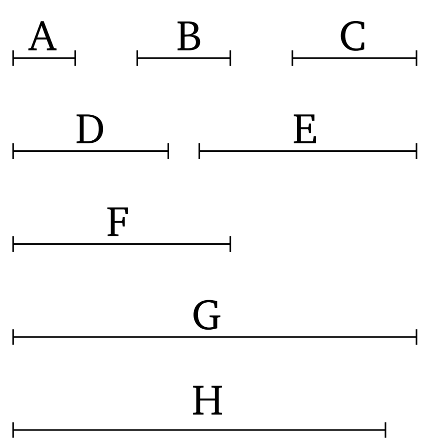


For let the number A have any part whatever, B. And let the [number] C be called the same as the part B (*i.e.*, B is the Cth part of A). I say that C measures A.

For since B is a part of A called the same as C, and the unit D is also a part of C called the same as it (*i.e.*, D is the Cth part of C), thus which(ever) part the unit D is of the number C, B is also the same part of A. Thus, the unit D measures the number C as many times as B (measures) A. Thus, alternately, the unit D measures the number B as many times as C (measures) A [Prop. 7.15]. Thus, C measures A. (Which is) the very thing it was required to show.

Proposition 39

To find the least number that will have given parts.



Let A, B, and C be the given parts. So it is required to find the least number which will have the parts A, B, and C (*i.e.*, an Ath part, a Bth part, and a Cth part).

For let D, E, and F be numbers having the same names as the parts A, B, and C (respectively). And let the least number, G, measured by D, E, and F, have been taken [Prop. 7.36].

Thus, G has parts called the same as D, E, and F [Prop. 7.37]. And A, B, and C are parts called the same as D, E, and F (respectively). Thus, G has the parts A, B, and C. So I say that (G) is also the least (number having the parts A, B, and C). For if not, there will be some number less than G which will have the parts A, B, and C. Let it be H. Since H has the parts A, B, and C, H will thus be measured by numbers called the same as the parts A, B, and C [Prop. 7.38]. And D, E, and F are numbers called the same as the parts A, B, and C (respectively). Thus, H is measured by D, E, and F. And (H) is less than G. The very thing is impossible. Thus, there cannot be some number less than G which will have the parts A, B, and C. (Which is) the very thing it was required to show.

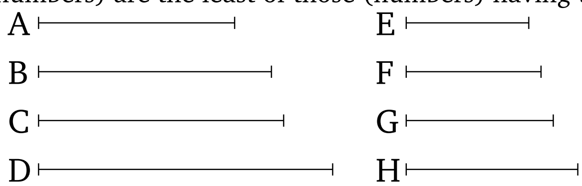
ELEMENTS BOOK 8

Continued Proportion†

† The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

Proposition 1

If there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them.



Let A, B, C, D be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, A and D, be prime to one another. I say that A, B, C, D are the least of those (numbers) having the same ratio as them.

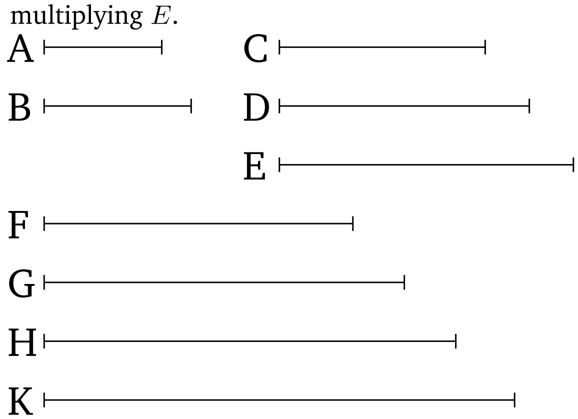
For if not, let E, F, G, H be less than A, B, C, D (respectively), being in the same ratio as them. And since A, B, C, D are in the same ratio as E, F, G, H, and the multitude [of A, B, C, D] is equal to the multitude [of E, F, G, H], thus, via equality, as A is to D, (so) E (is) to H [Prop. 7.14]. And A and D (are) prime (to one another). And prime (numbers are) also the least of those (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures E, the greater (measuring) the lesser. The very thing is impossible. Thus, E, F, G, H, being less than A, B, C, D, are not in the same ratio as them. Thus, A, B, C, D are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

Proposition 2

To find the least numbers, as many as may be prescribed, (which are) continuously proportional in a given ratio.

Let the given ratio, (expressed) in the least numbers, be that of A to B. So it is required to find the least numbers, as many as may be prescribed, (which are) in the ratio of A to B.

Let four (numbers) have been prescribed. And let A make C (by) multiplying itself, and let it make D (by) multiplying B. And, further, let B make E (by) multiplying itself. And, further, let A make F, G, H (by) multiplying C, D, E. And let B make K (by) multiplying E.



And since A has made C (by) multiplying itself, and has made D (by) multiplying B, thus as A is to B, [so] C (is) to D [Prop. 7.17]. Again, since A has made D (by) multiplying B, and B has made E (by) multiplying itself, A,

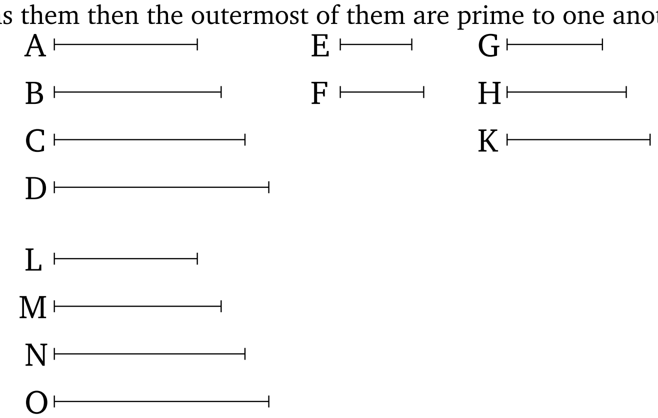
B have thus made D, E, respectively, (by) multiplying B. Thus, as A is to B, so D (is) to E [Prop. 7.18]. But, as A (is) to B, (so) C (is) to D. And thus as C (is) to D, (so) D (is) to E. And since A has made F, G (by) multiplying C, D, thus as C is to D, [so] F (is) to G [Prop. 7.17]. And as C (is) to D, so A was to B. And thus as A (is) to B, (so) F (is) to G. Again, since A has made G, H (by) multiplying D, E, thus as D is to E, (so) G (is) to H [Prop. 7.17]. But, as D (is) to E, (so) A (is) to B. And thus as A (is) to B, so G (is) to H. And since A, B have made H, K (by) multiplying E, thus as A is to B, so H (is) to K. But, as A (is) to B, so F (is) to G, and G to H. And thus as F (is) to G, so G (is) to H, and H to K. Thus, C, D, E and F, G, H, K are (both continuously) proportional in the ratio of A to B. So I say that (they are) also the least (sets of numbers continuously proportional in that ratio). For since A and B are the least of those (numbers) having the same ratio as them, and the least of those (numbers) having the same ratio are prime to one another [Prop. 7.22], A and B are thus prime to one another. And A, B have made C, E, respectively, (by) multiplying themselves, and have made F, K by multiplying C, E, respectively. Thus, C, E and F, K are prime to one another [Prop. 7.27]. And if there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them [Prop. 8.1]. Thus, C, D, E and F, G, H, K are the least of those (continuously proportional sets of numbers) having the same ratio as A and B. (Which is) the very thing it was required to show.

Corollary

So it is clear, from this, that if three continuously proportional numbers are the least of those (numbers) having the same ratio as them then the outermost of them are square, and, if four (numbers), cube.

Proposition 3

If there are any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them then the outermost of them are prime to one another.



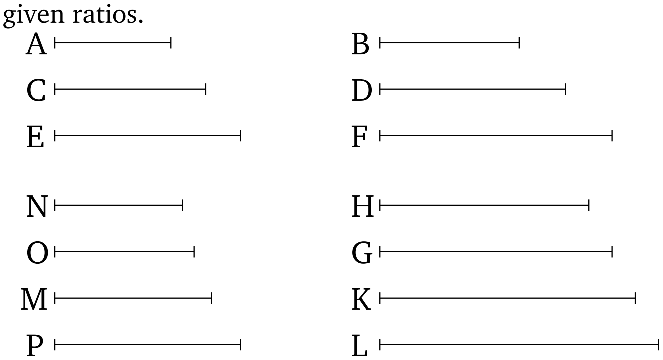
Let A, B, C, D be any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that the outermost of them, A and D, are prime to one another.

For let the two least (numbers) E, F (which are) in the same ratio as A, B, C, D have been taken [Prop. 7.33]. And the three (least numbers) G, H, K [Prop. 8.2]. And (so on), successively increasing by one, until the multitude of (numbers) taken is made equal to the multitude of A, B, C, D. Let them have been taken, and let them be L, M, N, O.

And since E and F are the least of those (numbers) having the same ratio as them they are prime to one another [Prop. 7.22]. And since E, F have made G, K, respectively, (by) multiplying themselves [Prop. 8.2 corr.], and have made L, O (by) multiplying G, K, respectively, G, K and L, O are thus also prime to one another [Prop. 7.27]. And since A, B, C, D are the least of those (numbers) having the same ratio as them, and L, M, N, O are also the least (of those numbers having the same ratio as them), being in the same ratio as A, B, C, D, and the multitude of A, B, C, D is equal to the multitude of L, M, N, O, thus A, B, C, D are equal to L, M, N, O, respectively. Thus, A is equal to L, and D to O. And L and O are prime to one another. Thus, A and D are also prime to one another. (Which is) the very thing it was required to show.

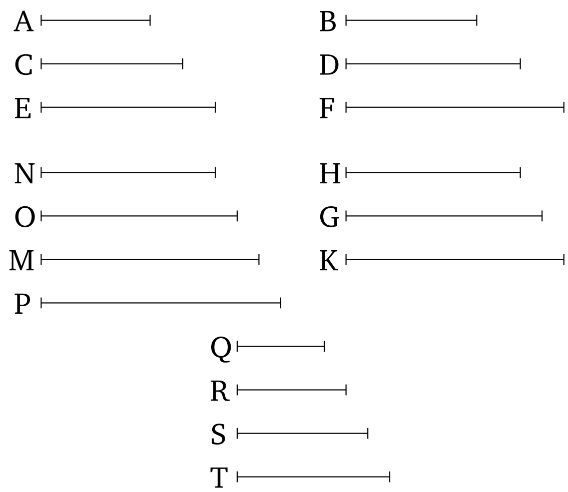
Proposition 4

For any multitude whatsoever of given ratios, (expressed) in the least numbers, to find the least numbers continuously proportional in these given ratios.



Let the given ratios, (expressed) in the least numbers, be the (ratios) of A to B, and of C to D, and, further, of E to F. So it is required to find the least numbers continuously proportional in the ratio of A to B, and of C to B, and, further, of E to F.

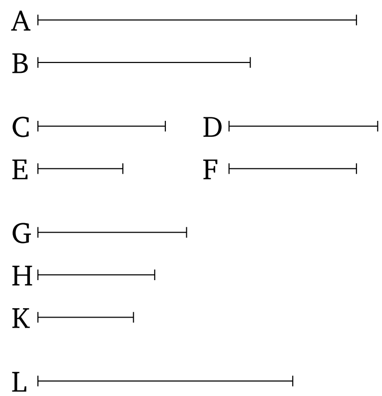
For let the least number, G, measured by (both) B and C have be taken [Prop. 7.34]. And as many times as B measures G, so many times let A also measure H. And as many times as C measures G, so many times let D also measure K. And E either measures, or does not measure, K. Let it, first of all, measure (K). And as many times as E measures K, so many times let F also measure L. And since A measures H the same number of times that B also (measures) G, thus as A is to B, so H (is) to G [Def. 7.20, Prop. 7.13]. And so, for the same (reasons), as C (is) to D, so G (is) to K, and, further, as E (is) to F, so K (is) to L. Thus, H, G, K, L are continuously proportional in the ratio of A to B, and of C to D, and, further, of E to F. So I say that (they are) also the least (numbers continuously proportional in these ratios). For if H, G, K, L are not the least numbers continuously proportional in the ratios of A to B, and of C to D, and of E to F, let N, O, M, P be (the least such numbers). And since as A is to B, so N (is) to O, and A and B are the least (numbers which have the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures O. So, for the same (reasons), C also measures O. Thus, B and C (both) measure O. Thus, the least number measured by (both) B and C will also measure O [Prop. 7.35]. And G (is) the least number measured by (both) B and C. Thus, G measures O, the greater (measuring) the lesser. The very thing is impossible. Thus, there cannot be any numbers less than H, G, K, L (which are) continuously (proportional) in the ratio of A to B, and of C to D, and, further, of E to F.



So let E not measure K. And let the least number, M, measured by (both) E and K have been taken [Prop. 7.34]. And as many times as K measures M, so many times let H, G also measure N, O, respectively. And as many times as E measures M, so many times let F also measure P. Since H measures N the same number of times as G (measures) O, thus as H is to G, so N (is) to O [Def. 7.20, Prop. 7.13]. And as H (is) to G, so A (is) to B. And thus as A (is) to B, so N (is) to O. And so, for the same (reasons), as C (is) to D, so O (is) to M. Again, since E measures M the same number of times as F (measures) P, thus as E is to F, so M (is) to P [Def. 7.20, Prop. 7.13]. Thus, N, O, M, P are continuously proportional in the ratios of A to B, and of C to D, and, further, of E to F. So I say that (they are) also the least (numbers) in the ratios of A B, C D, E F. For if not, then there will be some numbers less than N, O, M, P (which are) continuously proportional in the ratios of A B, C D, E F. Let them be Q, R, S, T. And since as Q is to R, so A (is) to B, and A and B (are) the least (numbers having the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures R. So, for the same (reasons), C also measures R. Thus, B and C (both) measure R. Thus, the least (number) measured by (both) B and C will also measure R [Prop. 7.35]. And G is the least number measured by (both) B and C. Thus, G measures R. And as G is to R, so K (is) to S. Thus, K also measures S [Def. 7.20]. And E also measures S [Prop. 7.20]. Thus, E and K (both) measure S. Thus, the least (number) measured by (both) E and K will also measure S [Prop. 7.35]. And M is the least (number) measured by (both) E and K. Thus, M measures S, the greater (measuring) the lesser. The very thing is impossible. Thus there cannot be any numbers less than N, O, M, P (which are) continuously proportional in the ratios of A to B, and of C to D, and, further, of E to F. Thus, N, O, M, P are the least (numbers) continuously proportional in the ratios of A B, C D, E F. (Which is) the very thing it was required to show.

Proposition 5

Plane numbers have to one another the ratio compounded† out of (the ratios of) their sides.



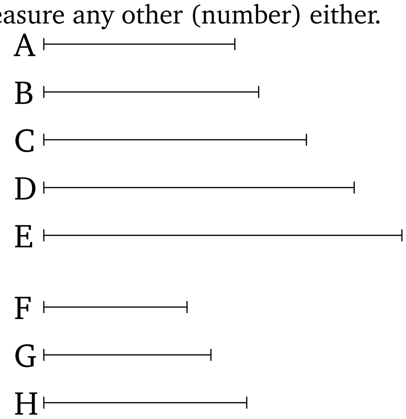
Let A and B be plane numbers, and let the numbers C, D be the sides of A, and (the numbers) E, F (the sides) of B. I say that A has to B the ratio compounded out of (the ratios of) their sides.

For given the ratios which C has to E, and D (has) to F, let the least numbers, G, H, K, continuously proportional in the ratios C E, D F have been taken [Prop. 8.4], so that as C is to E, so G (is) to H, and as D (is) to F, so H (is) to K. And let D make L (by) multiplying E.

And since D has made A (by) multiplying C, and has made L (by) multiplying E, thus as C is to E, so A (is) to L [Prop. 7.17]. And as C (is) to E, so G (is) to H. And thus as G (is) to H, so A (is) to L. Again, since E has made L (by) multiplying D [Prop. 7.16], but, in fact, has also made B (by) multiplying F, thus as D is to F, so L (is) to B [Prop. 7.17]. But, as D (is) to F, so H (is) to K. And thus as H (is) to K, so L (is) to B. And it was also shown that as G (is) to H, so A (is) to L. Thus, via equality, as G is to K, [so] A (is) to B [Prop. 7.14]. And G has to K the ratio compounded out of (the ratios of) the sides (of A and B). Thus, A also has to B the ratio compounded out of (the ratios of) the sides (of A and B). (Which is) the very thing it was required to show. † *i.e.*, multiplied.

Proposition 6

If there are any multitude whatsoever of continuously proportional numbers, and the first does not measure the second, then no other (number) will measure any other (number) either.

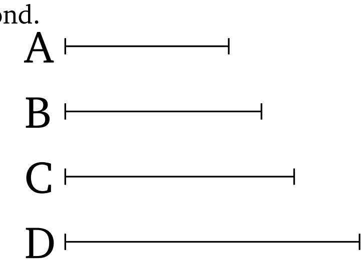


Let A, B, C, D, E be any multitude whatsoever of continuously proportional numbers, and let A not measure B. I say that no other (number) will measure any other (number) either.

Now, (it is) clear that A, B, C, D, E do not successively measure one another. For A does not even measure B. So I say that no other (number) will measure any other (number) either. For, if possible, let A measure C. And as many (numbers) as are A, B, C, let so many of the least numbers, F, G, H, have been taken of those (numbers) having the same ratio as A, B, C [Prop. 7.33]. And since F, G, H are in the same ratio as A, B, C, and the multitude of A, B, C is equal to the multitude of F, G, H, thus, via equality, as A is to C, so F (is) to H [Prop. 7.14]. And since as A is to B, so F (is) to G, and A does not measure B, F does not measure G either [Def. 7.20]. Thus, F is not a unit. For a unit measures all numbers. And F and H are prime to one another [Prop. 8.3] [and thus F does not measure H]. And as F is to H, so A (is) to C. And thus A does not measure C either [Def. 7.20]. So, similarly, we can show that no other (number) can measure any other (number) either. (Which is) the very thing it was required to show.

Proposition 7

If there are any multitude whatsoever of [continuously] proportional numbers, and the first measures the last, then (the first) will also measure the second.

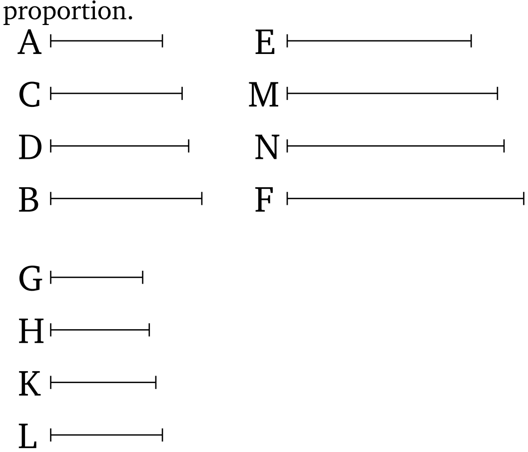


Let A, B, C, D be any number whatsoever of continuously proportional numbers. And let A measure D. I say that A also measures B.

For if A does not measure B then no other (number) will measure any other (number) either [Prop. 8.6]. But A measures D. Thus, A also measures B. (Which is) the very thing it was required to show.

Proposition 8

If between two numbers there fall (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall in between (any two numbers) having the same ratio [as them] in continued proportion.

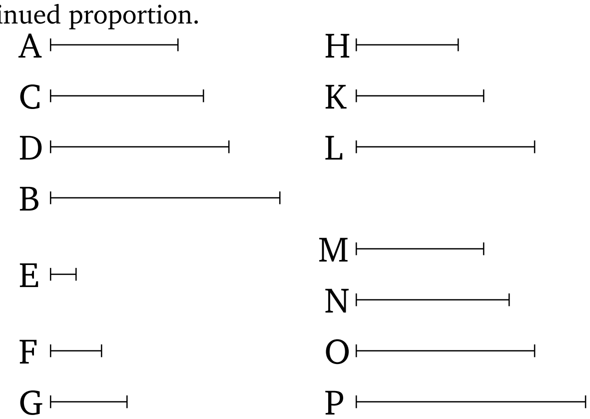


For let the numbers, C and D, fall between two numbers, A and B, in continued proportion, and let it have been contrived (that) as A (is) to B, so E (is) to F. I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall in between E and F in continued proportion.

For as many as A, B, C, D are in multitude, let so many of the least numbers, G, H, K, L, having the same ratio as A, B, C, D, have been taken [Prop. 7.33]. Thus, the outermost of them, G and L, are prime to one another [Prop. 8.3]. And since A, B, C, D are in the same ratio as G, H, K, L, and the multitude of A, B, C, D is equal to the multitude of G, H, K, L, thus, via equality, as A is to B, so G (is) to L [Prop. 7.14]. And as A (is) to B, so E (is) to F. And thus as G (is) to L, so E (is) to F. And G and L (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, G measures E the same number of times as L (measures) F. So as many times as G measures E, so many times let H, K also measure M, N, respectively. Thus, G, H, K, L measure E, M, N, F (respectively) an equal number of times. Thus, G, H, K, L are in the same ratio as E, M, N, F [Def. 7.20]. But, G, H, K, L are in the same ratio as A, C, D, B. Thus, A, C, D, B are also in the same ratio as E, M, N, F. And A, C, D, B are continuously proportional. Thus, E, M, N, F are also continuously proportional. Thus, as many numbers as have fallen in between A and B in continued proportion, so many numbers have also fallen in between E and F in continued proportion. (Which is) the very thing it was required to show.

Proposition 9

If two numbers are prime to one another and there fall in between them (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall between each of them and a unit in continued proportion.



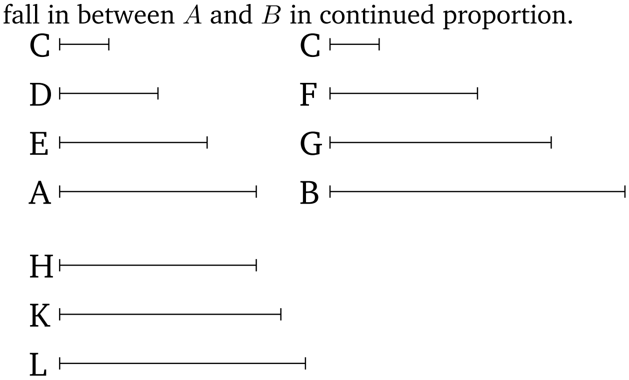
Let A and B be two numbers (which are) prime to one another, and let the (numbers) C and D fall in between them in continued proportion. And let the unit E be set out. I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall between each of A and B and the unit in continued proportion.

For let the least two numbers, F and G, which are in the ratio of A, C, D, B, have been taken [Prop. 7.33]. And the (least) three (numbers), H, K, L. And so on, successively increasing by one, until the multitude of the (least numbers taken) is made equal to the multitude of A, C, D, B [Prop. 8.2]. Let them have been taken, and let them be M, N, O, P. So (it is) clear that F has made H (by) multiplying itself, and has made M (by) multiplying H. And G has made L (by) multiplying itself, and has made P (by) multiplying L [Prop. 8.2 corr.]. And since M, N, O, P are the least of those (numbers) having the same ratio as F, G, and A, C, D, B are also the least of those (numbers) having the same ratio as F, G [Prop. 8.2], and the multitude of M, N, O, P is equal to the multitude of A, C, D, B, thus M, N, O, P are equal to A, C, D, B, respectively. Thus, M is equal to A, and P to B. And since F has made H (by) multiplying itself, F thus measures H according to the units in F [Def. 7.15]. And the unit E also measures F according to the units in it. Thus, the unit E measures the number F as many times as F (measures) H. Thus, as the unit E is to the number F, so F (is) to H [Def. 7.20]. Again, since F has made M (by) multiplying H, H thus measures M according to the units in F [Def. 7.15]. And the unit E also measures the number F according to the units in it. Thus, the unit E measures the number F as many times as H (measures) M. Thus, as the unit E is to the number F, so H (is) to M [Prop. 7.20]. And it was shown that as the unit E (is) to the number F, so F (is) to H. And thus as the unit E (is) to the number F, so F (is) to H, and H (is) to M. And M (is) equal to A. Thus, as the unit E is to the number F, so F (is) to H, and H to A. And so, for the same (reasons), as the unit E (is) to the number G, so G (is) to L, and L to B. Thus, as many (numbers) as have fallen in between A and B in continued proportion, so many numbers have also fallen between each of A and B and the unit E in continued proportion. (Which is) the very thing it was required to show.

Proposition 10

If (some) numbers fall between each of two numbers and a unit in continued proportion then, as many (numbers) as fall between each of the (two numbers) and the unit in continued proportion, so many (numbers) will also fall in between the (two numbers) themselves in continued proportion.

For let the numbers D, E and F, G fall between the numbers A and B (respectively) and the unit C in continued proportion. I say that, as many numbers as have fallen between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion.

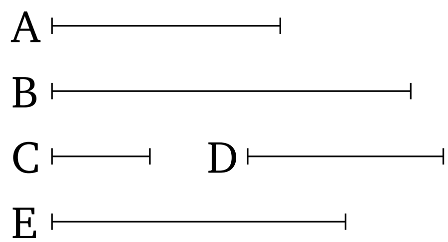


For let D make H (by) multiplying F. And let D, F make K, L, respectively, by multiplying H.

As since as the unit C is to the number D, so D (is) to E, the unit C thus measures the number D as many times as D (measures) E [Def. 7.20]. And the unit C measures the number D according to the units in D. Thus, the number D also measures E according to the units in D. Thus, D has made E (by) multiplying itself. Again, since as the [unit] C is to the number D, so E (is) to A, the unit C thus measures the number D as many times as E (measures) A [Def. 7.20]. And the unit C measures the number D according to the units in D. Thus, E also measures A according to the units in D. Thus, D has made A (by) multiplying E. And so, for the same (reasons), F has made G (by) multiplying itself, and has made B (by) multiplying G. And since D has made E (by) multiplying itself, and has made H (by) multiplying F, thus as D is to F, so E (is) to H [Prop 7.17]. And so, for the same reasons, as D (is) to F, so H (is) to G [Prop. 7.18]. And thus as E (is) to H, so H (is) to G. Again, since D has made A, K (by) multiplying E, H, respectively, thus as E is to H, so A (is) to K [Prop 7.17]. But, as E (is) to H, so D (is) to F. And thus as D (is) to F, so A (is) to K. Again, since D, F have made K, L, respectively, (by) multiplying H, thus as D is to F, so K (is) to L [Prop. 7.18]. But, as D (is) to F, so A (is) to K. And thus as A (is) to K, so K (is) to L. Further, since F has made L, B (by) multiplying H, G, respectively, thus as H is to G, so L (is) to B [Prop 7.17]. And as H (is) to G, so D (is) to F. And thus as D (is) to F, so L (is) to B. And it was also shown that as D (is) to F, so A (is) to K, and K to L. And thus as A (is) to K, so K (is) to L, and L to B. Thus, A, K, L, B are successively in continued proportion. Thus, as many numbers as fall between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion. (Which is) the very thing it was required to show.

Proposition 11

There exists one number in mean proportion to two (given) square numbers.† And (one) square (number) has to the (other) square (number) a squared‡ ratio with respect to (that) the side (of the former has) to the side (of the latter).



Let A and B be square numbers, and let C be the side of A, and D (the side) of B. I say that there exists one number in mean proportion to A and B, and that A has to B a squared ratio with respect to (that) C (has) to D.

For let C make E (by) multiplying D. And since A is square, and C is its side, C has thus made A (by) multiplying itself. And so, for the same (reasons), D has made B (by) multiplying itself. Therefore, since C has made A, E (by) multiplying C, D, respectively, thus as C is to D, so A (is) to E [Prop. 7.17]. And so, for the same (reasons), as C (is) to D, so E (is) to B [Prop. 7.18]. And thus as A (is) to E, so E (is) to B. Thus, one number (namely, E) is in mean proportion to A and B.

So I say that A also has to B a squared ratio with respect to (that) C (has) to D. For since A, E, B are three (continuously) proportional numbers, A thus has to B a squared ratio with respect to (that) A (has) to E [Def. 5.9]. And as A (is) to E, so C (is) to D. Thus, A has to B a squared ratio with respect to (that) side C (has) to (side) D. (Which is) the very thing it was required to show.

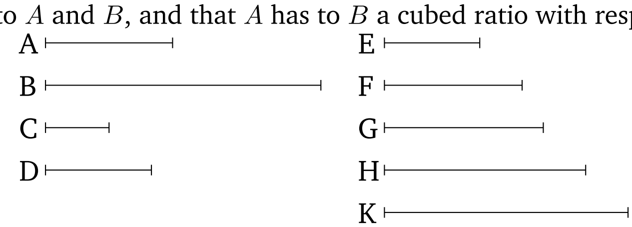
† In other words, between two given square numbers there exists a number in continued proportion.

‡ Literally, "double".

Proposition 12

There exist two numbers in mean proportion to two (given) cube numbers.† And (one) cube (number) has to the (other) cube (number) a cubed‡ ratio with respect to (that) the side (of the former has) to the side (of the latter).

Let A and B be cube numbers, and let C be the side of A, and D (the side) of B. I say that there exist two numbers in mean proportion to A and B, and that A has to B a cubed ratio with respect to (that) C (has) to D.



For let C make E (by) multiplying itself, and let it make F (by) multiplying D. And let D make G (by) multiplying itself, and let C, D make H, K, respectively, (by) multiplying F.

And since A is cube, and C (is) its side, and C has made E (by) multiplying itself, C has thus made E (by) multiplying itself, and has made A (by) multiplying E. And so, for the same (reasons), D has made G (by) multiplying itself, and has made B (by) multiplying G. And since C has made E, F (by) multiplying C, D, respectively, thus as C is to D, so E (is) to F [Prop. 7.17]. And so, for the same (reasons), as C (is) to D, so F (is) to G [Prop. 7.18].

Again, since C has made A, H (by) multiplying E, F, respectively, thus as E is to F, so A (is) to H [Prop. 7.17]. And as E (is) to F, so C (is) to D. And thus as C (is) to D, so A (is) to H. Again, since C, D have made H, K, respectively, (by) multiplying F, thus as C is to D, so H (is) to K [Prop. 7.18]. Again, since D has made K, B (by) multiplying F, G, respectively, thus as F is to G, so K (is) to B [Prop. 7.17]. And as F (is) to G, so C (is) to D. And thus as C (is) to D, so A (is) to H, and H to K, and K to B. Thus, H and K are two (numbers) in mean proportion to A and B.

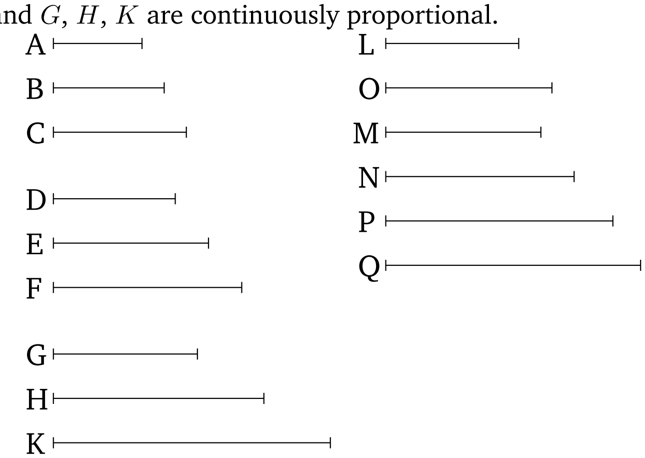
So I say that A also has to B a cubed ratio with respect to (that) C (has) to D. For since A, H, K, B are four (continuously) proportional numbers, A thus has to B a cubed ratio with respect to (that) A (has) to H [Def. 5.10]. And as A (is) to H, so C (is) to D. And [thus] A has to B a cubed ratio with respect to (that) C (has) to D. (Which is) the very thing it was required to show. † In other words, between two given cube numbers there exist two numbers in continued proportion.

‡ Literally, "triple".

Proposition 13

If there are any multitude whatsoever of continuously proportional numbers, and each makes some (number by) multiplying itself, then the (numbers) created from them will (also) be (continuously) proportional. And if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be (continuously) proportional [and this always happens with the extremes].

Let A, B, C be any multitude whatsoever of continuously proportional numbers, (such that) as A (is) to B, so B (is) to C. And let A, B, C make D, E, F (by) multiplying themselves, and let them make G, H, K (by) multiplying D, E, F. I say that D, E, F and G, H, K are continuously proportional.



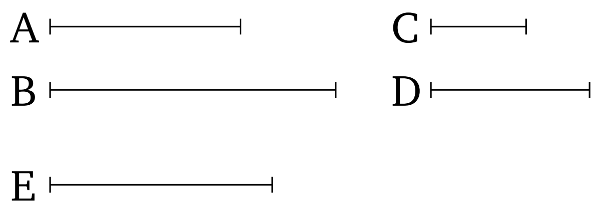
For let A make L (by) multiplying B. And let A, B make M, N, respectively, (by) multiplying L. And, again, let B make O (by) multiplying C. And let B, C make P, Q, respectively, (by) multplying O.

So, similarly to the above, we can show that D, L, E and G, M, N, H are continuously proportional in the ratio of A to B, and, further, (that) E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C. And as A is to B, so B (is) to C. And thus D, L, E are in the same ratio as E, O, F, and, further, G, M, N, H (are in the same ratio) as H, P, Q, K. And the multitude of D, L, E is equal to the multitude of E, O, F, and that of G, M, N, H to that of H, P, Q, K. Thus, via equality, as D is to E, so E (is) to F, and as G (is) to H, so H (is) to K [Prop. 7.14]. (Which is) the very thing it was required to show.

Proposition 14

If a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number).

Let A and B be square numbers, and let C and D be their sides (respectively). And let A measure B. I say that C also measures D.



For let C make E (by) multiplying D. Thus, A, E, B are continuously proportional in the ratio of C to D [Prop. 8.11]. And since A, E, B are continuously proportional, and A measures B, A thus also measures E [Prop. 8.7]. And as A is to E, so C (is) to D. Thus, C also measures D [Def. 7.20].

So, again, let C measure D. I say that A also measures B.

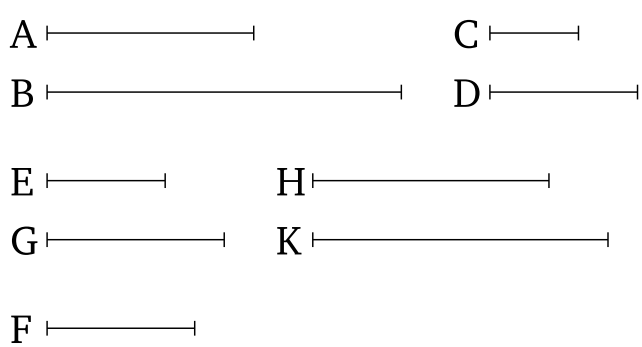
For similarly, with the same construction, we can show that A, E, B are continuously proportional in the ratio of C to D. And since as C is to D, so A (is) to E, and C measures D, A thus also measures E [Def. 7.20]. And A, E, B are continuously proportional. Thus, A also measures B.

Thus, if a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number). (Which is) the very thing it was required to show.

Proposition 15

If a cube number measures a(nother) cube number then the side (of the former) will also measure the side (of the latter). And if the side (of a cube number) measures the side (of another cube number) then the (former) cube (number) will also measure the (latter) cube (number).

For let the cube number A measure the cube (number) B, and let C be the side of A, and D (the side) of B. I say that C measures D.



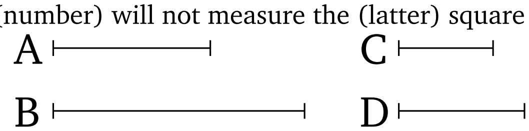
For let C make E (by) multiplying itself. And let D make G (by) multiplying itself. And, further, [let] C [make] F (by) multiplying D, and let C, D make H, K, respectively, (by) multiplying F. So it is clear that E, F, G and A, H, K, B are continuously proportional in the ratio of C to D [Prop. 8.12]. And since A, H, K, B are continuously proportional, and A measures B, (A) thus also measures H [Prop. 8.7]. And as A is to H, so C (is) to D. Thus, C also measures D [Def. 7.20].

And so let C measure D. I say that A will also measure B.

For similarly, with the same construction, we can show that A, H, K, B are continuously proportional in the ratio of C to D. And since C measures D, and as C is to D, so A (is) to H, A thus also measures H [Def. 7.20]. Hence, A also measures B. (Which is) the very thing it was required to show.

Proposition 16

If a square number does not measure a(nother) square number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a square number) does not measure the side (of another square number) then the (former) square (number) will not measure the (latter) square (number) either.



Let A and B be square numbers, and let C and D be their sides (respectively). And let A not measure B. I say that C does not measure D either.

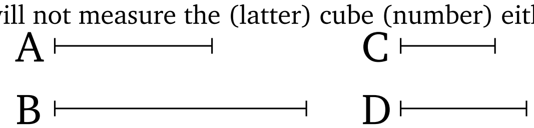
For if C measures D then A will also measure B [Prop. 8.14]. And A does not measure B. Thus, C will not measure D either.

[So], again, let C not measure D. I say that A will not measure B either.

For if A measures B then C will also measure D [Prop. 8.14]. And C does not measure D. Thus, A will not measure B either. (Which is) the very thing it was required to show.

Proposition 17

If a cube number does not measure a(nother) cube number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a cube number) does not measure the side (of another cube number) then the (former) cube (number) will not measure the (latter) cube (number) either.



For let the cube number A not measure the cube number B. And let C be the side of A, and D (the side) of B. I say that C will not measure D.

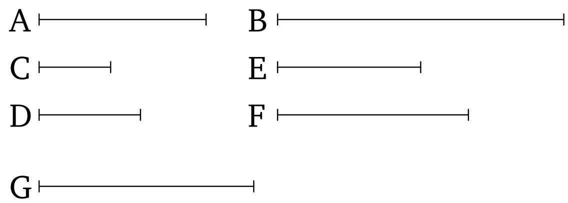
For if C measures D then A will also measure B [Prop. 8.15]. And A does not measure B. Thus, C does not measure D either.

And so let C not measure D. I say that A will not measure B either.

For if A measures B then C will also measure D [Prop. 8.15]. And C does not measure D. Thus, A will not measure B either. (Which is) the very thing it was required to show.

Proposition 18

There exists one number in mean proportion to two similar plane numbers. And (one) plane (number) has to the (other) plane (number) a squared[†] ratio with respect to (that) a corresponding side (of the former has) to a corresponding side (of the latter).



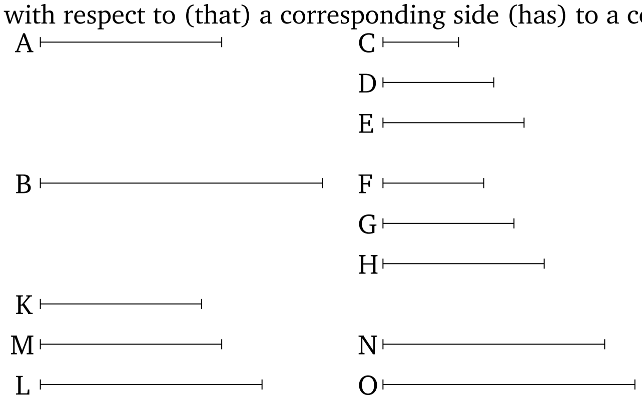
Let A and B be two similar plane numbers. And let the numbers C, D be the sides of A, and E, F (the sides) of B. And since similar numbers are those having proportional sides [Def. 7.21], thus as C is to D, so E (is) to F. Therefore, I say that there exists one number in mean proportion to A and B, and that A has to B a squared ratio with respect to that C (has) to E, or D to F—that is to say, with respect to (that) a corresponding side (has) to a corresponding [side].

For since as C is to D, so E (is) to F, thus, alternately, as C is to E, so D (is) to F [Prop. 7.13]. And since A is plane, and C, D its sides, D has thus made A (by) multiplying C. And so, for the same (reasons), E has made B (by) multiplying F. So let D make G (by) multiplying E. And since D has made A (by) multiplying C, and has made G (by) multiplying E, thus as C is to E, so A (is) to G [Prop. 7.17]. But as C (is) to E, [so] D (is) to F. And thus as D (is) to F, so A (is) to G. Again, since E has made G (by) multiplying D, and has made B (by) multiplying F, thus as D is to F, so G (is) to B [Prop. 7.17]. And it was also shown that as D (is) to F, so A (is) to G. And thus as A (is) to G, so G (is) to B. Thus, A, G, B are continously proportional. Thus, there exists one number (namely, G) in mean proportion to A and B.

So I say that A also has to B a squared ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) C (has) to E, or D to F. For since A, G, B are continuously proportional, A has to B a squared ratio with respect to (that A has) to G [Prop. 5.9]. And as A is to G, so C (is) to E, and D to F. And thus A has to B a squared ratio with respect to (that) C (has) to E, or D to F. (Which is) the very thing it was required to show. † Literally, "double".

Proposition 19

Two numbers fall (between) two similar solid numbers in mean proportion. And a solid (number) has to a similar solid (number) a cubed[†] ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let A and B be two similar solid numbers, and let C, D, E be the sides of A, and F, G, H (the sides) of B. And since similar solid (numbers) are those having proportional sides [Def. 7.21], thus as C is to D, so F (is) to G, and as D (is) to E, so G (is) to H. I say that two numbers fall (between) A and B in mean proportion, and (that) A has to B a cubed ratio with respect to (that) C (has) to F, and D to G, and, further, E to H.

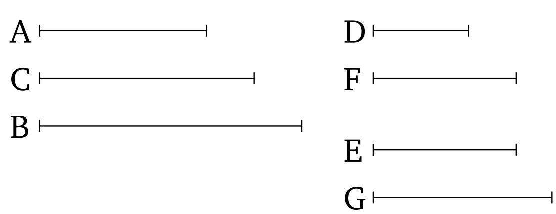
For let C make K (by) multiplying D, and let F make L (by) multiplying G. And since C, D are in the same ratio as F, G, and K is the (number created) from (multiplying) C, D, and L the (number created) from (multiplying) F, G, [thus] K and L are similar plane numbers [Def. 7.21]. Thus, there exits one number in mean proportion to K and L [Prop. 8.18]. Let it be M. Thus, M is the (number created) from (multiplying) D, F, as shown in the theorem before this (one). And since D has made K (by) multiplying C, and has made M (by) multiplying F, thus as C is to F, so K (is) to M [Prop. 7.17]. But, as K (is) to M, (so) M (is) to L. Thus, K, M, L are continuously proportional in the ratio of C to F. And since as C is to D, so F (is) to G, thus, alternately, as C is to F, so D (is) to G [Prop. 7.13]. And so, for the same (reasons), as D (is) to G, so E (is) to H. Thus, K, M, L are continuously proportional in the ratio of C to F, and of D to G, and, further, of E to H. So let E, H make N, O, respectively, (by) multiplying M. And since A is solid, and C, D, E are its sides, E has thus made A (by) multiplying the (number created) from (multiplying) C, D. And K is the (number created) from (multiplying) C, D. Thus, E has made A (by) multiplying K. And so, for the same (reasons), H has made B (by) multiplying L. And since E has made A (by) multiplying K, but has, in fact, also made N (by) multiplying M, thus as K is to M, so A (is) to N [Prop. 7.17]. And as K (is) to M, so C (is) to F, and D to G, and, further, E to H. And thus as C (is) to F, and D to G, and E to H, so A (is) to N. Again, since E, H have made N, O, respectively, (by) multiplying M, thus as E is to H, so N (is) to O [Prop. 7.18]. But, as E (is) to H, so C (is) to F, and D to G. And thus as C (is) to F, and D to G, and E to H, so (is) A to N, and N to O. Again, since H has made O (by) multiplying M, but has, in fact, also made B (by) multiplying L, thus as M (is) to L, so O (is) to B [Prop. 7.17]. But, as M (is) to L, so C (is) to F, and D to G, and E to H. And thus as C (is) to F, and D to G, and E to H, so not only (is) O to B, but also A to N, and N to O. Thus, A, N, O, B are continuously proportional in the aforementioned ratios of the sides.

So I say that A also has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C (has) to F, or D to G, and, further, E to H. For since A, N, O, B are four continuously proportional numbers, A thus has to B a cubed ratio with respect to (that) A (has) to N [Def. 5.10]. But, as A (is) to N, so it was shown (is) C to F, and D to G, and, further, E to H. And thus A has to B a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number C (has) to F, and D to G, and, further, E to H. (Which is) the very thing it was required to show. † Literally, "triple".

Proposition 20

If one number falls between two numbers in mean proportion then the numbers will be similar plane (numbers).

For let one number C fall between the two numbers A and B in mean proportion. I say that A and B are similar plane numbers.



[For] let the least numbers, D and E, having the same ratio as A and C have been taken [Prop. 7.33]. Thus, D measures A as many times as E (measures) C [Prop. 7.20]. So as many times as D measures A, so many units let there be in F. Thus, F has made A (by) multiplying D [Def. 7.15]. Hence, A is plane, and D, F (are) its sides. Again, since D and E are the least of those (numbers) having the same ratio as C and B, D thus measures C as many times as E (measures) B [Prop. 7.20]. So as many times as E measures B, so many units let there be in G.

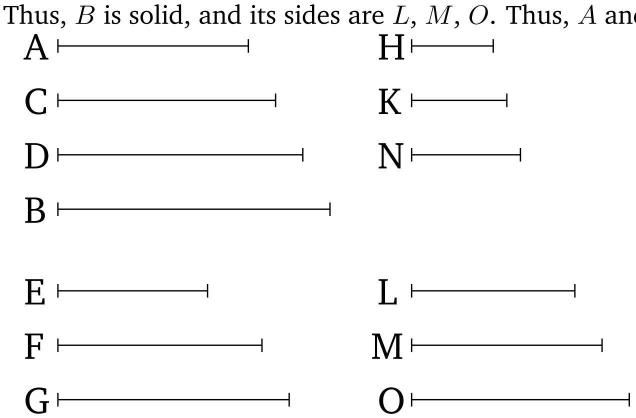
Thus, E measures B according to the units in G. Thus, G has made B (by) multiplying E [Def. 7.15]. Thus, B is plane, and E, G are its sides. Thus, A and B are (both) plane numbers. So I say that (they are) also similar. For since F has made A (by) multiplying D, and has made C (by) multiplying E, thus as D is to E, so A (is) to C—that is to say, C to B [Prop. 7.17].† Again, since E has made C, B (by) multiplying F, G, respectively, thus as F is to G, so C (is) to B [Prop. 7.17]. And as C (is) to B, so D (is) to E. And thus as D (is) to E, so F (is) to G. And, alternately, as D (is) to F, so E (is) to G [Prop. 7.13]. Thus, A and B are similar plane numbers. For their sides are proportional [Def. 7.21]. (Which is) the very thing it was required to show. † This part of the proof is defective, since it is not demonstrated that $F \times E = C$. Furthermore, it is not necessary to show that $D : E :: A : C$, because this is true by hypothesis.

Proposition 21

If two numbers fall between two numbers in mean proportion then the (latter) are similar solid (numbers).

For let the two numbers C and D fall between the two numbers A and B in mean proportion. I say that A and B are similar solid (numbers).

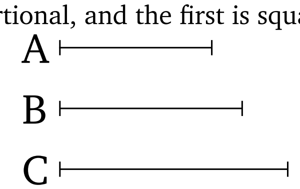
For let the three least numbers E, F, G having the same ratio as A, C, D have been taken [Prop. 8.2]. Thus, the outermost of them, E and G, are prime to one another [Prop. 8.3]. And since one number, F, has fallen (between) E and G in mean proportion, E and G are thus similar plane numbers [Prop. 8.20]. Therefore, let H, K be the sides of E, and L, M (the sides) of G. Thus, it is clear from the (proposition) before this (one) that E, F, G are continuously proportional in the ratio of H to L, and of K to M. And since E, F, G are the least (numbers) having the same ratio as A, C, D, and the multitude of E, F, G is equal to the multitude of A, C, D, thus, via equality, as E is to G, so A (is) to D [Prop. 7.14]. And E and G (are) prime (to one another), and prime (numbers) are also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures A the same number of times as G (measures) D. So as many times as E measures A, so many units let there be in N. Thus, N has made A (by) multiplying E [Def. 7.15]. And E is the (number created) from (multiplying) H and K. Thus, N has made A (by) multiplying the (number created) from (multiplying) H and K. Thus, A is solid, and its sides are H, K, N. Again, since E, F, G are the least (numbers) having the same ratio as C, D, B, thus E measures C the same number of times as G (measures) B [Prop. 7.20]. So as many times as E measures C, so many units let there be in O. Thus, G measures B according to the units in O. Thus, O has made B (by) multiplying G. And G is the (number created) from (multiplying) L and M. Thus, O has made B (by) multiplying the (number created) from (multiplying) L and M. Thus, B is solid, and its sides are L, M, O. Thus, A and B are (both) solid.



[So] I say that (they are) also similar. For since N, O have made A, C (by) multiplying E, thus as N is to O, so A (is) to C—that is to say, E to F [Prop. 7.18]. But, as E (is) to F, so H (is) to L, and K to M. And thus as H (is) to L, so K (is) to M, and N to O. And H, K, N are the sides of A, and L, M, O the sides of B. Thus, A and B are similar solid numbers [Def. 7.21]. (Which is) the very thing it was required to show. † The Greek text has "O, L, M", which is obviously a mistake.

Proposition 22

If three numbers are continuously proportional, and the first is square, then the third will also be square.

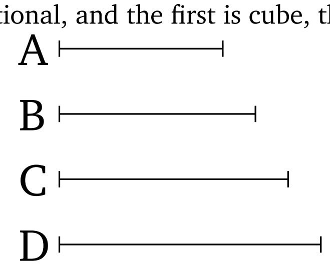


Let A, B, C be three continuously proportional numbers, and let the first A be square. I say that the third C is also square.

For since one number, B, is in mean proportion to A and C, A and C are thus similar plane (numbers) [Prop. 8.20]. And A is square. Thus, C is also square [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 23

If four numbers are continuously proportional, and the first is cube, then the fourth will also be cube.

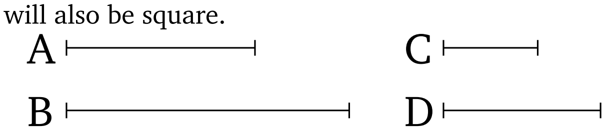


Let A, B, C, D be four continuously proportional numbers, and let A be cube. I say that D is also cube.

For since two numbers, B and C, are in mean proportion to A and D, A and D are thus similar solid numbers [Prop. 8.21]. And A (is) cube. Thus, D (is) also cube [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 24

If two numbers have to one another the ratio which a square number (has) to a(nother) square number, and the first is square, then the second will also be square.

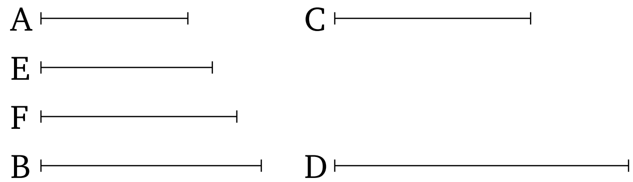


For let two numbers, A and B, have to one another the ratio which the square number C (has) to the square number D. And let A be square. I say that B is also square.

For since C and D are square, C and D are thus similar plane (numbers). Thus, one number falls (between) C and D in mean proportion [Prop. 8.18]. And as C is to D, (so) A (is) to B. Thus, one number also falls (between) A and B in mean proportion [Prop. 8.8]. And A is square. Thus, B is also square [Prop. 8.22]. (Which is) the very thing it was required to show.

Proposition 25

If two numbers have to one another the ratio which a cube number (has) to a(nother) cube number, and the first is cube, then the second will also be cube.

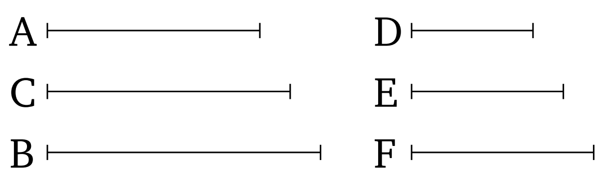


For let two numbers, A and B, have to one another the ratio which the cube number C (has) to the cube number D. And let A be cube. [So] I say that B is also cube.

For since C and D are cube (numbers), C and D are (thus) similar solid (numbers). Thus, two numbers fall (between) C and D in mean proportion [Prop. 8.19]. And as many (numbers) as fall in between C and D in continued proportion, so many also (fall) in (between) those (numbers) having the same ratio as them (in continued proportion) [Prop. 8.8]. And hence two numbers fall (between) A and B in mean proportion. Let E and F (so) fall. Therefore, since the four numbers A, E, F, B are continuously proportional, and A is cube, B (is) thus also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 26

Similar plane numbers have to one another the ratio which (some) square number (has) to a(nother) square number.

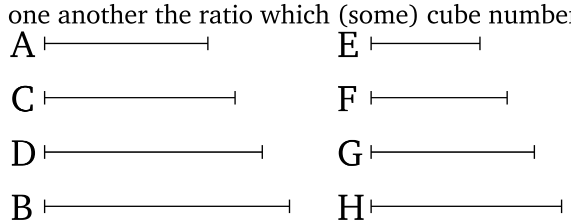


Let A and B be similar plane numbers. I say that A has to B the ratio which (some) square number (has) to a(nother) square number.

For since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion [Prop. 8.18]. Let it (so) fall, and let it be C. And let the least numbers, D, E, F, having the same ratio as A, C, B have been taken [Prop. 8.2]. The outermost of them, D and F, are thus square [Prop. 8.2 corr.]. And since as D is to F, so A (is) to B, and D and F are square, A thus has to B the ratio which (some) square number (has) to a(nother) square number. (Which is) the very thing it was required to show.

Proposition 27

Similar solid numbers have to one another the ratio which (some) cube number (has) to a(nother) cube number.



Let A and B be similar solid numbers. I say that A has to B the ratio which (some) cube number (has) to a(nother) cube number.

For since A and B are similar solid (numbers), two numbers thus fall (between) A and B in mean proportion [Prop. 8.19]. Let C and D have (so) fallen. And let the least numbers, E, F, G, H, having the same ratio as A, C, D, B, (and) equal in multitude to them, have been taken [Prop. 8.2]. Thus, the outermost of them, E and H, are cube [Prop. 8.2 corr.]. And as E is to H, so A (is) to B. And thus A has to B the ratio which (some) cube number (has) to a(nother) cube number. (Which is) the very thing it was required to show.

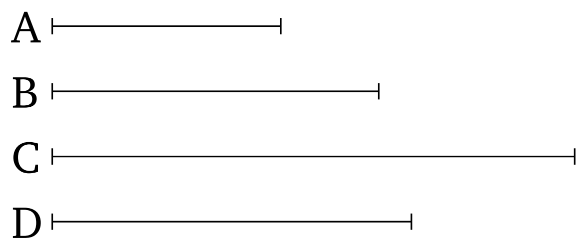
ELEMENTS BOOK 9

Applications of Number Theory[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

Proposition 1

If two similar plane numbers make some (number by) multiplying one another then the created (number) will be square.

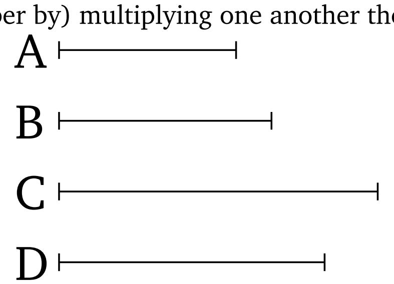


Let A and B be two similar plane numbers, and let A make C (by) multiplying B. I say that C is square.

For let A make D (by) multiplying itself. D is thus square. Therefore, since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D (is) to C [Prop. 7.17]. And since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion [Prop. 8.18]. And if (some) numbers fall between two numbers in continued proportion then, as many (numbers) as fall in (between) them (in continued proportion), so many also (fall) in (between numbers) having the same ratio (as them in continued proportion) [Prop. 8.8]. And hence one number falls (between) D and C in mean proportion. And D is square. Thus, C (is) also square [Prop. 8.22]. (Which is) the very thing it was required to show.

Proposition 2

If two numbers make a square (number by) multiplying one another then they are similar plane numbers.

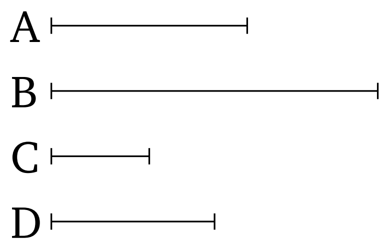


Let A and B be two numbers, and let A make the square (number) C (by) multiplying B. I say that A and B are similar plane numbers.

For let A make D (by) multiplying itself. Thus, D is square. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D (is) to C [Prop. 7.17]. And since D is square, and C (is) also, D and C are thus similar plane numbers. Thus, one (number) falls (between) D and C in mean proportion [Prop. 8.18]. And as D is to C, so A (is) to B. Thus, one (number) also falls (between) A and B in mean proportion [Prop. 8.8]. And if one (number) falls (between) two numbers in mean proportion then [the] numbers are similar plane (numbers) [Prop. 8.20]. Thus, A and B are similar plane (numbers). (Which is) the very thing it was required to show.

Proposition 3

If a cube number makes some (number by) multiplying itself then the created (number) will be cube.

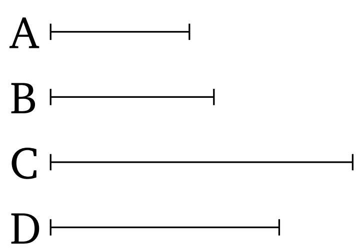


For let the cube number A make B (by) multiplying itself. I say that B is cube.

For let the side C of A have been taken. And let C make D by multiplying itself. So it is clear that C has made A (by) multiplying D. And since C has made D (by) multiplying itself, C thus measures D according to the units in it [Def. 7.15]. But, in fact, a unit also measures C according to the units in it [Def. 7.20]. Thus, as a unit is to C, so C (is) to D. Again, since C has made A (by) multiplying D, D thus measures A according to the units in C. And a unit also measures C according to the units in it. Thus, as a unit is to C, so D (is) to A. But, as a unit (is) to C, so C (is) to D. And thus as a unit (is) to C, so C (is) to D, and D to A. Thus, two numbers, C and D, have fallen (between) a unit and the number A in continued mean proportion. Again, since A has made B (by) multiplying itself, A thus measures B according to the units in it. And a unit also measures A according to the units in it. Thus, as a unit is to A, so A (is) to B. And two numbers have fallen (between) a unit and A in mean proportion. Thus two numbers will also fall (between) A and B in mean proportion [Prop. 8.8]. And if two (numbers) fall (between) two numbers in mean proportion, and the first (number) is cube, then the second will also be cube [Prop. 8.23]. And A is cube. Thus, B is also cube. (Which is) the very thing it was required to show.

Proposition 4

If a cube number makes some (number by) multiplying a(nother) cube number then the created (number) will be cube.

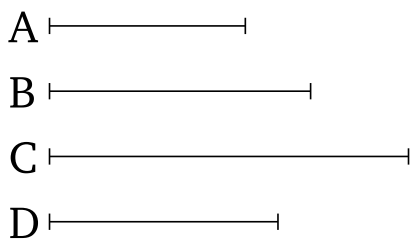


For let the cube number A make C (by) multiplying the cube number B. I say that C is cube.

For let A make D (by) multiplying itself. Thus, D is cube [Prop. 9.3]. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D (is) to C [Prop. 7.17]. And since A and B are cube, A and B are similar solid (numbers). Thus, two numbers fall (between) A and B in mean proportion [Prop. 8.19]. Hence, two numbers will also fall (between) D and C in mean proportion [Prop. 8.8]. And D is cube. Thus, C (is) also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 5

If a cube number makes a(nother) cube number (by) multiplying some (number) then the (number) multiplied will also be cube.

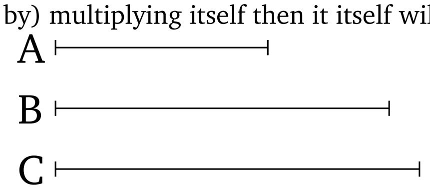


For let the cube number A make the cube (number) C (by) multiplying some number B. I say that B is cube.

For let A make D (by) multiplying itself. D is thus cube [Prop. 9.3]. And since A has made D (by) multiplying itself, and has made C (by) multiplying B, thus as A is to B, so D (is) to C [Prop. 7.17]. And since D and C are (both) cube, they are similar solid (numbers). Thus, two numbers fall (between) D and C in mean proportion [Prop. 8.19]. And as D is to C, so A (is) to B. Thus, two numbers also fall (between) A and B in mean proportion [Prop. 8.8]. And A is cube. Thus, B is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 6

If a number makes a cube (number by) multiplying itself then it itself will also be cube.

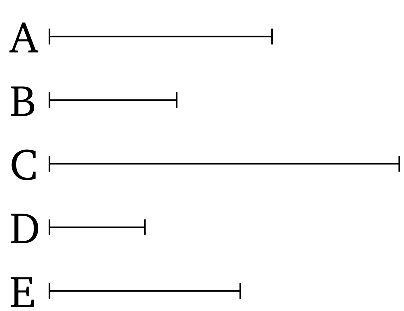


For let the number A make the cube (number) B (by) multiplying itself. I say that A is also cube.

For let A make C (by) multiplying B. Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B, C is thus cube. And since A has made B (by) multiplying itself, A thus measures B according to the units in (A). And a unit also measures A according to the units in it. Thus, as a unit is to A, so A (is) to B. And since A has made C (by) multiplying B, B thus measures C according to the units in A. And a unit also measures A according to the units in it. Thus, as a unit is to A, so B (is) to C. But, as a unit (is) to A, so A (is) to B. And thus as A (is) to B, (so) B (is) to C. And since B and C are cube, they are similar solid (numbers). Thus, there exist two numbers in mean proportion (between) B and C [Prop. 8.19]. And as B is to C, (so) A (is) to B. Thus, there also exist two numbers in mean proportion (between) A and B [Prop. 8.8]. And B is cube. Thus, A is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 7

If a composite number makes some (number by) multiplying some (other) number then the created (number) will be solid.

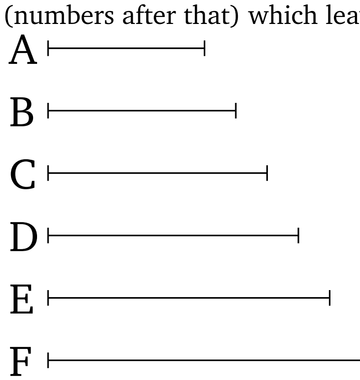


For let the composite number A make C (by) multiplying some number B. I say that C is solid.

For since A is a composite (number), it will be measured by some number. Let it be measured by D. And, as many times as D measures A, so many units let there be in E. Therefore, since D measures A according to the units in E, E has thus made A (by) multiplying D [Def. 7.15]. And since A has made C (by) multiplying B, and A is the (number created) from (multiplying) D, E, the (number created) from (multiplying) D, E has thus made C (by) multiplying B. Thus, C is solid, and its sides are D, E, B. (Which is) the very thing it was required to show.

Proposition 8

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then the third from the unit will be square, and (all) those (numbers after that) which leave an interval of one (number), and the fourth (will be) cube, and all those (numbers after that) which leave an interval of two (numbers), and the seventh (will be) both cube and square, and (all) those (numbers after that) which leave an interval of five (numbers).

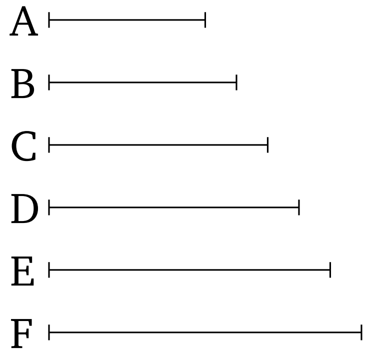


Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit. I say that the third from the unit, B, is square, and all those (numbers after that) which leave an interval of one (number). And the fourth (from the unit), C, (is) cube, and all those (numbers after that) which leave an interval of two (numbers). And the seventh (from the unit), F, (is) both cube and square, and all those (numbers after that) which leave an interval of five (numbers).

For since as the unit is to A, so A (is) to B, the unit thus measures the number A the same number of times as A (measures) B [Def. 7.20]. And the unit measures the number A according to the units in it. Thus, A also measures B according to the units in A. A has thus made B (by) multiplying itself [Def. 7.15]. Thus, B is square. And since B, C, D are continuously proportional, and B is square, D is thus also square [Prop. 8.22]. So, for the same (reasons), F is also square. So, similarly, we can also show that all those (numbers after that) which leave an interval of one (number) are square. So I also say that the fourth (number) from the unit, C, is cube, and all those (numbers after that) which leave an interval of two (numbers). For since as the unit is to A, so B (is) to C, the unit thus measures the number A the same number of times that B (measures) C. And the unit measures the number A according to the units in A. And thus B measures C according to the units in A. A has thus made C (by) multiplying B. Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B, C is thus cube. And since C, D, E, F are continuously proportional, and C is cube, F is thus also cube [Prop. 8.23]. And it was also shown (to be) square. Thus, the seventh (number) from the unit is (both) cube and square. So, similarly, we can show that all those (numbers after that) which leave an interval of five (numbers) are (both) cube and square. (Which is) the very thing it was required to show.

Proposition 9

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is square, then all the remaining (numbers) will also be square. And if the (number) after the unit is cube, then all the remaining (numbers) will also be cube.



Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit. And let the (number) after the unit, A, be square. I say that all the remaining (numbers) will also be square.

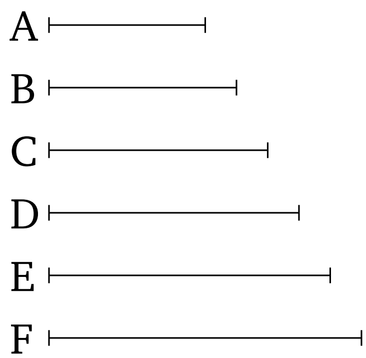
In fact, it has (already) been shown that the third (number) from the unit, B, is square, and all those (numbers after that) which leave an interval of one (number) [Prop. 9.8]. [So] I say that all the remaining (numbers) are also square. For since A, B, C are continuously proportional, and A (is) square, C is [thus] also square [Prop. 8.22]. Again, since B, C, D are [also] continuously proportional, and B is square, D is [thus] also square [Prop. 8.22]. So, similarly, we can show that all the remaining (numbers) are also square.

And so let A be cube. I say that all the remaining (numbers) are also cube.

In fact, it has (already) been shown that the fourth (number) from the unit, C, is cube, and all those (numbers after that) which leave an interval of two (numbers) [Prop. 9.8]. [So] I say that all the remaining (numbers) are also cube. For since as the unit is to A, so A (is) to B, the unit thus measures A the same number of times as A (measures) B. And the unit measures A according to the units in it. Thus, A also measures B according to the units in (A). A has thus made B (by) multiplying itself. And A is cube. And if a cube number makes some (number by) multiplying itself then the created (number) is cube [Prop. 9.3]. Thus, B is also cube. And since the four numbers A, B, C, D are continuously proportional, and A is cube, D is thus also cube [Prop. 8.23]. So, for the same (reasons), E is also cube, and, similarly, all the remaining (numbers) are cube. (Which is) the very thing it was required to show.

Proposition 10

If any multitude whatsoever of numbers is [continuously] proportional, (starting) from a unit, and the (number) after the unit is not square, then no other (number) will be square either, apart from the third from the unit, and all those (numbers after that) which leave an interval of one (number). And if the (number) after the unit is not cube, then no other (number) will be cube either, apart from the fourth from the unit, and all those (numbers after that) which leave an interval of two (numbers).



Let any multitude whatsoever of numbers, A, B, C, D, E, F, be continuously proportional, (starting) from a unit. And let the (number) after the unit, A, not be square. I say that no other (number) will be square either, apart from the third from the unit [and (all) those (numbers after that) which leave an interval of one (number)].

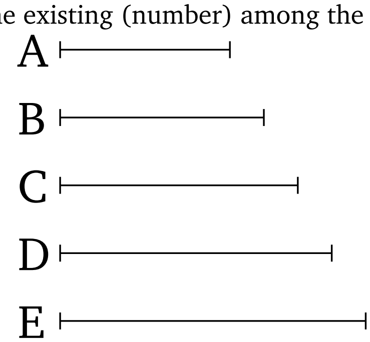
For, if possible, let C be square. And B is also square [Prop. 9.8]. Thus, B and C have to one another (the) ratio which (some) square number (has) to (some other) square number. And as B is to C, (so) A (is) to B. Thus, A and B have to one another (the) ratio which (some) square number has to (some other) square number. Hence, A and B are similar plane (numbers) [Prop. 8.26]. And B is square. Thus, A is also square. The very opposite thing was assumed. C is thus not square. So, similarly, we can show that no other (number is) square either, apart from the third from the unit, and (all) those (numbers after that) which leave an interval of one (number).

And so let A not be cube. I say that no other (number) will be cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers).

For, if possible, let D be cube. And C is also cube [Prop. 9.8]. For it is the fourth (number) from the unit. And as C is to D, (so) B (is) to C. And B thus has to C the ratio which (some) cube (number has) to (some other) cube (number). And C is cube. Thus, B is also cube [Props. 7.13, 8.25]. And since as the unit is to A, (so) A (is) to B, and the unit measures A according to the units in it, A thus also measures B according to the units in (A). Thus, A has made the cube (number) B (by) multiplying itself. And if a number makes a cube (number by) multiplying itself then it itself will be cube [Prop. 9.6]. Thus, A (is) also cube. The very opposite thing was assumed. Thus, D is not cube. So, similarly, we can show that no other (number) is cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers). (Which is) the very thing it was required to show.

Proposition 11

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then a lesser (number) measures a greater according to some existing (number) among the proportional numbers.



Let any multitude whatsoever of numbers, B, C, D, E, be continuously proportional, (starting) from the unit A. I say that, for B, C, D, E, the least (number), B, measures E according to some (one) of C, D.

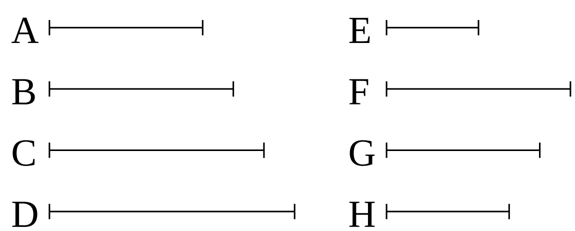
For since as the unit A is to B, so D (is) to E, the unit A thus measures the number B the same number of times as D (measures) E. Thus, alternately, the unit A measures D the same number of times as B (measures) E [Prop. 7.15]. And the unit A measures D according to the units in it. Thus, B also measures E according to the units in D. Hence, the lesser (number) B measures the greater E according to some existing number among the proportional numbers (namely, D).

Corollary

And (it is) clear that what(ever relative) place the measuring (number) has from the unit, the (number) according to which it measures has the same (relative) place from the measured (number), in (the direction of the number) before it. (Which is) the very thing it was required to show.

Proposition 12

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then however many prime numbers the last (number) is measured by, the (number) next to the unit will also be measured by the same (prime numbers).



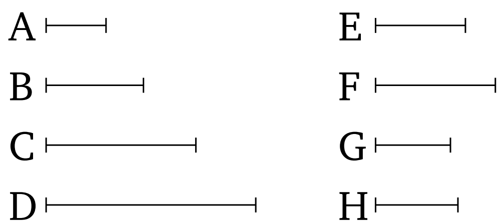
Let any multitude whatsoever of numbers, A, B, C, D, be (continuously) proportional, (starting) from a unit. I say that however many prime numbers D is measured by, A will also be measured by the same (prime numbers).

For let D be measured by some prime number E. I say that E measures A. For (suppose it does) not. E is prime, and every prime number is prime to every number which it does not measure [Prop. 7.29]. Thus, E and A are prime to one another. And since E measures D, let it measure it according to F. Thus, E has made D (by) multiplying F. Again, since A measures D according to the units in C [Prop. 9.11 corr.], A has thus made D (by) multiplying C. But, in fact, E has also made D (by) multiplying F. Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F. Thus, as A is to E, (so) F (is) to C [Prop. 7.19]. And A and E (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures C. Let it measure it according to G. Thus, E has made C (by) multiplying G. But, in fact, via the (proposition) before this, A has also made C (by) multiplying B [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A, B is equal to the (number created) from (multiplying) E, G. Thus, as A is to E, (so) G (is) to B [Prop. 7.19]. And A and E (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures B. Let it measure it according to H. Thus, E has made B (by) multiplying H. But, in fact, A has also made B (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) E, H is equal to the (square) on A. Thus, as E is to A, (so) A (is) to H [Prop. 7.19]. And A and E are prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures A, as the leading (measuring the) leading. But, in fact, (E) also does not measure (A). The very thing (is) impossible. Thus, E and A are not prime to one another. Thus, (they are) composite (to one another). And (numbers) composite (to one another) are (both) measured by some [prime] number [Def. 7.14]. And since E is assumed (to be) prime, and a prime (number) is not measured by another number (other) than itself [Def. 7.11], E thus measures (both) A and E. Hence, E measures A. And it also measures D. Thus, E measures (both) A and D. So, similarly, we can show that however many prime numbers D is measured by, A will also be measured by the same (prime numbers). (Which is) the very thing it was required to show.

Proposition 13

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is prime, then the greatest (number) will be measured by no [other] (numbers) except (numbers) existing among the proportional numbers.

Let any multitude whatsoever of numbers, A, B, C, D, be continuously proportional, (starting) from a unit. And let the (number) after the unit, A, be prime. I say that the greatest of them, D, will be measured by no other (numbers) except A, B, C.

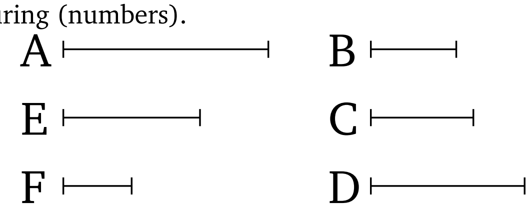


For, if possible, let it be measured by E, and let E not be the same as one of A, B, C. So it is clear that E is not prime. For if E is prime, and measures D, then it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, E is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, E is measured by some prime number. So I say that it will be measured by no other prime number than A. For if E is measured by another (prime number), and E measures D, then this (prime number) will thus also measure D. Hence, it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, A measures E. And since E measures D, let it measure it according to F. I say that F is not the same as one of A, B, C. For if F is the same as one of A, B, C, and measures D according to E, then one of A, B, C thus also measures D according to E. But one of A, B, C (only) measures D according to some (one) of A, B, C [Prop. 9.11]. And thus E is the same as one of A, B, C. The very opposite thing was assumed. Thus, F is not the same as one of A, B, C. Similarly, we can show that F is measured by A, (by) again showing that F is not prime. For if (F is prime), and measures D, then it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, F is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, F is measured by some prime number. So I say that it will be measured by no other prime number than A. For if some other prime (number) measures F, and F measures D, then this (prime number) will thus also measure D. Hence, it will also measure A, (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, A measures F. And since E measures D according to F, E has thus made D (by) multiplying F. But, in fact, A has also made D (by) multiplying C [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F. Thus, proportionally, as A is to E, so F (is) to C [Prop. 7.19]. And A measures E. Thus, F also measures C. Let it measure it according to G. So, similarly, we can show that G is not the same as one of A, B, and that it is measured by A. And since F measures C according to G, F has thus made C (by) multiplying G. But, in fact, A has also made C

(by) multiplying B [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A, B is equal to the (number created) from (multiplying) F, G. Thus, proportionally, as A (is) to F, so G (is) to B [Prop. 7.19]. And A measures F. Thus, G also measures B. Let it measure it according to H. So, similarly, we can show that H is not the same as A. And since G measures B according to H, G has thus made B (by) multiplying H. But, in fact, A has also made B (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) H, G is equal to the square on A. Thus, as H is to A, (so) A (is) to G [Prop. 7.19]. And A measures G. Thus, H also measures A, (despite A) being prime (and) not being the same as it. The very thing (is) absurd. Thus, the greatest (number) D cannot be measured by another (number) except (one of) A, B, C. (Which is) the very thing it was required to show.

Proposition 14

If a least number is measured by (some) prime numbers then it will not be measured by any other prime number except (one of) the original measuring (numbers).

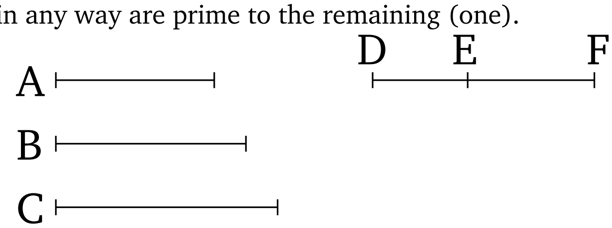


For let A be the least number measured by the prime numbers B, C, D. I say that A will not be measured by any other prime number except (one of) B, C, D.

For, if possible, let it be measured by the prime (number) E. And let E not be the same as one of B, C, D. And since E measures A, let it measure it according to F. Thus, E has made A (by) multiplying F. And A is measured by the prime numbers B, C, D. And if two numbers make some (number by) multiplying one another, and some prime number measures the number created from them, then (the prime number) will also measure one of the original (numbers) [Prop. 7.30]. Thus, B, C, D will measure one of E, F. In fact, they do not measure E. For E is prime, and not the same as one of B, C, D. Thus, they (all) measure F, which is less than A. The very thing (is) impossible. For A was assumed (to be) the least (number) measured by B, C, D. Thus, no prime number can measure A except (one of) B, C, D. (Which is) the very thing it was required to show.

Proposition 15

If three continuously proportional numbers are the least of those (numbers) having the same ratio as them then two (of them) added together in any way are prime to the remaining (one).

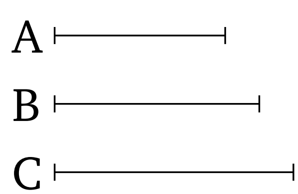


Let A, B, C be three continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that two of A, B, C added together in any way are prime to the remaining (one), (that is) A and B (prime) to C, B and C to A, and, further, A and C to B.

Let the two least numbers, DE and EF, having the same ratio as A, B, C, have been taken [Prop. 8.2]. So it is clear that DE has made A (by) multiplying itself, and has made B (by) multiplying EF, and, further, EF has made C (by) multiplying itself [Prop. 8.2]. And since DE, EF are the least (of those numbers having the same ratio as them), they are prime to one another [Prop. 7.22]. And if two numbers are prime to one another then the sum (of them) is also prime to each [Prop. 7.28]. Thus, DF is also prime to each of DE, EF. But, in fact, DE is also prime to EF. Thus, DF, DE are (both) prime to EF. And if two numbers are (both) prime to some number then the (number) created from (multiplying) them is also prime to the remaining (number) [Prop. 7.24]. Hence, the (number created) from (multiplying) FD, DE is prime to EF. Hence, the (number created) from (multiplying) FD, DE is also prime to the (square) on EF [Prop. 7.25]. [For if two numbers are prime to one another then the (number) created from (squaring) one of them is prime to the remaining (number).] But the (number created) from (multiplying) FD, DE is the (square) on DE plus the (number created) from (multiplying) DE, EF [Prop. 2.3]. Thus, the (square) on DE plus the (number created) from (multiplying) DE, EF is prime to the (square) on EF. And the (square) on DE is A, and the (number created) from (multiplying) DE, EF (is) B, and the (square) on EF (is) C. Thus, A, B summed is prime to C. So, similarly, we can show that B, C (summed) is also prime to A. So I say that A, C (summed) is also prime to B. For since DF is prime to each of DE, EF then the (square) on DF is also prime to the (number created) from (multiplying) DE, EF [Prop. 7.25]. But, the (sum of the squares) on DE, EF plus twice the (number created) from (multiplying) DE, EF is equal to the (square) on DF [Prop. 2.4]. And thus the (sum of the squares) on DE, EF plus twice the (rectangle contained) by DE, EF [is] prime to the (rectangle contained) by DE, EF. By separation, the (sum of the squares) on DE, EF plus once the (rectangle contained) by DE, EF is prime to the (rectangle contained) by DE, EF.† Again, by separation, the (sum of the squares) on DE, EF is prime to the (rectangle contained) by DE, EF. And the (square) on DE is A, and the (rectangle contained) by DE, EF (is) B, and the (square) on EF (is) C. Thus, A, C summed is prime to B. (Which is) the very thing it was required to show. † Since if $\alpha\,\beta$ measures $\alpha^2 + \beta^2 + 2\,\alpha\,\beta$ then it also measures $\alpha^2 + \beta^2 + \alpha\,\beta$, and *vice versa*.

Proposition 16

If two numbers are prime to one another then as the first is to the second, so the second (will) not (be) to some other (number).



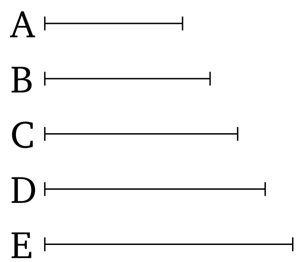
For let the two numbers A and B be prime to one another. I say that as A is to B, so B is not to some other (number).

For, if possible, let it be that as A (is) to B, (so) B (is) to C. And A and B (are) prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures B, as the leading (measuring) the leading. And (A) also measures itself. Thus, A measures A and B, which are prime to one another. The very thing (is) absurd. Thus, as A (is) to B, so B cannot be to C. (Which is) the very thing it was required to show.

Proposition 17

If any multitude whatsoever of numbers is continuously proportional, and the outermost of them are prime to one another, then as the first (is) to the second, so the last will not be to some other (number).

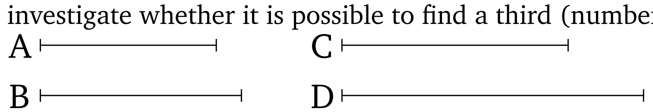
Let A, B, C, D be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, A and D, be prime to one another. I say that as A is to B, so D (is) not to some other (number).



For, if possible, let it be that as A (is) to B, so D (is) to E. Thus, alternately, as A is to D, (so) B (is) to E [Prop. 7.13]. And A and D are prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures B. And as A is to B, (so) B (is) to C. Thus, B also measures C. And hence A measures C [Def. 7.20]. And since as B is to C, (so) C (is) to D, and B measures C, C thus also measures D [Def. 7.20]. But, A was (found to be) measuring C. And hence A also measures D. And (A) also measures itself. Thus, A measures A and D, which are prime to one another. The very thing is impossible. Thus, as A (is) to B, so D cannot be to some other (number). (Which is) the very thing it was required to show.

Proposition 18

For two given numbers, to investigate whether it is possible to find a third (number) proportional to them.



Let A and B be the two given numbers. And let it be required to investigate whether it is possible to find a third (number) proportional to them.

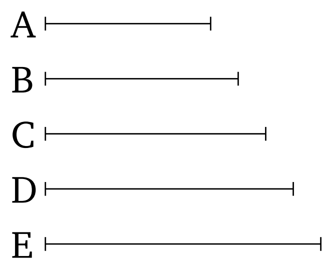
So A and B are either prime to one another, or not. And if they are prime to one another then it has (already) been show that it is impossible to find a third (number) proportional to them [Prop. 9.16].

And so let A and B not be prime to one another. And let B make C (by) multiplying itself. So A either measures, or does not measure, C. Let it first of all measure (C) according to D. Thus, A has made C (by) multiplying D. But, in fact, B has also made C (by) multiplying itself. Thus, the (number created) from (multiplying) A, D is equal to the (square) on B. Thus, as A is to B, (so) B (is) to D [Prop. 7.19]. Thus, a third number has been found proportional to A, B, (namely) D.

And so let A not measure C. I say that it is impossible to find a third number proportional to A, B. For, if possible, let it have been found, (and let it be) D. Thus, the (number created) from (multiplying) A, D is equal to the (square) on B [Prop. 7.19]. And the (square) on B is C. Thus, the (number created) from (multiplying) A, D is equal to C. Hence, A has made C (by) multiplying D. Thus, A measures C according to D. But (A) was, in fact, also assumed (to be) not measuring (C). The very thing (is) absurd. Thus, it is not possible to find a third number proportional to A, B when A does not measure C. (Which is) the very thing it was required to show.

Proposition 19†

For three given numbers, to investigate when it is possible to find a fourth (number) proportional to them.



Let A, B, C be the three given numbers. And let it be required to investigate when it is possible to find a fourth (number) proportional to them.

In fact, (A, B, C) are either not continuously proportional and the outermost of them are prime to one another, or are continuously proportional and the outermost of them are not prime to one another, or are neither continuously proportional nor are the outermost of them prime to one another, or are continuously proportional and the outermost of them are prime to one another.

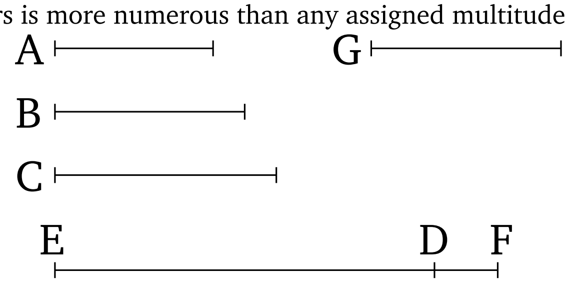
In fact, if A, B, C are continuously proportional, and the outermost of them, A and C, are prime to one another, (then) it has (already) been shown that it is impossible to find a fourth number proportional to them [Prop. 9.17]. So let A, B, C not be continuously proportional, (with) the outermost of them again being prime to one another. I say that, in this case, it is also impossible to find a fourth (number) proportional to them. For, if possible, let it have been found, (and let it be) D. Hence, it will be that as A (is) to B, (so) C (is) to D. And let it be contrived that as B (is) to C, (so) D (is) to E. And since as A is to B, (so) C (is) to D, and as B (is) to C, (so) D (is) to E, thus, via equality, as A (is) to C, (so) C (is) to E [Prop. 7.14]. And A and C (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least (numbers) measure those numbers having the same ratio as them (the same number of times), the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures C, (as) the leading (measuring) the leading. And it also measures itself. Thus, A measures A and C, which are prime to one another. The very thing is impossible. Thus, it is not possible to find a fourth (number) proportional to A, B, C.

And so let A, B, C again be continuously proportional, and let A and C not be prime to one another. I say that it is possible to find a fourth (number) proportional to them. For let B make D (by) multiplying C. Thus, A either measures or does not measure D. Let it, first of all, measure (D) according to E. Thus, A has made D (by) multiplying E. But, in fact, B has also made D (by) multiplying C. Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C. Thus, proportionally, as A [is] to B, (so) C (is) to E [Prop. 7.19]. Thus, a fourth (number) proportional to A, B, C has been found, (namely) E.

And so let A not measure D. I say that it is impossible to find a fourth number proportional to A, B, C. For, if possible, let it have been found, (and let it be) E. Thus, the (number created) from (multiplying) A, E is equal to the (number created) from (multiplying) B, C. But, the (number created) from (multiplying) B, C is D. And thus the (number created) from (multiplying) A, E is equal to D. Thus, A has made D (by) multiplying E. Thus, A measures D according to E. Hence, A measures D. But, it also does not measure (D). The very thing (is) absurd. Thus, it is not possible to find a fourth number proportional to A, B, C when A does not measure D. And so (let) A, B, C (be) neither continuously proportional, nor (let) the outermost of them (be) prime to one another. And let B make D (by) multiplying C. So, similarly, it can be show that if A measures D then it is possible to find a fourth (number) proportional to (A, B, C), and impossible if (A) does not measure (D). (Which is) the very thing it was required to show. † The proof of this proposition is incorrect. There are, in fact, only two cases. Either A, B, C are continuously proportional, with A and C prime to one another, or not. In the first case, it is impossible to find a fourth proportional number. In the second case, it is possible to find a fourth proportional number provided that A measures B times C. Of the four cases considered by Euclid, the proof given in the second case is incorrect, since it only demonstrates that if $A : B :: C : D$ then a number E cannot be found such that $B : C :: D : E$. The proofs given in the other three cases are correct.

Proposition 20

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.



Let A, B, C be the assigned prime numbers. I say that the (set of all) primes numbers is more numerous than A, B, C.

For let the least number measured by A, B, C have been taken, and let it be DE [Prop. 7.36]. And let the unit DF have been added to DE. So EF is either prime, or not. Let it, first of all, be prime. Thus, the (set of) prime numbers A, B, C, EF, (which is) more numerous than A, B, C, has been found.

And so let EF not be prime. Thus, it is measured by some prime number [Prop. 7.31]. Let it be measured by the prime (number) G. I say that G is not the same as any of A, B, C. For, if possible, let it be (the same). And A, B, C (all) measure DE. Thus, G will also measure DE. And it also measures EF. (So) G will also measure the remainder, unit DF, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C. And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found. (Which is) the very thing it was required to show.

Proposition 21

If any multitude whatsoever of even numbers is added together then the whole is even.

For let any multitude whatsoever of even numbers, AB, BC, CD, DE, lie together. I say that the whole, AE, is even.

For since everyone of AB, BC, CD, DE is even, it has a half part [Def. 7.6]. And hence the whole AE has a half part. And an even number is one (which can be) divided in half [Def. 7.6]. Thus, AE is even. (Which is) the very thing it was required to show.

Proposition 22

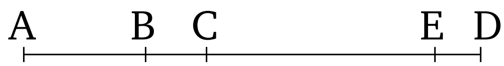
If any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.

For let any even multitude whatsoever of odd numbers, AB, BC, CD, DE, lie together. I say that the whole, AE, is even.

For since everyone of AB, BC, CD, DE is odd then, a unit being subtracted from each, everyone of the remainders will be (made) even [Def. 7.7]. And hence the sum of them will be even [Prop. 9.21]. And the multitude of the units is even. Thus, the whole AE is also even [Prop. 9.21]. (Which is) the very thing it was required to show.

Proposition 23

If any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.

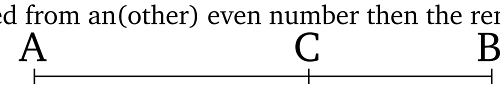


For let any multitude whatsoever of odd numbers, AB, BC, CD, lie together, and let the multitude of them be odd. I say that the whole, AD, is also odd.

For let the unit DE have been subtracted from CD. The remainder CE is thus even [Def. 7.7]. And CA is also even [Prop. 9.22]. Thus, the whole AE is also even [Prop. 9.21]. And DE is a unit. Thus, AD is odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 24

If an even (number) is subtracted from an(other) even number then the remainder will be even.

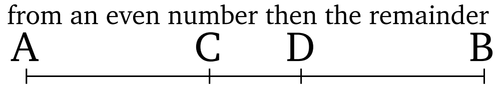


For let the even (number) BC have been subtracted from the even number AB. I say that the remainder CA is even.

For since AB is even, it has a half part [Def. 7.6]. So, for the same (reasons), BC also has a half part. And hence the remainder [CA has a half part]. [Thus,] AC is even. (Which is) the very thing it was required to show.

Proposition 25

If an odd (number) is subtracted from an even number then the remainder will be odd.

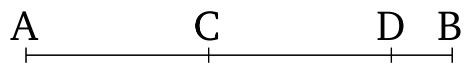


For let the odd (number) BC have been subtracted from the even number AB. I say that the remainder CA is odd.

For let the unit CD have been subtracted from BC. DB is thus even [Def. 7.7]. And AB is also even. And thus the remainder AD is even [Prop. 9.24]. And CD is a unit. Thus, CA is odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 26

If an odd (number) is subtracted from an odd number then the remainder will be even.

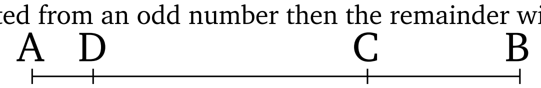


For let the odd (number) BC have been subtracted from the odd (number) AB. I say that the remainder CA is even.

For since AB is odd, let the unit BD have been subtracted (from it). Thus, the remainder AD is even [Def. 7.7]. So, for the same (reasons), CD is also even. And hence the remainder CA is even [Prop. 9.24]. (Which is) the very thing it was required to show.

Proposition 27

If an even (number) is subtracted from an odd number then the remainder will be odd.

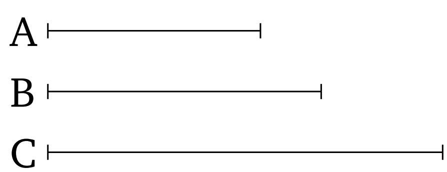


For let the even (number) BC have been subtracted from the odd (number) AB. I say that the remainder CA is odd.

[For] let the unit AD have been subtracted (from AB). DB is thus even [Def. 7.7]. And BC is also even. Thus, the remainder CD is also even [Prop. 9.24]. CA (is) thus odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 28

If an odd number makes some (number by) multiplying an even (number) then the created (number) will be even.

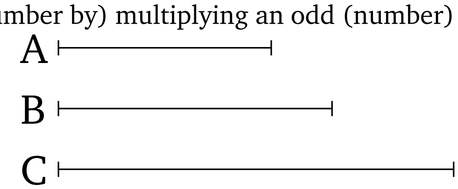


For let the odd number A make C (by) multiplying the even (number) B. I say that C is even.

For since A has made C (by) multiplying B, C is thus composed out of so many (magnitudes) equal to B, as many as (there) are units in A [Def. 7.15]. And B is even. Thus, C is composed out of even (numbers). And if any multitude whatsoever of even numbers is added together then the whole is even [Prop. 9.21]. Thus, C is even. (Which is) the very thing it was required to show.

Proposition 29

If an odd number makes some (number by) multiplying an odd (number) then the created (number) will be odd.

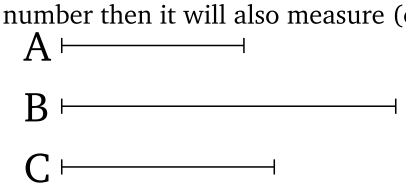


For let the odd number A make C (by) multiplying the odd (number) B. I say that C is odd.

For since A has made C (by) multiplying B, C is thus composed out of so many (magnitudes) equal to B, as many as (there) are units in A [Def. 7.15]. And each of A, B is odd. Thus, C is composed out of odd (numbers), (and) the multitude of them is odd. Hence C is odd [Prop. 9.23]. (Which is) the very thing it was required to show.

Proposition 30

If an odd number measures an even number then it will also measure (one) half of it.

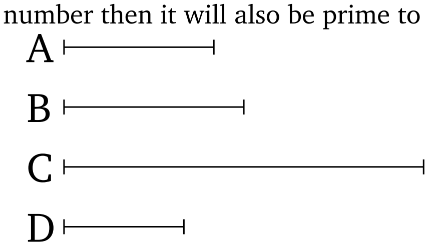


For let the odd number A measure the even (number) B. I say that (A) will also measure (one) half of (B).

For since A measures B, let it measure it according to C. I say that C is not odd. For, if possible, let it be (odd). And since A measures B according to C, A has thus made B (by) multiplying C. Thus, B is composed out of odd numbers, (and) the multitude of them is odd. B is thus odd [Prop. 9.23]. The very thing (is) absurd. For (B) was assumed (to be) even. Thus, C is not odd. Thus, C is even. Hence, A measures B an even number of times. So, on account of this, (A) will also measure (one) half of (B). (Which is) the very thing it was required to show.

Proposition 31

If an odd number is prime to some number then it will also be prime to its double.

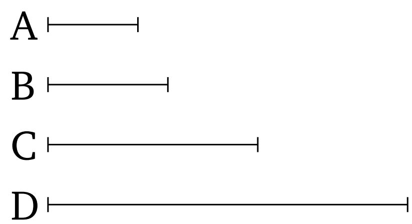


For let the odd number A be prime to some number B. And let C be double B. I say that A is [also] prime to C.

For if [A and C] are not prime (to one another) then some number will measure them. Let it measure (them), and let it be D. And A is odd. Thus, D (is) also odd. And since D, which is odd, measures C, and C is even, [D] will thus also measure half of C [Prop. 9.30]. And B is half of C. Thus, D measures B. And it also measures A. Thus, D measures (both) A and B, (despite) them being prime to one another. The very thing is impossible. Thus, A is not unprime to C. Thus, A and C are prime to one another. (Which is) the very thing it was required to show.

Proposition 32

Each of the numbers (which is continually) doubled, (starting) from a dyad, is an even-times-even (number) only.

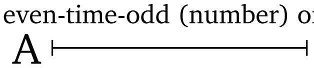


For let any multitude of numbers whatsoever, B, C, D, have been (continually) doubled, (starting) from the dyad A. I say that B, C, D are even-times-even (numbers) only.

In fact, (it is) clear that each [of B, C, D] is an even-times-even (number). For it is doubled from a dyad [Def. 7.8]. I also say that (they are even-times-even numbers) only. For let a unit be laid down. Therefore, since any multitude of numbers whatsoever are continuously proportional, starting from a unit, and the (number) A after the unit is prime, the greatest of A, B, C, D, (namely) D, will not be measured by any other (numbers) except A, B, C [Prop. 9.13]. And each of A, B, C is even. Thus, D is an even-time-even (number) only [Def. 7.8]. So, similarly, we can show that each of B, C is [also] an even-time-even (number) only. (Which is) the very thing it was required to show.

Proposition 33

If a number has an odd half then it is an even-time-odd (number) only.



For let the number A have an odd half. I say that A is an even-times-odd (number) only.

In fact, (it is) clear that (A) is an even-times-odd (number). For its half, being odd, measures it an even number of times [Def. 7.9]. So I also say that (it is an even-times-odd number) only. For if A is also an even-times-even (number) then it will be measured by an even (number) according to an even number [Def. 7.8]. Hence, its half will also be measured by an even number, (despite) being odd. The very thing is absurd. Thus, A is an even-times-odd (number) only. (Which is) the very thing it was required to show.

Proposition 34

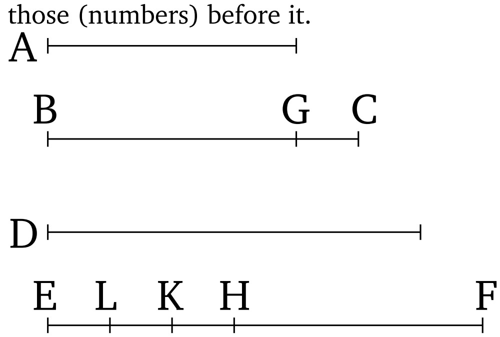
If a number is neither (one) of the (numbers) doubled from a dyad, nor has an odd half, then it is (both) an even-times-even and an even-times-odd (number).

For let the number A neither be (one) of the (numbers) doubled from a dyad, nor let it have an odd half. I say that A is (both) an even-times-even and an even-times-odd (number).

In fact, (it is) clear that A is an even-times-even (number) [Def. 7.8]. For it does not have an odd half. So I say that it is also an even-times-odd (number). For if we cut A in half, and (then cut) its half in half, and we do this continually, then we will arrive at some odd number which will measure A according to an even number. For if not, we will arrive at a dyad, and A will be (one) of the (numbers) doubled from a dyad. The very opposite thing (was) assumed. Hence, A is an even-times-odd (number) [Def. 7.9]. And it was also shown (to be) an even-times-even (number). Thus, A is (both) an even-times-even and an even-times-odd (number). (Which is) the very thing it was required to show.

Proposition 35†

If there is any multitude whatsoever of continually proportional numbers, and (numbers) equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those (numbers) before it.



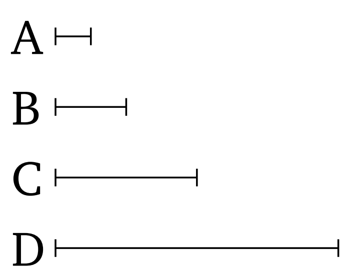
Let A, BC, D, EF be any multitude whatsoever of continuously proportional numbers, beginning from the least A. And let BG and FH, each equal to A, have been subtracted from BC and EF (respectively). I say that as GC is to A, so EH is to A, BC, D.

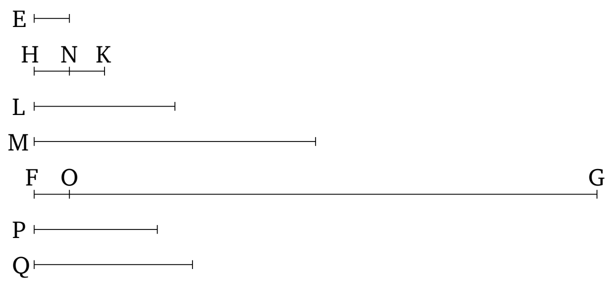
For let FK be made equal to BC, and FL to D. And since FK is equal to BC, of which FH is equal to BG, the remainder HK is thus equal to the remainder GC. And since as EF is to D, so D (is) to BC, and BC to A [Prop. 7.13], and D (is) equal to FL, and BC to FK, and A to FH, thus as EF is to FL, so LF (is) to FK, and FK to FH. By separation, as EL (is) to LF, so LK (is) to FK, and KH to FH [Props. 7.11, 7.13]. And thus as one of the leading (numbers) is to one of the following, so (the sum of) all of the leading (numbers is) to (the sum of) all of the following [Prop. 7.12]. Thus, as KH is to FH, so EL, LK, KH (are) to LF, FK, HF. And KH (is) equal to CG, and FH to A, and LF, FK, HF to D, BC, A. Thus, as CG is to A, so EH (is) to D, BC, A. Thus, as the excess of the second (number) is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it. (Which is) the very thing it was required to show. † This proposition allows us to sum a geometric series of the form a, $a\,r$, $a\,r^2$, $a\,r^3$, $\cdots a\,r^{n-1}$. According to Euclid, the sum S_n satisfies $(a\,r-a)/a = (a\,r^n - a)/S_n$. Hence, $S_n = a\,(r^n-1)/(r-1)$.

Proposition 36†

If any multitude whatsoever of numbers is set out continuously in a double proportion, (starting) from a unit, until the whole sum added together becomes prime, and the sum multiplied into the last (number) makes some (number), then the (number so) created will be perfect.

For let any multitude of numbers, A, B, C, D, be set out (continuouly) in a double proportion, until the whole sum added together is made prime. And let E be equal to the sum. And let E make FG (by) multiplying D. I say that FG is a perfect (number).





For as many as is the multitude of A, B, C, D, let so many (numbers), E, HK, L, M, have been taken in a double proportion, (starting) from E. Thus, via equality, as A is to D, so E (is) to M [Prop. 7.14]. Thus, the (number created) from (multiplying) E, D is equal to the (number created) from (multiplying) A, M. And FG is the (number created) from (multiplying) E, D. Thus, FG is also the (number created) from (multiplying) A, M [Prop. 7.19]. Thus, A has made FG (by) multiplying M. Thus, M measures FG according to the units in A. And A is a dyad. Thus, FG is double M. And M, L, HK, E are also continuously double one another. Thus, E, HK, L, M, FG are continuously proportional in a double proportion. So let HN and FO, each equal to the first (number) E, have been subtracted from the second (number) HK and the last FG (respectively). Thus, as the excess of the second number is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it [Prop. 9.35]. Thus, as NK is to E, so OG (is) to M, L, KH, E. And NK is equal to E. And thus OG is equal to M, L, HK, E. And FO is also equal to E, and E to A, B, C, D, and a unit. Thus, the whole of FG is equal to E, HK, L, M, and A, B, C, D, and a unit. And it is measured by them. I also say that FG will be measured by no other (numbers) except A, B, C, D, E, HK, L, M, and a unit. For, if possible, let some (number) P measure FG, and let P not be the same as any of A, B, C, D, E, HK, L, M. And as many times as P measures FG, so many units let there be in Q. Thus, Q has made FG (by) multiplying P. But, in fact, E has also made FG (by) multiplying D. Thus, as E is to Q, so P (is) to D [Prop. 7.19]. And since A, B, C, D are continually proportional, (starting) from a unit, D will thus not be measured by any other numbers except A, B, C [Prop. 9.13]. And P was assumed not (to be) the same as any of A, B, C. Thus, P does not measure D. But, as P (is) to D, so E (is) to Q. Thus, E does not measure Q either [Def. 7.20]. And E is a prime (number). And every prime number [is] prime to every (number) which it does not measure [Prop. 7.29]. Thus, E and Q are prime to one another. And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. And as E is to Q, (so) P (is) to D. Thus, E measures P the same number of times as Q (measures) D. And D is not measured by any other (numbers) except A, B, C. Thus, Q is the same as one of A, B, C. Let it be the same as B. And as many as is the multitude of B, C, D, let so many (of the set out numbers) have been taken, (starting) from E, (namely) E, HK, L. And E, HK, L are in the same ratio as B, C, D. Thus, via equality, as B (is) to D, (so) E (is) to L [Prop. 7.14]. Thus, the (number created) from (multiplying) B, L is equal to the (number created) from multiplying D, E [Prop. 7.19]. But, the (number created) from (multiplying) D, E is equal to the (number created) from (multiplying) Q, P. Thus, the (number created) from (multiplying) Q, P is equal to the (number created) from (multiplying) B, L. Thus, as Q is to B, (so) L (is) to P [Prop. 7.19]. And Q is the same as B. Thus, L is also the same as P. The very thing (is) impossible. For P was assumed not (to be) the same as any of the (numbers) set out. Thus, FG cannot be measured by any number except A, B, C, D, E, HK, L, M, and a unit. And FG was shown (to be) equal to (the sum of) A, B, C, D, E, HK, L, M, and a unit. And a perfect number is one which is equal to (the sum of) its own parts [Def. 7.22]. Thus, FG is a perfect (number). (Which is) the very thing it was required to show. † This proposition demonstrates that perfect numbers take the form $2^{n-1}\,(2^n-1)$ provided that 2^n-1 is a prime number. The ancient Greeks knew of four perfect numbers: 6, 28, 496, and 8128, which correspond to $n=2$, 3, 5, and 7, respectively.

ELEMENTS BOOK 10

Incommensurable Magnitudes†

†The theory of incommensurable magntidues set out in this book is generally attributed to Theaetetus of Athens. In the footnotes throughout this book, k, k', *etc.* stand for distinct ratios of positive integers.

Definitions

1. Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable.†

2. (Two) straight-lines are commensurable in square‡ when the squares on them are measured by the same area, but (are) incommensurable (in square) when no area admits to be a common measure of the squares on them.§

3. These things being assumed, it is proved that there exist an infinite multitude of straight-lines commensurable and incommensurable with an assigned straight-line—those (incommensurable) in length only, and those also (commensurable or incommensurable) in square.¶ Therefore, let the assigned straight-line be called rational. And (let) the (straight-lines) commensurable with it, either in length and square, or in square only, (also be called) rational. But let the (straight-lines) incommensurable with it be called irrational.*

4. And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their square-roots$ (also be called) irrational—the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure).‖

† In other words, two magnitudes α and β are commensurable if $\alpha : \beta :: 1 : k$, and incommensurable otherwise.

‡ Literally, "in power".

§ In other words, two straight-lines of length α and β are commensurable in square if $\alpha : \beta :: 1 : k^{1/2}$, and incommensurable in square otherwise. Likewise, the straight-lines are commensurable in length if $\alpha : \beta :: 1 : k$, and incommensurable in length otherwise.

¶ To be more exact, straight-lines can either be commensurable in square only, incommensurable in length only, or commenusrable/incommensurable in both length and square, with an assigned straight-line.

* Let the length of the assigned straight-line be unity. Then rational straight-lines have lengths expressible as k or $k^{1/2}$, depending on whether the lengths are commensurable in length, or in square only, respectively, with unity. All other straight-lines are irrational.

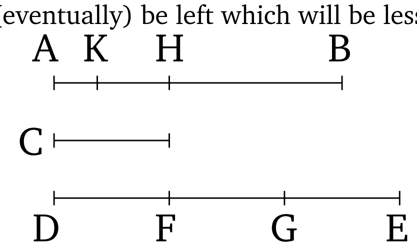
$ The square-root of an area is the length of the side of an equal area square.

‖ The area of the square on the assigned straight-line is unity. Rational areas are expressible as k. All other areas are irrational. Thus, squares whose sides are of rational length have rational areas, and *vice versa*.

Proposition 1†

If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude.

Let AB and C be two unequal magnitudes, of which (let) AB (be) the greater. I say that if (a part) greater than half is subtracted from AB, and (if a part) greater than half (is subtracted) from the remainder, and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the magnitude C.



For C, when multiplied (by some number), will sometimes be greater than AB [Def. 5.4]. Let it have been (so) multiplied. And let DE be (both) a multiple of C, and greater than AB. And let DE have been divided into the (divisions) DF, FG, GE, equal to C. And let BH, (which is) greater than half, have been subtracted from AB. And (let) HK, (which is) greater than half, (have been subtracted) from AH. And let this happen continually, until the divisions in AB become equal in number to the divisions in DE.

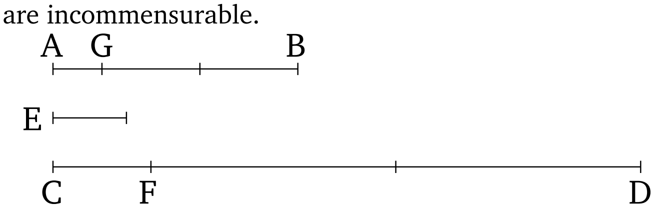
Therefore, let the divisions (in AB) be AK, KH, HB, being equal in number to DF, FG, GE. And since DE is greater than AB, and EG, (which is) less than half, has been subtracted from DE, and BH, (which is) greater than half, from AB, the remainder GD is thus greater than the remainder HA. And since GD is greater than HA, and the half GF has been subtracted from GD, and HK, (which is) greater than half, from HA, the remainder DF is thus greater than the remainder AK. And DF (is) equal to C. C is thus also greater than AK. Thus, AK (is) less than C.

Thus, the magnitude AK, which is less than the lesser laid out magnitude C, is left over from the magnitude AB. (Which is) the very thing it was required to show. — (The theorem) can similarly be proved even if the (parts) subtracted are halves. † This theorem is the basis of the so-called *method of exhaustion*, and is generally attributed to Eudoxus of Cnidus.

Proposition 2

If the remainder of two unequal magnitudes (which are) [laid out] never measures the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater, then the (original) magnitudes will be incommensurable.

For, AB and CD being two unequal magnitudes, and AB (being) the lesser, let the remainder never measure the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater. I say that the magnitudes AB and CD are incommensurable.

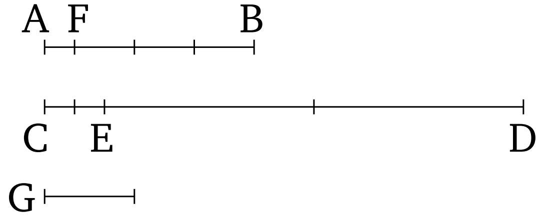


For if they are commensurable then some magnitude will measure them (both). If possible, let it (so) measure (them), and let it be E. And let AB leave CF less than itself (in) measuring FD, and let CF leave AG less than itself (in) measuring BG, and let this happen continually, until some magnitude which is less than E is left. Let (this) have occurred,† and let AG, (which is) less than E, have been left. Therefore, since E measures AB, but AB measures DF, E will thus also measure FD. And it also measures the whole (of) CD. Thus, it will also measure the remainder CF. But, CF measures BG. Thus, E also measures BG. And it also measures the whole (of) AB. Thus, it will also measure the remainder AG, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude cannot measure (both) the magnitudes AB and CD. Thus, the magnitudes AB and CD are incommensurable [Def. 10.1].

Thus, if . . . of two unequal magnitudes, and so on . . . . † The fact that this will eventually occur is guaranteed by Prop. 10.1.

Proposition 3

To find the greatest common measure of two given commensurable magnitudes.



Let AB and CD be the two given magnitudes, of which (let) AB (be) the lesser. So, it is required to find the greatest common measure of AB and CD.

For the magnitude AB either measures, or (does) not (measure), CD. Therefore, if it measures (CD), and (since) it also measures itself, AB is thus a common measure of AB and CD. And (it is) clear that (it is) also (the) greatest. For a (magnitude) greater than magnitude AB cannot measure AB.

So let AB not measure CD. And continually subtracting in turn the lesser (magnitude) from the greater, the remaining (magnitude) will (at) some time measure the (magnitude) before it, on account of AB and CD not being incommensurable [Prop. 10.2]. And let AB leave EC less than itself (in) measuring ED, and let EC leave AF less than itself (in) measuring FB, and let AF measure CE.

Therefore, since AF measures CE, but CE measures FB, AF will thus also measure FB. And it also measures itself. Thus, AF will also measure the whole (of) AB. But, AB measures DE. Thus, AF will also measure ED. And it also measures CE. Thus, it also measures the whole of CD. Thus, AF is a common measure of AB and CD. So I say that (it is) also (the) greatest (common measure). For, if not, there will be some magnitude, greater than AF, which will measure (both) AB and CD. Let it be G. Therefore, since G measures AB, but AB measures ED, G will thus also measure ED. And it also measures the whole of CD. Thus, G will also measure the remainder CE. But CE measures FB. Thus, G will also measure FB. And it also measures the whole (of) AB. And (so) it will measure the remainder AF, the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude greater than AF cannot measure (both) AB and CD. Thus, AF is the greatest common measure of AB and CD.

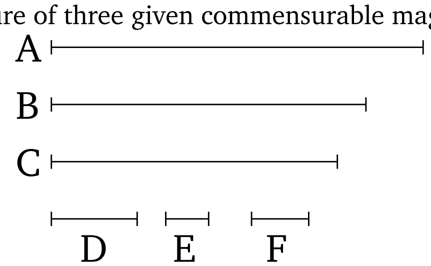
Thus, the greatest common measure of two given commensurable magnitudes, AB and CD, has been found. (Which is) the very thing it was required to show.

Corollary

So (it is) clear, from this, that if a magnitude measures two magnitudes then it will also measure their greatest common measure.

Proposition 4

To find the greatest common measure of three given commensurable magnitudes.



Let A, B, C be the three given commensurable magnitudes. So it is required to find the greatest common measure of A, B, C.

For let the greatest common measure of the two (magnitudes) A and B have been taken [Prop. 10.3], and let it be D. So D either measures, or [does] not [measure], C. Let it, first of all, measure (C). Therefore, since D measures C, and it also measures A and B, D thus measures A, B, C. Thus, D is a common measure of A, B, C. And (it is) clear that (it is) also (the) greatest (common measure). For no magnitude larger than D measures (both) A and B.

So let D not measure C. I say, first, that C and D are commensurable. For if A, B, C are commensurable then some magnitude will measure them which will clearly also measure A and B. Hence, it will also measure D, the greatest common measure of A and B [Prop. 10.3 corr.]. And it also measures C. Hence, the aforementioned magnitude will measure (both) C and D. Thus, C and D are commensurable [Def. 10.1]. Therefore, let their greatest common measure have been taken [Prop. 10.3], and let it be E. Therefore, since E measures D, but D measures (both) A and B, E will thus also measure A and B. And it also measures C. Thus, E measures A, B, C. Thus, E is a common measure of A, B, C. So I say that (it is) also (the) greatest (common measure). For, if possible, let F be some magnitude greater than E, and let it measure A, B, C. And since F measures A, B, C, it will thus also measure A and B, and will (thus) measure the greatest common measure of A and B [Prop. 10.3 corr.]. And D is the greatest common measure of A and B. Thus, F measures D. And it also measures C. Thus, F measures (both) C and D. Thus, F will also measure the greatest common measure of C and D [Prop. 10.3 corr.]. And it is E. Thus, F will measure E, the greater (measuring) the lesser. The very thing is impossible. Thus, some [magnitude] greater than the magnitude E cannot measure A, B, C. Thus, if D does not measure C then E is the greatest common measure of A, B, C. And if it does measure (C) then D itself (is the greatest common measure).

Thus, the greatest common measure of three given commensurable magnitudes has been found. [(Which is) the very thing it was required to show.]

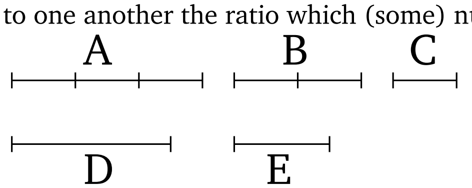
Corollary

So (it is) clear, from this, that if a magnitude measures three magnitudes then it will also measure their greatest common measure.

So, similarly, the greatest common measure of more (magnitudes) can also be taken, and the (above) corollary will go forward. (Which is) the very thing it was required to show.

Proposition 5

Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.



Let A and B be commensurable magnitudes. I say that A has to B the ratio which (some) number (has) to (some) number.

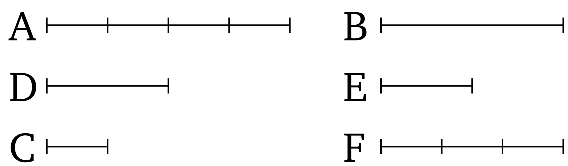
For if A and B are commensurable (magnitudes) then some magnitude will measure them. Let it (so) measure (them), and let it be C. And as many times as C measures A, so many units let there be in D. And as many times as C measures B, so many units let there be in E.

Therefore, since C measures A according to the units in D, and a unit also measures D according to the units in it, a unit thus measures the number D as many times as the magnitude C (measures) A. Thus, as C is to A, so a unit (is) to D [Def. 7.20].† Thus, inversely, as A (is) to C, so D (is) to a unit [Prop. 5.7 corr.]. Again, since C measures B according to the units in E, and a unit also measures E according to the units in it, a unit thus measures E the same number of times that C (measures) B. Thus, as C is to B, so a unit (is) to E [Def. 7.20]. And it was also shown that as A (is) to C, so D (is) to a unit. Thus, via equality, as A is to B, so the number D (is) to the (number) E [Prop. 5.22].

Thus, the commensurable magnitudes A and B have to one another the ratio which the number D (has) to the number E. (Which is) the very thing it was required to show. † There is a slight logical gap here, since Def. 7.20 applies to four numbers, rather than two number and two magnitudes.

Proposition 6

If two magnitudes have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be commensurable.



For let the two magnitudes A and B have to one another the ratio which the number D (has) to the number E. I say that the magnitudes A and B are commensurable.

For, as many units as there are in D, let A have been divided into so many equal (divisions). And let C be equal to one of them. And as many units as there are in E, let F be the sum of so many magnitudes equal to C.

Therefore, since as many units as there are in D, so many magnitudes equal to C are also in A, therefore whichever part a unit is of D, C is also the same part of A. Thus, as C is to A, so a unit (is) to D [Def. 7.20]. And a unit measures the number D. Thus, C also measures A. And since as C is to A, so a unit (is) to the [number] D, thus, inversely, as A (is) to C, so the number D (is) to a unit [Prop. 5.7 corr.]. Again, since as many units as there are in E, so many (magnitudes) equal to C are also in F, thus as C is to F, so a unit (is) to the [number] E [Def. 7.20]. And it was also shown that as A (is) to C, so D (is) to a unit. Thus, via equality, as A is to F, so D (is) to E [Prop. 5.22]. But, as D (is) to E, so A is to B. And thus as A (is) to B, so (it) also is to F [Prop. 5.11]. Thus, A has the same ratio to each of B and F. Thus, B is equal to F [Prop. 5.9]. And C measures F. Thus, it also measures B. But, in fact, (it) also (measures) A. Thus, C measures (both) A and B. Thus, A is commensurable with B [Def. 10.1].

Thus, if two magnitudes ... to one another, and so on ....

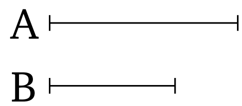
Corollary

So it is clear, from this, that if there are two numbers, like D and E, and a straight-line, like A, then it is possible to contrive that as the number D (is) to the number E, so the straight-line (is) to (another) straight-line (*i.e., F*). And if the mean proportion, (say) B, is taken of A and F, then as A is to F, so the (square) on A (will be) to the (square) on B. That is to say, as the first (is) to the third, so the (figure) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. But, as A (is) to F, so the number D is to the number E. Thus, it has also been contrived that as the number D (is) to the number E, so the (figure) on the straight-line A (is) to the (similar figure) on the straight-line B. (Which is) the very thing it was required to show.

Proposition 7

Incommensurable magnitudes do not have to one another the ratio which (some) number (has) to (some) number.

Let A and B be incommensurable magnitudes. I say that A does not have to B the ratio which (some) number (has) to (some) number.

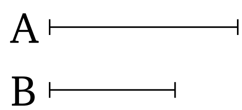


For if A has to B the ratio which (some) number (has) to (some) number then A will be commensurable with B [Prop. 10.6]. But it is not. Thus, A does not have to B the ratio which (some) number (has) to (some) number.

Thus, incommensurable numbers do not have to one another, and so on ....

Proposition 8

If two magnitudes do not have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be incommensurable.



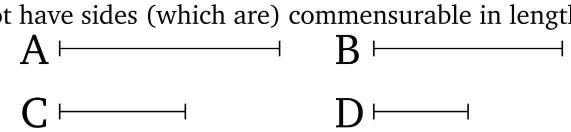
For let the two magnitudes A and B not have to one another the ratio which (some) number (has) to (some) number. I say that the magnitudes A and B are incommensurable.

For if they are commensurable, A will have to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. But it does not have (such a ratio). Thus, the magnitudes A and B are incommensurable.

Thus, if two magnitudes ... to one another, and so on ....

Proposition 9

Squares on straight-lines (which are) commensurable in length have to one another the ratio which (some) square number (has) to (some) square number. And squares having to one another the ratio which (some) square number (has) to (some) square number will also have sides (which are) commensurable in length. But squares on straight-lines (which are) incommensurable in length do not have to one another the ratio which (some) square number (has) to (some) square number. And squares not having to one another the ratio which (some) square number (has) to (some) square number will not have sides (which are) commensurable in length either.



For let A and B be (straight-lines which are) commensurable in length. I say that the square on A has to the square on B the ratio which (some) square number (has) to (some) square number.

For since A is commensurable in length with B, A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (that) which C (has) to D. Therefore, since as A is to B, so C (is) to D. But the (ratio) of the square on A to the square on B is the square of the ratio of A to B. For similar figures are in the squared ratio of (their) corresponding sides [Prop. 6.20 corr.]. And the (ratio) of the square on C to the square on D is the square of the ratio of the [number] C to the [number] D. For there exits one number in mean proportion to two square numbers, and (one) square (number) has to the (other) square [number] a squared ratio with respect to (that) the side (of the former has) to the side (of the latter) [Prop. 8.11]. And, thus, as the square on A is to the square on B, so the square [number] on the (number) C (is) to the square [number] on the [number] D.[†]

And so let the square on A be to the (square) on B as the square (number) on C (is) to the [square] (number) on D. I say that A is commensurable in length with B.

For since as the square on A is to the [square] on B, so the square (number) on C (is) to the [square] (number) on D. But, the ratio of the square on A to the (square) on B is the square of the (ratio) of A to B [Prop. 6.20 corr.]. And the (ratio) of the square [number] on the [number] C to the square [number] on the [number] D is the square of the ratio of the [number] C to the [number] D [Prop. 8.11]. Thus, as A is to B, so the [number] C also (is) to the [number] D. A, thus, has to B the ratio which the number C has to the number D. Thus, A is commensurable in length with B [Prop. 10.6].‡

And so let A be incommensurable in length with B. I say that the square on A does not have to the [square] on B the ratio which (some) square number (has) to (some) square number.

For if the square on A has to the [square] on B the ratio which (some) square number (has) to (some) square number then A will be commensurable (in length) with B. But it is not. Thus, the square on A does not have to the [square] on the B the ratio which (some) square number (has) to (some) square number.

So, again, let the square on A not have to the [square] on B the ratio which (some) square number (has) to (some) square number. I say that A is incommensurable in length with B.

For if A is commensurable (in length) with B then the (square) on A will have to the (square) on B the ratio which (some) square number (has) to (some) square number. But it does not have (such a ratio). Thus, A is not commensurable in length with B.

Thus, (squares) on (straight-lines which are) commensurable in length, and so on ....

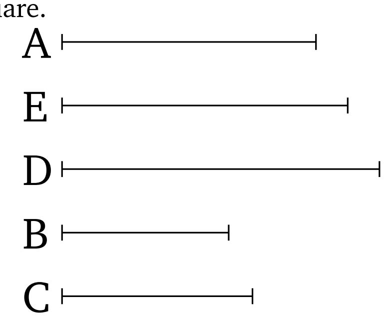
Corollary

And it will be clear, from (what) has been demonstrated, that (straight-lines) commensurable in length (are) always also (commensurable) in square, but (straight-lines commensurable) in square (are) not always also (commensurable) in length. † There is an unstated assumption here that if $\alpha : \beta :: \gamma : \delta$ then $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$.

‡ There is an unstated assumption here that if $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$ then $\alpha : \beta :: \gamma : \delta$.

Proposition 10†

To find two straight-lines incommensurable with a given straight-line, the one (incommensurable) in length only, the other also (incommensurable) in square.



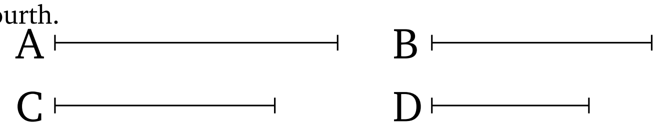
Let A be the given straight-line. So it is required to find two straight-lines incommensurable with A, the one (incommensurable) in length only, the other also (incommensurable) in square.

For let two numbers, B and C, not having to one another the ratio which (some) square number (has) to (some) square number—that is to say, not (being) similar plane (numbers)—have been taken. And let it be contrived that as B (is) to C, so the square on A (is) to the square on D. For we learned (how to do this) [Prop. 10.6 corr.]. Thus, the (square) on A (is) commensurable with the (square) on D [Prop. 10.6]. And since B does not have to C the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on D the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with D [Prop. 10.9]. Let the (straight-line) E (which is) in mean proportion to A and D have been taken [Prop. 6.13]. Thus, as A is to D, so the square on A (is) to the (square) on E [Def. 5.9]. And A is incommensurable in length with D. Thus, the square on A is also incommensurble with the square on E [Prop. 10.11]. Thus, A is incommensurable in square with E.

Thus, two straight-lines, D and E, (which are) incommensurable with the given straight-line A, have been found, the one, D, (incommensurable) in length only, the other, E, (incommensurable) in square, and, clearly, also in length. [(Which is) the very thing it was required to show.] † This whole proposition is regarded by Heiberg as an interpolation into the original text.

Proposition 11

If four magnitudes are proportional, and the first is commensurable with the second, then the third will also be commensurable with the fourth. And if the first is incommensurable with the second, then the third will also be incommensurable with the fourth.



Let A, B, C, D be four proportional magnitudes, (such that) as A (is) to B, so C (is) to D. And let A be commensurable with B. I say that C will also be commensurable with D.

For since A is commensurable with B, A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. And as A is to B, so C (is) to D. Thus, C also has to D the ratio which (some) number (has) to (some) number. Thus, C is commensurable with D [Prop. 10.6].

And so let A be incommensurable with B. I say that C will also be incommensurable with D. For since A is incommensurable with B, A thus does not have to B the ratio which (some) number (has) to (some) number [Prop. 10.7]. And as A is to B, so C (is) to D. Thus, C does not have to D the ratio which (some) number (has) to (some) number either. Thus, C is incommensurable with D [Prop. 10.8].

Thus, if four magnitudes, and so on ....

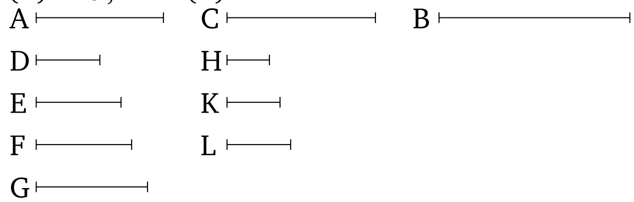
Proposition 12

(Magnitudes) commensurable with the same magnitude are also commensurable with one another.

For let A and B each be commensurable with C. I say that A is also commensurable with B.

For since A is commensurable with C, A thus has to C the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which D (has) to E. Again, since C is commensurable with B, C thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which F (has) to G. And for any multitude whatsoever of given ratios—(namely,) those which D has to E, and F to G—let the numbers

H, K, L (which are) continuously (proportional) in the(se) given ratios have been taken [Prop. 8.4]. Hence, as D is to E, so H (is) to K, and as F (is) to G, so K (is) to L.

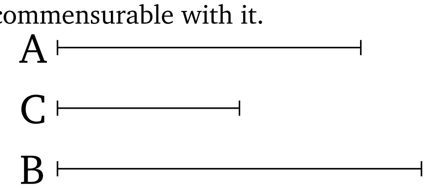


Therefore, since as A is to C, so D (is) to E, but as D (is) to E, so H (is) to K, thus also as A is to C, so H (is) to K [Prop. 5.11]. Again, since as C is to B, so F (is) to G, but as F (is) to G, [so] K (is) to L, thus also as C (is) to B, so K (is) to L [Prop. 5.11]. And also as A is to C, so H (is) to K. Thus, via equality, as A is to B, so H (is) to L [Prop. 5.22]. Thus, A has to B the ratio which the number H (has) to the number L. Thus, A is commensurable with B [Prop. 10.6].

Thus, (magnitudes) commensurable with the same magnitude are also commensurable with one another. (Which is) the very thing it was required to show.

Proposition 13

If two magnitudes are commensurable, and one of them is incommensurable with some magnitude, then the remaining (magnitude) will also be incommensurable with it.



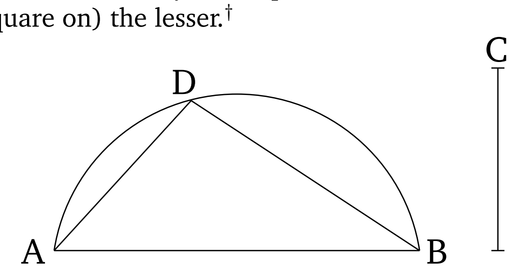
Let A and B be two commensurable magnitudes, and let one of them, A, be incommensurable with some other (magnitude), C. I say that the remaining (magnitude), B, is also incommensurable with C.

For if B is commensurable with C, but A is also commensurable with B, A is thus also commensurable with C [Prop. 10.12]. But, (it is) also incommensurable (with C). The very thing (is) impossible. Thus, B is not commensurable with C. Thus, (it is) incommensurable.

Thus, if two magnitudes are commensurable, and so on ....

Lemma

For two given unequal straight-lines, to find by (the square on) which (straight-line) the square on the greater (straight-line is) larger than (the square on) the lesser.†



Let AB and C be the two given unequal straight-lines, and let AB be the greater of them. So it is required to find by (the square on) which (straight-line) the square on AB (is) greater than (the square on) C.

Let the semi-circle ADB have been described on AB. And let AD, equal to C, have been inserted into it [Prop. 4.1]. And let DB have been joined. So (it is) clear that the angle ADB is a right-angle [Prop. 3.31], and that the square on AB (is) greater than (the square on) AD—that is to say, (the square on) C—by (the square on) DB [Prop. 1.47].

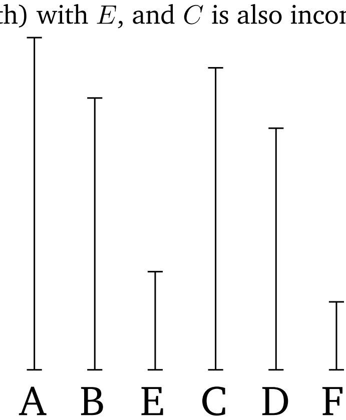
And, similarly, the square-root of (the sum of the squares on) two given straight-lines is also found likeso.

Let AD and DB be the two given straight-lines. And let it be necessary to find the square-root of (the sum of the squares on) them. For let them have been laid down such as to encompass a right-angle—(namely), that (angle encompassed) by AD and DB. And let AB have been joined. (It is) again clear that AB is the square-root of (the sum of the squares on) AD and DB [Prop. 1.47]. (Which is) the very thing it was required to show. † That is, if α and β are the lengths of two given straight-lines, with α being greater than β, to find a straight-line of length γ such that $\alpha^2 = \beta^2 + \gamma^2$. Similarly, we can also find γ such that $\gamma^2 = \alpha^2 + \beta^2$.

Proposition 14

If four straight-lines are proportional, and the square on the first is greater than (the square on) the second by the (square) on (some straight-line) commensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) commensurable [in length] with the third. And if the square on the first is greater than (the square on) the second by the (square) on (some straight-line) incommensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) incommensurable [in length] with the third.

Let A, B, C, D be four proportional straight-lines, (such that) as A (is) to B, so C (is) to D. And let the square on A be greater than (the square on) B by the (square) on E, and let the square on C be greater than (the square on) D by the (square) on F. I say that A is either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F.



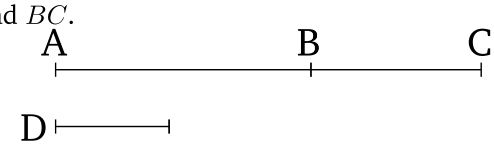
For since as A is to B, so C (is) to D, thus as the (square) on A is to the (square) on B, so the (square) on C (is) to the (square) on D [Prop. 6.22]. But the (sum of the squares) on E and B is equal to the (square) on A, and the (sum of the squares) on D and F is equal to the (square) on C. Thus, as the (sum of the squares) on E and B is to the (square) on B, so the (sum of the squares) on D and F (is) to the (square) on D. Thus, via separation, as the (square) on E is to the (square) on B, so the (square) on F (is) to the (square) on D [Prop. 5.17]. Thus, also, as E is to B, so F (is) to D [Prop. 6.22]. Thus, inversely, as B is to E, so D (is) to F [Prop. 5.7 corr.]. But, as A is to B, so C also (is) to D. Thus, via equality, as A is to E, so C (is) to F [Prop. 5.22]. Therefore, A is either commensurable (in length) with E, and C is also commensurable with F, or A is incommensurable (in length) with E, and C is also incommensurable with F [Prop. 10.11].

Thus, if, and so on ....

Proposition 15

If two commensurable magnitudes are added together then the whole will also be commensurable with each of them. And if the whole is commensurable with one of them then the original magnitudes will also be commensurable (with one another).

For let the two commensurable magnitudes AB and BC be laid down together. I say that the whole AC is also commensurable with each of AB and BC.



For since AB and BC are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be D. Therefore, since D measures (both) AB and BC, it will also measure the whole AC. And it also measures AB and BC. Thus, D measures AB, BC, and AC. Thus, AC is commensurable with each of AB and BC [Def. 10.1].

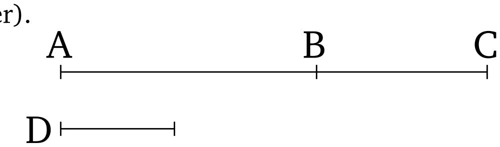
And so let AC be commensurable with AB. I say that AB and BC are also commensurable.

For since AC and AB are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be D. Therefore, since D measures (both) CA and AB, it will thus also measure the remainder BC. And it also measures AB. Thus, D will measure (both) AB and BC. Thus, AB and BC are commensurable [Def. 10.1].

Thus, if two magnitudes, and so on ....

Proposition 16

If two incommensurable magnitudes are added together then the whole will also be incommensurable with each of them. And if the whole is incommensurable with one of them then the original magnitudes will also be incommensurable (with one another).



For let the two incommensurable magnitudes AB and BC be laid down together. I say that that the whole AC is also incommensurable with each of AB and BC.

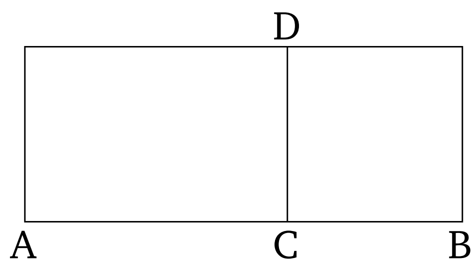
For if CA and AB are not incommensurable then some magnitude will measure [them]. If possible, let it (so) measure (them), and let it be D. Therefore, since D measures (both) CA and AB, it will thus also measure the remainder BC. And it also measures AB. Thus, D measures (both) AB and BC. Thus, AB and BC are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) CA and AB. Thus, CA and AB are incommensurable [Def. 10.1]. So, similarly, we can show that AC and CB are also incommensurable. Thus, AC is incommensurable with each of AB and BC.

And so let AC be incommensurable with one of AB and BC. So let it, first of all, be incommensurable with AB. I say that AB and BC are also incommensurable. For if they are commensurable then some magnitude will measure them. Let it (so) measure (them), and let it be D. Therefore, since D measures (both) AB and BC, it will thus also measure the whole AC. And it also measures AB. Thus, D measures (both) CA and AB. Thus, CA and AB are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) AB and BC. Thus, AB and BC are incommensurable [Def. 10.1].

Thus, if two... magnitudes, and so on ....

Lemma

If a parallelogram,† falling short by a square figure, is applied to some straight-line then the applied (parallelogram) is equal (in area) to the (rectangle contained) by the pieces of the straight-line created via the application (of the parallelogram).



For let the parallelogram AD, falling short by the square figure DB, have been applied to the straight-line AB. I say that AD is equal to the (rectangle contained) by AC and CB.

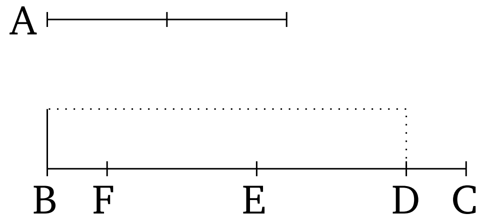
And it is immediately obvious. For since DB is a square, DC is equal to CB. And AD is the (rectangle contained) by AC and CD—that is to say, by AC and CB.

Thus, if ... to some straight-line, and so on .... † Note that this lemma only applies to rectangular parallelograms.

Proposition 17†

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable in length, then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater. And if the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) commensurable in length.

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater. And let a (rectangle) equal to the fourth part of the (square) on the lesser, A—that is, (equal) to the (square) on half of A—falling short by a square figure, have been applied to BC. And let it be the (rectangle contained) by BD and DC [see previous lemma]. And let BD be commensurable in length with DC. I say that that the square on BC is greater than the (square on) A by (the square on some straight-line) commensurable (in length) with (BC).



For let BC have been cut in half at the point E [Prop. 1.10]. And let EF be made equal to DE [Prop. 1.3]. Thus, the remainder DC is equal to BF. And since the straight-line BC has been cut into equal (pieces) at E, and into unequal (pieces) at D, the rectangle contained by BD and DC, plus the square on ED, is thus equal to the square on EC [Prop. 2.5]. (The same) also (for) the quadruples. Thus, four times the (rectangle contained) by BD and DC, plus the quadruple of the (square) on DE, is equal to four times the square on EC. But, the square on A is equal to the quadruple of the (rectangle contained) by BD and DC, and the square on DF is equal to the quadruple of the (square) on DE. For DF is double DE. And the square on BC is equal to the quadruple of the (square) on EC. For, again, BC is double CE. Thus, the (sum of the) squares on A and DF is equal to the square on BC. Hence, the (square) on BC is greater than the (square) on A by the (square) on DF. Thus, BC is greater in square than A by DF. It must also be shown that BC is commensurable (in length) with DF. For since BD is commensurable in length with DC, BC is thus also commensurable in length with CD [Prop. 10.15]. But, CD is commensurable in length with CD plus BF. For CD is equal to BF [Prop. 10.6]. Thus, BC is also commensurable in length with BF plus CD [Prop. 10.12]. Hence, BC is also commensurable in length with the remainder FD [Prop. 10.15]. Thus, the square on BC is greater than (the square on) A by the (square) on (some straight-line) commensurable (in length) with (BC).

And so let the square on BC be greater than the (square on) A by the (square) on (some straight-line) commensurable (in length) with (BC). And let a (rectangle) equal to the fourth (part) of the (square) on A, falling short by a square figure, have been applied to BC. And let it be the (rectangle contained) by BD and DC. It must be shown that BD is commensurable in length with DC.

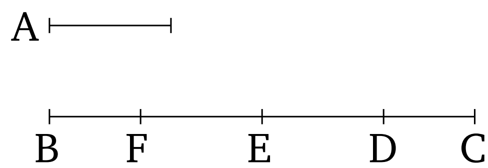
For, similarly, by the same construction, we can show that the square on BC is greater than the (square on) A by the (square) on FD. And the square on BC is greater than the (square on) A by the (square) on (some straight-line) commensurable (in length) with (BC). Thus, BC is commensurable in length with FD. Hence, BC is also commensurable in length with the remaining sum of BF and DC [Prop. 10.15]. But, the sum of BF and DC is commensurable [in length] with DC [Prop. 10.6]. Hence, BC is also commensurable in length with CD [Prop. 10.12]. Thus, via separation, BD is also commensurable in length with DC [Prop. 10.15].

Thus, if there are two unequal straight-lines, and so on .... † This proposition states that if $\alpha\,x - x^2 = \beta^2/4$ (where $\alpha = BC$, $x = DC$, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are commensurable when $\alpha - x$ are x are commensurable, and *vice versa*.

Proposition 18†

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable [in length], then the square on the greater will be larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater. And if the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) incommensurable [in length].

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater. And let a (rectangle) equal to the fourth [part] of the (square) on the lesser, A, falling short by a square figure, have been applied to BC. And let it be the (rectangle contained) by BDC. And let BD be incommensurable in length with DC. I say that that the square on BC is greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC).



For, similarly, by the same construction as before, we can show that the square on BC is greater than the (square on) A by the (square) on FD. [Therefore] it must be shown that BC is incommensurable in length with DF. For since BD is incommensurable in length with DC, BC is thus also incommensurable in length with CD [Prop. 10.16]. But, DC is commensurable (in length) with the sum of BF and DC [Prop. 10.6]. And, thus, BC is incommensurable (in length) with the sum of BF and DC [Prop. 10.13]. Hence, BC is also incommensurable in length with the remainder FD [Prop. 10.16]. And the square on BC is greater than the (square on) A by the (square) on FD. Thus, the square on BC is greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC).

So, again, let the square on BC be greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC). And let a (rectangle) equal to the fourth [part] of the (square) on A, falling short by a square figure, have been applied to BC. And let it be the (rectangle contained) by BD and DC. It must be shown that BD is incommensurable in length with DC.

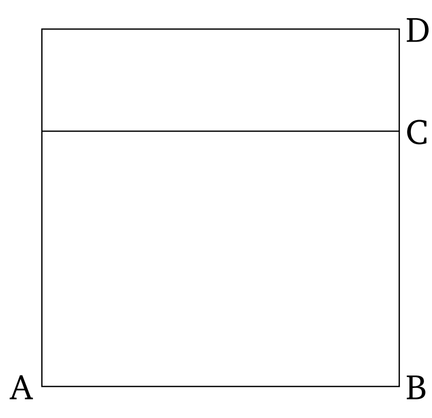
For, similarly, by the same construction, we can show that the square on BC is greater than the (square) on A by the (square) on FD. But, the square on BC is greater than the (square) on A by the (square) on (some straight-line) incommensurable (in length) with (BC). Thus, BC is incommensurable in length with FD. Hence, BC is also incommensurable (in length) with the remaining sum of BF and DC [Prop. 10.16]. But, the sum of BF and DC is commensurable in length with DC [Prop. 10.6]. Thus, BC is also incommensurable in length with DC [Prop. 10.13]. Hence, via separation, BD is also incommensurable in length with DC [Prop. 10.16].

Thus, if there are two ... straight-lines, and so on .... † This proposition states that if $\alpha x - x^2 = \beta^2/4$ (where $\alpha = BC$, $x = DC$, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are incommensurable when $\alpha - x$ are x are incommensurable, and *vice versa*.

Proposition 19

The rectangle contained by rational straight-lines (which are) commensurable in length is rational.

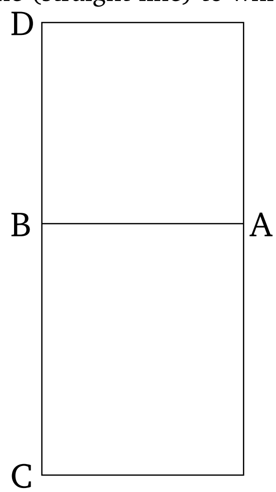
For let the rectangle AC have been enclosed by the rational straight-lines AB and BC (which are) commensurable in length. I say that AC is rational.



For let the square AD have been described on AB. AD is thus rational [Def. 10.4]. And since AB is commensurable in length with BC, and AB is equal to BD, BD is thus commensurable in length with BC. And as BD is to BC, so DA (is) to AC [Prop. 6.1]. Thus, DA is commensurable with AC [Prop. 10.11]. And DA (is) rational. Thus, AC is also rational [Def. 10.4]. Thus, the ... by rational straight-lines ... commensurable, and so on ....

Proposition 20

If a rational (area) is applied to a rational (straight-line) then it produces as breadth a (straight-line which is) rational, and commensurable in length with the (straight-line) to which it is applied.



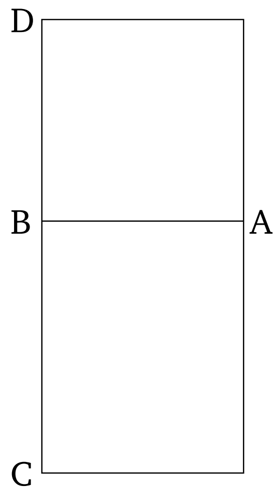
For let the rational (area) AC have been applied to the rational (straight-line) AB, producing the (straight-line) BC as breadth. I say that BC is rational, and commensurable in length with BA.

For let the square AD have been described on AB. AD is thus rational [Def. 10.4]. And AC (is) also rational. DA is thus commensurable with AC. And as DA is to AC, so DB (is) to BC [Prop. 6.1]. Thus, DB is also commensurable (in length) with BC [Prop. 10.11]. And DB (is) equal to BA. Thus, AB (is) also commensurable (in length) with BC. And AB is rational. Thus, BC is also rational, and commensurable in length with AB [Def. 10.3].

Thus, if a rational (area) is applied to a rational (straight-line), and so on ....

Proposition 21

The rectangle contained by rational straight-lines (which are) commensurable in square only is irrational, and its square-root is irrational—let it be called medial.†

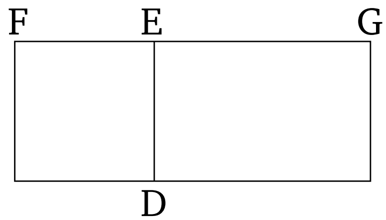


For let the rectangle AC be contained by the rational straight-lines AB and BC (which are) commensurable in square only. I say that AC is irrational, and its square-root is irrational—let it be called medial.

For let the square AD have been described on AB. AD is thus rational [Def. 10.4]. And since AB is incommensurable in length with BC. For they were assumed to be commensurable in square only. And AB (is) equal to BD. DB is thus also incommensurable in length with BC. And as DB is to BC, so AD (is) to AC [Prop. 6.1]. Thus, DA [is] incommensurable with AC [Prop. 10.11]. And DA (is) rational. Thus, AC is irrational [Def. 10.4]. Hence, its square-root [that is to say, the square-root of the square equal to it] is also irrational [Def. 10.4]—let it be called medial. (Which is) the very thing it was required to show. † Thus, a medial straight-line has a length expressible as $k^{1/4}$.

Lemma

If there are two straight-lines then as the first is to the second, so the (square) on the first (is) to the (rectangle contained) by the two straight-lines.

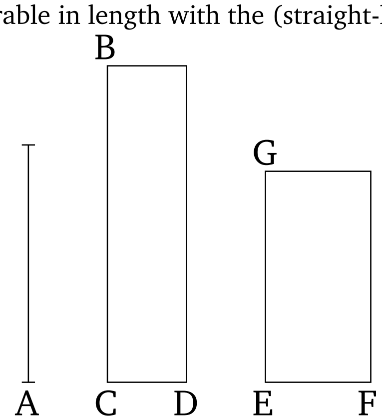


Let FE and EG be two straight-lines. I say that as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG.

For let the square DF have been described on FE. And let GD have been completed. Therefore, since as FE is to EG, so FD (is) to DG [Prop. 6.1], and FD is the (square) on FE, and DG the (rectangle contained) by DE and EG—that is to say, the (rectangle contained) by FE and EG—thus as FE is to EG, so the (square) on FE (is) to the (rectangle contained) by FE and EG. And also, similarly, as the (rectangle contained) by GE and EF is to the (square on) EF—that is to say, as GD (is) to FD—so GE (is) to EF. (Which is) the very thing it was required to show.

Proposition 22

The square on a medial (straight-line), being applied to a rational (straight-line), produces as breadth a (straight-line which is) rational, and incommensurable in length with the (straight-line) to which it is applied.



Let A be a medial (straight-line), and CB a rational (straight-line), and let the rectangular area BD, equal to the (square) on A, have been applied to BC, producing CD as breadth. I say that CD is rational, and incommensurable in length with CB.

For since A is medial, the square on it is equal to a (rectangular) area contained by rational (straight-lines which are) commensurable in square only [Prop. 10.21]. Let the square on (A) be equal to GF. And the square on (A) is also equal to BD. Thus, BD is equal to GF. And (BD) is also equiangular with (GF). And for equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, proportionally, as BC is to EG, so EF (is) to CD. And, also, as the (square) on BC is to the (square) on EG, so the (square) on EF (is) to the (square) on CD [Prop. 6.22]. And the (square) on CB is commensurable with the (square) on EG. For they are each rational. Thus, the (square) on EF is also commensurable with the (square) on CD [Prop. 10.11].

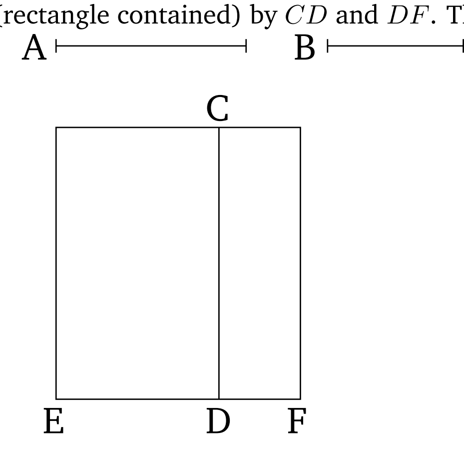
And the (square) on EF is rational. Thus, the (square) on CD is also rational [Def. 10.4]. Thus, CD is rational. And since EF is incommensurable in length with EG. For they are commensurable in square only. And as EF (is) to EG, so the (square) on EF (is) to the (rectangle contained) by FE and EG [see previous lemma]. The (square) on EF [is] thus incommensurable with the (rectangle contained) by FE and EG [Prop. 10.11]. But, the (square) on CD is commensurable with the (square) on EF. For they are rational in square. And the (rectangle contained) by DC and CB is commensurable with the (rectangle contained) by FE and EG. For they are (both) equal to the (square) on A. Thus, the (square) on CD is also incommensurable with the (rectangle contained) by DC and CB [Prop. 10.13]. And as the (square) on CD (is) to the (rectangle contained) by DC and CB, so DC is to CB [see previous lemma]. Thus, DC is incommensurable in length with CB [Prop. 10.11]. Thus, CD is rational, and incommensurable in length with CB. (Which is) the very thing it was required to show. † Literally, "rational".

Proposition 23

A (straight-line) commensurable with a medial (straight-line) is medial.

Let A be a medial (straight-line), and let B be commensurable with A. I say that B is also a medial (staight-line).

Let the rational (straight-line) CD be set out, and let the rectangular area CE, equal to the (square) on A, have been applied to CD, producing ED as width. ED is thus rational, and incommensurable in length with CD [Prop. 10.22]. And let the rectangular area CF, equal to the (square) on B, have been applied to CD, producing DF as width. Therefore, since A is commensurable with B, the (square) on A is also commensurable with the (square) on B. But, EC is equal to the (square) on A, and CF is equal to the (square) on B. Thus, EC is commensurable with CF. And as EC is to CF, so ED (is) to DF [Prop. 6.1]. Thus, ED is commensurable in length with DF [Prop. 10.11]. And ED is rational, and incommensurable in length with CD. DF is thus also rational [Def. 10.3], and incommensurable in length with DC [Prop. 10.13]. Thus, CD and DF are rational, and commensurable in square only. And the square-root of a (rectangle contained) by rational (straight-lines which are) commensurable in square only is medial [Prop. 10.21]. Thus, the square-root of the (rectangle contained) by CD and DF is medial. And the square on B is equal to the (rectangle contained) by CD and DF. Thus, B is a medial (straight-line).



Corollary

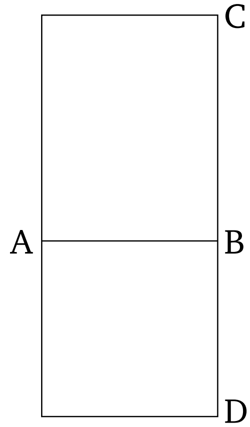
And (it is) clear, from this, that an (area) commensurable with a medial area† is medial. † A medial area is equal to the square on some medial straight-line. Hence, a medial area is expressible as $k^{1/2}$.

Proposition 24

A rectangle contained by medial straight-lines (which are) commensurable in length is medial.

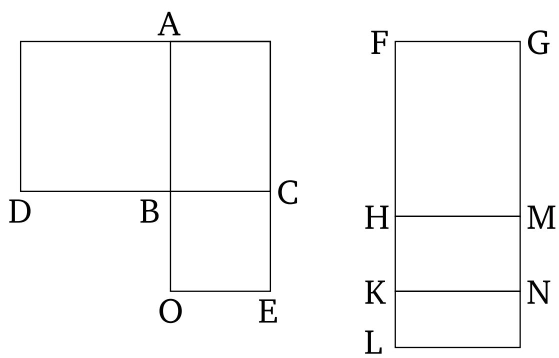
For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in length. I say that AC is medial.

For let the square AD have been described on AB. AD is thus medial [see previous footnote]. And since AB is commensurable in length with BC, and AB (is) equal to BD, DB is thus also commensurable in length with BC. Hence, DA is also commensurable with AC [Props. 6.1, 10.11]. And DA (is) medial. Thus, AC (is) also medial [Prop. 10.23 corr.]. (Which is) the very thing it was required to show.



Proposition 25

The rectangle contained by medial straight-lines (which are) commensurable in square only is either rational or medial.



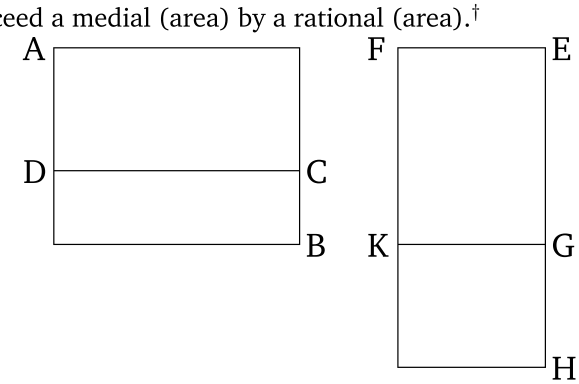
For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in square only. I say that AC is either rational or medial.

For let the squares AD and BE have been described on (the straight-lines) AB and BC (respectively). AD and BE are thus each medial. And let the rational (straight-line) FG be laid out. And let the rectangular parallelogram GH, equal to AD, have been applied to FG, producing FH as breadth. And let the rectangular parallelogram MK, equal to AC, have been applied to HM, producing HK as breadth. And, finally, let NL, equal to BE, have similarly been applied to KN, producing KL as breadth. Thus, FH, HK, and KL are in a straight-line. Therefore, since AD and BE are each medial, and AD is equal to GH, and BE to NL, GH and NL (are) thus each also medial. And they are applied to the rational (straight-line) FG. FH and KL are thus each rational, and incommensurable in length with FG [Prop. 10.22]. And since AD is commensurable with BE, GH is thus also commensurable with NL. And as GH is to NL, so FH (is) to KL [Prop. 6.1]. Thus, FH is commensurable in length with KL [Prop. 10.11]. Thus, FH and KL are rational (straight-lines which are) commensurable in length. Thus, the (rectangle contained) by FH and KL is rational [Prop. 10.19]. And since DB is equal to BA, and OB to BC, thus as DB is to BC, so AB (is) to BO. But, as DB (is) to BC, so DA (is) to AC [Props. 6.1]. And as AB (is) to BO, so AC (is) to CO [Prop. 6.1]. Thus, as DA is to AC, so AC (is) to CO. And AD is equal to GH, and AC to MK, and CO to NL. Thus, as GH is to MK, so MK (is) to NL. Thus, also, as FH is to HK, so HK (is) to KL [Props. 6.1, 5.11]. Thus, the (rectangle contained) by FH and KL is equal to the (square) on HK [Prop. 6.17]. And the (rectangle contained) by FH and KL (is) rational. Thus, the (square) on HK is also rational. Thus, HK is rational. And if it is commensurable in length with FG then HN is rational [Prop. 10.19]. And if it is incommensurable in length with FG then KH and HM are rational (straight-lines which are) commensurable in square only: thus, HN is medial [Prop. 10.21]. Thus, HN is either rational or medial. And HN (is) equal to AC. Thus, AC is either rational or medial.

Thus, the ... by medial straight-lines (which are) commensurable in square only, and so on ....

Proposition 26

A medial (area) does not exceed a medial (area) by a rational (area).†

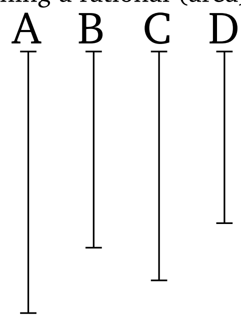


For, if possible, let the medial (area) AB exceed the medial (area) AC by the rational (area) DB. And let the rational (straight-line) EF be laid down. And let the rectangular parallelogram FH, equal to AB, have been applied to to EF, producing EH as breadth. And let FG, equal to AC, have been cut off (from FH). Thus, the remainder BD is equal to the remainder KH. And DB is rational. Thus, KH is also rational. Therefore, since AB and AC are each medial, and AB is equal to FH, and AC to FG, FH and FG are thus each also medial. And they are applied to the rational (straight-line) EF. Thus, HE and EG are each rational, and incommensurable in length with EF [Prop. 10.22]. And since DB is rational, and is equal to KH, KH is thus also rational. And (KH) is applied to the rational (straight-line) EF. GH is thus rational, and commensurable in length with EF [Prop. 10.20]. But, EG is also rational, and incommensurable in length with EF. Thus, EG is incommensurable in length with GH [Prop. 10.13]. And as EG is to GH, so the (square) on EG (is) to the (rectangle contained) by EG and GH [Prop. 10.13 lem.]. Thus, the (square) on EG is incommensurable with the (rectangle contained) by EG and GH [Prop. 10.11]. But, the (sum of the) squares on EG and GH is commensurable with the (square) on EG. For (EG and GH are) both rational. And twice the (rectangle contained) by EG and GH is commensurable with the (rectangle contained) by EG and GH [Prop. 10.6]. For (the former) is double the latter. Thus, the (sum of the squares) on EG and GH is incommensurable with twice the (rectangle contained) by EG and GH [Prop. 10.13]. And thus the sum of the (squares) on EG and GH plus twice the (rectangle contained) by EG and GH, that is the (square) on EH [Prop. 2.4], is incommensurable with the (sum of the squares) on EG and GH [Prop. 10.16]. And the (sum of the squares) on EG and GH (is) rational. Thus, the (square) on EH is irrational [Def. 10.4]. Thus, EH is irrational [Def. 10.4]. But, (it is) also rational. The very thing is impossible.

Thus, a medial (area) does not exceed a medial (area) by a rational (area). (Which is) the very thing it was required to show. † In other words, $\sqrt{k} - \sqrt{k'} \neq k''$.

Proposition 27

To find (two) medial (straight-lines), containing a rational (area), (which are) commensurable in square only.



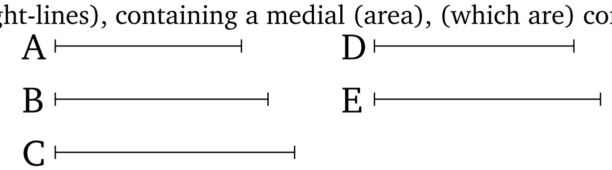
Let the two rational (straight-lines) A and B, (which are) commensurable in square only, be laid down. And let C—the mean proportional (straight-line) to A and B—have been taken [Prop. 6.13]. And let it be contrived that as A (is) to B, so C (is) to D [Prop. 6.12].

And since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on C [Prop. 6.17]—is thus medial [Prop 10.21]. Thus, C is medial [Prop. 10.21]. And since as A is to B, [so] C (is) to D, and A and B [are] commensurable in square only, C and D are thus also commensurable in square only [Prop. 10.11]. And C is medial. Thus, D is also medial [Prop. 10.23]. Thus, C and D are medial (straight-lines which are) commensurable in square only. I say that they also contain a rational (area). For since as A is to B, so C (is) to D, thus, alternately, as A is to C, so B (is) to D [Prop. 5.16]. But, as A (is) to C, (so) C (is) to B. And thus as C (is) to B, so B (is) to D [Prop. 5.11]. Thus, the (rectangle contained) by C and D is equal to the (square) on B [Prop. 6.17]. And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D [is] also rational.

Thus, (two) medial (straight-lines, C and D), containing a rational (area), (which are) commensurable in square only, have been found.† (Which is) the very thing it was required to show. † C and D have lengths $k^{1/4}$ and $k^{3/4}$ times that of A, respectively, where the length of B is $k^{1/2}$ times that of A.

Proposition 28

To find (two) medial (straight-lines), containing a medial (area), (which are) commensurable in square only.



Let the [three] rational (straight-lines) A, B, and C, (which are) commensurable in square only, be laid down. And let, D, the mean proportional (straight-line) to A and B, have been taken [Prop. 6.13]. And let it be contrived that as B (is) to C, (so) D (is) to E [Prop. 6.12].

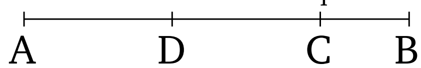
Since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B—that is to say, the (square) on D [Prop. 6.17]—is medial [Prop. 10.21]. Thus, D (is) medial [Prop. 10.21]. And since B and C are commensurable in square only, and as B is to C, (so) D (is) to E, D and E are thus commensurable in square only [Prop. 10.11]. And D (is) medial. E (is) thus also medial [Prop. 10.23]. Thus, D and E are medial (straight-lines which are) commensurable in square only. So, I say that they also enclose a medial (area). For since as B is to C, (so) D (is) to E, thus, alternately, as B (is) to D, (so) C (is) to E [Prop. 5.16]. And as B (is) to D, (so)

D (is) to A. And thus as D (is) to A, (so) C (is) to E. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by D and E [Prop. 6.16]. And the (rectangle contained) by A and C is medial [Prop. 10.21]. Thus, the (rectangle contained) by D and E (is) also medial.

Thus, (two) medial (straight-lines, D and E), containing a medial (area), (which are) commensurable in square only, have been found. (Which is) the very thing it was required to show. † D and E have lengths $k^{1/4}$ and $k'^{1/2}/k^{1/4}$ times that of A, respectively, where the lengths of B and C are $k^{1/2}$ and $k'^{1/2}$ times that of A, respectively.

Lemma I

To find two square numbers such that the sum of them is also square.

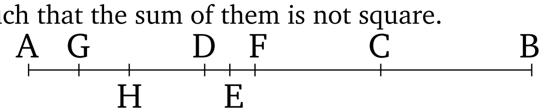


Let the two numbers AB and BC be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder AC is thus even. Let AC have been cut in half at D. And let AB and BC also be either similar plane (numbers), or square (numbers)—which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [Prop. 2.6]. And the (number created) from (multiplying) AB and BC is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1]. Thus, two square numbers have been found—(namely,) the (number created) from (multiplying) AB and BC, and the (square) on CD—which, (when) added (together), make the square on BD.

And (it is) clear that two square (numbers) have again been found—(namely,) the (square) on BD, and the (square) on CD—such that their difference—(namely,) the (rectangle) contained by AB and BC—is square whenever AB and BC are similar plane (numbers). But, when they are not similar plane numbers, two square (numbers) have been found—(namely,) the (square) on BD, and the (square) on DC—between which the difference—(namely,) the (rectangle) contained by AB and BC—is not square. (Which is) the very thing it was required to show.

Lemma II

To find two square numbers such that the sum of them is not square.



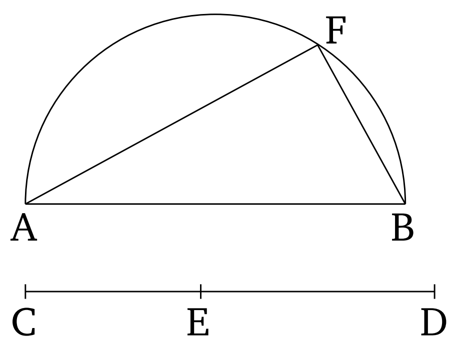
For let the (number created) from (multiplying) AB and BC, as we said, be square. And (let) CA (be) even. And let CA have been cut in half at D. So it is clear that the square (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [see previous lemma]. Let the unit DE have been subtracted (from BD). Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is less than the square on BD. I say, therefore, that the square (number created) from (multiplying) AB and BC, plus the (square) on CE, is not square.

For if it is square, it is either equal to the (square) on BE, or less than the (square) on BE, but cannot any more be greater (than the square on BE), lest the unit be divided. First of all, if possible, let the (number created) from (multiplying) AB and BC, plus the (square) on CE, be equal to the (square) on BE. And let GA be double the unit DE. Therefore, since the whole of AC is double the whole of CD, of which AG is double DE, the remainder GC is thus double the remainder EC. Thus, GC has been cut in half at E. Thus, the (number created) from (multiplying)

GB and BC, plus the (square) on CE, is equal to the square on BE [Prop. 2.6]. But, the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the square on BE. Thus, the (number created) from (multiplying) GB and BC, plus the (square) on CE, is equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE. And subtracting the (square) on CE from both, AB is inferred (to be) equal to GB. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to the (square) on BE. So I say that (it is) not less than the (square) on BE either. For, if possible, let it be equal to the (square) on BF. And (let) HA (be) double DF. And it can again be inferred that HC (is) double CF. Hence, CH has also been cut in half at F. And, on account of this, the (number created) from (multiplying) HB and BC, plus the (square) on FC, becomes equal to the (square) on BF [Prop. 2.6]. And the (number created) from (multiplying) AB and BC, plus the (square) on CE, was also assumed (to be) equal to the (square) on BF. Hence, the (number created) from (multiplying) HB and BC, plus the (square) on CF, will also be equal to the (number created) from (multiplying) AB and BC, plus the (square) on CE. The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC, plus the (square) on CE, is not equal to less than the (square) on BE. And it was shown that (is it) not equal to the (square) on BE either. Thus, the (number created) from (multiplying) AB and BC, plus the square on CE, is not square. (Which is) the very thing it was required to show.

Proposition 29

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line which is) commensurable in length with the greater.



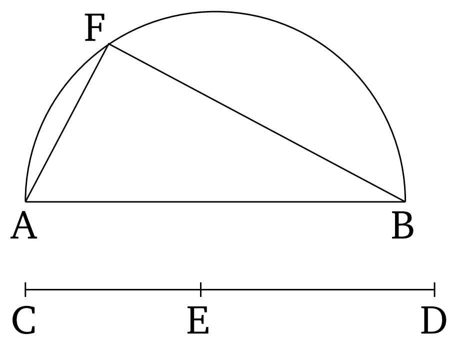
For let some rational (straight-line) AB be laid down, and two square numbers, CD and DE, such that the difference between them, CE, is not square [Prop. 10.28 lem. I]. And let the semi-circle AFB have been drawn on AB. And let it be contrived that as DC (is) to CE, so the square on BA (is) to the square on AF [Prop. 10.6 corr.]. And let FB have been joined.

[Therefore,] since as the (square) on BA is to the (square) on AF, so DC (is) to CE, the (square) on BA thus has to the (square) on AF the ratio which the number DC (has) to the number CE. Thus, the (square) on BA is commensurable with the (square) on AF [Prop. 10.6]. And the (square) on AB (is) rational [Def. 10.4]. Thus, the (square) on AF (is) also rational. Thus, AF (is) also rational. And since DC does not have to CE the ratio which (some) square number (has) to (some) square number, the (square) on BA thus does not have to the (square) on AF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with AF [Prop. 10.9]. Thus, the rational (straight-lines) BA and AF are commensurable in square only. And since as DC [is] to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on BF [Props. 5.19 corr., 3.31, 1.47]. And CD has to DE the ratio which (some) square number (has) to (some) square number. Thus, the (square) on AB also has to the (square) on BF the ratio which (some) square number has to (some) square number. AB is thus commensurable in length with BF [Prop. 10.9]. And the (square) on AB is equal to the (sum of the squares) on AF and FB [Prop. 1.47]. Thus, the square on AB is greater than (the square on) AF by (the square on) BF, (which is) commensurable (in length) with (AB).

Thus, two rational (straight-lines), BA and AF, commensurable in square only, have been found such that the square on the greater, AB, is larger than (the square on) the lesser, AF, by the (square) on BF, (which is) commensurable in length with (AB).† (Which is) the very thing it was required to show. † BA and AF have lengths 1 and $\sqrt{1-k^2}$ times that of AB, respectively, where $k = \sqrt{DE/CD}$.

Proposition 30

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (the square on) lesser by the (square) on (some straight-line which is) incommensurable in length with the greater.



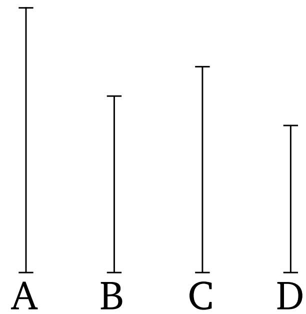
Let the rational (straight-line) AB be laid out, and the two square numbers, CE and ED, such that the sum of them, CD, is not square [Prop. 10.28 lem. II]. And let the semi-circle AFB have been drawn on AB. And let it be contrived that as DC (is) to CE, so the (square) on BA (is) to the (square) on AF [Prop. 10.6 corr]. And let FB have been joined.

So, similarly to the (proposition) before this, we can show that BA and AF are rational (straight-lines which are) commensurable in square only. And since as DC is to CE, so the (square) on BA (is) to the (square) on AF, thus, via conversion, as CD (is) to DE, so the (square) on AB (is) to the (square) on BF [Props. 5.19 corr., 3.31, 1.47]. And CD does not have to DE the ratio which (some) square number (has) to (some) square number. Thus, the (square) on AB does not have to the (square) on BF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with BF [Prop. 10.9]. And the square on AB is greater than the (square on) AF by the (square) on FB [Prop. 1.47], (which is) incommensurable (in length) with (AB).

Thus, AB and AF are rational (straight-lines which are) commensurable in square only, and the square on AB is greater than (the square on) AF by the (square) on FB, (which is) incommensurable (in length) with (AB).† (Which is) the very thing it was required to show. † AB and AF have lengths 1 and $1/\sqrt{1+k^2}$ times that of AB, respectively, where $k = \sqrt{DE/CE}$.

Proposition 31

To find two medial (straight-lines), commensurable in square only, (and) containing a rational (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable in length with the greater.



Let two rational (straight-lines), A and B, commensurable in square only, be laid out, such that the square on the greater A is larger than the (square on the) lesser B by the (square) on (some straight-line) commensurable in length with (A) [Prop. 10.29]. And let the (square) on C be equal to the (rectangle contained) by A and B. And the (rectangle contained by) A and B (is) medial [Prop. 10.21]. Thus, the (square) on C (is) also medial. Thus, C (is) also medial [Prop. 10.21]. And let the (rectangle contained) by C and D be equal to the (square) on B. And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D (is) also rational. And since as A is to B, so the (rectangle contained) by A and B (is) to the (square) on B [Prop. 10.21 lem.], but the (square) on C is equal to the (rectangle contained) by A and B, and the (rectangle contained) by C and D to the (square) on B, thus as A (is) to B, so the (square) on C (is) to the (rectangle contained) by C and D. And as the (square) on C (is) to the (rectangle contained) by C and D, so C (is) to D [Prop. 10.21 lem.]. And thus as A (is) to B, so C (is) to D. And A is commensurable in square only with B. Thus, C (is) also commensurable in square only with D [Prop. 10.11]. And C is medial. Thus, D (is) also medial [Prop. 10.23]. And since as A is to B, (so) C (is) to D, and the square on A is greater than (the square on) B by the (square) on (some straight-line) commensurable (in length) with (A), the square on C is thus also greater than (the square on) D by the (square) on (some straight-line) commensurable (in length) with (C) [Prop. 10.14].

Thus, two medial (straight-lines), C and D, commensurable in square only, (and) containing a rational (area), have been found. And the square on C is greater than (the square on) D by the (square) on (some straight-line) commensurable in length with (C).†

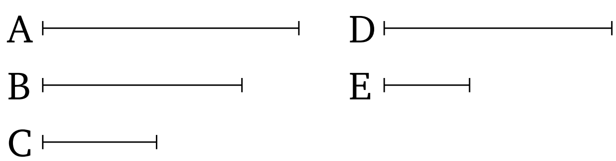
So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with C), provided that the square on A is greater than (the square on B) by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].‡

† C and D have lengths $(1-k^2)^{1/4}$ and $(1-k^2)^{3/4}$ times that of A, respectively, where k is defined in the footnote to Prop. 10.29.

‡ C and D would have lengths $1/(1+k^2)^{1/4}$ and $1/(1+k^2)^{3/4}$ times that of A, respectively, where k is defined in the footnote to Prop. 10.30.

Proposition 32

To find two medial (straight-lines), commensurable in square only, (and) containing a medial (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.



Let three rational (straight-lines), A, B and C, commensurable in square only, be laid out such that the square on A is greater than (the square on C) by the (square) on (some straight-line) commensurable (in length) with (A) [Prop. 10.29]. And let the (square) on D be equal to the (rectangle contained) by A and B. Thus, the (square) on

D (is) medial. Thus, D is also medial [Prop. 10.21]. And let the (rectangle contained) by D and E be equal to the (rectangle contained) by B and C. And since as the (rectangle contained) by A and B is to the (rectangle contained) by B and C, so A (is) to C [Prop. 10.21 lem.], but the (square) on D is equal to the (rectangle contained) by A and B, and the (rectangle contained) by D and E to the (rectangle contained) by B and C, thus as A is to C, so the (square) on D (is) to the (rectangle contained) by D and E. And as the (square) on D (is) to the (rectangle contained) by D and E, so D (is) to E [Prop. 10.21 lem.]. And thus as A (is) to C, so D (is) to E. And A (is) commensurable in square [only] with C. Thus, D (is) also commensurable in square only with E [Prop. 10.11]. And D (is) medial. Thus, E (is) also medial [Prop. 10.23]. And since as A is to C, (so) D (is) to E, and the square on A is greater than (the square on) C by the (square) on (some straight-line) commensurable (in length) with (A), the square on D will thus also be greater than (the square on) E by the (square) on (some straight-line) commensurable (in length) with (D) [Prop. 10.14]. So, I also say that the (rectangle contained) by D and E is medial. For since the (rectangle contained) by B and C is equal to the (rectangle contained) by D and E, and the (rectangle contained) by B and C (is) medial [for B and C are rational (straight-lines which are) commensurable in square only] [Prop. 10.21], the (rectangle contained) by D and E (is) thus also medial.

Thus, two medial (straight-lines), D and E, commensurable in square only, (and) containing a medial (area), have been found such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.†.

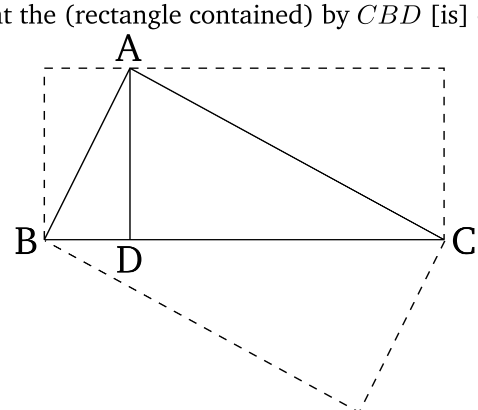
So, similarly, (the proposition) can again also be demonstrated for (some straight-line) incommensurable (in length with the greater), provided that the square on A is greater than (the square on) C by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].‡ † D and E have lengths $k'^{1/4}$ and $k'^{1/4}\sqrt{1-k^2}$ times that of A, respectively, where the length of B is $k'^{1/2}$ times that of A, and k is defined in the footnote to Prop. 10.29.

‡ D and E would have lengths $k'^{1/4}$ and $k'^{1/4}/\sqrt{1+k^2}$ times that of A, respectively, where the length of B is $k'^{1/2}$ times that of A, and k is defined in the footnote to Prop. 10.30.

Lemma

Let ABC be a right-angled triangle having the (angle) A a right-angle. And let the perpendicular AD have been drawn. I say that the (rectangle contained) by CBD is equal to the (square) on BA, and the (rectangle contained) by BCD (is) equal to the (square) on CA, and the (rectangle contained) by BD and DC (is) equal to the (square) on AD, and, further, the (rectangle contained) by BC and AD [is] equal to the (rectangle contained) by BA and AC.

And, first of all, (let us prove) that the (rectangle contained) by CBD [is] equal to the (square) on BA.



For since AD has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, ABD and ADC are thus triangles (which are) similar to the whole, ABC, and to one another [Prop. 6.8]. And since triangle

ABC is similar to triangle ABD, thus as CB is to BA, so BA (is) to BD [Prop. 6.4]. Thus, the (rectangle contained) by CBD is equal to the (square) on AB [Prop. 6.17].

So, for the same (reasons), the (rectangle contained) by BCD is also equal to the (square) on AC.

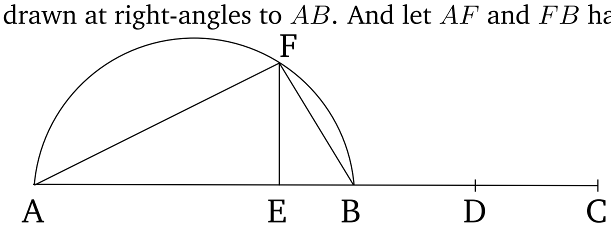
And since if a (straight-line) is drawn from the right-angle in a right-angled triangle, perpendicular to the base, the (straight-line so) drawn is the mean proportional to the pieces of the base [Prop. 6.8 corr.], thus as BD is to DA, so AD (is) to DC. Thus, the (rectangle contained) by BD and DC is equal to the (square) on DA [Prop. 6.17].

I also say that the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC. For since, as we said, ABC is similar to ABD, thus as BC is to CA, so BA (is) to AD [Prop. 6.4]. Thus, the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC [Prop. 6.16]. (Which is) the very thing it was required to show.

Proposition 33

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Let the two rational (straight-lines) AB and BC, (which are) commensurable in square only, be laid out such that the square on the greater, AB, is larger than (the square on) the lesser, BC, by the (square) on (some straight-line which is) incommensurable (in length) with (AB) [Prop. 10.30]. And let BC have been cut in half at D. And let a parallelogram equal to the (square) on either of BD or DC, (and) falling short by a square figure, have been applied to AB [Prop. 6.28], and let it be the (rectangle contained) by AEB. And let the semi-circle AFB have been drawn on AB. And let EF have been drawn at right-angles to AB. And let AF and FB have been joined.

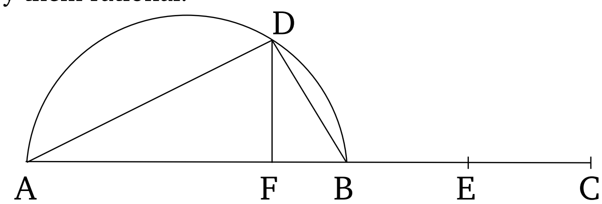


And since AB and BC are [two] unequal straight-lines, and the square on AB is greater than (the square on) BC by the (square) on (some straight-line which is) incommensurable (in length) with (AB). And a parallelogram, equal to one quarter of the (square) on BC—that is to say, (equal) to the (square) on half of it—(and) falling short by a square figure, has been applied to AB, and makes the (rectangle contained) by AEB. AE is thus incommensurable (in length) with EB [Prop. 10.18]. And as AE is to EB, so the (rectangle contained) by BA and AE (is) to the (rectangle contained) by AB and BE. And the (rectangle contained) by BA and AE (is) equal to the (square) on AF, and the (rectangle contained) by AB and BE to the (square) on BF [Prop. 10.32 lem.]. The (square) on AF is thus incommensurable with the (square) on FB [Prop. 10.11]. Thus, AF and FB are incommensurable in square. And since AB is rational, the (square) on AB is also rational. Hence, the sum of the (squares) on AF and FB is also rational [Prop. 1.47]. And, again, since the (rectangle contained) by AE and EB is equal to the (square) on EF, and the (rectangle contained) by AE and EB was assumed (to be) equal to the (square) on BD, FE is thus equal to BD. Thus, BC is double FE. And hence the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and EF [Prop. 10.6]. And the (rectangle contained) by AB and BC (is) medial [Prop. 10.21]. Thus, the (rectangle contained) by AB and EF (is) also medial [Prop. 10.23 corr.]. And the (rectangle contained) by AB and EF (is) equal to the (rectangle contained) by AF and FB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AF and FB (is) also medial. And the sum of the squares on them was also shown (to be) rational.

Thus, the two straight-lines, AF and FB, (which are) incommensurable in square, have been found, making the sum of the squares on them rational, and the (rectangle contained) by them medial. (Which is) the very thing it was required to show. † AF and FB have lengths $\sqrt{[1+k/(1+k^2)^{1/2}]/2}$ and $\sqrt{[1-k/(1+k^2)^{1/2}]/2}$ times that of AB, respectively, where k is defined in the footnote to Prop. 10.30.

Proposition 34

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.



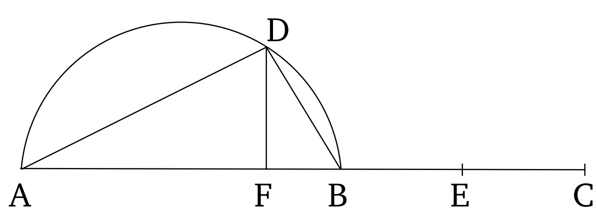
Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out having the (rectangle contained) by them rational, (and) such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.31]. And let the semi-circle ADB have been drawn on AB. And let BC have been cut in half at E. And let a (rectangular) parallelogram equal to the (square) on BE, (and) falling short by a square figure, have been applied to AB, (and let it be) the (rectangle contained by) AFB [Prop. 6.28]. Thus, AF [is] incommensurable in length with FB [Prop. 10.18]. And let FD have been drawn from F at right-angles to AB. And let AD and DB have been joined.

Since AF is incommensurable (in length) with FB, the (rectangle contained) by BA and AF is thus also incommensurable with the (rectangle contained) by AB and BF [Prop. 10.11]. And the (rectangle contained) by BA and AF (is) equal to the (square) on AD, and the (rectangle contained) by AB and BF to the (square) on DB [Prop. 10.32 lem.]. Thus, the (square) on AD is also incommensurable with the (square) on DB. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since BC is double DF [see previous proposition], the (rectangle contained) by AB and BC (is) thus also double the (rectangle contained) by AB and FD. And the (rectangle contained) by AB and BC (is) rational. Thus, the (rectangle contained) by AB and FD (is) also rational [Prop. 10.6, Def. 10.4]. And the (rectangle contained) by AB and FD (is) equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. And hence the (rectangle contained) by AD and DB is rational.

Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the squares on them medial, and the (rectangle contained) by them rational.† (Which is) the very thing it was required to show. † AD and DB have lengths $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]}$ and $\sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]}$ times that of AB, respectively, where k is defined in the footnote to Prop. 10.29.

Proposition 35

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.



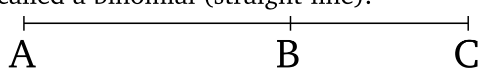
Let the two medial (straight-lines) AB and BC, (which are) commensurable in square only, be laid out containing a medial (area), such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.32]. And let the semi-circle ADB have been drawn on AB. And let the remainder (of the figure) be generated similarly to the above (proposition).

And since AF is incommensurable in length with FB [Prop. 10.18], AD is also incommensurable in square with DB [Prop. 10.11]. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since the (rectangle contained) by AF and FB is equal to the (square) on each of BE and DF, BE is thus equal to DF. Thus, BC (is) double FD. And hence the (rectangle contained) by AB and BC is double the (rectangle) contained by AB and FD. And the (rectangle contained) by AB and BC (is) medial. Thus, the (rectangle contained) by AB and FD (is) also medial. And it is equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AD and DB (is) also medial. And since AB is incommensurable in length with BC, and CB (is) commensurable (in length) with BE, AB (is) thus also incommensurable in length with BE [Prop. 10.13]. And hence the (square) on AB is also incommensurable with the (rectangle contained) by AB and BE [Prop. 10.11]. But the (sum of the squares) on AD and DB is equal to the (square) on AB [Prop. 1.47]. And the (rectangle contained) by AB and FD—that is to say, the (rectangle contained) by AD and DB—is equal to the (rectangle contained) by AB and BE. Thus, the sum of the (squares) on AD and DB is incommensurable with the (rectangle contained) by AD and DB.

Thus, two straight-lines, AD and DB, (which are) incommensurable in square, have been found, making the sum of the (squares) on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.† (Which is) the very thing it was required to show. † AD and DB have lengths $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2}$ and $k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2}$ times that of AB, respectively, where k and k' are defined in the footnote to Prop. 10.32.

Proposition 36

If two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is irrational—let it be called a binomial (straight-line).†



For let the two rational (straight-lines), AB and BC, (which are) commensurable in square only, be laid down together. I say that the whole (straight-line), AC, is irrational. For since AB is incommensurable in length with BC—for they are commensurable in square only—and as AB (is) to BC, so the (rectangle contained) by ABC (is) to the (square) on BC, the (rectangle contained) by AB and BC is thus incommensurable with the (square) on BC [Prop. 10.11]. But, twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. And (the sum of) the (squares) on AB and BC is commensurable with the (square) on BC—for the rational (straight-lines) AB and BC are commensurable in square only [Prop. 10.15]. Thus, twice the (rectangle contained) by AB and BC is incommensurable with (the sum of) the (squares) on AB and BC [Prop. 10.13]. And, via composition, twice the (rectangle contained) by AB and BC, plus (the sum of) the (squares) on AB and BC—that is to say, the (square) on AC [Prop. 2.4]—is incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16]. And the sum of the (squares) on AB and BC (is) rational. Thus, the (square) on AC [is] irrational [Def. 10.4]. Hence, AC is also irrational [Def. 10.4]—let it be called a binomial (straight-line).‡ (Which is) the very thing it was required to show. † Literally, "from two names".

‡ Thus, a binomial straight-line has a length expressible as $1 + k^{1/2}$ [or, more generally, $\rho\,(1 + k^{1/2})$, where ρ is rational—the same proviso applies to the definitions in the following propositions]. The binomial and the corresponding apotome, whose length is expressible as $1 - k^{1/2}$ (see Prop. 10.73), are the positive roots of the quartic $x^4 - 2\,(1+k)\,x^2 + (1-k)^2 = 0$.

Proposition 37

If two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together then the whole (straight-line) is irrational—let it be called a first bimedial (straight-line).[†]

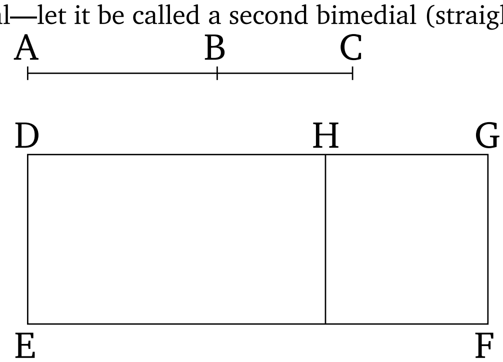
For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a rational (area), be laid down together. I say that the whole (straight-line), AC, is irrational.

For since AB is incommensurable in length with BC, (the sum of) the (squares) on AB and BC is thus also incommensurable with twice the (rectangle contained) by AB and BC [see previous proposition]. And, via composition, (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC—that is, the (square) on AC [Prop. 2.4]—is incommensurable with the (rectangle contained) by AB and BC [Prop. 10.16]. And the (rectangle contained) by AB and BC (is) rational—for AB and BC were assumed to enclose a rational (area). Thus, the (square) on AC (is) irrational. Thus, AC (is) irrational [Def. 10.4]—let it be called a first bimedial (straight-line).[‡] (Which is) the very thing it was required to show. [†] Literally, "first from two medials".

[‡] Thus, a first bimedial straight-line has a length expressible as $k^{1/4} + k^{3/4}$. The first bimedial and the corresponding first apotome of a medial, whose length is expressible as $k^{1/4} - k^{3/4}$ (see Prop. 10.74), are the positive roots of the quartic $x^4 - 2\sqrt{k}\,(1+k)\,x^2 + k\,(1-k)^2 = 0$.

Proposition 38

If two medial (straight-lines), commensurable in square only, which contain a medial (area), are added together then the whole (straight-line) is irrational—let it be called a second bimedial (straight-line).



For let the two medial (straight-lines), AB and BC, commensurable in square only, (and) containing a medial (area), be laid down together [Prop. 10.28]. I say that AC is irrational.

For let the rational (straight-line) DE be laid down, and let (the rectangle) DF, equal to the (square) on AC, have been applied to DE, making DG as breadth [Prop. 1.44]. And since the (square) on AC is equal to (the sum of) the (squares) on AB and BC, plus twice the (rectangle contained) by AB and BC [Prop. 2.4], so let (the rectangle) EH, equal to (the sum of) the squares on AB and BC, have been applied to DE. The remainder HF is thus equal to twice the (rectangle contained) by AB and BC. And since AB and BC are each medial, (the sum of) the squares on AB and BC is thus also medial.[‡] And twice the (rectangle contained) by AB and BC was also assumed (to be) medial. And EH is equal to (the sum of) the squares on AB and BC, and FH (is) equal to twice the (rectangle contained) by AB and BC. Thus, EH and HF (are) each medial. And they were applied to the rational (straight-line) DE. Thus, DH and HG are each rational, and incommensurable in length with DE [Prop. 10.22]. Therefore, since AB is incommensurable in length with BC, and as AB is to BC, so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11]. But, the sum of the squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. Thus, the sum of the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC [Prop. 10.13]. But, EH is equal to (the sum of) the squares on AB and BC, and HF is equal to twice the (rectangle) contained by AB and BC. Thus, EH is incommensurable with HF. Hence, DH is also incommensurable in length with HG [Props. 6.1, 10.11]. Thus, DH and HG are rational (straight-lines which are) commensurable in square only. Hence, DG is irrational [Prop. 10.36]. And DE (is) rational. And the rectangle contained by irrational and rational (straight-lines) is irrational [Prop. 10.20]. The area DF is thus irrational, and (so) the square-root [of it] is irrational [Def. 10.4]. And AC is the square-root of DF. AC is thus irrational—let it be called a second bimedial (straight-line).[§] (Which is) the very thing it was required to show. [†] Literally, "second from two medials".

[‡] Since, by hypothesis, the squares on AB and BC are commensurable—see Props. 10.15, 10.23.

[§] Thus, a second bimedial straight-line has a length expressible as $k^{1/4}+k'^{1/2}/k^{1/4}$. The second bimedial and the corresponding second apotome of a medial, whose length is expressible as $k^{1/4}-k'^{1/2}/k^{1/4}$ (see Prop. 10.75), are the positive roots of the quartic $x^4-2\,[(k+k')/\sqrt{k}\,]\,x^2+[(k-k')^2/k]=0$.

Proposition 39

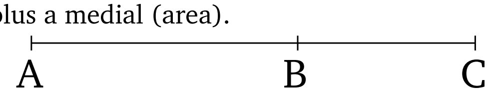
If two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together then the whole straight-line is irrational—let it be called a major (straight-line).

For let the two straight-lines, AB and BC, incommensurable in square, and fulfilling the prescribed (conditions), be laid down together [Prop. 10.33]. I say that AC is irrational.

For since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BC is [thus] also medial [Props. 10.6, 10.23 corr.]. And the sum of the (squares) on AB and BC (is) rational. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the sum of the (squares) on AB and BC [Def. 10.4]. Hence, (the sum of) the squares on AB and BC, plus twice the (rectangle contained) by AB and BC—that is, the (square) on AC [Prop. 2.4]—is also incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16] [and the sum of the (squares) on AB and BC (is) rational]. Thus, the (square) on AC is irrational. Hence, AC is also irrational [Def. 10.4]—let it be called a major (straight-line).[†] (Which is) the very thing it was required to show. [†] Thus, a major straight-line has a length expressible as $\sqrt{[1+k/(1+k^2)^{1/2}]/2}+\sqrt{[1-k/(1+k^2)^{1/2}]/2}$. The major and the corresponding minor, whose length is expressible as $\sqrt{[1+k/(1+k^2)^{1/2}]/2}-\sqrt{[1-k/(1+k^2)^{1/2}]/2}$ (see Prop. 10.76), are the positive roots of the quartic $x^4-2\,x^2+k^2/(1+k^2)=0$.

Proposition 40

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational, are added together then the whole straight-line is irrational—let it be called the square-root of a rational plus a medial (area).



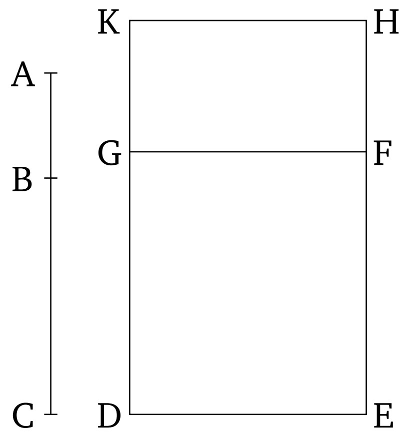
For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.34]. I say that AC is irrational.

For since the sum of the (squares) on AB and BC is medial, and twice the (rectangle contained) by AB and BC (is) rational, the sum of the (squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC. Hence, the (square) on AC is also incommensurable with twice the (rectangle contained) by AB

and BC [Prop. 10.16]. And twice the (rectangle contained) by AB and BC (is) rational. The (square) on AC (is) thus irrational. Thus, AC (is) irrational [Def. 10.4]—let it be called the square-root of a rational plus a medial (area).† (Which is) the very thing it was required to show. † Thus, the square-root of a rational plus a medial (area) has a length expressible as $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]}+\sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]}$. This and the corresponding irrational with a minus sign, whose length is expressible as $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]}-\sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]}$ (see Prop. 10.77), are the positive roots of the quartic $x^4-(2/\sqrt{1+k^2})\,x^2+k^2/(1+k^2)^2=0$.

Proposition 41

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them, are added together then the whole straight-line is irrational—let it be called the square-root of (the sum of) two medial (areas).

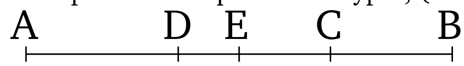


For let the two straight-lines, AB and BC, incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.35]. I say that AC is irrational.

Let the rational (straight-line) DE be laid out, and let (the rectangle) DF, equal to (the sum of) the (squares) on AB and BC, and (the rectangle) GH, equal to twice the (rectangle contained) by AB and BC, have been applied to DE. Thus, the whole of DH is equal to the square on AC [Prop. 2.4]. And since the sum of the (squares) on AB and BC is medial, and is equal to DF, DF is thus also medial. And it is applied to the rational (straight-line) DE. Thus, DG is rational, and incommensurable in length with DE [Prop. 10.22]. So, for the same (reasons), GK is also rational, and incommensurable in length with GF—that is to say, DE. And since (the sum of) the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC, DF is incommensurable with GH. Hence, DG is also incommensurable (in length) with GK [Props. 6.1, 10.11]. And they are rational. Thus, DG and GK are rational (straight-lines which are) commensurable in square only. Thus, DK is irrational, and that (straight-line which is) called binomial [Prop. 10.36]. And DE (is) rational. Thus, DH is irrational, and its square-root is irrational [Def. 10.4]. And AC (is) the square-root of HD. Thus, AC is irrational—let it be called the square-root of (the sum of) two medial (areas).† (Which is) the very thing it was required to show. † Thus, the square-root of (the sum of) two medial (areas) has a length expressible as $k'^{1/4}\left(\sqrt{[1+k/(1+k^2)^{1/2}]/2}+\sqrt{[1-k/(1+k^2)^{1/2}]/2}\right)$. This and the corresponding irrational with a minus sign, whose length is expressible as $k'^{1/4}\left(\sqrt{[1+k/(1+k^2)^{1/2}]/2}-\sqrt{[1-k/(1+k^2)^{1/2}]/2}\right)$ (see Prop. 10.78), are the positive roots of the quartic $x^4-2\,k'^{1/2}\,x^2+k'\,k^2/(1+k^2)=0$.

Lemma

We will now demonstrate that the aforementioned irrational (straight-lines) are uniquely divided into the straight-lines of which they are the sum, and which produce the prescribed types, (after) setting forth the following lemma.

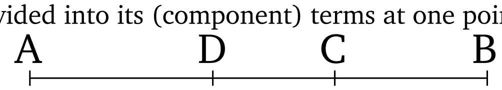


Let the straight-line AB be laid out, and let the whole (straight-line) have been cut into unequal parts at each of the (points) C and D. And let AC be assumed (to be) greater than DB. I say that (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB.

For let AB have been cut in half at E. And since AC is greater than DB, let DC have been subtracted from both. Thus, the remainder AD is greater than the remainder CB. And AE (is) equal to EB. Thus, DE (is) less than EC. Thus, points C and D are not equally far from the point of bisection. And since the (rectangle contained) by AC and CB, plus the (square) on EC, is equal to the (square) on EB [Prop. 2.5], but, moreover, the (rectangle contained) by AD and DB, plus the (square) on DE, is also equal to the (square) on EB [Prop. 2.5], the (rectangle contained) by AC and CB, plus the (square) on EC, is thus equal to the (rectangle contained) by AD and DB, plus the (square) on DE. And, of these, the (square) on DE is less than the (square) on EC. And, thus, the remaining (rectangle contained) by AC and CB is less than the (rectangle contained) by AD and DB. And, hence, twice the (rectangle contained) by AC and CB is less than twice the (rectangle contained) by AD and DB. And thus the remaining sum of the (squares) on AC and CB is greater than the sum of the (squares) on AD and DB.† (Which is) the very thing it was required to show. † Since, $AC^2 + CB^2 + 2\,AC\,CB = AD^2 + DB^2 + 2\,AD\,DB = AB^2$.

Proposition 42

A binomial (straight-line) can be divided into its (component) terms at one point only.†



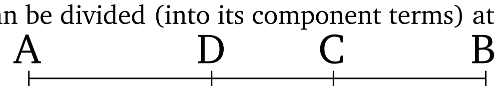
Let AB be a binomial (straight-line) which has been divided into its (component) terms at C. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. I say that AB cannot be divided at another point into two rational (straight-lines which are) commensurable in square only.

For, if possible, let it also have been divided at D, such that AD and DB are also rational (straight-lines which are) commensurable in square only. So, (it is) clear that AC is not the same as DB. For, if possible, let it be (the same). So, AD will also be the same as CB. And as AC will be to CB, so BD (will be) to DA. And AB will (thus) also be divided at D in the same (manner) as the division at C. The very opposite was assumed. Thus, AC is not the same as DB. So, on account of this, points C and D are not equally far from the point of bisection. Thus, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount)—on account of both (the sum of) the (squares) on AC and CB, plus twice the (rectangle contained) by AC and CB, and (the sum of) the (squares) on AD and DB, plus twice the (rectangle contained) by AD and DB, being equal to the (square) on AB [Prop. 2.4]. But, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by a rational (area, despite both) being medial (areas) [Prop. 10.21]. The very thing is absurd. For a medial (area) cannot exceed a medial (area) by a rational (area) [Prop. 10.26].

Thus, a binomial (straight-line) cannot be divided (into its component terms) at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show. † In other words, $k + k'^{1/2} = k'' + k'''^{1/2}$ has only one solution: *i.e.*, $k'' = k$ and $k''' = k'$. Likewise, $k^{1/2} + k'^{1/2} = k''^{1/2} + k'''^{1/2}$ has only one solution: *i.e.*, $k'' = k$ and $k''' = k'$ (or, equivalently, $k'' = k'$ and $k''' = k$).

Proposition 43

A first bimedial (straight-line) can be divided (into its component terms) at one point only.†



Let AB be a first bimedial (straight-line) which has been divided at C, such that AC and CB are medial (straight-lines), commensurable in square only, (and) containing a rational (area) [Prop. 10.37]. I say that AB cannot be (so) divided at another point.

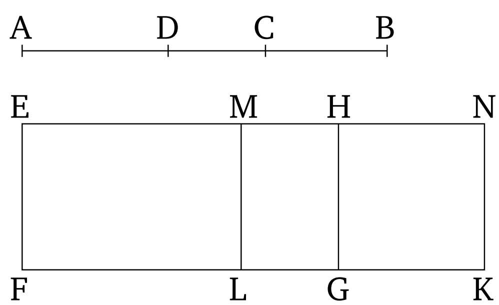
For, if possible, let it also have been divided at D, such that AD and DB are also medial (straight-lines), commensurable in square only, (and) containing a rational (area). Since, therefore, by whatever (amount) twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by this (same amount) [Prop. 10.41 lem.]. And twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB by a rational (area). For (they are) both rational (areas). (The sum of) the (squares) on AC and CB thus differs from (the sum of) the (squares) on AD and DB by a rational (area, despite both) being medial (areas). The very thing is absurd [Prop. 10.26].

Thus, a first bimedial (straight-line) cannot be divided into its (component) terms at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show. † In other words, $k^{1/4} + k^{3/4} = k'^{1/4} + k'^{3/4}$ has only one solution: *i.e.*, $k' = k$.

Proposition 44

A second bimedial (straight-line) can be divided (into its component terms) at one point only.†

Let AB be a second bimedial (straight-line) which has been divided at C, so that AC and BC are medial (straight-lines), commensurable in square only, (and) containing a medial (area) [Prop. 10.38]. So, (it is) clear that C is not (located) at the point of bisection, since (AC and BC) are not commensurable in length. I say that AB cannot be (so) divided at another point.

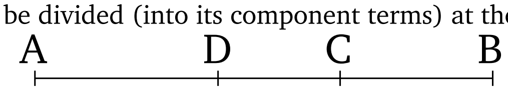


For, if possible, let it also have been (so) divided at D, so that AC is not the same as DB, but AC (is), by hypothesis, greater. So, (it is) clear that (the sum of) the (squares) on AD and DB is also less than (the sum of) the (squares) on AC and CB, as we showed above [Prop. 10.41 lem.]. And AD and DB are medial (straight-lines), commensurable in square only, (and) containing a medial (area). And let the rational (straight-line) EF be laid down. And let the rectangular parallelogram EK, equal to the (square) on AB, have been applied to EF. And let EG, equal to (the sum of) the (squares) on AC and CB, have been cut off (from EK). Thus, the remainder, HK, is equal to twice the (rectangle contained) by AC and CB [Prop. 2.4]. So, again, let EL, equal to (the sum of) the (squares) on AD and DB—which was shown (to be) less than (the sum of) the (squares) on AC and CB—have been cut off (from EK). And, thus, the remainder, MK, (is) equal to twice the (rectangle contained) by AD and

DB. And since (the sum of) the (squares) on AC and CB is medial, EG (is) thus [also] medial. And it is applied to the rational (straight-line) EF. Thus, EH is rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF. And since AC and CB are medial (straight-lines which are) commensurable in square only, AC is thus incommensurable in length with CB. And as AC (is) to CB, so the (square) on AC (is) to the (rectangle contained) by AC and CB [Prop. 10.21 lem.]. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11]. But, (the sum of) the (squares) on AC and CB is commensurable with the (square) on AC. For, AC and CB are commensurable in square [Prop. 10.15]. And twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6]. And thus (the sum of) the squares on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13]. But, EG is equal to (the sum of) the (squares) on AC and CB, and HK equal to twice the (rectangle contained) by AC and CB. Thus, EG is incommensurable with HK. Hence, EH is also incommensurable in length with HN [Props. 6.1, 10.11]. And (they are) rational (straight-lines). Thus, EH and HN are rational (straight-lines which are) commensurable in square only. And if two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is that irrational called binomial [Prop. 10.36]. Thus, EN is a binomial (straight-line) which has been divided (into its component terms) at H. So, according to the same (reasoning), EM and MN can be shown (to be) rational (straight-lines which are) commensurable in square only. And EN will (thus) be a binomial (straight-line) which has been divided (into its component terms) at the different (points) H and M (which is absurd [Prop. 10.42]). And EH is not the same as MN, since (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB. But, (the sum of) the (squares) on AD and DB is greater than twice the (rectangle contained) by AD and DB [Prop. 10.59 lem.]. Thus, (the sum of) the (squares) on AC and CB—that is to say, EG—is also much greater than twice the (rectangle contained) by AD and DB—that is to say, MK. Hence, EH is also greater than MN [Prop. 6.1]. Thus, EH is not the same as MN. (Which is) the very thing it was required to show. † In other words, $k^{1/4} + k'^{1/2}/k^{1/4} = k''^{1/4} + k'''^{1/2}/k''^{1/4}$ has only one solution: *i.e.*, $k'' = k$ and $k''' = k'$.

Proposition 45

A major (straight-line) can only be divided (into its component terms) at the same point.†



Let AB be a major (straight-line) which has been divided at C, so that AC and CB are incommensurable in square, making the sum of the squares on AC and CB rational, and the (rectangle contained) by AC and CD medial [Prop. 10.39]. I say that AB cannot be (so) divided at another point.

For, if possible, let it also have been divided at D, such that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB rational, and the (rectangle contained) by them medial. And since, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount). But, (the sum of) the (squares) on AC and CB exceeds (the sum of) the (squares) on AD and DB by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, a major (straight-line) cannot be divided (into its component terms) at different points. Thus, it can only be (so) divided at the same (point). (Which is) the very thing it was required to show. † In other words, $\sqrt{[1+k/(1+k^2)^{1/2}]/2} + \sqrt{[1-k/(1+k^2)^{1/2}]/2} = \sqrt{[1+k'/(1+k'^2)^{1/2}]/2} + \sqrt{[1-k'/(1+k'^2)^{1/2}]/2}$ has only one solution: *i.e.*, $k' = k$.

Proposition 46

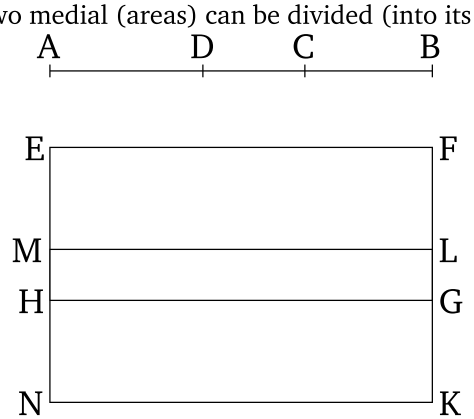
The square-root of a rational plus a medial (area) can be divided (into its component terms) at one point only.†

Let AB be the square-root of a rational plus a medial (area) which has been divided at C, so that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and twice the (rectangle contained) by AC and CB rational [Prop. 10.40]. I say that AB cannot be (so) divided at another point.

For, if possible, let it also have been divided at D, so that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB medial, and twice the (rectangle contained) by AD and DB rational. Therefore, since by whatever (amount) twice the (rectangle contained) by AC and CB differs from twice the (rectangle contained) by AD and DB, (the sum of) the (squares) on AD and DB also differs from (the sum of) the (squares) on AC and CB by this (same amount). And twice the (rectangle contained) by AC and CB exceeds twice the (rectangle contained) by AD and DB by a rational (area). (The sum of) the (squares) on AD and DB thus also exceeds (the sum of) the (squares) on AC and CB by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, the square-root of a rational plus a medial (area) cannot be divided (into its component terms) at different points. Thus, it can be (so) divided at one point (only). (Which is) the very thing it was required to show. † In other words, $\sqrt{[(1+k^2)^{1/2}+k]/[2\,(1+k^2)]}+\sqrt{[(1+k^2)^{1/2}-k]/[2\,(1+k^2)]} = \sqrt{[(1+k'^2)^{1/2}+k']/[2\,(1+k'^2)]}$
$+\sqrt{[(1+k'^2)^{1/2}-k']/[2\,(1+k'^2)]}$ has only one solution: *i.e.*, $k'=k$.

Proposition 47

The square-root of (the sum of) two medial (areas) can be divided (into its component terms) at one point only.†



Let AB be [the square-root of (the sum of) two medial (areas)] which has been divided at C, such that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and the (rectangle contained) by AC and CB medial, and, moreover, incommensurable with the sum of the (squares) on (AC and CB) [Prop. 10.41]. I say that AB cannot be divided at another point fulfilling the prescribed (conditions).

For, if possible, let it have been divided at D, such that AC is again manifestly not the same as DB, but AC (is), by hypothesis, greater. And let the rational (straight-line) EF be laid down. And let EG, equal to (the sum of) the (squares) on AC and CB, and HK, equal to twice the (rectangle contained) by AC and CB, have been applied to EF. Thus, the whole of EK is equal to the square on AB [Prop. 2.4]. So, again, let EL, equal to (the sum of) the (squares) on AD and DB, have been applied to EF. Thus, the remainder—twice the (rectangle contained) by AD and DB—is equal to the remainder, MK. And since the sum of the (squares) on AC and CB was assumed (to be) medial, EG is also medial. And it is applied to the rational (straight-line) EF. HE is thus rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is thus also incommensurable with GN. Hence, EH is also incommensurable with HN [Props. 6.1, 10.11]. And they are (both) rational (straight-lines). Thus, EH and HN are rational (straight-lines which are) commensurable in square only. Thus, EN is a binomial (straight-line) which has been divided (into its component terms) at H [Prop. 10.36]. So, similarly, we can show that it has also been (so) divided at M. And EH is not the same as MN. Thus, a binomial (straight-line) has been divided (into its component terms) at different points. The very thing is absurd [Prop. 10.42]. Thus, the square-root of (the sum of) two medial (areas) cannot be divided (into its component terms) at different points. Thus, it can be (so) divided at one [point] only. †

† In other words, $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2}+k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2} = k'''^{1/4}\sqrt{[1+k''/(1+k''^2)^{1/2}]/2}$ $+k'''^{1/4}\sqrt{[1-k''/(1+k''^2)^{1/2}]/2}$ has only one solution: *i.e.*, $k''=k$ and $k'''=k'$.

Definitions II

5. Given a rational (straight-line), and a binomial (straight-line) which has been divided into its (component) terms, of which the square on the greater term is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let [the whole] (straight-line) be called a first binomial (straight-line).

6. And if the lesser term is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a second binomial (straight-line).

7. And if neither of the terms is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a third binomial (straight-line).

8. So, again, if the square on the greater term is larger than (the square on) [the lesser] by the (square) on (some straight-line) incommensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let (the whole straight-line) be called a fourth binomial (straight-line).

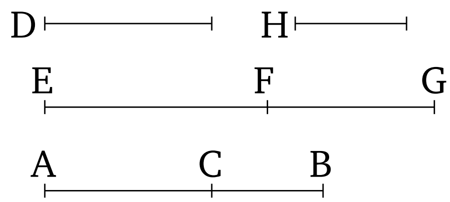
9. And if the lesser (term is commensurable), a fifth (binomial straight-line).

10. And if neither (term is commensurable), a sixth (binomial straight-line).

Proposition 48

To find a first binomial (straight-line).

Let two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to CA the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let some rational (straight-line) D be laid down. And let EF be commensurable in length with D. EF is thus also rational [Def. 10.3]. And let it have been contrived that as the number BA (is) to AC, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And AB has to AC the ratio which (some) number (has) to (some) number. Thus, the (square) on EF also has to the (square) on FG the ratio which (some) number (has) to (some) number. Hence, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. And EF is rational. Thus, FG (is) also rational. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, thus the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop 10.9]. EF and FG are thus rational (straight-lines which are) commensurable in square only. Thus, EG is a binomial (straight-line) [Prop. 10.36]. I say that (it is) also a first (binomial straight-line).

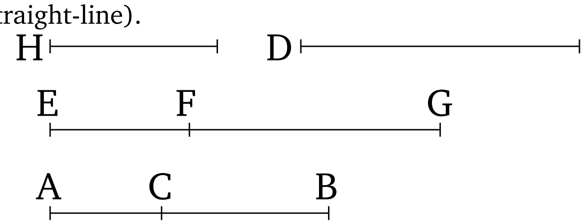


For since as the number BA is to AC, so the (square) on EF (is) to the (square) on FG, and BA (is) greater than AC, the (square) on EF (is) thus also greater than the (square) on FG [Prop. 5.14]. Therefore, let (the sum of) the (squares) on FG and H be equal to the (square) on EF. And since as BA is to AC, so the (square) on EF (is) to the (square) on FG, thus, via conversion, as AB is to BC, so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.]. And AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, EF is commensurable in length with H [Prop. 10.9]. Thus, the square on EF is greater than (the square on) FG by the (square) on (some straight-line) commensurable (in length) with (EF). And EF and FG are rational (straight-lines). And EF (is) commensurable in length with D.

Thus, EG is a first binomial (straight-line) [Def. 10.5].† (Which is) the very thing it was required to show. †If the rational straight-line has unit length then the length of a first binomial straight-line is $k + k\sqrt{1-k'^{\,2}}$. This, and the first apotome, whose length is $k - k\sqrt{1-k'^{\,2}}$ [Prop. 10.85], are the roots of $x^2 - 2\,k\,x + k^2\,k'^{\,2} = 0$.

Proposition 49

To find a second binomial (straight-line).



Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) D be laid down. And let EF be commensurable in length with D. EF is thus a rational (straight-line). So, let it also have been contrived that as the number CA (is) to AB, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. Thus, FG is also a rational (straight-line). And since the number CA does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. EF and FG are thus rational (straight-lines which are) commensurable in square only. Thus, EG is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).

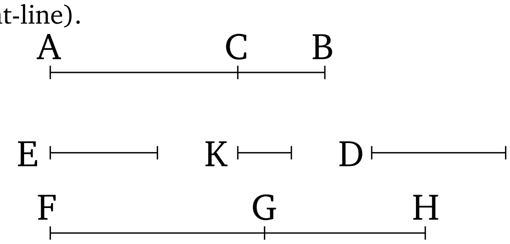
For since, inversely, as the number BA is to AC, so the (square) on GF (is) to the (square) on FE [Prop. 5.7 corr.], and BA (is) greater than AC, the (square) on GF (is) thus [also] greater than the (square) on FE [Prop. 5.14]. Let (the sum of) the (squares) on EF and H be equal to the (square) on GF. Thus, via conversion, as AB is to BC, so the (square) on FG (is) to the (square) on H [Prop. 5.19 corr.]. But, AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, FG is commensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) commensurable in length with (FG). And FG and FE are rational (straight-lines which are) commensurable in square only. And the lesser term EF is commensurable in length with the rational (straight-line) D (previously) laid down.

Thus, EG is a second binomial (straight-line) [Def. 10.6].† (Which is) the very thing it was required to show. †

If the rational straight-line has unit length then the length of a second binomial straight-line is $k/\sqrt{1-k'^2}+k$. This, and the second apotome, whose length is $k/\sqrt{1-k'^2}-k$ [Prop. 10.86], are the roots of $x^2-(2\,k/\sqrt{1-k'^2})\,x+k^2\,[k'^2/(1-k'^2)]=0$.

Proposition 50

To find a third binomial (straight-line).



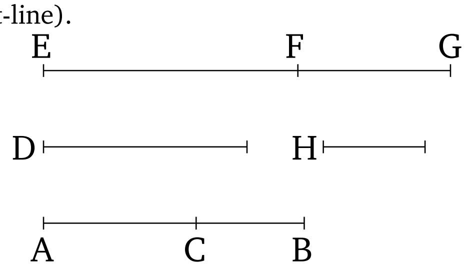
Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number. And let some other non-square number D also be laid down, and let it not have to each of BA and AC the ratio which (some) square number (has) to (some) square number. And let some rational straight-line E be laid down, and let it have been contrived that as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E is commensurable with the (square) on FG [Prop. 10.6]. And E is a rational (straight-line). Thus, FG is also a rational (straight-line). And since D does not have to AB the ratio which (some) square number has to (some) square number, the (square) on E does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. E is thus incommensurable in length with FG [Prop. 10.9]. So, again, let it have been contrived that as the number BA (is) to AC, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.]. Thus, the (square) on FG is commensurable with the (square) on GH [Prop. 10.6]. And FG (is) a rational (straight-line). Thus, GH (is) also a rational (straight-line). And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on HG the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. FG and GH are thus rational (straight-lines which are) commensurable in square only. Thus, FH is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a third (binomial straight-line).

For since as D is to AB, so the (square) on E (is) to the (square) on FG, and as BA (is) to AC, so the (square) on FG (is) to the (square) on GH, thus, via equality, as D (is) to AC, so the (square) on E (is) to the (square) on GH [Prop. 5.22]. And D does not have to AC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on E does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, E is incommensurable in length with GH [Prop. 10.9]. And since as BA is to AC, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14]. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG. Thus, via conversion, as AB [is] to BC, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number (has) to (some) square number. Thus, FG [is] commensurable in length with K [Prop. 10.9]. Thus, the square on FG is greater than (the square on) GH by the (square) on (some straight-line) commensurable (in length) with (FG). And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with E.

Thus, FH is a third binomial (straight-line) [Def. 10.7].† (Which is) the very thing it was required to show. † If the rational straight-line has unit length then the length of a third binomial straight-line is $k^{1/2}\,(1+\sqrt{1-k'^{\,2}})$. This, and the third apotome, whose length is $k^{1/2}\,(1-\sqrt{1-k'^{\,2}})$ [Prop. 10.87], are the roots of $x^2-2\,k^{1/2}\,x+k\,k'^{\,2}=0$.

Proposition 51

To find a fourth binomial (straight-line).



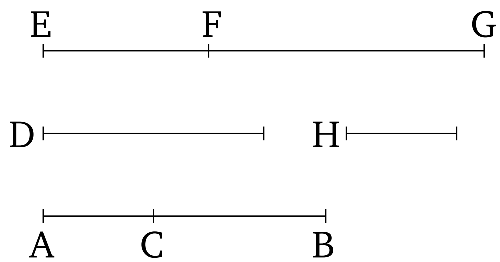
Let the two numbers AC and CB be laid down such that AB does not have to BC, or to AC either, the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) D be laid down. And let EF be commensurable in length with D. Thus, EF is also a rational (straight-line). And let it have been contrived that as the number BA (is) to AC, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. Thus, FG is also a rational (straight-line). And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. Thus, EF and FG are rational (straight-lines which are) commensurable in square only. Hence, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fourth (binomial straight-line).

For since as BA is to AC, so the (square) on EF (is) to the (square) on FG [and BA (is) greater than AC], the (square) on EF (is) thus greater than the (square) on FG [Prop. 5.14]. Therefore, let (the sum of) the squares on FG and H be equal to the (square) on EF. Thus, via conversion, as the number AB (is) to BC, so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with H [Prop. 10.9]. Thus, the square on EF is greater than (the square on) GF by the (square) on (some straight-line) incommensurable (in length) with (EF). And EF and FG are rational (straight-lines which are) commensurable in square only. And EF is commensurable in length with D.

Thus, EG is a fourth binomial (straight-line) [Def. 10.8].† (Which is) the very thing it was required to show. † If the rational straight-line has unit length then the length of a fourth binomial straight-line is $k\,(1+1/\sqrt{1+k'})$. This, and the fourth apotome, whose length is $k\,(1-1/\sqrt{1+k'})$ [Prop. 10.88], are the roots of $x^2-2\,k\,x+k^2\,k'/(1+k')=0$.

Proposition 52

To find a fifth binomial straight-line.



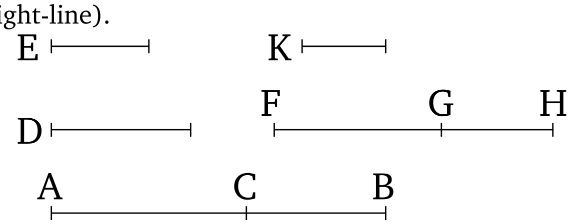
Let the two numbers AC and CB be laid down such that AB does not have to either of them the ratio which (some) square number (has) to (some) square number [Prop. 10.38 lem.]. And let some rational straight-line D be laid down. And let EF be commensurable [in length] with D. Thus, EF (is) a rational (straight-line). And let it have been contrived that as CA (is) to AB, so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And CA does not have to AB the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF and FG are rational (straight-lines which are) commensurable in square only [Prop. 10.9]. Thus, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For since as CA is to AB, so the (square) on EF (is) to the (square) on FG, inversely, as BA (is) to AC, so the (square) on FG (is) to the (square) on FE [Prop. 5.7 corr.]. Thus, the (square) on GF (is) greater than the (square) on FE [Prop. 5.14]. Therefore, let (the sum of) the (squares) on EF and H be equal to the (square) on GF. Thus, via conversion, as the number AB is to BC, so the (square) on GF (is) to the (square) on H [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) incommensurable (in length) with (FG). And GF and FE are rational (straight-lines which are) commensurable in square only. And the lesser term EF is commensurable in length with the rational (straight-line previously) laid down, D.

Thus, EG is a fifth binomial (straight-line).† (Which is) the very thing it was required to show. † If the rational straight-line has unit length then the length of a fifth binomial straight-line is $k\,(\sqrt{1+k'}+1)$. This, and the fifth apotome, whose length is $k\,(\sqrt{1+k'}-1)$ [Prop. 10.89], are the roots of $x^2 - 2\,k\,\sqrt{1+k'}\,x + k^2\,k' = 0$.

Proposition 53

To find a sixth binomial (straight-line).



Let the two numbers AC and CB be laid down such that AB does not have to each of them the ratio which (some) square number (has) to (some) square number. And let D also be another number, which is not square, and does not have to each of BA and AC the ratio which (some) square number (has) to (some) square number either [Prop. 10.28 lem. I]. And let some rational straight-line E be laid down. And let it have been contrived that as D (is) to AB, so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E (is) commensurable with the (square) on FG [Prop. 10.6]. And E is rational. Thus, FG (is) also rational. And since D does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on E

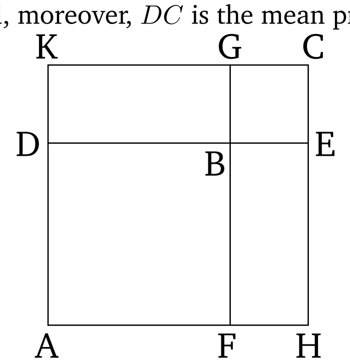
thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, E (is) incommensurable in length with FG [Prop. 10.9]. So, again, let it have be contrived that as BA (is) to AC, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.]. The (square) on FG (is) thus commensurable with the (square) on HG [Prop. 10.6]. The (square) on HG (is) thus rational. Thus, HG (is) rational. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. Thus, FG and GH are rational (straight-lines which are) commensurable in square only. Thus, FH is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a sixth (binomial straight-line).

For since as D is to AB, so the (square) on E (is) to the (square) on FG, and also as BA is to AC, so the (square) on FG (is) to the (square) on GH, thus, via equality, as D is to AC, so the (square) on E (is) to the (square) on GH [Prop. 5.22]. And D does not have to AC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on E does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. E is thus incommensurable in length with GH [Prop. 10.9]. And (E) was also shown (to be) incommensurable (in length) with FG. Thus, FG and GH are each incommensurable in length with E. And since as BA is to AC, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14]. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG. Thus, via conversion, as AB (is) to BC, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Hence, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with K [Prop. 10.9]. The square on FG is thus greater than (the square on) GH by the (square) on (some straight-line which is) incommensurable (in length) with (FG). And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with the rational (straight-line) E (previously) laid down.

Thus, FH is a sixth binomial (straight-line) [Def. 10.10].† (Which is) the very thing it was required to show. † If the rational straight-line has unit length then the length of a sixth binomial straight-line is $\sqrt{k} + \sqrt{k'}$. This, and the sixth apotome, whose length is $\sqrt{k} - \sqrt{k'}$ [Prop. 10.90], are the roots of $x^2 - 2\sqrt{k}\,x + (k - k') = 0$.

Lemma

Let AB and BC be two squares, and let them be laid down such that DB is straight-on to BE. FB is, thus, also straight-on to BG. And let the parallelogram AC have been completed. I say that AC is a square, and that DG is the mean proportional to AB and BC, and, moreover, DC is the mean proportional to AC and CB.



For since DB is equal to BF, and BE to BG, the whole of DE is thus equal to the whole of FG. But DE is equal to each of AH and KC, and FG is equal to each of AK and HC [Prop. 1.34]. Thus, AH and KC are also equal to AK and HC, respectively. Thus, the parallelogram AC is equilateral. And (it is) also right-angled. Thus, AC is a square.

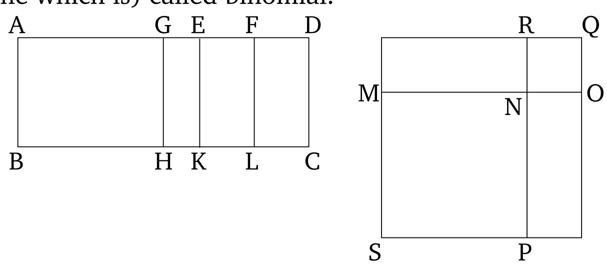
And since as FB is to BG, so DB (is) to BE, but as FB (is) to BG, so AB (is) to DG, and as DB (is) to BE, so DG (is) to BC [Prop. 6.1], thus also as AB (is) to DG, so DG (is) to BC [Prop. 5.11]. Thus, DG is the mean proportional to AB and BC.

So I also say that DC [is] the mean proportional to AC and CB.

For since as AD is to DK, so KG (is) to GC. For [they are] respectively equal. And, via composition, as AK (is) to KD, so KC (is) to CG [Prop. 5.18]. But as AK (is) to KD, so AC (is) to CD, and as KC (is) to CG, so DC (is) to CB [Prop. 6.1]. Thus, also, as AC (is) to DC, so DC (is) to BC [Prop. 5.11]. Thus, DC is the mean proportional to AC and CB. Which (is the very thing) it was prescribed to show.

Proposition 54

If an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called binomial.†



For let the area AC be contained by the rational (straight-line) AB and by the first binomial (straight-line) AD. I say that square-root of area AC is the irrational (straight-line which is) called binomial.

For since AD is a first binomial (straight-line), let it have been divided into its (component) terms at E, and let AE be the greater term. So, (it is) clear that AE and ED are rational (straight-lines which are) commensurable in square only, and that the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and that AE is commensurable (in length) with the rational (straight-line) AB (first) laid out [Def. 10.5]. So, let ED have been cut in half at point F. And since the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), thus if a (rectangle) equal to the fourth part of the (square) on the lesser (term)—that is to say, the (square) on EF—falling short by a square figure, is applied to the greater (term) AE, then it divides it into (terms which are) commensurable (in length) [Prop 10.17]. Therefore, let the (rectangle contained) by AG and GE, equal to the (square) on EF, have been applied to AE. AG is thus commensurable in length with EG. And let GH, EK, and FL have been drawn from (points) G, E, and F (respectively), parallel to either of AB or CD. And let the square SN, equal to the parallelogram AH, have been constructed, and (the square) NQ, equal to (the parallelogram) GK [Prop. 2.14]. And let MN be laid down so as to be straight-on to NO. RN is thus also straight-on to NP. And let the parallelogram SQ have been completed. SQ is thus a square [Prop. 10.53 lem.]. And since the (rectangle contained) by AG and GE is equal to the (square) on EF, thus as AG is to EF, so FE (is) to EG [Prop. 6.17]. And thus as AH (is) to EL, (so) EL (is) to KG [Prop. 6.1]. Thus, EL is the mean proportional to AH and GK. But, AH is equal to SN, and GK (is) equal to NQ. EL is thus the mean proportional to SN and NQ. And MR is also the mean proportional to the same—(namely), SN and NQ [Prop. 10.53 lem.]. EL is thus equal to MR. Hence, it is also equal to PO [Prop. 1.43]. And AH plus GK is equal to SN plus NQ. Thus, the whole of AC is equal to the whole of SQ—that is to say, to the square on MO. Thus, MO (is) the square-root of (area) AC. I say that MO is a binomial (straight-line).

For since AG is commensurable (in length) with GE, AE is also commensurable (in length) with each of AG and GE [Prop. 10.15]. And AE was also assumed (to be) commensurable (in length) with AB. Thus, AG and GE are also commensurable (in length) with AB [Prop. 10.12]. And AB is rational. AG and GE are thus each also rational.

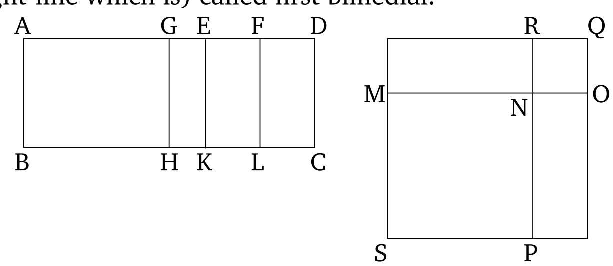
Thus, AH and GK are each rational (areas), and AH is commensurable with GK [Prop. 10.19]. But, AH is equal to SN, and GK to NQ. SN and NQ—that is to say, the (squares) on MN and NO (respectively)—are thus also rational and commensurable. And since AE is incommensurable in length with ED, but AE is commensurable (in length) with AG, and DE (is) commensurable (in length) with EF, AG (is) thus also incommensurable (in length) with EF [Prop. 10.13]. Hence, AH is also incommensurable with EL [Props. 6.1, 10.11]. But, AH is equal to SN, and EL to MR. Thus, SN is also incommensurable with MR. But, as SN (is) to MR, (so) PN (is) to NR [Prop. 6.1]. PN is thus incommensurable (in length) with NR [Prop. 10.11]. And PN (is) equal to MN, and NR to NO. Thus, MN is incommensurable (in length) with NO. And the (square) on MN is commensurable with the (square) on NO, and each (is) rational. MN and NO are thus rational (straight-lines which are) commensurable in square only.

Thus, MO is (both) a binomial (straight-line) [Prop. 10.36], and the square-root of AC. (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a first binomial straight-line is a binomial straight-line: *i.e.*, a first binomial straight-line has a length $k + k\sqrt{1-k'^2}$ whose square-root can be written $\rho\,(1+\sqrt{k''})$, where $\rho = \sqrt{k\,(1+k')/2}$ and $k'' = (1-k')/(1+k')$. This is the length of a binomial straight-line (see Prop. 10.36), since ρ is rational.

Proposition 55

If an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called first bimedial.†



For let the area $ABCD$ be contained by the rational (straight-line) AB and by the second binomial (straight-line) AD. I say that the square-root of area AC is a first bimedial (straight-line).

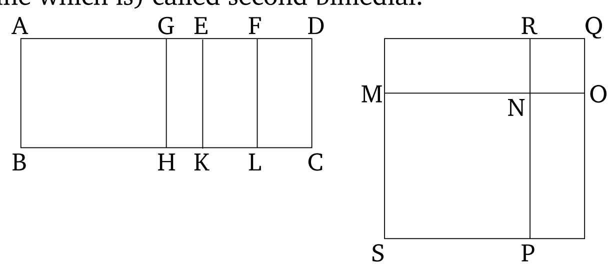
For since AD is a second binomial (straight-line), let it have been divided into its (component) terms at E, such that AE is the greater term. Thus, AE and ED are rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and the lesser term ED is commensurable in length with AB [Def. 10.6]. Let ED have been cut in half at F. And let the (rectangle contained) by AGE, equal to the (square) on EF, have been applied to AE, falling short by a square figure. AG (is) thus commensurable in length with GE [Prop. 10.17]. And let GH, EK, and FL have been drawn through (points) G, E, and F (respectively), parallel to AB and CD. And let the square SN, equal to the parallelogram AH, have been constructed, and the square NQ, equal to GK. And let MN be laid down so as to be straight-on to NO. Thus, RN [is] also straight-on to NP. And let the square SQ have been completed. So, (it is) clear from what has been previously demonstrated [Prop. 10.53 lem.] that MR is the mean proportional to SN and NQ, and (is) equal to EL, and that MO is the square-root of the area AC. So, we must show that MO is a first bimedial (straight-line).

Since AE is incommensurable in length with ED, and ED (is) commensurable (in length) with AB, AE (is) thus incommensurable (in length) with AB [Prop. 10.13]. And since AG is commensurable (in length) with EG, AE is also commensurable (in length) with each of AG and GE [Prop. 10.15]. But, AE is incommensurable in length with AB. Thus, AG and GE are also (both) incommensurable (in length) with AB [Prop. 10.13]. Thus, BA, AG, and (BA, and) GE are (pairs of) rational (straight-lines which are) commensurable in square only. And, hence, each of AH and GK is a medial (area) [Prop. 10.21]. Hence, each of SN and NQ is also a medial (area). Thus, MN and NO are medial (straight-lines). And since AG (is) commensurable in length with GE, AH is also commensurable with GK—that is to say, SN with NQ—that is to say, the (square) on MN with the (square) on NO [hence, MN and NO are commensurable in square] [Props. 6.1, 10.11]. And since AE is incommensurable in length with ED, but AE is commensurable (in length) with AG, and ED commensurable (in length) with EF, AG (is) thus incommensurable (in length) with EF [Prop. 10.13]. Hence, AH is also incommensurable with EL—that is to say, SN with MR—that is to say, PN with NR—that is to say, MN is incommensurable in length with NO [Props. 6.1, 10.11]. But MN and NO have also been shown to be medial (straight-lines) which are commensurable in square. Thus, MN and NO are medial (straight-lines which are) commensurable in square only. So, I say that they also contain a rational (area). For since DE was assumed (to be) commensurable (in length) with each of AB and EF, EF (is) thus also commensurable with EK [Prop. 10.12]. And they (are) each rational. Thus, EL—that is to say, MR—(is) rational [Prop. 10.19]. And MR is the (rectangle contained) by MNO. And if two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together, then the whole is (that) irrational (straight-line which is) called first bimedial [Prop. 10.37].

Thus, MO is a first bimedial (straight-line). (Which is) the very thing it was required to show. † If the rational straight-line has unit length then this proposition states that the square-root of a second binomial straight-line is a first bimedial straight-line: *i.e.*, a second binomial straight-line has a length $k/\sqrt{1-k'^2}+k$ whose square-root can be written $\rho\,(k''^{1/4}+k''^{3/4})$, where $\rho=\sqrt{(k/2)\,(1+k')/(1-k')}$ and $k''=(1-k')/(1+k')$. This is the length of a first bimedial straight-line (see Prop. 10.37), since ρ is rational.

Proposition 56

If an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called second bimedial.†



For let the area $ABCD$ be contained by the rational (straight-line) AB and by the third binomial (straight-line) AD, which has been divided into its (component) terms at E, of which AE is the greater. I say that the square-root of area AC is the irrational (straight-line which is) called second bimedial.

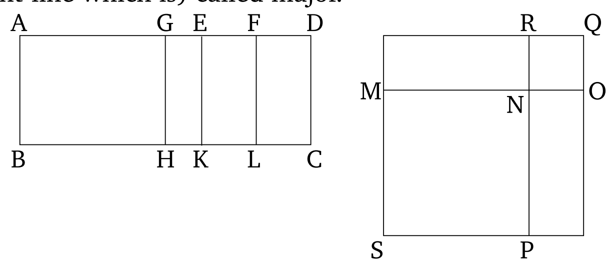
For let the same construction be made as previously. And since AD is a third binomial (straight-line), AE and ED are thus rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE), and neither of AE and ED [is] commensurable in length with AB [Def. 10.7]. So, similarly to that which has been previously demonstrated, we can show that MO is the square-root of area AC, and MN and NO are medial (straight-lines which are) commensurable in square only. Hence, MO is bimedial. So, we must show that (it is) also second (bimedial).

[And] since DE is incommensurable in length with AB—that is to say, with EK—and DE (is) commensurable (in length) with EF, EF is thus incommensurable in length with EK [Prop. 10.13]. And they are (both) rational (straight-lines). Thus, FE and EK are rational (straight-lines which are) commensurable in square only. EL—that is to say, MR—[is] thus medial [Prop. 10.21]. And it is contained by MNO. Thus, the (rectangle contained) by MNO is medial.

Thus, MO is a second bimedial (straight-line) [Prop. 10.38]. (Which is) the very thing it was required to show.
† If the rational straight-line has unit length then this proposition states that the square-root of a third binomial straight-line is a second bimedial straight-line: *i.e.*, a third binomial straight-line has a length $k^{1/2}\,(1+\sqrt{1-k'^{\,2}})$ whose square-root can be written $\rho\,(k^{1/4}+k''^{1/2}/k^{1/4})$, where $\rho=\sqrt{(1+k')/2}$ and $k''=k\,(1-k')/(1+k')$. This is the length of a second bimedial straight-line (see Prop. 10.38), since ρ is rational.

Proposition 57

If an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called major.†



For let the area AC be contained by the rational (straight-line) AB and the fourth binomial (straight-line) AD, which has been divided into its (component) terms at E, of which let AE be the greater. I say that the square-root of AC is the irrational (straight-line which is) called major.

For since AD is a fourth binomial (straight-line), AE and ED are thus rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) incommensurable (in length) with (AE), and AE [is] commensurable in length with AB [Def. 10.8]. Let DE have been cut in half at F, and let the parallelogram (contained by) AG and GE, equal to the (square) on EF, (and falling short by a square figure) have been applied to AE. AG is thus incommensurable in length with GE [Prop. 10.18]. Let GH, EK, and FL have been drawn parallel to AB, and let the rest (of the construction) have been made the same as the (proposition) before this. So, it is clear that MO is the square-root of area AC. So, we must show that MO is the irrational (straight-line which is) called major.

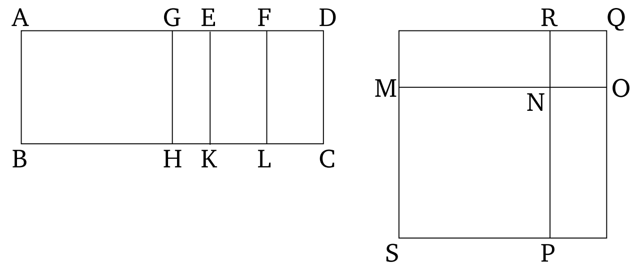
Since AG is incommensurable in length with EG, AH is also incommensurable with GK—that is to say, SN with NQ [Props. 6.1, 10.11]. Thus, MN and NO are incommensurable in square. And since AE is commensurable in length with AB, AK is rational [Prop. 10.19]. And it is equal to the (sum of the squares) on MN and NO. Thus, the sum of the (squares) on MN and NO [is] also rational. And since DE [is] incommensurable in length with AB [Prop. 10.13]—that is to say, with EK—but DE is commensurable (in length) with EF, EF (is) thus incommensurable in length with EK [Prop. 10.13]. Thus, EK and EF are rational (straight-lines which are) commensurable in square only. LE—that is to say, MR—(is) thus medial [Prop. 10.21]. And it is contained by MN and NO. The (rectangle contained) by MN and NO is thus medial. And the [sum] of the (squares) on MN and NO (is) rational, and MN and NO are incommensurable in square. And if two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together, then the whole is the irrational (straight-line which is) called major [Prop. 10.39].

Thus, MO is the irrational (straight-line which is) called major. And (it is) the square-root of area AC. (Which is) the very thing it was required to show. † If the rational straight-line has unit length then this proposition states that the square-root of a fourth binomial straight-line is a major straight-line: *i.e.*, a fourth binomial straight-line has a length $k\,(1+1/\sqrt{1+k'})$ whose square-root can be written $\rho\,\sqrt{[1+k''/(1+k''^{\,2})^{1/2}]/2}+\rho\,\sqrt{[1-k''/(1+k''^{\,2})^{1/2}]/2}$, where $\rho=\sqrt{k}$ and $k''^{\,2}=k'$. This is the length of a major straight-line (see Prop. 10.39), since ρ is rational.

Proposition 58

If an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).†

For let the area AC be contained by the rational (straight-line) AB and the fifth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term. [So] I say that the square-root of area AC is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).



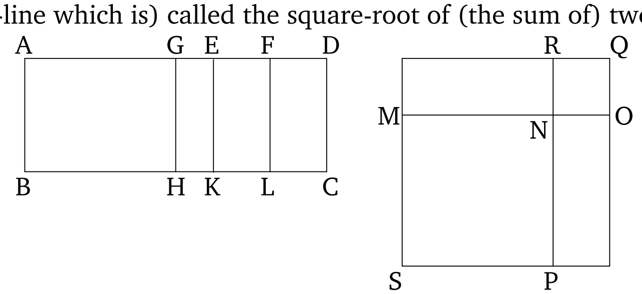
For let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of area AC. So, we must show that MO is the square-root of a rational plus a medial (area).

For since AG is incommensurable (in length) with GE [Prop. 10.18], AH is thus also incommensurable with HE—that is to say, the (square) on MN with the (square) on NO [Props. 6.1, 10.11]. Thus, MN and NO are incommensurable in square. And since AD is a fifth binomial (straight-line), and ED [is] its lesser segment, ED (is) thus commensurable in length with AB [Def. 10.9]. But, AE is incommensurable (in length) with ED. Thus, AB is also incommensurable in length with AE [BA and AE are rational (straight-lines which are) commensurable in square only] [Prop. 10.13]. Thus, AK—that is to say, the sum of the (squares) on MN and NO—is medial [Prop. 10.21]. And since DE is commensurable in length with AB—that is to say, with EK—but, DE is commensurable (in length) with EF, EF is thus also commensurable (in length) with EK [Prop. 10.12]. And EK (is) rational. Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—(is) also rational [Prop. 10.19]. MN and NO are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

Thus, MO is the square-root of a rational plus a medial (area) [Prop. 10.40]. And (it is) the square-root of area AC. (Which is) the very thing it was required to show. † If the rational straight-line has unit length then this proposition states that the square-root of a fifth binomial straight-line is the square root of a rational plus a medial area: *i.e.*, a fifth binomial straight-line has a length $k\,(\sqrt{1+k'}+1)$ whose square-root can be written $\rho\,\sqrt{[(1+k''^{\,2})^{1/2}+k'']/[2\,(1+k''^{\,2})]}+\rho\,\sqrt{[(1+k''^{\,2})^{1/2}-k'']/[2\,(1+k''^{\,2})]}$, where $\rho=\sqrt{k\,(1+k''^{\,2})}$ and $k''^{\,2}=k'$. This is the length of the square root of a rational plus a medial area (see Prop. 10.40), since ρ is rational.

Proposition 59

If an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of (the sum of) two medial (areas).†



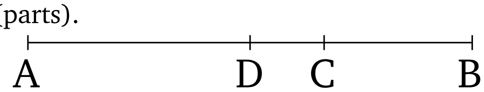
For let the area $ABCD$ be contained by the rational (straight-line) AB and the sixth binomial (straight-line) AD, which has been divided into its (component) terms at E, such that AE is the greater term. So, I say that the square-root of AC is the square-root of (the sum of) two medial (areas).

[For] let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of AC, and that MN is incommensurable in square with NO. And since EA is incommensurable in length with AB [Def. 10.10], EA and AB are thus rational (straight-lines which are) commensurable in square only. Thus, AK—that is to say, the sum of the (squares) on MN and NO—is medial [Prop. 10.21]. Again, since ED is incommensurable in length with AB [Def. 10.10], FE is thus also incommensurable (in length) with EK [Prop. 10.13]. Thus, FE and EK are rational (straight-lines which are) commensurable in square only. Thus, EL—that is to say, MR—that is to say, the (rectangle contained) by MNO—is medial [Prop. 10.21]. And since AE is incommensurable (in length) with EF, AK is also incommensurable with EL [Props. 6.1, 10.11]. But, AK is the sum of the (squares) on MN and NO, and EL is the (rectangle contained) by MNO. Thus, the sum of the (squares) on MNO is incommensurable with the (rectangle contained) by MNO. And each of them is medial. And MN and NO are incommensurable in square.

Thus, MO is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. And (it is) the square-root of AC. (Which is) the very thing it was required to show. † If the rational straight-line has unit length then this proposition states that the square-root of a sixth binomial straight-line is the square root of the sum of two medial areas: *i.e.*, a sixth binomial straight-line has a length $\sqrt{k}+\sqrt{k'}$ whose square-root can be written $k^{1/4}\left(\sqrt{[1+k''/(1+k''^{2})^{1/2}]/2}+\sqrt{[1-k''/(1+k''^{2})^{1/2}]/2}\right)$, where $k''^{2}=(k-k')/k'$. This is the length of the square-root of the sum of two medial areas (see Prop. 10.41).

Lemma

If a straight-line is cut unequally then (the sum of) the squares on the unequal (parts) is greater than twice the rectangle contained by the unequal (parts).

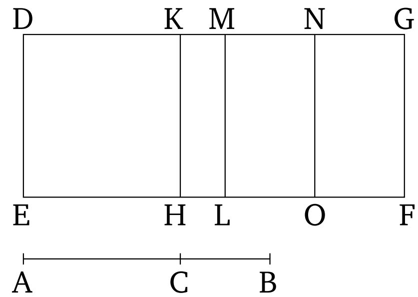


Let AB be a straight-line, and let it have been cut unequally at C, and let AC be greater (than CB). I say that (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB.

For let AB have been cut in half at D. Therefore, since a straight-line has been cut into equal (parts) at D, and into unequal (parts) at C, the (rectangle contained) by AC and CB, plus the (square) on CD, is thus equal to the (square) on AD [Prop. 2.5]. Hence, the (rectangle contained) by AC and CB is less than the (square) on AD. Thus, twice the (rectangle contained) by AC and CB is less than double the (square) on AD. But, (the sum of) the (squares) on AC and CB [is] double (the sum of) the (squares) on AD and DC [Prop. 2.9]. Thus, (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB. (Which is) the very thing it was required to show.

Proposition 60

The square on a binomial (straight-line) applied to a rational (straight-line) produces as breadth a first binomial (straight-line).†



Let AB be a binomial (straight-line), having been divided into its (component) terms at C, such that AC is the greater term. And let the rational (straight-line) DE be laid down. And let the (rectangle) $DEFG$, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a first binomial (straight-line).

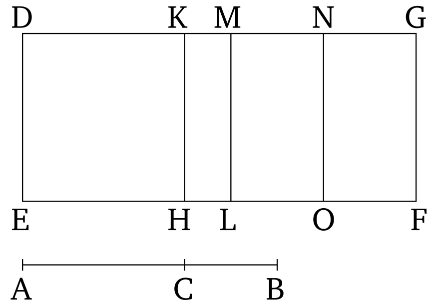
For let DH, equal to the (square) on AC, and KL, equal to the (square) on BC, have been applied to DE. Thus, the remaining twice the (rectangle contained) by AC and CB is equal to MF [Prop. 2.4]. Let MG have been cut in half at N, and let NO have been drawn parallel [to each of ML and GF]. MO and NF are thus each equal to once the (rectangle contained) by ACB. And since AB is a binomial (straight-line), having been divided into its (component) terms at C, AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Thus, the (squares) on AC and CB are rational, and commensurable with one another. And hence the sum of the (squares) on AC and CB (is rational) [Prop. 10.15], and is equal to DL. Thus, DL is rational. And it is applied to the rational (straight-line) DE. DM is thus rational, and commensurable in length with DE [Prop. 10.20]. Again, since AC and CB are rational (straight-lines which are) commensurable in square only, twice the (rectangle contained) by AC and CB—that is to say, MF—is thus medial [Prop. 10.21]. And it is applied to the rational (straight-line) ML. MG is thus also rational, and incommensurable in length with ML—that is to say, with DE [Prop. 10.22]. And MD is also rational, and commensurable in length with DE. Thus, DM is incommensurable in length with MG [Prop. 10.13]. And they are rational. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a first (binomial straight-line).

Since the (rectangle contained) by ACB is the mean proportional to the squares on AC and CB [Prop. 10.53 lem.], MO is thus also the mean proportional to DH and KL. Thus, as DH is to MO, so MO (is) to KL—that is to say, as DK (is) to MN, (so) MN (is) to MK [Prop. 6.1]. Thus, the (rectangle contained) by DK and KM is equal to the (square) on MN [Prop. 6.17]. And since the (square) on AC is commensurable with the (square) on CB, DH is also commensurable with KL. Hence, DK is also commensurable with KM [Props. 6.1, 10.11]. And since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59 lem.], DL (is) thus also greater than MF. Hence, DM is also greater than MG [Props. 6.1, 5.14]. And the (rectangle contained) by DK and KM is equal to the (square) on MN—that is to say, to one quarter the (square) on MG. And DK (is) commensurable (in length) with KM. And if there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable (in length), then the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable (in length) with the greater [Prop. 10.17]. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM). And DM and MG are rational. And DM, which is the greater term, is commensurable in length with the (previously) laid down rational (straight-line) DE.

Thus, DG is a first binomial (straight-line) [Def. 10.5]. (Which is) the very thing it was required to show. † In other words, the square of a binomial is a first binomial. See Prop. 10.54.

Proposition 61

The square on a first bimedial (straight-line) applied to a rational (straight-line) produces as breadth a second binomial (straight-line).†



Let AB be a first bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, of which AC (is) the greater. And let the rational (straight-line) DE be laid down. And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a second binomial (straight-line).

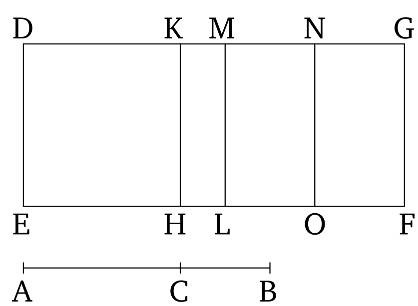
For let the same construction have been made as in the (proposition) before this. And since AB is a first bimedial (straight-line), having been divided at C, AC and CB are thus medial (straight-lines) commensurable in square only, and containing a rational (area) [Prop. 10.37]. Hence, the (squares) on AC and CB are also medial [Prop. 10.21]. Thus, DL is medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) DE. MD is thus rational, and incommensurable in length with DE [Prop. 10.22]. Again, since twice the (rectangle contained) by AC and CB is rational, MF is also rational. And it is applied to the rational (straight-line) ML. Thus, MG [is] also rational, and commensurable in length with ML—that is to say, with DE [Prop. 10.20]. DM is thus incommensurable in length with MG [Prop. 10.13]. And they are rational. DM and MG are thus rational, and commensurable in square only. DG is thus a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).

For since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59], DL (is) thus also greater than MF. Hence, DM (is) also (greater) than MG [Prop. 6.1]. And since the (square) on AC is commensurable with the (square) on CB, DH is also commensurable with KL. Hence, DK is also commensurable (in length) with KM [Props. 6.1, 10.11]. And the (rectangle contained) by DKM is equal to the (square) on MN. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17]. And MG is commensurable in length with DE.

Thus, DG is a second binomial (straight-line) [Def. 10.6]. †In other words, the square of a first bimedial is a second binomial. See Prop. 10.55.

Proposition 62

The square on a second bimedial (straight-line) applied to a rational (straight-line) produces as breadth a third binomial (straight-line).†



Let AB be a second bimedial (straight-line) having been divided into its (component) medial (straight-lines) at C, such that AC is the greater segment. And let DE be some rational (straight-line). And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a third binomial (straight-line).

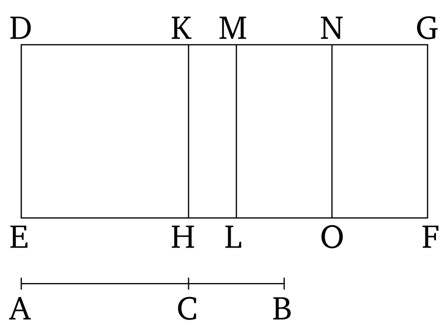
Let the same construction be made as that shown previously. And since AB is a second bimedial (straight-line), having been divided at C, AC and CB are thus medial (straight-lines) commensurable in square only, and containing a medial (area) [Prop. 10.38]. Hence, the sum of the (squares) on AC and CB is also medial [Props. 10.15, 10.23 corr.]. And it is equal to DL. Thus, DL (is) also medial. And it is applied to the rational (straight-line) DE. MD is thus also rational, and incommensurable in length with DE [Prop. 10.22]. So, for the same (reasons), MG is also rational, and incommensurable in length with ML—that is to say, with DE. Thus, DM and MG are each rational, and incommensurable in length with DE. And since AC is incommensurable in length with CB, and as AC (is) to CB, so the (square) on AC (is) to the (rectangle contained) by ACB [Prop. 10.21 lem.], the (square) on AC (is) also incommensurable with the (rectangle contained) by ACB [Prop. 10.11]. And hence the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by ACB—that is to say, DL with MF [Props. 10.12, 10.13]. Hence, DM is also incommensurable (in length) with MG [Props. 6.1, 10.11]. And they are rational. DG is thus a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a third (binomial straight-line).

So, similarly to the previous (propositions), we can conclude that DM is greater than MG, and DK (is) commensurable (in length) with KM. And the (rectangle contained) by DKM is equal to the (square) on MN. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17]. And neither of DM and MG is commensurable in length with DE.

Thus, DG is a third binomial (straight-line) [Def. 10.7]. (Which is) the very thing it was required to show. † In other words, the square of a second bimedial is a third binomial. See Prop. 10.56.

Proposition 63

The square on a major (straight-line) applied to a rational (straight-line) produces as breadth a fourth binomial (straight-line).†



Let AB be a major (straight-line) having been divided at C, such that AC is greater than CB, and (let) DE (be) a rational (straight-line). And let the parallelogram DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a fourth binomial (straight-line).

Let the same construction be made as that shown previously. And since AB is a major (straight-line), having been divided at C, AC and CB are incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. Therefore, since the sum of the (squares) on AC and CB is rational, DL is thus rational. Thus, DM (is) also rational, and commensurable in length with DE [Prop. 10.20]. Again, since twice the (rectangle contained) by AC and CB—that is to say, MF—is medial, and is (applied to) the rational (straight-line) ML, MG is thus also rational, and incommensurable in length with DE [Prop. 10.22]. DM is thus also incommensurable in length with MG [Prop. 10.13]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a fourth (binomial straight-line).

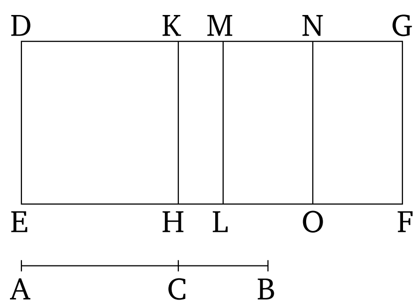
So, similarly to the previous (propositions), we can show that DM is greater than MG, and that the (rectangle contained) by DKM is equal to the (square) on MN. Therefore, since the (square) on AC is incommensurable with the (square) on CB, DH is also incommensurable with KL. Hence, DK is also incommensurable with KM [Props. 6.1, 10.11]. And if there are two unequal straight-lines, and a parallelogram equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable (in length), then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) incommensurable in length with the greater [Prop. 10.18]. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM). And DM and MG are rational (straight-lines which are) commensurable in square only. And DM is commensurable (in length) with the (previously) laid down rational (straight-line) DE.

Thus, DG is a fourth binomial (straight-line) [Def. 10.8]. (Which is) the very thing it was required to show. † In other words, the square of a major is a fourth binomial. See Prop. 10.57.

Proposition 64

The square on the square-root of a rational plus a medial (area) applied to a rational (straight-line) produces as breadth a fifth binomial (straight-line).†

Let AB be the square-root of a rational plus a medial (area) having been divided into its (component) straight-lines at C, such that AC is greater. And let the rational (straight-line) DE be laid down. And let the (parallelogram) DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a fifth binomial straight-line.



Let the same construction be made as in the (propositions) before this. Therefore, since AB is the square-root of a rational plus a medial (area), having been divided at C, AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. Therefore, since the sum of the (squares) on AC and CB is medial, DL is thus medial. Hence, DM is rational and incommensurable in length with DE [Prop. 10.22]. Again, since twice the (rectangle contained) by ACB—that is to say, MF—is rational, MG (is) thus rational and commensurable (in length) with DE [Prop. 10.20]. DM (is) thus incommensurable (in length) with MG [Prop. 10.13]. Thus, DM and MG are rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For, similarly (to the previous propositions), it can be shown that the (rectangle contained) by DKM is equal to the (square) on MN, and DK (is) incommensurable in length with KM. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM) [Prop. 10.18].

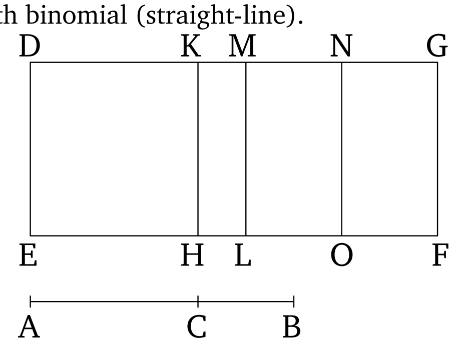
And DM and MG are [rational] (straight-lines which are) commensurable in square only, and the lesser MG is commensurable in length with DE.

Thus, DG is a fifth binomial (straight-line) [Def. 10.9]. (Which is) the very thing it was required to show. † In other words, the square of the square-root of a rational plus medial is a fifth binomial. See Prop. 10.58.

Proposition 65

The square on the square-root of (the sum of) two medial (areas) applied to a rational (straight-line) produces as breadth a sixth binomial (straight-line).†

Let AB be the square-root of (the sum of) two medial (areas), having been divided at C. And let DE be a rational (straight-line). And let the (parallelogram) DF, equal to the (square) on AB, have been applied to DE, producing DG as breadth. I say that DG is a sixth binomial (straight-line).



For let the same construction be made as in the previous (propositions). And since AB is the square-root of (the sum of) two medial (areas), having been divided at C, AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.41]. Hence, according to what has been previously demonstrated, DL and MF are each medial. And they are applied to the rational (straight-line) DE. Thus, DM and MG are each rational, and incommensurable in length with DE [Prop. 10.22]. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, DL is thus incommensurable with MF. Thus, DM (is) also incommensurable (in length) with MG [Props. 6.1, 10.11]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a sixth (binomial straight-line).

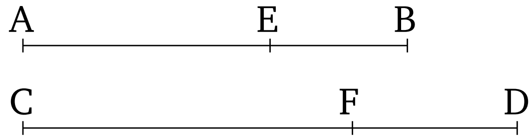
So, similarly (to the previous propositions), we can again show that the (rectangle contained) by DKM is equal to the (square) on MN, and that DK is incommensurable in length with KM. And so, for the same (reasons), the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable in length with (DM) [Prop. 10.18]. And neither of DM and MG is commensurable in length with the (previously) laid down rational (straight-line) DE.

Thus, DG is a sixth binomial (straight-line) [Def. 10.10]. (Which is) the very thing it was required to show. † In other words, the square of the square-root of two medials is a sixth binomial. See Prop. 10.59.

Proposition 66

A (straight-line) commensurable in length with a binomial (straight-line) is itself also binomial, and the same in order.

Let AB be a binomial (straight-line), and let CD be commensurable in length with AB. I say that CD is a binomial (straight-line), and (is) the same in order as AB.



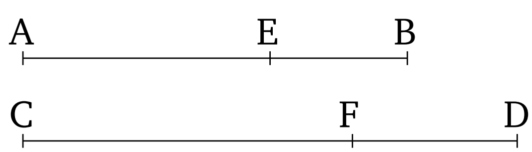
For since AB is a binomial (straight-line), let it have been divided into its (component) terms at E, and let AE be the greater term. AE and EB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Let it have been contrived that as AB (is) to CD, so AE (is) to CF [Prop. 6.12]. Thus, the remainder EB is also to the remainder FD, as AB (is) to CD [Props. 6.16, 5.19 corr.]. And AB (is) commensurable in length with CD. Thus, AE is also commensurable (in length) with CF, and EB with FD [Prop. 10.11]. And AE and EB are rational. Thus, CF and FD are also rational. And as AE is to CF, (so) EB (is) to FD [Prop. 5.11]. Thus, alternately, as AE is to EB, (so) CF (is) to FD [Prop. 5.16]. And AE and EB [are] commensurable in square only. Thus, CF and FD are also commensurable in square only [Prop. 10.11]. And they are rational. CD is thus a binomial (straight-line) [Prop. 10.36]. So, I say that it is the same in order as AB.

For the square on AE is greater than (the square on) EB by the (square) on (some straight-line) either commensurable or incommensurable (in length) with (AE). Therefore, if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with (some previously) laid down rational (straight-line) then CF will also be commensurable (in length) with it [Prop. 10.12]. And, on account of this, AB and CD are each first binomial (straight-lines) [Def. 10.5]—that is to say, the same in order. And if EB is commensurable (in length) with the (previously) laid down rational (straight-line) then FD is also commensurable (in length) with it [Prop. 10.12], and, again, on account of this, (CD) will be the same in order as AB. For each of them will be second binomial (straight-lines) [Def. 10.6]. And if neither of AE and EB is commensurable (in length) with the (previously) laid down rational (straight-line) then neither of CF and FD will be commensurable (in length) with it [Prop. 10.13], and each (of AB and CD) is a third (binomial straight-line) [Def. 10.7]. And if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) incommensurable (in length) with (AE) then the square on CF is also greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with the (previously) laid down rational (straight-line) then CF is also commensurable (in length) with it [Prop. 10.12], and each (of AB and CD) is a fourth (binomial straight-line) [Def. 10.8]. And if EB (is commensurable in length with the previously laid down rational straight-line) then FD (is) also (commensurable in length with it), and each (of AB and CD) will be a fifth (binomial straight-line) [Def. 10.9]. And if neither of AE and EB (is commensurable in length with the previously laid down rational straight-line) then also neither of CF and FD is commensurable (in length) with the laid down rational (straight-line), and each (of AB and CD) will be a sixth (binomial straight-line) [Def. 10.10].

Hence, a (straight-line) commensurable in length with a binomial (straight-line) is a binomial (straight-line), and the same in order. (Which is) the very thing it was required to show.

Proposition 67

A (straight-line) commensurable in length with a bimedial (straight-line) is itself also bimedial, and the same in order.



Let AB be a bimedial (straight-line), and let CD be commensurable in length with AB. I say that CD is bimedial, and the same in order as AB.

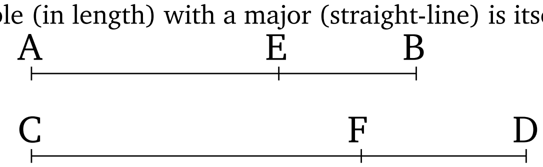
For since AB is a bimedial (straight-line), let it have been divided into its (component) medial (straight-lines) at E. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.37, 10.38]. And let it have been contrived that as AB (is) to CD, (so) AE (is) to CF [Prop. 6.12]. And thus as the remainder EB is to the remainder FD, so AB (is) to CD [Props. 5.19 corr., 6.16]. And AB (is) commensurable in length with CD. Thus, AE and EB are also commensurable (in length) with CF and FD, respectively [Prop. 10.11]. And AE and EB (are) medial. Thus, CF and FD (are) also medial [Prop. 10.23]. And since as AE is to EB, (so) CF (is) to FD, and AE and EB are commensurable in square only, CF and FD are [thus] also commensurable in square only [Prop. 10.11]. And they were also shown (to be) medial. Thus, CD is a bimedial (straight-line). So, I say that it is also the same in order as AB.

For since as AE is to EB, (so) CF (is) to FD, thus also as the (square) on AE (is) to the (rectangle contained) by AEB, so the (square) on CF (is) to the (rectangle contained) by CFD [Prop. 10.21 lem.]. Alternately, as the (square) on AE (is) to the (square) on CF, so the (rectangle contained) by AEB (is) to the (rectangle contained) by CFD [Prop. 5.16]. And the (square) on AE (is) commensurable with the (square) on CF. Thus, the (rectangle contained) by AEB (is) also commensurable with the (rectangle contained) by CFD [Prop. 10.11]. Therefore, either the (rectangle contained) by AEB is rational, and the (rectangle contained) by CFD is rational [and, on account of this, (AE and CD) are first bimedial (straight-lines)], or (the rectangle contained by AEB is) medial, and (the rectangle contained by CFD is) medial, and (AB and CD) are each second (bimedial straight-lines) [Props. 10.23, 10.37, 10.38].

And, on account of this, CD will be the same in order as AB. (Which is) the very thing it was required to show.

Proposition 68

A (straight-line) commensurable (in length) with a major (straight-line) is itself also major.



Let AB be a major (straight-line), and let CD be commensurable (in length) with AB. I say that CD is a major (straight-line).

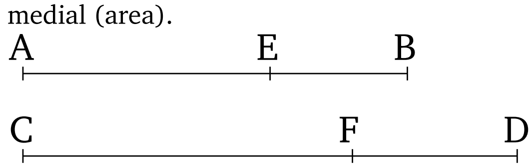
Let AB have been divided (into its component terms) at E. AE and EB are thus incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. And let (the) same (things) have been contrived as in the previous (propositions). And since as AB is to CD, so AE (is) to CF and EB to FD, thus also as AE (is) to CF, so EB (is) to FD [Prop. 5.11]. And AB (is) commensurable (in length) with CD. Thus, AE and EB (are) also commensurable (in length) with CF and FD, respectively [Prop. 10.11]. And since as AE is to CF, so EB (is) to FD, also, alternately, as AE (is) to EB, so CF (is) to FD [Prop. 5.16], and thus, via composition, as AB is to BE, so CD (is) to DF [Prop. 5.18]. And thus as the (square) on AB (is) to the (square) on BE, so the (square) on CD (is) to the (square) on DF [Prop. 6.20]. So, similarly, we can also show that as the (square) on AB (is) to the (square) on AE, so the (square) on CD (is) to the (square) on CF. And thus as the (square) on AB (is) to (the sum of) the (squares) on AE and EB, so the (square) on CD (is) to (the sum of) the (squares) on CF and FD. And thus, alternately, as the (square) on AB is to the (square) on CD, so (the sum of) the (squares) on AE and EB (is) to (the sum of) the (squares) on CF and FD [Prop. 5.16]. And the (square) on AB (is) commensurable with the (square) on CD. Thus, (the sum of) the (squares) on AE and EB (is) also commensurable with (the sum of) the (squares) on CF and FD [Prop. 10.11]. And the (squares) on AE and EB

(added) together are rational. The (squares) on CF and FD (added) together (are) thus also rational. So, similarly, twice the (rectangle contained) by AE and EB is also commensurable with twice the (rectangle contained) by CF and FD. And twice the (rectangle contained) by AE and EB is medial. Therefore, twice the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.]. CF and FD are thus (straight-lines which are) incommensurable in square [Prop 10.13], simultaneously making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. The whole, CD, is thus that irrational (straight-line) called major [Prop. 10.39].

Thus, a (straight-line) commensurable (in length) with a major (straight-line) is major. (Which is) the very thing it was required to show.

Proposition 69

A (straight-line) commensurable (in length) with the square-root of a rational plus a medial (area) is [itself also] the square-root of a rational plus a medial (area).



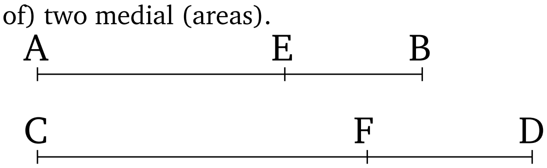
Let AB be the square-root of a rational plus a medial (area), and let CD be commensurable (in length) with AB. We must show that CD is also the square-root of a rational plus a medial (area).

Let AB have been divided into its (component) straight-lines at E. AE and EB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that CF and FD are also incommensurable in square, and that the sum of the (squares) on AE and EB (is) commensurable with the sum of the (squares) on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. And hence the sum of the squares on CF and FD is medial, and the (rectangle contained) by CF and FD (is) rational.

Thus, CD is the square-root of a rational plus a medial (area) [Prop. 10.40]. (Which is) the very thing it was required to show.

Proposition 70

A (straight-line) commensurable (in length) with the square-root of (the sum of) two medial (areas) is (itself also) the square-root of (the sum of) two medial (areas).



Let AB be the square-root of (the sum of) two medial (areas), and (let) CD (be) commensurable (in length) with AB. We must show that CD is also the square-root of (the sum of) two medial (areas).

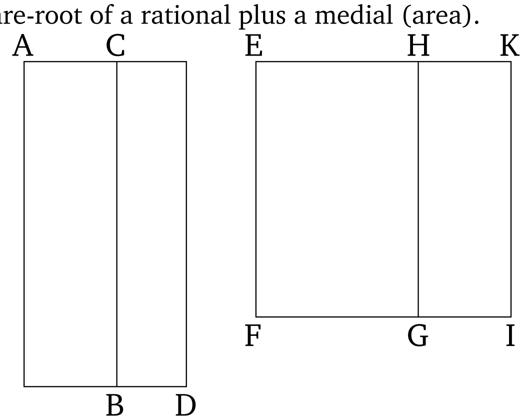
For since AB is the square-root of (the sum of) two medial (areas), let it have been divided into its (component) straight-lines at E. Thus, AE and EB are incommensurable in square, making the sum of the [squares] on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the (squares) on AE and EB incommensurable with the (rectangle) contained by AE and EB [Prop. 10.41]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that CF and FD are also incommensurable in square, and (that) the sum of the (squares) on AE and EB (is) commensurable with the sum of the (squares) on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Hence, the sum of the squares on CF and FD is also medial, and the (rectangle contained) by CF and FD (is) medial, and, moreover, the sum of the squares on CF and FD (is) incommensurable with the (rectangle contained) by CF and FD.

Thus, CD is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. (Which is) the very thing it was required to show.

Proposition 71

When a rational and a medial (area) are added together, four irrational (straight-lines) arise (as the square-roots of the total area)—either a binomial, or a first bimedial, or a major, or the square-root of a rational plus a medial (area).

Let AB be a rational (area), and CD a medial (area). I say that the square-root of area AD is either binomial, or first bimedial, or major, or the square-root of a rational plus a medial (area).



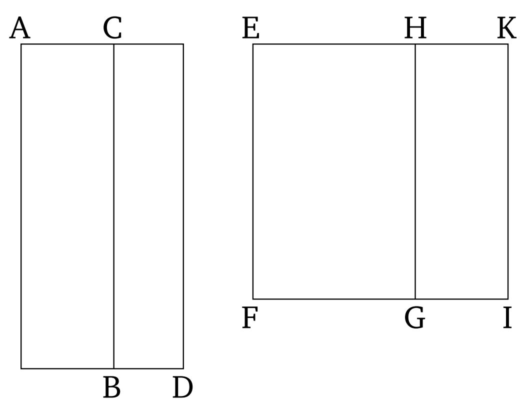
For AB is either greater or less than CD. Let it, first of all, be greater. And let the rational (straight-line) EF be laid down. And let (the rectangle) EG, equal to AB, have been applied to EF, producing EH as breadth. And let (the recatangle) HI, equal to DC, have been applied to EF, producing HK as breadth. And since AB is rational, and is equal to EG, EG is thus also rational. And it has been applied to the [rational] (straight-line) EF, producing EH as breadth. EH is thus rational, and commensurable in length with EF [Prop. 10.20]. Again, since CD is medial, and is equal to HI, HI is thus also medial. And it is applied to the rational (straight-line) EF, producing HK as breadth. HK is thus rational, and incommensurable in length with EF [Prop. 10.22]. And since CD is medial, and AB rational, AB is thus incommensurable with CD. Hence, EG is also incommensurable with HI. And as EG (is) to HI, so EH is to HK [Prop. 6.1]. Thus, EH is also incommensurable in length with HK [Prop. 10.11]. And they are both rational. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line), having been divided (into its component terms) at H [Prop. 10.36]. And since AB is greater than CD, and AB (is) equal to EG, and CD to HI, EG (is) thus also greater than HI. Thus, EH is also greater than HK [Prop. 5.14]. Therefore, the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable in length with (EH), or by the (square) on (some straight-line) incommensurable (in length with EH). Let it, first of all, be greater by the (square) on (some straight-line) commensurable (in length with EH). And the greater (of the two components of EK) HE is commensurable (in length) with the (previously) laid down (straight-line) EF. EK is thus a first binomial (straight-line) [Def. 10.5]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is a binomial (straight-line) [Prop. 10.54]. Thus, the square-root of EI is a binomial (straight-line). Hence the square-root of AD is also a binomial (straight-line). And, so, let the square on EH be greater than (the square on) HK by the (square) on (some straight-line) incommensurable (in length) with (EH). And the greater (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a fourth binomial (straight-line) [Def. 10.8]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line) called major [Prop. 10.57]. Thus, the square-root of area EI is a major (straight-line). Hence, the square-root of AD is also major.

And so, let AB be less than CD. Thus, EG is also less than HI. Hence, EH is also less than HK [Props. 6.1, 5.14]. And the square on HK is greater than (the square on) EH either by the (square) on (some straight-line) commensurable (in length) with (HK), or by the (square) on (some straight-line) incommensurable (in length) with (HK). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (HK). And the lesser (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a second binomial (straight-line) [Def. 10.6]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is a first bimedial (straight-line) [Prop. 10.55]. Thus, the square-root of area EI is a first bimedial (straight-line). Hence, the square-root of AD is also a first bimedial (straight-line). And so, let the square on HK be greater than (the square on) HE by the (square) on (some straight-line) incommensurable (in length) with (HK). And the lesser (of the two components of EK) EH is commensurable (in length) with the (previously) laid down rational (straight-line) EF. Thus, EK is a fifth binomial (straight-line) [Def. 10.9]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the square-root of a rational plus a medial (area) [Prop. 10.58]. Thus, the square-root of area EI is the square-root of a rational plus a medial (area). Hence, the square-root of area AD is also the square-root of a rational plus a medial (area).

Thus, when a rational and a medial area are added together, four irrational (straight-lines) arise (as the square-roots of the total area)—either a binomial, or a first bimedial, or a major, or the square-root of a rational plus a medial (area). (Which is) the very thing it was required to show.

Proposition 72

When two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second bimedial, or the square-root of (the sum of) two medial (areas).



For let the two medial (areas) AB and CD, (which are) incommensurable with one another, have been added together. I say that the square-root of area AD is either a second bimedial, or the square-root of (the sum of) two medial (areas).

For AB is either greater than or less than CD. By chance, let AB, first of all, be greater than CD. And let the rational (straight-line) EF be laid down. And let EG, equal to AB, have been applied to EF, producing EH as breadth, and HI, equal to CD, producing HK as breadth. And since AB and CD are each medial, EG and HI (are) thus also each medial. And they are applied to the rational straight-line FE, producing EH and HK (respectively) as breadth. Thus, EH and HK are each rational (straight-lines which are) incommensurable in length with EF [Prop. 10.22]. And since AB is incommensurable with CD, and AB is equal to EG, and CD to HI, EG is thus also incommensurable with HI. And as EG (is) to HI, so EH is to HK [Prop. 6.1]. EH is thus incommensurable in length with HK [Prop. 10.11]. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line) [Prop. 10.36]. And the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable (in length) with (EH), or by the (square) on (some straight-line) incommensurable (in length with EH). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (EH). And neither of EH or HK is commensurable in length with the (previously) laid down rational (straight-line) EF. Thus, EK is a third binomial (straight-line) [Def. 10.7]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is a second bimedial (straight-line) [Prop. 10.56]. Thus, the square-root of EI—that is to say, of AD—is a second bimedial. And so, let the square on EH be greater than (the square) on HK by the (square) on (some straight-line) incommensurable in length with (EH). And EH and HK are each incommensurable in length with EF. Thus, EK is a sixth binomial (straight-line) [Def. 10.10]. And if an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the square-root of (the sum of) two medial (areas) [Prop. 10.59]. Hence, the square-root of area AD is also the square-root of (the sum of) two medial (areas).

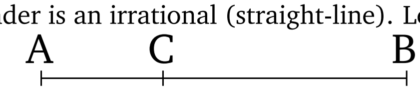
[So, similarly, we can show that, even if AB is less than CD, the square-root of area AD is either a second bimedial or the square-root of (the sum of) two medial (areas).]

Thus, when two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second bimedial, or the square-root of (the sum of) two medial (areas).

A binomial (straight-line), and the (other) irrational (straight-lines) after it, are neither the same as a medial (straight-line) nor (the same) as one another. For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) also incommensurable in length with (the straight-line) to which it is applied [Prop. 10.22]. And the (square) on a binomial (straight-line), applied to a rational (straight-line), produces as breadth a first binomial [Prop. 10.60]. And the (square) on a first bimedial (straight-line), applied to a rational (straight-line), produces as breadth a second binomial [Prop. 10.61]. And the (square) on a second bimedial (straight-line), applied to a rational (straight-line), produces as breadth a third binomial [Prop. 10.62]. And the (square) on a major (straight-line), applied to a rational (straight-line), produces as breadth a fourth binomial [Prop. 10.63]. And the (square) on the square-root of a rational plus a medial (area), applied to a rational (straight-line), produces as breadth a fifth binomial [Prop. 10.64]. And the (square) on the square-root of (the sum of) two medial (areas), applied to a rational (straight-line), produces as breadth a sixth binomial [Prop. 10.65]. And the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational—and from one another, because they are not the same in order. Hence, the (previously mentioned) irrational (straight-lines) themselves also differ from one another.

Proposition 73

If a rational (straight-line), which is commensurable in square only with the whole, is subtracted from a(nother) rational (straight-line) then the remainder is an irrational (straight-line). Let it be called an apotome.

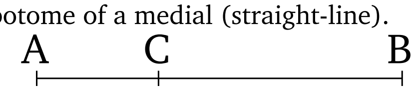


For let the rational (straight-line) BC, which commensurable in square only with the whole, have been subtracted from the rational (straight-line) AB. I say that the remainder AC is that irrational (straight-line) called an apotome.

For since AB is incommensurable in length with BC, and as AB is to BC, so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11]. But, the (sum of the) squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. And, inasmuch as the (sum of the squares) on AB and BC is equal to twice the (rectangle contained) by AB and BC plus the (square) on CA [Prop. 2.7], the (sum of the squares) on AB and BC is thus also incommensurable with the remaining (square) on AC [Props. 10.13, 10.16]. And the (sum of the squares) on AB and BC is rational. AC is thus an irrational (straight-line) [Def. 10.4]. And let it be called an apotome.† (Which is) the very thing it was required to show. † See footnote to Prop. 10.36.

Proposition 74

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a rational (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a first apotome of a medial (straight-line).



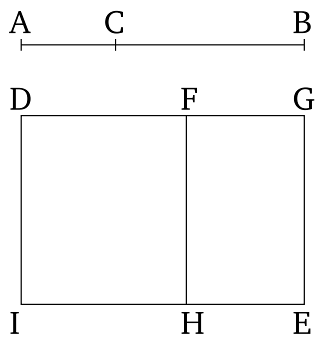
For let the medial (straight-line) BC, which is commensurable in square only with AB, and which makes with AB the rational (rectangle contained) by AB and BC, have been subtracted from the medial (straight-line) AB [Prop. 10.27]. I say that the remainder AC is an irrational (straight-line). Let it be called the first apotome of a medial (straight-line).

For since AB and BC are medial (straight-lines), the (sum of the squares) on AB and BC is also medial. And twice the (rectangle contained) by AB and BC (is) rational. The (sum of the squares) on AB and BC (is) thus incommensurable with twice the (rectangle contained) by AB and BC. Thus, twice the (rectangle contained) by AB and BC is also incommensurable with the remaining (square) on AC [Prop. 2.7], since if the whole is incommensurable with one of the (constituent magnitudes) then the original magnitudes will also be incommensurable (with one another) [Prop. 10.16]. And twice the (rectangle contained) by AB and BC (is) rational. Thus, the (square) on AC is irrational. Thus, AC is an irrational (straight-line) [Def. 10.4]. Let it be called a first apotome of a medial (straight-line).† † See footnote to Prop. 10.37.

Proposition 75

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a medial (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).

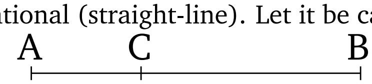
For let the medial (straight-line) CB, which is commensurable in square only with the whole, AB, and which contains with the whole, AB, the medial (rectangle contained) by AB and BC, have been subtracted from the medial (straight-line) AB [Prop. 10.28]. I say that the remainder AC is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).



For let the rational (straight-line) DI be laid down. And let DE, equal to the (sum of the squares) on AB and BC, have been applied to DI, producing DG as breadth. And let DH, equal to twice the (rectangle contained) by AB and BC, have been applied to DI, producing DF as breadth. The remainder FE is thus equal to the (square) on AC [Prop. 2.7]. And since the (squares) on AB and BC are medial and commensurable (with one another), DE (is) thus also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) DI, producing DG as breadth. Thus, DG is rational, and incommensurable in length with DI [Prop. 10.22]. Again, since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BC is thus also medial [Prop. 10.23 corr.]. And it is equal to DH. Thus, DH is also medial. And it has been applied to the rational (straight-line) DI, producing DF as breadth. DF is thus rational, and incommensurable in length with DI [Prop. 10.22]. And since AB and BC are commensurable in square only, AB is thus incommensurable in length with BC. Thus, the square on AB (is) also incommensurable with the (rectangle contained) by AB and BC [Props. 10.21 lem., 10.11]. But, the (sum of the squares) on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the (sum of the squares) on AB and BC [Prop. 10.13]. And DE is equal to the (sum of the squares) on AB and BC, and DH to twice the (rectangle contained) by AB and BC. Thus, DE [is] incommensurable with DH. And as DE (is) to DH, so GD (is) to DF [Prop. 6.1]. Thus, GD is incommensurable with DF [Prop. 10.11]. And they are both rational (straight-lines). Thus, GD and DF are rational (straight-lines which are) commensurable in square only. Thus, FG is an apotome [Prop. 10.73]. And DI (is) rational. And the (area) contained by a rational and an irrational (straight-line) is irrational [Prop. 10.20], and its square-root is irrational. And AC is the square-root of FE. Thus, AC is an irrational (straight-line) [Def. 10.4]. And let it be called the second apotome of a medial (straight-line).† (Which is) the very thing it was required to show. † See footnote to Prop. 10.38.

Proposition 76

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the (squares) on them (added) together rational, and the (rectangle contained) by them medial, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called a minor (straight-line).



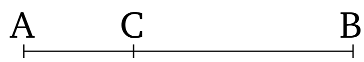
For let the straight-line BC, which is incommensurable in square with the whole, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.33]. I say that the remainder AC is that irrational (straight-line) called minor.

For since the sum of the squares on AB and BC is rational, and twice the (rectangle contained) by AB and BC (is) medial, the (sum of the squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC. And, via conversion, the (sum of the squares) on AB and BC is incommensurable with the remaining (square) on AC [Props. 2.7, 10.16]. And the (sum of the squares) on AB and BC (is) rational. The (square) on AC

(is) thus irrational. Thus, AC (is) an irrational (straight-line) [Def. 10.4]. Let it be called a minor (straight-line).† (Which is) the very thing it was required to show. † See footnote to Prop. 10.39.

Proposition 77

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called that which makes with a rational (area) a medial whole.

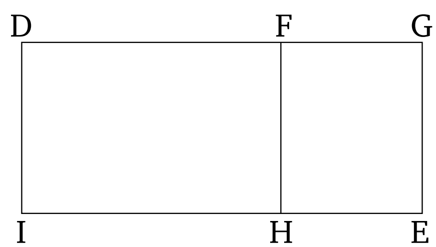


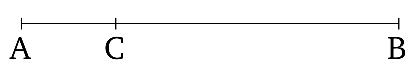
For let the straight-line BC, which is incommensurable in square with AB, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.34]. I say that the remainder AC is the aforementioned irrational (straight-line).

For since the sum of the squares on AB and BC is medial, and twice the (rectangle contained) by AB and BC rational, the (sum of the squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC. Thus, the remaining (square) on AC is also incommensurable with twice the (rectangle contained) by AB and BC [Props. 2.7, 10.16]. And twice the (rectangle contained) by AB and BC is rational. Thus, the (square) on AC is irrational. Thus, AC is an irrational (straight-line) [Def. 10.4]. And let it be called that which makes with a rational (area) a medial whole.† (Which is) the very thing it was required to show. † See footnote to Prop. 10.40.

Proposition 78

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called that which makes with a medial (area) a medial whole.





For let the straight-line BC, which is incommensurable in square AB, and fulfils the (other) prescribed (conditions), have been subtracted from the (straight-line) AB [Prop. 10.35]. I say that the remainder AC is the irrational (straight-line) called that which makes with a medial (area) a medial whole.

For let the rational (straight-line) DI be laid down. And let DE, equal to the (sum of the squares) on AB and BC, have been applied to DI, producing DG as breadth. And let DH, equal to twice the (rectangle contained) by AB and BC, have been subtracted (from DE) [producing DF as breadth]. Thus, the remainder FE is equal to the (square) on AC [Prop. 2.7]. Hence, AC is the square-root of FE. And since the sum of the squares on AB

and BC is medial, and is equal to DE, DE [is] thus medial. And it is applied to the rational (straight-line) DI, producing DG as breadth. Thus, DG is rational, and incommensurable in length with DI [Prop 10.22]. Again, since twice the (rectangle contained) by AB and BC is medial, and is equal to DH, DH is thus medial. And it is applied to the rational (straight-line) DI, producing DF as breadth. Thus, DF is also rational, and incommensurable in length with DI [Prop. 10.22]. And since the (sum of the squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC, DE (is) also incommensurable with DH. And as DE (is) to DH, so DG also is to DF [Prop. 6.1]. Thus, DG (is) incommensurable (in length) with DF [Prop. 10.11]. And they are both rational. Thus, GD and DF are rational (straight-lines which are) commensurable in square only. Thus, FG is an apotome [Prop. 10.73]. And FH (is) rational. And the [rectangle] contained by a rational (straight-line) and an apotome is irrational [Prop. 10.20], and its square-root is irrational. And AC is the square-root of FE. Thus, AC is irrational. Let it be called that which makes with a medial (area) a medial whole.† (Which is) the very thing it was required to show. † See footnote to Prop. 10.41.

Proposition 79

[Only] one rational straight-line, which is commensurable in square only with the whole, can be attached to an apotome.†

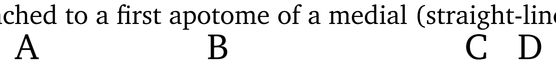
Let AB be an apotome, with BC (so) attached to it. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.73]. I say that another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area). For both exceed by the same (area)—(namely), the (square) on AB [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB [also] exceeds twice the (rectangle contained) by AC and CB by this (same area). And the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB by a rational (area). For both (are) rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.21], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26]. Thus, another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB.

Thus, only one rational (straight-line), which is commensurable in square only with the whole, can be attached to an apotome. (Which is) the very thing it was required to show. † This proposition is equivalent to Prop. 10.42, with minus signs instead of plus signs.

Proposition 80

Only one medial straight-line, which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line).†



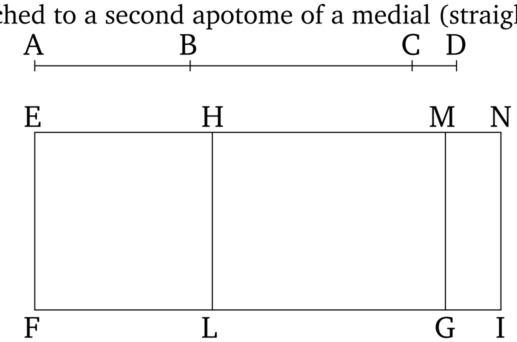
For let AB be a first apotome of a medial (straight-line), and let BC be (so) attached to AB. Thus, AC and CB are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that contained) by AC and CB [Prop. 10.74]. I say that a(nother) medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, cannot be attached to AB.

For, if possible, let DB also be (so) attached to AB. Thus, AD and DB are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that) contained by AD and DB [Prop. 10.74]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB, the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area). For [again] both exceed by the same (area)—(namely), the (square) on AB [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area). And twice the (rectangle contained) by AD and DB exceeds twice the (rectangle contained) by AC and CB by a rational (area). For both (are) rational (areas). Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the) [squares] on AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Props. 10.15, 10.23 corr.], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26].

Thus, only one medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line). (Which is) the very thing it was required to show. † This proposition is equivalent to Prop. 10.43, with minus signs instead of plus signs.

Proposition 81

Only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line).†



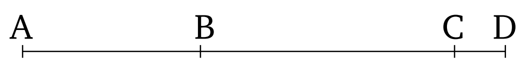
Let AB be a second apotome of a medial (straight-line), with BC (so) attached to AB. Thus, AC and CB are medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by AC and CB [Prop. 10.75]. I say that a(nother) medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, cannot be attached to AB.

For, if possible, let BD be (so) attached. Thus, AD and DB are also medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by AD and DB [Prop. 10.75]. And let the rational (straight-line) EF be laid down. And let EG, equal to the (sum of the squares) on AC and CB, have been applied to EF, producing EM as breadth. And let HG, equal to twice the (rectangle contained) by AC and CB, have been subtracted (from EG), producing HM as breadth. The remainder EL is thus equal to the (square) on AB [Prop. 2.7]. Hence, AB is the square-root of EL. So, again, let EI, equal to the (sum of the squares) on AD and DB have been applied to EF, producing EN as breadth. And EL is also equal to the square on AB. Thus, the remainder HI is equal to twice the (rectangle contained) by AD and DB [Prop. 2.7]. And since AC and CB are (both) medial (straight-lines), the (sum of the squares) on AC and CB is also medial. And it is equal to EG. Thus, EG is also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) EF, producing EM as breadth. Thus, EM is rational, and incommensurable in length with EF [Prop. 10.22]. Again, since the (rectangle contained) by AC and CB is medial, twice the (rectangle contained) by AC and CB is also medial [Prop. 10.23 corr.]. And it is equal to HG. Thus, HG is also medial. And it is applied to the rational (straight-line) EF, producing HM as breadth. Thus, HM is also rational, and incommensurable in length with EF [Prop. 10.22]. And since AC and CB are commensurable in square only, AC is thus incommensurable in length with CB. And as AC (is) to CB, so the (square) on AC is to the (rectangle contained) by AC and CB [Prop. 10.21 corr.]. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11]. But, the (sum of the squares) on AC and CB is commensurable with the (square) on AC, and twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6]. Thus, the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13]. And EG is equal to the (sum of the squares) on AC and CB. And GH is equal to twice the (rectangle contained) by AC and CB. Thus, EG is incommensurable with HG. And as EG (is) to HG, so EM is to HM [Prop. 6.1]. Thus, EM is incommensurable in length with MH [Prop. 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], and HM (is) attached to it. So, similarly, we can show that HN (is) also (commensurable in square only with EN and is) attached to (EH). Thus, different straight-lines, which are commensurable in square only with the whole, are attached to an apotome. The very thing is impossible [Prop. 10.79].

Thus, only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line). (Which is) the very thing it was required to show. † This proposition is equivalent to Prop. 10.44, with minus signs instead of plus signs.

Proposition 82

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the (sum of the) squares on them rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line).



Let AB be a minor (straight-line), and let BC be (so) attached to AB. Thus, AC and CB are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial [Prop. 10.76]. I say that another another straight-line fulfilling the same (conditions) cannot be attached to AB.

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also (straight-lines which are) incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.76]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area) [Prop. 2.7]. And the (sum of the) squares on AD and DB exceeds the (sum of the) squares on AC and CB by a rational (area). For both are rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, only one straight-line, which is incommensurable in square with the whole, and (with the whole) makes the squares on them (added) together rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line). (Which is) the very thing it was required to show. † This proposition is equivalent to Prop. 10.45, with minus signs instead of plus signs.

Proposition 83

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, can be attached to that (straight-line) which with a rational (area) makes a medial whole.†

Let AB be a (straight-line) which with a rational (area) makes a medial whole, and let BC be (so) attached to AB. Thus, AC and CB are (straight-lines which are) incommensurable in square, fulfilling the (other) proscribed (conditions) [Prop. 10.77]. I say that another (straight-line) fulfilling the same (conditions) cannot be attached to AB.

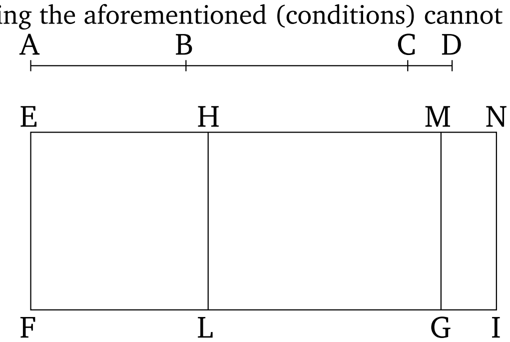
For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also straight-lines (which are) incommensurable in square, fulfilling the (other) prescribed (conditions) [Prop. 10.77]. Therefore, analogously to the (propositions) before this, since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area). And twice the (rectangle contained) by AD and DB exceeds twice the (rectangle contained) by AC and CB by a rational (area). For they are (both) rational (areas). Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the squares) on AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, another straight-line cannot be attached to AB, which is incommensurable in square with the whole, and fulfills the (other) aforementioned (conditions) with the whole. Thus, only one (such straight-line) can be (so) attached. (Which is) the very thing it was required to show. † This proposition is equivalent to Prop. 10.46, with minus signs instead of plus signs.

Proposition 84

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the (squares) on them, can be attached to that (straight-line) which with a medial (area) makes a medial whole.†

Let AB be a (straight-line) which with a medial (area) makes a medial whole, BC being (so) attached to it. Thus, AC and CB are incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.78]. I say that a(nother) (straight-line) fulfilling the aforementioned (conditions) cannot be attached to AB.



For, if possible, let BD be (so) attached. Hence, AD and DB are also (straight-lines which are) incommensurable in square, making the squares on AD and DB (added) together medial, and twice the (rectangle contained) by AD and DB medial, and, moreover, the (sum of the squares) on AD and DB incommensurable with twice the (rectangle contained) by AD and DB [Prop. 10.78]. And let the rational (straight-line) EF be laid down. And let EG, equal to the (sum of the squares) on AC and CB, have been applied to EF, producing EM as breadth. And let HG, equal to twice the (rectangle contained) by AC and CB, have been applied to EF, producing HM as breadth. Thus, the remaining (square) on AB is equal to EL [Prop. 2.7]. Thus, AB is the square-root of EL. Again, let EI, equal to the (sum of the squares) on AD and DB, have been applied to EF, producing EN as breadth. And the (square) on AB is also equal to EL. Thus, the remaining twice the (rectangle contained) by AD and DB [is] equal to HI [Prop. 2.7]. And since the sum of the (squares) on AC and CB is medial, and is equal to EG, EG is thus also medial. And it is applied to the rational (straight-line) EF, producing EM as breadth. EM is thus rational, and incommensurable in length with EF [Prop. 10.22]. Again, since twice the (rectangle contained) by AC and CB is medial, and is equal to HG, HG is thus also medial. And it is applied to the rational (straight-line) EF, producing HM as breadth. HM is thus rational, and incommensurable in length with EF [Prop. 10.22]. And since the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB, EG is also incommensurable with HG. Thus, EM is also incommensurable in length with MH [Props. 6.1, 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], with HM attached to it. So, similarly, we can show that EH is again an apotome, with HN attached to it. Thus, different rational (straight-lines), which are commensurable in square only with the whole, are attached to an apotome. The very thing was shown (to be) impossible [Prop. 10.79]. Thus, another straight-line cannot be (so) attached to AB.

Thus, only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the squares on them (added) together medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with the (rectangle contained) by them, can be attached to AB. (Which is) the very thing it was required to show. † This proposition is equivalent to Prop. 10.47, with minus signs instead of plus signs.

Definitions III

11. Given a rational (straight-line) and an apotome, if the square on the whole is greater than the (square on a straight-line) attached (to the apotome) by the (square) on (some straight-line) commensurable in length with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a first apotome.

12. And if the attached (straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a second apotome.

13. And if neither of (the whole or the attached straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a third apotome.

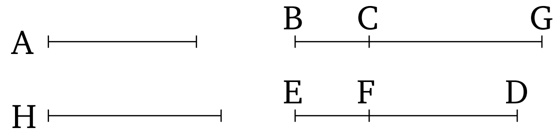
14. Again, if the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) incommensurable [in length] with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a fourth apotome.

15. And if the attached (straight-line is commensurable), a fifth (apotome).

16. And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

Proposition 85

To find a first apotome.



Let the rational (straight-line) A be laid down. And let BG be commensurable in length with A. BG is thus also a rational (straight-line). And let two square numbers DE and EF be laid down, and let their difference FD be not square [Prop. 10.28 lem. I]. Thus, ED does not have to DF the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as ED (is) to DF, so the square on BG (is) to the square on GC [Prop. 10.6. corr.]. Thus, the (square) on BG is commensurable with the (square) on GC [Prop. 10.6]. And the (square) on BG (is) rational. Thus, the (square) on GC (is) also rational. Thus, GC is also rational. And since ED does not have to DF the ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). Thus, BG and GC are rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

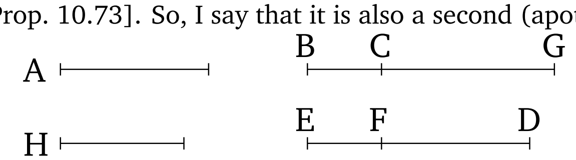
Let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.]. And since as ED is to FD, so the (square) on BG (is) to the (square) on GC, thus, via conversion, as DE is to EF, so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.]. And DE has to EF the ratio which (some) square-number (has) to (some) square-number. For each is a square (number). Thus, the (square) on GB also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, BG is commensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on BG is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with (BG). And the whole, BG, is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, BC is a first apotome [Def. 10.11].

Thus, the first apotome BC has been found. (Which is) the very thing it was required to find. † See footnote to Prop. 10.48.

Proposition 86

To find a second apotome.

Let the rational (straight-line) A, and GC (which is) commensurable in length with A, be laid down. Thus, GC is a rational (straight-line). And let the two square numbers DE and EF be laid down, and let their difference DF be not square [Prop. 10.28 lem. I]. And let it have been contrived that as FD (is) to DE, so the square on CG (is) to the square on GB [Prop. 10.6 corr.]. Thus, the square on CG is commensurable with the square on GB [Prop. 10.6]. And the (square) on CG (is) rational. Thus, the (square) on GB [is] also rational. Thus, BG is a rational (straight-line). And since the square on GC does not have to the (square) on GB the ratio which (some) square number (has) to (some) square number, CG is incommensurable in length with GB [Prop. 10.9]. And they are both rational (straight-lines). Thus, CG and GB are rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that it is also a second (apotome).



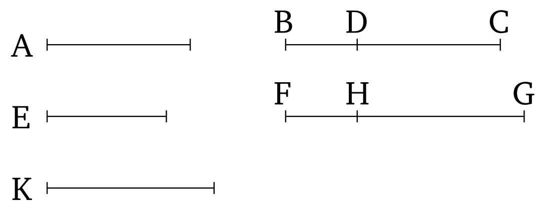
For let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.]. Therefore, since as the (square) on BG is to the (square) on GC, so the number ED (is) to the number DF, thus, also, via conversion, as the (square) on BG is to the (square) on H, so DE (is) to EF

[Prop. 5.19 corr.]. And DE and EF are each square (numbers). Thus, the (square) on BG has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, BG is commensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on BG is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with (BG). And the attachment CG is commensurable (in length) with the (prevously) laid down rational (straight-line) A. Thus, BC is a second apotome [Def. 10.12].†

Thus, the second apotome BC has been found. (Which is) the very thing it was required to show. † See footnote to Prop. 10.49.

Proposition 87

To find a third apotome.



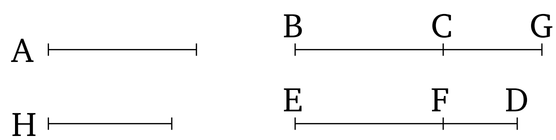
Let the rational (straight-line) A be laid down. And let the three numbers, E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. And let CB have to BD the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as E (is) to BC, so the square on A (is) to the square on FG, and as BC (is) to CD, so the square on FG (is) to the (square) on GH [Prop. 10.6 corr.]. Therefore, since as E is to BC, so the square on A (is) to the square on FG, the square on A is thus commensurable with the square on FG [Prop. 10.6]. And the square on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the square on A thus does not have to the [square] on FG the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with FG [Prop. 10.9]. Again, since as BC is to CD, so the square on FG is to the (square) on GH, the square on FG is thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH (is) also rational. Thus, GH is a rational (straight-line). And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straight-lines). FG and GH are thus rational (straight-lines which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since as E is to BC, so the square on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on HG, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on HG [Prop. 5.22]. And E does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. A (is) thus incommensurable in length with GH [Prop. 10.9]. Thus, neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.]. Therefore, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, thus, via conversion, as BC is to BD, so the square on FG (is) to the square on K [Prop. 5.19 corr.]. And BC has to BD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number (has) to (some) square number. FG is thus commensurable in length with K [Prop. 10.9]. And the square on FG is (thus) greater than (the square on) GH by the (square) on (some straight-line) commensurable (in length) with (FG). And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, FH is a third apotome [Def. 10.13].

Thus, the third apotome FH has been found. (Which is) very thing it was required to show. † See footnote to Prop. 10.50.

Proposition 88

To find a fourth apotome.



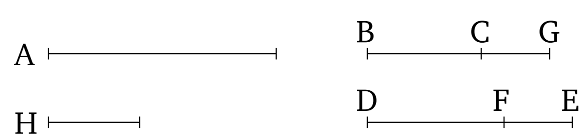
Let the rational (straight-line) A, and BG (which is) commensurable in length with A, be laid down. Thus, BG is also a rational (straight-line). And let the two numbers DF and FE be laid down such that the whole, DE, does not have to each of DF and EF the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as DE (is) to EF, so the square on BG (is) to the (square) on GC [Prop. 10.6 corr.]. The (square) on BG is thus commensurable with the (square) on GC [Prop. 10.6]. And the (square) on BG (is) rational. Thus, the (square) on GC (is) also rational. Thus, GC (is) a rational (straight-line). And since DE does not have to EF the ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). Thus, BG and GC are rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. [So, I say that (it is) also a fourth (apotome).]

Now, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.]. Therefore, since as DE is to EF, so the (square) on BG (is) to the (square) on GC, thus, also, via conversion, as ED is to DF, so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.]. And ED does not have to DF the ratio which (some) square number (has) to (some) square number. Thus, the (square) on GB does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on BG is greater than (the square) on GC by the (square) on (some straight-line) incommensurable (in length) with (BG). And the whole, BG, is commensurable in length with the the (previously) laid down rational (straight-line) A. Thus, BC is a fourth apotome [Def. 10.14].†

Thus, a fourth apotome has been found. (Which is) the very thing it was required to show. † See footnote to Prop. 10.51.

Proposition 89

To find a fifth apotome.



Let the rational (straight-line) A be laid down, and let CG be commensurable in length with A. Thus, CG [is] a rational (straight-line). And let the two numbers DF and FE be laid down such that DE again does not have to each of DF and FE the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as FE (is) to ED, so the (square) on CG (is) to the (square) on GB. Thus, the (square) on GB (is) also rational [Prop. 10.6]. Thus, BG is also rational. And since as DE is to EF, so the (square) on BG (is) to the (square) on GC. And DE does not have to EF the ratio which (some) square number (has) to (some) square number. The (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). BG and GC are thus rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

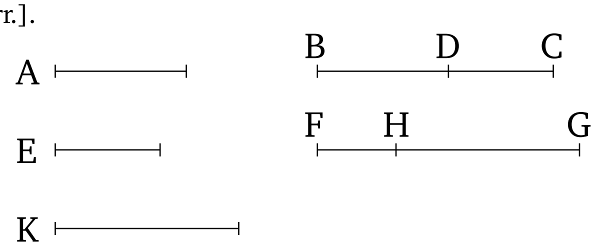
For, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.]. Therefore, since as the (square) on BG (is) to the (square) on GC, so DE (is) to EF, thus, via conversion, as ED is to DF, so the (square) on BG (is) to the (square) on H [Prop. 5.19 corr.]. And ED does not have to DF the ratio which (some) square number (has) to (some) square number. Thus, the (square) on BG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H. Thus, the square on GB is greater than (the square on) GC by the (square) on (some straight-line) incommensurable in length with (GB). And the attachment CG is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, BC is a fifth apotome [Def. 10.15].†

Thus, the fifth apotome BC has been found. (Which is) the very thing it was required to show. † See footnote to Prop. 10.52.

Proposition 90

To find a sixth apotome.

Let the rational (straight-line) A, and the three numbers E, BC, and CD, not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. Furthermore, let CB also not have to BD the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as E (is) to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.].



Therefore, since as E is to BC, so the (square) on A (is) to the (square) on FG, the (square) on A (is) thus commensurable with the (square) on FG [Prop. 10.6]. And the (square) on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is also a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with FG [Prop. 10.9]. Again, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, the (square) on FG (is) thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH (is) also rational. Thus, GH (is) also rational. And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square (number) has to (some) square (number) either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straight-lines). Thus, FG and GH are rational (straight-lines which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

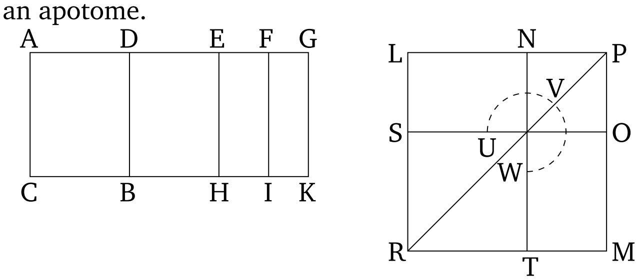
For since as E is to BC, so the (square) on A (is) to the (square) on FG, and as BC (is) to CD, so the (square) on FG (is) to the (square) on GH, thus, via equality, as E is to CD, so the (square) on A (is) to the (square) on GH [Prop. 5.22]. And E does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) GH the ratio which (some) square number (has) to (some) square number either. A is thus incommensurable in length with GH [Prop. 10.9]. Thus, neither of FG and GH is commensurable in length with the rational (straight-line) A. Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.]. Therefore, since as BC is to CD, so the (square) on FG (is) to the (square) on GH, thus, via conversion, as CB is to BD, so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And CB does not have to BD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. FG is thus incommensurable in length with K [Prop. 10.9]. And the square on FG is greater than (the square on) GH by the (square) on K. Thus, the square on FG is greater than (the square on) GH by the (square) on (some straight-line) incommensurable in length with (FG). And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A. Thus, FH is a sixth apotome [Def. 10.16].

Thus, the sixth apotome FH has been found. (Which is) the very thing it was required to show. † See footnote to Prop. 10.53.

Proposition 91

If an area is contained by a rational (straight-line) and a first apotome then the square-root of the area is an apotome.

For let the area AB have been contained by the rational (straight-line) AC and the first apotome AD. I say that the square-root of area AB is an apotome.



For since AD is a first apotome, let DG be its attachment. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And the whole, AG, is commensurable (in length) with the (previously) laid down rational (straight-line) AC, and the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.11]. Thus, if (an area) equal to the fourth part of the (square) on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Let DG have been cut in half at E. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. AF is thus commensurable (in length) with FG. And let EH, FI, and GK have been drawn through points E, F, and G (respectively), parallel to AC.

And since AF is commensurable in length with FG, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. But AG is commensurable (in length) with AC. Thus, each of AF and FG is also commensurable in length with AC [Prop. 10.12]. And AC is a rational (straight-line). Thus, AF and FG (are) each also rational (straight-lines). Hence, AI and FK are also each rational (areas) [Prop. 10.19]. And since DE is commensurable in length with EG, DG is thus also commensurable in length with each of DE and EG [Prop. 10.15]. And DG (is) rational, and incommensurable in length with AC. DE and EG (are) thus each rational, and incommensurable in length with AC [Prop. 10.13]. Thus, DH and EK are each medial (areas) [Prop. 10.21].

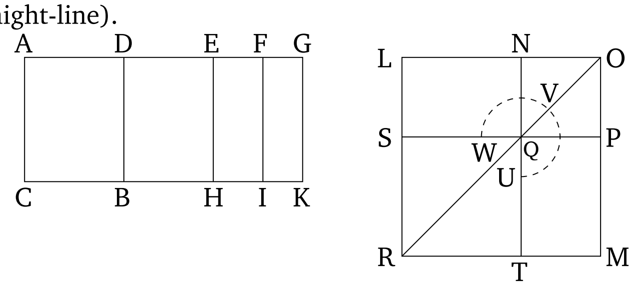
So let the square LM, equal to AI, be laid down. And let the square NO, equal to FK, have been subtracted (from LM), having with it the common angle LPM. Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the rectangle contained by AF and FG is equal to the square EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI (is) to EK, and as EG (is) to FG, so EK is to KF [Prop. 6.1]. Thus, EK is the mean proportional to AI and KF [Prop. 5.11]. And MN is also the mean proportional to LM and NO, as shown before [Prop. 10.53 lem.]. And AI is equal to the square LM, and KF to NO. Thus, MN is also equal to EK. But, EK is equal to DH, and MN to LO [Prop. 1.43]. Thus, DK is equal to the gnomon UVW and NO. And AK is also equal to (the sum of) the squares LM and NO. Thus, the remainder AB is equal to ST. And ST is the square on LN. Thus, the square on LN is equal to AB. Thus, LN is the square-root of AB. So, I say that LN is an apotome.

For since AI and FK are each rational (areas), and are equal to LM and NO (respectively), thus LM and NO—that is to say, the (squares) on each of LP and PN (respectively)—are also each rational (areas). Thus, LP and PN are also each rational (straight-lines). Again, since DH is a medial (area), and is equal to LO, LO is thus also a medial (area). Therefore, since LO is medial, and NO rational, LO is thus incommensurable with NO. And as LO (is) to NO, so LP is to PN [Prop. 6.1]. LP is thus incommensurable in length with PN [Prop. 10.11]. And they are both rational (straight-lines). Thus, LP and PN are rational (straight-lines which are) commensurable in square only. Thus, LN is an apotome [Prop. 10.73]. And it is the square-root of area AB. Thus, the square-root of area AB is an apotome.

Thus, if an area is contained by a rational (straight-line), and so on ....

Proposition 92

If an area is contained by a rational (straight-line) and a second apotome then the square-root of the area is a first apotome of a medial (straight-line).



For let the area AB have been contained by the rational (straight-line) AC and the second apotome AD. I say that the square-root of area AB is the first apotome of a medial (straight-line).

For let DG be an attachment to AD. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment DG is commensurable (in length) with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, GD, by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.12]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG), thus if (an area) equal to the fourth part of the (square) on GD is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let DG have been cut in half at E. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. Thus, AF is commensurable in length with FG. AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. And AG (is) a rational (straight-line), and incommensurable in length with AC. AF and FG are thus also each rational (straight-lines), and incommensurable in length with AC [Prop. 10.13]. Thus, AI and FK are each medial (areas) [Prop. 10.21]. Again, since DE is commensurable (in length) with EG, thus DG is also commensurable (in length) with each of DE and EG [Prop. 10.15]. But, DG is commensurable in length with AC [thus, DE and EG are also each rational, and commensurable in length with AC]. Thus, DH and EK are each rational (areas) [Prop. 10.19].

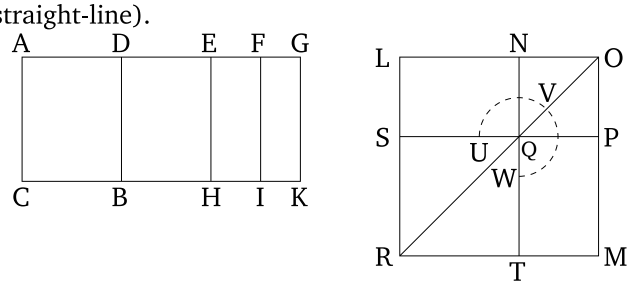
Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, which is about the same angle LPM as LM, have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since AI and FK are medial (areas), and are equal to the (squares) on LP and PN (respectively), [thus] the (squares) on LP and PN are also medial. Thus, LP and PN are also medial (straight-lines which are) commensurable in square only.† And since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus as AF is to EG, so EG (is) to FG [Prop. 10.17]. But, as AF (is) to EG, so AI (is) to EK. And as EG (is) to FG, so EK [is] to FK [Prop. 6.1]. Thus, EK is the mean proportional to AI and FK [Prop. 5.11]. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.]. And AI is equal to LM, and FK to NO. Thus, MN is also equal to EK. But, DH [is] equal to EK, and LO equal to MN [Prop. 1.43]. Thus, the whole (of) DK is equal to the gnomon UVW and NO. Therefore, since the whole (of) AK is equal to LM and NO, of which DK is equal to the gnomon UVW and NO, the remainder AB is thus equal to TS. And TS is the (square) on LN. Thus, the (square) on LN is equal to the area AB. LN is thus the square-root of area AB. [So], I say that LN is the first apotome of a medial (straight-line).

For since EK is a rational (area), and is equal to LO, LO—that is to say, the (rectangle contained) by LP and PN—is thus a rational (area). And NO was shown (to be) a medial (area). Thus, LO is incommensurable with NO. And as LO (is) to NO, so LP is to PN [Prop. 6.1]. Thus, LP and PN are incommensurable in length [Prop. 10.11]. LP and PN are thus medial (straight-lines which are) commensurable in square only, and which contain a rational (area). Thus, LN is the first apotome of a medial (straight-line) [Prop. 10.74]. And it is the square-root of area AB.

Thus, the square root of area AB is the first apotome of a medial (straight-line). (Which is) the very thing it was required to show. † There is an error in the argument here. It should just say that LP and PN are commensurable in square, rather than in square only, since LP and PN are only shown to be incommensurable in length later on.

Proposition 93

If an area is contained by a rational (straight-line) and a third apotome then the square-root of the area is a second apotome of a medial (straight-line).



For let the area AB have been contained by the rational (straight-line) AC and the third apotome AD. I say that the square-root of area AB is the second apotome of a medial (straight-line).

For let DG be an attachment to AD. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of AG and GD is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment,

DG, by the (square) on (some straight-line) commensurable (in length) with (AG) [Def. 10.13]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG), thus if (an area) equal to the fourth part of the square on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let DG have been cut in half at E. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. And let EH, FI, and GK have been drawn through points E, F, and G (respectively), parallel to AC. Thus, AF and FG are commensurable (in length). AI (is) thus also commensurable with FK [Props. 6.1, 10.11]. And since AF and FG are commensurable in length, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. And AG (is) rational, and incommensurable in length with AC. Hence, AF and FG (are) also (rational, and incommensurable in length with AC) [Prop. 10.13]. Thus, AI and FK are each medial (areas) [Prop. 10.21]. Again, since DE is commensurable in length with EG, DG is also commensurable in length with each of DE and EG [Prop. 10.15]. And GD (is) rational, and incommensurable in length with AC. Thus, DE and EG (are) each also rational, and incommensurable in length with AC [Prop. 10.13]. DH and EK are thus each medial (areas) [Prop. 10.21]. And since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD. But, AG is commensurable in length with AF, and DG with EG. Thus, AF is incommensurable in length with EG [Prop. 10.13]. And as AF (is) to EG, so AI is to EK [Prop. 6.1]. Thus, AI is incommensurable with EK [Prop. 10.11].

Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, which is about the same angle as LM, have been subtracted (from LM). Thus, LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus as AF is to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI is to EK [Prop. 6.1]. And as EG (is) to FG, so EK is to FK [Prop. 6.1]. And thus as AI (is) to EK, so EK (is) to FK [Prop. 5.11]. Thus, EK is the mean proportional to AI and FK. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.]. And AI is equal to LM, and FK to NO. Thus, EK is also equal to MN. But, MN is equal to LO, and EK [is] equal to DH [Prop. 1.43]. And thus the whole of DK is equal to the gnomon UVW and NO. And AK (is) also equal to LM and NO. Thus, the remainder AB is equal to ST—that is to say, to the square on LN. Thus, LN is the square-root of area AB. I say that LN is the second apotome of a medial (straight-line).

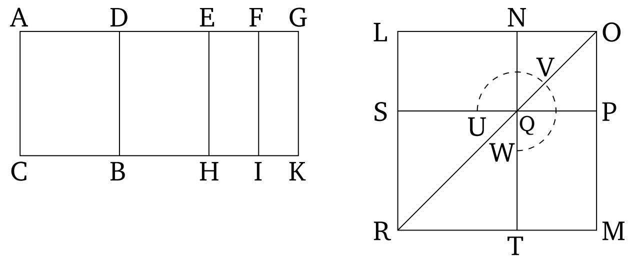
For since AI and FK were shown (to be) medial (areas), and are equal to the (squares) on LP and PN (respectively), the (squares) on each of LP and PN (are) thus also medial. Thus, LP and PN (are) each medial (straight-lines). And since AI is commensurable with FK [Props. 6.1, 10.11], the (square) on LP (is) thus also commensurable with the (square) on PN. Again, since AI was shown (to be) incommensurable with EK, LM is thus also incommensurable with MN—that is to say, the (square) on LP with the (rectangle contained) by LP and PN. Hence, LP is also incommensurable in length with PN [Props. 6.1, 10.11]. Thus, LP and PN are medial (straight-lines which are) commensurable in square only. So, I say that they also contain a medial (area).

For since EK was shown (to be) a medial (area), and is equal to the (rectangle contained) by LP and PN, the (rectangle contained) by LP and PN is thus also medial. Hence, LP and PN are medial (straight-lines which are) commensurable in square only, and which contain a medial (area). Thus, LN is the second apotome of a medial (straight-line) [Prop. 10.75]. And it is the square-root of area AB.

Thus, the square-root of area AB is the second apotome of a medial (straight-line). (Which is) the very thing it was required to show.

Proposition 94

If an area is contained by a rational (straight-line) and a fourth apotome then the square-root of the area is a minor (straight-line).



For let the area AB have been contained by the rational (straight-line) AC and the fourth apotome AD. I say that the square-root of area AB is a minor (straight-line). For let DG be an attachment to AD. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and AG is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the square on (some straight-line) incommensurable in length with (AG) [Def. 10.14]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG), thus if (some area), equal to the fourth part of the (square) on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been cut in half at E, and let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure, and let it be the (rectangle contained) by AF and FG. Thus, AF is incommensurable in length with FG. Therefore, let EH, FI, and GK have been drawn through E, F, and G (respectively), parallel to AC and BD. Therefore, since AG is rational, and commensurable in length with AC, the whole (area) AK is thus rational [Prop. 10.19]. Again, since DG is incommensurable in length with AC, and both are rational (straight-lines), DK is thus a medial (area) [Prop. 10.21]. Again, since AF is incommensurable in length with FG, AI (is) thus also incommensurable with FK [Props. 6.1, 10.11].

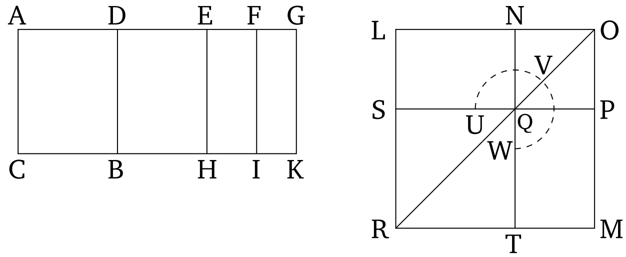
Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, (and) about the same angle, LPM, have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG, thus, proportionally, as AF is to EG, so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG, so AI is to EK, and as EG (is) to FG, so EK is to FK [Prop. 6.1]. Thus, EK is the mean proportional to AI and FK [Prop. 5.11]. And MN is also the mean proportional to the squares LM and NO [Prop. 10.13 lem.], and AI is equal to LM, and FK to NO. EK is thus also equal to MN. But, DH is equal to EK, and LO is equal to MN [Prop. 1.43]. Thus, the whole of DK is equal to the gnomon UVW and NO. Therefore, since the whole of AK is equal to the (sum of the) squares LM and NO, of which DK is equal to the gnomon UVW and the square NO, the remainder AB is thus equal to ST—that is to say, to the square on LN. Thus, LN is the square-root of area AB. I say that LN is the irrational (straight-line which is) called minor.

For since AK is rational, and is equal to the (sum of the) squares LP and PN, the sum of the (squares) on LP and PN is thus rational. Again, since DK is medial, and DK is equal to twice the (rectangle contained) by LP and PN, thus twice the (rectangle contained) by LP and PN is medial. And since AI was shown (to be) incommensurable with FK, the square on LP (is) thus also incommensurable with the square on PN. Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. LN is thus the irrational (straight-line) called minor [Prop. 10.76]. And it is the square-root of area AB.

Thus, the square-root of area AB is a minor (straight-line). (Which is) the very thing it was required to show.

Proposition 95

If an area is contained by a rational (straight-line) and a fifth apotome then the square-root of the area is that (straight-line) which with a rational (area) makes a medial whole.



For let the area AB have been contained by the rational (straight-line) AC and the fifth apotome AD. I say that the square-root of area AB is that (straight-line) which with a rational (area) makes a medial whole.

For let DG be an attachment to AD. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment GD is commensurable in length the the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straight-line) incommensurable (in length) with (AG) [Def. 10.15]. Thus, if (some area), equal to the fourth part of the (square) on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been divided in half at point E, and let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure, and let it be the (rectangle contained) by AF and FG. Thus, AF is incommensurable in length with FG. And since AG is incommensurable in length with CA, and both are rational (straight-lines), AK is thus a medial (area) [Prop. 10.21]. Again, since DG is rational, and commensurable in length with AC, DK is a rational (area) [Prop. 10.19].

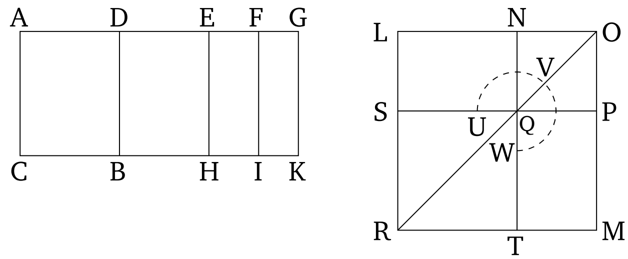
Therefore, let the square LM, equal to AI, have been constructed. And let the square NO, equal to FK, (and) about the same angle, LPM, have been subtracted (from NO). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly (to the previous propositions), we can show that LN is the square-root of area AB. I say that LN is that (straight-line) which with a rational (area) makes a medial whole.

For since AK was shown (to be) a medial (area), and is equal to (the sum of) the squares on LP and PN, the sum of the (squares) on LP and PN is thus medial. Again, since DK is rational, and is equal to twice the (rectangle contained) by LP and PN, (the latter) is also rational. And since AI is incommensurable with FK, the (square) on LP is thus also incommensurable with the (square) on PN. Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational. Thus, the remainder LN is the irrational (straight-line) called that which with a rational (area) makes a medial whole [Prop. 10.77]. And it is the square-root of area AB.

Thus, the square-root of area AB is that (straight-line) which with a rational (area) makes a medial whole. (Which is) the very thing it was required to show.

Proposition 96

If an area is contained by a rational (straight-line) and a sixth apotome then the square-root of the area is that (straight-line) which with a medial (area) makes a medial whole.



For let the area AB have been contained by the rational (straight-line) AC and the sixth apotome AD. I say that the square-root of area AB is that (straight-line) which with a medial (area) makes a medial whole.

For let DG be an attachment to AD. Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of them is commensurable in length with the (previously) laid down rational (straight-line) AC, and the square on the whole, AG, is greater than (the square on) the attachment, DG, by the (square) on (some straight-line) incommensurable in length with (AG) [Def. 10.16]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG), thus if (some area), equal to the fourth part of square on DG, is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been cut in half at [point] E. And let (some area), equal to the (square) on EG, have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. AF is thus incommensurable in length with FG. And as AF (is) to FG, so AI is to FK [Prop. 6.1]. Thus, AI is incommensurable with FK [Prop. 10.11]. And since AG and AC are rational (straight-lines which are) commensurable in square only, AK is a medial (area) [Prop. 10.21]. Again, since AC and DG are rational (straight-lines which are) incommensurable in length, DK is also a medial (area) [Prop. 10.21]. Therefore, since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD. And as AG (is) to GD, so AK is to KD [Prop. 6.1]. Thus, AK is incommensurable with KD [Prop. 10.11].

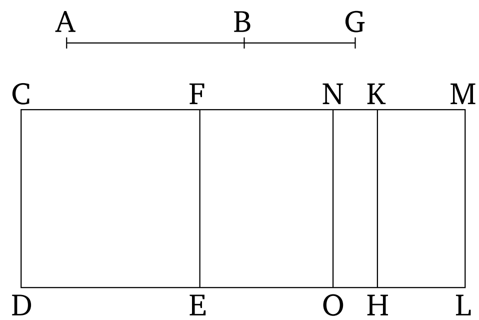
Therefore, let the square LM, equal to AI, have been constructed. And let NO, equal to FK, (and) about the same angle, have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly to the above, we can show that LN is the square-root of area AB. I say that LN is that (straight-line) which with a medial (area) makes a medial whole.

For since AK was shown (to be) a medial (area), and is equal to the (sum of the) squares on LP and PN, the sum of the (squares) on LP and PN is medial. Again, since DK was shown (to be) a medial (area), and is equal to twice the (rectangle contained) by LP and PN, twice the (rectangle contained) by LP and PN is also medial. And since AK was shown (to be) incommensurable with DK, [thus] the (sum of the) squares on LP and PN is also incommensurable with twice the (rectangle contained) by LP and PN. And since AI is incommensurable with FK, the (square) on LP (is) thus also incommensurable with the (square) on PN. Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by medial, and, furthermore, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them. Thus, LN is the irrational (straight-line) called that which with a medial (area) makes a medial whole [Prop. 10.78]. And it is the square-root of area AB.

Thus, the square-root of area (AB) is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to show.

Proposition 97

The (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth.



Let AB be an apotome, and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a first apotome.

For let BG be an attachment to AB. Thus, AG and GB are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let CH, equal to the (square) on AG, and KL, (equal) to the (square) on BG, have been applied to CD. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB. The remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Let FM have been cut in half at point N. And let NO have been drawn through N, parallel to CD. Thus, FO and LN are each equal to the (rectangle contained) by AG and GB. And since the (sum of the squares) on AG and GB is rational, and DM is equal to the (sum of the squares) on AG and GB, DM is thus rational. And it has been applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM is rational, and commensurable in length with CD [Prop. 10.20]. Again, since twice the (rectangle contained) by AG and GB is medial, and FL (is) equal to twice the (rectangle contained) by AG and GB, FL (is) thus a medial (area). And it is applied to the rational (straight-line) CD, producing FM as breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since the (sum of the squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB. And CL is equal to the (sum of the squares) on AG and GB, and FL to twice the (rectangle contained) by AG and GB. DM is thus incommensurable with FL. And as DM (is) to FL, so CM is to FM [Prop. 6.1]. CM is thus incommensurable in length with FM [Prop. 10.11]. And both are rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

For since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on BG, and NL to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to KM [Prop. 6.1]. Thus, the (rectangle contained) by CK and KM is equal to the (square) on NM—that is to say, to the fourth part of the (square) on FM [Prop. 6.17]. And since the (square) on AG is commensurable with the (square) on GB, CH [is] also commensurable with KL. And as CH (is) to KL, so CK (is) to KM [Prop. 6.1]. CK is thus commensurable (in length) with KM [Prop. 10.11]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and CK is commensurable (in length) with KM, the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. And CM is commensurable in length with the (previously) laid down rational (straight-line) CD. Thus, CF is a first apotome [Def. 10.15].

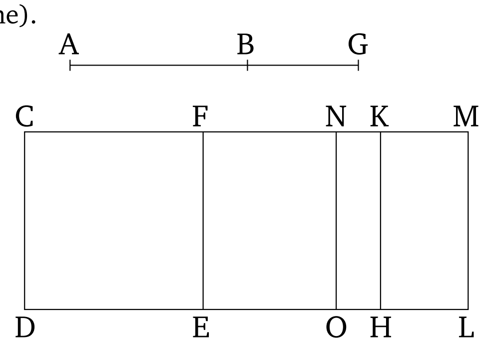
Thus, the (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth. (Which is) the very thing it was required to show.

Proposition 98

The (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth.

Let AB be a first apotome of a medial (straight-line), and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a second apotome.

For let BG be an attachment to AB. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a rational (area) [Prop. 10.74]. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on GB, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB. Thus, CL (is) also a medial (area) [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) CD, producing CM as breadth. CM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since CL is equal to the (sum of the squares) on AG and GB, of which the (square) on AB is equal to CE, the remainder, twice the (rectangle contained) by AG and GB, is thus equal to FL [Prop. 2.7]. And twice the (rectangle contained) by AG and GB [is] rational. Thus, FL (is) rational. And it is applied to the rational (straight-line) FE, producing FM as breadth. FM is thus also rational, and commensurable in length with CD [Prop. 10.20]. Therefore, since the (sum of the squares) on AG and GB—that is to say, CL—is medial, and twice the (rectangle contained) by AG and GB—that is to say, FL—(is) rational, CL is thus incommensurable with FL. And as CL (is) to FL, so CM is to FM [Prop. 6.1]. Thus, CM (is) incommensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a second (apotome).

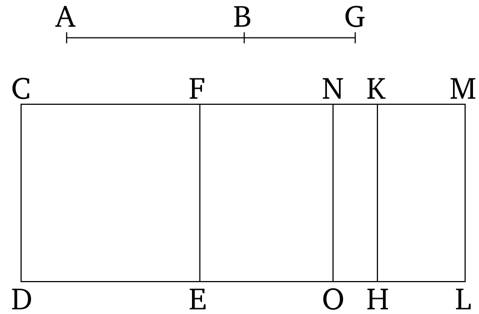


For let FM have been cut in half at N. And let NO have been drawn through (point) N, parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since the (rectangle contained) by AG and GB is the mean proportional to the squares on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (rectangle contained) by AG and GB to NL, and the (square) on BG to KL, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL [Prop. 5.11]. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to MK [Prop. 6.1]. Thus, as CK (is) to NM, so NM is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on NM [Prop. 6.17]—that is to say, to the fourth part of the (square) on FM [and since the (square) on AG is commensurable with the (square) on BG, CH is also commensurable with KL—that is to say, CK with KM]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to the greater CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. The attachment FM is also commensurable in length with the (previously) laid down rational (straight-line) CD. CF is thus a second apotome [Def. 10.16].

Thus, the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth. (Which is) the very thing it was required to show.

Proposition 99

The (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth.



Let AB be the second apotome of a medial (straight-line), and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a third apotome.

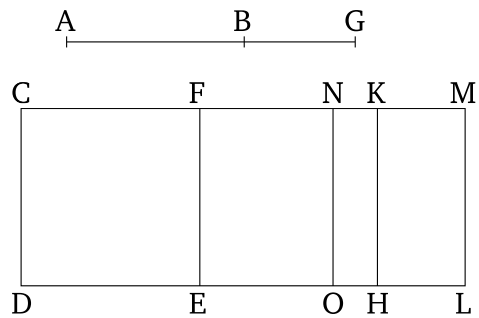
For let BG be an attachment to AB. Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a medial (area) [Prop. 10.75]. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth. And let KL, equal to the (square) on BG, have been applied to KH, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB [and the (sum of the squares) on AG and GB is medial]. CL (is) thus also medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder LF is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at point N. And let NO have been drawn parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial. And it is applied to the rational (straight-line) EF, producing FM as breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since AG and GB are commensurable in square only, AG [is] thus incommensurable in length with GB. Thus, the (square) on AG is also incommensurable with the (rectangle contained) by AG and GB [Props. 6.1, 10.11]. But, the (sum of the squares) on AG and GB is commensurable with the (square) on AG, and twice the (rectangle contained) by AG and GB with the (rectangle contained) by AG and GB. The (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.13]. But, CL is equal to the (sum of the squares) on AG and GB, and FL is equal to twice the (rectangle contained) by AG and GB. Thus, CL is incommensurable with FL. And as CL (is) to FL, so CM is to FM [Prop. 6.1]. CM is thus incommensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since the (square) on AG is commensurable with the (square) on GB, CH (is) thus also commensurable with KL. Hence, CK (is) also (commensurable in length) with KM [Props. 6.1, 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on GB, and NL equal to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM (is) to KM [Prop. 6.1]. Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11]. Thus, the (rectangle contained) by CK and KM is equal to the [(square) on MN—that is to say, to the] fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable (in length) with (CM) [Prop. 10.17]. And neither of CM and MF is commensurable in length with the (previously) laid down rational (straight-line) CD. CF is thus a third apotome [Def. 10.13].

Thus, the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth. (Which is) the very thing it was required to show.

Proposition 100

The (square) on a minor (straight-line), applied to a rational (straight-line), produces a fourth apotome as breadth.



Let AB be a minor (straight-line), and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to the rational (straight-line) CD, producing CF as breadth. I say that CF is a fourth apotome.

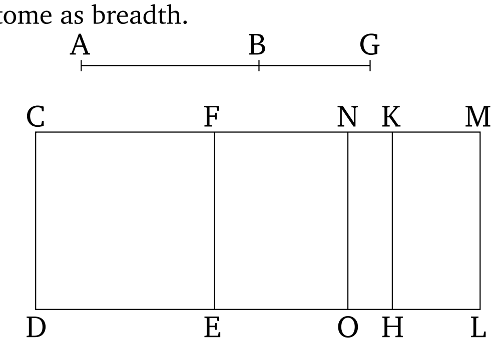
For let BG be an attachment to AB. Thus, AG and GB are incommensurable in square, making the sum of the squares on AG and GB rational, and twice the (rectangle contained) by AG and GB medial [Prop. 10.76]. And let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on BG, producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB. And the sum of the (squares) on AG and GB is rational. CL is thus also rational. And it is applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM (is) also rational, and commensurable in length with CD [Prop. 10.20]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at point N. And let NO have been drawn through N, parallel to either of CD or ML. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since twice the (rectangle contained) by AG and GB is medial, and is equal to FL, FL is thus also medial. And it is applied to the rational (straight-line) FE, producing FM as breadth. Thus, FM is rational, and incommensurable in length with CD [Prop. 10.22]. And since the sum of the (squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is [thus] incommensurable with twice the (rectangle contained) by AG and GB. And CL (is) equal to the (sum of the squares) on AG and GB, and FL equal to twice the (rectangle contained) by AG and GB. CL [is] thus incommensurable with FL. And as CL (is) to FL, so CM is to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. [So], I say that (it is) also a fourth (apotome).

For since AG and GB are incommensurable in square, the (square) on AG (is) thus also incommensurable with the (square) on GB. And CH is equal to the (square) on AG, and KL equal to the (square) on GB. Thus, CH is incommensurable with KL. And as CH (is) to KL, so CK is to KM [Prop. 6.1]. CK is thus incommensurable in length with KM [Prop. 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH, and the (square) on GB to KL, and the (rectangle contained) by AG and GB to NL, NL is thus the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. But, as CH (is) to NL, so CK is to NM, and as NL (is) to KL, so NM is to KM [Prop. 6.1]. Thus, as CK (is) to MN, so MN is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on MN—that is to say, to the fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM, equal to the fourth part of the (square) on MF, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18]. And the whole of CM is commensurable in length with the (previously) laid down rational (straight-line) CD. Thus, CF is a fourth apotome [Def. 10.14].

Thus, the (square) on a minor, and so on . . .

Proposition 101

The (square) on that (straight-line) which with a rational (area) makes a medial whole, applied to a rational (straight-line), produces a fifth apotome as breadth.



Let AB be that (straight-line) which with a rational (area) makes a medial whole, and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a fifth apotome.

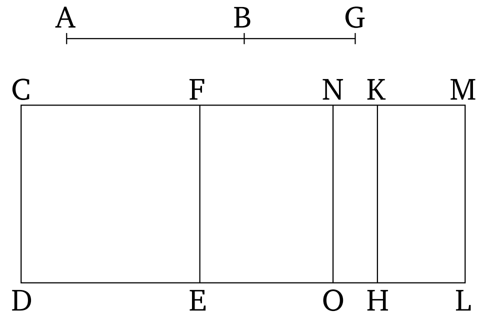
Let BG be an attachment to AB. Thus, the straight-lines AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational [Prop. 10.77]. And let CH, equal to the (square) on AG, have been applied to CD, and KL, equal to the (square) on GB. The whole of CL is thus equal to the (sum of the squares) on AG and GB. And the sum of the (squares) on AG and GB together is medial. Thus, CL is medial. And it has been applied to the rational (straight-line) CD, producing CM as breadth. CM is thus rational, and incommensurable (in length) with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at N. And let NO have been drawn through N, parallel to either of CD or ML. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since twice the (rectangle contained) by AG and GB is rational, and [is] equal to FL, FL is thus rational. And it is applied to the rational (straight-line) EF, producing FM as breadth. Thus, FM is rational, and commensurable in length with CD [Prop. 10.20]. And since CL is medial, and FL rational, CL is thus incommensurable with FL. And as CL (is) to FL, so CM (is) to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational. Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

For, similarly (to the previous propositions), we can show that the (rectangle contained) by CKM is equal to the (square) on NM—that is to say, to the fourth part of the (square) on FM. And since the (square) on AG is incommensurable with the (square) on GB, and the (square) on AG (is) equal to CH, and the (square) on GB to KL, CH (is) thus incommensurable with KL. And as CH (is) to KL, so CK (is) to KM [Prop. 6.1]. Thus, CK (is) incommensurable in length with KM [Prop. 10.11]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM, has been applied to CM, falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18]. And the attachment

FM is commensurable with the (previously) laid down rational (straight-line) CD. Thus, CF is a fifth apotome [Def. 10.15]. (Which is) the very thing it was required to show.

Proposition 102

The (square) on that (straight-line) which with a medial (area) makes a medial whole, applied to a rational (straight-line), produces a sixth apotome as breadth.



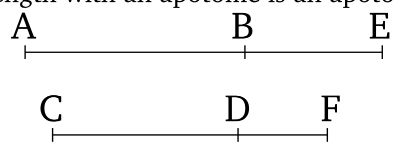
Let AB be that (straight-line) which with a medial (area) makes a medial whole, and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a sixth apotome.

For let BG be an attachment to AB. Thus, AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by AG and GB medial, and the (sum of the squares) on AG and GB incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.78]. Therefore, let CH, equal to the (square) on AG, have been applied to CD, producing CK as breadth, and KL, equal to the (square) on BG. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB. CL [is] thus also medial. And it is applied to the rational (straight-line) CD, producing CM as breadth. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22]. Therefore, since CL is equal to the (sum of the squares) on AG and GB, of which CE (is) equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. And twice the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial. And it is applied to the rational (straight-line) FE, producing FM as breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since the (sum of the squares) on AG and GB is incommensurable with twice the (rectangle contained) by AG and GB, and CL equal to the (sum of the squares) on AG and GB, and FL equal to twice the (rectangle contained) by AG and GB, CL [is] thus incommensurable with FL. And as CL (is) to FL, so CM is to MF [Prop. 6.1]. Thus, CM is incommensurable in length with MF [Prop. 10.11]. And they are both rational. Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since FL is equal to twice the (rectangle contained) by AG and GB, let FM have been cut in half at N, and let NO have been drawn through N, parallel to CD. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since AG and GB are incommensurable in square, the (square) on AG is thus incommensurable with the (square) on GB. But, CH is equal to the (square) on AG, and KL is equal to the (square) on GB. Thus, CH is incommensurable with KL. And as CH (is) to KL, so CK is to KM [Prop. 6.1]. Thus, CK is incommensurable (in length) with KM [Prop. 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG, and KL equal to the (square) on GB, and NL equal to the (rectangle contained) by AG and GB, NL is thus also the mean proportional to CH and KL. Thus, as CH is to NL, so NL (is) to KL. And for the same (reasons as the preceding propositions), the square on CM is greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18]. And neither of them is commensurable with the (previously) laid down rational (straight-line) CD. Thus, CF is a sixth apotome [Def. 10.16]. (Which is) the very thing it was required to show.

Proposition 103

A (straight-line) commensurable in length with an apotome is an apotome, and (is) the same in order.



Let AB be an apotome, and let CD be commensurable in length with AB. I say that CD is also an apotome, and (is) the same in order as AB.

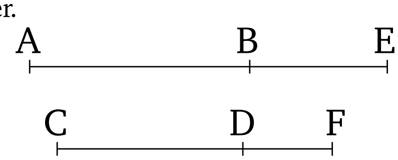
For since AB is an apotome, let BE be an attachment to it. Thus, AE and EB are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let it have been contrived that the (ratio) of BE to DF is the same as the ratio of AB to CD [Prop. 6.12]. Thus, also, as one is to one, (so) all [are] to all [Prop. 5.12]. And thus as the whole AE is to the whole CF, so AB (is) to CD. And AB (is) commensurable in length with CD. AE (is) thus also commensurable (in length) with CF, and BE with DF [Prop. 10.11]. And AE and BE are rational (straight-lines which are) commensurable in square only. Thus, CF and FD are also rational (straight-lines which are) commensurable in square only [Prop. 10.13]. [CD is thus an apotome. So, I say that (it is) also the same in order as AB.]

Therefore, since as AE is to CF, so BE (is) to DF, thus, alternately, as AE is to EB, so CF (is) to FD [Prop. 5.16]. So, the square on AE is greater than (the square on) EB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (AE). Therefore, if the (square) on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable in length with a (previously) laid down rational (straight-line) then so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF, and if neither of AE or EB (are commensurable), neither (are) either of CF or FD [Prop. 10.13]. And if the (square) on AE is greater [than (the square on) EB] by the (square) on (some straight-line) incommensurable (in length) with (AE) then the (square) on CF will also be greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable in length with a (previously) laid down rational (straight-line), so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF, and if neither of AE or EB (are commensurable), neither (are) either of CF or FD [Prop. 10.13].

Thus, CD is an apotome, and (is) the same in order as AB [Defs. 10.11—10.16]. (Which is) the very thing it was required to show.

Proposition 104

A (straight-line) commensurable (in length) with an apotome of a medial (straight-line) is an apotome of a medial (straight-line), and (is) the same in order.



Let AB be an apotome of a medial (straight-line), and let CD be commensurable in length with AB. I say that CD is also an apotome of a medial (straight-line), and (is) the same in order as AB.

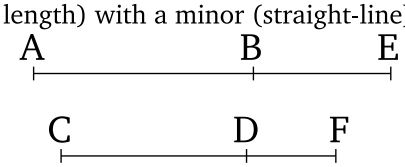
For since AB is an apotome of a medial (straight-line), let EB be an attachment to it. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.74, 10.75]. And let it have been contrived that as AB is to CD, so BE (is) to DF [Prop. 6.12]. Thus, AE [is] also commensurable (in length) with CF, and BE with DF [Props. 5.12, 10.11]. And AE and EB are medial (straight-lines which are) commensurable in square only. CF and FD are thus also medial (straight-lines which are) commensurable in square only [Props. 10.23, 10.13]. Thus, CD is an apotome of a medial (straight-line) [Props. 10.74, 10.75]. So, I say that it is also the same in order as AB.

[For] since as AE is to EB, so CF (is) to FD [Props. 5.12, 5.16] [but as AE (is) to EB, so the (square) on AE (is) to the (rectangle contained) by AE and EB, and as CF (is) to FD, so the (square) on CF (is) to the (rectangle contained) by CF and FD], thus as the (square) on AE is to the (rectangle contained) by AE and EB, so the (square) on CF also (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.] [and, alternately, as the (square) on AE (is) to the (square) on CF, so the (rectangle contained) by AE and EB (is) to the (rectangle contained) by CF and FD]. And the (square) on AE (is) commensurable with the (square) on CF. Thus, the (rectangle contained) by AE and EB is also commensurable with the (rectangle contained) by CF and FD [Props. 5.16, 10.11]. Therefore, either the (rectangle contained) by AE and EB is rational, and the (rectangle contained) by CF and FD will also be rational [Def. 10.4], or the (rectangle contained) by AE and EB [is] medial, and the (rectangle contained) by CF and FD [is] also medial [Prop. 10.23 corr.].

Therefore, CD is the apotome of a medial (straight-line), and is the same in order as AB [Props. 10.74, 10.75]. (Which is) the very thing it was required to show.

Proposition 105

A (straight-line) commensurable (in length) with a minor (straight-line) is a minor (straight-line).



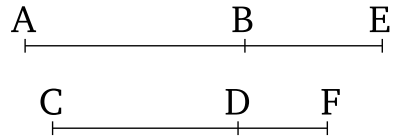
For let AB be a minor (straight-line), and (let) CD (be) commensurable (in length) with AB. I say that CD is also a minor (straight-line).

For let the same things have been contrived (as in the former proposition). And since AE and EB are (straight-lines which are) incommensurable in square [Prop. 10.76], CF and FD are thus also (straight-lines which are) incommensurable in square [Prop. 10.13]. Therefore, since as AE is to EB, so CF (is) to FD [Props. 5.12, 5.16], thus also as the (square) on AE is to the (square) on EB, so the (square) on CF (is) to the (square) on FD [Prop. 6.22]. Thus, via composition, as the (sum of the squares) on AE and EB is to the (square) on EB, so the (sum of the squares) on CF and FD (is) to the (square) on FD [Prop. 5.18], [also alternately]. And the (square) on BE is commensurable with the (square) on DF [Prop. 10.104]. The sum of the squares on AE and EB (is) thus also commensurable with the sum of the squares on CF and FD [Prop. 5.16, 10.11]. And the sum of the (squares) on AE and EB is rational [Prop. 10.76]. Thus, the sum of the (squares) on CF and FD is also rational [Def. 10.4]. Again, since as the (square) on AE is to the (rectangle contained) by AE and EB, so the (square) on CF (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.], and the square on AE (is) commensurable with the square on CF, the (rectangle contained) by AE and EB is thus also commensurable with the (rectangle contained) by CF and FD. And the (rectangle contained) by AE and EB (is) medial [Prop. 10.76]. Thus, the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.]. CF and FD are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Thus, CD is a minor (straight-line) [Prop. 10.76]. (Which is) the very thing it was required to show.

Proposition 106

A (straight-line) commensurable (in length) with a (straight-line) which with a rational (area) makes a medial whole is a (straight-line) which with a rational (area) makes a medial whole.



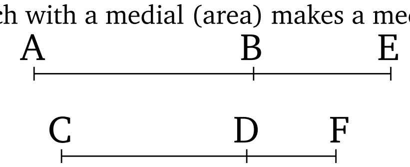
Let AB be a (straight-line) which with a rational (area) makes a medial whole, and (let) CD (be) commensurable (in length) with AB. I say that CD is also a (straight-line) which with a rational (area) makes a medial (whole).

For let BE be an attachment to AB. Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on AE and EB medial, and the (rectangle contained) by them rational [Prop. 10.77]. And let the same construction have been made (as in the previous propositions). So, similarly to the previous (propositions), we can show that CF and FD are in the same ratio as AE and EB, and the sum of the squares on AE and EB is commensurable with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Hence, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on CF and FD medial, and the (rectangle contained) by them rational.

CD is thus a (straight-line) which with a rational (area) makes a medial whole [Prop. 10.77]. (Which is) the very thing it was required to show.

Proposition 107

A (straight-line) commensurable (in length) with a (straight-line) which with a medial (area) makes a medial whole is itself also a (straight-line) which with a medial (area) makes a medial whole.



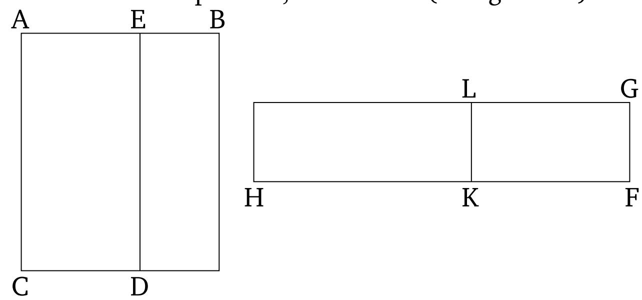
Let AB be a (straight-line) which with a medial (area) makes a medial whole, and let CD be commensurable (in length) with AB. I say that CD is also a (straight-line) which with a medial (area) makes a medial whole.

For let BE be an attachment to AB. And let the same construction have been made (as in the previous propositions). Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.78]. And, as was shown (previously), AE and EB are commensurable (in length) with CF and FD (respectively), and the sum of the squares on AE and EB with the sum of the squares on CF and FD, and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD. Thus, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the [squares] on them incommensurable with the (rectangle contained) by them.

Thus, CD is a (straight-line) which with a medial (area) makes a medial whole [Prop. 10.78]. (Which is) the very thing it was required to show.

Proposition 108

A medial (area) being subtracted from a rational (area), one of two irrational (straight-lines) arise (as) the square-root of the remaining area—either an apotome, or a minor (straight-line).



For let the medial (area) BD have been subtracted from the rational (area) BC. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either an apotome, or a minor (straight-line).

For let the rational (straight-line) FG have been laid out, and let the right-angled parallelogram GH, equal to BC, have been applied to FG, and let GK, equal to DB, have been subtracted (from GH). Thus, the remainder EC is equal to LH. Therefore, since BC is a rational (area), and BD a medial (area), and BC (is) equal to GH, and BD to GK, GH is thus a rational (area), and GK a medial (area). And they are applied to the rational (straight-line) FG. Thus, FH (is) rational, and commensurable in length with FG [Prop. 10.20], and FK (is) also rational, and incommensurable in length with FG [Prop. 10.22]. Thus, FH is incommensurable in length with FK [Prop. 10.13]. FH and FK are thus rational (straight-lines which are) commensurable in square only. Thus, KH is an apotome [Prop. 10.73], and KF an attachment to it. So, the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) either commensurable, or not (commensurable), (in length with HF).

First, let the square (on it) be (greater) by the (square) on (some straight-line which is) commensurable (in length with HF). And the whole of HF is commensurable in length with the (previously) laid down rational (straight-line) FG. Thus, KH is a first apotome [Def. 10.1]. And the square-root of an (area) contained by a rational (straight-line) and a first apotome is an apotome [Prop. 10.91]. Thus, the square-root of LH—that is to say, (of) EC—is an apotome.

And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) incommensurable (in length) with (HF), and (since) the whole of FH is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fourth apotome [Prop. 10.14]. And the square-root of an (area) contained by a rational (straight-line) and a fourth apotome is a minor (straight-line) [Prop. 10.94]. (Which is) the very thing it was required to show.

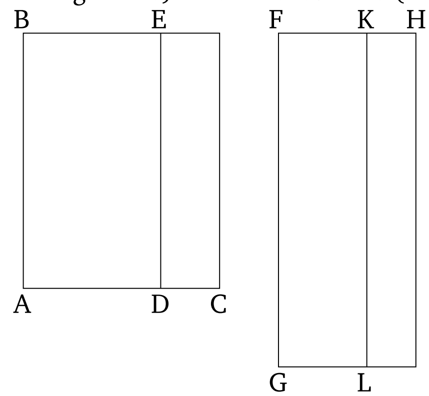
Proposition 109

A rational (area) being subtracted from a medial (area), two other irrational (straight-lines) arise (as the square-root of the remaining area)—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (area) BD have been subtracted from the medial (area) BC. I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (straight-line) FG be laid down, and let similar areas (to the preceding proposition) have been applied (to it). So, accordingly, FH is rational, and incommensurable in length with FG, and KF (is) also rational, and commensurable in length with FG. Thus, FH and FK are rational (straight-lines which are) commensurable in square only [Prop. 10.13]. KH is thus an apotome [Prop. 10.73], and FK an attachment to it. So, the square on

HF is greater than (the square on) FK either by the (square) on (some straight-line) commensurable (in length) with (HF), or by the (square) on (some straight-line) incommensurable (in length with HF).



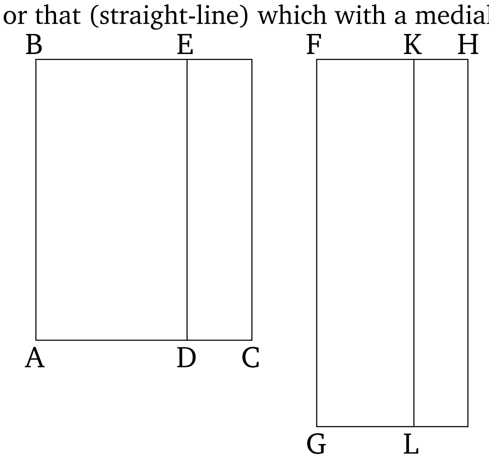
Therefore, if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a second apotome [Def. 10.12]. And FG (is) rational. Hence, the square-root of LH—that is to say, (of) EC—is a first apotome of a medial (straight-line) [Prop. 10.92].

And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) incommensurable (in length with HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a fifth apotome [Def. 10.15]. Hence, the square-root of EC is that (straight-line) which with a rational (area) makes a medial whole [Prop. 10.95]. (Which is) the very thing it was required to show.

Proposition 110

A medial (area), incommensurable with the whole, being subtracted from a medial (area), the two remaining irrational (straight-lines) arise (as) the (square-root of the area)—either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.

For, as in the previous figures, let the medial (area) BD, incommensurable with the whole, have been subtracted from the medial (area) BC. I say that the square-root of EC is one of two irrational (straight-lines)—either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.



For since BC and BD are each medial (areas), and BC (is) incommensurable with BD, accordingly, FH and FK will each be rational (straight-lines), and incommensurable in length with FG [Prop. 10.22]. And since BC

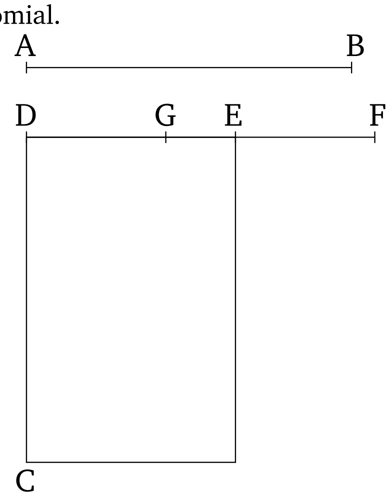
is incommensurable with BD—that is to say, GH with GK—HF (is) also incommensurable (in length) with FK [Props. 6.1, 10.11]. Thus, FH and FK are rational (straight-lines which are) commensurable in square only. KH is thus as apotome [Prop. 10.73], [and FK an attachment (to it). So, the square on FH is greater than (the square on) FK either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (FH).]

So, if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (FH), and (since) neither of FH and FK is commensurable in length with the (previously) laid down rational (straight-line) FG, KH is a third apotome [Def. 10.3]. And KL (is) rational. And the rectangle contained by a rational (straight-line) and a third apotome is irrational, and the square-root of it is that irrational (straight-line) called a second apotome of a medial (straight-line) [Prop. 10.93]. Hence, the square-root of LH—that is to say, (of) EC—is a second apotome of a medial (straight-line).

And if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) incommensurable [in length] with (FH), and (since) neither of HF and FK is commensurable in length with FG, KH is a sixth apotome [Def. 10.16]. And the square-root of the (rectangle contained) by a rational (straight-line) and a sixth apotome is that (straight-line) which with a medial (area) makes a medial whole [Prop. 10.96]. Thus, the square-root of LH—that is to say, (of) EC—is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to show.

Proposition 111

An apotome is not the same as a binomial.



Let AB be an apotome. I say that AB is not the same as a binomial.

For, if possible, let it be (the same). And let a rational (straight-line) DC be laid down. And let the rectangle CE, equal to the (square) on AB, have been applied to CD, producing DE as breadth. Therefore, since AB is an apotome, DE is a first apotome [Prop. 10.97]. Let EF be an attachment to it. Thus, DF and FE are rational (straight-lines which are) commensurable in square only, and the square on DF is greater than (the square on) FE by the (square) on (some straight-line) commensurable (in length) with (DF), and DF is commensurable in length with the (previously) laid down rational (straight-line) DC [Def. 10.10]. Again, since AB is a binomial, DE is thus a first binomial [Prop. 10.60]. Let (DE) have been divided into its (component) terms at G, and let DG be the greater term. Thus, DG and GE are rational (straight-lines which are) commensurable in square only, and the square on DG is greater than (the square on) GE by the (square) on (some straight-line) commensurable (in length) with (DG), and the greater (term) DG is commensurable in length with the (previously) laid down rational (straight-line) DC [Def. 10.5]. Thus, DF is also commensurable in length with DG [Prop. 10.12]. The remainder GF is thus commensurable in length with DF [Prop. 10.15]. [Therefore, since DF is commensurable with GF, and DF is rational, GF is thus also rational. Therefore, since DF is commensurable in length with GF,] DF (is) incommensurable in length with EF. Thus, FG is also incommensurable in length with EF [Prop. 10.13]. GF and FE [are] thus rational (straight-lines which are) commensurable in square only. Thus, EG is an apotome [Prop. 10.73]. But, (it is) also rational. The very thing is impossible.

Thus, an apotome is not the same as a binomial. (Which is) the very thing it was required to show.

[Corollary]

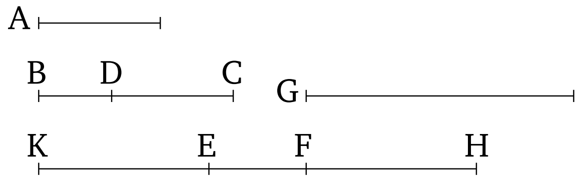
The apotome and the irrational (straight-lines) after it are neither the same as a medial (straight-line) nor (the same) as one another.

For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) incommensurable in length with the (straight-line) to which (the area) is applied [Prop. 10.22]. And the (square) on an apotome, applied to a rational (straight-line), produces as breadth a first apotome [Prop. 10.97]. And the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a second apotome [Prop. 10.98]. And the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a third apotome [Prop. 10.99]. And (square) on a minor (straight-line), applied to a rational (straight-line), produces as breadth a fourth apotome [Prop. 10.100]. And (square) on that (straight-line) which with a rational (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a fifth apotome [Prop. 10.101]. And (square) on that (straight-line) which with a medial (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a sixth apotome [Prop. 10.102]. Therefore, since the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational, and from one another since they are not the same in order—clearly, the irrational (straight-lines) themselves also differ from one another. And since it has been shown that an apotome is not the same as a binomial [Prop. 10.111], and (that) the (irrational straight-lines) after the apotome, being applied to a rational (straight-line), produce as breadth, each according to its own (order), apotomes, and (that) the (irrational straight-lines) after the binomial themselves also (produce as breadth), according (to their) order, binomials, the (irrational straight-lines) after the apotome are thus different, and the (irrational straight-lines) after the binomial (are also) different, so that there are, in order, 13 irrational (straight-lines) in all:

Medial,
Binomial,
First bimedial,
Second bimedial,
Major,
Square-root of a rational plus a medial (area),
Square-root of (the sum of) two medial (areas),
Apotome,
First apotome of a medial,
Second apotome of a medial,
Minor,
That which with a rational (area) produces a medial whole,
That which with a medial (area) produces a medial whole.

Proposition 112†

The (square) on a rational (straight-line), applied to a binomial (straight-line), produces as breadth an apotome whose terms are commensurable (in length) with the terms of the binomial, and, furthermore, in the same ratio. Moreover, the created apotome will have the same order as the binomial.



Let A be a rational (straight-line), and BC a binomial (straight-line), of which let DC be the greater term. And let the (rectangle contained) by BC and EF be equal to the (square) on A. I say that EF is an apotome whose terms are commensurable (in length) with CD and DB, and in the same ratio, and, moreover, that EF will have the same order as BC.

For, again, let the (rectangle contained) by BD and G be equal to the (square) on A. Therefore, since the (rectangle contained) by BC and EF is equal to the (rectangle contained) by BD and G, thus as CB is to BD, so G (is) to EF [Prop. 6.16]. And CB (is) greater than BD. Thus, G is also greater than EF [Props. 5.16, 5.14]. Let EH be equal to G. Thus, as CB is to BD, so HE (is) to EF. Thus, via separation, as CD is to BD, so HF (is) to FE [Prop. 5.17]. Let it have been contrived that as HF (is) to FE, so FK (is) to KE. And, thus, the whole HK is to the whole KF, as FK (is) to KE. For as one of the leading (proportional magnitudes is) to one of the following, so all of the leading (magnitudes) are to all of the following [Prop. 5.12]. And as FK (is) to KE, so CD is to DB [Prop. 5.11]. And, thus, as HK (is) to KF, so CD is to DB [Prop. 5.11]. And the (square) on CD (is) commensurable with the (square) on DB [Prop. 10.36]. The (square) on HK is thus also commensurable with the (square) on KF [Props. 6.22, 10.11]. And as the (square) on HK is to the (square) on KF, so HK (is) to KE, since the three (straight-lines) HK, KF, and KE are proportional [Def. 5.9]. HK is thus commensurable in length with KE [Prop. 10.11]. Hence, HE is also commensurable in length with EK [Prop. 10.15]. And since the (square) on A is equal to the (rectangle contained) by EH and BD, and the (square) on A is rational, the (rectangle contained) by EH and BD is thus also rational. And it is applied to the rational (straight-line) BD. Thus, EH is rational, and commensurable in length with BD [Prop. 10.20]. And, hence, the (straight-line) commensurable (in length) with it, EK, is also rational [Def. 10.3], and commensurable in length with BD [Prop. 10.12]. Therefore, since as CD is to DB, so FK (is) to KE, and CD and DB are (straight-lines which are) commensurable in square only, FK and KE are also commensurable in square only [Prop. 10.11]. And KE is rational. Thus, FK is also rational. FK and KE are thus rational (straight-lines which are) commensurable in square only. Thus, EF is an apotome [Prop. 10.73].

And the square on CD is greater than (the square on) DB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (CD).

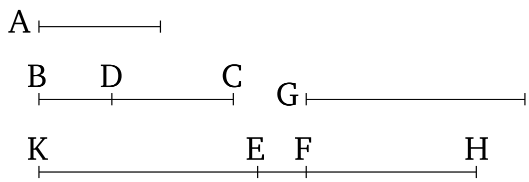
Therefore, if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) commensurable (in length) with [CD] then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) commensurable (in length) with (FK) [Prop. 10.14]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, 10.12]. And if BD (is commensurable), (so) also (is) KE [Prop. 10.12]. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE.

And if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) incommensurable (in length) with (CD) then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) incommensurable (in length) with (FK) [Prop. 10.14]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, 10.12]. And if BD (is commensurable), (so) also (is) KE [Prop. 10.12]. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE. Hence, FE is an apotome whose terms, FK and KE, are commensurable (in length) with the terms, CD and DB, of the binomial, and in the same ratio. And (FE) has the same order as BC

[Defs. 10.5—10.10]. (Which is) the very thing it was required to show. † Heiberg considers this proposition, and the succeeding ones, to be relatively early interpolations into the original text.

Proposition 113

The (square) on a rational (straight-line), applied to an apotome, produces as breadth a binomial whose terms are commensurable with the terms of the apotome, and in the same ratio. Moreover, the created binomial has the same order as the apotome.



Let A be a rational (straight-line), and BD an apotome. And let the (rectangle contained) by BD and KH be equal to the (square) on A, such that the square on the rational (straight-line) A, applied to the apotome BD, produces KH as breadth. I say that KH is a binomial whose terms are commensurable with the terms of BD, and in the same ratio, and, moreover, that KH has the same order as BD.

For let DC be an attachment to BD. Thus, BC and CD are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let the (rectangle contained) by BC and G also be equal to the (square) on A. And the (square) on A (is) rational. The (rectangle contained) by BC and G (is) thus also rational. And it has been applied to the rational (straight-line) BC. Thus, G is rational, and commensurable in length with BC [Prop. 10.20]. Therefore, since the (rectangle contained) by BC and G is equal to the (rectangle contained) by BD and KH, thus, proportionally, as CB is to BD, so KH (is) to G [Prop. 6.16]. And BC (is) greater than BD. Thus, KH (is) also greater than G [Prop. 5.16, 5.14]. Let KE be made equal to G. KE is thus commensurable in length with BC. And since as CB is to BD, so HK (is) to KE, thus, via conversion, as BC (is) to CD, so KH (is) to HE [Prop. 5.19 corr.]. Let it have been contrived that as KH (is) to HE, so HF (is) to FE. And thus the remainder KF is to FH, as KH (is) to HE—that is to say, [as] BC (is) to CD [Prop. 5.19]. And BC and CD [are] commensurable in square only. KF and FH are thus also commensurable in square only [Prop. 10.11]. And since as KH is to HE, (so) KF (is) to FH, but as KH (is) to HE, (so) HF (is) to FE, thus, also as KF (is) to FH, (so) HF (is) to FE [Prop. 5.11]. And hence as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9]. And thus as KF (is) to FE, so the (square) on KF (is) to the (square) on FH. And the (square) on KF is commensurable with the (square) on FH. For KF and FH are commensurable in square. Thus, KF is also commensurable in length with FE [Prop. 10.11]. Hence, KF [is] also commensurable in length with KE [Prop. 10.15]. And KE is rational, and commensurable in length with BC. Thus, KF (is) also rational, and commensurable in length with BC [Prop. 10.12]. And since as BC is to CD, (so) KF (is) to FH, alternately, as BC (is) to KF, so DC (is) to FH [Prop. 5.16]. And BC (is) commensurable (in length) with KF. Thus, FH (is) also commensurable in length with CD [Prop. 10.11]. And BC and CD are rational (straight-lines which are) commensurable in square only. KF and FH are thus also rational (straight-lines which are) commensurable in square only [Def. 10.3, Prop. 10.13]. Thus, KH is a binomial [Prop. 10.36].

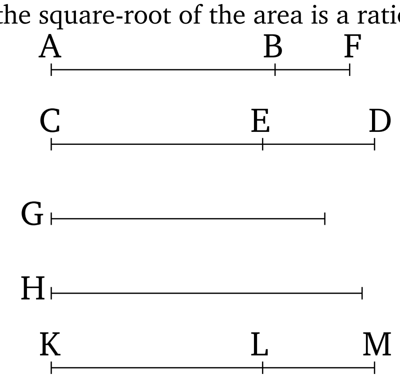
Therefore, if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) commensurable (in length) with (BC), then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) commensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

And if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) incommensurable (in length) with (BC) then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) incommensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable, (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

KH is thus a binomial whose terms, KF and FH, [are] commensurable (in length) with the terms, BC and CD, of the apotome, and in the same ratio. Moreover, KH will have the same order as BC [Defs. 10.5—10.10]. (Which is) the very thing it was required to show.

Proposition 114

If an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome then the square-root of the area is a rational (straight-line).



For let an area, the (rectangle contained) by AB and CD, have been contained by the apotome AB, and the binomial CD, of which let the greater term be CE. And let the terms of the binomial, CE and ED, be commensurable with the terms of the apotome, AF and FB (respectively), and in the same ratio. And let the square-root of the (rectangle contained) by AB and CD be G. I say that G is a rational (straight-line).

For let the rational (straight-line) H be laid down. And let (some rectangle), equal to the (square) on H, have been applied to CD, producing KL as breadth. Thus, KL is an apotome, of which let the terms, KM and ML, be commensurable with the terms of the binomial, CE and ED (respectively), and in the same ratio [Prop. 10.112]. But, CE and ED are also commensurable with AF and FB (respectively), and in the same ratio. Thus, as AF is to FB, so KM (is) to ML. Thus, alternately, as AF is to KM, so BF (is) to LM [Prop. 5.16]. Thus, the remainder AB is also to the remainder KL as AF (is) to KM [Prop. 5.19]. And AF (is) commensurable with KM [Prop. 10.12]. AB is thus also commensurable with KL [Prop. 10.11]. And as AB is to KL, so the (rectangle contained) by CD and AB (is) to the (rectangle contained) by CD and KL [Prop. 6.1]. Thus, the (rectangle contained) by CD and AB is also commensurable with the (rectangle contained) by CD and KL [Prop. 10.11]. And the (rectangle contained) by CD and KL (is) equal to the (square) on H. Thus, the (rectangle contained) by CD and AB is commensurable with the (square) on H. And the (square) on G is equal to the (rectangle contained) by CD and AB. The (square) on G is thus commensurable with the (square) on H. And the (square) on H (is) rational. Thus, the (square) on G is also rational. G is thus rational. And it is the square-root of the (rectangle contained) by CD and AB.

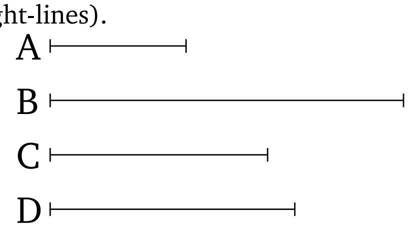
Thus, if an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome, then the square-root of the area is a rational (straight-line).

Corollary

And it has also been made clear to us, through this, that it is possible for a rational area to be contained by irrational straight-lines. (Which is) the very thing it was required to show.

Proposition 115

An infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and none of them is the same as any of the preceding (straight-lines).



Let A be a medial (straight-line). I say that an infinite (series) of irrational (straight-lines) can be created from A, and that none of them is the same as any of the preceding (straight-lines).

Let the rational (straight-line) B be laid down. And let the (square) on C be equal to the (rectangle contained) by B and A. Thus, C is irrational [Def. 10.4]. For an (area contained) by an irrational and a rational (straight-line) is irrational [Prop. 10.20]. And (C is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces a medial (straight-line) as breadth. So, again, let the (square) on D be equal to the (rectangle contained) by B and C. Thus, the (square) on D is irrational [Prop. 10.20]. D is thus irrational [Def. 10.4]. And (D is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces C as breadth. So, similarly, this arrangement being advanced to infinity, it is clear that an infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and that none of them is the same as any of the preceding (straight-lines). (Which is) the very thing it was required to show.

ELEMENTS BOOK 11

Elementary Stereometry

Definitions

1. A solid is a (figure) having length and breadth and depth.

2. The extremity of a solid (is) a surface.

3. A straight-line is at right-angles to a plane when it makes right-angles with all of the straight-lines joined to it which are also in the plane.

4. A plane is at right-angles to a(nother) plane when (all of) the straight-lines drawn in one of the planes, at right-angles to the common section of the planes, are at right-angles to the remaining plane.

5. The inclination of a straight-line to a plane is the angle contained by the drawn and standing (straight-lines), when a perpendicular is lead to the plane from the end of the (standing) straight-line raised (out of the plane), and a straight-line is (then) joined from the point (so) generated to the end of the (standing) straight-line (lying) in the plane.

6. The inclination of a plane to a(nother) plane is the acute angle contained by the (straight-lines), (one) in each of the planes, drawn at right-angles to the common segment (of the planes), at the same point.

7. A plane is said to have been similarly inclined to a plane, as another to another, when the aforementioned angles of inclination are equal to one another.

8. Parallel planes are those which do not meet (one another).

9. Similar solid figures are those contained by equal numbers of similar planes (which are similarly arranged).

10. But equal and similar solid figures are those contained by similar planes equal in number and in magnitude (which are similarly arranged).

11. A solid angle is the inclination (constituted) by more than two lines joining one another (at the same point), and not being in the same surface, to all of the lines. Otherwise, a solid angle is that contained by more than two plane angles, not being in the same plane, and constructed at one point.

12. A pyramid is a solid figure, contained by planes, (which is) constructed from one plane to one point.

13. A prism is a solid figure, contained by planes, of which the two opposite (planes) are equal, similar, and parallel, and the remaining (planes are) parallelograms.

14. A sphere is the figure enclosed when, the diameter of a semicircle remaining (fixed), the semicircle is carried around, and again established at the same (position) from which it began to be moved.

15. And the axis of the sphere is the fixed straight-line about which the semicircle is turned.

16. And the center of the sphere is the same as that of the semicircle.

17. And the diameter of the sphere is any straight-line which is drawn through the center and terminated in both directions by the surface of the sphere.

18. A cone is the figure enclosed when, one of the sides of a right-angled triangle about the right-angle remaining (fixed), the triangle is carried around, and again established at the same (position) from which it began to be moved. And if the fixed straight-line is equal to the remaining (straight-line) about the right-angle, (which is) carried around, then the cone will be right-angled, and if less, obtuse-angled, and if greater, acute-angled.

19. And the axis of the cone is the fixed straight-line about which the triangle is turned.

20. And the base (of the cone is) the circle described by the (remaining) straight-line (about the right-angle which is) carried around (the axis).

21. A cylinder is the figure enclosed when, one of the sides of a right-angled parallelogram about the right-angle remaining (fixed), the parallelogram is carried around, and again established at the same (position) from which it began to be moved.

22. And the axis of the cylinder is the stationary straight-line about which the parallelogram is turned.

23. And the bases (of the cylinder are) the circles described by the two opposite sides (which are) carried around.

24. Similar cones and cylinders are those for which the axes and the diameters of the bases are proportional.

25. A cube is a solid figure contained by six equal squares.

26. An octahedron is a solid figure contained by eight equal and equilateral triangles.

27. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

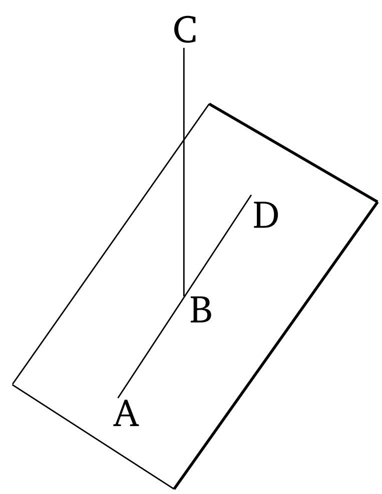
28. A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

Proposition 1[†]

Some part of a straight-line cannot be in a reference plane, and some part in a more elevated (plane).

For, if possible, let some part, AB, of the straight-line ABC be in a reference plane, and some part, BC, in a more elevated (plane).

In the reference plane, there will be some straight-line continuous with, and straight-on to, AB.[‡] Let it be BD. Thus, AB is a common segment of the two (different) straight-lines ABC and ABD. The very thing is impossible, inasmuch as if we draw a circle with center B and radius AB then the diameters (ABD and ABC) will cut off unequal circumferences of the circle.



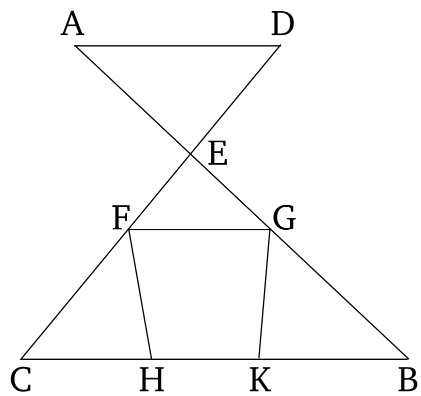
Thus, some part of a straight-line cannot be in a reference plane, and (some part) in a more elevated (plane). (Which is) the very thing it was required to show.

[†] The proofs of the first three propositions in this book are not at all rigorous.

Hence, these three propositions should properly be regarded as additional axioms.
‡ This assumption essentially presupposes the validity of the proposition under discussion.

Proposition 2

If two straight-lines cut one another then they are in one plane, and every triangle (formed using segments of both lines) is in one plane.

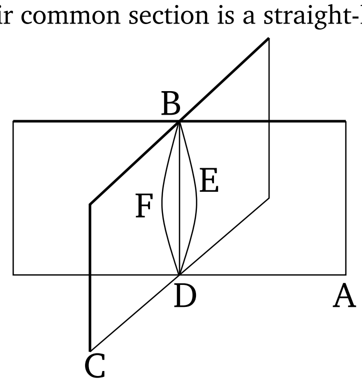


For let the two straight-lines AB and CD have cut one another at point E. I say that AB and CD are in one plane, and that every triangle (formed using segments of both lines) is in one plane.

For let the random points F and G have been taken on EC and EB (respectively). And let CB and FG have been joined, and let FH and GK have been drawn across. I say, first of all, that triangle ECB is in one (reference) plane. For if part of triangle ECB, either FHC or GBK, is in the reference [plane], and the remainder in a different (plane) then a part of one the straight-lines EC and EB will also be in the reference plane, and (a part) in a different (plane). And if the part $FCBG$ of triangle ECB is in the reference plane, and the remainder in a different (plane) then parts of both of the straight-lines EC and EB will also be in the reference plane, and (parts) in a different (plane). The very thing was shown to be absurb [Prop. 11.1]. Thus, triangle ECB is in one plane. And in whichever (plane) triangle ECB is (found), in that (plane) EC and EB (will) each also (be found). And in whichever (plane) EC and EB (are) each (found), in that (plane) AB and CD (will) also (be found) [Prop. 11.1]. Thus, the straight-lines AB and CD are in one plane, and every triangle (formed using segments of both lines) is in one plane. (Which is) the very thing it was required to show.

Proposition 3

If two planes cut one another then their common section is a straight-line.



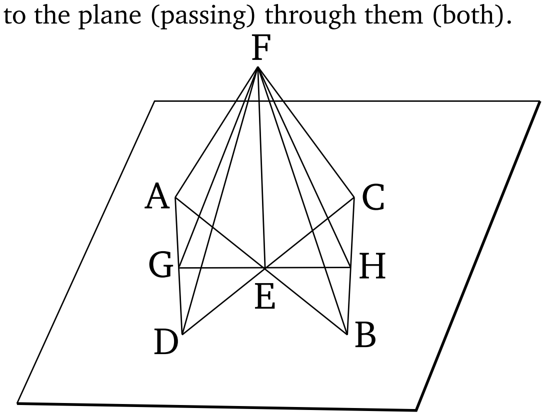
For let the two planes AB and BC cut one another, and let their common section be the line DB. I say that the line DB is straight.

For, if not, let the straight-line DEB have been joined from D to B in the plane AB, and the straight-line DFB in the plane BC. So two straight-lines, DEB and DFB, will have the same ends, and they will clearly enclose an area. The very thing (is) absurd. Thus, DEB and DFB are not straight-lines. So, similarly, we can show than no other straight-line can be joined from D to B except DB, the common section of the planes AB and BC.

Thus, if two planes cut one another then their common section is a straight-line. (Which is) the very thing it was required to show.

Proposition 4

If a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both).



For let some straight-line EF have (been) set up at right-angles to two straight-lines, AB and CD, cutting one another at point E, at E. I say that EF is also at right-angles to the plane (passing) through AB and CD.

For let AE, EB, CE and ED have been cut off from (the two straight-lines so as to be) equal to one another. And let GEH have been drawn, at random, through E (in the plane passing through AB and CD). And let AD and CB have been joined. And, furthermore, let FA, FG, FD, FC, FH, and FB have been joined from the random (point) F (on EF).

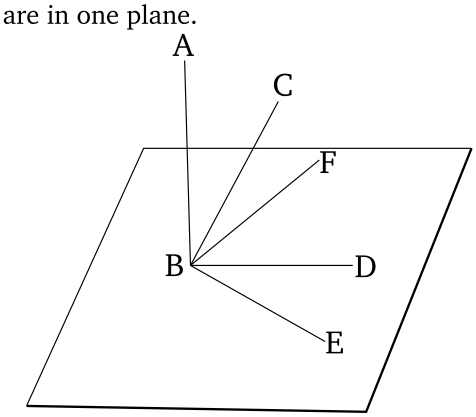
For since the two (straight-lines) AE and ED are equal to the two (straight-lines) CE and EB, and they enclose equal angles [Prop. 1.15], the base AD is thus equal to the base CB, and triangle AED will be equal to triangle CEB [Prop. 1.4]. Hence, the angle DAE [is] equal to the angle EBC. And the angle AEG (is) also equal to the angle BEH [Prop. 1.15]. So AGE and BEH are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), those by the equal angles, AE and EB. Thus, they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, GE (is) equal to EH, and AG to BH. And since AE is equal to EB, and FE is common and at right-angles, the base FA is thus equal to the base FB [Prop. 1.4]. So, for the same (reasons), FC is also equal to FD. And since AD is equal to CB, and FA is also equal to FB, the two (straight-lines) FA and AD are equal to the two (straight-lines) FB and BC, respectively. And the base FD was shown (to be) equal to the base FC. Thus, the angle FAD is also equal to the angle FBC [Prop. 1.8]. And, again, since AG was shown (to be) equal to BH, but FA (is) also equal to FB, the two (straight-lines) FA and AG are equal to the two (straight-lines) FB and BH (respectively). And the angle FAG was shown (to be) equal to the angle FBH. Thus, the base FG is equal to the base FH [Prop. 1.4]. And, again, since GE was shown (to be) equal to EH, and EF (is) common, the two (straight-lines) GE and EF are equal to the two (straight-lines) HE

and EF (respectively). And the base FG (is) equal to the base FH. Thus, the angle GEF is equal to the angle HEF [Prop. 1.8]. Each of the angles GEF and HEF (are) thus right-angles [Def. 1.10]. Thus, FE is at right-angles to GH, which was drawn at random through E (in the reference plane passing though AB and AC). So, similarly, we can show that FE will make right-angles with all straight-lines joined to it which are in the reference plane. And a straight-line is at right-angles to a plane when it makes right-angles with all straight-lines joined to it which are in the plane [Def. 11.3]. Thus, FE is at right-angles to the reference plane. And the reference plane is that (passing) through the straight-lines AB and CD. Thus, FE is at right-angles to the plane (passing) through AB and CD.

Thus, if a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both). (Which is) the very thing it was required to show.

Proposition 5

If a straight-line is set up at right-angles to three straight-lines cutting one another, at the common point of section, then the three straight-lines are in one plane.



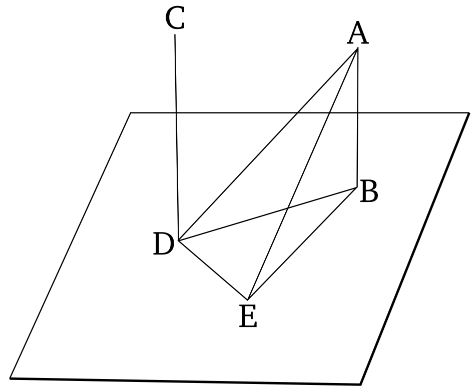
For let some straight-line AB have been set up at right-angles to three straight-lines BC, BD, and BE, at the (common) point of section B. I say that BC, BD, and BE are in one plane.

For (if) not, and if possible, let BD and BE be in the reference plane, and BC in a more elevated (plane). And let the plane through AB and BC have been produced. So it will make a straight-line as a common section with the reference plane [Def. 11.3]. Let it make BF. Thus, the three straight-lines AB, BC, and BF are in one plane—(namely), that drawn through AB and BC. And since AB is at right-angles to each of BD and BE, AB is thus also at right-angles to the plane (passing) through BD and BE [Prop. 11.4]. And the plane (passing) through BD and BE is the reference plane. Thus, AB is at right-angles to the reference plane. Hence, AB will also make right-angles with all straight-lines joined to it which are also in the reference plane [Def. 11.3]. And BF, which is in the reference plane, is joined to it. Thus, the angle ABF is a right-angle. And ABC was also assumed to be a right-angle. Thus, angle ABF (is) equal to ABC. And they are in one plane. The very thing is impossible. Thus, BC is not in a more elevated plane. Thus, the three straight-lines BC, BD, and BE are in one plane.

Thus, if a straight-line is set up at right-angles to three straight-lines cutting one another, at the (common) point of section, then the three straight-lines are in one plane. (Which is) the very thing it was required to show.

Proposition 6

If two straight-lines are at right-angles to the same plane then the straight-lines will be parallel.[†]



For let the two straight-lines AB and CD be at right-angles to a reference plane. I say that AB is parallel to CD.

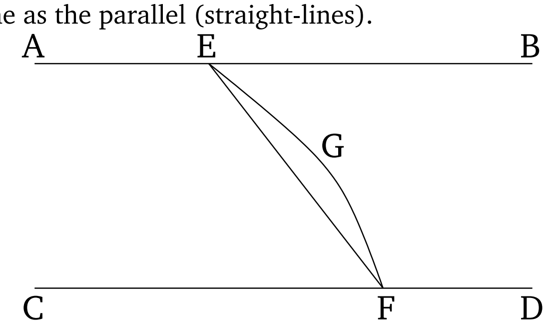
For let them meet the reference plane at points B and D (respectively). And let the straight-line BD have been joined. And let DE have been drawn at right-angles to BD in the reference plane. And let DE be made equal to AB. And let BE, AE, and AD have been joined.

And since AB is at right-angles to the reference plane, it will [thus] also make right-angles with all straight-lines joined to it which are in the reference plane [Def. 11.3]. And BD and BE, which are in the reference plane, are each joined to AB. Thus, each of the angles ABD and ABE are right-angles. So, for the same (reasons), each of the angles CDB and CDE are also right-angles. And since AB is equal to DE, and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And they contain right-angles. Thus, the base AD is equal to the base BE [Prop. 1.4]. And since AB is equal to DE, and AD (is) also (equal) to BE, the two (straight-lines) AB and BE are thus equal to the two (straight-lines) ED and DA (respectively). And their base AE (is) common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. Thus, EDA (is) also a right-angle. ED is thus at right-angles to DA. And it is also at right-angles to each of BD and DC. Thus, ED is standing at right-angles to the three straight-lines BD, DA, and DC at the (common) point of section. Thus, the three straight-lines BD, DA, and DC are in one plane [Prop. 11.5]. And in which(ever) plane DB and DA (are found), in that (plane) AB (will) also (be found). For every triangle is in one plane [Prop. 11.2]. And each of the angles ABD and BDC is a right-angle. Thus, AB is parallel to CD [Prop. 1.28].

Thus, if two straight-lines are at right-angles to the same plane then the straight-lines will be parallel. (Which is) the very thing it was required to show. † In other words, the two straight-lines lie in the same plane, and never meet when produced in either direction.

Proposition 7

If there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines).



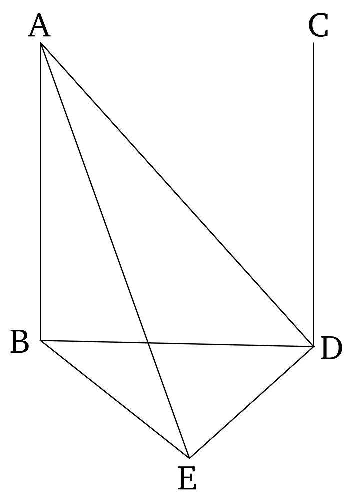
Let AB and CD be two parallel straight-lines, and let the random points E and F have been taken on each of them (respectively). I say that the straight-line joining points E and F is in the same (reference) plane as the parallel (straight-lines).

For (if) not, and if possible, let it be in a more elevated (plane), such as EGF. And let a plane have been drawn through EGF. So it will make a straight cutting in the reference plane [Prop. 11.3]. Let it make EF. Thus, two straight-lines (with the same end-points), EGF and EF, will enclose an area. The very thing is impossible. Thus, the straight-line joining E to F is not in a more elevated plane. The straight-line joining E to F is thus in the plane through the parallel (straight-lines) AB and CD.

Thus, if there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines). (Which is) the very thing it was required to show.

Proposition 8

If two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane.



Let AB and CD be two parallel straight-lines, and let one of them, AB, be at right-angles to a reference plane. I say that the remaining (one), CD, will also be at right-angles to the same plane.

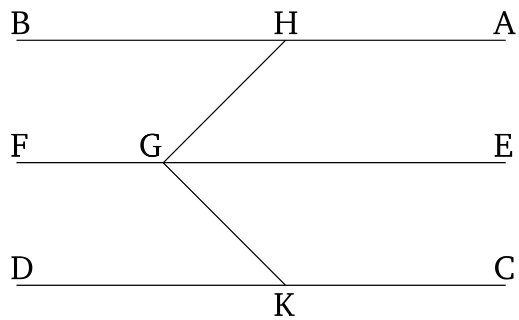
For let AB and CD meet the reference plane at points B and D (respectively). And let BD have been joined. AB, CD, and BD are thus in one plane [Prop. 11.7]. Let DE have been drawn at right-angles to BD in the reference plane, and let DE be made equal to AB, and let BE, AE, and AD have been joined.

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straight-lines joined to it which are in the reference plane [Def. 11.3]. Thus, the angles ABD and ABE [are] each right-angles. And since the straight-line BD has met the parallel (straight-lines) AB and CD, the (sum of the) angles ABD and CDB is thus equal to two right-angles [Prop. 1.29]. And ABD (is) a right-angle. Thus, CDB (is) also a right-angle. CD is thus at right-angles to BD. And since AB is equal to DE, and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And angle ABD (is) equal to angle EDB. For each (is) a right-angle. Thus, the base AD (is) equal to the base BE [Prop. 1.4]. And since AB is equal to DE, and BE to AD, the two (sides) AB, BE are equal to the two (sides) ED, DA, respectively. And their base AE is common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. EDA (is) thus also a right-angle. Thus, ED is at right-angles to AD. And it is also at right-angles to DB. Thus, ED is also at right-angles to the plane through BD and DA [Prop. 11.4]. And ED will thus make right-angles with all of the straight-lines joined to it which are also in the plane through BDA. And DC is in the plane through BDA, inasmuch as AB and BD are in the plane through BDA [Prop. 11.2], and in which(ever plane) AB and BD (are found), DC is also (found). Thus, ED is at right-angles to DC. Hence, CD is also at right-angles to DE. And CD is also at right-angles to BD. Thus, CD is standing at right-angles to two straight-lines, DE and DB, which meet one another, at the (point) of section, D. Hence, CD is also at right-angles to the plane through DE and DB [Prop. 11.4]. And the plane through DE and DB is the reference (plane). CD is thus at right-angles to the reference plane.

Thus, if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 9

(Straight-lines) parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another.



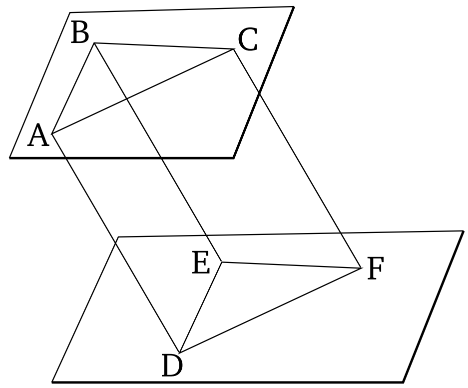
For let AB and CD each be parallel to EF, not being in the same plane as it. I say that AB is parallel to CD.

For let some point G have been taken at random on EF. And from it let GH have been drawn at right-angles to EF in the plane through EF and AB. And let GK have been drawn, again at right-angles to EF, in the plane through FE and CD.

And since EF is at right-angles to each of GH and GK, EF is thus also at right-angles to the plane through GH and GK [Prop. 11.4]. And EF is parallel to AB. Thus, AB is also at right-angles to the plane through HGK [Prop. 11.8]. So, for the same (reasons), CD is also at right-angles to the plane through HGK. Thus, AB and CD are each at right-angles to the plane through HGK. And if two straight-lines are at right–angles to the same plane then the straight-lines are parallel [Prop. 11.6]. Thus, AB is parallel to CD. (Which is) the very thing it was required to show.

Proposition 10

If two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles.



For let the two straight-lines joined to one another, AB and BC, be (respectively) parallel to the two straight-lines joined to one another, DE and EF, (but) not in the same plane. I say that angle ABC is equal to (angle) DEF.

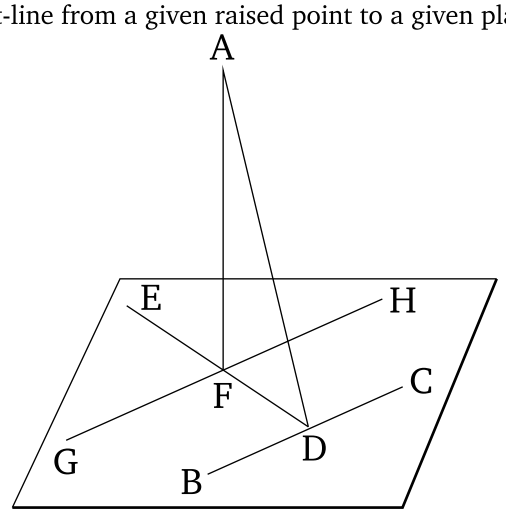
For let BA, BC, ED, and EF have been cut off (so as to be, respectively) equal to one another. And let AD, CF, BE, AC, and DF have been joined.

And since BA is equal and parallel to ED, AD is thus also equal and parallel to BE [Prop. 1.33]. So, for the same reasons, CF is also equal and parallel to BE. Thus, AD and CF are each equal and parallel to BE. And straight-lines parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another [Prop. 11.9]. Thus, AD is parallel and equal to CF. And AC and DF join them. Thus, AC is also equal and parallel to DF [Prop. 1.33]. And since the two (straight-lines) AB and BC are equal to the two (straight-lines) DE and EF (respectvely), and the base AC (is) equal to the base DF, the angle ABC is thus equal to the (angle) DEF [Prop. 1.8].

Thus, if two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles. (Which is) the very thing it was required to show.

Proposition 11

To draw a perpendicular straight-line from a given raised point to a given plane.



Let A be the given raised point, and the given plane the reference (plane). So, it is required to draw a perpendicular straight-line from point A to the reference plane.

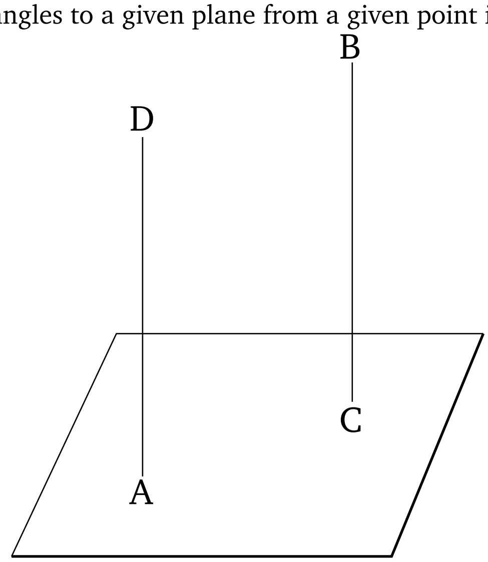
Let some random straight-line BC have been drawn across in the reference plane, and let the (straight-line) AD have been drawn from point A perpendicular to BC [Prop. 1.12]. If, therefore, AD is also perpendicular to the reference plane then that which was prescribed will have occurred. And, if not, let DE have been drawn in the reference plane from point D at right-angles to BC [Prop. 1.11], and let the (straight-line) AF have been drawn from A perpendicular to DE [Prop. 1.12], and let GH have been drawn through point F, parallel to BC [Prop. 1.31].

And since BC is at right-angles to each of DA and DE, BC is thus also at right-angles to the plane through EDA [Prop. 11.4]. And GH is parallel to it. And if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (straight-line) will also be at right-angles to the same plane [Prop. 11.8]. Thus, GH is also at right-angles to the plane through ED and DA. And GH is thus at right-angles to all of the straight-lines joined to it which are also in the plane through ED and AD [Def. 11.3]. And AF, which is in the plane through ED and DA, is joined to it. Thus, GH is at right-angles to FA. Hence, FA is also at right-angles to HG. And AF is also at right-angles to DE. Thus, AF is at right-angles to each of GH and DE. And if a straight-line is set up at right-angles to two straight-lines cutting one another, at the point of section, then it will also be at right-angles to the plane through them [Prop. 11.4]. Thus, FA is at right-angles to the plane through ED and GH. And the plane through ED and GH is the reference (plane). Thus, AF is at right-angles to the reference plane.

Thus, the straight-line AF has been drawn from the given raised point A perpendicular to the reference plane. (Which is) the very thing it was required to do.

Proposition 12

To set up a straight-line at right-angles to a given plane from a given point in it.



Let the given plane be the reference (plane), and A a point in it. So, it is required to set up a straight-line at right-angles to the reference plane at point A.

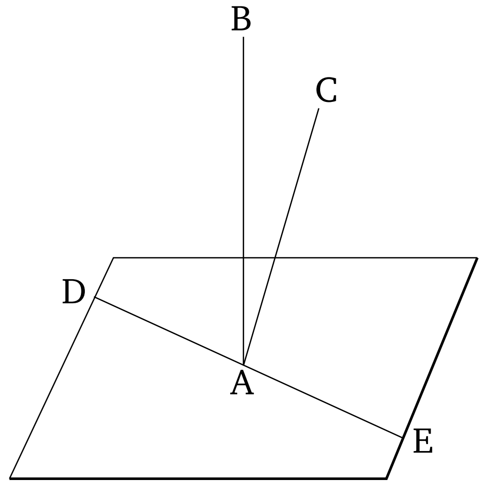
Let some raised point B have been assumed, and let the perpendicular (straight-line) BC have been drawn from B to the reference plane [Prop. 11.11]. And let AD have been drawn from point A parallel to BC [Prop. 1.31].

Therefore, since AD and CB are two parallel straight-lines, and one of them, BC, is at right-angles to the reference plane, the remaining (one) AD is thus also at right-angles to the reference plane [Prop. 11.8].

Thus, AD has been set up at right-angles to the given plane, from the point in it, A. (Which is) the very thing it was required to do.

Proposition 13

Two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side.



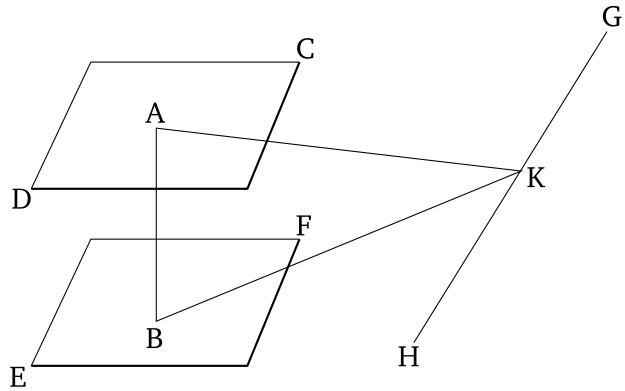
For, if possible, let the two straight-lines AB and AC have been set up at the same point A at right-angles to the reference plane, on the same side. And let the plane through BA and AC have been drawn. So it will make a straight cutting (passing) through (point) A in the reference plane [Prop. 11.3]. Let it have made DAE. Thus, AB, AC, and DAE are straight-lines in one plane. And since CA is at right-angles to the reference plane, it will thus also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. And DAE, which is in the reference plane, is joined to it. Thus, angle CAE is a right-angle. So, for the same (reasons), BAE is also a right-angle. Thus, CAE (is) equal to BAE. And they are in one plane. The very thing is impossible.

Thus, two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side. (Which is) the very thing it was required to show.

Proposition 14

Planes to which the same straight-line is at right-angles will be parallel planes.

For let some straight-line AB be at right-angles to each of the planes CD and EF. I say that the planes are parallel.



For, if not, being produced, they will meet. Let them have met. So they will make a straight-line as a common section [Prop. 11.3]. Let them have made GH. And let some random point K have been taken on GH. And let AK and BK have been joined.

And since AB is at right-angles to the plane EF, AB is thus also at right-angles to BK, which is a straight-line in the produced plane EF [Def. 11.3]. Thus, angle ABK is a right-angle. So, for the same (reasons), BAK is also a right-angle. So the (sum of the) two angles ABK and BAK in the triangle ABK is equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, planes CD and EF, being produced, will not meet. Planes CD and EF are thus parallel [Def. 11.8].

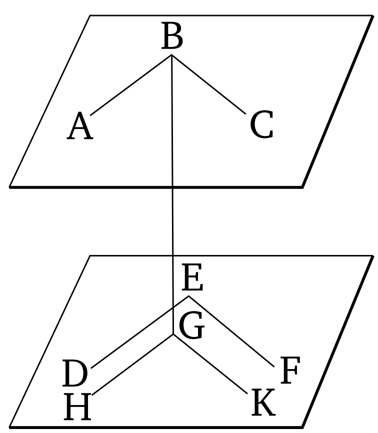
Thus, planes to which the same straight-line is at right-angles are parallel planes. (Which is) the very thing it was required to show.

Proposition 15

If two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another).

For let the two straight-lines joined to one another, AB and BC, be parallel to the two straight-lines joined to one another, DE and EF (respectively), not being in the same plane. I say that the planes through AB, BC and DE, EF will not meet one another (when) produced.

For let BG have been drawn from point B perpendicular to the plane through DE and EF [Prop. 11.11], and let it meet the plane at point G. And let GH have been drawn through G parallel to ED, and GK (parallel) to EF [Prop. 1.31].



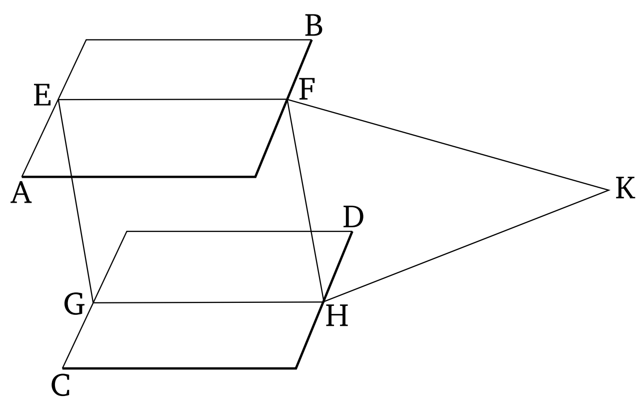
And since BG is at right-angles to the plane through DE and EF, it will thus also make right-angles with all of the straight-lines joined to it, which are also in the plane through DE and EF [Def. 11.3]. And each of GH and GK, which are in the plane through DE and EF, are joined to it. Thus, each of the angles BGH and BGK are right-angles. And since BA is parallel to GH [Prop. 11.9], the (sum of the) angles GBA and BGH is equal to two right-angles [Prop. 1.29]. And BGH (is) a right-angle. GBA (is) thus also a right-angle. Thus, GB is at right-angles to BA. So, for the same (reasons), GB is also at right-angles to BC. Therefore, since the straight-line GB has been set up at right-angles to two straight-lines, BA and BC, cutting one another, GB is thus at right-angles to the plane through BA and BC [Prop. 11.4]. [So, for the same (reasons), BG is also at right-angles to the plane through GH and GK. And the plane through GH and GK is the (plane) through DE and EF. And it was also shown that GB is at right-angles to the plane through AB and BC.] And planes to which the same straight-line is at right-angles are parallel planes [Prop. 11.14]. Thus, the plane through AB and BC is parallel to the (plane) through DE and EF.

Thus, if two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another). (Which is) the very thing it was required to show.

Proposition 16

If two parallel planes are cut by some plane then their common sections are parallel.

For let the two parallel planes AB and CD have been cut by the plane $EFGH$. And let EF and GH be their common sections. I say that EF is parallel to GH.



For, if not, being produced, EF and GH will meet either in the direction of F, H, or of E, G. Let them be produced, as in the direction of F, H, and let them, first of all, have met at K. And since EFK is in the plane AB, all of the points on EFK are thus also in the plane AB [Prop. 11.1]. And K is one of the points on EFK. Thus, K is in the plane AB. So, for the same (reasons), K is also in the plane CD. Thus, the planes AB and CD, being produced, will meet. But they do not meet, on account of being (initially) assumed (to be mutually) parallel. Thus, the straight-lines EF and GH, being produced in the direction of F, H, will not meet. So, similarly, we can show that the straight-lines EF and GH, being produced in the direction of E, G, will not meet either. And (straight-lines in one plane which), being produced, do not meet in either direction are parallel [Def. 1.23]. EF is thus parallel to GH.

Thus, if two parallel planes are cut by some plane then their common sections are parallel. (Which is) the very thing it was required to show.

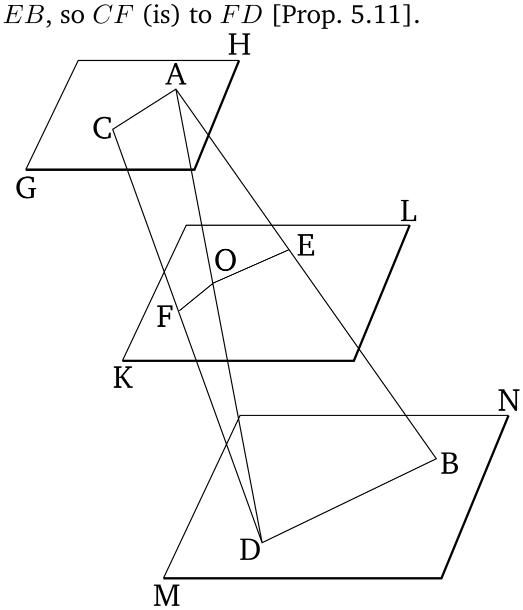
Proposition 17

If two straight-lines are cut by parallel planes then they will be cut in the same ratios.

For let the two straight-lines AB and CD be cut by the parallel planes GH, KL, and MN at the points A, E, B, and C, F, D (respectively). I say that as the straight-line AE is to EB, so CF (is) to FD.

For let AC, BD, and AD have been joined, and let AD meet the plane KL at point O, and let EO and OF have been joined.

And since two parallel planes KL and MN are cut by the plane $EBDO$, their common sections EO and BD are parallel [Prop. 11.16]. So, for the same (reasons), since two parallel planes GH and KL are cut by the plane $AOFC$, their common sections AC and OF are parallel [Prop. 11.16]. And since the straight-line EO has been drawn parallel to one of the sides BD of triangle ABD, thus, proportionally, as AE is to EB, so AO (is) to OD [Prop. 6.2]. Again, since the straight-line OF has been drawn parallel to one of the sides AC of triangle ADC, proportionally, as AO is to OD, so CF (is) to FD [Prop. 6.2]. And it was also shown that as AO (is) to OD, so AE (is) to EB. And thus as AE (is) to EB, so CF (is) to FD [Prop. 5.11].



Thus, if two straight-lines are cut by parallel planes then they will be cut in the same ratios. (Which is) the very thing it was required to show.

Proposition 18

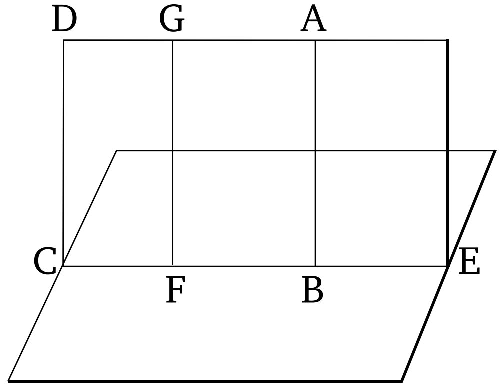
If a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane.

For let some straight-line AB be at right-angles to a reference plane. I say that all of the planes (passing) through AB are also at right-angles to the reference plane.

For let the plane DE have been produced through AB. And let CE be the common section of the plane DE and the reference (plane). And let some random point F have been taken on CE. And let FG have been drawn from F, at right-angles to CE, in the plane DE [Prop. 1.11].

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Hence, it is also at right-angles to CE. Thus, angle

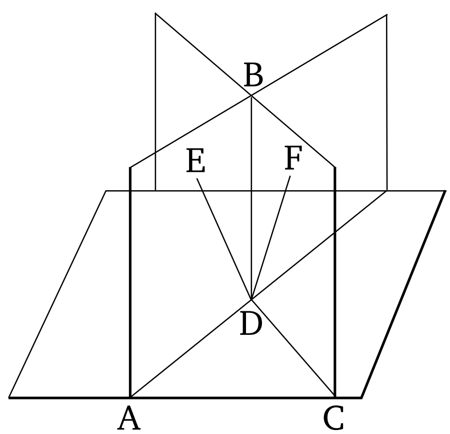
ABF is a right-angle. And GFB is also a right-angle. Thus, AB is parallel to FG [Prop. 1.28]. And AB is at right-angles to the reference plane. Thus, FG is also at right-angles to the reference plane [Prop. 11.8]. And a plane is at right-angles to a(nother) plane when the straight-lines drawn at right-angles to the common section of the planes, (and lying) in one of the planes, are at right-angles to the remaining plane [Def. 11.4]. And FG, (which was) drawn at right-angles to the common section of the planes, CE, in one of the planes, DE, was shown to be at right-angles to the reference plane. Thus, plane DE is at right-angles to the reference (plane). So, similarly, it can be shown that all of the planes (passing) at random through AB (are) at right-angles to the reference plane.



Thus, if a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 19

If two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane.



For let the two planes AB and BC be at right-angles to a reference plane, and let their common section be BD. I say that BD is at right-angles to the reference plane.

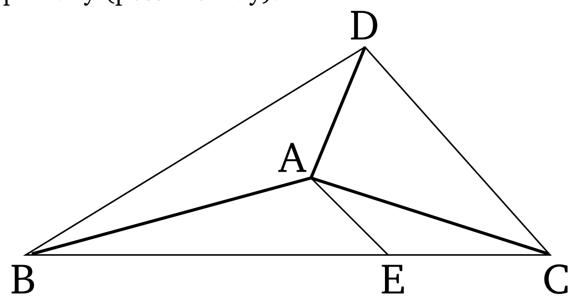
For (if) not, let DE also have been drawn from point D, in the plane AB, at right-angles to the straight-line AD, and DF, in the plane BC, at right-angles to CD.

And since the plane AB is at right-angles to the reference (plane), and DE has been drawn at right-angles to their common section AD, in the plane AB, DE is thus at right-angles to the reference plane [Def. 11.4]. So, similarly, we can show that DF is also at right-angles to the reference plane. Thus, two (different) straight-lines are set up, at the same point D, at right-angles to the reference plane, on the same side. The very thing is impossible [Prop. 11.13]. Thus, no (other straight-line) except the common section DB of the planes AB and BC can be set up at point D, at right-angles to the reference plane.

Thus, if two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 20

If a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way).



For let the solid angle A have been contained by the three plane angles BAC, CAD, and DAB. I say that (the sum of) any two of the angles BAC, CAD, and DAB is greater than the remaining (one), (the angles) being taken up in any (possible way).

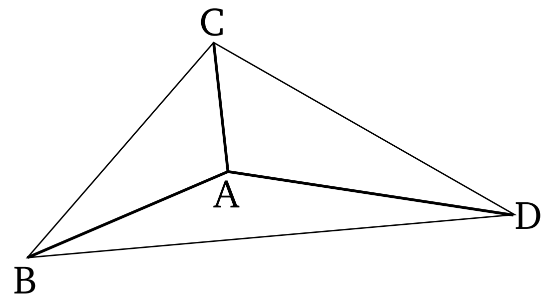
For if the angles BAC, CAD, and DAB are equal to one another then (it is) clear that (the sum of) any two is greater than the remaining (one). But, if not, let BAC be greater (than CAD or DAB). And let (angle) BAE, equal to the angle DAB, have been constructed in the plane through BAC, on the straight-line AB, at the point A on it. And let AE be made equal to AD. And BEC being drawn across through point E, let it cut the straight-lines AB and AC at points B and C (respectively). And let DB and DC have been joined.

And since DA is equal to AE, and AB (is) common, the two (straight-lines AD and AB are) equal to the two (straight-lines EA and AB, respectively). And angle DAB (is) equal to angle BAE. Thus, the base DB is equal to the base BE [Prop. 1.4]. And since the (sum of the) two (straight-lines) BD and DC is greater than BC [Prop. 1.20], of which DB was shown (to be) equal to BE, the remainder DC is thus greater than the remainder EC. And since DA is equal to AE, but AC (is) common, and the base DC is greater than the base EC, the angle DAC is thus greater than the angle EAC [Prop. 1.25]. And DAB was also shown (to be) equal to BAE. Thus, (the sum of) DAB and DAC is greater than BAC. So, similarly, we can also show that the remaining (angles), being taken in pairs, are greater than the remaining (one).

Thus, if a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way). (Which is) the very thing it was required to show.

Proposition 21

Any solid angle is contained by plane angles (whose sum is) less [than] four right-angles.†



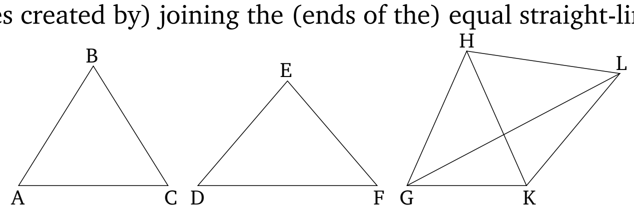
Let the solid angle A be contained by the plane angles BAC, CAD, and DAB. I say that (the sum of) BAC, CAD, and DAB is less than four right-angles.

For let the random points B, C, and D have been taken on each of (the straight-lines) AB, AC, and AD (respectively). And let BC, CD, and DB have been joined. And since the solid angle at B is contained by the three plane angles CBA, ABD, and CBD, (the sum of) any two is greater than the remaining (one) [Prop. 11.20]. Thus, (the sum of) CBA and ABD is greater than CBD. So, for the same (reasons), (the sum of) BCA and ACD is also greater than BCD, and (the sum of) CDA and ADB is greater than CDB. Thus, the (sum of the) six angles CBA, ABD, BCA, ACD, CDA, and ADB is greater than the (sum of the) three (angles) CBD, BCD, and CDB. But, the (sum of the) three (angles) CBD, BDC, and BCD is equal to two right-angles [Prop. 1.32]. Thus, the (sum of the) six angles CBA, ABD, BCA, ACD, CDA, and ADB is greater than two right-angles. And since the (sum of the) three angles of each of the triangles ABC, ACD, and ADB is equal to two right-angles, the (sum of the) nine angles CBA, ACB, BAC, ACD, CDA, CAD, ADB, DBA, and BAD of the three triangles is equal to six right-angles, of which the (sum of the) six angles ABC, BCA, ACD, CDA, ADB, and DBA is greater than two right-angles. Thus, the (sum of the) remaining three [angles] BAC, CAD, and DAB, containing the solid angle, is less than four right-angles.

Thus, any solid angle is contained by plane angles (whose sum is) less [than] four right-angles. (Which is) the very thing it was required to show. † This proposition is only proved for the case of a solid angle contained by three plane angles. However, the generalization to a solid angle contained by more than three plane angles is straightforward.

Proposition 22

If there are three plane angles, of which (the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way), and if equal straight-lines contain them, then it is possible to construct a triangle from (the straight-lines created by) joining the (ends of the) equal straight-lines.

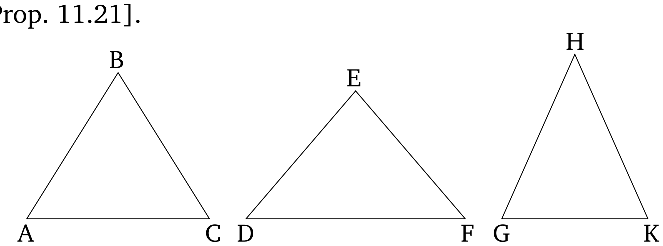


Let ABC, DEF, and GHK be three plane angles, of which the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way)—(that is), ABC and DEF (greater) than GHK, DEF and GHK (greater) than ABC, and, further, GHK and ABC (greater) than DEF. And let AB, BC, DE, EF, GH, and HK be equal straight-lines. And let AC, DF, and GK have been joined. I say that that it is possible to construct a triangle out of (straight-lines) equal to AC, DF, and GK—that is to say, that (the sum of) any two of AC, DF, and GK is greater than the remaining (one).

Now, if the angles ABC, DEF, and GHK are equal to one another then (it is) clear that, (with) AC, DF, and GK also becoming equal, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. And if not, let them be unequal, and let KHL, equal to angle ABC, have been constructed on the straight-line HK, at the point H on it. And let HL be made equal to one of AB, BC, DE, EF, GH, and HK. And let KL and GL have been joined. And since the two (straight-lines) AB and BC are equal to the two (straight-lines) KH and HL (respectively), and the angle at B (is) equal to KHL, the base AC is thus equal to the base KL [Prop. 1.4]. And since (the sum of) ABC and GHK is greater than DEF, and ABC equal to KHL, GHL is thus greater than DEF. And since the two (straight-lines) GH and HL are equal to the two (straight-lines) DE and EF (respectively), and angle GHL (is) greater than DEF, the base GL is thus greater than the base DF [Prop. 1.24]. But, (the sum of) GK and KL is greater than GL [Prop. 1.20]. Thus, (the sum of) GK and KL is much greater than DF. And KL (is) equal to AC. Thus, (the sum of) AC and GK is greater than the remaining (straight-line) DF. So, similarly, we can show that (the sum of) AC and DF is greater than GK, and, further, that (the sum of) DF and GK is greater than AC. Thus, it is possible to construct a triangle from (straight-lines) equal to AC, DF, and GK. (Which is) the very thing it was required to show.

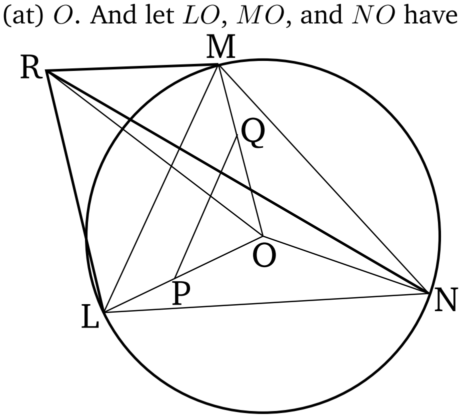
Proposition 23

To construct a solid angle from three (given) plane angles, (the sum of) two of which is greater than the remaining (one, the angles) being taken up in any (possible way). So, it is necessary for the (sum of the) three (angles) to be less than four right-angles [Prop. 11.21].



Let ABC, DEF, and GHK be the three given plane angles, of which let (the sum of) two be greater than the remaining (one, the angles) being taken up in any (possible way), and, further, (let) the (sum of the) three (be) less than four right-angles. So, it is necessary to construct a solid angle from (plane angles) equal to ABC, DEF, and GHK.

Let AB, BC, DE, EF, GH, and HK be cut off (so as to be) equal (to one another). And let AC, DF, and GK have been joined. It is, thus, possible to construct a triangle from (straight-lines) equal to AC, DF, and GK [Prop. 11.22]. Let (such a triangle), LMN, have be constructed, such that AC is equal to LM, DF to MN, and, further, GK to NL. And let the circle LMN have been circumscribed about triangle LMN [Prop. 4.5]. And let its center have been found, and let it be (at) O. And let LO, MO, and NO have been joined.

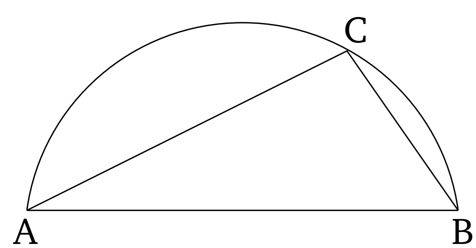


I say that AB is greater than LO. For, if not, AB is either equal to, or less than, LO. Let it, first of all, be equal. And since AB is equal to LO, but AB is equal to BC, and OL to OM, so the two (straight-lines) AB and BC are equal to the two (straight-lines) LO and OM, respectively. And the base AC was assumed (to be) equal to the base LM. Thus, angle ABC is equal to angle LOM [Prop. 1.8]. So, for the same (reasons), DEF is also equal to MON, and, further, GHK to NOL. Thus, the three angles ABC, DEF, and GHK are equal to the three angles LOM, MON, and NOL, respectively. But, the (sum of the) three angles LOM, MON, and NOL is equal to four right-angles. Thus, the (sum of the) three angles ABC, DEF, and GHK is also equal to four right-angles. And it was also assumed (to be) less than four right-angles. The very thing (is) absurd. Thus, AB is not equal to LO. So, I say that AB is not less than LO either. For, if possible, let it be (less). And let OP be made equal to AB, and OQ equal to BC, and let PQ have been joined. And since AB is equal to BC, OP is also equal to OQ. Hence, the remainder LP is also equal to (the remainder) QM. LM is thus parallel to PQ [Prop. 6.2], and (triangle) LMO (is) equiangular with (triangle) PQO [Prop. 1.29]. Thus, as OL is to LM, so OP (is) to PQ [Prop. 6.4]. Alternately, as LO (is) to OP, so LM (is) to PQ [Prop. 5.16]. And LO (is) greater than OP. Thus, LM (is) also greater than PQ [Prop. 5.14]. But LM was made equal to AC. Thus, AC is also greater than PQ. Therefore, since the two (straight-lines) AB and BC are equal to the two (straight-lines) PO and OQ (respectively), and the base AC is greater than the base PQ, the angle ABC is thus greater than the angle POQ [Prop. 1.25]. So, similarly, we can show that DEF is also greater than MON, and GHK than NOL. Thus, the (sum of the) three angles ABC, DEF, and GHK is greater than the (sum of the) three angles LOM, MON, and NOL. But, (the sum of) ABC, DEF, and GHK was assumed (to be) less than four right-angles. Thus, (the sum of) LOM, MON, and NOL is much less than four right-angles. But, (it is) also equal (to four right-angles). The very thing is absurd. Thus, AB is not less than LO. And it was shown (to be) not equal either. Thus, AB (is) greater than LO.

So let OR have been set up at point O at right-angles to the plane of circle LMN [Prop. 11.12]. And let the (square) on OR be equal to that (area) by which the square on AB is greater than the (square) on LO [Prop. 11.23 lem.]. And let RL, RM, and RN have been joined.

And since RO is at right-angles to the plane of circle LMN, RO is thus also at right-angles to each of LO, MO, and NO. And since LO is equal to OM, and OR is common and at right-angles, the base RL is thus equal to the base RM [Prop. 1.4]. So, for the same (reasons), RN is also equal to each of RL and RM. Thus, the three (straight-lines) RL, RM, and RN are equal to one another. And since the (square) on OR was assumed to be equal to that (area) by which the (square) on AB is greater than the (square) on LO, the (square) on AB is thus equal to the (sum of the squares) on LO and OR. And the (square) on LR is equal to the (sum of the squares) on LO and OR. For LOR (is) a right-angle [Prop. 1.47]. Thus, the (square) on AB is equal to the (square) on RL. Thus, AB (is) equal to RL. But, each of BC, DE, EF, GH, and HK is equal to AB, and each of RM and RN equal to RL. Thus, each of AB, BC, DE, EF, GH, and HK is equal to each of RL, RM, and RN. And since the two (straight-lines) LR and RM are equal to the two (straight-lines) AB and BC (respectively), and the base LM was assumed (to be) equal to the base AC, the angle LRM is thus equal to the angle ABC [Prop. 1.8]. So, for the same (reasons), MRN is also equal to DEF, and LRN to GHK.

Thus, the solid angle R, contained by the angles LRM, MRN, and LRN, has been constructed out of the three plane angles LRM, MRN, and LRN, which are equal to the three given (plane angles) ABC, DEF, and GHK (respectively). (Which is) the very thing it was required to do.

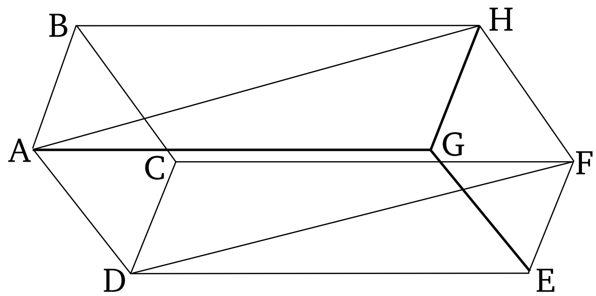


Lemma

And we can demonstrate, thusly, in which manner to take the (square) on OR equal to that (area) by which the (square) on AB is greater than the (square) on LO. Let the straight-lines AB and LO be set out, and let AB be greater, and let the semicircle ABC have been drawn around it. And let AC, equal to the straight-line LO, which is not greater than the diameter AB, have been inserted into the semicircle ABC [Prop. 4.1]. And let CB have been joined. Therefore, since the angle ACB is in the semicircle ACB, ACB is thus a right-angle [Prop. 3.31]. Thus, the (square) on AB is equal to the (sum of the) squares on AC and CB [Prop. 1.47]. Hence, the (square) on AB is greater than the (square) on AC by the (square) on CB. And AC (is) equal to LO. Thus, the (square) on AB is greater than the (square) on LO by the (square) on CB. Therefore, if we take OR equal to BC then the (square) on AB will be greater than the (square) on LO by the (square) on OR. (Which is) the very thing it was prescribed to do.

Proposition 24

If a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic.



For let the solid (figure) $CDHG$ have been contained by the parallel planes AC, GF, and AH, DF, and BF, AE. I say that its opposite planes are both equal and parallelogrammic.

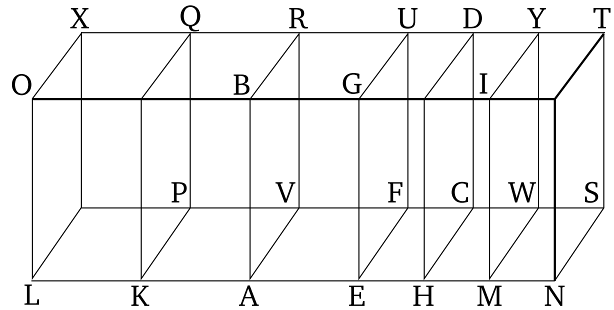
For since the two parallel planes BG and CE are cut by the plane AC, their common sections are parallel [Prop. 11.16]. Thus, AB is parallel to DC. Again, since the two parallel planes BF and AE are cut by the plane AC, their common sections are parallel [Prop. 11.16]. Thus, BC is parallel to AD. And AB was also shown (to be) parallel to DC. Thus, AC is a parallelogram. So, similarly, we can also show that DF, FG, GB, BF, and AE are each parallelograms.

Let AH and DF have been joined. And since AB is parallel to DC, and BH to CF, so the two (straight-lines) joining one another, AB and BH, are parallel to the two straight-lines joining one another, DC and CF (respectively), not (being) in the same plane. Thus, they will contain equal angles [Prop. 11.10]. Thus, angle ABH (is) equal to (angle) DCF. And since the two (straight-lines) AB and BH are equal to the two (straight-lines) DC and CF (respectively) [Prop. 1.34], and angle ABH is equal to angle DCF, the base AH is thus equal to the base DF, and triangle ABH is equal to triangle DCF [Prop. 1.4]. And parallelogram BG is double (triangle) ABH, and parallelogram CE double (triangle) DCF [Prop. 1.34]. Thus, parallelogram BG (is) equal to parallelogram CE. So, similarly, we can show that AC is also equal to GF, and AE to BF.

Thus, if a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic. (Which is) the very thing it was required to show.

Proposition 25

If a parallelipiped solid is cut by a plane which is parallel to the opposite planes (of the parallelipiped) then as the base (is) to the base, so the solid will be to the solid.



For let the parallelipiped solid $ABCD$ have been cut by the plane FG which is parallel to the opposite planes RA and DH. I say that as the base $AEFV$ (is) to the base $EHCF$, so the solid $ABFU$ (is) to the solid $EGCD$.

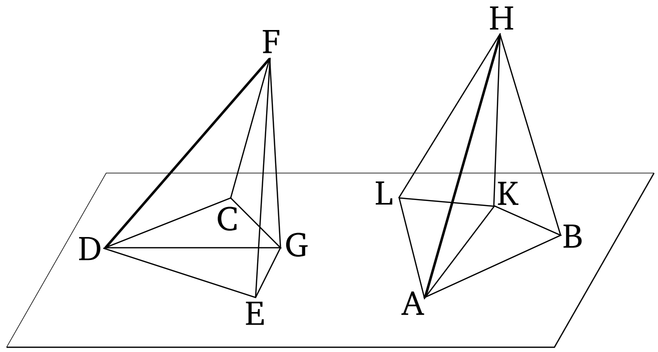
For let AH have been produced in each direction. And let any number whatsoever (of lengths), AK and KL, be made equal to AE, and any number whatsoever (of lengths), HM and MN, equal to EH. And let the parallelograms LP, KV, HW, and MS have been completed, and the solids LQ, KR, DM, and MT.

And since the straight-lines LK, KA, and AE are equal to one another, the parallelograms LP, KV, and AF are also equal to one another, and KO, KB, and AG (are equal) to one another, and, further, LX, KQ, and AR (are equal) to one another. For (they are) opposite [Prop. 11.24]. So, for the same (reasons), the parallelograms EC, HW, and MS are also equal to one another, and HG, HI, and IN are equal to one another, and, further, DH, MY, and NT (are equal to one another). Thus, three planes of (one of) the solids LQ, KR, and AU are equal to the (corresponding) three planes (of the others). But, the three planes (in one of the soilds) are equal to the three opposite planes [Prop. 11.24]. Thus, the three solids LQ, KR, and AU are equal to one another [Def. 11.10]. So, for the same (reasons), the three solids ED, DM, and MT are also equal to one another. Thus, as many multiples as the base LF is of the base AF, so many multiples is the solid LU also of the the solid AU. So, for the same (reasons), as many multiples as the base NF is of the base FH, so many multiples is the solid NU also of the solid HU. And if the base LF is equal to the base NF then the solid LU is also equal to the solid NU.† And if the base LF exceeds the base NF then the solid LU also exceeds the solid NU. And if (LF) is less than (NF) then (LU) is (also) less than (NU). So, there are four magnitudes, the two bases AF and FH, and the two solids AU and UH, and equal multiples have been taken of the base AF and the solid AU— (namely), the base LF and the solid LU—and of the base HF and the solid HU—(namely), the base NF and the solid NU. And it has been shown that if the base LF exceeds the base FN then the solid LU also exceeds the [solid] NU, and if (LF is) equal (to FN) then (LU is) equal (to NU), and if (LF is) less than (FN) then (LU is) less than (NU). Thus, as the base AF is to the base FH, so the solid AU (is) to the solid UH [Def. 5.5]. (Which is) the very thing it was required to show. † Here, Euclid assumes that $LF \gtreqless NF$ implies $LU \gtreqless NU$. This is easily demonstrated.

Proposition 26

To construct a solid angle equal to a given solid angle on a given straight-line, and at a given point on it.

Let AB be the given straight-line, and A the given point on it, and D the given solid angle, contained by the plane angles EDC, EDF, and FDC. So, it is necessary to construct a solid angle equal to the solid angle D on the straight-line AB, and at the point A on it.



For let some random point F have been taken on DF, and let FG have been drawn from F perpendicular to the plane through ED and DC [Prop. 11.11], and let it meet the plane at G, and let DG have been joined. And let BAL, equal to the angle EDC, and BAK, equal to EDG, have been constructed on the straight-line AB at the point A on it [Prop. 1.23]. And let AK be made equal to DG. And let KH have been set up at the point K at right-angles to the plane through BAL [Prop. 11.12]. And let KH be made equal to GF. And let HA have been joined. I say that the solid angle at A, contained by the (plane) angles BAL, BAH, and HAL, is equal to the solid angle at D, contained by the (plane) angles EDC, EDF, and FDC.

For let AB and DE have been cut off (so as to be) equal, and let HB, KB, FE, and GE have been joined. And since FG is at right-angles to the reference plane (EDC), it will also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Thus, the angles FGD and FGE are right-angles. So, for the same (reasons), the angles HKA and HKB are also right-angles. And since the two (straight-lines) KA and AB are equal to the two (straight-lines) GD and DE, respectively, and they contain equal angles, the base KB is thus equal to the base GE [Prop. 1.4]. And KH is also equal to GF. And they contain right-angles (with the respective bases). Thus, HB (is) also equal to FE [Prop. 1.4]. Again, since the two (straight-lines) AK and KH are equal to the two (straight-lines) DG and GF (respectively), and they contain right-angles, the base AH is thus equal to the base FD [Prop. 1.4]. And AB (is) also equal to DE. So, the two (straight-lines) HA and AB are equal to the two (straight-lines) DF and DE (respectively). And the base HB (is) equal to the base FE. Thus, the angle BAH is equal to the angle EDF [Prop. 1.8]. So, for the same (reasons), HAL is also equal to FDC. And BAL is also equal to EDC.

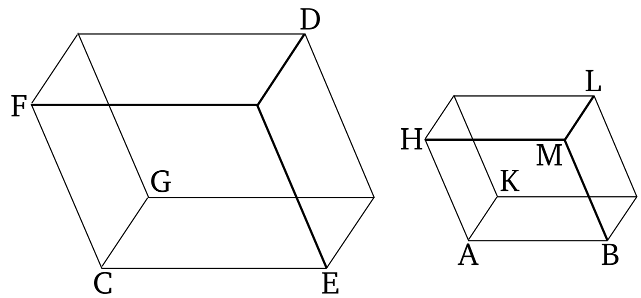
Thus, (a solid angle) has been constructed, equal to the given solid angle at D, on the given straight-line AB, at the given point A on it. (Which is) the very thing it was required to do.

Proposition 27

To describe a parallelepiped solid similar, and similarly laid out, to a given parallelepiped solid on a given straight-line.

Let the given straight-line be AB, and the given parallelepiped solid CD. So, it is necessary to describe a parallelepiped solid similar, and similarly laid out, to the given parallelepiped solid CD on the given straight-line AB.

For, let a (solid angle) contained by the (plane angles) BAH, HAK, and KAB have been constructed, equal to solid angle at C, on the straight-line AB at the point A on it [Prop. 11.26], such that angle BAH is equal to ECF, and BAK to ECG, and KAH to GCF. And let it have been contrived that as EC (is) to CG, so BA (is) to AK, and as GC (is) to CF, so KA (is) to AH [Prop. 6.12]. And thus, via equality, as EC is to CF, so BA (is) to AH [Prop. 5.22]. And let the parallelogram HB have been completed, and the solid AL.

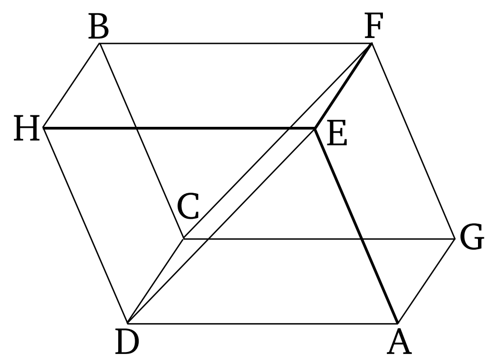


And since as EC is to CG, so BA (is) to AK, and the sides about the equal angles ECG and BAK are (thus) proportional, the parallelogram GE is thus similar to the parallelogram KB. So, for the same (reasons), the parallelogram KH is also similar to the parallelogram GF, and, further, FE (is similar) to HB. Thus, three of the parallelograms of solid CD are similar to three of the parallelograms of solid AL. But, the (former) three are equal and similar to the three opposite, and the (latter) three are equal and similar to the three opposite. Thus, the whole solid CD is similar to the whole solid AL [Def. 11.9].

Thus, AL, similar, and similarly laid out, to the given parallelepiped solid CD, has been described on the given straight-lines AB. (Which is) the very thing it was required to do.

Proposition 28

If a parallelepiped solid is cut by a plane (passing) through the diagonals of (a pair of) opposite planes then the solid will be cut in half by the plane.



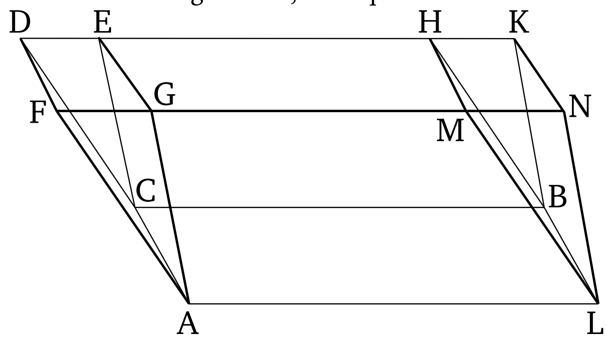
For let the parallelepiped solid AB have been cut by the plane $CDEF$ (passing) through the diagonals of the opposite planes CF and DE.† I say that the solid AB will be cut in half by the plane $CDEF$.

For since triangle CGF is equal to triangle CFB, and ADE (is equal) to DEH [Prop. 1.34], and parallelogram CA is also equal to EB—for (they are) opposite [Prop. 11.24]—and GE (equal) to CH, thus the prism contained by the two triangles CGF and ADE, and the three parallelograms GE, AC, and CE, is also equal to the prism contained by the two triangles CFB and DEH, and the three parallelograms CH, BE, and CE. For they are contained by planes (which are) equal in number and in magnitude [Def. 11.10].‡ Thus, the whole of solid AB is cut in half by the plane $CDEF$. (Which is) the very thing it was required to show. † Here, it is assumed that the two diagonals lie in the same plane. The proof is easily supplied.

‡ However, strictly speaking, the prisms are not similarly arranged, being mirror images of one another.

Proposition 29

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are on the same straight-lines, are equal to one another.



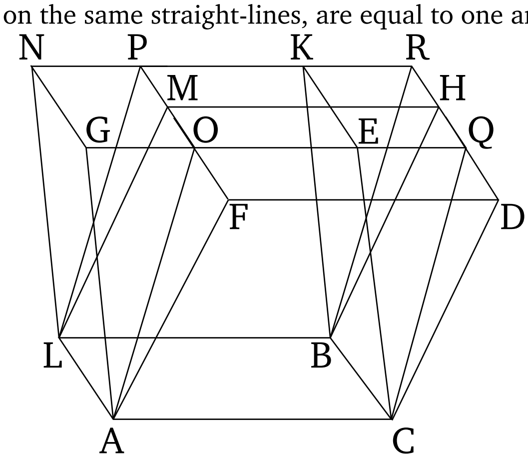
For let the parallelepiped solids CM and CN be on the same base AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AG, AF, LM, LN, CD, CE, BH, and BK, be on the same straight-lines, FN and DK. I say that solid CM is equal to solid CN.

For since CH and CK are each parallelograms, CB is equal to each of DH and EK [Prop. 1.34]. Hence, DH is also equal to EK. Let EH have been subtracted from both. Thus, the remainder DE is equal to the remainder HK. Hence, triangle DCE is also equal to triangle HBK [Props. 1.4, 1.8], and parallelogram DG to parallelogram HN [Prop. 1.36]. So, for the same (reasons), traingle AFG is also equal to triangle MLN. And parallelogram CF is also equal to parallelogram BM, and CG to BN [Prop. 11.24]. For they are opposite. Thus, the prism contained by the two triangles AFG and DCE, and the three parallelograms AD, DG, and CG, is equal to the prism contained by the two triangles MLN and HBK, and the three parallelograms BM, HN, and BN. Let the solid whose base (is) parallelogram AB, and (whose) opposite (face is) $GEHM$, have been added to both (prisms). Thus, the whole parallelepiped solid CM is equal to the whole parallelepiped solid CN.

Thus, parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up (are) on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

Proposition 30

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another.



Let the parallelepiped solids CM and CN be on the same base, AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AF, AG, LM, LN, CD, CE, BH, and BK, not be on the same straight-lines. I say that the solid CM is equal to the solid CN.

For let NK and DH have been produced, and let them have joined one another at R. And, further, let FM and GE have been produced to P and Q (respectively). And let AO, LP, CQ, and BR have been joined. So, solid CM, whose base (is) parallelogram $ACBL$, and opposite (face) $FDHM$, is equal to solid CP, whose base (is) parallelogram $ACBL$, and opposite (face) $OQRP$. For they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, AF, AO, LM, LP, CD, CQ, BH, and BR, are on the same straight-lines, FP and DR [Prop. 11.29]. But, solid CP, whose base is parallelogram $ACBL$, and opposite (face) $OQRP$, is equal to solid CN, whose base (is) parallelogram $ACBL$, and opposite (face) $GEKN$. For, again, they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, AG, AO, CE, CQ, LN, LP, BK, and BR, are on the same straight-lines, GQ and NR [Prop. 11.29]. Hence, solid CM is also equal to solid CN.

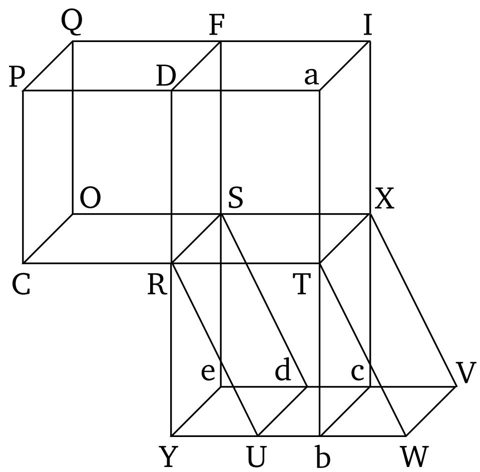
Thus, parallelepiped solids (which are) on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

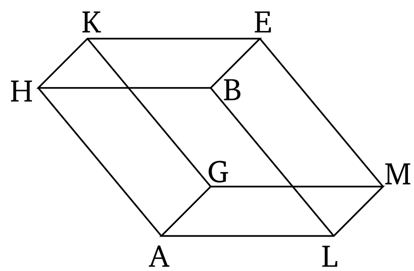
Proposition 31

Parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another.

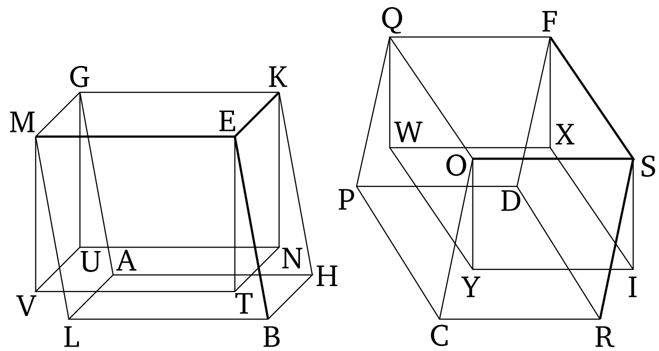
Let the parallelepiped solids AE and CF be on the equal bases AB and CD (respectively), and (have) the same height. I say that solid AE is equal to solid CF.

So, let the (straight-lines) standing up, HK, BE, AG, LM, PQ, DF, CO, and RS, first of all, be at right-angles to the bases AB and CD. And let RT have been produced in a straight-line with CR. And let (angle) TRU, equal to angle ALB, have been constructed on the straight-line RT, at the point R on it [Prop. 1.23]. And let RT be made equal to AL, and RU to LB. And let the base RW, and the solid XU, have been completed.





And since the two (straight-lines) TR and RU are equal to the two (straight-lines) AL and LB (respectively), and they contain equal angles, parallelogram RW is thus equal and similar to parallelogram HL [Prop. 6.14]. And, again, since AL is equal to RT, and LM to RS, and they contain right-angles, parallelogram RX is thus equal and similar to parallelogram AM [Prop. 6.14]. So, for the same (reasons), LE is also equal and similar to SU. Thus, three parallelograms of solid AE are equal and similar to three parallelograms of solid XU. But, the three (faces of the former solid) are equal and similar to the three opposite (faces), and the three (faces of the latter solid) to the three opposite (faces) [Prop. 11.24]. Thus, the whole parallelepiped solid AE is equal to the whole parallelepiped solid XU [Def. 11.10]. Let DR and WU have been drawn across, and let them have met one another at Y. And let aTb have been drawn through T parallel to DY. And let PD have been produced to a. And let the solids YX and RI have been completed. So, solid XY, whose base is parallelogram RX, and opposite (face) Yc, is equal to solid XU, whose base (is) parallelogram RX, and opposite (face) UV. For they are on the same base RX, and (have) the same height, and the (ends of the straight-lines) standing up in them, RY, RU, Tb, TW, Se, Sd, Xc and XV, are on the same straight-lines, YW and eV [Prop. 11.29]. But, solid XU is equal to AE. Thus, solid XY is also equal to solid AE. And since parallelogram $RUWT$ is equal to parallelogram YT. For they are on the same base RT, and between the same parallels RT and YW [Prop. 1.35]. But, $RUWT$ is equal to CD, since (it is) also (equal) to AB. Parallelogram YT is thus also equal to CD. And DT is another (parallelogram). Thus, as base CD is to DT, so YT (is) to DT [Prop. 5.7]. And since the parallelepiped solid CI has been cut by the plane RF, which is parallel to the opposite planes (of CI), as base CD is to base DT, so solid CF (is) to solid RI [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid YI has been cut by the plane RX, which is parallel to the opposite planes (of YI), as base YT is to base TD, so solid YX (is) to solid RI [Prop. 11.25]. But, as base CD (is) to DT, so YT (is) to DT. And, thus, as solid CF (is) to solid RI, so solid YX (is) to solid RI. Thus, solids CF and YX each have the same ratio to RI [Prop. 5.11]. Thus, solid CF is equal to solid YX [Prop. 5.9]. But, YX was show (to be) equal to AE. Thus, AE is also equal to CF.

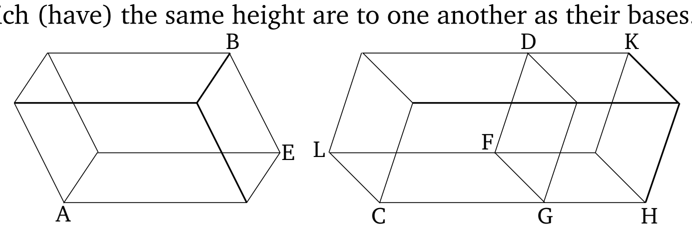


And so let the (straight-lines) standing up, AG, HK, BE, LM, CO, PQ, DF, and RS, not be at right-angles to the bases AB and CD. Again, I say that solid AE (is) equal to solid CF. For let KN, ET, GU, MV, QW, FX, OY, and SI have been drawn from points K, E, G, M, Q, F, O, and S (respectively) perpendicular to the reference plane (*i.e.*, the plane of the bases AB and CD), and let them have met the plane at points N, T, U, V, W, X, Y, and I (respectively). And let NT, NU, UV, TV, WX, WY, YI, and IX have been joined. So solid KV is equal to solid QI. For they are on the equal bases KM and QS, and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid KV is equal to solid AE, and QI to CF. For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid AE is also equal to solid CF.

Thus, parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.

Proposition 32

Parallelepiped solids which (have) the same height are to one another as their bases.



Let AB and CD be parallelepiped solids (having) the same height. I say that the parallelepiped solids AB and CD are to one another as their bases. That is to say, as base AE is to base CF, so solid AB (is) to solid CD.

For let FH, equal to AE, have been applied to FG (in the angle FGH equal to angle LCG) [Prop. 1.45]. And let the parallelepiped solid GK, (having) the same height as CD, have been completed on the base FH. So solid AB is equal to solid GK. For they are on the equal bases AE and FH, and (have) the same height [Prop. 11.31]. And since the parallelepiped solid CK has been cut by the plane DG, which is parallel to the opposite planes (of CK), thus as the base CF is to the base FH, so the solid CD (is) to the solid DH [Prop. 11.25]. And base FH (is) equal to base AE, and solid GK to solid AB. And thus as base AE is to base CF, so solid AB (is) to solid CD.

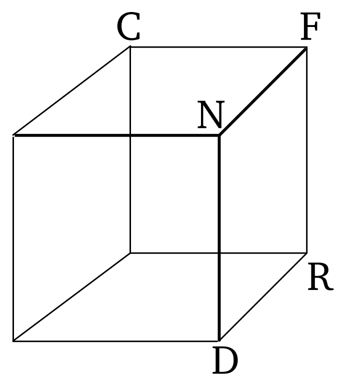
Thus, parallelepiped solids which (have) the same height are to one another as their bases. (Which is) the very thing it was required to show.

Proposition 33

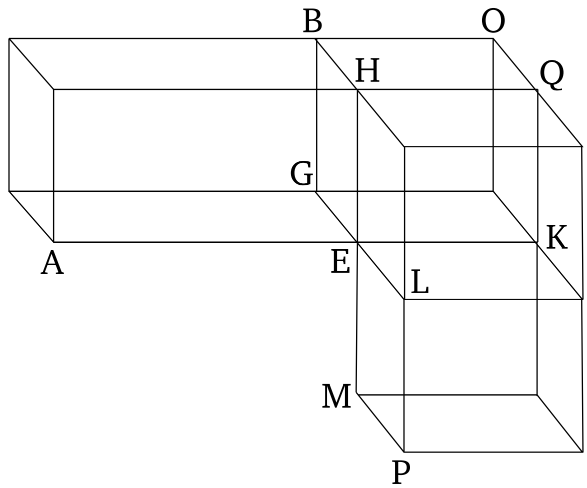
Similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides.

Let AB and CD be similar parallelepiped solids, and let AE correspond to CF. I say that solid AB has to solid CD the cubed ratio that AE (has) to CF.

For let EK, EL, and EM have been produced in a straight-line with AE, GE, and HE (respectively). And let EK be made equal to CF, and EL equal to FN, and, further, EM equal to FR. And let the parallelogram KL have been completed, and the solid KP.



And since the two (straight-lines) KE and EL are equal to the two (straight-lines) CF and FN, but angle KEL is also equal to angle CFN, inasmuch as AEG is also equal to CFN, on account of the similarity of the solids AB and CD, parallelogram KL is thus equal [and similar] to parallelogram CN. So, for the same (reasons), parallelogram KM is also equal and similar to [parallelogram] CR, and, further, EP to DF. Thus, three parallelograms of solid KP are equal and similar to three parallelograms of solid CD. But the three (former parallelograms) are equal and similar to the three opposite (parallelograms), and the three (latter parallelograms) are equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the whole of solid KP is equal and similar to the whole of solid CD [Def. 11.10]. Let parallelogram GK have been completed. And let the the solids EO and LQ, with bases the parallelograms GK and KL (respectively), and with the same height as AB, have been completed. And since, on account of the similarity of solids AB and CD, as AE is to CF, so EG (is) to FN, and EH to FR [Defs. 6.1, 11.9], and CF (is) equal to EK, and FN to EL, and FR to EM, thus as AE is to EK, so GE (is) to EL, and HE to EM. But, as AE (is) to EK, so [parallelogram] AG (is) to parallelogram GK, and as GE (is) to EL, so GK (is) to KL, and as HE (is) to EM, so QE (is) to KM [Prop. 6.1]. And thus as parallelogram AG (is) to GK, so GK (is) to KL, and QE (is) to KM. But, as AG (is) to GK, so solid AB (is) to solid EO, and as GK (is) to KL, so solid OE (is) to solid QL, and as QE (is) to KM, so solid QL (is) to solid KP [Prop. 11.32]. And, thus, as solid AB is to EO, so EO (is) to QL, and QL to KP. And if four magnitudes are continuously proportional then the first has to the fourth the cubed ratio that (it has) to the second [Def. 5.10]. Thus, solid AB has to KP the cubed ratio which AB (has) to EO. But, as AB (is) to EO, so parallelogram AG (is) to GK, and the straight-line AE to EK [Prop. 6.1]. Hence, solid AB also has to KP the cubed ratio that AE (has) to EK. And solid KP (is) equal to solid CD, and straight-line EK to CF. Thus, solid AB also has to solid CD the cubed ratio which its corresponding side AE (has) to the corresponding side CF.



Thus, similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides. (Which is) the very thing it was required to show.

Corollary

So, (it is) clear, from this, that if four straight-lines are (continuously) proportional then as the first is to the fourth, so the parallelepiped solid on the first will be to the similar, and similarly described, parallelepiped solid on the second, since the first also has to the fourth the cubed ratio that (it has) to the second.

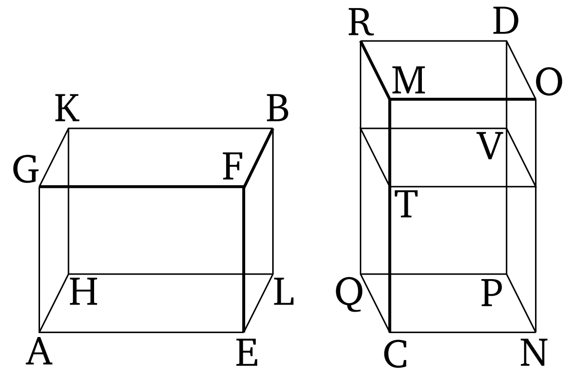
Proposition 34†

The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let AB and CD be equal parallelepiped solids. I say that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

For, first of all, let the (straight-lines) standing up, AG, EF, LB, HK, CM, NO, PD, and QR, be at right-angles to their bases. I say that as base EH is to base NQ, so CM (is) to AG.

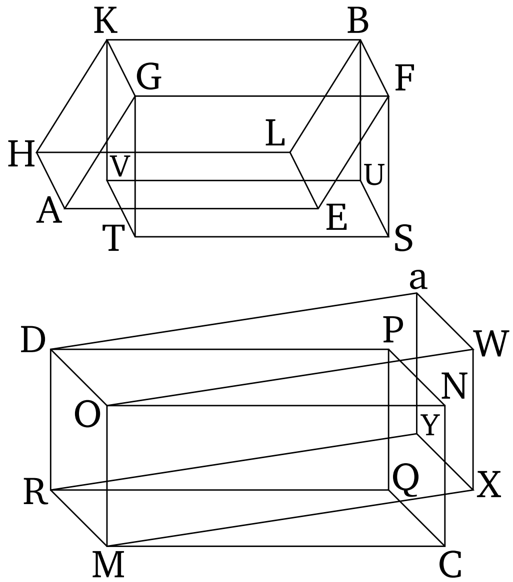
Therefore, if base EH is equal to base NQ, and solid AB is also equal to solid CD, CM will also be equal to AG. For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base EH (is) to NQ, so CM will be to AG. And (so it is) clear that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.



So let base EH not be equal to base NQ, but let EH be greater. And solid AB is also equal to solid CD. Thus, CM is also greater than AG. Therefore, let CT be made equal to AG. And let the parallelepiped solid VC have been completed on the base NQ, with height CT. And since solid AB is equal to solid CD, and CV (is) extrinsic (to them), and equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7], thus as solid AB is to solid CV, so solid CD (is) to solid CV. But, as solid AB (is) to solid CV, so base EH (is) to base NQ. For the solids AB and CV (are) of equal height [Prop. 11.32]. And as solid CD (is) to solid CV, so base MQ (is) to base TQ [Prop. 11.25], and CM to CT [Prop. 6.1]. And, thus, as base EH is to base NQ, so MC (is) to AG. And CT (is) equal to AG. And thus as base EH (is) to base NQ, so MC (is) to AG. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepipid solids AB and CD be reciprocally proportional to their heights, and let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD. [For] let the (straight-lines) standing up again be at right-angles to the bases. And if base EH is equal to base NQ, and as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, the height of solid CD is thus also equal to the height of solid AB. And parallelepiped solids on equal bases, and also with the same height, are equal to one another [Prop. 11.31]. Thus, solid AB is equal to solid CD.

So, let base EH not be equal to [base] NQ, but let EH be greater. Thus, the height of solid CD is also greater than the height of solid AB, that is to say CM (greater) than AG. Let CT again be made equal to AG, and let the solid CV have been similarly completed. Since as base EH is to base NQ, so MC (is) to AG, and AG (is) equal to CT, thus as base EH (is) to base NQ, so CM (is) to CT. But, as [base] EH (is) to base NQ, so solid AB (is) to solid CV. For solids AB and CV are of equal heights [Prop. 11.32]. And as CM (is) to CT, so (is) base MQ to base QT [Prop. 6.1], and solid CD to solid CV [Prop. 11.25]. And thus as solid AB (is) to solid CV, so solid CD (is) to solid CV. Thus, AB and CD each have the same ratio to CV. Thus, solid AB is equal to solid CD [Prop. 5.9].



So, let the (straight-lines) standing up, FE, BL, GA, KH, ON, DP, MC, and RQ, not be at right-angles to their bases. And let perpendiculars have been drawn to the planes through EH and NQ from points F, G, B, K, O, M, R, and D, and let them have joined the planes at (points) S, T, U, V, W, X, Y, and a (respectively). And let the solids FV and OY have been completed. In this case, also, I say that the solids AB and CD being equal, their bases are reciprocally proportional to their heights, and (so) as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB.

Since solid AB is equal to solid CD, but AB is equal to BT. For they are on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid CD is equal is equal to DX. For, again, they are on the same base RO, and (have) the same height [Props. 11.29, 11.30]. Solid BT is thus also equal to solid DX. Thus, as base FK (is) to base OR, so the height of solid DX (is) to the height of solid BT (see first part of proposition). And base FK (is) equal to base EH, and base OR to NQ. Thus, as base EH is to base NQ, so the height of solid DX (is) to the height of solid BT. And solids DX, BT are the same height as (solids) DC, BA (respectively). Thus, as base EH is to base NQ, so the height of solid DC (is) to the height of solid AB. Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

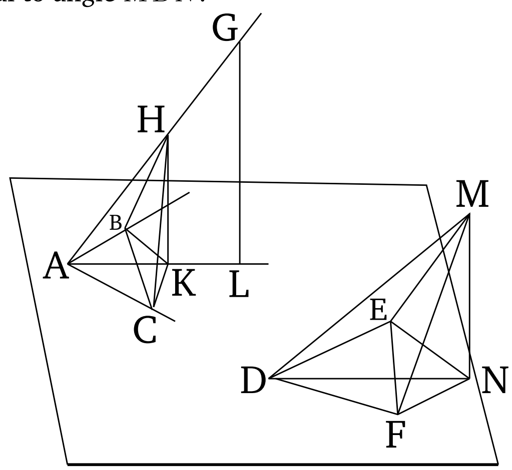
So, again, let the bases of the parallelepiped solids AB and CD be reciprocally proportional to their heights, and (so) let base EH be to base NQ, as the height of solid CD (is) to the height of solid AB. I say that solid AB is equal to solid CD.

For, with the same construction (as before), since as base EH is to base NQ, so the height of solid CD (is) to the height of solid AB, and base EH (is) equal to base FK, and NQ to OR, thus as base FK is to base OR, so the height of solid CD (is) to the height of solid AB. And solids AB, CD are the same height as (solids) BT, DX (respectively). Thus, as base FK is to base OR, so the height of solid DX (is) to the height of solid BT. Thus, the bases of the parallelepiped solids BT and DX are reciprocally proportional to their heights. Thus, solid BT is equal to solid DX (see first part of proposition). But, BT is equal to BA. For [they are] on the same base FK, and (have) the same height [Props. 11.29, 11.30]. And solid DX is equal to solid DC [Props. 11.29, 11.30]. Thus, solid AB is also equal to solid CD. (Which is) the very thing it was required to show. † This proposition assumes that (a) if two parallelepipeds are equal, and have equal bases, then their heights are equal, and (b) if the bases of two equal parallelepipeds are unequal, then that solid which has the lesser base has the greater height.

Proposition 35

If there are two equal plane angles, and raised straight-lines are stood on the apexes of them, containing equal angles respectively with the original straight-lines (forming the angles), and random points are taken on the raised (straight-lines), and perpendiculars are drawn from them to the planes in which the original angles are, and straight-lines are joined from the points created in the planes to the (vertices of the) original angles, then they will enclose equal angles with the raised (straight-lines).

Let BAC and EDF be two equal rectilinear angles. And let the raised straight-lines AG and DM have been stood on points A and D, containing equal angles respectively with the original straight-lines. (That is) MDE (equal) to GAB, and MDF (to) GAC. And let the random points G and M have been taken on AG and DM (respectively). And let the GL and MN have been drawn from points G and M perpendicular to the planes through BAC and EDF (respectively). And let them have joined the planes at points L and N (respectively). And let LA and ND have been joined. I say that angle GAL is equal to angle MDN.



Let AH be made equal to DM. And let HK have been drawn through point H parallel to GL. And GL is perpendicular to the plane through BAC. Thus, HK is also perpendicular to the plane through BAC [Prop. 11.8]. And let KC, NF, KB, and NE have been drawn from points K and N perpendicular to the straight-lines AC, DF, AB, and DE. And let HC, CB, MF, and FE have been joined. Since the (square) on HA is equal to the (sum of the squares) on HK and KA [Prop. 1.47], and the (sum of the squares) on KC and CA is equal to the (square) on KA [Prop. 1.47], thus the (square) on HA is equal to the (sum of the squares) on HK, KC, and CA. And the (square) on HC is equal to the (sum of the squares) on HK and KC [Prop. 1.47]. Thus, the (square) on HA is equal to the (sum of the squares) on HC and CA. Thus, angle HCA is a right-angle [Prop. 1.48]. So, for the same (reasons), angle DFM is also a right-angle. Thus, angle ACH is equal to (angle) DFM. And HAC is also equal to MDF. So, MDF and HAC are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that subtending one of the equal angles —(that is), HA (equal) to MD. Thus, they will also have the remaining sides equal to the remaining sides, respectively [Prop. 1.26]. Thus, AC is equal to DF. So, similarly, we can show that AB is also equal to DE. Therefore, since AC is equal to DF, and AB to DE, so the two (straight-lines) CA and AB are equal to the two (straight-lines) FD and DE (respectively). But, angle CAB is also equal to angle FDE. Thus, base BC is equal to base EF, and triangle (ACB) to triangle (DFE), and the remaining angles to the remaining angles (respectively) [Prop. 1.4]. Thus, angle ACB (is) equal to DFE. And the right-angle ACK is also equal to the right-angle DFN. Thus, the remainder BCK is equal to the remainder EFN. So, for the same (reasons), CBK is also equal to FEN. So, BCK and EFN are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that by the equal angles—(that is), BC (equal) to EF. Thus, they will also have the remaining sides equal to the remaining sides (respectively) [Prop. 1.26]. Thus, CK is equal to FN. And AC (is) also equal to DF. So, the two (straight-lines) AC and CK are equal to the two (straight-lines) DF and FN (respectively). And they enclose right-angles. Thus, base AK is equal to base DN [Prop. 1.4]. And since AH is equal to DM, the (square) on AH is also equal to the (square) on DM. But, the the (sum of the squares) on AK and KH is equal to the (square) on AH. For angle AKH (is) a right-angle [Prop. 1.47].

And the (sum of the squares) on DN and NM (is) equal to the square on DM. For angle DNM (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AK and KH is equal to the (sum of the squares) on DN and NM, of which the (square) on AK is equal to the (square) on DN. Thus, the remaining (square) on KH is equal to the (square) on NM. Thus, HK (is) equal to MN. And since the two (straight-lines) HA and AK are equal to the two (straight-lines) MD and DN, respectively, and base HK was shown (to be) equal to base MN, angle HAK is thus equal to angle MDN [Prop. 1.8].

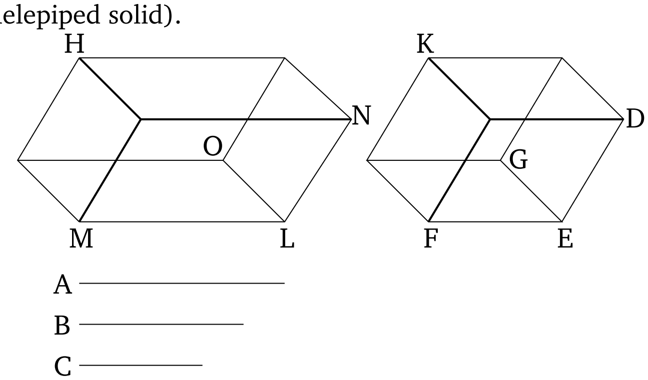
Thus, if there are two equal plane angles, and so on of the proposition. [(Which is) the very thing it was required to show].

Corollary

So, it is clear, from this, that if there are two equal plane angles, and equal raised straight-lines are stood on them (at their apexes), containing equal angles respectively with the original straight-lines (forming the angles), then the perpendiculars drawn from (the raised ends of) them to the planes in which the original angles lie are equal to one another. (Which is) the very thing it was required to show.

Proposition 36

If three straight-lines are (continuously) proportional then the parallelepiped solid (formed) from the three (straight-lines) is equal to the equilateral parallelepiped solid on the middle (straight-line which is) equiangular to the aforementioned (parallelepiped solid).

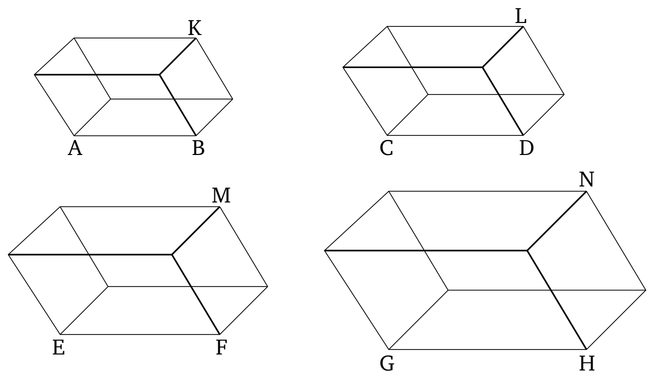


Let A, B, and C be three (continuously) proportional straight-lines, (such that) as A (is) to B, so B (is) to C. I say that the (parallelepiped) solid (formed) from A, B, and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid).

Let the solid angle at E, contained by DEG, GEF, and FED, be set out. And let DE, GE, and EF each be made equal to B. And let the parallelepiped solid EK have been completed. And (let) LM (be made) equal to A. And let the solid angle contained by NLO, OLM, and MLN have been constructed on the straight-line LM, and at the point L on it, (so as to be) equal to the solid angle E [Prop. 11.23]. And let LO be made equal to B, and LN equal to C. And since as A (is) to B, so B (is) to C, and A (is) equal to LM, and B to each of LO and ED, and C to LN, thus as LM (is) to EF, so DE (is) to LN. And (so) the sides around the equal angles NLM and DEF are reciprocally proportional. Thus, parallelogram MN is equal to parallelogram DF [Prop. 6.14]. And since the two plane rectilinear angles DEF and NLM are equal, and the raised straight-lines stood on them (at their apexes), LO and EG, are equal to one another, and contain equal angles respectively with the original straight-lines (forming the angles), the perpendiculars drawn from points G and O to the planes through NLM and DEF (respectively) are thus equal to one another [Prop. 11.35 corr.]. Thus, the solids LH and EK (have) the same height. And parallelepiped solids on equal bases, and with the same height, are equal to one another [Prop. 11.31]. Thus, solid HL is equal to solid EK. And LH is the solid (formed) from A, B, and C, and EK the solid on B. Thus, the parallelepiped solid (formed) from A, B, and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid). (Which is) the very thing it was required to show.

Proposition 37†

If four straight-lines are proportional then the similar, and similarly described, parallelepiped solids on them will also be proportional. And if the similar, and similarly described, parallelepiped solids on them are proportional then the straight-lines themselves will be proportional.



Let AB, CD, EF, and GH, be four proportional straight-lines, (such that) as AB (is) to CD, so EF (is) to GH. And let the similar, and similarly laid out, parallelepiped solids KA, LC, ME and NG have been described on AB, CD, EF, and GH (respectively). I say that as KA is to LC, so ME (is) to NG.

For since the parallelepiped solid KA is similar to LC, KA thus has to LC the cubed ratio that AB (has) to CD [Prop. 11.33]. So, for the same (reasons), ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33]. And since as AB is to CD, so EF (is) to GH, thus, also, as AK (is) to LC, so ME (is) to NG.

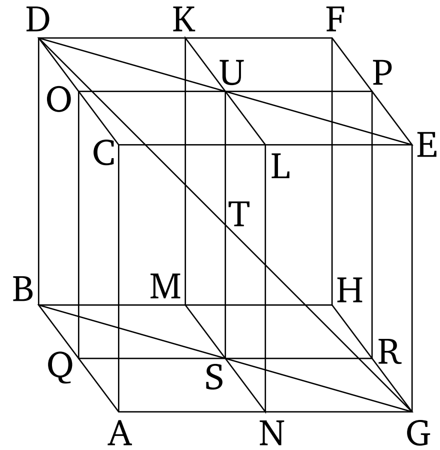
And so let solid AK be to solid LC, as solid ME (is) to NG. I say that as straight-line AB is to CD, so EF (is) to GH.

For, again, since KA has to LC the cubed ratio that AB (has) to CD [Prop. 11.33], and ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33], and as KA is to LC, so ME (is) to NG, thus, also, as AB (is) to CD, so EF (is) to GH.

Thus, if four straight-lines are proportional, and so on of the proposition. (Which is) the very thing it was required to show. † This proposition assumes that if two ratios are equal then the cube of the former is also equal to the cube of the latter, and *vice versa*.

Proposition 38

If the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half.



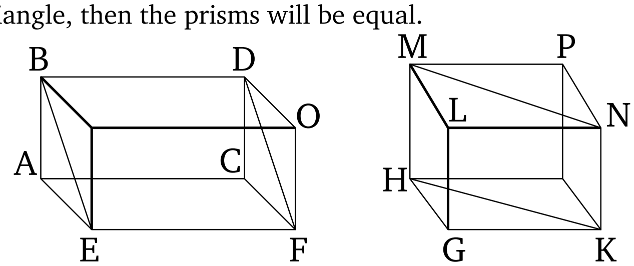
For let the opposite planes CF and AH of the cube AF have been cut in half at the points K, L, M, N, O, Q, P, and R. And let the planes KN and OR have been produced through the pieces. And let US be the common section of the planes, and DG the diameter of cube AF. I say that UT is equal to TS, and DT to TG.

For let DU, UE, BS, and SG have been joined. And since DO is parallel to PE, the alternate angles DOU and UPE are equal to one another [Prop. 1.29]. And since DO is equal to PE, and OU to UP, and they contain equal angles, base DU is thus equal to base UE, and triangle DOU is equal to triangle PUE, and the remaining angles (are) equal to the remaining angles [Prop. 1.4]. Thus, angle OUD (is) equal to angle PUE. So, for this (reason), DUE is a straight-line [Prop. 1.14]. So, for the same (reason), BSG is also a straight-line, and BS equal to SG. And since CA is equal and parallel to DB, but CA is also equal and parallel to EG, DB is thus also equal and parallel to EG [Prop. 11.9]. And the straight-lines DE and BG join them. DE is thus parallel to BG [Prop. 1.33]. Thus, angle EDT (is) equal to BGT. For (they are) alternate [Prop. 1.29]. And (angle) DTU (is equal) to GTS [Prop. 1.15]. So, DTU and GTS are two triangles having two angles equal to two angles, and one side equal to one side—(namely), that subtended by one of the equal angles—(that is), DU (equal) to GS. For they are halves of DE and BG (respectively). (Thus), they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, DT (is) equal to TG, and UT to TS.

Thus, if the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half. (Which is) the very thing it was required to show.

Proposition 39

If there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms will be equal.



Let $ABCDEF$ and $GHKLMN$ be two equal height prisms, and let the former have the parallelogram AF, and the latter the triangle GHK, as a base. And let parallelogram AF be twice triangle GHK. I say that prism $ABCDEF$ is equal to prism $GHKLMN$.

For let the solids AO and GP have been completed. Since parallelogram AF is double triangle GHK, and parallelogram HK is also double triangle GHK [Prop. 1.34], parallelogram AF is thus equal to parallelogram HK. And parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another [Prop. 11.31]. Thus, solid AO is equal to solid GP. And prism $ABCDEF$ is half of solid AO, and prism $GHKLMN$ half of solid GP [Prop. 11.28]. Prism $ABCDEF$ is thus equal to prism $GHKLMN$.

Thus, if there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms are equal. (Which is) the very thing it was required to show.

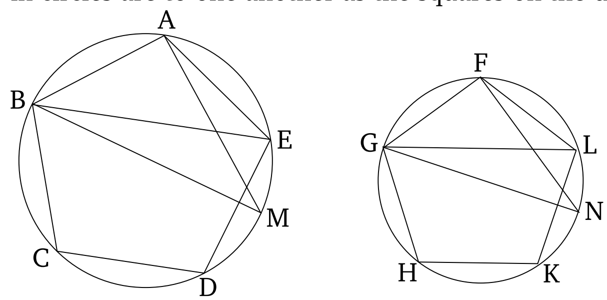
ELEMENTS BOOK 12

Proportional Stereometry[†]

[†]The novel feature of this book is the use of the so-called *method of exhaustion* (see Prop. 10.1), a precursor to integration which is generally attributed to Eudoxus of Cnidus.

Proposition 1

Similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles).



Let ABC and FGH be circles, and let $ABCDE$ and $FGHKL$ be similar polygons (inscribed) in them (respectively), and let BM and GN be the diameters of the circles (respectively). I say that as the square on BM is to the square on GN, so polygon $ABCDE$ (is) to polygon $FGHKL$.

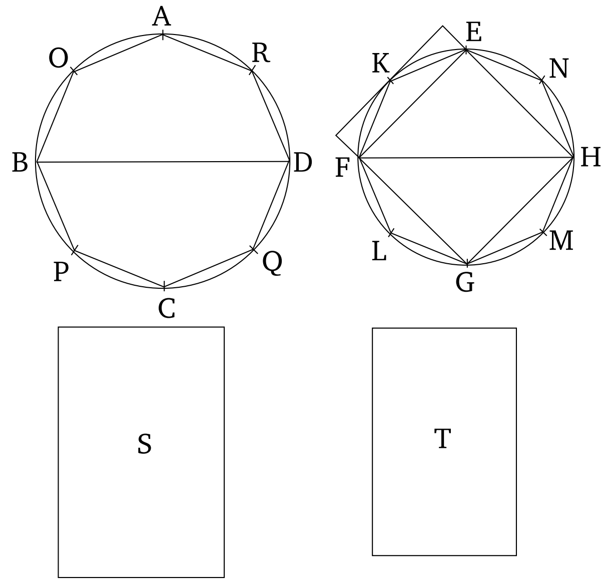
For let BE, AM, GL, and FN have been joined. And since polygon $ABCDE$ (is) similar to polygon $FGHKL$, angle BAE is also equal to (angle) GFL, and as BA is to AE, so GF (is) to FL [Def. 6.1]. So, BAE and GFL are two triangles having one angle equal to one angle, (namely), BAE (equal) to GFL, and the sides around the equal angles proportional. Triangle ABE is thus equiangular with triangle FGL [Prop. 6.6]. Thus, angle AEB is equal to (angle) FLG. But, AEB is equal to AMB, and FLG to FNG, for they stand on the same circumference [Prop. 3.27]. Thus, AMB is also equal to FNG. And the right-angle BAM is also equal to the right-angle GFN [Prop. 3.31]. Thus, the remaining (angle) is also equal to the remaining (angle) [Prop. 1.32]. Thus, triangle ABM is equiangular with triangle FGN. Thus, proportionally, as BM is to GN, so BA (is) to GF [Prop. 6.4]. But, the (ratio) of the square on BM to the square on GN is the square of the ratio of BM to GN, and the (ratio) of polygon $ABCDE$ to polygon $FGHKL$ is the square of the (ratio) of BA to GF [Prop. 6.20]. And, thus, as the square on BM (is) to the square on GN, so polygon $ABCDE$ (is) to polygon $FGHKL$.

Thus, similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles). (Which is) the very thing it was required to show.

Proposition 2

Circles are to one another as the squares on (their) diameters.

Let $ABCD$ and $EFGH$ be circles, and [let] BD and FH [be] their diameters. I say that as circle $ABCD$ is to circle $EFGH$, so the square on BD (is) to the square on FH.



For if the circle $ABCD$ is not to the (circle) $EFGH$, as the square on BD (is) to the (square) on FH, then as the (square) on BD (is) to the (square) on FH, so circle $ABCD$ will be to some area either less than, or greater than, circle $EFGH$. Let it, first of all, be (in that ratio) to (some) lesser (area), S. And let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. So the inscribed square is greater than half of circle $EFGH$, inasmuch as if we draw tangents to the circle through the points E, F, G, and H, then square $EFGH$ is half of the square circumscribed about the circle [Prop. 1.47], and the circle is less than the circumscribed square. Hence, the inscribed square $EFGH$ is greater than half of circle $EFGH$. Let the circumferences EF, FG, GH, and HE have been cut in half at points K, L, M, and N (respectively), and let EK, KF, FL, LG, GM, MH, HN, and NE have been joined. And, thus, each of the triangles EKF, FLG, GMH, and HNE is greater than half of the segment of the circle about it, inasmuch as if we draw tangents to the circle through points K, L, M, and N, and complete the parallelograms on the straight-lines EF, FG, GH, and HE, then each of the triangles EKF, FLG, GMH, and HNE will be half of the parallelogram about it, but the segment about it is less than the parallelogram. Hence, each of the triangles EKF, FLG, GMH, and HNE is greater than half of the segment of the circle about it. So, by cutting the circumferences remaining behind in half, and joining straight-lines, and doing this continually, we will (eventually) leave behind some segments of the circle whose (sum) will be less than the excess by which circle $EFGH$ exceeds the area S. For we showed in the first theorem of the tenth book that if two unequal magnitudes are laid out, and if (a part) greater than a half is subtracted from the greater, and (if from) the remainder (a part) greater than a half (is subtracted), and this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude [Prop. 10.1]. Therefore, let the (segments) have been left, and let the (sum of the) segments of the circle $EFGH$ on EK, KF, FL, LG, GM, MH, HN, and NE be less than the excess by which circle $EFGH$ exceeds area S. Thus, the remaining polygon $EKFLGMHN$ is greater than area S. And let the polygon $AOBPCQDR$, similar to the polygon $EKFLGMHN$, have been inscribed in circle $ABCD$. Thus, as the square on BD is to the square on FH, so polygon $AOBPCQDR$ (is) to polygon $EKFLGMHN$ [Prop. 12.1]. But, also, as the square on BD (is) to the square on FH, so circle $ABCD$ (is) to area S. And, thus, as circle $ABCD$ (is) to area S, so polygon $AOBPGQDR$ (is) to polygon $EKFLGMHN$ [Prop. 5.11]. Thus, alternately, as circle $ABCD$ (is) to the polygon (inscribed) within it, so area S (is) to polygon $EKFLGMHN$ [Prop. 5.16]. And circle $ABCD$ (is) greater than the polygon (inscribed) within it. Thus, area S is also greater than polygon $EKFLGMHN$. But, (it is) also less. The very thing is impossible. Thus, the square on BD is not to the (square) on FH, as circle $ABCD$ (is) to some area less than circle $EFGH$. So, similarly, we can show that the (square) on FH (is) not to the (square) on BD as circle $EFGH$ (is) to some area less than circle $ABCD$ either.

So, I say that neither (is) the (square) on BD to the (square) on FH, as circle $ABCD$ (is) to some area greater than circle $EFGH$.

For, if possible, let it be (in that ratio) to (some) greater (area), S. Thus, inversely, as the square on FH [is] to the (square) on DB, so area S (is) to circle $ABCD$ [Prop. 5.7 corr.]. But, as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$ (see lemma). And, thus, as the (square) on FH (is) to the (square) on BD, so circle $EFGH$ (is) to some area less than circle $ABCD$ [Prop. 5.11]. The very thing was shown (to be) impossible. Thus, as the square on BD is to the (square) on FH, so circle $ABCD$ (is) not to some area greater than circle $EFGH$. And it was shown that neither (is it in that ratio) to (some) lesser (area). Thus, as the square on BD is to the (square) on FH, so circle $ABCD$ (is) to circle $EFGH$.

Thus, circles are to one another as the squares on (their) diameters. (Which is) the very thing it was required to show.

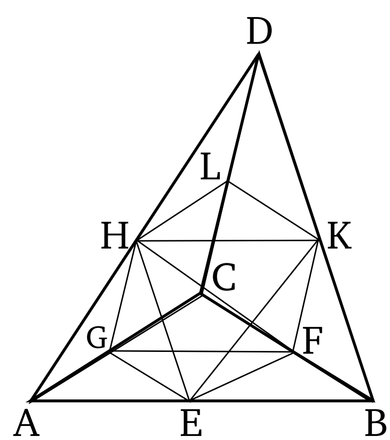
Lemma

So, I say that, area S being greater than circle $EFGH$, as area S is to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$.

For let it have been contrived that as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to area T. I say that area T is less than circle $ABCD$. For since as area S is to circle $ABCD$, so circle $EFGH$ (is) to area T, alternately, as area S is to circle $EFGH$, so circle $ABCD$ (is) to area T [Prop. 5.16]. And area S (is) greater than circle $EFGH$. Thus, circle $ABCD$ (is) also greater than area T [Prop. 5.14]. Hence, as area S is to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$. (Which is) the very thing it was required to show.

Proposition 3

Any pyramid having a triangular base is divided into two pyramids having triangular bases (which are) equal, similar to one another, and [similar] to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.



Let there be a pyramid whose base is triangle ABC, and (whose) apex (is) point D. I say that pyramid $ABCD$ is divided into two pyramids having triangular bases (which are) equal to one another, and similar to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.

For let AB, BC, CA, AD, DB, and DC have been cut in half at points E, F, G, H, K, and L (respectively). And let HE, EG, GH, HK, KL, LH, KF, and FG have been joined. Since AE is equal to EB, and AH to DH,

EH is thus parallel to DB [Prop. 6.2]. So, for the same (reasons), HK is also parallel to AB. Thus, $HEBK$ is a parallelogram. Thus, HK is equal to EB [Prop. 1.34]. But, EB is equal to EA. Thus, AE is also equal to HK. And AH is also equal to HD. So the two (straight-lines) EA and AH are equal to the two (straight-lines) KH and HD, respectively. And angle EAH (is) equal to angle KHD [Prop. 1.29]. Thus, base EH is equal to base KD [Prop. 1.4]. Thus, triangle AEH is equal and similar to triangle HKD [Prop. 1.4]. So, for the same (reasons), triangle AHG is also equal and similar to triangle HLD. And since EH and HG are two straight-lines joining one another (which are respectively) parallel to two straight-lines joining one another, KD and DL, not being in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle EHG is equal to angle KDL. And since the two straight-lines EH and HG are equal to the two straight-lines KD and DL, respectively, and angle EHG is equal to angle KDL, base EG [is] thus equal to base KL [Prop. 1.4]. Thus, triangle EHG is equal and similar to triangle KDL. So, for the same (reasons), triangle AEG is also equal and similar to triangle HKL. Thus, the pyramid whose base is triangle AEG, and apex the point H, is equal and similar to the pyramid whose base is triangle HKL, and apex the point D [Def. 11.10]. And since HK has been drawn parallel to one of the sides, AB, of triangle ADB, triangle ADB is equiangular to triangle DHK [Prop. 1.29], and they have proportional sides. Thus, triangle ADB is similar to triangle DHK [Def. 6.1]. So, for the same (reasons), triangle DBC is also similar to triangle DKL, and ADC to DLH. And since two straight-lines joining one another, BA and AC, are parallel to two straight-lines joining one another, KH and HL, not in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle BAC is equal to (angle) KHL. And as BA is to AC, so KH (is) to HL. Thus, triangle ABC is similar to triangle HKL [Prop. 6.6]. And, thus, the pyramid whose base is triangle ABC, and apex the point D, is similar to the pyramid whose base is triangle HKL, and apex the point D [Def. 11.9]. But, the pyramid whose base [is] triangle HKL, and apex the point D, was shown (to be) similar to the pyramid whose base is triangle AEG, and apex the point H. Thus, each of the pyramids $AEGH$ and $HKLD$ is similar to the whole pyramid $ABCD$.

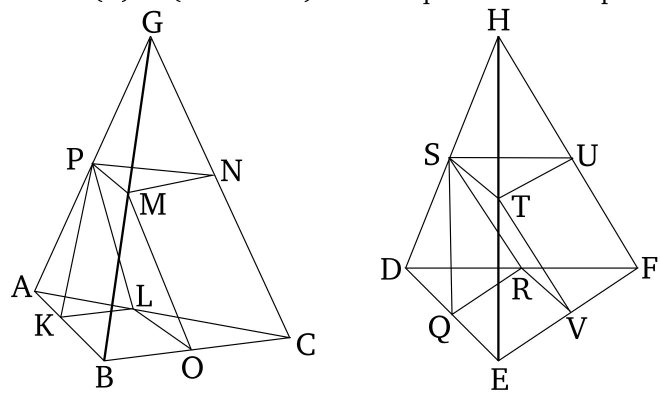
And since BF is equal to FC, parallelogram $EBFG$ is double triangle GFC [Prop. 1.41]. And since, if two prisms (have) equal heights, and the former has a parallelogram as a base, and the latter a triangle, and the parallelogram (is) double the triangle, then the prisms are equal [Prop. 11.39], the prism contained by the two triangles BKF and EHG, and the three parallelograms $EBFG$, $EBKH$, and $HKFG$, is thus equal to the prism contained by the two triangles GFC and HKL, and the three parallelograms $KFCL$, $LCGH$, and $HKFG$. And (it is) clear that each of the prisms whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK, and whose base (is) triangle GFC, and opposite (plane) triangle HKL, is greater than each of the pyramids whose bases are triangles AEG and HKL, and apexes the points H and D (respectively), inasmuch as, if we [also] join the straight-lines EF and EK then the prism whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK, is greater than the pyramid whose base (is) triangle EBF, and apex the point K. But the pyramid whose base (is) triangle EBF, and apex the point K, is equal to the pyramid whose base is triangle AEG, and apex point H. For they are contained by equal and similar planes. And, hence, the prism whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK, is greater than the pyramid whose base (is) triangle AEG, and apex the point H. And the prism whose base is parallelogram $EBFG$, and opposite (side) straight-line HK, (is) equal to the prism whose base (is) triangle GFC, and opposite (plane) triangle HKL. And the pyramid whose base (is) triangle AEG, and apex the point H, is equal to the pyramid whose base (is) triangle HKL, and apex the point D. Thus, the (sum of the) aforementioned two prisms is greater than the (sum of the) aforementioned two pyramids, whose bases (are) triangles AEG and HKL, and apexes the points H and D (respectively).

Thus, the whole pyramid, whose base (is) triangle ABC, and apex the point D, has been divided into two pyramids (which are) equal to one another [and similar to the whole], and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid. (Which is) the very thing it was required to show.

Proposition 4

If there are two pyramids with the same height, having trianglular bases, and each of them is divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms then as the base of one pyramid (is) to the base of the other pyramid, so (the sum of) all the prisms in one pyramid will be to (the sum of all) the equal number of prisms in the other pyramid.

Let there be two pyramids with the same height, having the triangular bases ABC and DEF, (with) apexes the points G and H (respectively). And let each of them have been divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms [Prop. 12.3]. I say that as base ABC is to base DEF, so (the sum of) all the prisms in pyramid $ABCG$ (is) to (the sum of) all the equal number of prisms in pyramid $DEFH$.



For since BO is equal to OC, and AL to LC, LO is thus parallel to AB, and triangle ABC similar to triangle LOC [Prop. 12.3]. So, for the same (reasons), triangle DEF is also similar to triangle RVF. And since BC is double CO, and EF (double) FV, thus as BC (is) to CO, so EF (is) to FV. And the similar, and similarly laid out, rectilinear (figures) ABC and LOC have been described on BC and CO (respectively), and the similar, and similarly laid out, [rectilinear] (figures) DEF and RVF on EF and FV (respectively). Thus, as triangle ABC is to triangle LOC, so triangle DEF (is) to triangle RVF [Prop. 6.22]. Thus, alternately, as triangle ABC is to [triangle] DEF, so [triangle] LOC (is) to triangle RVF [Prop. 5.16]. But, as triangle LOC (is) to triangle RVF, so the prism whose base [is] triangle LOC, and opposite (plane) PMN, (is) to the prism whose base (is) triangle RVF, and opposite (plane) STU (see lemma). And, thus, as triangle ABC (is) to triangle DEF, so the prism whose base (is) triangle LOC, and opposite (plane) PMN, (is) to the prism whose base (is) triangle RVF, and opposite (plane) STU. And as the aforementioned prisms (are) to one another, so the prism whose base (is) parallelogram $KBOL$, and opposite (side) straight-line PM, (is) to the prism whose base (is) parallelogram $QEVR$, and opposite (side) straight-line ST [Props. 11.39, 12.3]. Thus, also, (is) the (sum of the) two prisms—that whose base (is) parallelogram $KBOL$, and opposite (side) PM, and that whose base (is) LOC, and opposite (plane) PMN—to (the sum of) the (two) prisms—that whose base (is) $QEVR$, and opposite (side) straight-line ST, and that whose base (is) triangle RVF, and opposite (plane) STU [Prop. 5.12]. And, thus, as base ABC (is) to base DEF, so the (sum of the first) aforementioned two prisms (is) to the (sum of the second) aforementioned two prisms.

And, similarly, if pyramids $PMNG$ and $STUH$ are divided into two prisms, and two pyramids, as base PMN (is) to base STU, so (the sum of) the two prisms in pyramid $PMNG$ will be to (the sum of) the two prisms in pyramid $STUH$. But, as base PMN (is) to base STU, so base ABC (is) to base DEF. For the triangles PMN and STU (are) equal to LOC and RVF, respectively. And, thus, as base ABC (is) to base DEF, so (the sum of) the four prisms (is) to (the sum of) the four prisms [Prop. 5.12]. So, similarly, even if we divide the pyramids left behind into two pyramids and into two prisms, as base ABC (is) to base DEF, so (the sum of) all the prisms in pyramid $ABCG$ will be to (the sum of) all the equal number of prisms in pyramid $DEFH$. (Which is) the very thing it was required to show.

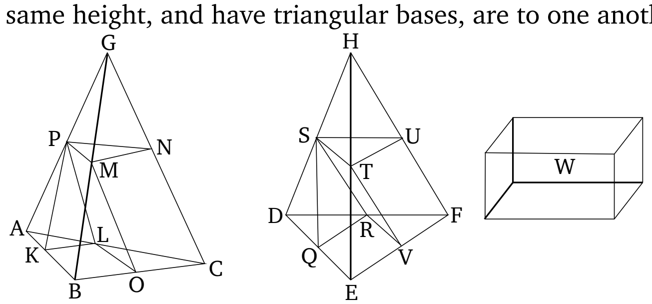
Lemma

And one may show, as follows, that as triangle LOC is to triangle RVF, so the prism whose base (is) triangle LOC, and opposite (plane) PMN, (is) to the prism whose base (is) [triangle] RVF, and opposite (plane) STU.

For, in the same figure, let perpendiculars have been conceived (drawn) from (points) G and H to the planes ABC and DEF (respectively). These clearly turn out to be equal, on account of the pyramids being assumed (to be) of equal height. And since two straight-lines, GC and the perpendicular from G, are cut by the parallel planes ABC and PMN they will be cut in the same ratios [Prop. 11.17]. And GC was cut in half by the plane PMN at N. Thus, the perpendicular from G to the plane ABC will also be cut in half by the plane PMN. So, for the same (reasons), the perpendicular from H to the plane DEF will also be cut in half by the plane STU. And the perpendiculars from G and H to the planes ABC and DEF (respectively) are equal. Thus, the perpendiculars from the triangles PMN and STU to ABC and DEF (respectively, are) also equal. Thus, the prisms whose bases are triangles LOC and RVF, and opposite (sides) PMN and STU (respectively), [are] of equal height. And, hence, the parallelepiped solids described on the aforementioned prisms [are] of equal height and (are) to one another as their bases [Prop. 11.32]. Likewise, the halves (of the solids) [Prop. 11.28]. Thus, as base LOC is to base RVF, so the aforementioned prisms (are) to one another. (Which is) the very thing it was required to show.

Proposition 5

Pyramids which are of the same height, and have triangular bases, are to one another as their bases.



Let there be pyramids of the same height whose bases (are) the triangles ABC and DEF, and apexes the points G and H (respectively). I say that as base ABC is to base DEF, so pyramid $ABCG$ (is) to pyramid $DEFH$.

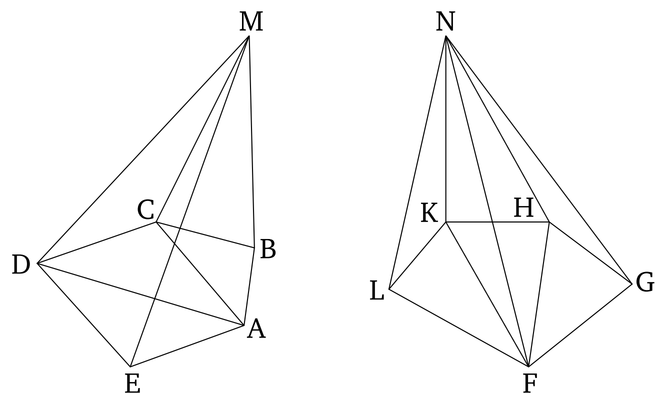
For if base ABC is not to base DEF, as pyramid $ABCG$ (is) to pyramid $DEFH$, then base ABC will be to base DEF, as pyramid $ABCG$ (is) to some solid either less than, or greater than, pyramid $DEFH$. Let it, first of all, be (in this ratio) to (some) lesser (solid), W. And let pyramid $DEFH$ have been divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms. So, the (sum of the) two prisms is greater than half of the whole pyramid [Prop. 12.3]. And, again, let the pyramids generated by the division have been similarly divided, and let this be done continually until some pyramids are left from pyramid $DEFH$ which (when added together) are less than the excess by which pyramid $DEFH$ exceeds the solid W [Prop. 10.1]. Let them have been left, and, for the sake of argument, let them be $DQRS$ and $STUH$. Thus, the (sum of the) remaining prisms within pyramid $DEFH$ is greater than solid W. Let pyramid $ABCG$ also have been divided similarly, and a similar number of times, as pyramid $DEFH$. Thus, as base ABC is to base DEF, so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 12.4]. But, also, as base ABC (is) to base DEF, so pyramid $ABCG$ (is) to solid W. And, thus, as pyramid $ABCG$ (is) to solid W, so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.11]. Thus, alternately, as pyramid $ABCG$ (is) to the (sum of the) prisms within it, so solid W (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.16]. And pyramid $ABCG$ (is) greater than the (sum of the) prisms within it. Thus, solid W (is) also greater than the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.14]. But, (it is) also less. This very thing is impossible. Thus, as base ABC is to base DEF, so pyramid $ABCG$ (is) not to some solid less than pyramid $DEFH$. So, similarly, we can show that base DEF is not to base ABC, as pyramid $DEFH$ (is) to some solid less than pyramid $ABCG$ either.

So, I say that neither is base ABC to base DEF, as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$.

For, if possible, let it be (in this ratio) to some greater (solid), W. Thus, inversely, as base DEF (is) to base ABC, so solid W (is) to pyramid $ABCG$ [Prop. 5.7. corr.]. And as solid W (is) to pyramid $ABCG$, so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$, as shown before [Prop. 12.2 lem.]. And, thus, as base DEF (is) to base ABC, so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$ [Prop. 5.11]. The very thing was shown (to be) absurd. Thus, base ABC is not to base DEF, as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$. And, it was shown that neither (is it in this ratio) to a lesser (solid). Thus, as base ABC is to base DEF, so pyramid $ABCG$ (is) to pyramid $DEFH$. (Which is) the very thing it was required to show.

Proposition 6

Pyramids which are of the same height, and have polygonal bases, are to one another as their bases.

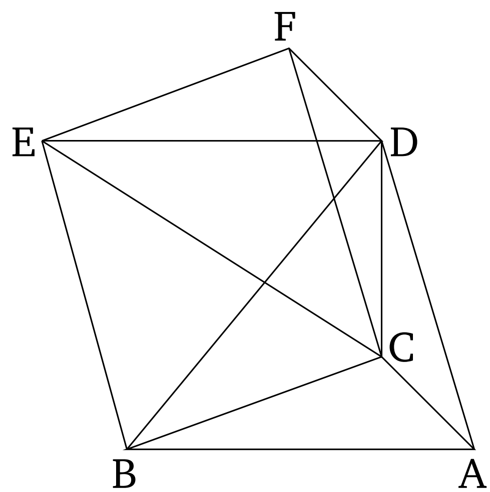


Let there be pyramids of the same height whose bases (are) the polygons $ABCDE$ and $FGHKL$, and apexes the points M and N (respectively). I say that as base $ABCDE$ is to base $FGHKL$, so pyramid $ABCDEM$ (is) to pyramid $FGHKLN$.

For let AC, AD, FH, and FK have been joined. Therefore, since $ABCM$ and $ACDM$ are two pyramids having triangular bases and equal height, they are to one another as their bases [Prop. 12.5]. Thus, as base ABC is to base ACD, so pyramid $ABCM$ (is) to pyramid $ACDM$. And, via composition, as base $ABCD$ (is) to base ACD, so pyramid $ABCDM$ (is) to pyramid $ACDM$ [Prop. 5.18]. But, as base ACD (is) to base ADE, so pyramid $ACDM$ (is) also to pyramid $ADEM$ [Prop. 12.5]. Thus, via equality, as base $ABCD$ (is) to base ADE, so pyramid $ABCDM$ (is) to pyramid $ADEM$ [Prop. 5.22]. And, again, via composition, as base $ABCDE$ (is) to base ADE, so pyramid $ABCDEM$ (is) to pyramid $ADEM$ [Prop. 5.18]. So, similarly, it can also be shown that as base $FGHKL$ (is) to base FGH, so pyramid $FGHKLN$ (is) also to pyramid $FGHN$. And since $ADEM$ and $FGHN$ are two pyramids having triangular bases and equal height, thus as base ADE (is) to base FGH, so pyramid $ADEM$ (is) to pyramid $FGHN$ [Prop. 12.5]. But, as base ADE (is) to base $ABCDE$, so pyramid $ADEM$ (was) to pyramid $ABCDEM$. Thus, via equality, as base $ABCDE$ (is) to base FGH, so pyramid $ABCDEM$ (is) also to pyramid $FGHN$ [Prop. 5.22]. But, furthermore, as base FGH (is) to base $FGHKL$, so pyramid $FGHN$ was also to pyramid $FGHKLN$. Thus, via equality, as base $ABCDE$ (is) to base $FGHKL$, so pyramid $ABCDEM$ (is) also to pyramid $FGHKLN$ [Prop. 5.22]. (Which is) the very thing it was required to show.

Proposition 7

Any prism having a triangular base is divided into three pyramids having triangular bases (which are) equal to one another.



Let there be a prism whose base (is) triangle ABC, and opposite (plane) DEF. I say that prism $ABCDEF$ is divided into three pyramids having triangular bases (which are) equal to one another.

For let BD, EC, and CD have been joined. Since $ABED$ is a parallelogram, and BD is its diagonal, triangle ABD is thus equal to triangle EBD [Prop. 1.34]. And, thus, the pyramid whose base (is) triangle ABD, and apex the point C, is equal to the pyramid whose base is triangle DEB, and apex the point C [Prop. 12.5]. But, the pyramid whose base is triangle DEB, and apex the point C, is the same as the pyramid whose base is triangle EBC, and apex the point D. For they are contained by the same planes. And, thus, the pyramid whose base is ABD, and apex the point C, is equal to the pyramid whose base is EBC and apex the point D. Again, since $FCBE$ is a parallelogram, and CE is its diagonal, triangle CEF is equal to triangle CBE [Prop. 1.34]. And, thus, the pyramid whose base is triangle BCE, and apex the point D, is equal to the pyramid whose base is triangle ECF, and apex the point D [Prop. 12.5]. And the pyramid whose base is triangle BCE, and apex the point D, was shown (to be) equal to the pyramid whose base is triangle ABD, and apex the point C. Thus, the pyramid whose base is triangle CEF, and apex the point D, is also equal to the pyramid whose base [is] triangle ABD, and apex the point C. Thus, the prism $ABCDEF$ has been divided into three pyramids having triangular bases (which are) equal to one another.

And since the pyramid whose base is triangle ABD, and apex the point C, is the same as the pyramid whose base is triangle CAB, and apex the point D. For they are contained by the same planes. And the pyramid whose base (is) triangle ABD, and apex the point C, was shown (to be) a third of the prism whose base is triangle ABC, and opposite (plane) DEF, thus the pyramid whose base is triangle ABC, and apex the point D, is also a third of the pyramid having the same base, triangle ABC, and opposite (plane) DEF.

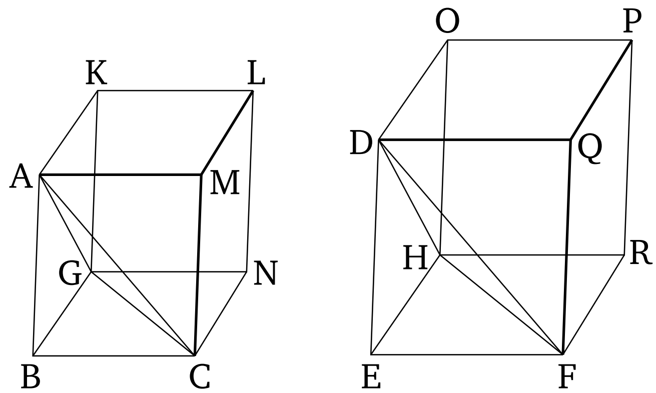
Corollary

And, from this, (it is) clear that any pyramid is the third part of the prism having the same base as it, and an equal height. (Which is) the very thing it was required to show.

Proposition 8

Similar pyramids which also have triangular bases are in the cubed ratio of their corresponding sides.

Let there be similar, and similarly laid out, pyramids whose bases are triangles ABC and DEF, and apexes the points G and H (respectively). I say that pyramid $ABCG$ has to pyramid $DEFH$ the cubed ratio of that BC (has) to EF.



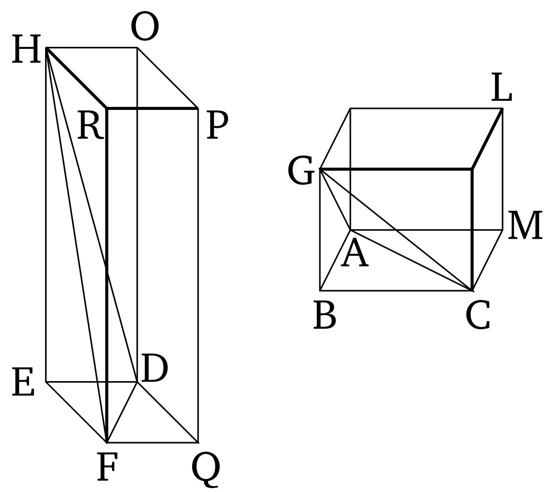
For let the parallelepiped solids $BGML$ and $EHQP$ have been completed. And since pyramid $ABCG$ is similar to pyramid $DEFH$, angle ABC is thus equal to angle DEF, and GBC to HEF, and ABG to DEH. And as AB is to DE, so BC (is) to EF, and BG to EH [Def. 11.9]. And since as AB is to DE, so BC (is) to EF, and (so) the sides around equal angles are proportional, parallelogram BM is thus similar to paralleleogram EQ. So, for the same (reasons), BN is also similar to ER, and BK to EO. Thus, the three (parallelograms) MB, BK, and BN are similar to the three (parallelograms) EQ, EO, ER (respectively). But, the three (parallelograms) MB, BK, and BN are (both) equal and similar to the three opposite (parallelograms), and the three (parallelograms) EQ, EO, and ER are (both) equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the solids $BGML$ and $EHQP$ are contained by equal numbers of similar (and similarly laid out) planes. Thus, solid $BGML$ is similar to solid $EHQP$ [Def. 11.9]. And similar parallelepiped solids are in the cubed ratio of corresponding sides [Prop. 11.33]. Thus, solid $BGML$ has to solid $EHQP$ the cubed ratio that the corresponding side BC (has) to the corresponding side EF. And as solid $BGML$ (is) to solid $EHQP$, so pyramid $ABCG$ (is) to pyramid $DEFH$, inasmuch as the pyramid is the sixth part of the solid, on account of the prism, being half of the parallelepiped solid [Prop. 11.28], also being three times the pyramid [Prop. 12.7]. Thus, pyramid $ABCG$ also has to pyramid $DEFH$ the cubed ratio that BC (has) to EF. (Which is) the very thing it was required to show.

Corollary

So, from this, (it is) also clear that similar pyramids having polygonal bases (are) to one another as the cubed ratio of their corresponding sides. For, dividing them into the pyramids (contained) within them which have triangular bases, with the similar polygons of the bases also being divided into similar triangles (which are) both equal in number, and corresponding, to the wholes [Prop. 6.20]. As one pyramid having a triangular base in the former (pyramid having a polygonal base is) to one pyramid having a triangular base in the latter (pyramid having a polygonal base), so (the sum of) all the pyramids having triangular bases in the former pyramid will also be to (the sum of) all the pyramids having triangular bases in the latter pyramid [Prop. 5.12]—that is to say, the (former) pyramid itself having a polygonal base to the (latter) pyramid having a polygonal base. And a pyramid having a triangular base is to a (pyramid) having a triangular base in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, a (pyramid) having a polygonal base also has to to a (pyramid) having a similar base the cubed ratio of a (corresponding) side to a (corresponding) side.

Proposition 9

The bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids which have triangular bases whose bases are reciprocally proportional to their heights are equal.



For let there be (two) equal pyramids having the triangular bases ABC and DEF, and apexes the points G and H (respectively). I say that the bases of the pyramids $ABCG$ and $DEFH$ are reciprocally proportional to their heights, and (so) that as base ABC is to base DEF, so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$.

For let the parallelepiped solids $BGML$ and $EHQP$ have been completed. And since pyramid $ABCG$ is equal to pyramid $DEFH$, and solid $BGML$ is six times pyramid $ABCG$ (see previous proposition), and solid $EHQP$ (is) six times pyramid $DEFH$, solid $BGML$ is thus equal to solid $EHQP$. And the bases of equal parallelepiped solids are reciprocally proportional to their heights [Prop. 11.34]. Thus, as base BM is to base EQ, so the height of solid $EHQP$ (is) to the height of solid $BGML$. But, as base BM (is) to base EQ, so triangle ABC (is) to triangle DEF [Prop. 1.34]. And, thus, as triangle ABC (is) to triangle DEF, so the height of solid $EHQP$ (is) to the height of solid $BGML$ [Prop. 5.11]. But, the height of solid $EHQP$ is the same as the height of pyramid $DEFH$, and the height of solid $BGML$ is the same as the height of pyramid $ABCG$. Thus, as base ABC is to base DEF, so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$. Thus, the bases of pyramids $ABCG$ and $DEFH$ are reciprocally proportional to their heights.

And so, let the bases of pyramids $ABCG$ and $DEFH$ be reciprocally proportional to their heights, and (thus) let base ABC be to base DEF, as the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$. I say that pyramid $ABCG$ is equal to pyramid $DEFH$.

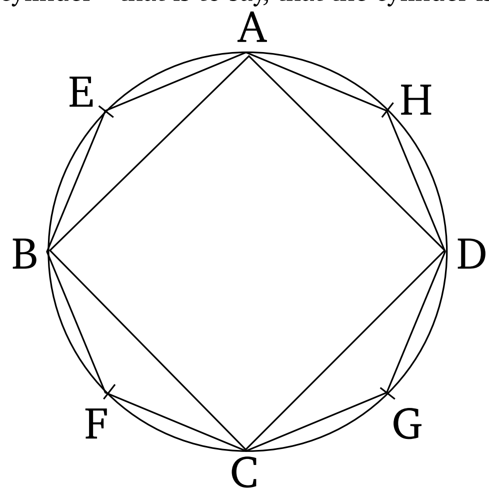
For, with the same construction, since as base ABC is to base DEF, so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$, but as base ABC (is) to base DEF, so parallelogram BM (is) to parallelogram EQ [Prop. 1.34], thus as parallelogram BM (is) to parallelogram EQ, so the height of pyramid $DEFH$ (is) also to the height of pyramid $ABCG$ [Prop. 5.11]. But, the height of pyramid $DEFH$ is the same as the height of parallelepiped $EHQP$, and the height of pyramid $ABCG$ is the same as the height of parallelepiped $BGML$. Thus, as base BM is to base EQ, so the height of parallelepiped $EHQP$ (is) to the height of parallelepiped $BGML$. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal [Prop. 11.34]. Thus, the parallelepiped solid $BGML$ is equal to the parallelepiped solid $EHQP$. And pyramid $ABCG$ is a sixth part of $BGML$, and pyramid $DEFH$ a sixth part of parallelepiped $EHQP$. Thus, pyramid $ABCG$ is equal to pyramid $DEFH$.

Thus, the bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids having triangular bases whose bases are reciprocally proportional to their heights are equal. (Which is) the very thing it was required to show.

Proposition 10

Every cone is the third part of the cylinder which has the same base as it, and an equal height.

For let there be a cone (with) the same base as a cylinder, (namely) the circle $ABCD$, and an equal height. I say that the cone is the third part of the cylinder—that is to say, that the cylinder is three times the cone.



For if the cylinder is not three times the cone then the cylinder will be either more than three times, or less than three times, (the cone). Let it, first of all, be more than three times (the cone). And let the square $ABCD$ have been inscribed in circle $ABCD$ [Prop. 4.6]. So, square $ABCD$ is more than half of circle $ABCD$ [Prop. 12.2]. And let a prism of equal height to the cylinder have been set up on square $ABCD$. So, the prism set up is more than half of the cylinder, inasmuch as if we also circumscribe a square around circle $ABCD$ [Prop. 4.7] then the square inscribed in circle $ABCD$ is half of the circumscribed (square). And the solids set up on them are parallelepiped prisms of equal height. And parallelepiped solids having the same height are to one another as their bases [Prop. 11.32]. And, thus, the prism set up on square $ABCD$ is half of the prism set up on the square circumscribed about circle $ABCD$. And the cylinder is less than the prism set up on the square circumscribed about circle $ABCD$. Thus, the prism set up on square $ABCD$ of the same height as the cylinder is more than half of the cylinder. Let the circumferences AB, BC, CD, and DA have been cut in half at points E, F, G, and H. And let AE, EB, BF, FC, CG, GD, DH, and HA have been joined. And thus each of the triangles AEB, BFC, CGD, and DHA is more than half of the segment of circle $ABCD$ about it, as was shown previously [Prop. 12.2]. Let prisms of equal height to the cylinder have been set up on each of the triangles AEB, BFC, CGD, and DHA. And each of the prisms set up is greater than the half part of the segment of the cylinder about it—inasmuch as if we draw (straight-lines) parallel to AB, BC, CD, and DA through points E, F, G, and H (respectively), and complete the parallelograms on AB, BC, CD, and DA, and set up parallelepiped solids of equal height to the cylinder on them, then the prisms on triangles AEB, BFC, CGD, and DHA are each half of the set up (parallelepipeds). And the segments of the cylinder are less than the set up parallelepiped solids. Hence, the prisms on triangles AEB, BFC, CGD, and DHA are also greater than half of the segments of the cylinder about them. So (if) the remaining circumferences are cut in half, and straight-lines are joined, and prisms of equal height to the cylinder are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cylinder whose (sum) is less than the excess by which the cylinder exceeds three times the cone [Prop. 10.1]. Let them have been left, and let them be AE, EB, BF, FC, CG, GD, DH, and HA. Thus, the remaining prism whose base (is) polygon $AEBFCGDH$, and height the same as the cylinder, is greater than three times the cone. But, the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder, is three times the pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone [Prop. 12.7 corr.]. And thus the pyramid whose base [is] polygon $AEBFCGDH$, and apex the same as the cone, is greater than the cone having (as) base circle $ABCD$. But (it is) also less. For it is encompassed by it. The very thing (is) impossible. Thus, the cylinder is not more than three times the cone.

So, I say that neither (is) the cylinder less than three times the cone.

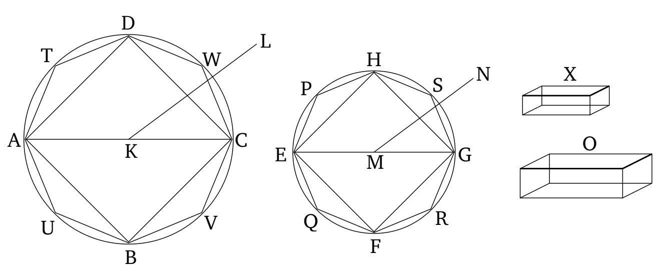
For, if possible, let the cylinder be less than three times the cone. Thus, inversely, the cone is greater than the third part of the cylinder. So, let the square $ABCD$ have been inscribed in circle $ABCD$ [Prop. 4.6]. Thus, square $ABCD$ is greater than half of circle $ABCD$. And let a pyramid having the same apex as the cone have been set up on square $ABCD$. Thus, the pyramid set up is greater than the half part of the cone, inasmuch as we showed previously that if we circumscribe a square about the circle [Prop. 4.7] then the square $ABCD$ will be half of the square circumscribed about the circle [Prop. 12.2]. And if we set up on the squares parallelepiped solids—which are also called prisms—of the same height as the cone, then the (prism) set up on square $ABCD$ will be half of the (prism) set up on the square circumscribed about the circle. For they are to one another as their bases [Prop. 11.32]. Hence, (the same) also (goes for) the thirds. Thus, the pyramid whose base is square $ABCD$ is half of the pyramid set up on the square circumscribed about the circle [Prop. 12.7 corr.]. And the pyramid set up on the square circumscribed about the circle is greater than the cone. For it encompasses it. Thus, the pyramid whose base is square $ABCD$, and apex the same as the cone, is greater than half of the cone. Let the circumferences AB, BC, CD, and DA have been cut in half at points E, F, G, and H (respectively). And let AE, EB, BF, FC, CG, GD, DH, and HA have been joined. And, thus, each of the triangles AEB, BFC, CGD, and DHA is greater than the half part of the segment of circle $ABCD$ about it [Prop. 12.2]. And let pyramids having the same apex as the cone have been set up on each of the triangles AEB, BFC, CGD, and DHA. And, thus, in the same way, each of the pyramids set up is more than the half part of the segment of the cone about it. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which the cone exceeds the third part of the cylinder [Prop. 10.1]. Let them have been left, and let them be the (segments) on AE, EB, BF, FC, CG, GD, DH, and HA. Thus, the remaining pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone, is greater than the third part of the cylinder. But, the pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone, is the third part of the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder [Prop. 12.7 corr.]. Thus, the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder, is greater than the cylinder whose base is circle $ABCD$. But, (it is) also less. For it is encompassed by it. The very thing is impossible. Thus, the cylinder is not less than three times the cone. And it was shown that neither (is it) greater than three times (the cone). Thus, the cylinder (is) three times the cone. Hence, the cone is the third part of the cylinder.

Thus, every cone is the third part of the cylinder which has the same base as it, and an equal height. (Which is) the very thing it was required to show.

Proposition 11

Cones and cylinders having the same height are to one another as their bases.

Let there be cones and cylinders of the same height whose bases [are] the circles $ABCD$ and $EFGH$, axes KL and MN, and diameters of the bases AC and EG (respectively). I say that as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN.



For if not, then as circle $ABCD$ (is) to circle $EFGH$, so cone AL will be to some solid either less than, or greater than, cone EN. Let it, first of all, be (in this ratio) to (some) lesser (solid), O. And let solid X be equal to that (magnitude) by which solid O is less than cone EN. Thus, cone EN is equal to (the sum of) solids O and X. Let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. Thus, the square is greater than half of the circle [Prop. 12.2]. Let a pyramid of the same height as the cone have been set up on square $EFGH$. Thus, the pyramid set up is greater than half of the cone, inasmuch as, if we circumscribe a square about the circle [Prop. 4.7], and set up on it a pyramid of the same height as the cone, then the inscribed pyramid is half of the circumscribed pyramid. For they are to one another as their bases [Prop. 12.6]. And the cone (is) less than the circumscribed pyramid. Let the circumferences EF, FG, GH, and HE have been cut in half at points P, Q, R, and S. And let HP, PE, EQ, QF, FR, RG, GS, and SH have been joined. Thus, each of the triangles HPE, EQF, FRG, and GSH is greater than half of the segment of the circle about it [Prop. 12.2]. Let pyramids of the same height as the cone have been set up on each of the triangles HPE, EQF, FRG, and GSH. And, thus, each of the pyramids set up is greater than half of the segment of the cone about it [Prop. 12.10]. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids of equal height to the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone (the sum of) which is less than solid X [Prop. 10.1]. Let them have been left, and let them be the (segments) on HPE, EQF, FRG, and GSH. Thus, the remaining pyramid whose base is polygon $HPEQFRGS$, and height the same as the cone, is greater than solid O [Prop. 6.18]. And let the polygon $DTAUBVCW$, similar, and similarly laid out, to polygon $HPEQFRGS$, have been inscribed in circle $ABCD$. And on it let a pyramid of the same height as cone AL have been set up. Therefore, since as the (square) on AC is to the (square) on EG, so polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$ [Prop. 12.1], and as the (square) on AC (is) to the (square) on EG, so circle $ABCD$ (is) to circle $EFGH$ [Prop. 12.2], thus as circle $ABCD$ (is) to circle $EFGH$, so polygon $DTAUBVCW$ also (is) to polygon $HPEQFRGS$. And as circle $ABCD$ (is) to circle $EFGH$, so cone AL (is) to solid O. And as polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$, so the pyramid whose base is polygon $DTAUBVCW$, and apex the point L, (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 12.6]. And, thus, as cone AL (is) to solid O, so the pyramid whose base is $DTAUBVCW$, and apex the point L, (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 5.11]. Thus, alternately, as cone AL is to the pyramid within it, so solid O (is) to the pyramid within cone EN [Prop. 5.16]. But, cone AL (is) greater than the pyramid within it. Thus, solid O (is) also greater than the pyramid within cone EN [Prop. 5.14]. But, (it is) also less. The very thing (is) absurd. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid less than cone EN. So, similarly, we can show that neither is circle $EFGH$ to circle $ABCD$, as cone EN (is) to some solid less than cone AL.

So, I say that neither is circle $ABCD$ to circle $EFGH$, as cone AL (is) to some solid greater than cone EN.

For, if possible, let it be (in this ratio) to (some) greater (solid), O. Thus, inversely, as circle $EFGH$ is to circle $ABCD$, so solid O (is) to cone AL [Prop. 5.7 corr.]. But, as solid O (is) to cone AL, so cone EN (is) to some solid less than cone AL [Prop. 12.2 lem.]. And, thus, as circle $EFGH$ (is) to circle $ABCD$, so cone EN (is) to some solid less than cone AL. The very thing was shown (to be) impossible. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid greater than cone EN. And, it was shown that neither (is it in this ratio) to (some) lesser (solid). Thus, as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN.

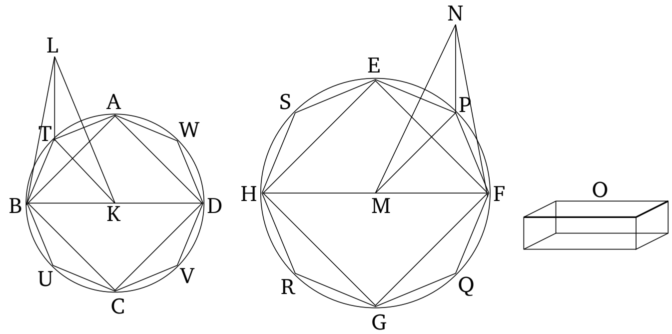
But, as the cone (is) to the cone, (so) the cylinder (is) to the cylinder. For each (is) three times each [Prop. 12.10]. Thus, circle $ABCD$ (is) also to circle $EFGH$, as (the ratio of the cylinders) on them (having) the same height.

Thus, cones and cylinders having the same height are to one another as their bases. (Which is) the very thing it was required to show.

Proposition 12

Similar cones and cylinders are to one another in the cubed ratio of the diameters of their bases.

Let there be similar cones and cylinders of which the bases (are) the circles $ABCD$ and $EFGH$, the diameters of the bases (are) BD and FH, and the axes of the cones and cylinders (are) KL and MN (respectively). I say that the cone whose base [is] circle $ABCD$, and apex the point L, has to the cone whose base [is] circle $EFGH$, and apex the point N, the cubed ratio that BD (has) to FH.



For if cone $ABCDL$ does not have to cone $EFGHN$ the cubed ratio that BD (has) to FH then cone $ABCDL$ will have the cubed ratio to some solid either less than, or greater than, cone $EFGHN$. Let it, first of all, have (such a ratio) to (some) lesser (solid), O. And let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. Thus, square $EFGH$ is greater than half of circle $EFGH$ [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on square $EFGH$. Thus, the pyramid set up is greater than the half part of the cone [Prop. 12.10]. So, let the circumferences EF, FG, GH, and HE have been cut in half at points P, Q, R, and S (respectively). And let EP, PF, FQ, QG, GR, RH, HS, and SE have been joined. And, thus, each of the triangles EPF, FQG, GRH, and HSE is greater than the half part of the segment of circle $EFGH$ about it [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on each of the triangles EPF, FQG, GRH, and HSE. And thus each of the pyramids set up is greater than the half part of the segment of the cone about it [Prop. 12.10]. So, (if) the the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which cone $EFGHN$ exceeds solid O [Prop. 10.1]. Let them have been left, and let them be the (segments) on EP, PF, FQ, QG, GR, RH, HS, and SE. Thus, the remaining pyramid whose base is polygon $EPFQGRHS$, and apex the point N, is greater than solid O. And let the polygon $ATBUCVDW$, similar, and similarly laid out, to polygon $EPFQGRHS$, have been inscribed in circle $ABCD$ [Prop. 6.18]. And let a pyramid having the same apex as the cone have been set up on polygon $ATBUCVDW$. And let LBT be one of the triangles containing the pyramid whose base is polygon $ATBUCVDW$, and apex the point L. And let NFP be one of the triangles containing the pyramid whose base is triangle $EPFQGRHS$, and apex the point N. And let KT and MP have been joined. And since cone $ABCDL$ is similar to cone $EFGHN$, thus as BD is to FH, so axis KL (is) to axis MN [Def. 11.24]. And as BD (is) to FH, so BK (is) to FM. And, thus, as BK (is) to FM, so KL (is) to MN. And, alternately, as BK (is) to KL, so FM (is) to MN [Prop. 5.16]. And the sides around the equal angles BKL and FMN are proportional. Thus, triangle BKL is similar to triangle FMN [Prop. 6.6]. Again, since as BK (is) to KT, so FM (is) to MP, and (they are) about the equal angles BKT and FMP, inasmuch as whatever part angle BKT is of the four right-angles at the center K, angle FMP is also the same part of the four right-angles at the center M. Therefore, since the sides about equal angles are proportional, triangle BKT is thus similar to traingle FMP [Prop. 6.6]. Again, since it was shown that as BK (is) to KL, so FM (is) to MN, and BK (is) equal to KT, and FM to PM, thus as TK (is) to KL, so PM (is) to MN. And the sides about the equal angles TKL and PMN—for (they are both) right-angles—are proportional. Thus, triangle LKT (is) similar to triangle NMP [Prop. 6.6]. And since, on account of the similarity of triangles LKB and NMF, as LB (is) to BK, so NF (is) to FM, and, on account of the similarity of triangles BKT and FMP, as KB (is) to BT, so MF (is) to FP [Def. 6.1], thus, via equality, as LB (is) to BT, so NF (is) to FP [Prop. 5.22]. Again, since, on account of the similarity of triangles LTK and NPM, as LT (is) to TK, so NP (is) to PM, and, on account of the similarity of triangles TKB and PMF, as KT (is) to TB, so MP (is) to PF, thus, via equality, as LT (is) to TB, so NP (is) to PF [Prop. 5.22]. And it was shown that as TB (is) to BL, so PF (is) to FN. Thus, via equality, as TL (is) to LB, so PN (is) to NF [Prop. 5.22]. Thus, the sides of triangles LTB and NPF are proportional. Thus, triangles LTB and NPF are equiangular [Prop. 6.5]. And, hence, (they are) similar [Def. 6.1]. And, thus, the pyramid whose base is triangle BKT, and apex the point L, is similar to the pyramid whose base is triangle FMP, and apex the point N. For they are contained by equal numbers of similar planes [Def. 11.9]. And similar pyramids which also have triangular bases are in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, pyramid $BKTL$ has to pyramid $FMPN$ the cubed ratio that BK (has) to FM. So, similarly, joining straight-lines from (points) A, W, D, V, C, and U to (center) K, and from (points) E, S, H, R, G, and Q to (center) M, and setting up pyramids having the same apexes as the cones on each of the triangles (so formed), we can also show that each of the pyramids (on base $ABCD$ taken) in order will have to each of the pyramids (on base $EFGH$ taken) in order the cubed ratio that the corresponding side BK (has) to the corresponding side FM—that is to say, that BD (has) to FH. And (for two sets of proportional magnitudes) as one of the leading (magnitudes is) to one of the following, so (the sum of) all of the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. And, thus, as pyramid $BKTL$ (is) to pyramid $FMPN$, so the whole pyramid whose base is polygon $ATBUCVDW$, and apex the point L, (is) to the whole pyramid whose base is polygon $EPFQGRHS$, and apex the point N. And, hence, the pyramid whose base is polygon $ATBUCVDW$, and apex the point L, has to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N, the cubed ratio that BD (has) to FH. And it was also assumed that the cone whose base is circle $ABCD$, and apex the point L, has to solid O the cubed ratio that BD (has) to FH. Thus, as the cone whose base is circle $ABCD$, and apex the point L, is to solid O, so the pyramid whose base (is) [polygon] $ATBUCVDW$, and apex the point L, (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N. Thus, alternately, as the cone whose base (is) circle $ABCD$, and apex the point L, (is) to the pyramid within it whose base (is) the polygon $ATBUCVDW$, and apex the point L, so the [solid] O (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N [Prop. 5.16]. And the aforementioned cone (is) greater than the pyramid within it. For it encompasses it. Thus, solid O (is) also greater than the pyramid whose base is polygon $EPFQGRHS$, and apex the point N. But, (it is) also less. The very thing is impossible. Thus, the cone whose base (is) circle $ABCD$, and apex the [point] L, does not have to some solid less than the cone whose base (is) circle $EFGH$, and apex the point N, the cubed ratio that BD (has) to EH. So, similarly, we can show that neither does cone $EFGHN$ have to some solid less than cone $ABCDL$ the cubed ratio that FH (has) to BD.

So, I say that neither does cone $ABCDL$ have to some solid greater than cone $EFGHN$ the cubed ratio that BD (has) to FH.

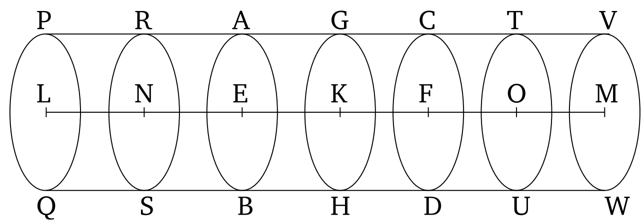
For, if possible, let it have (such a ratio) to a greater (solid), O. Thus, inversely, solid O has to cone $ABCDL$ the cubed ratio that FH (has) to BD [Prop. 5.7 corr.]. And as solid O (is) to cone $ABCDL$, so cone $EFGHN$ (is) to some solid less than cone $ABCDL$ [12.2 lem.]. Thus, cone $EFGHN$ also has to some solid less than cone $ABCDL$ the cubed ratio that FH (has) to BD. The very thing was shown (to be) impossible. Thus, cone $ABCDL$ does not have to some solid greater than cone $EFGHN$ the cubed ratio than BD (has) to FH. And it was shown that neither (does it have such a ratio) to a lesser (solid). Thus, cone $ABCDL$ has to cone $EFGHN$ the cubed ratio that BD (has) to FG.

And as the cone (is) to the cone, so the cylinder (is) to the cylinder. For a cylinder is three times a cone on the same base as the cone, and of the same height as it [Prop. 12.10]. Thus, the cylinder also has to the cylinder the cubed ratio that BD (has) to FH.

Thus, similar cones and cylinders are in the cubed ratio of the diameters of their bases. (Which is) the very thing it was required to show.

Proposition 13

If a cylinder is cut by a plane which is parallel to the opposite planes (of the cylinder) then as the cylinder (is) to the cylinder, so the axis will be to the axis.

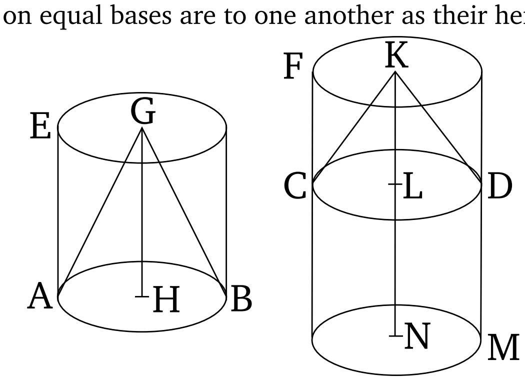


For let the cylinder AD have been cut by the plane GH which is parallel to the opposite planes (of the cylinder), AB and CD. And let the plane GH have met the axis at point K. I say that as cylinder BG is to cylinder GD, so axis EK (is) to axis KF.

For let axis EF have been produced in each direction to points L and M. And let any number whatsoever (of lengths), EN and NL, equal to axis EK, be set out (on the axis EL), and any number whatsoever (of lengths), FO and OM, equal to (axis) FK, (on the axis KM). And let the cylinder PW, whose bases (are) the circles PQ and VW, have been conceived on axis LM. And let planes parallel to AB, CD, and the bases of cylinder PW, have been produced through points N and O, and let them have made the circles RS and TU around the centers N and O (respectively). And since axes LN, NE, and EK are equal to one another, the cylinders QR, RB, and BG are to one another as their bases [Prop. 12.11]. But the bases are equal. Thus, the cylinders QR, RB, and BG (are) also equal to one another. Therefore, since the axes LN, NE, and EK are equal to one another, and the cylinders QR, RB, and BG are also equal to one another, and the number (of the former) is equal to the number (of the latter), thus as many multiples as axis KL is of axis EK, so many multiples is cylinder QG also of cylinder GB. And so, for the same (reasons), as many multiples as axis MK is of axis KF, so many multiples is cylinder WG also of cylinder GD. And if axis KL is equal to axis KM then cylinder QG will also be equal to cylinder GW, and if the axis (is) greater than the axis then the cylinder (will also be) greater than the cylinder, and if (the axis is) less then (the cylinder will also be) less. So, there are four magnitudes—the axes EK and KF, and the cylinders BG and GD—and equal multiples have been taken of axis EK and cylinder BG—(namely), axis LK and cylinder QG—and of axis KF and cylinder GD—(namely), axis KM and cylinder GW. And it has been shown that if axis KL exceeds axis KM then cylinder QG also exceeds cylinder GW, and if (the axes are) equal then (the cylinders are) equal, and if (KL is) less then (QG is) less. Thus, as axis EK is to axis KF, so cylinder BG (is) to cylinder GD [Def. 5.5]. (Which is) the very thing it was required to show.

Proposition 14

Cones and cylinders which are on equal bases are to one another as their heights.

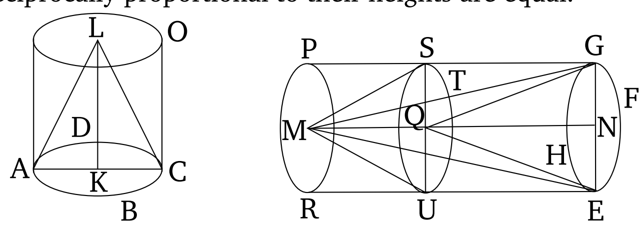


For let EB and FD be cylinders on equal bases, (namely) the circles AB and CD (respectively). I say that as cylinder EB is to cylinder FD, so axis GH (is) to axis KL.

For let the axis KL have been produced to point N. And let LN be made equal to axis GH. And let the cylinder CM have been conceived about axis LN. Therefore, since cylinders EB and CM have the same height they are to one another as their bases [Prop. 12.11]. And the bases are equal to one another. Thus, cylinders EB and CM are also equal to one another. And since cylinder FM has been cut by the plane CD, which is parallel to its opposite planes, thus as cylinder CM is to cylinder FD, so axis LN (is) to axis KL [Prop. 12.13]. And cylinder CM is equal to cylinder EB, and axis LN to axis GH. Thus, as cylinder EB is to cylinder FD, so axis GH (is) to axis KL. And as cylinder EB (is) to cylinder FD, so cone ABG (is) to cone CDK [Prop. 12.10]. Thus, also, as axis GH (is) to axis KL, so cone ABG (is) to cone CDK, and cylinder EB to cylinder FD. (Which is) the very thing it was required to show.

Proposition 15

The bases of equal cones and cylinders are reciprocally proportional to their heights. And, those cones and cylinders whose bases (are) reciprocally proportional to their heights are equal.



Let there be equal cones and cylinders whose bases are the circles $ABCD$ and $EFGH$, and the diameters of (the bases) AC and EG, and (whose) axes (are) KL and MN, which are also the heights of the cones and cylinders (respectively). And let the cylinders AO and EP have been completed. I say that the bases of cylinders AO and EP are reciprocally proportional to their heights, and (so) as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL.

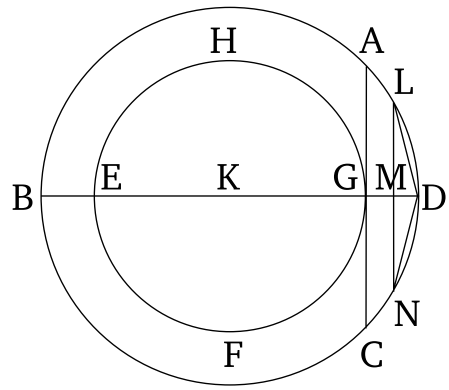
For height LK is either equal to height MN, or not. Let it, first of all, be equal. And cylinder AO is also equal to cylinder EP. And cones and cylinders having the same height are to one another as their bases [Prop. 12.11]. Thus, base $ABCD$ (is) also equal to base $EFGH$. And, hence, reciprocally, as base $ABCD$ (is) to base $EFGH$, so height MN (is) to height KL. And so, let height LK not be equal to MN, but let MN be greater. And let QN, equal to KL, have been cut off from height MN. And let the cylinder EP have been cut, through point Q, by the plane TUS (which is) parallel to the planes of the circles $EFGH$ and RP. And let cylinder ES have been conceived, with base the circle $EFGH$, and height NQ. And since cylinder AO is equal to cylinder EP, thus, as cylinder AO (is) to cylinder ES, so cylinder EP (is) to cylinder ES [Prop. 5.7]. But, as cylinder AO (is) to cylinder ES, so base $ABCD$ (is) to base $EFGH$. For cylinders AO and ES (have) the same height [Prop. 12.11]. And as cylinder EP (is) to (cylinder) ES, so height MN (is) to height QN. For cylinder EP has been cut by a plane which is parallel to its opposite planes [Prop. 12.13]. And, thus, as base $ABCD$ is to base $EFGH$, so height MN (is) to height QN [Prop. 5.11]. And height QN (is) equal to height KL. Thus, as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL. Thus, the bases of cylinders AO and EP are reciprocally proportional to their heights.

And, so, let the bases of cylinders AO and EP be reciprocally proportional to their heights, and (thus) let base $ABCD$ be to base $EFGH$, as height MN (is) to height KL. I say that cylinder AO is equal to cylinder EP.

For, with the same construction, since as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL, and height KL (is) equal to height QN, thus, as base $ABCD$ (is) to base $EFGH$, so height MN will be to height QN. But, as base $ABCD$ (is) to base $EFGH$, so cylinder AO (is) to cylinder ES. For they are the same height [Prop. 12.11]. And as height MN (is) to [height] QN, so cylinder EP (is) to cylinder ES [Prop. 12.13]. Thus, as cylinder AO is to cylinder ES, so cylinder EP (is) to (cylinder) ES [Prop. 5.11]. Thus, cylinder AO (is) equal to cylinder EP [Prop. 5.9]. In the same manner, (the proposition can) also (be demonstrated) for the cones. (Which is) the very thing it was required to show.

Proposition 16

There being two circles about the same center, to inscribe an equilateral and even-sided polygon in the greater circle, not touching the lesser circle.

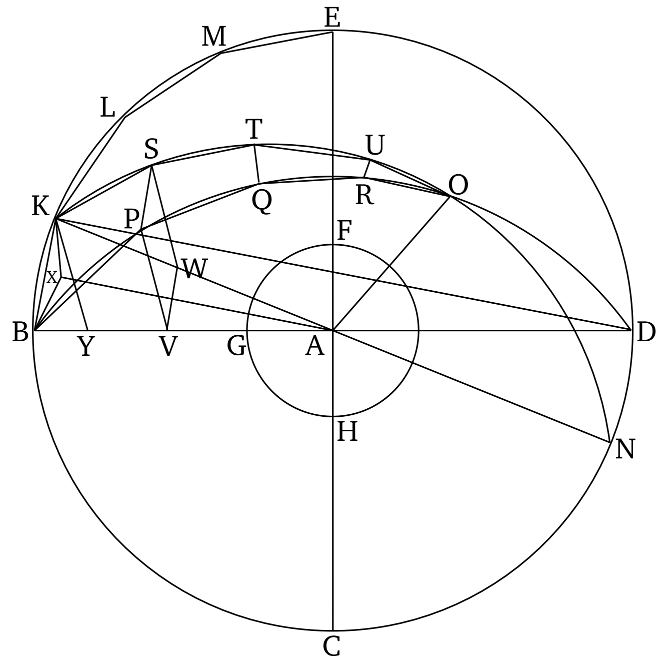


Let $ABCD$ and $EFGH$ be the given two circles, about the same center, K. So, it is necessary to inscribe an equilateral and even-sided polygon in the greater circle $ABCD$, not touching circle $EFGH$.

Let the straight-line BKD have been drawn through the center K. And let GA have been drawn, at right-angles to the straight-line BD, through point G, and let it have been drawn through to C. Thus, AC touches circle $EFGH$ [Prop. 3.16 corr.]. So, (by) cutting circumference BAD in half, and the half of it in half, and doing this continually, we will (eventually) leave a circumference less than AD [Prop. 10.1]. Let it have been left, and let it be LD. And let LM have been drawn, from L, perpendicular to BD, and let it have been drawn through to N. And let LD and DN have been joined. Thus, LD is equal to DN [Props. 3.3, 1.4]. And since LN is parallel to AC [Prop. 1.28], and AC touches circle $EFGH$, LN thus does not touch circle $EFGH$. Thus, even more so, LD and DN do not touch circle $EFGH$. And if we continuously insert (straight-lines) equal to straight-line LD into circle $ABCD$ [Prop. 4.1] then an equilateral and even-sided polygon, not touching the lesser circle $EFGH$, will have been inscribed in circle $ABCD$.† (Which is) the very thing it was required to do. † Note that the chord of the polygon, LN, does not touch the inner circle either.

Proposition 17

There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.



Let two spheres have been conceived about the same center, A. So, it is necessary to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Let the spheres have been cut by some plane through the center. So, the sections will be circles, inasmuch as a sphere is generated by the diameter remaining behind, and a semi-circle being carried around [Def. 11.14]. And, hence, whatever position we conceive (of for) the semi-circle, the plane produced through it will make a circle on the surface of the sphere. And (it is) clear that (it is) also a great (circle), inasmuch as the diameter of the sphere, which is also manifestly the diameter of the semi-circle and the circle, is greater than all of the (other) [straight-lines] drawn across in the circle or the sphere [Prop. 3.15]. Therefore, let $BCDE$ be the circle in the greater sphere, and FGH the circle in the lesser sphere. And let two diameters of them have been drawn at right-angles to one another, (namely), BD and CE. And there being two circles about the same center—(namely), $BCDE$ and FGH—let an equilateral and even-sided polygon have been inscribed in the greater circle, $BCDE$, not touching the lesser circle, FGH [Prop. 12.16], of which let the sides in the quadrant BE be BK, KL, LM, and ME. And, KA being joined, let it have been drawn across to N. And let AO have been set up at point A, at right-angles to the plane of circle $BCDE$. And let it meet the surface of the (greater) sphere at O. And let planes have been produced through AO and each of BD and KN. So, according to the aforementioned (discussion), they will make great circles on the surface of the (greater) sphere. Let them make (great circles), of which let BOD and KON be semi-circles on the diameters BD and KN (respectively). And since OA is at right-angles to the plane of circle $BCDE$, all of the planes through OA are thus also at right-angles to the plane of circle $BCDE$ [Prop. 11.18]. And, hence, the semi-circles BOD and KON are also at right-angles to the plane of circle $BCDE$. And since semi-circles BED, BOD, and KON are equal—for (they are) on the equal diameters BD and KN [Def. 3.1]—the quadrants BE, BO, and KO are also equal to one another. Thus, as many sides of the polygon as are in quadrant BE, so many are also in quadrants BO and KO equal to the straight-lines BK, KL, LM, and ME. Let them have been inscribed, and let them be BP, PQ, QR, RO, KS, ST, TU, and UO. And let SP, TQ, and UR have been joined. And let perpendiculars have been drawn from P and S to the plane of circle $BCDE$ [Prop. 11.11]. So, they will fall on the common sections of the planes BD and KN (with $BCDE$), inasmuch as the planes of BOD and KON are also at right-angles to the plane of circle $BCDE$ [Def. 11.4]. Let them have fallen, and let them be PV and SW. And let WV have been joined. And since BP and KS are equal (circumferences) having been cut off in the equal semi-circles BOD and KON [Def. 3.28], and PV and SW are perpendiculars having been drawn (from them), PV is [thus] equal to SW, and BV to KW [Props. 3.27, 1.26]. And the whole of BA is also equal to the whole of KA. And, thus, as BV is to VA, so KW (is) to WA. WV is thus parallel to KB [Prop. 6.2]. And since PV and SW are each at right-angles to the plane of circle $BCDE$, PV is thus parallel to SW [Prop. 11.6]. And it was also shown (to be) equal to it. And, thus, WV and SP are equal and parallel [Prop. 1.33]. And since WV is parallel to SP, but WV is parallel to KB, SP is thus also parallel to KB [Prop. 11.1]. And BP and KS join them. Thus, the quadrilateral $KBPS$ is in one plane, inasmuch as if there are two parallel straight-lines, and a random point is taken on each of them, then the straight-line joining the points is in the same plane as the parallel (straight-lines) [Prop. 11.7]. So, for the same (reasons), each of the quadrilaterals $SPQT$ and $TQRU$ is also in one plane. And triangle URO is also in one plane [Prop. 11.2]. So, if we conceive straight-lines joining points P, S, Q, T, R, and U to A then some solid polyhedral figure will have been constructed between the circumferences BO and KO, being composed of pyramids whose bases (are) the quadrilaterals $KBPS$, $SPQT$, $TQRU$, and the triangle URO, and apex the point A. And if we also make the same construction on each of the sides KL, LM, and ME, just as on BK, and, further, (repeat the construction) in the remaining three quadrants, then some polyhedral figure which has been inscribed in the sphere will have been constructed, being contained by pyramids whose bases (are) the aforementioned quadrilaterals, and triangle URO, and the (quadrilaterals and triangles) similarly arranged to them, and apex the point A.

So, I say that the aforementioned polyhedron will not touch the lesser sphere on the surface on which the circle FGH is (situated).

Let the perpendicular (straight-line) AX have been drawn from point A to the plane $KBPS$, and let it meet the plane at point X [Prop. 11.11]. And let XB and XK have been joined. And since AX is at right-angles to the plane of quadrilateral $KBPS$, it is thus also at right-angles to all of the straight-lines joined to it which are also in the plane of the quadrilateral [Def. 11.3]. Thus, AX is at right-angles to each of BX and XK. And since AB is equal to AK, the (square) on AB is also equal to the (square) on AK. And the (sum of the squares) on AX and XB is equal to the (square) on AB. For the angle at X (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on AX and XK is equal to the (square) on AK [Prop. 1.47]. Thus, the (sum of the squares) on AX and XB is equal to the (sum of the squares) on AX and XK. Let the (square) on AX have been subtracted from both. Thus, the remaining (square) on BX is equal to the remaining (square) on XK. Thus, BX (is) equal to XK. So, similarly, we can show that the straight-lines joined from X to P and S are equal to each of BX and XK. Thus, a circle drawn (in the plane of the quadrilateral) with center X, and radius one of XB or XK, will also pass through P and S, and the quadrilateral $KBPS$ will be inside the circle.

And since KB is greater than WV, and WV (is) equal to SP, KB (is) thus greater than SP. And KB (is) equal to each of KS and BP. Thus, KS and BP are each greater than SP. And since quadrilateral $KBPS$ is in a circle, and KB, BP, and KS are equal (to one another), and PS (is) less (than them), and BX is the radius of the circle, the (square) on KB is thus greater than double the (square) on BX.[†] Let the perpendicular KY have been drawn from K to BV.[‡] And since BD is less than double DY, and as BD is to DY, so the (rectangle contained) by DB and BY (is) to the (rectangle contained) by DY and YB—a square being described on BY, and a (rectangular) parallelogram (with short side equal to BY) completed on YD—the (rectangle contained) by DB and BY is thus also less than double the (rectangle contained) by DY and YB. And, KD being joined, the (rectangle contained) by DB and BY is equal to the (square) on BK, and the (rectangle contained) by DY and YB equal to the (square) on KY [Props. 3.31, 6.8 corr.]. Thus, the (square) on KB is less than double the (square) on KY. But, the (square) on KB is greater than double the (square) on BX. Thus, the (square) on KY (is) greater than the (square) on BX. And since BA is equal to KA, the (square) on BA is equal to the (square) on AK. And the (sum of the squares) on BX and XA is equal to the (square) on BA, and the (sum of the squares) on KY and YA (is) equal to the (square) on KA [Prop. 1.47]. Thus, the (sum of the squares) on BX and XA is equal to the (sum of the squares) on KY and YA, of which the (square) on KY (is) greater than the (square) on BX. Thus, the remaining (square) on YA is less than the (square) on XA. Thus, AX (is) greater than AY. Thus, AX is much greater than AG.[§] And AX is (a perpendicular) on one of the bases of the polyhedron, and AG (is a perpendicular) on the surface of the lesser sphere. Hence, the polyhedron will not touch the lesser sphere on its surface.

Thus, there being two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere on its surface. (Which is) the very thing it was required to do. [†] Since KB, BP, and KS are greater than the sides of an inscribed square, which are each of length $\sqrt{2}\,BX$.

‡ Note that points Y and V are actually identical.

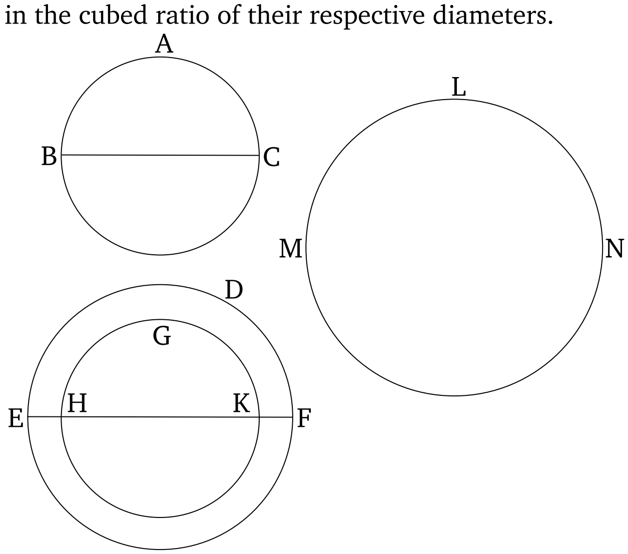
§ This conclusion depends on the fact that the chord of the polygon in proposition 12.16 does not touch the inner circle.

Corollary

And, also, if a similar polyhedral solid to that in sphere $BCDE$ is inscribed in another sphere then the polyhedral solid in sphere $BCDE$ has to the polyhedral solid in the other sphere the cubed ratio that the diameter of sphere $BCDE$ has to the diameter of the other sphere. For if the solids are divided into similarly numbered, and similarly situated, pyramids, then the pyramids will be similar. And similar pyramids are in the cubed ratio of corresponding sides [Prop. 12.8 corr.]. Thus, the pyramid whose base is quadrilateral $KBPS$, and apex the point A, will have to the similarly situated pyramid in the other sphere the cubed ratio that a corresponding side (has) to a corresponding side. That is to say, that of radius AB of the sphere about center A to the radius of the other sphere. And, similarly, each pyramid in the sphere about center A will have to each similarly situated pyramid in the other sphere the cubed ratio that AB (has) to the radius of the other sphere. And as one of the leading (magnitudes is) to one of the following (in two sets of proportional magnitudes), so (the sum of) all the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. Hence, the whole polyhedral solid in the sphere about center A will have to the whole polyhedral solid in the other [sphere] the cubed ratio that (radius) AB (has) to the radius of the other sphere. That is to say, that diameter BD (has) to the diameter of the other sphere. (Which is) the very thing it was required to show.

Proposition 18

Spheres are to one another in the cubed ratio of their respective diameters.



Let the spheres ABC and DEF have been conceived, and (let) their diameters (be) BC and EF (respectively). I say that sphere ABC has to sphere DEF the cubed ratio that BC (has) to EF.

For if sphere ABC does not have to sphere DEF the cubed ratio that BC (has) to EF then sphere ABC will have to some (sphere) either less than, or greater than, sphere DEF the cubed ratio that BC (has) to EF. Let it, first of all, have (such a ratio) to a lesser (sphere), GHK. And let DEF have been conceived about the same center as GHK. And let a polyhedral solid have been inscribed in the greater sphere DEF, not touching the lesser sphere GHK on its surface [Prop. 12.17]. And let a polyhedral solid, similar to the polyhedral solid in sphere DEF, have also been inscribed in sphere ABC. Thus, the polyhedral solid in sphere ABC has to the polyhedral solid in sphere DEF the cubed ratio that BC (has) to EF [Prop. 12.17 corr.]. And sphere ABC also has to sphere GHK the cubed ratio that BC (has) to EF. Thus, as sphere ABC is to sphere GHK, so the polyhedral solid in sphere ABC (is) to the polyhedral solid is sphere DEF. [Thus], alternately, as sphere ABC (is) to the polygon within it, so sphere GHK (is) to the polyhedral solid within sphere DEF [Prop. 5.16]. And sphere ABC (is) greater than the polyhedron within it. Thus, sphere GHK (is) also greater than the polyhedron within sphere DEF [Prop. 5.14]. But, (it is) also less. For it is encompassed by it. Thus, sphere ABC does not have to (a sphere) less than sphere DEF the cubed ratio that diameter BC (has) to EF. So, similarly, we can show that sphere DEF does not have to (a sphere) less than sphere ABC the cubed ratio that EF (has) to BC either.

So, I say that sphere ABC does not have to some (sphere) greater than sphere DEF the cubed ratio that BC (has) to EF either.

For, if possible, let it have (the cubed ratio) to a greater (sphere), LMN. Thus, inversely, sphere LMN (has) to sphere ABC the cubed ratio that diameter EF (has) to diameter BC [Prop. 5.7 corr.]. And as sphere LMN (is) to sphere ABC, so sphere DEF (is) to some (sphere) less than sphere ABC, inasmuch as LMN is greater than DEF, as was shown before [Prop. 12.2 lem.]. And, thus, sphere DEF has to some (sphere) less than sphere ABC the cubed ratio that EF (has) to BC. The very thing was shown (to be) impossible. Thus, sphere ABC does not have to some (sphere) greater than sphere DEF the cubed ratio that BC (has) to EF. And it was shown that neither (does it have such a ratio) to a lesser (sphere). Thus, sphere ABC has to sphere DEF the cubed ratio that BC (has) to EF. (Which is) the very thing it was required to show.

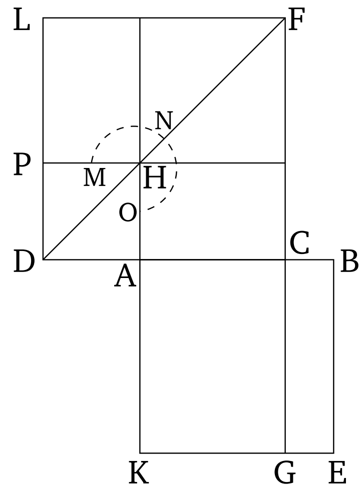
ELEMENTS BOOK 13

The Platonic Solids†

†The five regular solids—the cube, tetrahedron (*i.e.*, pyramid), octahedron, icosahedron, and dodecahedron—were problably discovered by the school of Pythagoras. They are generally termed “Platonic” solids because they feature prominently in Plato’s famous dialogue *Timaeus*. Many of the theorems contained in this book—particularly those which pertain to the last two solids—are ascribed to Theaetetus of Athens.

Proposition 1

If a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half.



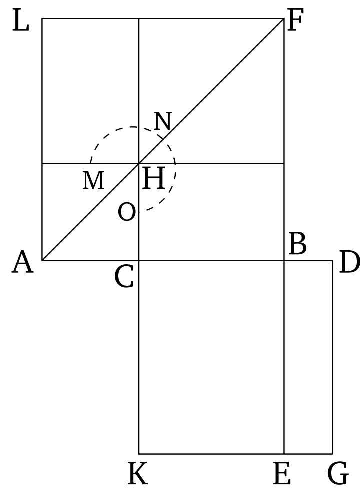
For let the straight-line AB have been cut in extreme and mean ratio at point C, and let AC be the greater piece. And let the straight-line AD have been produced in a straight-line with CA. And let AD be made (equal to) half of AB. I say that the (square) on CD is five times the (square) on DA.

For let the squares AE and DF have been described on AB and DC (respectively). And let the figure in DF have been drawn. And let FC have been drawn across to G. And since AB has been cut in extreme and mean ratio at C, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC, and FH the (square) on AC. Thus, CE (is) equal to FH. And since BA is double AD, and BA (is) equal to KA, and AD to AH, KA (is) thus also double AH. And as KA (is) to AH, so CK (is) to CH [Prop. 6.1]. Thus, CK (is) double CH. And LH plus HC is also double CH [Prop. 1.43]. Thus, KC (is) equal to LH plus HC. And CE was also shown (to be) equal to HF. Thus, the whole square AE is equal to the gnomon MNO. And since BA is double AD, the (square) on BA is four times the (square) on AD—that is to say, AE (is four times) DH. And AE (is) equal to gnomon MNO. And, thus, gnomon MNO is also four times AP. Thus, the whole of DF is five times AP. And DF is the (square) on DC, and AP the (square) on DA. Thus, the (square) on CD is five times the (square) on DA.

Thus, if a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half. (Which is) the very thing it was required to show.

Proposition 2

If the square on a straight-line is five times the (square) on a piece of it, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line.



For let the square on the straight-line AB be five times the (square) on the piece of it, AC. And let CD be double AC. I say that if CD is cut in extreme and mean ratio then the greater piece is CB.

For let the squares AF and CG have been described on each of AB and CD (respectively). And let the figure in AF have been drawn. And let BE have been drawn across. And since the (square) on BA is five times the (square) on AC, AF is five times AH. Thus, gnomon MNO (is) four times AH. And since DC is double CA, the (square) on DC is thus four times the (square) on CA—that is to say, CG (is four times) AH. And the gnomon MNO was also shown (to be) four times AH. Thus, gnomon MNO (is) equal to CG. And since DC is double CA, and DC (is) equal to CK, and AC to CH, [KC (is) thus also double CH], (and) KB (is) also double BH [Prop. 6.1]. And LH plus HB is also double HB [Prop. 1.43]. Thus, KB (is) equal to LH plus HB. And the whole gnomon MNO was also shown (to be) equal to the whole of CG. Thus, the remainder HF is also equal to (the remainder) BG. And BG is the (rectangle contained) by CDB. For CD (is) equal to DG. And HF (is) the square on CB. Thus, the (rectangle contained) by CDB is equal to the (square) on CB. Thus, as DC is to CB, so CB (is) to BD [Prop. 6.17]. And DC (is) greater than CB (see lemma). Thus, CB (is) also greater than BD [Prop. 5.14]. Thus, if the straight-line CD is cut in extreme and mean ratio then the greater piece is CB.

Thus, if the square on a straight-line is five times the (square) on a piece of itself, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line. (Which is) the very thing it was required to show.

Lemma

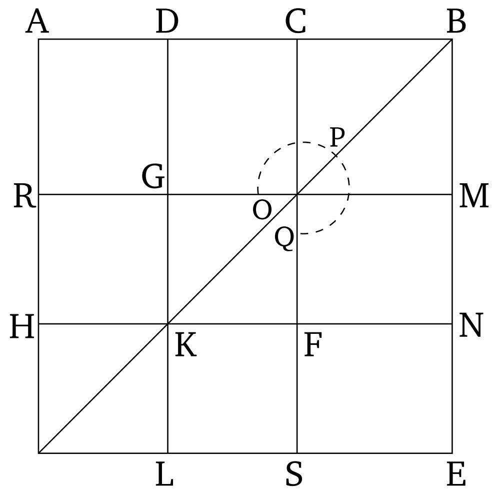
And it can be shown that double AC (*i.e.*, DC) is greater than BC, as follows.

For if (double AC is) not (greater than BC), if possible, let BC be double CA. Thus, the (square) on BC (is) four times the (square) on CA. Thus, the (sum of) the (squares) on BC and CA (is) five times the (square) on CA. And the (square) on BA was assumed (to be) five times the (square) on CA. Thus, the (square) on BA is equal to the (sum of) the (squares) on BC and CA. The very thing (is) impossible [Prop. 2.4]. Thus, CB is not double AC. So, similarly, we can show that a (straight-line) less than CB is not double AC either. For (in this case) the absurdity is much [greater].

Thus, double AC is greater than CB. (Which is) the very thing it was required to show.

Proposition 3

If a straight-line is cut in extreme and mean ratio then the square on the lesser piece added to half of the greater piece is five times the square on half of the greater piece.

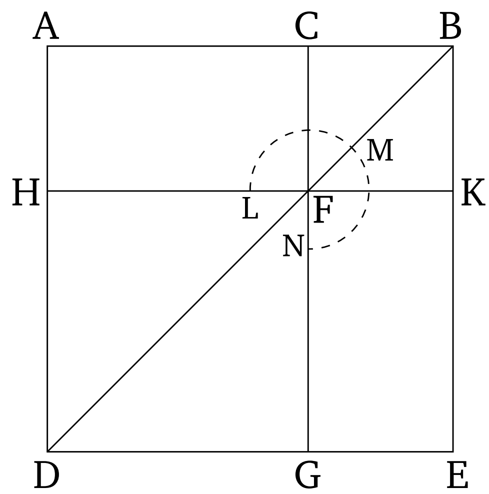


For let some straight-line AB have been cut in extreme and mean ratio at point C. And let AC be the greater piece. And let AC have been cut in half at D. I say that the (square) on BD is five times the (square) on DC.

For let the square AE have been described on AB. And let the figure have been drawn double. Since AC is double DC, the (square) on AC (is) thus four times the (square) on DC—that is to say, RS (is four times) FG. And since the (rectangle contained) by ABC is equal to the (square) on AC [Def. 6.3, Prop. 6.17], and CE is the (rectangle contained) by ABC, CE is thus equal to RS. And RS (is) four times FG. Thus, CE (is) also four times FG. Again, since AD is equal to DC, HK is also equal to KF. Hence, square GF is also equal to square HL. Thus, GK (is) equal to KL—that is to say, MN to NE. Hence, MF is also equal to FE. But, MF is equal to CG. Thus, CG is also equal to FE. Let CN have been added to both. Thus, gnomon OPQ is equal to CE. But, CE was shown (to be) equal to four times GF. Thus, gnomon OPQ is also four times square FG. Thus, gnomon OPQ plus square FG is five times FG. But, gnomon OPQ plus square FG is (square) DN. And DN is the (square) on DB, and GF the (square) on DC. Thus, the (square) on DB is five times the (square) on DC. (Which is) the very thing it was required to show.

Proposition 4

If a straight-line is cut in extreme and mean ratio then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece.

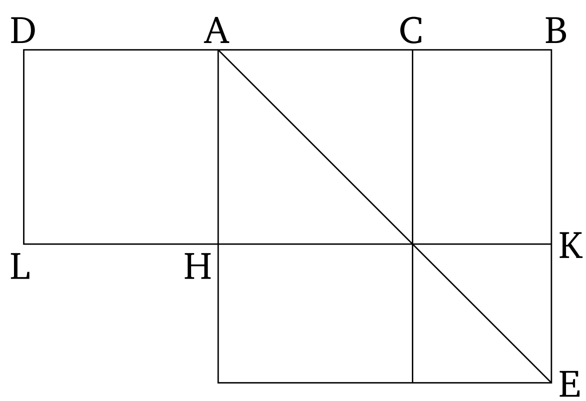


Let AB be a straight-line, and let it have been cut in extreme and mean ratio at C, and let AC be the greater piece. I say that the (sum of the squares) on AB and BC is three times the (square) on CA.

For let the square $ADEB$ have been described on AB, and let the (remainder of the) figure have been drawn. Therefore, since AB has been cut in extreme and mean ratio at C, and AC is the greater piece, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And AK is the (rectangle contained) by ABC, and HG the (square) on AC. Thus, AK is equal to HG. And since AF is equal to FE [Prop. 1.43], let CK have been added to both. Thus, the whole of AK is equal to the whole of CE. Thus, AK plus CE is double AK. But, AK plus CE is the gnomon LMN plus the square CK. Thus, gnomon LMN plus square CK is double AK. But, indeed, AK was also shown (to be) equal to HG. Thus, gnomon LMN plus [square CK is double HG. Hence, gnomon LMN plus] the squares CK and HG is three times the square HG. And gnomon LMN plus the squares CK and HG is the whole of AE plus CK—which are the squares on AB and BC (respectively)—and GH (is) the square on AC. Thus, the (sum of the) squares on AB and BC is three times the square on AC. (Which is) the very thing it was required to show.

Proposition 5

If a straight-line is cut in extreme and mean ratio, and a (straight-line) equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.



For let the straight-line AB have been cut in extreme and mean ratio at point C. And let AC be the greater piece.

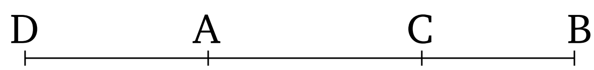
And let AD be [made] equal to AC. I say that the straight-line DB has been cut in extreme and mean ratio at A, and that the original straight-line AB is the greater piece.

For let the square AE have been described on AB, and let the (remainder of the) figure have been drawn. And since AB has been cut in extreme and mean ratio at C, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC, and CH the (square) on AC. But, HE is equal to CE [Prop. 1.43], and DH equal to HC. Thus, DH is also equal to HE. [Let HB have been added to both.] Thus, the whole of DK is equal to the whole of AE. And DK is the (rectangle contained) by BD and DA. For AD (is) equal to DL. And AE (is) the (square) on AB. Thus, the (rectangle contained) by BDA is equal to the (square) on AB. Thus, as DB (is) to BA, so BA (is) to AD [Prop. 6.17]. And DB (is) greater than BA. Thus, BA (is) also greater than AD [Prop. 5.14].

Thus, DB has been cut in extreme and mean ratio at A, and the greater piece is AB. (Which is) the very thing it was required to show.

Proposition 6

If a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.



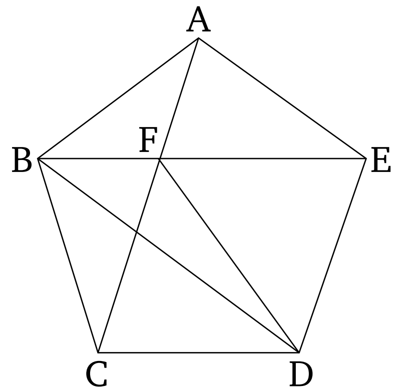
Let AB be a rational straight-line cut in extreme and mean ratio at C, and let AC be the greater piece. I say that AC and CB is each that irrational (straight-line) called an apotome.

For let BA have been produced, and let AD be made (equal) to half of BA. Therefore, since the straight-line AB has been cut in extreme and mean ratio at C, and AD, which is half of AB, has been added to the greater piece AC, the (square) on CD is thus five times the (square) on DA [Prop. 13.1]. Thus, the (square) on CD has to the (square) on DA the ratio which a number (has) to a number. The (square) on CD (is) thus commensurable with the (square) on DA [Prop. 10.6]. And the (square) on DA (is) rational. For DA [is] rational, being half of AB, which is rational. Thus, the (square) on CD (is) also rational [Def. 10.4]. Thus, CD is also rational. And since the (square) on CD does not have to the (square) on DA the ratio which a square number (has) to a square number, CD (is) thus incommensurable in length with DA [Prop. 10.9]. Thus, CD and DA are rational (straight-lines which are) commensurable in square only. Thus, AC is an apotome [Prop. 10.73]. Again, since AB has been cut in extreme and mean ratio, and AC is the greater piece, the (rectangle contained) by AB and BC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. Thus, the (square) on the apotome AC, applied to the rational (straight-line) AB, makes BC as width. And the (square) on an apotome, applied to a rational (straight-line), makes a first apotome as width [Prop. 10.97]. Thus, CB is a first apotome. And CA was also shown (to be) an apotome.

Thus, if a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

Proposition 7

If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.



For let three angles of the equilateral pentagon $ABCDE$—first of all, the consecutive (angles) at A, B, and C—-be equal to one another. I say that pentagon $ABCDE$ is equiangular.

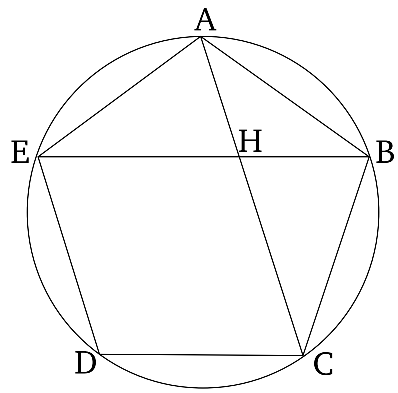
For let AC, BE, and FD have been joined. And since the two (straight-lines) CB and BA are equal to the two (straight-lines) BA and AE, respectively, and angle CBA is equal to angle BAE, base AC is thus equal to base BE, and triangle ABC equal to triangle ABE, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4], (that is), BCA (equal) to BEA, and ABE to CAB. And hence side AF is also equal to side BF [Prop. 1.6]. And the whole of AC was also shown (to be) equal to the whole of BE. Thus, the remainder FC is also equal to the remainder FE. And CD is also equal to DE. So, the two (straight-lines) FC and CD are equal to the two (straight-lines) FE and ED (respectively). And FD is their common base. Thus, angle FCD is equal to angle FED [Prop. 1.8]. And BCA was also shown (to be) equal to AEB. And thus the whole of BCD (is) equal to the whole of AED. But, (angle) BCD was assumed (to be) equal to the angles at A and B. Thus, (angle) AED is also equal to the angles at A and B. So, similarly, we can show that angle CDE is also equal to the angles at A, B, C. Thus, pentagon $ABCDE$ is equiangular.

And so let consecutive angles not be equal, but let the (angles) at points A, C, and D be equal. I say that pentagon $ABCDE$ is also equiangular in this case.

For let BD have been joined. And since the two (straight-lines) BA and AE are equal to the (straight-lines) BC and CD, and they contain equal angles, base BE is thus equal to base BD, and triangle ABE is equal to triangle BCD, and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle AEB is equal to (angle) CDB. And angle BED is also equal to (angle) BDE, since side BE is also equal to side BD [Prop. 1.5]. Thus, the whole angle AED is also equal to the whole (angle) CDE. But, (angle) CDE was assumed (to be) equal to the angles at A and C. Thus, angle AED is also equal to the (angles) at A and C. So, for the same (reasons), (angle) ABC is also equal to the angles at A, C, and D. Thus, pentagon $ABCDE$ is equiangular. (Which is) the very thing it was required to show.

Proposition 8

If straight-lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio, and their greater pieces are equal to the sides of the pentagon.

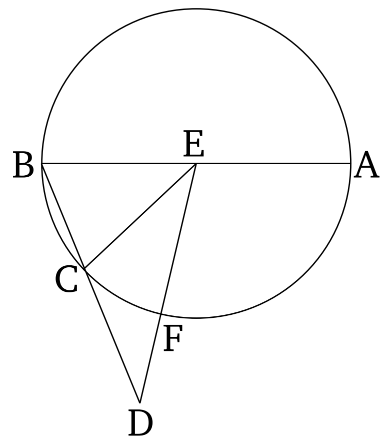


For let the two straight-lines, AC and BE, cutting one another at point H, have subtended two consecutive angles, at A and B (respectively), of the equilateral and equiangular pentagon $ABCDE$. I say that each of them has been cut in extreme and mean ratio at point H, and that their greater pieces are equal to the sides of the pentagon.

For let the circle $ABCDE$ have been circumscribed about pentagon $ABCDE$ [Prop. 4.14]. And since the two straight-lines EA and AB are equal to the two (straight-lines) AB and BC (respectively), and they contain equal angles, the base BE is thus equal to the base AC, and triangle ABE is equal to triangle ABC, and the remaining angles will be equal to the remaining angles, respectively, which the equal sides subtend [Prop. 1.4]. Thus, angle BAC is equal to (angle) ABE. Thus, (angle) AHE (is) double (angle) BAH [Prop. 1.32]. And EAC is also double BAC, inasmuch as circumference EDC is also double circumference CB [Props. 3.28, 6.33]. Thus, angle HAE (is) equal to (angle) AHE. Hence, straight-line HE is also equal to (straight-line) EA—that is to say, to (straight-line) AB [Prop. 1.6]. And since straight-line BA is equal to AE, angle ABE is also equal to AEB [Prop. 1.5]. But, ABE was shown (to be) equal to BAH. Thus, BEA is also equal to BAH. And (angle) ABE is common to the two triangles ABE and ABH. Thus, the remaining angle BAE is equal to the remaining (angle) AHB [Prop. 1.32]. Thus, triangle ABE is equiangular to triangle ABH. Thus, proportionally, as EB is to BA, so AB (is) to BH [Prop. 6.4]. And BA (is) equal to EH. Thus, as BE (is) to EH, so EH (is) to HB. And BE (is) greater than EH. EH (is) thus also greater than HB [Prop. 5.14]. Thus, BE has been cut in extreme and mean ratio at H, and the greater piece HE is equal to the side of the pentagon. So, similarly, we can show that AC has also been cut in extreme and mean ratio at H, and that its greater piece CH is equal to the side of the pentagon. (Which is) the very thing it was required to show.

Proposition 9

If the side of a hexagon and of a decagon inscribed in the same circle are added together then the whole straight-line has been cut in extreme and mean ratio (at the junction point), and its greater piece is the side of the hexagon.†

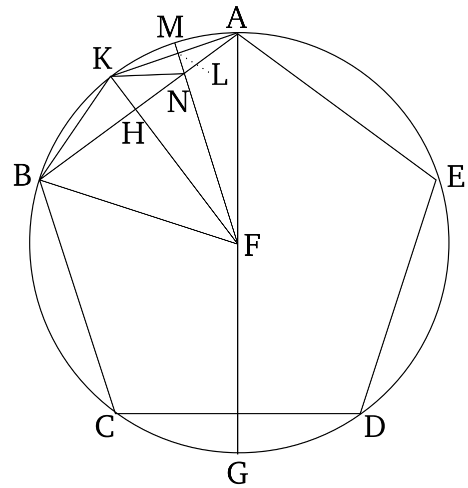


Let ABC be a circle. And of the figures inscribed in circle ABC, let BC be the side of a decagon, and CD (the side) of a hexagon. And let them be (laid down) straight-on (to one another). I say that the whole straight-line BD has been cut in extreme and mean ratio (at C), and that CD is its greater piece.

For let the center of the circle, point E, have been found [Prop. 3.1], and let EB, EC, and ED have been joined, and let BE have been drawn across to A. Since BC is a side on an equilateral decagon, circumference ACB (is) thus five times circumference BC. Thus, circumference AC (is) four times CB. And as circumference AC (is) to CB, so angle AEC (is) to CEB [Prop. 6.33]. Thus, (angle) AEC (is) four times CEB. And since angle EBC (is) equal to ECB [Prop. 1.5], angle AEC is thus double ECB [Prop. 1.32]. And since straight-line EC is equal to CD—for each of them is equal to the side of the hexagon [inscribed] in circle ABC [Prop. 4.15 corr.]— angle CED is also equal to angle CDE [Prop. 1.5]. Thus, angle ECB (is) double EDC [Prop. 1.32]. But, AEC was shown (to be) double ECB. Thus, AEC (is) four times EDC. And AEC was also shown (to be) four times BEC. Thus, EDC (is) equal to BEC. And angle EBD (is) common to the two triangles BEC and BED. Thus, the remaining (angle) BED is equal to the (remaining angle) ECB [Prop. 1.32]. Thus, triangle EBD is equiangular to triangle EBC. Thus, proportionally, as DB is to BE, so EB (is) to BC [Prop. 6.4]. And EB (is) equal to CD. Thus, as BD is to DC, so DC (is) to CB. And BD (is) greater than DC. Thus, DC (is) also greater than CB [Prop. 5.14]. Thus, the straight-line BD has been cut in extreme and mean ratio [at C], and DC is its greater piece. (Which is), the very thing it was required to show.† If the circle is of unit radius then the side of the hexagon is 1, whereas the side of the decagon is $(1/2)\,(\sqrt{5}-1)$.

Proposition 10

If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.†



Let $ABCDE$ be a circle. And let the equilateral pentagon $ABCDE$ have been inscribed in circle $ABCDE$. I say that the square on the side of pentagon $ABCDE$ is the (sum of the squares) on the sides of the hexagon and of the decagon inscribed in circle $ABCDE$.

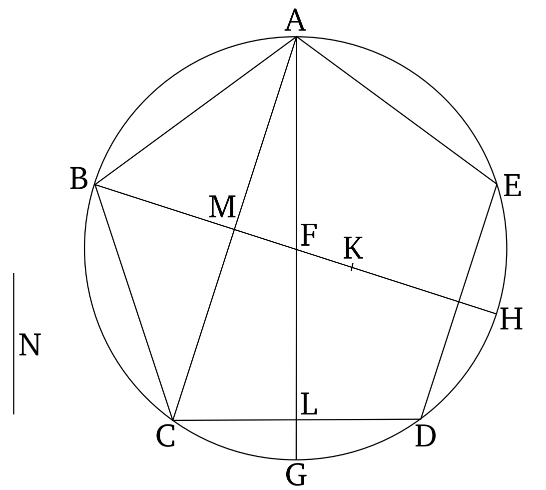
For let the center of the circle, point F, have been found [Prop. 3.1]. And, AF being joined, let it have been drawn across to point G. And let FB have been joined. And let FH have been drawn from F perpendicular to AB. And let it have been drawn across to K. And let AK and KB have been joined. And, again, let FL have been drawn from F perpendicular to AK. And let it have been drawn across to M. And let KN have been joined.

Since circumference $ABCG$ is equal to circumference $AEDG$, of which ABC is equal to AED, the remaining circumference CG is thus equal to the remaining (circumference) GD. And CD (is the side) of the pentagon. CG (is) thus (the side) of the decagon. And since FA is equal to FB, and FH is perpendicular (to AB), angle AFK (is) thus also equal to KFB [Props. 1.5, 1.26]. Hence, circumference AK is also equal to KB [Prop. 3.26]. Thus, circumference AB (is) double circumference BK. Thus, straight-line AK is the side of the decagon. So, for the same (reasons, circumference) AK is also double KM. And since circumference AB is double circumference BK, and circumference CD (is) equal to circumference AB, circumference CD (is) thus also double circumference BK. And circumference CD is also double CG. Thus, circumference CG (is) equal to circumference BK. But, BK is double KM, since KA (is) also (double KM). Thus, (circumference) CG is also double KM. But, indeed, circumference CB is also double circumference BK. For circumference CB (is) equal to BA. Thus, the whole circumference GB is also double BM. Hence, angle GFB [is] also double angle BFM [Prop. 6.33]. And GFB (is) also double FAB. For FAB (is) equal to ABF. Thus, BFN is also equal to FAB. And angle ABF (is) common to the two triangles ABF and BFN. Thus, the remaining (angle) AFB is equal to the remaining (angle) BNF [Prop. 1.32]. Thus, triangle ABF is equiangular to triangle BFN. Thus, proportionally, as straight-line AB (is) to BF, so FB (is) to BN [Prop. 6.4]. Thus, the (rectangle contained) by ABN is equal to the (square) on BF [Prop. 6.17]. Again, since AL is equal to LK, and LN is common and at right-angles (to KA), base KN is thus equal to base AN [Prop. 1.4]. And, thus, angle LKN is equal to angle LAN. But, LAN is equal to KBN [Props. 3.29, 1.5]. Thus, LKN is also equal to KBN. And the (angle) at A (is) common to the two triangles AKB and AKN. Thus, the remaining (angle) AKB is equal to the remaining (angle) KNA [Prop. 1.32]. Thus, triangle KBA is equiangular to triangle KNA. Thus, proportionally, as straight-line BA is to AK, so KA (is) to AN [Prop. 6.4]. Thus, the (rectangle contained) by BAN is equal to the (square) on AK [Prop. 6.17]. And the (rectangle contained) by ABN was also shown (to be) equal to the (square) on BF. Thus, the (rectangle contained) by ABN plus the (rectangle contained) by BAN, which is the (square) on BA [Prop. 2.2], is equal to the (square) on BF plus the (square) on AK. And BA is the side of the pentagon, and BF (the side) of the hexagon [Prop. 4.15 corr.], and AK (the side) of the decagon.

Thus, the square on the side of the pentagon (inscribed in a circle) is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle. † If the circle is of unit radius then the side of the pentagon is $(1/2)\sqrt{10-2\sqrt{5}}$.

Proposition 11

If an equilateral pentagon is inscribed in a circle which has a rational diameter then the side of the pentagon is that irrational (straight-line) called minor.



For let the equilateral pentagon $ABCDE$ have been inscribed in the circle $ABCDE$ which has a rational diameter. I say that the side of pentagon [$ABCDE$] is that irrational (straight-line) called minor.

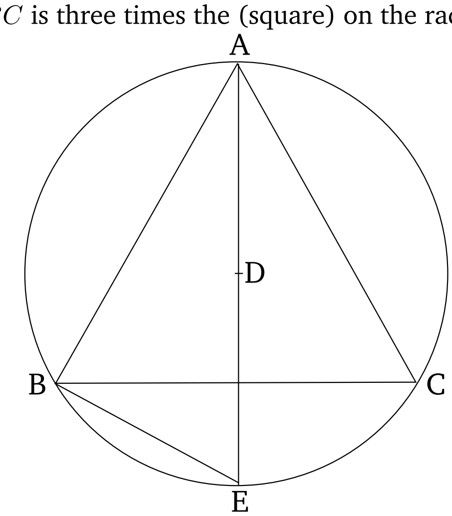
For let the center of the circle, point F, have been found [Prop. 3.1]. And let AF and FB have been joined. And let them have been drawn across to points G and H (respectively). And let AC have been joined. And let FK made (equal) to the fourth part of AF. And AF (is) rational. FK (is) thus also rational. And BF is also rational. Thus, the whole of BK is rational. And since circumference ACG is equal to circumference ADG, of which ABC is equal to AED, the remainder CG is thus equal to the remainder GD. And if we join AD then the angles at L are inferred (to be) right-angles, and CD (is inferred to be) double CL [Prop. 1.4]. So, for the same (reasons), the (angles) at M are also right-angles, and AC (is) double CM. Therefore, since angle ALC (is) equal to AMF, and (angle) LAC (is) common to the two triangles ACL and AMF, the remaining (angle) ACL is thus equal to the remaining (angle) MFA [Prop. 1.32]. Thus, triangle ACL is equiangular to triangle AMF. Thus, proportionally, as LC (is) to CA, so MF (is) to FA [Prop. 6.4]. And (we can take) the doubles of the leading (magnitudes). Thus, as double LC (is) to CA, so double MF (is) to FA. And as double MF (is) to FA, so MF (is) to half of FA. And, thus, as double LC (is) to CA, so MF (is) to half of FA. And (we can take) the halves of the following (magnitudes). Thus, as double LC (is) to half of CA, so MF (is) to the fourth of FA. And DC is double LC, and CM half of CA, and FK the fourth part of FA. Thus, as DC is to CM, so MF (is) to FK. Via composition, as the sum of DCM (*i.e.*, DC and CM) (is) to CM, so MK (is) to KF [Prop. 5.18]. And, thus, as the (square) on the sum of DCM (is) to the (square) on CM, so the (square) on MK (is) to the (square) on KF. And since the greater piece of a (straight-line) subtending two sides of a pentagon, such as AC, (which is) cut in extreme and mean ratio is equal to the side of the pentagon [Prop. 13.8]—that is to say, to DC—-and the square on the greater piece added to half of the whole is five times the (square) on half of the whole [Prop. 13.1], and CM (is) half of the whole, AC, thus the (square) on DCM, (taken) as one, is five times the (square) on CM. And the (square) on DCM, (taken) as one, (is) to the (square) on CM, so the (square) on MK was shown (to be) to the (square) on KF. Thus, the (square) on MK (is) five times the (square) on KF. And the square on KF (is) rational. For the diameter (is) rational. Thus, the (square) on MK (is) also rational. Thus, MK is rational [in square only]. And since BF is four times FK, BK is thus five times KF. Thus, the (square) on BK (is) twenty-five times the (square) on KF. And the (square) on MK (is) five times the square on KF. Thus, the (square) on BK (is) five times the (square) on KM. Thus, the (square) on BK does not have to the (square) on KM the ratio which a square number (has) to a square number. Thus, BK is incommensurable in length with KM [Prop. 10.9]. And each of them is a rational (straight-line). Thus, BK and KM are rational (straight-lines which are) commensurable in square only. And if from a rational (straight-line) a rational (straight-line) is subtracted, which is commensurable in square only with the whole, then the remainder is that irrational (straight-line called) an apotome [Prop. 10.73]. Thus, MB is an apotome, and MK its attachment. So, I say that (it is) also a fourth (apotome). So, let the (square) on N be (made) equal to that (magnitude) by which the (square) on BK is greater than the (square) on KM. Thus, the square on BK is greater than the (square) on KM by the (square) on N. And since KF is commensurable (in length) with FB then, via composition, KB is also commensurable (in length) with FB [Prop. 10.15]. But, BF is commensurable (in length) with BH. Thus, BK is also commensurable (in length) with BH [Prop. 10.12]. And since the (square) on BK is five times the (square) on KM, the (square) on BK thus has to the (square) on KM the ratio which 5 (has) to one. Thus, via conversion, the (square) on BK has to the (square) on N the ratio which 5 (has) to 4 [Prop. 5.19 corr.], which is not (that) of a square (number) to a square (number). BK is thus incommensurable (in length) with N [Prop. 10.9]. Thus, the square on BK is greater than the (square) on KM by the (square) on (some straight-line which is) incommensurable (in length) with (BK). Therefore, since the square on the whole, BK, is greater than the (square) on the attachment, KM, by the (square) on (some straight-line which is) incommensurable (in length) with (BK), and the whole, BK, is commensurable (in length) with the (previously) laid down rational (straight-line) BH, MB is thus a fourth apotome [Def. 10.14]. And the rectangle contained by a rational (straight-line) and a fourth apotome is irrational, and its square-root is that irrational (straight-line) called minor [Prop. 10.94]. And the square on AB is the rectangle contained by HBM, on account of joining AH, (so that) triangle ABH becomes equiangular with triangle ABM [Prop. 6.8], and (proportionally) as HB is to BA, so AB (is) to BM.

Thus, the side AB of the pentagon is that irrational (straight-line) called minor.† (Which is) the very thing it was required to show. † If the circle has unit radius then the side of the pentagon is $(1/2)\sqrt{10-2\sqrt{5}}$. However, this length can be written in the "minor" form (see Prop. 10.94) $(\rho/\sqrt{2})\sqrt{1+k/\sqrt{1+k^2}} - (\rho/\sqrt{2})\sqrt{1-k/\sqrt{1+k^2}}$, with $\rho = \sqrt{5/2}$ and $k = 2$.

Proposition 12

If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the (square) on the radius of the circle.

Let there be a circle ABC, and let the equilateral triangle ABC have been inscribed in it [Prop. 4.2]. I say that the square on one side of triangle ABC is three times the (square) on the radius of circle ABC.



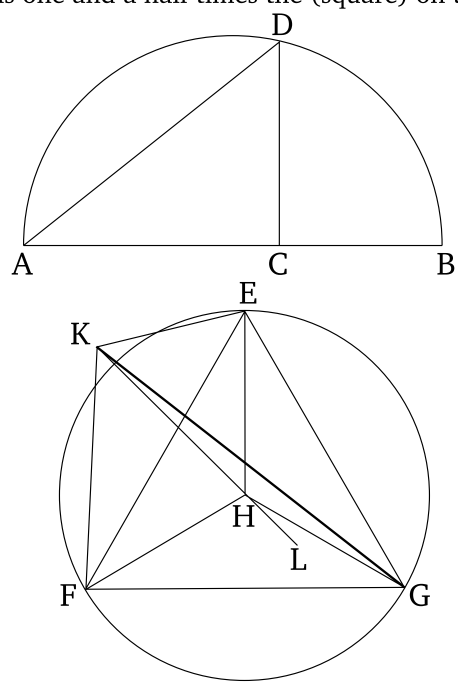
For let the center, D, of circle ABC have been found [Prop. 3.1]. And AD (being) joined, let it have been drawn across to E. And let BE have been joined.

And since triangle ABC is equilateral, circumference BEC is thus the third part of the circumference of circle ABC. Thus, circumference BE is the sixth part of the circumference of the circle. Thus, straight-line BE is (the side) of a hexagon. Thus, it is equal to the radius DE [Prop. 4.15 corr.]. And since AE is double DE, the (square) on AE is four times the (square) on ED—that is to say, of the (square) on BE. And the (square) on AE (is) equal to the (sum of the squares) on AB and BE [Props. 3.31, 1.47]. Thus, the (sum of the squares) on AB and BE is four times the (square) on BE. Thus, via separation, the (square) on AB is three times the (square) on BE. And BE (is) equal to DE. Thus, the (square) on AB is three times the (square) on DE.

Thus, the square on the side of the triangle is three times the (square) on the radius [of the circle]. (Which is) the very thing it was required to show.

Proposition 13

To construct a (regular) pyramid (*i.e.*, a tetrahedron), and to enclose (it) in a given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.



Let the diameter AB of the given sphere be laid out, and let it have been cut at point C such that AC is double CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB. And let CD have been drawn from point C at right-angles to AB. And let DA have been joined. And let the circle EFG be laid down having a radius equal to DC, and let the equilateral triangle EFG have been inscribed in circle EFG [Prop. 4.2]. And let the center of the circle, point H, have been found [Prop. 3.1]. And let EH, HF, and HG have been joined. And let HK have been set up, at point H, at right-angles to the plane of circle EFG [Prop. 11.12]. And let HK, equal to the straight-line AC, have been cut off from HK. And let KE, KF, and KG have been joined. And since KH is at right-angles to the plane of circle EFG, it will thus also make right-angles with all of the straight-lines joining it (which are) also in the plane of circle EFG [Def. 11.3]. And HE, HF, and HG each join it. Thus, HK is at right-angles to each of HE, HF, and HG. And since AC is equal to HK, and CD to HE, and they contain right-angles, the base DA is thus equal to the base KE [Prop. 1.4]. So, for the same (reasons), KF and KG is each equal to DA. Thus, the three (straight-lines) KE, KF, and KG are equal to one another. And since AC is double CB, AB (is) thus triple BC. And as AB (is) to BC, so the (square) on AD (is) to the (square) on DC, as will be shown later [see lemma]. Thus, the (square) on AD (is) three times the (square) on DC. And the (square) on FE is also three times the (square) on EH [Prop. 13.12], and DC is equal to EH. Thus, DA (is) also equal to EF. But, DA was shown (to be) equal to each of KE, KF, and KG. Thus, EF, FG, and GE are equal to KE, KF, and KG, respectively. Thus, the four triangles EFG, KEF, KFG, and KEG are equilateral. Thus, a pyramid, whose base is triangle EFG, and apex the point K, has been constructed from four equilateral triangles.

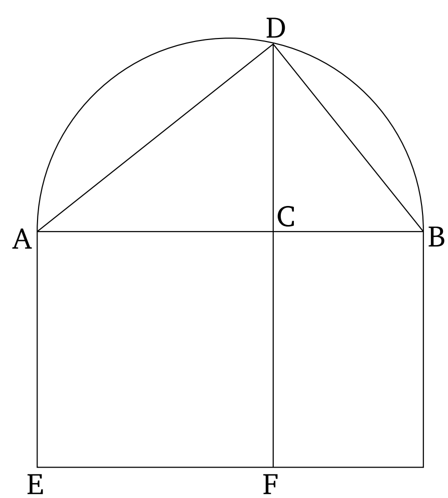
So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For let the straight-line HL have been produced in a straight-line with KH, and let HL be made equal to CB. And since as AC (is) to CD, so CD (is) to CB [Prop. 6.8 corr.], and AC (is) equal to KH, and CD to HE, and CB to HL, thus as KH is to HE, so EH (is) to HL. Thus, the (rectangle contained) by KH and HL is equal to the (square) on EH [Prop. 6.17]. And each of the angles KHE and EHL is a right-angle. Thus, the semi-circle drawn on KL will also pass through E [inasmuch as if we join EL then the angle LEK becomes a right-angle, on account of triangle ELK becoming equiangular to each of the triangles ELH and EHK [Props. 6.8, 3.31]]. So, if KL remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points F and G, (because) if FL and LG are joined, the angles at F and G will similarly become right-angles. And the pyramid will have been enclosed by the given sphere. For the diameter, KL, of the sphere is equal to the diameter, AB, of the given sphere—inasmuch as KH was made equal to AC, and HL to CB.

So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since AC is double CB, AB is thus triple BC. Thus, via conversion, BA is one and a half times AC. And as BA (is) to AC, so the (square) on BA (is) to the (square) on AD [inasmuch as if DB is joined then as BA is to AD, so DA (is) to AC, on account of the similarity of triangles DAB and DAC. And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on BA (is) also one and a half times the (square) on AD. And BA is the diameter of the given sphere, and AD (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.† (Which is) the very thing it was required to show. † If the radius of the sphere is unity then the side of the pyramid (*i.e.*, tetrahedron) is $\sqrt{8/3}$.



Lemma

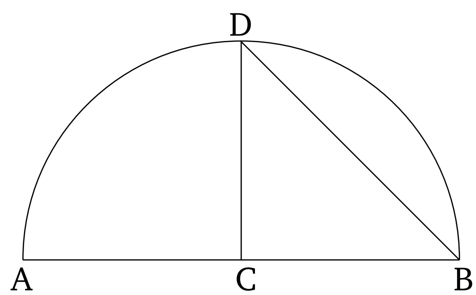
It must be shown that as AB is to BC, so the (square) on AD (is) to the (square) on DC.

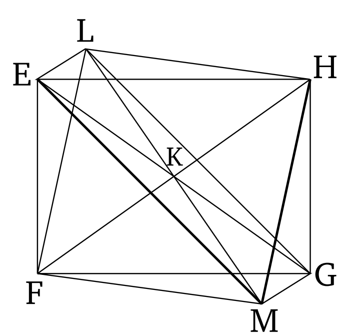
For, let the figure of the semi-circle have been set out, and let DB have been joined. And let the square EC have been described on AC. And let the parallelogram FB have been completed. Therefore, since, on account of triangle DAB being equiangular to triangle DAC [Props. 6.8, 6.4], (proportionally) as BA is to AD, so DA (is) to AC, the (rectangle contained) by BA and AC is thus equal to the (square) on AD [Prop. 6.17]. And since as AB is to BC, so EB (is) to BF [Prop. 6.1]. And EB is the (rectangle contained) by BA and AC—for EA (is) equal to AC. And BF the (rectangle contained) by AC and CB. Thus, as AB (is) to BC, so the (rectangle contained) by BA and AC (is) to the (rectangle contained) by AC and CB. And the (rectangle contained) by BA and AC is equal to the (square) on AD, and the (rectangle contained) by ACB (is) equal to the (square) on DC. For the perpendicular DC is the mean proportional to the pieces of the base, AC and CB, on account of ADB being a right-angle [Prop. 6.8 corr.]. Thus, as AB (is) to BC, so the (square) on AD (is) to the (square) on DC. (Which is) the very thing it was required to show.

Proposition 14

To construct an octahedron, and to enclose (it) in a (given) sphere, like in the preceding (proposition), and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

Let the diameter AB of the given sphere be laid out, and let it have been cut in half at C. And let the semi-circle ADB have been drawn on AB. And let CD be drawn from C at right-angles to AB. And let DB have been joined. And let the square $EFGH$, having each of its sides equal to DB, be laid out. And let HF and EG have been joined. And let the straight-line KL have been set up, at point K, at right-angles to the plane of square $EFGH$ [Prop. 11.12]. And let it have been drawn across on the other side of the plane, like KM. And let KL and KM, equal to one of EK, FK, GK, and HK, have been cut off from KL and KM, respectively. And let LE, LF, LG, LH, ME, MF, MG, and MH have been joined.





And since KE is equal to KH, and angle EKH is a right-angle, the (square) on the HE is thus double the (square) on EK [Prop. 1.47]. Again, since LK is equal to KE, and angle LKE is a right-angle, the (square) on EL is thus double the (square) on EK [Prop. 1.47]. And the (square) on HE was also shown (to be) double the (square) on EK. Thus, the (square) on LE is equal to the (square) on EH. Thus, LE is equal to EH. So, for the same (reasons), LH is also equal to HE. Triangle LEH is thus equilateral. So, similarly, we can show that each of the remaining triangles, whose bases are the sides of the square $EFGH$, and apexes the points L and M, are equilateral. Thus, an octahedron contained by eight equilateral triangles has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

For since the three (straight-lines) LK, KM, and KE are equal to one another, the semi-circle drawn on LM will thus also pass through E. And, for the same (reasons), if LM remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through points F, G, and H, and the octahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since LK is equal to KM, and KE (is) common, and they contain right-angles, the base LE is thus equal to the base EM [Prop. 1.4]. And since angle LEM is a right-angle—for (it is) in a semi-circle [Prop. 3.31]—the (square) on LM is thus double the (square) on LE [Prop. 1.47]. Again, since AC is equal to CB, AB is double BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BD. And the (square) on LM was also shown (to be) double the (square) on LE. And the (square) on DB is equal to the (square) on LE. For EH was made equal to DB. Thus, the (square) on AB (is) also equal to the (square) on LM. Thus, AB (is) equal to LM. And AB is the diameter of the given sphere. Thus, LM is equal to the diameter of the given sphere.

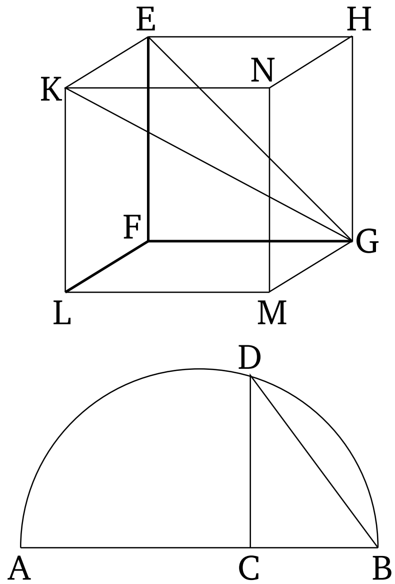
Thus, the octahedron has been enclosed by the given sphere, and it has been simultaneously proved that the square on the diameter of the sphere is double the (square) on the side of the octahedron.† (Which is) the very thing it was required to show. † If the radius of the sphere is unity then the side of octahedron is $\sqrt{2}$.

Proposition 15

To construct a cube, and to enclose (it) in a sphere, like in the (case of the) pyramid, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is double CB. And let the semi-circle ADB have been drawn on AB. And let CD have been drawn from C at right-angles to AB. And let DB have been joined. And let the square $EFGH$, having (its) side equal to DB, be laid out. And let EK, FL, GM, and HN have been drawn from (points) E, F, G, and H, (respectively), at right-angles to the plane of square $EFGH$. And let EK, FL, GM, and HN, equal to one of EF, FG, GH, and HE, have been cut off from EK, FL, GM, and HN, respectively. And let KL, LM, MN, and NK have been joined. Thus, a cube contained by six equal squares has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

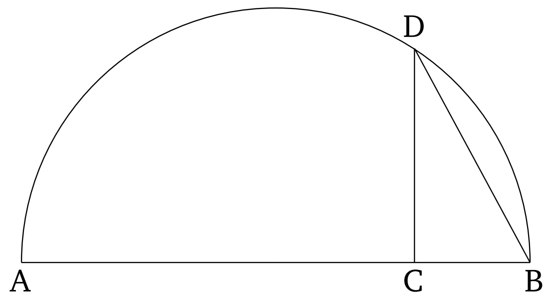


For let KG and EG have been joined. And since angle KEG is a right-angle—on account of KE also being at right-angles to the plane EG, and manifestly also to the straight-line EG [Def. 11.3]—the semi-circle drawn on KG will thus also pass through point E. Again, since GF is at right-angles to each of FL and FE, GF is thus also at right-angles to the plane FK [Prop. 11.4]. Hence, if we also join FK then GF will also be at right-angles to FK. And, again, on account of this, the semi-circle drawn on GK will also pass through point F. Similarly, it will also pass through the remaining (angular) points of the cube. So, if KG remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then the cube will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since GF is equal to FE, and the angle at F is a right-angle, the (square) on EG is thus double the (square) on EF [Prop. 1.47]. And EF (is) equal to EK. Thus, the (square) on EG is double the (square) on EK. Hence, the (sum of the squares) on GE and EK—that is to say, the (square) on GK [Prop. 1.47]—is three times the (square) on EK. And since AB is three times BC, and as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9], the (square) on AB (is) thus three times the (square) on BD. And the (square) on GK was also shown (to be) three times the (square) on KE. And KE was made equal to DB. Thus, KG (is) also equal to AB. And AB is the radius of the given sphere. Thus, KG is also equal to the diameter of the given sphere.

Thus, the cube has been enclosed by the given sphere. And it has simultaneously been shown that the square on the diameter of the sphere is three times the (square) on the side of the cube.† (Which is) the very thing it was required to show. † If the radius of the sphere is unity then the side of the cube is $\sqrt{4/3}$.

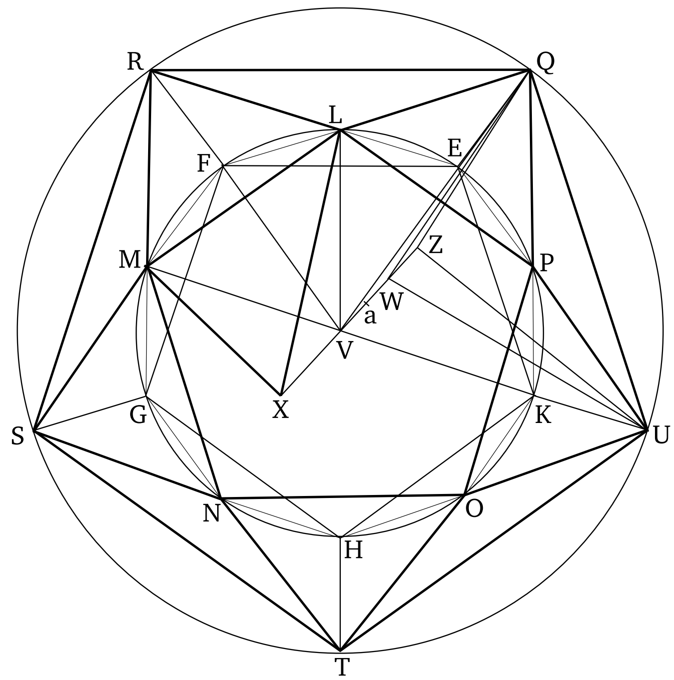
Proposition 16

To construct an icosahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the icosahedron is that irrational (straight-line) called minor.



Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is four times CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB. And let the straight-line CD have been drawn from C at right-angles to AB. And let DB have been joined. And let the circle $EFGHK$ be set down, and let its radius be equal to DB. And let the equilateral and equiangular pentagon $EFGHK$ have been inscribed in circle $EFGHK$ [Prop. 4.11]. And let the circumferences EF, FG, GH, HK, and KE have been cut in half at points L, M, N, O, and P (respectively). And let LM, MN, NO, OP, PL, and EP have been joined. Thus, pentagon $LMNOP$ is also equilateral, and EP (is) the side of the decagon (inscribed in the circle). And let the straight-lines EQ, FR, GS, HT, and KU, which are equal to the radius of circle $EFGHK$, have been set up at right-angles to the plane of the circle, at points E, F, G, H, and K (respectively). And let QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, and PQ have been joined.

And since EQ and KU are each at right-angles to the same plane, EQ is thus parallel to KU [Prop. 11.6]. And it is also equal to it. And straight-lines joining equal and parallel (straight-lines) on the same side are (themselves) equal and parallel [Prop. 1.33]. Thus, QU is equal and parallel to EK. And EK (is the side) of an equilateral pentagon (inscribed in circle $EFGHK$). Thus, QU (is) also the side of an equilateral pentagon inscribed in circle $EFGHK$. So, for the same (reasons), QR, RS, ST, and TU are also the sides of an equilateral pentagon inscribed in circle $EFGHK$. Pentagon $QRSTU$ (is) thus equilateral. And side QE is (the side) of a hexagon (inscribed in circle $EFGHK$), and EP (the side) of a decagon, and (angle) QEP is a right-angle, thus QP is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), PU is also the side of a pentagon. And QU is also (the side) of a pentagon. Thus, triangle QPU is equilateral. So, for the same (reasons), (triangles) QLR, RMS, SNT, and TOU are each also equilateral. And since QL and QP were each shown (to be the sides) of a pentagon, and LP is also (the side) of a pentagon, triangle QLP is thus equilateral. So, for the same (reasons), triangles LRM, MSN, NTO, and OUP are each also equilateral.



Let the center, point V, of circle $EFGHK$ have been found [Prop. 3.1]. And let VZ have been set up, at (point) V, at right-angles to the plane of the circle. And let it have been produced on the other side (of the circle), like VX. And let VW have been cut off (from XZ so as to be equal to the side) of a hexagon, and each of VX and WZ (so as to be equal to the side) of a decagon. And let QZ, QW, UZ, EV, LV, LX, and XM have been joined.

And since VW and QE are each at right-angles to the plane of the circle, VW is thus parallel to QE [Prop. 11.6]. And they are also equal. EV and QW are thus equal and parallel (to one another) [Prop. 1.33]. And EV (is the side) of a hexagon. Thus, QW (is) also (the side) of a hexagon. And since QW is (the side) of a hexagon, and WZ (the side) of a decagon, and angle QWZ is a right-angle [Def. 11.3, Prop. 1.29], QZ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), UZ is also (the side) of a pentagon—inasmuch as, if we join VK and WU then they will be equal and opposite. And VK, being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, WU (is) also the side of a hexagon. And WZ (is the side) of a decagon, and (angle) UWZ (is) a right-angle. Thus, UZ (is the side) of a pentagon [Prop. 13.10]. And QU is also (the side) of a pentagon. Triangle QUZ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines QR, RS, ST, and TU, and apexes the point Z, are also equilateral. Again, since VL (is the side) of a hexagon, and VX (the side) of a decagon, and angle LVX is a right-angle, LX is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join MV, which is (the side) of a hexagon, MX is also inferred (to be the side) of a pentagon. And LM is also (the side) of a pentagon. Thus, triangle LMX is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) MN, NO, OP, and PL, and apexes the point X, are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since VW is (the side) of a hexagon, and WZ (the side) of a decagon, VZ has thus been cut in extreme and mean ratio at W, and VW is its greater piece [Prop. 13.9]. Thus, as ZV is to VW, so VW (is) to WZ. And VW (is) equal to VE, and WZ to VX. Thus, as ZV is to VE, so EV (is) to VX. And angles ZVE and EVX are right-angles. Thus, if we join straight-line EZ then angle XEZ will be a right-angle, on account of the similarity of triangles XEZ and VEZ. [Prop. 6.8]. So, for the same (reasons), since as ZV is to VW, so VW (is) to WZ, and

ZV (is) equal to XW, and VW to WQ, thus as XW is to WQ, so QW (is) to WZ. And, again, on account of this, if we join QX then the angle at Q will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on XZ will also pass through Q [Prop. 3.31]. And if XZ remains fixed, and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through (point) Q, and (through) the remaining (angular) points of the icosahedron. And the icosahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let VW have been cut in half at a. And since the straight-line VZ has been cut in extreme and mean ratio at W, and ZW is its lesser piece, then the square on ZW added to half of the greater piece, Wa, is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on Za is five times the (square) on aW. And ZX is double Za, and VW double aW. Thus, the (square) on ZX is five times the (square) on WV. And since AC is four times CB, AB is thus five times BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is five times the (square) on BD. And the (square) on ZX was also shown (to be) five times the (square) on VW. And DB is equal to VW. For each of them is equal to the radius of circle $EFGHK$. Thus, AB (is) also equal to XZ. And AB is the diameter of the given sphere. Thus, XZ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

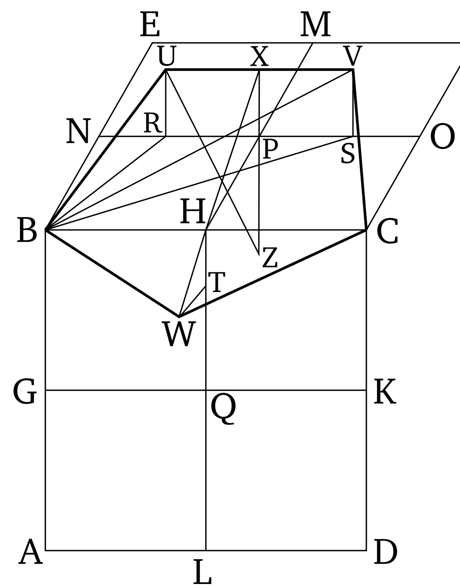
So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle $EFGHK$, the radius of circle $EFGHK$ is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon $EFGHK$ is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle.† † If the radius of the sphere is unity then the radius of the circle is $2/\sqrt{5}$, and the sides of the hexagon, decagon, and pentagon/icosahedron are $2/\sqrt{5}$, $1-1/\sqrt{5}$, and $(1/\sqrt{5})\sqrt{10-2\sqrt{5}}$, respectively.

Proposition 17

To construct a dodecahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.



Let two planes of the aforementioned cube [Prop. 13.15], $ABCD$ and $CBEF$, (which are) at right-angles to one another, be laid out. And let the sides AB, BC, CD, DA, EF, EB, and FC have each been cut in half at points G, H, K, L, M, N, and O (respectively). And let GK, HL, MH, and NO have been joined. And let NP, PO, and HQ have each been cut in extreme and mean ratio at points R, S, and T (respectively). And let their greater pieces be RP, PS, and TQ (respectively). And let RU, SV, and TW have been set up on the exterior side of the cube, at points R, S, and T (respectively), at right-angles to the planes of the cube. And let them be made equal to RP, PS, and TQ. And let UB, BW, WC, CV, and VU have been joined.

I say that the pentagon $UBWCV$ is equilateral, and in one plane, and, further, equiangular. For let RB, SB, and VB have been joined. And since the straight-line NP has been cut in extreme and mean ratio at R, and RP is the greater piece, the (sum of the squares) on PN and NR is thus three times the (square) on RP [Prop. 13.4]. And PN (is) equal to NB, and PR to RU. Thus, the (sum of the squares) on BN and NR is three times the (square) on RU. And the (square) on BR is equal to the (sum of the squares) on BN and NR [Prop. 1.47]. Thus, the (square) on BR is three times the (square) on RU. Hence, the (sum of the squares) on BR and RU is four times the (square) on RU. And the (square) on BU is equal to the (sum of the squares) on BR and RU [Prop. 1.47]. Thus, the (square) on BU is four times the (square) on UR. Thus, BU is double RU. And VU is also double UR, inasmuch as SR is also double PR—that is to say, RU. Thus, BU (is) equal to UV. So, similarly, it can be shown that each of BW, WC, CV is equal to each of BU and UV. Thus, pentagon $BUVCW$ is equilateral. So, I say that it is also in one plane. For let PX have been drawn from P, parallel to each of RU and SV, on the exterior side of the cube. And let XH and HW have been joined. I say that XHW is a straight-line. For since HQ has been cut in extreme and mean ratio at T, and QT is its greater piece, thus as HQ is to QT, so QT (is) to TH. And HQ (is) equal to HP, and QT to each of TW and PX. Thus, as HP is to PX, so WT (is) to TH. And HP is parallel to TW. For of each of them is at right-angles to the plane BD [Prop. 11.6]. And TH (is parallel) to PX. For each of them is at right-angles to the plane BF [Prop. 11.6]. And if two triangles, like XPH and HTW, having two sides proportional to two sides, are placed together at a single angle such that their corresponding sides are also parallel then the remaining sides will be straight-on (to one another) [Prop. 6.32]. Thus, XH is straight-on to HW. And every straight-line is in one plane [Prop. 11.1]. Thus, pentagon $UBWCV$ is in one plane.

So, I say that it is also equiangular.

For since the straight-line NP has been cut in extreme and mean ratio at R, and PR is the greater piece [thus as the sum of NP and PR is to PN, so NP (is) to PR], and PR (is) equal to PS [thus as SN is to NP, so NP (is) to PS], NS has thus also been cut in extreme and mean ratio at P, and NP is the greater piece [Prop. 13.5].

Thus, the (sum of the squares) on NS and SP is three times the (square) on NP [Prop. 13.4]. And NP (is) equal to NB, and PS to SV. Thus, the (sum of the) squares on NS and SV is three times the (square) on NB. Hence, the (sum of the squares) on VS, SN, and NB is four times the (square) on NB. And the (square) on SB is equal to the (sum of the squares) on SN and NB [Prop. 1.47]. Thus, the (sum of the squares) on BS and SV—that is to say, the (square) on BV [for angle VSB (is) a right-angle]—is four times the (square) on NB [Def. 11.3, Prop. 1.47]. Thus, VB is double BN. And BC (is) also double BN. Thus, BV is equal to BC. And since the two (straight-lines) BU and UV are equal to the two (straight-lines) BW and WC (respectively), and the base BV (is) equal to the base BC, angle BUV is thus equal to angle BWC [Prop. 1.8]. So, similarly, we can show that angle UVC is equal to angle BWC. Thus, the three angles BWC, BUV, and UVC are equal to one another. And if three angles of an equilateral pentagon are equal to one another then the pentagon is equiangular [Prop. 13.7]. Thus, pentagon $BUVCW$ is equiangular. And it was also shown (to be) equilateral. Thus, pentagon $BUVCW$ is equilateral and equiangular, and it is on one of the sides, BC, of the cube. Thus, if we make the same construction on each of the twelve sides of the cube then some solid figure contained by twelve equilateral and equiangular pentagons will have been constructed, which is called a dodecahedron.

So, it is necessary to enclose it in the given sphere, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

For let XP have been produced, and let (the produced straight-line) be XZ. Thus, PZ meets the diameter of the cube, and they cut one another in half. For, this has been proved in the penultimate theorem of the eleventh book [Prop. 11.38]. Let them cut (one another) at Z. Thus, Z is the center of the sphere enclosing the cube, and ZP (is) half the side of the cube. So, let UZ have been joined. And since the straight-line NS has been cut in extreme and mean ratio at P, and its greater piece is NP, the (sum of the squares) on NS and SP is thus three times the (square) on NP [Prop. 13.4]. And NS (is) equal to XZ, inasmuch as NP is also equal to PZ, and XP to PS. But, indeed, PS (is) also (equal) to XU, since (it is) also (equal) to RP. Thus, the (sum of the squares) on ZX and XU is three times the (square) on NP. And the (square) on UZ is equal to the (sum of the squares) on ZX and XU [Prop. 1.47]. Thus, the (square) on UZ is three times the (square) on NP. And the square on the radius of the sphere enclosing the cube is also three times the (square) on half the side of the cube. For it has previously been demonstrated (how to) construct the cube, and to enclose (it) in a sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube [Prop. 13.15]. And if the (square on the) whole (is three times) the (square on the) whole, then the (square on the) half (is) also (three times) the (square on the) half. And NP is half of the side of the cube. Thus, UZ is equal to the radius of the sphere enclosing the cube. And Z is the center of the sphere enclosing the cube. Thus, point U is on the surface of the sphere. So, similarly, we can show that each of the remaining angles of the dodecahedron is also on the surface of the sphere. Thus, the dodecahedron has been enclosed by the given sphere.

So, I say that the side of the dodecahedron is that irrational straight-line called an apotome.

For since RP is the greater piece of NP, which has been cut in extreme and mean ratio, and PS is the greater piece of PO, which has been cut in extreme and mean ratio, RS is thus the greater piece of the whole of NO, which has been cut in extreme and mean ratio. [Thus, since as NP is to PR, (so) PR (is) to RN, and (the same is also true) of the doubles. For parts have the same ratio as similar multiples (taken in corresponding order) [Prop. 5.15]. Thus, as NO (is) to RS, so RS (is) to the sum of NR and SO. And NO (is) greater than RS. Thus, RS (is) also greater than the sum of NR and SO [Prop. 5.14]. Thus, NO has been cut in extreme and mean ratio, and RS is its greater piece.] And RS (is) equal to UV. Thus, UV is the greater piece of NO, which has been cut in extreme and mean ratio. And since the diameter of the sphere is rational, and the square on it is three times the (square) on the side of the cube, NO, which is the side of the cube, is thus rational. And if a rational (straight)-line is cut in extreme and mean ratio then each of the pieces is the irrational (straight-line called) an apotome.

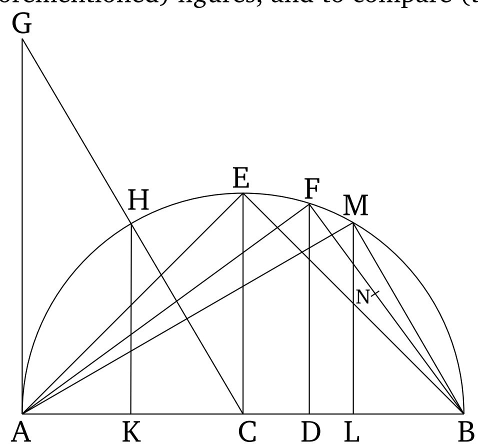
Thus, UV, which is the side of the dodecahedron, is the irrational (straight-line called) an apotome [Prop. 13.6].

Corollary

So, (it is) clear, from this, that the side of the dodecahedron is the greater piece of the side of the cube, when it is cut in extreme and mean ratio.† (Which is) the very thing it was required to show. † If the radius of the circumscribed sphere is unity then the side of the cube is $\sqrt{4/3}$, and the side of the dodecahedron is $(1/3)\,(\sqrt{15}-\sqrt{3})$.

Proposition 18

To set out the sides of the five (aforementioned) figures, and to compare (them) with one another.†



Let the diameter, AB, of the given sphere be laid out. And let it have been cut at C, such that AC is equal to CB, and at D, such that AD is double DB. And let the semi-circle AEB have been drawn on AB. And let CE and DF have been drawn from C and D (respectively), at right-angles to AB. And let AF, FB, and EB have been joined. And since AD is double DB, AB is thus triple BD. Thus, via conversion, BA is one and a half times AD. And as BA (is) to AD, so the (square) on BA (is) to the (square) on AF [Def. 5.9]. For triangle AFB is equiangular to triangle AFD [Prop. 6.8]. Thus, the (square) on BA is one and a half times the (square) on AF. And the square on the diameter of the sphere is also one and a half times the (square) on the side of the pyramid [Prop. 13.13]. And AB is the diameter of the sphere. Thus, AF is equal to the side of the pyramid.

Again, since AD is double DB, AB is thus triple BD. And as AB (is) to BD, so the (square) on AB (is) to the (square) on BF [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is three times the (square) on BF. And the square on the diameter of the sphere is also three times the (square) on the side of the cube [Prop. 13.15]. And AB is the diameter of the sphere. Thus, BF is the side of the cube.

And since AC is equal to CB, AB is thus double BC. And as AB (is) to BC, so the (square) on AB (is) to the (square) on BE [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BE. And the square on the diameter of the sphere is also double the (square) on the side of the octagon [Prop. 13.14]. And AB is the diameter of the given sphere. Thus, BE is the side of the octagon.

So let AG have been drawn from point A at right-angles to the straight-line AB. And let AG be made equal to AB. And let GC have been joined. And let HK have been drawn from H, perpendicular to AB. And since GA is double AC. For GA (is) equal to AB. And as GA (is) to AC, so HK (is) to KC [Prop. 6.4]. HK (is) thus also double KC. Thus, the (square) on HK is four times the (square) on KC. Thus, the (sum of the squares) on HK and KC, which is the (square) on HC [Prop. 1.47], is five times the (square) on KC. And HC (is) equal to CB. Thus, the (square) on BC (is) five times the (square) on CK. And since AB is double CB, of which AD is double DB, the remainder BD is thus double the remainder DC. BC (is) thus triple CD. The (square) on BC (is) thus nine times the (square) on CD. And the (square) on BC (is) five times the (square) on CK. Thus, the (square) on CK (is) greater than the (square) on CD. CK is thus greater than CD. Let CL be made equal to CK. And let LM have been drawn from L at right-angles to AB. And let MB have been joined. And since the (square) on BC is five times the (square) on CK, and AB is double BC, and KL double CK, the (square) on AB is thus five times the (square) on KL. And the square on the diameter of the sphere is also five times the (square) on the radius of the circle from which the icosahedron has been described [Prop. 13.16 corr.]. And AB is the diameter of the sphere. Thus, KL is the radius of the circle from which the icosahedron has been described. Thus, KL is (the side) of the hexagon (inscribed) in the aforementioned circle [Prop. 4.15 corr.]. And since the diameter of the sphere is composed of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the aforementioned circle, and AB is the diameter of the sphere, and KL the side of the hexagon, and AK (is) equal to LB, thus AK and LB are each sides of the decagon inscribed in the circle from which the icosahedron has been described. And since LB is (the side) of the decagon. And ML (is the side) of the hexagon—for (it is) equal to KL, since (it is) also (equal) to HK, for they are equally far from the center. And HK and KL are each double KC. MB is thus (the side) of the pentagon (inscribed in the circle) [Props. 13.10, 1.47]. And (the side) of the pentagon is (the side) of the icosahedron [Prop. 13.16]. Thus, MB is (the side) of the icosahedron.

And since FB is the side of the cube, let it have been cut in extreme and mean ratio at N, and let NB be the greater piece. Thus, NB is the side of the dodecahedron [Prop. 13.17 corr.].

And since the (square) on the diameter of the sphere was shown (to be) one and a half times the square on the side, AF, of the pyramid, and twice the square on (the side), BE, of the octagon, and three times the square on (the side), FB, of the cube, thus, of whatever (parts) the (square) on the diameter of the sphere (makes) six, of such (parts) the (square) on (the side) of the pyramid (makes) four, and (the square) on (the side) of the octagon three, and (the square) on (the side) of the cube two. Thus, the (square) on the side of the pyramid is one and a third times the square on the side of the octagon, and double the square on (the side) of the cube. And the (square) on (the side) of the octahedron is one and a half times the square on (the side) of the cube. Therefore, the aforementioned sides of the three figures—I mean, of the pyramid, and of the octahedron, and of the cube—are in rational ratios to one another. And (the sides of) the remaining two (figures)—I mean, of the icosahedron, and of the dodecahedron—are neither in rational ratios to one another, nor to the (sides) of the aforementioned (three figures). For they are irrational (straight-lines): (namely), a minor [Prop. 13.16], and an apotome [Prop. 13.17].

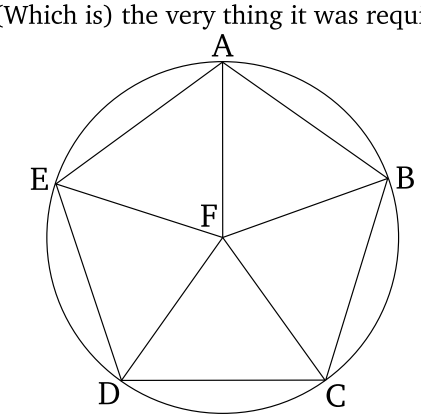
(And), we can show that the side, MB, of the icosahedron is greater that the (side), NB, or the dodecahedron, as follows.

For, since triangle FDB is equiangular to triangle FAB [Prop. 6.8], proportionally, as DB is to BF, so BF (is) to BA [Prop. 6.4]. And since three straight-lines are (continually) proportional, as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9, Prop. 6.20 corr.]. Thus, as DB is to BA, so the (square) on DB (is) to the (square) on BF. Thus, inversely, as AB (is) to BD, so the (square) on FB (is) to the (square) on BD. And AB (is) triple BD. Thus, the (square) on FB (is) three times the (square) on BD. And the (square) on AD is also four times the (square) on DB. For AD (is) double DB. Thus, the (square) on AD (is) greater than the (square) on FB. Thus, AD (is) greater than FB. Thus, AL is much greater than FB. And KL is the greater piece of AL, which is cut in extreme and mean ratio—inasmuch as LK is (the side) of the hexagon, and KA (the side) of the decagon [Prop. 13.9]. And NB is the greater piece of FB, which is cut in extreme and mean ratio. Thus, KL (is) greater than NB. And KL (is) equal to LM. Thus, LM (is) greater than NB [and MB is greater than LM]. Thus, MB, which is (the side) of the icosahedron, is much greater than NB, which is (the side) of the dodecahedron. (Which is) the very thing it was required to show. † If the radius of the given sphere is unity then the sides of the pyramid (*i.e.*, tetrahedron), octahedron, cube, icosahedron, and dodecahedron, respectively, satisfy the following inequality: $\sqrt{8/3} > \sqrt{2} > \sqrt{4/3} > (1/\sqrt{5})\sqrt{10-2\sqrt{5}} > (1/3)(\sqrt{15}-\sqrt{3})$.

So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.

For a solid angle cannot be constructed from two triangles, or indeed (two) planes (of any sort) [Def. 11.11]. And (the solid angle) of the pyramid (is constructed) from three (equiangular) triangles, and (that) of the octahedron from four (triangles), and (that) of the icosahedron from (five) triangles. And a solid angle cannot be (made) from six equilateral and equiangular triangles set up together at one point. For, since the angles of a equilateral triangle are (each) two-thirds of a right-angle, the (sum of the) six (plane) angles (containing the solid angle) will be four right-angles. The very thing (is) impossible. For every solid angle is contained by (plane angles whose sum is) less than four right-angles [Prop. 11.21]. So, for the same (reasons), a solid angle cannot be constructed from more than six plane angles (equal to two-thirds of a right-angle) either. And the (solid) angle of a cube is contained by three squares. And (a solid angle contained) by four (squares is) impossible. For, again, the (sum of the plane angles containing the solid angle) will be four right-angles. And (the solid angle) of a dodecahedron (is contained) by three equilateral and equiangular pentagons. And (a solid angle contained) by four (equiangular pentagons is) impossible. For, the angle of an equilateral pentagon being one and one-fifth of right-angle, four (such) angles will be greater (in sum) than four right-angles. The very thing (is) impossible. And, on account of the same absurdity, a solid angle cannot be constructed from any other (equiangular) polygonal figures either.

Thus, beside the five aforementioned figures, no other solid figure can be constructed (which is) contained by equilateral and equiangular (planes). (Which is) the very thing it was required to show.



Lemma

It can be shown that the angle of an equilateral and equiangular pentagon is one and one-fifth of a right-angle, as follows.

For let $ABCDE$ be an equilateral and equiangular pentagon, and let the circle $ABCDE$ have been circumscribed about it [Prop. 4.14]. And let its center, F, have been found [Prop. 3.1]. And let FA, FB, FC, FD, and FE have been joined. Thus, they cut the angles of the pentagon in half at (points) A, B, C, D, and E [Prop. 1.4]. And since the five angles at F are equal (in sum) to four right-angles, and are also equal (to one another), (any) one of them, like AFB, is thus one less a fifth of a right-angle. Thus, the (sum of the) remaining (angles in triangle ABF), FAB and ABF, is one plus a fifth of a right-angle [Prop. 1.32]. And FAB (is) equal to FBC. Thus, the whole angle, ABC, of the pentagon is also one and one-fifth of a right-angle. (Which is) the very thing it was required to show.

GREEK-ENGLISH LEXICON

ABBREVIATIONS: *act* - active; *adj* - adjective; *adv* - adverb; *conj* - conjunction; *fut* - future; *gen* - genitive; *imperat* - imperative; *impf* - imperfect; *ind* - indeclinable; *indic* - indicative; *intr* - intransitive; *mid* - middle; *neut* - neuter; *no* - noun; *par* - particle; *part* - participle; *pass* - passive; *perf* - perfect; *pre* - preposition; *pres* - present; *pro* - pronoun; *sg* - singular; *tr* - transitive; *vb* - verb.

, : *vb*, lead, draw (a line).

: *adj*, impossible.

: *adv*, always, for ever.

: *vb*, grasp.

: *vb*, postulate.

: *no*, postulate.

: *adj*, analogous, consequent on, in conformity with.

: *adj*, outermost, end, extreme.

: *conj*, but, otherwise.

: *adj*, irrational.

: *adv*, at once, at the same time, together.

: *adj*, obtuse-angled; , *no*, obtuse angle.

: *adj*, obtuse.

: *pro*, both.

: *vb*, describe (a figure); see .

: *no*, proportion, (geometric) progression.

: *adj*, proportional.

: *adv*, inverse(ly).

: *vb*, fill up.

: *vb*, turn upside down, convert (ratio); see .

: *no*, turning upside down, conversion (of ratio).

: *vb*, take away in turn; see .

: *vb*, set up; see .

: *adj*, unequal, uneven.

: *vb*, be reciprocally proportional; see .

: *vb*, axis.

: *adv*, once.

: *adj*, quite all, the whole.

: *adj*, infinite.

: *ind*, opposite.

: *vb*, be far from, be away from; see .

: *adj*, without breadth.

: *no*, proof.

: *vb*, re-establish, restore; see .

: *vb*, take from, subtract from, cut off from; see .

: *vb*, cut off, subtend.

: *no*, piece cut off, segment.

: *vb*, piece cut off, apotome.

—, , — : *vb*, touch, join, meet.

: *adj*, further off.

: *par*, thus, as it seems (inferential).

: *no*, number.

: *adv*, an even number of times.

: *adj*, having a even number of sides.

: *vb*, rule; *mid.*, begin.

: *adj*, incommensurable.

: *adj*, not touching, not meeting.

: *adj*, even, perfect.

: *adj*, uncut.

: *adj*, absurd, paradoxical.

: *adv*, immediately, obviously.

: *vb*, take from, subtract from, cut off from; see .

: *no*, point of contact.

: *no*, depth, height.

—, — : *vb*, walk; *perf*, stand (of angle).

: *vb*, throw.

: *no*, base (of a triangle).

: *conj*, for (explanatory).

— : *vb*, happen, become.

: *no*, gnomon.

: *no*, line.

/ : *vb*, draw (a figure).

: *no*, angle.

: *vb*, be necessary; , it is necessary; , it was necassary; , being necessary.

: *vb*, show, demonstrate.

: *ind*, one must show.

: *no*, proof.

: *adj*, ten-sided; , *no*, decagon.

—, : *vb*, receive, accept.

: *conj*, so (explanatory).

: *ind*, quite clear, manifest.

: *adj*, clear.

: *adv*, manifestly.

: *vb*, carry over, draw through, draw across; see .

: *adj*, diagonal.

: *vb*, leave an interval between.

: *adj*, diametrical; , *no*, diameter, diagonal.

: *no*, division, separation.

: *vb*, divide (in two); , *adj*, separated (ratio); see .

: *no*, radius.

: *vb*, differ; see .

: *vb*, give.

: *adj*, two-thirds.

: *vb*, double.

: *adj*, double, twofold.

: *adj*, double, twofold.

: *adj*, double.

: *adv*, twice.

: *adv*, in two, in half.

: *no*, point of bisection.

: *no*, the number two, dyad.

: *vb*, be able, be capable, generate, square, be when squared; , *no*, square-root (of area)—*i.e.*, straight-line whose square is equal to a given area.

: *no*, power (usually 2nd power when used in mathematical sence, hence), square.

: *adj*, possible.

: *adj*, twelve-sided.

: *adj*, of him/her/it/self, his/her/its/own.

: *adj*, nearer, nearest.

: *vb*, inscribe; see .

: *no*, figure, form, shape.

: *adj*, twenty-sided.

// : *vb*, say, speak; *per pass part*, , *adj*, said, aforementioned.

... : *ind*, either ... or.

: *pro*, each, every one.

: *pro*, each (of two).

: *vb*, produce (a line); see .

: *vb*, set out.

: *vb*, be set out, be taken; see .

: *vb*, set out; see .

: *pre* + *gen*, outside, external.

: *adj*, less, lesser.

: *adj*, least.

: *vb*, be less than, fall short of.

: *vb*, meet (of lines), fall on; see .

: *adv*, in front.

: *adv*, alternate(ly).

: *vb*, insert; *perf indic pass 3rd sg*, .

: *vb*, admit, allow.

: *ind*, on account of, for the sake of.

: *adj*, nine-fold, nine-times.

: *no*, notion.

: *vb*, encompass; see .

: see .

: *pre* + *gen*, inside, interior, within, internal.

: *adj*, hexagonal; , *no*, hexagon.

: *adj*, sixfold.

: *adv*, in order, successively, consecutively.

: *adv*, outside, extrinsic.

: *adv*, above.

: *no*, point of contact.

: *conj*, since (causal).

: *ind*, inasmuch as, seeing that.

, —, : *vb*, join (by a line).

: *vb*, conclude.

: *vb*, think of, contrive.

: *adj*, level, flat, plane; , *no*, plane.

: *vb*, investigate.

: *no*, inspection, investigation.

: *vb*, put upon, enjoin; , *no*, the (thing) prescribed; see .

: *adj*, one and a third times.

: *no*, surface.

: *vb*, follow.

, —, — : *vb*, come, go.

: *adj*, outermost, uttermost, last.

: *adj*, oblong; , *no*, rectangle.

: *adj*, other (of two).

: *par*, yet, still, besides.

: *adj*, rectilinear; , *no*, rectilinear figure.

: *adj*, straight; , *no*, straight-line; , in a straight-line, straight-on.

: *vb*, find.

: *vb*, bind to; *mid*, touch; , *no*, tangent; see .

: *vb*, coincide; *pass*, be applied.

: *adv*, in order, adjacent.

: *vb*, set, stand, place upon; see .

, — : *vb*, have.

, —, : *vb*, lead.

: *ind*, already, now.

, —, —, —, — : *vb*, have come, be present.

: *no*, semi-circle.

: *adj*, containing one and a half, one and a half times.

: *adj*, half.

= + : *conj*, than, than indeed.

... : *par*, surely, either ... or; in fact, either ... or.

: *no*, placing, setting, position.

: *no*, theorem.

: *adj*, one's own.

: *adv*, the same number of times; , the same multiples, equal multiples.

: *adj*, equiangular.

: *adj*, equilateral.

: *adj*, equal in number.

: *adj*, equal; , equally, evenly.

: *adj*, isosceles.

, —, —, : *vb tr*, stand (something).

: *vb intr*, stand up (oneself); Note: perfect *I have stood up* can be taken to mean present *I am standing*.

: *adj*, of equal height.
: *ind*, according as, just as.
: *adj*, perpendicular.
: *adv*, on the whole, in general.
: *vb*, call.
= .
= : *ind*, even if, and if.
: *no*, diagram, figure.
: *vb*, describe/draw, inscribe (a figure); see .
: *vb*, follow after.
: *vb*, leave behind; see ; , *no*, remainder.
: *adj*, in succession, in corresponding order.
: *vb*, measure (exactly).
: *vb*, come to, arrive at.
: *vb*, furnish, construct.
, —, —, —, — : *vb*, have been placed, lie, be made; see .
: *no*, center.
: *vb*, break off, inflect.
: *vb*, lean, incline.
: *no*, inclination, bending.
: *adj*, hollow, concave.
: *no*, top, summit, apex; , vertically opposite (of angles).
: *vb*, judge.
: *no*, cube.
: *no*, circle.
: *no*, cylinder.
: *adj*, convex.
: *no*, cone.
: *vb*, take.
: *vb*, say; *pres pass part*, , *adj*, so-called; see .
: *vb*, leave, leave behind.
: *no*, diminutive of .
: *no*, lemma.
: *no*, taking, catching.
: *no*, ratio, proportion, argument.
: *adj*, remaining.
, —, — : *vb*, learn.
: *no*, magnitude, size.
: *adj*, greater.
, —, — : *vb*, stay, remain.
: *no*, part, direction, side.
: *adj*, middle, mean, medial; , bimedial.
: *vb*, take up.
: *adv*, between.
: *adj*, raised off the ground.
: *vb*, measure.
: *no*, measure.
: *adj*, not even one, (neut.) nothing.
: *adv*, never.
: *pro*, neither (of two).
: *no*, length.
: *par*, truely, indeed.
: *no*, unit, unity.
: *adj*, unique.
: *adv*, uniquely.
: *adj*, alone.
, —, : *vb*, apprehend, conceive.
: *pre*, such as, of what sort.
: *adj*, eight-sided.
: *adj*, whole.
: *adj*, of the same kind.
: *adj*, similar.
: *adj*, similar in number.
: *adj*, similarly arranged.
: *no* similarity.
: *adv*, similarly.
: *adj*, corresponding, homologous.
: *adj*, ranged in the same row or line.
: *adj*, having the same name.
: *no*, name; , binomial.
: *adj*, acute-angled; , *no*, acute angle.
: *adj*, acute.
= + : *adj*, of whatever kind, any kind whatsoever.
: *pro*, as many, as many as.
= + + + : *adj*, of whatever number, any number whatsoever.
= + : *adj*, of whatever number, any number whatsoever.
: *pro*, either (of two), which (of two).
: *no*, rectangle, right-angle.
: *adj*, straight, right-angled, perpendicular; , at right-angles.
: *no*, boundary, definition, term (of a ratio).
= + + + : *ind*, any number whatsoever.
: *ind*, as many times as, as often as.
: *pro*, as many times as.
: *pro*, as many as.
: *pro*, the very man who, the very thing which.
: *pro*, anyone who, anything which.
: *adv*, when, whenever.
: *ind*, whatsoever.
: *pro*, not one, nothing.
: *pro*, not either.
: see .
: *ind*, nothing.
: *adv*, therefore, in fact.

: *adv*, thusly, in this case.
: *adv*, back, again.
: *adv*, in all ways.
: *prep* + *acc*, parallel to.
: *vb*, apply (a figure); see .
: *no*, application.
: *vb*, lie beside, apply (a figure); see .
—, , —, — : *vb*, miss, fall awry.
: *adj*, with parallel surfaces; , *no*, parallelepiped.
: *adj*, bounded by parallel lines; , *no*, parallelogram.
: *adj*, parallel; , *no*, parallel, parallel-line.
: *no*, complement (of a parallelogram).
: *adj*, penultimate.
: *prep* + *gen*, except.
: *vb*, insert; see .
, —, — : *vb*, suffer.
: *adj*, pentagonal; , *no*, pentagon.
: *adj*, five-fold, five-times.
: *no*, fifteen-sided figure.
: *adj*, finite, limited; see .
, —, : *vb*, bring to end, finish, complete; *pass*, be finite.
: *no*, end, extremity.
, —, —, —, —, — : *vb*, bring to an end.
: *vb*, circumscribe; see .
: *vb*, encompass, surround, contain, comprise; see .
: *vb*, enclose; see .
: *adv*, an odd number of times.
: *adj*, odd.
: *no*, circumference.
: *vb*, carry round; see .
: *no*, magnitude, size.
, —, — : *vb*, fall.
: *no*, breadth, width.
: *adj*, more, several.
: *no*, side.
: *no*, great number, multitude, number.
: *adv* & *prep* + *gen*, more than.
: *adj*, of a certain nature, kind, quality, type.
: *vb*, multiply.
: *no*, multiplication.
: *no*, multiple.
: *adj*, polyhedral; , *no*, polyhedron.
: *adj*, polygonal; , *no*, polygon.
: *adj*, multilateral.
: *no*, corollary.
: *ind*, at some time.
: *no*, prism.
: *vb*, step forward, advance.
: *vb*, show previously; see .
: *vb*, set forth beforehand; see .
: *vb*, say beforehand; *perf pass part*, , *adj*, aforementioned; see
: *vb*, fill up, complete.
: *vb*, complete (tracing of); see .
: *vb*, fit to, attach to.
: *vb*, produce (a line); see .
: *vb*, find besides, find; see .
: *vb*, add.
: *vb*, set before, prescribe; see .
: *vb*, be laid on, have been added to; see .
: *vb*, fall on, fall toward, meet; see .
: *no*, proposition.
: *vb*, prescribe, enjoin; , *no*, the thing prescribed; see .
: *vb*, add; see .
: *adj*, first (comparative), before, former.
: *vb*, assign; see .
: *vb*, go/come forward, advance.
: *adj*, first, prime.
: *no*, pyramid.
: *adj*, expressible, rational.
: *adj*, rhomboidal; , *no*, romboid.
no, rhombus.
: *no*, point.
: *adj*, scalene.
: *adj*, solid; , *no*, solid, solid body.
: *no*, element.
—, : *vb*, turn.
: *vb*, lie together, be the sum of, be composed;
, *adj*, composed (ratio), compounded; see .
: *vb*, compare; see .
: *vb*, come to pass, happen, follow; see .
: *vb*, throw together, meet; see .
: *adj*, commensurable.
: *no*, sum, whole.
: *vb*, meet together (of lines); see .
: *vb*, complete (a figure), fill in.
: *vb*, conclude, infer; see .
: *adj*, both together; , *no*, sum (of two things).
: *no*, demonstrate together; see .
: *no*, point of junction.
: *no*, two together, in pairs.
: *adj*, continuous; , continuously.
: *no*, putting together, composition.

: *adj*, composite.

: *vb*, construct (a figure), set up together; *perf imperat pass 3rd sg*, ; see .

: *vb*, put together, add together, compound (ratio); see .

: *no*, state, condition.

: *no*, figure.

: *no*, sphere.

: *no*, arrangement, order.

—, —, : *vb*, stir, trouble, disturbe; , *adj*, disturbed, perturbed.

: *vb*, arrange, draw up.

: *adj*, perfect.

: *vb*, cut; *pres/fut indic act 3rd sg*, .

: *no*, quadrant.

: *adj*, square; , *no*, square.

: *adv*, four times.

: *adj*, quadruple.

: *adj*, quadrilateral.

: *adj*, fourfold.

: *vb*, place, put.

: *no*, part cut off, piece, segment.

: *par*, accordingly.

: *pro*, such as this.

: *no*, sector (of circle).

: *no*, cutting, stump, piece.

: *no*, place, space.

: *adv*, so many times.

: *pro*, so many times.

: *pro*, so many.

= : *par*, that is to say.

: *no*, trapezium.

: *adj*, triangular; , *no*, triangle.

: *adj*, triple, threefold.

: *adj*, trilateral.

: *adj*, triple.

: *no*, way.

: *vb*, hit, happen to be at (a place).

: *vb*, begin, be, exist; see .

: *no*, removal.

: *vb*, overshoot, exceed; see .

: *no*, excess, difference.

: *vb*, exceed; see .

: *no*, hypothesis.

: *vb*, underlie, be assumed (as hypothesis); see .

: *vb*, leave remaining.

: *vb*, subtend.

: *no*, height.

: *adj*, visible, manifest.

, —, —, — : *vb*, say; , we said.

: *vb*, carry.

: *no*, place, spot, area, figure.

: *pre + gen*, apart from.

: *vb*, touch.

: *par*, as, like, for instance.

: *par*, at random.

: *adv*, in the same manner, just so.

: *conj*, so that (causal), hence.