PRELIMINARY RESULTS

Output matrix:

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Assumption:

C_2 has full rank $\Rightarrow$ $C_2 B$ invertible

Let $V_1 \in \mathbb{R}^{n\times(n-m)}$ whose column space is $\mathcal{N}(C_2)$

$\Rightarrow$ rank$(V_1) = n - m$

$$\overline{V}_1 = (V_1^\top V_1)^{-1/2} V_1^\top \ , \ \overline{C}_2 = (C_2 C_2^\top)^{-1/2} C_2$$

$$V = \begin{bmatrix} \overline{V}_1 \\ \overline{C}_2 \end{bmatrix} \in \mathbb{R}^{n\times n} \quad \text{orthogonal}$$

Transformation 1:

$$x \mapsto Vx = \text{col}(x_1, x_2)$$

$$\boxed{\begin{aligned} \dot{x}_1 &= \overline{A}_{11} x_1 + \overline{A}_{12} x_2 + \overline{B}_1 (u + d) \\ \dot{x}_2 &= \overline{A}_{21} x_1 + \overline{A}_{22} x_2 + \overline{B}_2 (u + d) \end{aligned} \qquad (2)}$$

where $x_1 \in \mathbb{R}^{n-m}$ and $x_2 \in \mathbb{R}^m$

$$\overline{A} = VAV^\top = \begin{bmatrix} \overline{V}_1 A \overline{V}_1^\top & \overline{V}_1 A \overline{C}_2^\top \\ \overline{C}_2 A \overline{V}_1^\top & \overline{C}_2 A \overline{C}_2^\top \end{bmatrix}$$

$$\overline{B} = VB = \begin{bmatrix} \overline{V}_1 B \\ \overline{C}_2 B \end{bmatrix} \qquad \overline{C} = CV^\top = \begin{bmatrix} C_1 \overline{V}_1^\top & C_1 \overline{C}_2^\top \\ 0 & C_2 \overline{C}_2^\top \end{bmatrix}$$

PRELIMINARY RESULTS

Transformation 2: (nonsingular)

$$z = U\text{col}(x_1, x_2) = \text{col}(z_1, z_2)$$

Where $U = \begin{bmatrix} I_{n-m} & -\overline{B}_1\overline{B}_2^{-1} \\ 0 & \overline{B}_2^{-1} \end{bmatrix}$

$$\dot{z}_1 = \widetilde{A}_{11}z_1 + \widetilde{A}_{12}z_2 \qquad (3a)$$

$$\dot{z}_2 = \widetilde{A}_{21}z_1 + \widetilde{A}_{22}z_2 + u + d \qquad (3b)$$

$$y = \widetilde{C}z = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{bmatrix} z = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} \\ 0 & \widetilde{C}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \qquad (3c)$$

System and input matrix:

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} \quad \text{and} \quad \widetilde{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

Lemma 1:

If the pair $(\widetilde{A}_{11}, \widetilde{C}_{11})$ is observable,

then the pair $(\widetilde{A}, \widetilde{C})$ is observable.

$$\Big\downarrow \; C_2V_1 = C_2AV_1 = 0$$

The pair $(\widetilde{A}, \widetilde{C})$ is observable iff

the pair $(\widetilde{A}_{11}, \widetilde{C}_{11})$ is observable