$$y = \widetilde{C}z = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{bmatrix} z = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} \\ 0 & \widetilde{C}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \quad \text{(3c)}$$

A new output function:

$$\widetilde{C}_{11} z_1 = y_1 - \widetilde{C}_{12} \widetilde{C}_{22}^{-1} y_2$$

Observer:

$$\boxed{\begin{aligned} \dot{\hat{z}}_1 &= \widetilde{A}_{11}\hat{z}_1 + \widetilde{A}_{12}\hat{z}_2 + L_1 \widetilde{C}_{11}(z_1 - \hat{z}_1) + \widetilde{A}_{12}(z_2 - \hat{z}_2) \quad \text{(5a)} \\ \dot{\hat{z}}_2 &= \widetilde{A}_{21}\hat{z}_1 + u + \widetilde{A}_{22} z_2 + K\mathrm{sign}(z_2 - \hat{z}_2) \quad \text{(5b)} \end{aligned}}$$

where $L_1 \in \mathbb{R}^{(n-m)\times(p-m)}$ is the linear observer gain, $K > 0$ is some scalar and sign function is given by $\mathrm{sign}(\gamma) = [\ \mathrm{sign}(\gamma_1) \quad \mathrm{sign}(\gamma_2) \quad \cdots \quad \mathrm{sign}(\gamma_m)\ ]^\top$ for any vector $\gamma = [\ \gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_m\ ]^\top$. Here, $\hat{z} = \mathrm{col}(\hat{z}_1, \hat{z}_2)$ denotes the estimate of $z = \mathrm{col}(z_1, z_2)$.

Lemma 1:

If the pair $(\widetilde{A}_{11}, \widetilde{C}_{11})$ is observable, then the pair $(\widetilde{A}, \widetilde{C})$ is observable.

↓

- L_1 exists
- The eigenvalues of $\widetilde{A}_1 := \widetilde{A}_{11} - L_1 \widetilde{C}_{11}$ are stable

Lyapunov equation:

$$\widetilde{A}_1^\top P_1 + P_1 \widetilde{A}_1 = -Q_1$$

SLIDING MODE OBSERVER

$$\widetilde{A}_1^\top P_1 + P_1 \widetilde{A}_1 = -Q_1$$

for any $Q_1 = Q_1^\top > 0$. With this, define the sets $\Omega_1(c_1) = \{z_1 \in \mathbb{R}^{n-m} : z_1^\top P_1 z_1 \leqslant c_1\}$ and $\Omega_2(c_2) = \{z_2 \in \mathbb{R}^m : \|z_2\| \leqslant c_2\}$ for any $c_1 > 0$ and $c_2 > 0$. Also, denote $\Omega(c_1, c_2) = \{z \in \mathbb{R}^n : z_1 \in \Omega_1(c_1) \text{ and } z_2 \in \Omega_2(c_2)\}$.

$$\dot{x} = Ax + B(u+d)$$
$$y = Cx$$

$$\dot{z}_1 = \widetilde{A}_{11} z_1 + \widetilde{A}_{12} z_2 \qquad (3a)$$
$$\dot{z}_2 = \widetilde{A}_{21} z_1 + \widetilde{A}_{22} z_2 + u + d \qquad (3b)$$

Define $\tilde{z}_1 := z_1 - \hat{z}_1$ and $\tilde{z}_2 := z_2 - \hat{z}_2$ as the estimation errors.

$$\dot{\hat{z}}_1 = \widetilde{A}_{11}\hat{z}_1 + \widetilde{A}_{12}\hat{z}_2 + L_1 \widetilde{C}_{11}(z_1 - \hat{z}_1) + \widetilde{A}_{12}(z_2 - \hat{z}_2) \qquad (5a)$$
$$\dot{\hat{z}}_2 = \widetilde{A}_{21}\hat{z}_1 + u + \widetilde{A}_{22} z_2 + K\text{sign}(z_2 - \hat{z}_2) \qquad (5b)$$

estimation error dynamics：

$$\dot{\tilde{z}}_1 = \widetilde{A}_1 \tilde{z}_1 \qquad (7a)$$
$$\dot{\tilde{z}}_2 = \widetilde{A}_{21}\tilde{z}_1 - K\text{sign}(\tilde{z}_2) + d\,. \qquad (7b)$$

Theorem 1: Consider the system (3) and the observer (5). Let $c_1 > 0$ and $c_2 > 0$ be any scalars. Assume that the $(\widetilde{A}_{11}, \widetilde{C}_{11})$ pair is observable and $z(0), \hat{z}(0) \in \Omega(c_1, c_2)$. Then, there exists a gain $K > 0$ such that the estimation error $\tilde{z}(t) := z(t) - \hat{z}(t)$ is bounded for all $t \geqslant 0$, and $\tilde{z}(t) \to 0$ as $t \to +\infty$.