

This is a derivation for the multi-slice mathematical framework for refractive-index reconstruction from electric-field measurements.

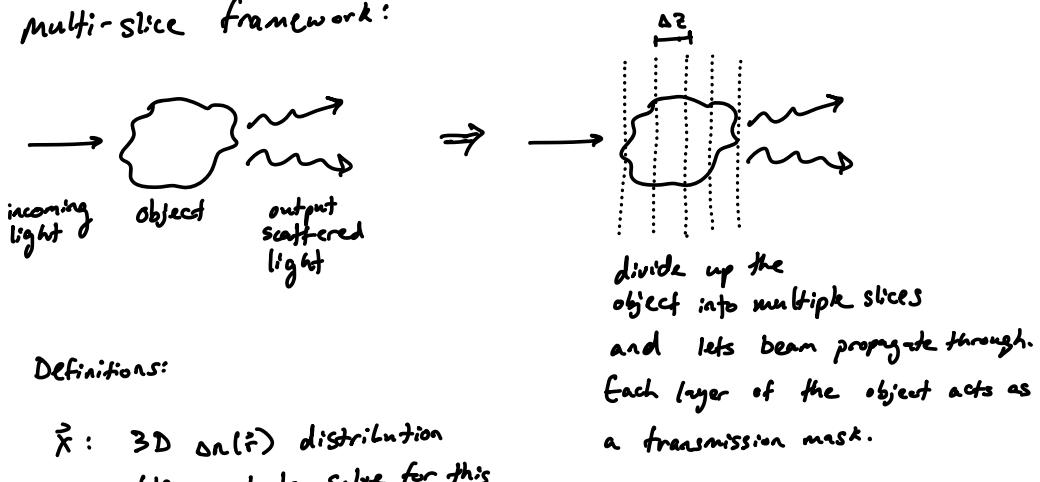
This follows the methodology introduced in Kamilov's paper:

"Optical Tomographic Image Reconstruction based on Beam propagation and Sparse regularization"

We make some slight tweaks, but the main incentive for this document is for beginner students to see a more thorough step-by-step of the derivation.

- Shreerajit Chowdhury

multi-slice framework:



\hat{y} : 2D complex-valued electric field measurements.
 $\hat{y} \in \mathbb{C}^m$, where $m < n$

S : nonlinear and nonconvex forward model that predicts 2D efield measurement from known \hat{x}

$$\hat{y} = S(\hat{x})$$

Remember that the $S\{\dots\}$ operator is $\mathbb{R}^n \rightarrow \mathbb{C}^m$

$$S_k(\hat{x}) = \text{diag}(P_k(\hat{x})) \cdot H \cdot S_{k-1}(\hat{x})$$

↓
propagation kernel
by some distance Δz

↑
Efield at
K-th layer,
as predicted
by $S\{\dots\}$

↑
transmission
mask at k-th
layer of object

↑
Efield at
 $k-1$ layer,
as predicted
by $S\{\dots\}$

↑
 $P_k(\hat{x}) = \epsilon_{eff}(j\lambda_0 \Delta z \cdot \hat{x}_k)$
corresponds to object's RI values
at the object's K-th layer

Say object has a total of K layers

Then to solve for \hat{x} , write the following optimization

$$\hat{x} = \underset{\hat{x}}{\operatorname{argmin}} \left\{ D(\hat{x}) + \gamma R(\hat{x}) \right\}$$

↓
Data fidelity
term, which we
try to minimize

[] Regularizer

$$D(\vec{x}) = \frac{1}{L} \sum_{k=1}^L D_k(\vec{x})$$

$$= \frac{1}{2L} \sum_{k=1}^L \| \vec{y}_k^l - S_k^l(\vec{x}) \|_{\ell^2}^2$$

here, $S_k(\vec{x})$ is the forward model's prediction of the electric field at last object layer.

y_k is the raw measurement of the electric field at the last object layer.

l is the acquisition number. This is an important parameter since you will have to take various measurements to solve this high-dimensional problem.

Definition:

ℓ^2 norm

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \| \vec{x} \|_{\ell^2}^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

$$= \vec{x}^H \cdot \vec{x}$$

$$= \langle \vec{x}, \vec{x} \rangle$$

From ℓ^2 norm definition we see that

$$D(\vec{x}) = \frac{1}{L} \sum_{k=1}^L D_k(\vec{x}) = \frac{1}{2L} \sum_{k=1}^L \| \vec{y}_k^l - S_k^l(\vec{x}) \|_{\ell^2}^2$$

is minimized whenever each component $D_k(\vec{x})$ is minimized. So lets find the minimum of just each individual $D_k(\vec{x})$, for which we just need to find gradient of:

$$D_k(\vec{x}) = \frac{1}{2} \| \vec{y}_k^l - S_k(\vec{x}) \|_{\ell^2}^2$$

for notational sake, I will refer to $D_k(\vec{x})$ as $D(\vec{x})$, and generally ignore the " l " parameter.

Derivation of gradient of Data-fidelity term:

$$D(\hat{x}) = \frac{1}{2} \left\| \hat{y}_k - S_k(\hat{x}) \right\|_F^2$$

$$= \frac{1}{2} \left[(\hat{y}_k - S_k(\hat{x}))^H \cdot (\hat{y}_k - S_k(\hat{x})) \right]$$

$$= \frac{1}{2} \left[(\hat{y}_k^H - S_k(\hat{x})^H) \cdot (\hat{y}_k - S_k(\hat{x})) \right]$$

$$= \frac{1}{2} \left[\hat{y}_k^H \cdot \hat{y}_k - \hat{y}_k^H S_k(\hat{x}) - S_k(\hat{x})^H \cdot \hat{y}_k + S_k(\hat{x})^H \cdot S_k(\hat{x}) \right] \dots \textcircled{1}$$

$$= \frac{1}{2} \langle \hat{y}_k, \hat{y}_k \rangle - \operatorname{Re} \{ S_k(\hat{x}), \hat{y}_k^H \} + \frac{1}{2} \langle S_k(\hat{x}), S_k(\hat{x}) \rangle$$

Define:

$$\frac{\partial}{\partial x_j} S_k(\hat{x}) = \begin{bmatrix} \frac{\partial}{\partial x_j} [S_k(\hat{x})]_1 \\ \vdots \\ \frac{\partial}{\partial x_j} [S_k(\hat{x})]_m \end{bmatrix}$$

$$\nabla D(\hat{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} D(\hat{x}) & \cdots & \frac{\partial}{\partial x_n} D(\hat{x}) \end{bmatrix}$$

Substitute into each column

$$\frac{\partial}{\partial \hat{x}} S_k(\hat{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} S_k(\hat{x}) & \cdots & \frac{\partial}{\partial x_n} S_k(\hat{x}) \end{bmatrix}$$

Hessian Matrix

$$\begin{bmatrix} \frac{\partial}{\partial x_1} [S_k(\hat{x})]_1 & \cdots & \frac{\partial}{\partial x_n} [S_k(\hat{x})]_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} [S_k(\hat{x})]_m & \cdots & \frac{\partial}{\partial x_n} [S_k(\hat{x})]_m \end{bmatrix}_{m \times n}$$

Starting from ①

$$\begin{aligned}
 D(\tilde{x}) &= \frac{1}{2} \cdot \left[\tilde{y}_K^H \cdot \tilde{y}_K - \tilde{y}_K^H S_K(\tilde{x}) - S_K(\tilde{x})^H \cdot \tilde{y}_K + S_K(\tilde{x})^H \cdot S_K(\tilde{x}) \right] \\
 &= \frac{1}{2} \cdot \left[\tilde{y}_K^H \cdot \tilde{y}_K - 2 \operatorname{Re} \{ \tilde{y}_K^H S_K(\tilde{x}) \} + S_K(\tilde{x})^H \cdot S_K(\tilde{x}) \right] \\
 &= \frac{1}{2} \tilde{y}_K^H \cdot \tilde{y}_K - \operatorname{Re} \{ \tilde{y}_K^H \cdot S_K(\tilde{x}) \} + \frac{1}{2} S_K(\tilde{x})^H \cdot S_K(\tilde{x})
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial}{\partial x_j} D(\tilde{x}) &= \frac{1}{2} \frac{\partial}{\partial x_j} \tilde{y}_K^H \cdot \tilde{y}_K - \frac{\partial}{\partial x_j} \operatorname{Re} \{ \tilde{y}_K^H \cdot S_K(\tilde{x}) \} + \frac{1}{2} \frac{\partial}{\partial x_j} [S_K(\tilde{x})^H \cdot S_K(\tilde{x})] \\
 &\quad \text{= 0, since } \tilde{y}_K^H \text{ is not dependent on } \tilde{x} \\
 &= -\operatorname{Re} \left\{ \frac{\partial}{\partial x_j} \tilde{y}_K^H \cdot S_K(\tilde{x}) \right\} + \frac{1}{2} \frac{\partial}{\partial x_j} [S_K(\tilde{x})^H \cdot S_K(\tilde{x})] \\
 &\quad \text{Real valued} \qquad \text{how do we solve this?} \\
 &\quad \downarrow \qquad \text{lets do derivation for single function value} \\
 S(x) &= a(x) + j b(x) \\
 S^H(x) &= S^*(x) = a(x) - j b(x)
 \end{aligned}$$

Therefore:

$$\begin{aligned}
 S^H(x) \cdot S(x) &= a^2(x) + b^2(x) \\
 \frac{\partial}{\partial x} [S^H(x) \cdot S(x)] &= \frac{\partial}{\partial x} [a^2(x) + b^2(x)] \\
 &= 2a(x) \frac{\partial}{\partial x} a(x) + 2b(x) \frac{\partial}{\partial x} b(x) \dots ②
 \end{aligned}$$

Now consider:

$$\begin{aligned}
 &2 S^H(x) \cdot \frac{\partial}{\partial x} S(x) \\
 &= 2 [a(x) - j b(x)] \cdot \frac{\partial}{\partial x} [a(x) + j b(x)] \\
 &= 2 [a(x) - j b(x)] \cdot \left[\frac{\partial}{\partial x} a(x) + j \frac{\partial}{\partial x} b(x) \right] \\
 &= 2 \left[a(x) \frac{\partial}{\partial x} a(x) + j a(x) \frac{\partial}{\partial x} b(x) - j b(x) \frac{\partial}{\partial x} a(x) + b(x) \frac{\partial}{\partial x} b(x) \right] \\
 &= 2a(x) \frac{\partial}{\partial x} a(x) + 2b(x) \frac{\partial}{\partial x} b(x) + 2j \left[a(x) \frac{\partial}{\partial x} b(x) - b(x) \frac{\partial}{\partial x} a(x) \right] \quad ③
 \end{aligned}$$

Real component

Imaginary component

If we compare ② and ③, we see that:

$$\frac{\partial}{\partial x} [S^H(x) \cdot S(x)] = \operatorname{Re} \left\{ 2 S^H(x) \cdot \frac{\partial}{\partial x} S(x) \right\} \dots ④$$

We use this property below

$$\begin{aligned} \frac{\partial}{\partial x_j} D(\tilde{x}) &= -\operatorname{Re} \left\{ \frac{\partial}{\partial x_j} \tilde{y}_k^H \cdot S_k(\tilde{x}) \right\} + \frac{1}{2} \frac{\partial}{\partial x_j} \left[S_k(\tilde{x})^H \cdot S_k(\tilde{x}) \right] \\ &= -\operatorname{Re} \left\{ \frac{\partial}{\partial x_j} \tilde{y}_k^H \cdot S_k(\tilde{x}) \right\} + \operatorname{Re} \left\{ S_k^H(\tilde{x}) \cdot \frac{\partial}{\partial x_j} S_k(\tilde{x}) \right\} \\ &\quad \text{constant with respect to } x_j \\ &= -\operatorname{Re} \left\{ \tilde{y}_k^H \cdot \frac{\partial}{\partial x_j} S_k(\tilde{x}) \right\} + \operatorname{Re} \left\{ S_k^H(\tilde{x}) \cdot \frac{\partial}{\partial x_j} S_k(\tilde{x}) \right\} \\ &= \operatorname{Re} \left\{ S_k^H(\tilde{x}) \cdot \frac{\partial}{\partial x_j} S_k(\tilde{x}) - \tilde{y}_k^H \cdot \frac{\partial}{\partial x_j} S_k(\tilde{x}) \right\} \\ \frac{\partial}{\partial x_j} D(\tilde{x}) &= \operatorname{Re} \left\{ (S_k^H(\tilde{x}) - \tilde{y}_k^H) \cdot \frac{\partial}{\partial x_j} S_k(\tilde{x}) \right\} \end{aligned}$$

$$\begin{aligned} \nabla D(\tilde{x}) &= \frac{\partial}{\partial \tilde{x}} D(\tilde{x}) \\ &= \left[\frac{\partial}{\partial x_1} D(\tilde{x}) \quad \frac{\partial}{\partial x_2} D(\tilde{x}) \quad \dots \quad \frac{\partial}{\partial x_n} D(\tilde{x}) \right] \\ &= \operatorname{Re} \left\{ (S_k(\tilde{x}) - \tilde{y}_k)^H \cdot \frac{\partial}{\partial \tilde{x}} S_k(\tilde{x}) \right\} \end{aligned}$$

$$[\nabla D(\tilde{x})]^H = \operatorname{Re} \left\{ \left[\frac{\partial}{\partial \tilde{x}} S_k(\tilde{x}) \right]^H \cdot (S_k(\tilde{x}) - \tilde{y}_k) \right\} \dots ⑤$$

This will be used for gradient descent protocol

$$[\nabla D(\vec{x})]^H = \text{Re} \left\{ \underbrace{\left[\frac{\partial}{\partial \vec{x}} S_k(\vec{x}) \right]^H}_{\text{how to compute this term?}} \cdot (S_k(\vec{x}) - \vec{j}_k) \right\}$$

Recall the recursive method to compute electric-fields at each object layer k

$$S_k(\vec{x}) = \text{diag}(P_k(\vec{x})) \cdot H \cdot S_{k-1}(\vec{x})$$

Therefore:

$$\begin{aligned} \frac{\partial}{\partial \vec{x}} S_k(\vec{x}) &= \frac{\partial}{\partial \vec{x}} \left[\text{diag}(P_k(\vec{x})) \cdot H \cdot S_{k-1}(\vec{x}) \right] \\ &= \text{diag}(P_k(\vec{x})) \cdot H \cdot \frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x}) + \left(\frac{\partial}{\partial \vec{x}} \text{diag}(P_k(\vec{x})) \right) \cdot H \cdot S_{k-1}(\vec{x}) \\ &= \text{diag}(P_k(\vec{x})) \cdot H \cdot \frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x}) + \text{diag}(H \cdot S_{k-1}(\vec{x})) \cdot \frac{\partial}{\partial \vec{x}} P_k(\vec{x}) \end{aligned}$$

Therefore:

$$\begin{aligned} \underline{\left[\frac{\partial}{\partial \vec{x}} S_k(\vec{x}) \right]^H} &= \left[\text{diag}(P_k(\vec{x})) \cdot H \cdot \frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x}) \right]^H + \left[\text{diag}(H \cdot S_{k-1}(\vec{x})) \cdot \frac{\partial}{\partial \vec{x}} P_k(\vec{x}) \right]^H \\ &= \underline{\left[\frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x}) \right]^H} \cdot H^H \cdot \text{diag}(\overline{P_k(\vec{x})}) + \left[\frac{\partial}{\partial \vec{x}} P_k(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{k-1}(\vec{x})}) \end{aligned}$$

Hence, this sets a recursive relationship between
the electric-field derivative at a layer and the electric-field derivative
at the layer before it ... ⑥

This means that we need an initial condition:

$$\left[\frac{\partial}{\partial \vec{x}} S_0(\vec{x}) \right]^H = \vec{0}$$

→ electric field at the very first layer of the object

using equation ⑥, we can write:

layer

$$k=K \quad \left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x}) \right]^H = \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x}) \right]^H \cdot H^H \cdot \text{diag}(\overline{P_K(\vec{x})}) + \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-1}(\vec{x})}) \\ \dots \quad \textcircled{7}$$

$$k=K-1 \quad \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x}) \right]^H = \left[\frac{\partial}{\partial \vec{x}} S_{K-2}(\vec{x}) \right]^H \cdot H^H \cdot \text{diag}(\overline{P_{K-1}(\vec{x})}) + \left[\frac{\partial}{\partial \vec{x}} P_{K-1}(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-2}(\vec{x})}) \\ \dots \quad \textcircled{8}$$

$$\begin{aligned} k=K-2 \quad & \left[\frac{\partial}{\partial \vec{x}} S_{K-2}(\vec{x}) \right]^H = \left[\frac{\partial}{\partial \vec{x}} S_{K-3}(\vec{x}) \right]^H \cdot H^H \cdot \text{diag}(\overline{P_{K-2}(\vec{x})}) + \left[\frac{\partial}{\partial \vec{x}} P_{K-2}(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-3}(\vec{x})}) \\ \vdots & \\ k=0 \quad & \left[\frac{\partial}{\partial \vec{x}} S_0(\vec{x}) \right]^H = \vec{0} \quad \dots \quad \textcircled{10} \end{aligned}$$

Recall equation ⑤:

$$\begin{aligned} \left[\nabla D(\vec{x}) \right]^H &= \text{Re} \left\{ \left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x}) \right]^H \cdot \underbrace{(S_K(\vec{x}) - \vec{y}_K)}_{\text{initialize } \vec{r}_K = S_K(\vec{x}) - \vec{y}_K} \right\} \\ &= \text{Re} \left\{ \left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x}) \right]^H \cdot \vec{r}_K \right\} \\ &\quad \text{let's expand this term using } \textcircled{7} \\ &\quad \downarrow \qquad \qquad \qquad \rightarrow \vec{r}_{K-1} = H^H \cdot \text{diag}(\overline{P_K(\vec{x})}) \cdot \vec{r}_K \\ \left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x}) \right]^H \cdot \vec{r}_K &= \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x}) \right]^H \cdot H^H \cdot \text{diag}(\overline{P_K(\vec{x})}) \cdot \vec{r}_K \\ &\quad + \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-1}(\vec{x})}) \cdot \vec{r}_K \end{aligned}$$

$$\begin{aligned} &= \underbrace{\left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x}) \right]^H \cdot \vec{r}_{K-1}}_{\text{expand this term using } \textcircled{8}} + \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-1}(\vec{x})}) \cdot \vec{r}_K \\ &\quad \dots \quad \textcircled{11} \end{aligned}$$

$$\begin{aligned}
& \vec{r}_{k-2} = H^H \cdot \text{diag}(\overline{P_{k-1}(x)}) \cdot \vec{r}_{k-1} \\
\left[\frac{\partial}{\partial x} S_{k-1}(x) \right]^H \cdot \vec{r}_{k-1} &= \left[\frac{\partial}{\partial x} S_{k-2}(x) \right]^H \cdot H^H \cdot \underbrace{\text{diag}(\overline{P_{k-1}(x)}) \cdot \vec{r}_{k-1}}_{\vec{r}_{k-2}} \\
&\quad + \left[\frac{\partial}{\partial x} P_{k-1}(x) \right]^H \cdot \text{diag}(\overline{H \cdot S_{k-2}(x)}) \cdot \vec{r}_{k-1} \\
&= \underbrace{\left[\frac{\partial}{\partial x} S_{k-2}(x) \right]^H \cdot \vec{r}_{k-2}}_{\text{expand this term using } ⑨} + \left[\frac{\partial}{\partial x} P_{k-1}(x) \right]^H \cdot \text{diag}(\overline{H \cdot S_{k-2}(x)}) \cdot \vec{r}_{k-1} \dots ⑫ \\
&\quad \vec{r}_{k-3} = H^H \cdot \text{diag}(\overline{P_{k-2}(x)}) \cdot \vec{r}_{k-2} \\
\left[\frac{\partial}{\partial x} S_{k-2}(x) \right]^H \cdot \vec{r}_{k-2} &= \left[\frac{\partial}{\partial x} S_{k-3}(x) \right]^H \cdot H^H \cdot \underbrace{\text{diag}(\overline{P_{k-2}(x)}) \cdot \vec{r}_{k-2}}_{\vec{r}_{k-3}} \\
&\quad + \left[\frac{\partial}{\partial x} P_{k-2}(x) \right]^H \cdot \text{diag}(\overline{H \cdot S_{k-3}(x)}) \cdot \vec{r}_{k-2} \\
&= \underbrace{\left[\frac{\partial}{\partial x} S_{k-3}(x) \right]^H \cdot \vec{r}_{k-3}}_{\text{expand this term using } ⑨} + \left[\frac{\partial}{\partial x} P_{k-2}(x) \right]^H \cdot \text{diag}(\overline{H \cdot S_{k-3}(x)}) \cdot \vec{r}_{k-2} \dots ⑬
\end{aligned}$$

And this pattern goes on and on till boundary condition is reached:

lets start back at ⑪ and do successive expansions to see this pattern more clearly:

$$\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x}) \right]^H \cdot \vec{r}_K = \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x}) \right]^H \cdot \vec{r}_{K-1} \rightarrow \text{expand this with } ⑯$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_k(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-1}(\vec{x})}) \cdot \vec{r}_K$$

$$= \left[\frac{\partial}{\partial \vec{x}} S_{K-2}(\vec{x}) \right]^H \cdot \vec{r}_{K-2} \rightarrow \text{expand this with } ⑯$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_{K-1}(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-2}(\vec{x})}) \cdot \vec{r}_{K-1}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_k(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-1}(\vec{x})}) \cdot \vec{r}_K$$

$$= \left[\frac{\partial}{\partial \vec{x}} S_{K-3}(\vec{x}) \right]^H \cdot \vec{r}_{K-3} \rightarrow \text{expand this using the same pattern as above.}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_{K-2}(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-3}(\vec{x})}) \cdot \vec{r}_{K-2}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_{K-1}(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-2}(\vec{x})}) \cdot \vec{r}_{K-1}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_k(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-1}(\vec{x})}) \cdot \vec{r}_K$$

⋮

$$= \underbrace{\left[\frac{\partial}{\partial \vec{x}} S_0(\vec{x}) \right]^H \cdot \vec{r}_0}_{=0} + \sum_{m=0} \left[\frac{\partial}{\partial \vec{x}} P_{K-m}(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-m-1}(\vec{x})}) \cdot \vec{r}_{K-m}$$

$$\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x}) \right]^H \cdot \vec{r}_K = \sum_{m=0} \left[\frac{\partial}{\partial \vec{x}} P_{K-m}(\vec{x}) \right]^H \cdot \text{diag}(\overline{H \cdot S_{K-m-1}(\vec{x})}) \cdot \vec{r}_{K-m} \dots ⑯$$

Recall:

$$\begin{aligned} [\nabla D(\vec{x})]^H &= \operatorname{Re} \left\{ \left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x}) \right]^H \cdot \vec{r}_K \right\} \\ &= \operatorname{Re} \left\{ \sum_{m=0}^{K-1} \left[\frac{\partial}{\partial \vec{x}} P_{K-m}(\vec{x}) \right]^H \cdot \operatorname{diag} \left(\overline{H \cdot S_{K-m-1}(\vec{x})} \right) \cdot \vec{r}_{K-m} \right\} \\ &\dots \textcircled{15} \end{aligned}$$

where

$$\vec{r}_{m-1} = H^H \cdot \operatorname{diag} (\overline{P_m(\vec{x})}) \cdot \vec{r}_m$$

This constitutes back-propagation algorithm to compute derivative function.

Electric-fields $S_m(\vec{x})$ are computed for each layer $m=0, 1, \dots, K$ through the object via the multi-slice forward model. These fields are stored.

Compute residual term \vec{r}_m for each m -th layer. However, since \vec{r}_{m-1} can only be calculated after calculating \vec{r}_m , this process has to be done iteratively and backwards through the object, hence the term "back-propagation" for Eq. 15.

After computing $[\nabla D(\vec{x})]^H$, standard gradient-descent methods can be used to minimize the Data-fidelity term