This is a derivation for the multi-slice mathematical framework for refractive-index reconstruction from electric-field measurements.

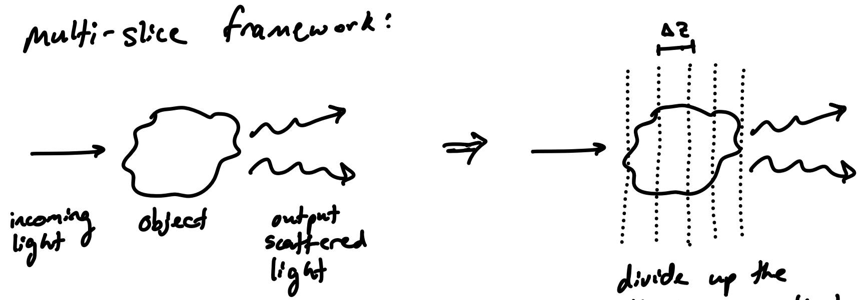
This follows the methodology introduced in Kamilov's paper:

"Optical Tomographic Image Reconstruction based on Beam propagation and Sparse regularization"

We make some slight tweaks, but the main incentive for this document is for beginner students to see a more thorough step-by-step of the derivation.

– Shwetadwip Chowdhury

multi-slice framework:



divide up the object into multiple slices and lets beam propagate through. Each layer of the object acts as a transmission mask.

Definitions:

$\vec{x}$: 3D $\Delta n(\vec{r})$ distribution
We want to solve for this
$\vec{x} \in \mathbb{R}^n$

$\vec{y}$: 2D complex-valued electric field measurements.
$\vec{y} \in \mathbb{C}^m$, where $m < n$

S: nonlinear and nonconvex forward model that predicts 2D efield measurement from known $\vec{x}$

$$\vec{y} = S(\vec{x})$$

Remember that the $S\{\dots\}$ operator is $\mathbb{R}^n \rightarrow \mathbb{C}^m$

propagation kernel by some distance Δz

$$S_k(\vec{x}) = \text{diag}\left(P_k(\vec{x})\right) \cdot H \cdot S_{k-1}(\vec{x})$$

Efield at k-th layer, as predicted by $S\{\dots\}$

transmission mask at k-th layer of object

Efield at $k-1$ layer, as predicted by $S\{\dots\}$

$$P_k(\vec{x}) = \exp(j k_0 \Delta z \cdot \vec{x}_k)$$

corresponds to object's RI values at the object's k-th layer

Say object has a total of K layers
Then to solve for $\vec{x}$, write the following optimization

$$\hat{x} = \underset{\vec{x}}{\text{argmin}} \left\{ \underbrace{D(\vec{x})}_{\text{Data fidelity term, which we try to minimize}} + \tau \underbrace{R(\vec{x})}_{\text{Regularizer}} \right\}$$

$$D(\vec{x}) = \frac{1}{L}\sum_{l=1}^{L} D_l(\vec{x})$$

$$= \frac{1}{2L}\sum_{l=1}^{L} \| \vec{y}_k^{\,l} - S_k^l(\vec{x}) \|_{l^2}^2$$

here, $S_k(\vec{x})$ is the forward model's prediction of the electric field at last object layer.

y_k is the raw measurement of the electric field at the last object layer.

l is the acquisition number. This is an important parameter since you will have to take various measurements to solve this high-dimensional problem.

Definition:

<u>l2 norm</u>

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \|\vec{x}\|_{l2}^2 = x_1^2 + x_2^2 + \dots x_n^2$$

$$= \vec{x}^H \cdot \vec{x}$$

$$= \langle \vec{x}, \vec{x} \rangle$$

From l^2 norm definition, we see that

$$D(\vec{x}) = \frac{1}{L}\sum_{l=1}^{L} D_l(\vec{x}) = \frac{1}{2L}\sum_{l=1}^{L} \| \vec{y}_k^{\,l} - S_k^l(\vec{x}) \|_{l^2}^2$$

is minimized whenever each component $D_l(\vec{x})$ is minimized. So lets find the minimum of just each individual $D_l(\vec{x})$, for which we just need to find gradient of:

$$D_l(\vec{x}) = \frac{1}{2} \| \vec{y}_k - S_k(\vec{x}) \|_{l2}^2$$

for notational sake, I will refer to $D_l(\vec{x})$ as $D(\vec{x})$, and generally ignore the "l" parameter.

Derivation of gradient of Data-fidelity term:

$$D(\vec{x}) = \frac{1}{2}\left\|\vec{y}_k - S_k(\vec{x})\right\|_{\ell_2}^2$$

$$= \frac{1}{2}\left[\left(\vec{y}_k - S_k(\vec{x})\right)^H \cdot \left(\vec{y}_k - S_k(\vec{x})\right)\right]$$

$$= \frac{1}{2}\left[\left(\vec{y}_k^H - S_k(\vec{x})^H\right) \cdot \left(\vec{y}_k - S_k(\vec{x})\right)\right]$$

$$= \frac{1}{2}\left[\vec{y}_k^H \cdot \vec{y}_k - \vec{y}_k^H S_k(\vec{x}) - S_k(\vec{x})^H \cdot \vec{y}_k + S_k(\vec{x})^H \cdot S_k(\vec{x})\right] \quad \cdots ①$$

$$= \frac{1}{2}\langle \vec{y}_k, \vec{y}_k \rangle - \mathrm{Re}\left\{ S_k(\vec{x}), \vec{y}_k^H \right\} + \frac{1}{2}\langle S_k(\vec{x}), S_k(\vec{x}) \rangle$$

Define:

$$\frac{\partial}{\partial x_j} S_k(\vec{x}) = \begin{bmatrix} \frac{\partial}{\partial x_j}\left[S_k(\vec{x})\right]_1 \\ \vdots \\ \frac{\partial}{\partial x_j}\left[S_k(\vec{x})\right]_m \end{bmatrix} \qquad \nabla D(\vec{x}) = \left[\frac{\partial}{\partial x_1} D(\vec{x}) \quad \cdots \quad \frac{\partial}{\partial x_n} D(\vec{x})\right]$$

substitute into each column

Hessian Matrix

$$\frac{\partial}{\partial \vec{x}} S_k(\vec{x}) = \left[\frac{\partial}{\partial x_1} S_k(\vec{x}) \quad \cdots \quad \frac{\partial}{\partial x_n} S_k(\vec{x})\right] = \begin{bmatrix} \frac{\partial}{\partial x_1}\left[S_k(\vec{x})\right]_1 & \cdots & \frac{\partial}{\partial x_n}\left[S_k(\vec{x})\right]_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}\left[S_k(\vec{x})\right]_m & \cdots & \frac{\partial}{\partial x_n}\left[S_k(\vec{x})\right]_m \end{bmatrix}_{m\times n}$$

Starting from ①

$$D(\vec{x}) = \frac{1}{2}\cdot\left[\vec{y}_k^H\cdot\vec{y}_k - \vec{y}_k^H S_k(\vec{x}) - S_k(\vec{x})^H\cdot\vec{y}_k + S_k(\vec{x})^H\cdot S_k(\vec{x})\right]$$

$$= \frac{1}{2}\cdot\left[\vec{y}_k^H\cdot\vec{y}_k - 2\,\mathrm{Re}\{\vec{y}_k^H\, S_k(\vec{x})\} + S_k(\vec{x})^H\cdot S_k(\vec{x})\right]$$

$$= \frac{1}{2}\,\vec{y}_k^H\cdot\vec{y}_k - \mathrm{Re}\{\vec{y}_k^H\cdot S_k(\vec{x})\} + \frac{1}{2} S_k(\vec{x})^H\cdot S_k(\vec{x})$$

$$\frac{\partial}{\partial x_j}D(\vec{x}) = \underbrace{\frac{1}{2}\frac{\partial}{\partial x_j}\vec{y}_k^H\cdot\vec{y}_k}_{=0,\ \text{since } \vec{y}_k \text{ is not dependent on } \vec{x}} - \frac{\partial}{\partial x_j}\cdot\mathrm{Re}\{\vec{y}_k^H\cdot S_k(\vec{x})\} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[S_k(\vec{x})^H\cdot S_k(\vec{x})\right]$$

$$= -\mathrm{Re}\left\{\frac{\partial}{\partial x_j}\vec{y}_k^H\cdot S_k(\vec{x})\right\} + \frac{1}{2}\underbrace{\frac{\partial}{\partial x_j}\left[S_k(\vec{x})^H\cdot S_k(\vec{x})\right]}_{\text{how do we solve this?}}$$

lets do derivation for single function value

Real valued

$$s(x) = a(x) + j\,b(x)$$

$$s^H(x) = s^*(x) = a(x) - j\,b(x)$$

Therefore:

$$s^H(x)\cdot s(x) = a^2(x) + b^2(x)$$

$$\frac{\partial}{\partial x}\left[s^H(x)\cdot s(x)\right] = \frac{\partial}{\partial x}\left[a^2(x) + b^2(x)\right]$$

$$= 2a(x)\frac{\partial}{\partial x}a(x) + 2b(x)\frac{\partial}{\partial x}b(x) \quad \ldots \text{②}$$

Now consider:

$$2\, s^H(x)\cdot\frac{\partial}{\partial x}s(x)$$

$$= 2\left[a(x) - j\,b(x)\right]\cdot\frac{\partial}{\partial x}\left[a(x) + j\,b(x)\right]$$

$$= 2\left[a(x) - j\,b(x)\right]\cdot\left[\frac{\partial}{\partial x}a(x) + j\frac{\partial}{\partial x}b(x)\right]$$

$$= 2\left[a(x)\frac{\partial}{\partial x}a(x) + j\,a(x)\frac{\partial}{\partial x}b(x) - j\,b(x)\frac{\partial}{\partial x}a(x) + b(x)\frac{\partial}{\partial x}b(x)\right]$$

$$= \underbrace{2a(x)\frac{\partial}{\partial x}a(x) + 2\,b(x)\frac{\partial}{\partial x}b(x)}_{\text{Real component}} + 2j\underbrace{\left[a(x)\frac{\partial}{\partial x}b(x) - b(x)\frac{\partial}{\partial x}a(x)\right]}_{\text{Imaginary component}} \quad \text{③}$$

If we compare ② and ③, we see that:

$$\frac{\partial}{\partial x}\left[s^H(x)\cdot s(x)\right] = \mathrm{Re}\left\{2\, s^H(x)\cdot \frac{\partial}{\partial x} s(x)\right\} \quad \cdots ④$$

We use this property below

$$\frac{\partial}{\partial x_j} D(\vec{x}) = -\mathrm{Re}\left\{\frac{\partial}{\partial x_j}\, \vec{y}_K^H \cdot S_K(\vec{x})\right\} + \frac{1}{2}\frac{\partial}{\partial x_j}\left[S_K(\vec{x})^H \cdot S_K(\vec{x})\right]$$

$$= -\mathrm{Re}\left\{\frac{\partial}{\partial x_j}\, \underbrace{\vec{y}_K^H}_{\text{constant with respect to } x_j} \cdot S_K(\vec{x})\right\} + \mathrm{Re}\left\{S_K^H(\vec{x})\cdot \frac{\partial}{\partial x_j} S_K(\vec{x})\right\}$$

$$= -\mathrm{Re}\left\{\vec{y}_K^H \cdot \frac{\partial}{\partial x_j} S_K(\vec{x})\right\} + \mathrm{Re}\left\{S_K^H(\vec{x})\cdot \frac{\partial}{\partial x_j} S_K(\vec{x})\right\}$$

$$= \mathrm{Re}\left\{\vec{S}_K^H(\vec{x})\cdot \frac{\partial}{\partial x_j} S_K(\vec{x}) - \vec{y}_K^H \cdot \frac{\partial}{\partial x_j} S_K(\vec{x})\right\}$$

$$\frac{\partial}{\partial x_j} D(\vec{x}) = \mathrm{Re}\left\{\left(S_K^H(\vec{x}) - \vec{y}_K^H\right)\cdot \frac{\partial}{\partial x_j} S_K(\vec{x})\right\}$$

$$\nabla D(\vec{x}) = \frac{\partial}{\partial \vec{x}} D(\vec{x})$$

$$= \left[\frac{\partial}{\partial x_1} D(\vec{x}) \quad \frac{\partial}{\partial x_2} D(\vec{x}) \quad \cdots \quad \frac{\partial}{\partial x_n} D(\vec{x})\right]$$

$$= \mathrm{Re}\left\{\left(S_K(\vec{x}) - y_K\right)^H \cdot \frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right\}$$

$$\left[\nabla D(\vec{x})\right]^H = \mathrm{Re}\left\{\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H \cdot \left(S_K(\vec{x}) - \vec{y}_K\right)\right\} \quad \cdots ⑤$$

This will be used for gradient descent protocol

$$\left[\nabla D(\vec{x})\right]^H = Re\left\{ \underbrace{\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]}_{\text{how to compute this term?}}{}^H \cdot \left(S_K(\vec{x}) - \vec{y}_K\right) \right\}$$

Recall the recursive method to compute electric-fields at each object layer k

$$S_k(\vec{x}) = diag\left(p_k(\vec{x})\right) \cdot H \cdot S_{k-1}(\vec{x})$$

Therefore:

$$\frac{\partial}{\partial \vec{x}} S_k(\vec{x}) = \frac{\partial}{\partial \vec{x}} \left[diag\left(p_k(\vec{x})\right) \cdot H \cdot S_{k-1}(\vec{x}) \right]$$

$$= diag\left(p_k(\vec{x})\right) \cdot H \cdot \frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x}) + \left(\frac{\partial}{\partial \vec{x}} diag\left(p_k(\vec{x})\right)\right) \cdot H \cdot S_{k-1}(\vec{x})$$

$$= diag\left(p_k(\vec{x})\right) \cdot H \cdot \frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x}) + diag\left(H \cdot S_{k-1}(\vec{x})\right) \cdot \frac{\partial}{\partial \vec{x}} p_k(\vec{x})$$

Therefore:

$$\underline{\left[\frac{\partial}{\partial \vec{x}} S_k(\vec{x})\right]^H} = \left[diag\left(p_k(\vec{x})\right) \cdot H \cdot \frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x})\right]^H + \left[diag\left(H \cdot S_{k-1}(\vec{x})\right) \cdot \frac{\partial}{\partial \vec{x}} p_k(\vec{x})\right]^H$$

$$= \underline{\left[\frac{\partial}{\partial \vec{x}} S_{k-1}(\vec{x})\right]^H} \cdot H^H \cdot diag\left(\overline{p_k(\vec{x})}\right) + \left[\frac{\partial}{\partial \vec{x}} p_k(\vec{x})\right]^H \cdot diag\left(\overline{H \cdot S_{k-1}(\vec{x})}\right) \quad \cdots (6)$$

Hence, this sets a recursive relationship between the electric-fied derivative of a layer and the electric-fied derivative of the layer before it

This means that we need an initial condition:

$$\left[\frac{\partial}{\partial \vec{x}} S_0(\vec{x})\right]^H = \vec{0}$$

electric field at the very first layer of the object using equation 6, we can write:

layer

$k=K$ $\quad \left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H = \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x})\right]^H \cdot H^H \cdot \mathrm{diag}\left(\overline{p_K(\vec{x})}\right) + \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-1}(\vec{x})}\right)$ ... 7

$k=K-1$ $\quad \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x})\right]^H = \left[\frac{\partial}{\partial \vec{x}} S_{K-2}(\vec{x})\right]^H \cdot H^H \cdot \mathrm{diag}\left(\overline{p_{K-1}(\vec{x})}\right) + \left[\frac{\partial}{\partial \vec{x}} P_{K-1}(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-2}(\vec{x})}\right)$ ... 8

$k=K-2$ $\quad \left[\frac{\partial}{\partial \vec{x}} S_{K-2}(\vec{x})\right]^H = \left[\frac{\partial}{\partial \vec{x}} S_{K-3}(\vec{x})\right]^H \cdot H^H \cdot \mathrm{diag}\left(\overline{p_{K-2}(\vec{x})}\right) + \left[\frac{\partial}{\partial \vec{x}} P_{K-2}(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-3}(\vec{x})}\right)$ ... 9

⋮

$k=0$ $\quad \left[\frac{\partial}{\partial \vec{x}} S_0(\vec{x})\right]^H = \vec{0}$ ... 10

Recall equation 5:

$$\left[\nabla D(\vec{x})\right]^H = \mathrm{Re}\left\{\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H \cdot \underbrace{\left(S_K(\vec{x}) - \vec{y}_K\right)}_{\text{initialize } \vec{r}_K = S_K(\vec{x}) - \vec{y}_K}\right\}$$

$$= \mathrm{Re}\left\{\underbrace{\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H \cdot \vec{r}_K}_{\text{let's expand this term using } 7}\right\}$$

$$\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H \cdot r_K = \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x})\right]^H \cdot \overbrace{H^H \cdot \mathrm{diag}\left(\overline{p_K(\vec{x})}\right) \cdot \vec{r}_K}^{\vec{r}_{K-1} = H^H \cdot \mathrm{diag}\left(\overline{p_K(\vec{x})}\right) \cdot \vec{r}_K}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-1}(\vec{x})}\right) \cdot \vec{r}_K$$

$$= \underbrace{\left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x})\right]^H \cdot \vec{r}_{K-1}}_{\text{expand this term using } 8} + \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-1}(\vec{x})}\right) \cdot \vec{r}_K$$ ... 11

$$\left[\frac{\partial}{\partial\vec{x}} S_{K-1}(\vec{x})\right]^H \cdot \vec{r}_{K-1} = \left[\frac{\partial}{\partial\vec{x}} S_{K-2}(\vec{x})\right]^H \cdot \underbrace{H^H \cdot \mathrm{diag}\left(\overline{p_{K-1}(\vec{x})}\right) \cdot \vec{r}_{K-1}}_{\vec{r}_{K-2} = H^H \cdot \mathrm{diag}\left(\overline{p_{K-1}(\vec{x})}\right) \cdot \vec{r}_{K-1}}$$

$$+ \left[\frac{\partial}{\partial\vec{x}} P_{K-1}(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-2}(\vec{x})}\right) \cdot \vec{r}_{K-1}$$

$$= \underbrace{\left[\frac{\partial}{\partial\vec{x}} S_{K-2}(\vec{x})\right]^H \cdot \vec{r}_{K-2}}_{\text{expand this term using (9)}} + \left[\frac{\partial}{\partial\vec{x}} P_{K-1}(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-2}(\vec{x})}\right) \cdot \vec{r}_{K-1} \quad \ldots (12)$$

$$\left[\frac{\partial}{\partial\vec{x}} S_{K-2}(\vec{x})\right]^H \cdot \vec{r}_{K-2} = \left[\frac{\partial}{\partial\vec{x}} S_{K-3}(\vec{x})\right]^H \cdot \underbrace{H^H \cdot \mathrm{diag}\left(\overline{p_{K-2}(\vec{x})}\right) \cdot \vec{r}_{K-2}}_{\vec{r}_{K-3} = H^H \cdot \mathrm{diag}\left(\overline{p_{K-2}(\vec{x})}\right) \cdot \vec{r}_{K-2}}$$

$$+ \left[\frac{\partial}{\partial\vec{x}} P_{K-2}(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-3}(\vec{x})}\right) \cdot \vec{r}_{K-2}$$

$$= \underbrace{\left[\frac{\partial}{\partial\vec{x}} S_{K-3}(\vec{x})\right]^H \cdot \vec{r}_{K-3}}_{\text{expand this term using (9)}} + \left[\frac{\partial}{\partial\vec{x}} P_{K-2}(\vec{x})\right]^H \cdot \mathrm{diag}\left(\overline{H \cdot S_{K-3}(\vec{x})}\right) \cdot \vec{r}_{K-2} \quad \ldots (13)$$

And this pattern goes on and on till boundary condition is reached:

Lets start back at (11) and do successive expansions to see this pattern more clearly:

$$\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H \cdot r_K = \left[\frac{\partial}{\partial \vec{x}} S_{K-1}(\vec{x})\right]^H \cdot \vec{r}_{K-1} \longrightarrow \text{expand this with (12)}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-1}(\vec{x})}\right) \cdot \vec{r}_K$$

$$= \left[\frac{\partial}{\partial \vec{x}} S_{K-2}(\vec{x})\right]^H \cdot \vec{r}_{K-2} \longrightarrow \text{expand this with (13)}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_{K-1}(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-2}(\vec{x})}\right) \cdot \vec{r}_{K-1}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-1}(\vec{x})}\right) \cdot \vec{r}_K$$

$$= \left[\frac{\partial}{\partial \vec{x}} S_{K-3}(\vec{x})\right]^H \cdot \vec{r}_{K-3} \longrightarrow \text{expand this using the same pattern as above.}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_{K-2}(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-3}(\vec{x})}\right) \cdot \vec{r}_{K-2}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_{K-1}(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-2}(\vec{x})}\right) \cdot \vec{r}_{K-1}$$

$$+ \left[\frac{\partial}{\partial \vec{x}} P_K(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-1}(\vec{x})}\right) \cdot \vec{r}_K$$

$$\vdots$$

$$= \underbrace{\left[\frac{\partial}{\partial \vec{x}} S_0(\vec{x})\right]^H}_{=0} \cdot \vec{r}_0 + \sum_{m \geq 0} \left[\frac{\partial}{\partial \vec{x}} P_{K-m}(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-m-1}(\vec{x})}\right) \cdot \vec{r}_{K-m}$$

$$\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H \cdot r_K = \sum_{m \geq 0} \left[\frac{\partial}{\partial \vec{x}} P_{K-m}(\vec{x})\right]^H \cdot \operatorname{diag}\left(\overline{H \cdot S_{K-m-1}(\vec{x})}\right) \cdot \vec{r}_{K-m} \quad \cdots \text{(14)}$$

Recall:

$$\left[\nabla D(\vec{x})\right]^H = Re\left\{\left[\frac{\partial}{\partial \vec{x}} S_K(\vec{x})\right]^H \cdot r_K\right\}$$

$$= Re\left\{\sum_{m=0}^{K-1}\left[\frac{\partial}{\partial \vec{x}} P_{K-m}(\vec{x})\right]^H \cdot diag\left(\overline{H \cdot S_{K-m-1}(\vec{x})}\right) \cdot \vec{r}_{K-m}\right\} \quad \dots (15)$$

where

$$\vec{r}_{m-1} = H^H \cdot diag\left(\overline{P_m(\vec{x})}\right) \cdot \vec{r}_m$$

This constitutes back-propagation algorithm to compute derivative function.

Electric-fields $S_m(\vec{x})$ are computed for each layer $m = 0, 1, \dots, K$ through the object via the multi-slice forward model. These fields are stored.

Compute residual term $\vec{r}_m$ for each m-th layer. However, since $\vec{r}_{m-1}$ can only be calculated after calculating $\vec{r}_m$, this process has to be done iteratively and backwards through the object, hence the term "back-propagation" for Eq. (15).

After computing $[\nabla D(\vec{x})]^H$, standard gradient-descent methods can be used to minimize the Data-fidelity term

How can we implement multislice when we only detect intensity-only measurements? i.e., non-interferometric measurements

In this scenario, we can rewrite the Data-fidelity term:

$$D_\ell(\underset{\sim}{x}) = \frac{1}{2} \| y_K^\ell - W_K^\ell(\underset{\sim}{x}) \|_{\ell^2}^2 \quad \cdots \text{(16)}$$

y_K is the raw amplitude measurement at the last object layer. This no longer denotes complex-field

ℓ is the acquisition number, same as before.

$W_K^\ell(\underset{\sim}{x})$ is the multi-slice forward model's prediction of the amplitude distribution at the last object layer.

This is different than the previously-introduced $S_K(\underset{\sim}{x})$, which was the forward model for <u>complex-field</u>.

Note that $W_K^\ell(\underset{\sim}{x}) = |S_K^\ell(\underset{\sim}{x})|$, which means $S_K^\ell(\underset{\sim}{x}) = \text{diag}\left(\exp\left(j \angle S_K^\ell(\underset{\sim}{x})\right)\right) \cdot W_K^\ell(x)$

To find derivative, consider single function value

$$\omega(x) = |s(x)| = [s^*(x)\, s(x)]^{1/2}$$

Use chain rule:

$$\frac{d}{dx}\omega(x) = \frac{d\omega}{ds} \cdot \frac{ds}{dx}$$

$$= \frac{1}{2}[s^*(x)\, s(x)]^{-1/2} \cdot s^*(x) \cdot \frac{ds}{dx}$$

$$= \frac{1}{2} \cdot \frac{s^*(x)}{|s(x)|} \cdot \frac{ds}{dx}$$

$$= \frac{1}{2} \cdot \exp[-j \angle s(x)] \cdot \frac{ds}{dx}$$

Thus, in matrix form:

$$\frac{\partial}{\partial \underset{\sim}{x}} W_K^\ell(\underset{\sim}{x}) = \frac{1}{2} \cdot \text{diag}\left(\exp(-j \angle S_K^\ell(\underset{\sim}{x})\right) \cdot \frac{\partial}{\partial \underset{\sim}{x}} S_K^\ell(\underset{\sim}{x}) \quad \cdots \text{(17)}$$

Let's return back to Data-fidelity term for intensity-only imaging in Eq 16

$$D_\ell(\vec{x}) = \frac{1}{2} \left\| y_k^\ell - W_k^\ell(\vec{x}) \right\|_{\ell^2}^2$$

Similarly to proof for Eq 5, we can show that:

$$[\nabla D_\ell(\vec{x})]^H = Re\left\{ \left[\frac{\partial}{\partial \vec{x}} W_k^\ell(\vec{x}) \right]^H \cdot \left(W_k^\ell(\vec{x}) - \vec{y}_k^\ell \right) \right\} \quad \cdots \quad 18$$

Substituting Eq. 17 and disregarding constant coefficients, we get:

$$\propto Re\left\{ \left[diag\left(exp(-j \angle S_k^\ell(\vec{x}))\right) \cdot \frac{\partial}{\partial \vec{x}} S_k^\ell(\vec{x}) \right]^H \cdot \left(W_k^\ell(\vec{x}) - \vec{y}_k^\ell \right) \right\}$$

$$\propto Re\left\{ \left[\frac{\partial}{\partial \vec{x}} S_k^\ell(\vec{x}) \right]^H \cdot diag\left(exp(j \angle S_k^\ell(\vec{x}))\right) \cdot \left(W_k^\ell(\vec{x}) - \vec{y}_k^\ell \right) \right\}$$

$$\propto Re\left\{ \left[\frac{\partial}{\partial \vec{x}} S_k^\ell(\vec{x}) \right]^H \cdot \left[\underbrace{diag\left(exp(j \angle S_k^\ell(\vec{x}))\right) \cdot W_k^\ell(\vec{x})} - diag\left(exp(j \angle S_k^\ell(\vec{x}))\right) \cdot \vec{y}_k^\ell \right] \right\}$$

Recall:

$$S_k^\ell(\vec{x}) = diag\left[exp(j \angle S_k^\ell(\vec{x})) \right] \cdot W_k^\ell(\vec{x})$$

$$[\nabla D_\ell(\vec{x})]^H \propto Re\left\{ \left[\frac{\partial}{\partial \vec{x}} S_k^\ell(\vec{x}) \right]^H \cdot \left(S_k^\ell(\vec{x}) - diag\left[exp(j \angle S_k^\ell(\vec{x})) \right] \cdot y_k^\ell \right) \right\} \quad 19$$

with amplitude measurements

Compare to gradient term with field measurements in Eq 5

$$[\nabla D(\vec{x})]^H = Re\left\{ \left[\frac{\partial}{\partial \vec{x}} S_k(\vec{x}) \right]^H \cdot \left(S_k(\vec{x}) - \vec{y}_k \right) \right\} \quad \cdots \quad 5$$

Eq. 5 and 19 are identical except for the $diag[exp[j \angle S_k^\ell(\vec{x})]]$ term in Eq 19. This simply manifests in the initial residual $\vec{r}_k = S_k^\ell(\vec{x}) - diag[exp[j \angle S_k^\ell(\vec{x})]]$.

Otherwise, the same process as outlined in Eqs. 6, 7, 8, 9, 10, 11, 12, 13, 14, and 15 can be used to compute the gradient for amplitude-only measurements!