Theoretical Physics 3
Group Project 2018 (4029P)

Analytical and numerical solution of electrostatic problems

1 Course Organisation

Please see the Group Project Guide for details of the organisation and assessment of your group project, including the group presentation and reports.

This project has both theoretical and computational aspects. You are presented with different electrostatic problems that you will solve analytically and/or numerically. The numerical solutions will be found using a software package that your group develops from first principles.

2 The Electrostatic Problems

Electromagnetism is one of the fundamental forces of nature and its application to the physical world is essential understanding for any physicist. One of the simplest examples of electromagnetism is *electrostatics*, the phenomena and properties of stationary or slow-moving electric charges, which has many applications in industry ranging from laser printers to defibrillators. In this project we will examine how the electrostatic potential (and thus the electric field) is modified by the presence of solid objects.

We first present two electrostatic problems that you will solve both analytically and numerically. These are common to all the project groups and can be used to benchmark your numerical solver. They are described in Section 2.1.

We will then present electrostatic problems taken from real world problems that do not have any analytical solution and hence you have to fully rely on your numerical solver. Finally, we ask you to extend the software package such that it can solve a generic electrostatic problem. They are described in Sections 2.2–2.4 and the groups will solve one of those problems each.

2.1 Problems with Analytic Solutions

The first problem consist of two concentrical cylinders, the inner one on ground potential and the outer one kept at potential $+V$. The configuration is illustrated in Figure 1 (left). The analytical solution to this problem can calculated easily.

The second problem is a bit more challenging since the outer cylinder is replaced by two infinite planes kept at potentials $+V$ and $-V$, as illustrated in Figure 1 (right). Solve this problem analytically using methods found in literature. Discuss any approximations you make and the existence or non-existence of an exact solution to this problem.

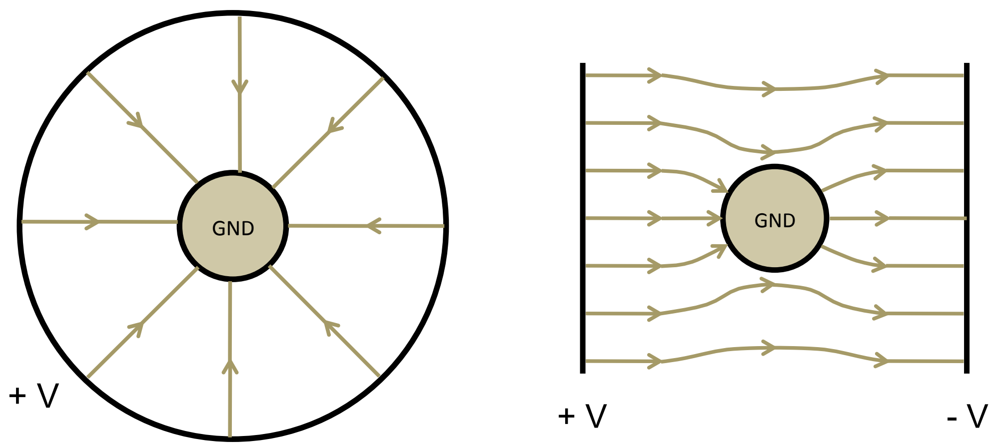

Figure 1: Illustration of the two electrostatic problems to be solved both analytically and numerically.

2.2 Multi-Wire Proportional Chambers

Multi-wire proportional chambers (MWPC) are commonly used for tracking detectors in particle physics. They consist of an array of wires held at ground potential between two plates held at high voltage $-V$, as shown in Figure 2. The gas between the two plates is ionised by traversing particles and the electrons will drift towards the wires. The electrons with gain sufficient kinetic energy in the high-field region around the wire to ionise the gas and generate more free electrons that will drift towards the wire. This process is known as gas multiplication.

Calculate the static electric field for the configuration shown in Figure 2. This problem cannot be solved analytically hence it has to be solved numerically.

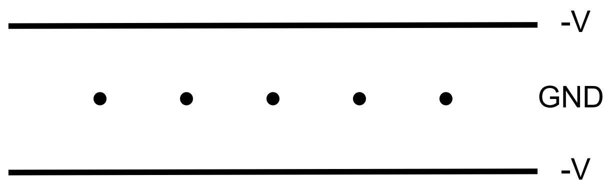

Figure 2: Illustration of a multi-wire proportional chamber. A cross-sectional view of an array of wires held at ground potential between two plates held at potential $-V$.

2.3 Silicon Microstrip Detectors

Silicon detectors are commonly used in tracking and vertexing detectors in particle physics. A schematic cross-sectional view of a silicon detector is shown in Figure 3. The detectors are normally made from silicon wafers with segmented doped implants on one side and a uniform doped implant on the other side, referred to as the backplane. The segmented implants are kept at ground potential and the backplane at the voltage $+V$. Ionising radiation will create electron-hole pairs in the silicon that will drift in the field and induce a signal on the segmented implants.

Calculate the static electric field for the configuration shown in Figure 3. This problem cannot be solved analytically hence it has to be solved numerically.

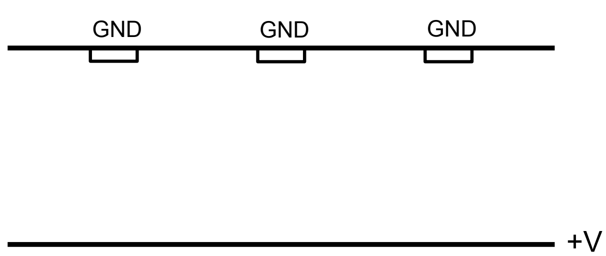

Figure 3: Illustration of a silicon microstrip detector. Cross-section of a silicon detector with segmented implants on one side and a uniform implant on the other side. The figure shows an array of three segmented implants, but for a real detector it would extend to more than 100 implants.

2.4 Edge-Coupled Stripline

High-speed serial data links are common both in consumer electronics and in particle physics experiments. Figure 4 shows a cross-sectional view of a high-speed link being developed for a future particle physics detector. The data link is implemented as a flat cable with a ground plane on each side and four rectangular conductors between them. The two outer conductors are held at ground potential and the two central conductors are at potentials $+V$ and $-V$ respectively. The transmitted signal is differential, which means that the receiver reads the difference in potential between the two central conductors.

Calculate the static electric field for the configuration shown in Figure 4. This problem cannot be solved analytically hence it has to be solved numerically.

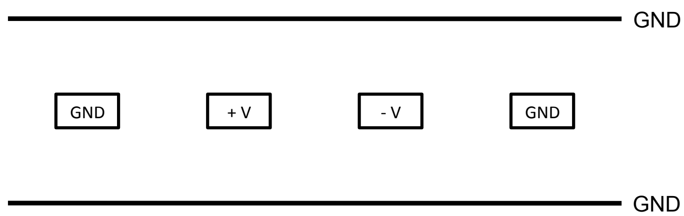

Figure 4: Illustration of an edge-coupled stripline. Cross-section of a high-speed data link implemented in a flat cable. The real cable has a repeating pattern of multiple data links next to each other.

3 Solving Laplace's equation analytically

Gauss's Law tells us that the divergence of the electric field is proportional to the charge density ρ. That is,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \tag{1}$$

where ε_0 is the permittivity of free space. For space outside the conductor, there is no charge, so tells us that the divergence is zero,

$$\vec{\nabla} \cdot \vec{E} = 0. \tag{2}$$

Furthermore, once equilibrium is reached and the fields attain a steady state, the electric field will be *irrotational* ($\vec{\nabla} \times \vec{E} = 0$ from the steady state version of the Maxwell-Faraday equation) and can therefore be written in terms of an *electrostatic potential* ϕ,

$$\vec{E} = -\vec{\nabla}\phi. \tag{3}$$

and Equation 2 becomes Laplace's equation,

$$\nabla^2\phi = 0. \tag{4}$$

Solving this equation and applying the boundary conditions gives the form of the potential in all space, and consequently the Electric field. You may assume that the cylinders and planes are perfect conductors. Knowing the analytical form for the electric field, you can find the analytical form for the potential. For simplicity, note that you can solve the equation in two dimensions.

Incidentally, Laplace's equation pops up in all sorts of places in physics, in fluid mechanics, thermodynamics, quantum mechanics, wave mechanics, and many other places, so the ability to solve it (either analytically or numerically) is an essential skill for a theoretical physicist.

4 Solving Laplace's equation numerically

Most real-life examples cannot be solved analytically, so it is useful to understand how to solve Laplace's equation **numerically** using C++. It is up to you how you do this, but we would suggest starting with the *Finite Difference Method* (go look this up!). If this is converging too slowly, you could try introducing a *relaxation parameter*, or try another method.

We provide you with a template of a C++ package example `http://ppewww.physics.gla.ac.uk/~abuzatu/BuzatuCode/CPP.html`, which is able to help you compile both shared objects and executables, both on Mac and on Linux. It also provides examples of C++ usages, that can be used as a C++ refresher and tutorial. If you know and want to use the CERN-based C++ libraries called ROOT for some mathematical calculations and plot making, you can use the following tutorial `http://ppewww.physics.gla.ac.uk/~abuzatu/BuzatuCode/CPPROOT.html`. You will have to install ROOT on your computer first. However, also feel free to experiment with your own Makefile if you want to learn more.

You need to use a version control system to share the code in a team. For this we recommend using Git. For the plotting, we recommend using gnuplot or ROOT. You can find very nice tutorials on programming about the Unix Shell, Version Control (Git), Makefile (Make) at the Software Carpentry website http://software-carpentry.org/lessons/.

5 Project Goals

During this project you will be expected to complete the following goals.

5.1 Team work

You need to work as a team and manage a collaborative computing project. You need to present a diagram of your software package code flow and the few persons that worked on each item. You need to think relatively early on in a project to assign the roles of project leader, code librarian, main editor of the group report, presenter of the group presentation on behalf of the group. However, each of you has to contribute to a distinct part of the C++ code and be able to explain what is its context, its goal, why this particular C++ logic was used and what its output is. It is not enough to work only on the group presentation and the group report. Each of you has to be able to check out the code, compile it, run it, produce the plots and numbers. The code librarian needs to provide instructions of how this can be done. Though you will focus on your on a specific part of the project, you have to learn about the other parts of the projects from your colleagues.

While the total project lasts for 10 weeks, you need to plan a full week to study how to prepare a presentation, then prepare your individual talk and implement the comments from your supervisors to it. Another week for your individual report. Another week for your group talk or group report. Another week to do the error analysis of your software. The first week is just the introduction to the lab, while the theoretical solution takes you the following week. It means there are only four weeks remaining to do the actual software coding for the basic problem and then find improvements to it. Time flies quickly, so please plan accordingly. Divide tasks in the group. You will meet your supervisor every week to follow your progress and to answer questions. If you encounter difficulties at any time that you cannot easily overcome, seek help from your project supervisor.

5.2 Systems with analytical solution

To understand the physics, the first task is to first solve the two systems with both analytical and numerical solutions and compare the results. This will also validate your numerical software package. Develop the code to be modular and generic, so that it allows to solve numerically any shape afterwards. You will solve only one of the systems that can be solved only numerically, but in principle you would be able to solve each one of them. To guide you through the process here are an outline of steps you are expected to go through:

- Solve the two systems analytically, using the assumption of infinite lengths and the boundary conditions given.
- Solve the system numerically, in 2D using the boundary conditions given.
- Plot the potential as a function of a grid of x and y coordinates for analytical and numerical solutions separately.
- Draw the equipotential lines for the numerical solution.
- Draw the electric field lines for the numerical solution.
- Quantify the accuracy as the numerical method as the absolute difference between the exact analytical value and the approximate numerical solution for the potential on the grid of x and y coordinates.
- Study the precision of your numerical method increases with the number of iterations and varies with different mesh sizes.
- Quantify how the difference in potential between two consecutive steps, the CPU consumption, and the time vary with the number of iterations.

5.3 System without analytical solution

Now that you have sorted out the simplest case and are equipped with a code that solves a generic problem, it is time to investigate some more systems that are not so amenable to analytic calculation, but which are used in real physics applications. Of course, no analytical solution is possible to these more complex cases, so only the numerical solution should be presented.

- Solve the system numerically in 2D with the boundary conditions given.

- Draw the electric field lines for the numerical solution.
- Draw the equipotential lines for the numerical solution.
- Quantify how the difference in potential between two consecutive steps, the CPU consumption, and the time vary with the number of iterations.

5.4 Generic system and extensions to the project

For maximum grades, you need to go beyond these tasks. You can develop the software to solve even more generic systems, or evaluate better the precision of the software, or improve the numerical simulation to be more accurate, faster, or both, to improve the user-friendliness.

Here are some examples of further extensions of the functionality that can be implemented:

- Alternative numerical methods for solving differential equations.
- Any non-uniform, adaptive grid (mesh) spacing. In a real life example, one would use a fine grid in regions of quickly changing potential and larger steps in regions where the field varies slowly.
- Any boundary conditions drawn pixel-by-pixel in programs such as Paint.
- Any arbitrary boundary conditions given with some pre-defined commands.
- A user interface, e.g. in C++'s Qt or similar, for the user to provide the input parameters and display the results and plots.
- Further profiling and performance evaluation of the program, to quantitively compare e.g. different numerical methods or grid sizes.

5.5 Presentations

Full details of the requirements of the presentations and the deadlines are given in the project guide.

The individual presentations will be given in sequence during one session and should be coordinated within the group. The set of individual presentations should cover all work performed by the group without overlap. You have to agree within the group who will present which part and in which order. The presentations are likely to be better if each group member is assigned a topic they are familiar with and have done a large fraction of the work on.

You will prepare the group presentation jointly and select one or two representatives of the group who will give the presentation in front of the lab heads and the other P3 students.

The presentations should be written in LaTeX (e.g. Beamer), Powerpoint, Keynote or similar. Please make sure you have all your material first, so that you have a clear view of what the main message is of your talk. Please plan for several days of work on the presentation alone. From previous years' experience, it takes several iterations with your supervisor to get the presentation perfect.

Some advice on how to give good science presentations are given by Jean-Luc Doumont in this one-hour video-conference http://www.youtube.com/watch?v=meBXuTIPJQk. An example of the structure of his talk is summarised on his website at http://treesmapsandtheorems.com/pdfs/TM&Th-3.0-summary.pdf.

5.6 Reports

You should document your results by producing a joint group report and one individual report per group member. The reports should be written at the level of quality of a journal or conference paper. These reports should include an introduction, a description of the analytic work and the numerical methods used, the results and conclusions. In your individual reports, each of you should focus on the contribution to the project that **you** have made, though you should also describe in briefer terms the work of the group and results generally. Results should be presented clearly and concisely, using diagrams to illustrate the potential and their corresponding electric fields. Don't forget to quantify your errors, and draw appropriate conclusions. The reports should be written in LaTeX (not Word). The reports need to be designed in a clear scientific way, which is very well exemplified by Jean-luc Dumont's book "Trees, maps and theorems", which is summarised at http://www.treesmapsandtheorems.com/pdfs/TM&Th-2.0-summary.pdf.