Combinatória I 2018: Lista 2

1. Determine the threshold for the event $\{K_r \subset G(n,p)\}$. Is the threshold sharp?

2. Prove that there exists a tournament of order n containing at least $2^{-n}(n-1)!$ directed hamiltonian cycles. (An ordering $(v_1, \ldots, v_n)$ of the vertices is a directed hamiltonian cycle if v_i beat v_{i+1} for every $i \in \mathbb{Z}_n$.)

3. Determine, for each n, the largest integer m such that there exists a graph with the following properties: $|V(G)| = n$, $e(G) = m$, and it is possible to red-blue colour the edges of G without creating a monochromatic triangle.

4. Say that a k-uniform hypergraph G is said to be 2-*colourable* if there exists a partition $V(G) = A \cup B$ with no edges entirely contained in either A or B. Let $b(k)$ denote the minimum number of edges in a k-uniform hypergraph that is not 2-colourable.
(a) Show that $b(2) = 3$.
(b) By considering a random colouring, show that $b(k) \geqslant 2^{k-1}$.
(c) By considering a random hypergraph, prove an upper bound for $b(k)$.

5. Prove that if T is a tree with k vertices, then $r(K_r, T) = (r-1)(k-1)+1$.

6. Prove that $k^{5/4} \leqslant r(C_4, K_k) \leqslant k^2$.

7. Prove that $\mathbf{ex}(n,H) = \Theta(n^2)$ if and only if $\chi(H) > 2$, and that $\mathbf{ex}(n,H) = \Theta(n)$ if and only if H is acyclic. Prove that otherwise there exists $\varepsilon = \varepsilon(H) > 0$ such that

$$n^{1+\varepsilon} \leqslant \mathbf{ex}(n,H) \leqslant n^{2-\varepsilon}.$$

8. Prove that, for every partition $\mathbb{N} = A_1 \cup \cdots \cup A_r$ of the positive integers into a finite number of parts, one of the parts contains a non-trivial[1] solution to the equation $x + y = 2z$.

9. Show that every graph H with k vertices and average degree $\gg \log k$ has super-polynomial Ramsey number, i.e., $r(H) \gg k^C$ for every constant $C > 0$.

10. Show that every bipartite graph H with k vertices and maximum degree at most $\log k$ has polynomial Ramsey number, i.e., $r(H) \leqslant k^C$ for some constant $C > 0$.

[1] That is, there exist distinct $x, y, z \in A_i$ (for some i) such that $x + y = 2z$.