

# COVERING 3-COLOURED RANDOM GRAPHS WITH MONOCHROMATIC TREES

YOSHIHARU KOHAYAKAWA, WALNER MENDONÇA, GUILHERME MOTA,  
AND BJARNE SCHÜLKE

ABSTRACT. We investigate the problem of determining how many monochromatic trees are necessary to cover the vertices of an edge-coloured random graph. More precisely, we show that for  $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$ , in any 3-colouring of the random graph  $G(n, p)$  we can find 3 monochromatic trees such that their union covers all vertices. This improves, for three colours, a result of Bucić, Korándi and Sudakov [*Covering random graphs by monochromatic trees and Helly-type results for hypergraphs*, [arXiv:1902.05055](https://arxiv.org/abs/1902.05055)].

## §1. INTRODUCTION

Given a graph  $G$  and a positive integer  $r$ , let  $\text{tc}_r(G)$  denote the minimum number  $k$  such that in any  $r$ -edge-colouring of  $G$ , there are  $k$  monochromatic trees  $T_1, \dots, T_k$  such that their vertex set cover  $V(G)$ , i.e.,

$$V(G) = V(T_1) \cup \dots \cup V(T_k).$$

We define  $\text{tp}_r(G)$  analogously by requiring the union above to be disjoint.

It is easy to see that  $\text{tp}_2(K_n) = 1$  for all  $n \geq 1$ , and Erdős, Gyárfás and Pyber [8] proved that  $\text{tp}_3(K_n) = 2$  for all  $n \geq 1$ , and conjectured that  $\text{tp}_r(K_n) = r - 1$  for every  $n$  and  $r$ . Haxell and Kohayakawa [10] showed that  $\text{tp}_r(K_n) \leq r$  for all sufficiently large  $n \geq n_0(r)$ . We remark that it is easy to see that  $\text{tc}_r(K_n) \leq r$  (just pick any vertex  $v \in V(K_n)$  and let  $T_i$ , for  $i \in [r]$ , be a maximal monochromatic tree of colour  $i$  containing  $v$ ), but it is not even known whether or not  $\text{tc}_r(K_n) \leq r - 1$  for every  $n$  and  $r$  (as would be implied by the conjecture of Erdős, Gyárfás and Pyber).

Concerning general graphs instead of complete graphs, Gyárfás [9] noted that a well-known conjecture of Ryser on matchings and transversal sets on hypergraphs is equivalent to the following bound on  $\text{tc}_r(G)$ .

---

2010 *Mathematics Subject Classification*. Primary: . Secondary:

Y. Kohayakawa was partially supported by FAPESP (2013/03447-6) and CNPq (311412/2018-1, 423833/2018-9). G. O. Mota was partially supported by FAPESP (2018/04876-1) and CNPq (304733/2017-2, 428385/2018-4). W. Mendonça was partially supported by CAPES. B. Schülke was partially supported by G.I.F. Grant Agreements No. I-1358-304.6/2016. The cooperation of the authors was supported by a joint CAPES/DAAD PROBRAL project (Proj. 430/15, 57350402, 57391197). This study was financed in part by CAPES, Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brasil, Finance Code 001. FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

**Conjecture 1.1** (Gyárfás' reformulation of Ryser's conjecture). *For any graph  $G$  and  $r \geq 2$ ,*

$$\text{tc}_r(G) \leq (r-1)\alpha(G). \quad (1.1)$$

In particular, Ryser's conjecture implies that  $\text{tc}_r(K_n) \leq r-1$ , for all  $n \geq 1$ , and  $r \geq 2$ . Ryser's conjecture was proved in the case  $r = 3$  by Aharoni [1], but for  $r \geq 4$  very little is known. For example, Haxell and Scott [11] proved (in the context of the original Ryser's conjecture) that there exists  $\varepsilon > 0$  such that for  $r \in \{4, 5\}$ , we have  $\text{tc}_r(G) \leq (r - \varepsilon)\alpha(G)$ , for any graph  $G$ .

Bal and DeBiasio [2] initiated the study of covering and partitioning random graphs by monochromatic trees. They proved that if

$$p \ll \left( \frac{\ln n}{n} \right)^{1/r},$$

then  $\text{tc}_r(G(n, p)) \rightarrow \infty$ . They conjectured that for any  $r \geq 2$ , this was the correct threshold for the event  $\text{tp}_r(G(n, p)) \leq r$ . Kohayakawa, Mota and Schacht [14] proved that such conjecture holds for  $r = 2$ , while Ebsen, Mota and Schnitzer<sup>1</sup> showed that it does not hold for more than two colours. Furthermore, Bucić, Korándi and Sudakov [6] proved the following theorem, which implies that the threshold for the event  $\text{tc}_r(G) \leq r$  is in fact significantly larger when  $r$  is large.

**Theorem 1.2** (Bucić, Korándi and Sudakov). *For any positive integer  $r$  there are constants  $c$  and  $C$  such that, for  $G = G(n, p)$ ,*

- (i) *if  $p < \left( \frac{c \ln n}{n} \right)^{\sqrt{r}/2^{r-2}}$ , then w.h.p.  $\text{tc}_r(G) \geq r + 1$ , and*
- (ii) *if  $p > \left( \frac{C \ln n}{n} \right)^{1/2^r}$ , then w.h.p.  $\text{tc}_r(G) \leq r$ .*

They also proved the following result that (roughly) determines the typical size of  $\text{tc}_r(G(n, p))$ .

**Theorem 1.3** (Bucić, Korándi and Sudakov). *For any integers  $k > r \geq 2$  there are constants  $c$  and  $C$  such that, for  $G = G(n, p)$ , if  $\left( \frac{C \ln n}{n} \right)^{1/k} < p < \left( \frac{c \ln n}{n} \right)^{1/(k+1)}$ , then w.h.p.  $\frac{r^2}{20 \ln k} \leq \text{tc}_r(G) \leq \frac{16r^2 \ln r}{\ln k}$ .*

Considering colourings with 3 colours, note that Theorem 1.2 implies that if  $p \gg \left( \frac{\ln n}{n} \right)^{1/8}$ , then w.h.p.  $\text{tc}_3(G(n, p)) \leq 3$ , while Theorem 1.3 implies that if  $\left( \frac{\ln n}{n} \right)^{1/6} \ll p \ll \left( \frac{\ln n}{n} \right)^{1/7}$ , then w.h.p.  $\text{tc}_3(G(n, p)) \leq 88$ . Our main result improves these results for 3 colours.

**Theorem 1.4.** *If  $p = p(n)$  satisfies  $p \gg \left( \frac{\ln n}{n} \right)^{1/6}$ , then*

$$\text{tc}_3(G(n, p)) = 3,$$

*with high probability as  $n \rightarrow \infty$ .*

---

<sup>1</sup>A description of this construction can be found in [14].

It can easily be seen that for  $p$  in the range as in the theorem, w.h.p. there is a colouring for which 3 monochromatic trees are needed to cover all vertices. Hence, we only need to show  $\text{tc}_3(G(n, p)) \leq 3$ . From the example described in [14], we know that for  $p \ll \left(\frac{\ln n}{n}\right)^{1/4}$  we have w.h.p.  $\text{tc}_3(G(n, p)) \geq 4$ . It would be very interesting to describe the behaviour of  $\text{tc}_3(G(n, p))$  when  $\left(\frac{\ln n}{n}\right)^{1/4} \ll p \ll \left(\frac{\ln n}{n}\right)^{1/6}$ .

This paper is organized as the following. We give a sketch of the proof of Theorem 1.4 in Section 2, while in Section 3 we present some auxiliary results. Finally, the details of the proof of Theorem 1.4 are given in Section 4.

## §2. SKETCH OF THE PROOF OF THEOREM 1.4

In this section we will give an overview of the proof of Theorem 1.4. Let  $G = G(n, p)$ , with  $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$ , and let  $c : E(G) \rightarrow \{\text{red, green, blue}\}$  be any 3-edge-colouring of  $G$ . We consider an auxiliary graph  $F$ , with  $V(F) = V(G)$  and  $ij \in E(F)$  if and only if there is, in the colouring  $c$ , a monochromatic path in  $G$  connecting  $i$  to  $j$ . Then we define a 3-edge-colouring  $c'$  of  $F$  with  $c'(ij)$  being the color of any monochromatic path in  $G$  connecting  $i$  to  $j$ . Note that any covering of  $F$  with monochromatic trees with respect to the colouring  $c'$  corresponds to a covering of  $G$  with monochromatic trees with respect to the colouring  $c$  with the same number of trees.

Next, we consider different cases depending on the value of  $\alpha(F)$ . If  $\alpha(F) = 1$ , then  $F$  is a complete 3-coloured graph and by Theorem 3.1, there exists a partition of  $V(F)$  into 2 monochromatic trees. The remaining proof now is divided into two cases:

- (i)  $\alpha(F) \geq 3$ ;
- (ii)  $\alpha(F) = 2$ .

Let us consider the case (i) first. Since  $\alpha(F) \geq 3$ , there exist three vertices  $r, b, g \in V(G)$  that pairwise do not have any monochromatic path connecting them. They w.h.p. have a common neighbourhood of size at least  $np^3/2$ . Let  $J$  be the largest subset of this common neighbourhood such that for each  $i \in \{r, b, g\}$ , the edges from  $i$  to  $J$  are all coloured with one colour. Then  $|J| \geq np^3/12$  and since there are no monochromatic paths between any two of  $r, b, g$ , we may assume that all edges from  $r$  to  $J$  are red, all from  $b$  to  $J$  are blue and those from  $g$  to  $J$  are green. Now we notice that all vertices that have a neighbour in  $J$  are covered by the union of the spanning trees of the red component of  $r$ , the blue component of  $b$  and the green component of  $g$ .

The proof would be finished if every vertex of  $G$  has a neighbour in  $J$ . If this is not the case, then let  $Y = V \setminus N(J)$  be the set of those vertices that have no neighbour in  $J$ . By the value of  $p$ , we get that w.h.p. every vertex  $y \in Y$  will have many common neighbours with  $r, g$  and  $b$  that are also neighbours of some vertex in  $J$ . With some careful analysis of the possible colourings of the edges to those common neighbours, we are able to show that for some  $i \in \{r, g, b\}$  (let us say, w.l.o.g.,  $i = r$ ), every vertex  $y \in Y$  can be connected

to  $r$  by a monochromatic path in colour red or either to  $g$  or  $b$  by a monochromatic path in the colour blue or green, respectively.

This already gives us that all vertices in  $G$  can be covered by 5 monochromatic trees, since all the vertices in  $N(J)$  lie in the red component of  $r$ , or the green component of  $g$ , or in the blue component of  $b$  and every vertex in  $V \setminus N(J)$  lies in the red component of  $r$ , in the blue component of  $g$  or in the green component of  $b$ . By analysing the colours of edges to the common neighbourhood of carefully chosen vertices, we are able to show that actually three of those five trees already cover all the vertices of  $G$ .

Now let us consider case (ii), where we have  $\alpha(F) = 2$ . Given a graph  $G$  and an  $r$ -edge-colouring of  $G$ , let us consider a hypergraph  $\mathcal{H}$  defined as follows (such a construction is inspired by a construction of Gyárfás [9]). The vertices of  $\mathcal{H}$  are the monochromatic components of  $F$  and  $r$  vertices form a hyperedge if the corresponding  $r$  components have a non-empty intersection (in particular they must be of different colours). Hence  $\mathcal{H}$  is an  $r$ -uniform  $r$ -partite hypergraph. Now observe that if  $A$  is a vertex cover of  $\mathcal{H}$ , then the monochromatic components associated with the vertices in  $A$  cover all the vertices of  $G$ . In fact, if  $v \in V(G)$  is not covered by those monochromatic components associated to the vertex cover of  $\mathcal{H}$ , then the monochromatic components of each colour containing  $v$  form a hyperedge of  $\mathcal{H}$  which does not intersect  $A$ , contradicting the fact that  $A$  is a vertex cover of  $\mathcal{H}$ . Further, notice that  $\nu(\mathcal{H}) \leq \alpha(G)$  because for each matching  $E_1, \dots, E_k$  in  $\mathcal{H}$  we can choose distinct vertices  $v_1, \dots, v_k$ , each  $v_i$  belonging to the intersection of the  $r$  monochromatic components associated to  $E_i$ . Then if we had  $k > \alpha(G)$ , two vertices among  $v_1, \dots, v_k$  would be adjacent and would therefore share one monochromatic component. But that would mean that their corresponding hyperedges intersect. The two observations above give us that Ryser's conjecture implies that  $\text{tc}_r(G) \leq (r-1)\alpha(G)$ .

Now, back to the sketch of proof, we are considering the case in which we have  $\alpha(F) = 2$ . The observations in the previous paragraph already imply that  $\text{tc}_3(F) \leq 4$ . But we want to prove that  $\text{tc}_3(F) \leq 3$ . To this aim, we analyse the hypergraph  $\mathcal{H}$  more carefully, reducing the situation to few possible settings of components covering all vertices. In each of those, we can again analyse the possible colouring of edges of common neighbours of specific vertices, inferring that indeed 3 monochromatic components cover all vertices.

### §3. PRELIMINARIES

Most of our notation is standard (see [3, 5, 7] and [4, 13]). However, we will mention in the following some definitions regarding hypergraphs that will play a major role in our proof just for completeness.

We say that a set  $A$  of vertices in a hypergraph  $\mathcal{H}$  is a *vertex cover* if every hyperedge of  $\mathcal{H}$  contains at least one element of  $A$ . The *covering number* of  $\mathcal{H}$ , denoted by  $\tau(\mathcal{H})$ , is the smallest size of a vertex cover in  $\mathcal{H}$ . A *matching* in  $\mathcal{H}$  is a collection of disjoint hyperedges in  $\mathcal{H}$ . The *matching number* of  $\mathcal{H}$ , denoted by  $\nu(\mathcal{H})$ , is the largest size of a matching in

$\mathcal{H}$ . An immediate relationship between  $\tau(\mathcal{H})$  and  $\nu(\mathcal{H})$  is the inequality  $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$ . If additionally  $\mathcal{H}$  is  $r$ -uniform, then we have  $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$ . A conjecture due Ryser, which is equivalent to Conjecture 1.1 (and it first appeared in the thesis of his Ph.D. student, Henderson [12]) states that for every  $r$ -uniform  $r$ -partite hypergraph  $\mathcal{H}$ , we have  $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$ . Note that König's theorem corresponds to Ryser's conjecture for  $r = 2$ . Aharoni [1] proved that Ryser's conjecture holds for  $r = 3$ , but the conjecture remains open for  $r \geq 4$ .

Given a vertex  $v$  in a 3-uniform hypergraph  $\mathcal{H}$ , the *link graph* of  $\mathcal{H}$  with respect to  $v$  is the graph  $L_v = (V, E)$  with vertex set  $V = V(\mathcal{H})$  and edge set  $E = \{xy : \{x, y, v\} \subseteq \mathcal{H}\}$ .

We will use the following theorem due to Erdős, Gyárfás and Pyber [8] in the proof of our main result.

**Theorem 3.1** (Erdős, Gyárfás and Pyber). *For any 3-edge-colouring of a complete graph  $K_n$ , there exists a partition of  $V(K_n)$  into 2 monochromatic trees.*

The following lemma is a simple application of Chernoff's inequality. For a proof for the first see [15, Lemma 3.8]. The second item is an immediate corollary of [15, Lemma 3.10].

**Lemma 3.2.** *Let  $\varepsilon > 0$ . If  $p = p(n) \gg ((\ln n)/n)^{1/6}$ , then w.h.p.  $G \in G(n, p)$  satisfies the following properties.*

(i) *For any disjoint sets  $X, Y \subseteq V(G)$  with  $|X|, |Y| \gg \ln n/p$ , we have*

$$|E_G(X, Y)| = (1 \pm \varepsilon)p|X||Y|.$$

(ii) *Every vertex  $v \in V(G)$  has degree  $d_G(v) = (1 \pm \varepsilon)pn$  and every set of  $i \leq 6$  vertices has  $(1 \pm \varepsilon)p^i n$  common neighbours.*

#### §4. PROOF OF THE MAIN RESULT

Instead of analysing the colouring of the graph  $G = G(n, p)$ , it will be helpful to analyse the following auxiliary graph.

**Definition 4.1** (Shortcut graph). *Let  $G$  be a graph and  $\varphi$  be a colouring of  $E(G)$ . The shortcut graph of  $G$  (with respect to  $\varphi$ ) is the graph  $F(G, \varphi)$  that has  $V(G)$  as the vertex set and the following set as the edge set:*

$$\{uv \in V(G) : u \text{ and } v \text{ are connected in } G \text{ by a path monochromatic under } \varphi\}.$$

We can consider a natural edge colouring  $\varphi'$  of  $F(G, \varphi)$  by assigning to an edge  $uv \in E(F(G, \varphi))$  the colour of any monochromatic path connecting  $u$  and  $v$  in  $G$  under the colouring  $\varphi$ . We will say that  $\varphi'$  is an *inherited colouring* of  $F(G, \varphi)$ . Let  $\text{tc}(F, \varphi')$  be the minimum number of monochromatic components (under the colouring  $\varphi'$ ) covering all the vertices of  $F$ . Note that any covering of  $F(G, \varphi)$  with monochromatic trees under  $\varphi'$  corresponds to a covering of  $G$  with monochromatic trees under the colouring  $\varphi$ . In

particular,  $\text{tc}_r(G) \leq \sup_{\varphi'} \text{tc}(F, \varphi')$ , where the supremum is over all the inherited colouring of  $F$ . Therefore, Theorem 1.4 follows from the following lemma.

**Lemma 4.2.** *Let  $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$  and let  $G = G(n, p)$ . The following holds w.h.p.. For any 3-edge-colouring  $\varphi$  of  $G$  and an inherited colouring  $\varphi'$  of the shortcut graph  $F = F(G, \varphi)$ , we have  $\text{tc}(F, \varphi') \leq 3$ .*

The proof of Lemma 4.2 is divided into two different cases, depending on the independence number of  $F$ . Subsections 4.1 and 4.2 are devoted, respectively, to the proof of Lemma 4.2 when  $\alpha(F) \geq 3$  and  $\alpha(F) \leq 2$ .

From now on, we fix  $\varepsilon > 0$  and assume that  $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$  and  $n$  is sufficiently large. Then, by Lemma 3.2, we may assume that w.h.p. the following holds:

- (1) There is an edge between any two sets of size  $\omega(\ln n/p)$ .
- (2) Every vertex  $v \in V(G)$  has degree  $d_G(v) = (1 \pm \varepsilon)pn$ .
- (3) Every set of  $i \leq 6$  vertices has  $(1 \pm \varepsilon)p^i n$  common neighbours.

#### 4.1. Shortcut graphs with independence number at least three.

*Proof of Lemma 4.2 for  $\alpha(F) \geq 3$ .* Since  $\alpha(F) \geq 3$ , there exist three vertices  $r, b, g \in V(G)$  that pairwise do not have any monochromatic path connecting them in  $G$ . In particular, if  $v$  is a common neighbour of  $r, b$  and  $g$  in  $G$ , then the edges  $vr, vb$  and  $vg$  have all different colours. The common neighbourhood of  $r, b$  and  $g$  in  $G$  has size at least  $np^3/2$ . Let  $X_{rbg}$  be the largest subset of this common neighbourhood such that for each  $i \in \{r, b, g\}$ , the edges between  $i$  and the vertices of  $X_{rbg}$  are all coloured with the same colour in  $G$ . Then  $|X_{rbg}| \geq np^3/12$ . Without loss of generality, assume that all edges between  $r$  and the vertices of  $X_{rbg}$  are red, between  $b$  and the vertices of  $X_{rbg}$  are blue and those between  $g$  and the vertices of  $X_{rbg}$  are green. Let  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$  and  $C_{\text{green}}(g)$  be respectively the red, blue and green components in  $G$  containing  $r, g$  and  $b$ .

Notice that all vertices of  $F$  that have a neighbour in  $X_{rbg}$  are covered by  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$  or  $C_{\text{green}}(g)$ . Therefore, the proof would be finished if every vertex has a neighbour in  $X_{rbg}$ . If this is not the case, we fix an arbitrary vertex  $y \in V \setminus N_G(X_{rbg})$ . By our choice of  $p$ , there are at least  $np^4/2$  common neighbours of  $y, r, b$  and  $g$ . Let  $X_{yrbg}$  be the largest subset of the common neighbourhood of  $y, r, b$  and  $g$  such that for each  $i \in \{r, b, g\}$ , the edges between  $i$  and  $X_{yrbg}$  are all coloured the same. Then  $|X_{yrbg}| \geq np^4/12$ . Note that since  $y \in V \setminus N_G(X_{rbg})$ , the sets  $X_{yrbg}$  and  $X_{rbg}$  are disjoint. Furthermore, since  $|X_{yrbg}|, |X_{rbg}| \gg \ln n/p$ , we have

$$|E_G(X_{yrbg}, X_{rbg})| \geq 1.$$

We now analyse the colours between  $r, b, g$  and the set  $X_{yrbg}$ . Again, since there is no monochromatic path connecting any two of  $r, b$  and  $g$ , all  $i \in \{r, b, g\}$  have to connect to  $X_{yrbg}$  in different colours. Since  $X_{yrbg}$  is disjoint of  $X_{rbg}$ , we cannot have  $r, b$  and  $g$



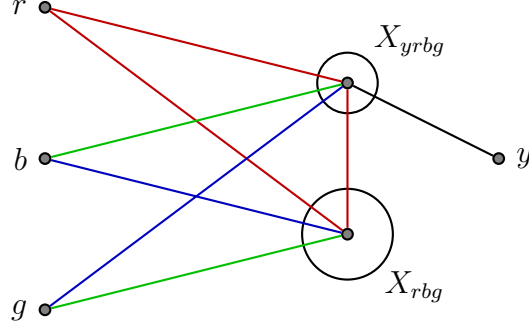


FIGURE 4.1. Analysis of the colouring of the edges incident on  $X_{rbg}$  and on  $X_{yrbg}$ .

being connected to  $X_{yrbg}$  by red, blue and green edges, respectively. Assume first that for each  $i \in \{r, b, g\}$ , the edges between  $i$  and  $X_{yrbg}$  have different colours from the edges between  $i$  and  $X_{rbg}$ . Then let  $uv$  be an edge between  $X_{yrbg}$  and  $X_{rbg}$  and notice that whatever the colour of  $uv$  is, we will have a monochromatic path connecting two of the vertices in  $\{r, g, b\}$ . Therefore, we can assume that for some  $i \in \{r, g, b\}$ , we have that all the edges between  $i$  and  $X_{rbg}$  and all the edges between  $i$  and  $X_{yrbg}$  are coloured the same. Without loss of generality, we may say that such  $i$  is  $r$ . In this case, the edges between  $b$  and  $X_{yrbg}$  are green and the edges between  $g$  and  $X_{yrbg}$  are blue, otherwise  $X_{yrbg}$  and  $X_{rbg}$  would not be disjoint sets. Finally, all the edges between  $X_{yrbg}$  and  $X_{rbg}$  are red, otherwise we would be able to connect  $b$  and  $g$  by some monochromatic path. Figure 4.1 shows the colouring of the edges that we have analysed so far.

Let us now consider any further vertex  $x \in V \setminus N(X_{rbg})$  with  $x \neq y$ , if such a vertex exists. We define  $X_{xrbg}$  analogously to  $X_{yrbg}$  and observe that the colour pattern from  $r, b, g$  to  $X_{xrbg}$  must be the same as the one to  $X_{yrbg}$ . In fact, if this is not the case, then a similar analysis of the colours of the edges between  $\{r, b, g\}$  and  $X_{xrbg}$  yields that for some  $i \in \{b, g\}$ , we know that the edges connecting  $i$  to  $X_{xrbg}$  are of the same colour as the edges connecting  $i$  to  $X_{rbg}$ . Without loss of generality, let us say that  $i$  is  $g$ . Then the edges between  $b$  and  $X_{xrbg}$  are red and the edges between  $r$  and  $X_{xrbg}$  are green, otherwise  $X_{xrbg}$  and  $X_{rbg}$  would not be disjoint sets. Figure 4.2 shows the colouring of the edges incident to  $X_{yrbg}$  and  $X_{xrbg}$ . Since  $|X_{yrbg}|, |X_{xrbg}| \gg \ln n/p$ , we have that there is some edge  $uv$  between  $X_{yrbg}$  and  $X_{xrbg}$ . But then however we colour  $uv$ , we will get a monochromatic path connecting two vertices in  $\{r, b, g\}$ , which is a contradiction. Thus, the colour pattern of edges between  $\{r, b, g\}$  and  $X_{xrbg}$  is the same as the colour pattern of the edges between  $\{r, b, g\}$  and  $X_{yrbg}$ .

Therefore, we have that each vertex in  $N(X_{rbg})$  belongs to one of the monochromatic components  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$  or  $C_{\text{green}}(g)$ , while a vertex in  $V(G) \setminus N(X_{rbg})$  belongs to one of the monochromatic components  $C_{\text{red}}(r)$ ,  $C_{\text{green}}(b)$  or  $C_{\text{blue}}(g)$  where the latter two are the green component containing  $b$  and the blue component containing  $g$ , respectively.

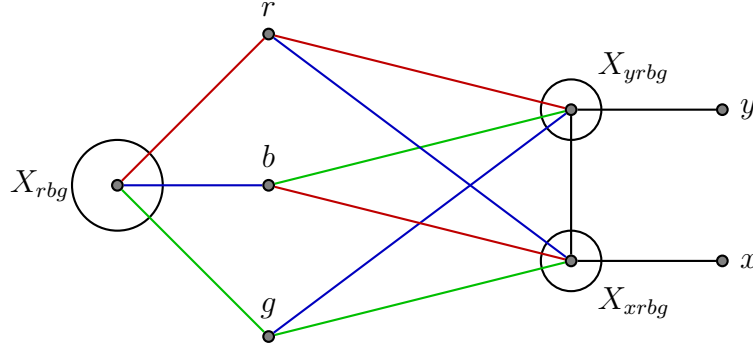


FIGURE 4.2. Analysis of the color of the edges incident on  $X_{yrbg}$  and on  $X_{xrbg}$ .

This gives a covering of  $G$  with five trees. Next we will show that actually three of those trees already cover all the vertices.

Suppose that at least 4 among the components  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$ ,  $C_{\text{green}}(b)$ ,  $C_{\text{green}}(g)$ , and  $C_{\text{green}}(b)$  are needed to cover all vertices. Since there does not exist any monochromatic path between any two of  $r, b, g$ , we know that for each  $i \in \{r, b, g\}$ , any monochromatic component containing  $i$  does not intersect  $\{r, g, b\} \setminus \{i\}$ . Hence, among those at least 4 components, we have for each  $i \in \{r, b, g\}$  one component containing it and without restriction two containing  $b$ . That is, three components of those at least 4 components needed to cover all the vertices are  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$  and  $C_{\text{green}}(b)$ . Now there are two cases regarding the fourth component: we need  $C_{\text{green}}(g)$  as the fourth component or we need  $C_{\text{blue}}(g)$  (those two cases might intersect).

We begin with the first case, where we need the components  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$ ,  $C_{\text{green}}(b)$  and  $C_{\text{green}}(g)$  to cover all the vertices of  $G$ . Let

$$\tilde{b} \in C_{\text{blue}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{green}}(b) \cup C_{\text{green}}(g))$$

and let

$$\tilde{g} \in C_{\text{green}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{blue}}(b) \cup C_{\text{green}}(g)).$$

Then let  $X_{\tilde{b}\tilde{g}rbg}$  be the maximum set of common neighbours of  $\tilde{b}, \tilde{g}, r, g, b$  such that for each  $i \in \{\tilde{b}, \tilde{g}, r, b, g\}$ , the edges from  $i$  to  $X_{\tilde{b}\tilde{g}rbg}$  are all coloured the same. Since  $|X_{\tilde{b}\tilde{g}rbg}| \geq np^5/108 \gg \frac{\ln n}{p}$ , we have

$$|E_G(X_{\tilde{b}\tilde{g}rbg}, X_{yrbg})| \geq 1 \quad \text{and} \quad |E_G(X_{\tilde{b}\tilde{g}rbg}, X_{xrbg})| \geq 1.$$

We will analyse the possible colours of the edges between the specified vertices and  $X_{\tilde{b}\tilde{g}rbg}$ . If for each of  $r, b, g$ , the colour it sends to  $X_{\tilde{b}\tilde{g}rbg}$  is different from the colour it sends to  $X_{rbg}$ , then any edge between  $X_{\tilde{b}\tilde{g}rbg}$  and  $X_{rbg}$  ensures a monochromatic path between two of  $r, b, g$  (in the colour of that edge). Similarly, it cannot happen that for each of  $r, b, g$ , the colour it sends to  $X_{\tilde{b}\tilde{g}rbg}$  is different from the colour it sends to  $X_{yrbg}$ . Thus, since  $r$  sends red to both  $X_{rbg}$  and  $X_{yrbg}$  while the colours from  $b$  (and  $g$ ) to  $X_{rbg}$  and  $X_{yrbg}$  are switched, the colour of the edges between  $r$  and  $X_{\tilde{b}\tilde{g}rbg}$  is red.



Now note, that by the choice of  $\tilde{b}$  and  $\tilde{g}$ , the edges between one of them and  $X_{\tilde{b}\tilde{g}rbg}$  can not be red. Further, the choice implies that an edge between  $\tilde{b}$  and  $X_{\tilde{b}\tilde{g}rbg}$  can not be of the same colour (green or blue) as an edge between  $\tilde{g}$  and  $X_{\tilde{b}\tilde{g}rbg}$ . If  $g$  would send blue (and hence  $b$  would send green) edges to  $X_{\tilde{b}\tilde{g}rbg}$ , there would either be a blue path between  $b$  and  $g$  (if the edges between  $\tilde{b}$  and  $X_{\tilde{b}\tilde{g}rbg}$  are blue) or  $\tilde{b}$  would lie in  $C_{\text{green}}(b)$  (if the edges between  $\tilde{b}$  and  $X_{\tilde{b}\tilde{g}rbg}$  are green). Since both those situations would mean a contradiction, we may assume that each of  $r, b, g$  sends edges with that colour to  $X_{\tilde{b}\tilde{g}rbg}$  as it does to  $X_{rbg}$ . But then  $X_{\tilde{b}\tilde{g}rbg}$  is actually a subset of  $X_{rbg}$  and therefore  $\tilde{g}$ , having an edge to  $X_{rbg}$ , lies in one of  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$ , or  $C_{\text{green}}(g)$ , a contradiction.

In the case where the forth component that we need is  $C_{\text{blue}}(g)$ , we repeat the construction of  $X_{\tilde{b}\tilde{g}rbg}$  similarly as before by letting

$$\tilde{b} \in C_{\text{blue}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{green}}(b) \cup C_{\text{blue}}(g))$$

and

$$\tilde{g} \in C_{\text{green}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{blue}}(b) \cup C_{\text{blue}}(g)).$$

Also as before, we end up with  $X_{\tilde{b}\tilde{g}rbg}$  being part of  $X_{rbg}$ . From the choice of  $\tilde{g}$ , the edges it sends to  $X_{\tilde{b}\tilde{g}rbg}$  have to be green, since otherwise it would be in  $C_{\text{red}}(r)$  or  $C_{\text{blue}}(b)$ . But that gives a green path between  $b$  and  $g$ , a contradiction.

Summarised, we infer that three of the components  $C_{\text{red}}(r)$ ,  $C_{\text{blue}}(b)$ ,  $C_{\text{green}}(b)$ ,  $C_{\text{green}}(g)$  and  $C_{\text{blue}}(g)$  cover the vertex set of  $G$ .  $\square$

#### 4.2. Shortcut graphs with independence number at most two.

*Proof of Lemma 4.2 for  $\alpha(F) \leq 2$ .* We start by noticing that if  $\alpha(F) = 1$ , then the graph  $F$  together with the colouring  $\varphi'$  is a complete 3-coloured graph and therefore, by Theorem 3.1, there exists a partition of  $V(F)$  into 2 monochromatic trees. Thus, we may assume that  $\alpha(F) = 2$ .

Let  $\mathcal{H}$  be the 3-uniform hypergraph with  $V(\mathcal{H})$  being the collection of all the monochromatic components of  $F$  under the colouring  $\varphi'$  and three monochromatic components form a hyperedge in  $\mathcal{H}$  if their intersection is non-empty. Notice that  $\mathcal{H}$  is 3-partite, since distinct monochromatic components of the same colour do not have a common vertex and therefore they can not belong to the same hyperedge. In other words, the colour of each component give us a 3-partition of the vertex set of  $\mathcal{H}$ . We denote by  $V_{\text{red}}, V_{\text{blue}}$  and  $V_{\text{green}}$  the set of vertices of  $V(\mathcal{H})$  that correspond to, respectively, red, blue and green components. Such construction was inspired by a construction due to Gyárfás [9].

Note that every vertex  $v$  of  $F$  is contained in a monochromatic component for each one of the colours (a monochromatic component could consist only of  $v$ ). Therefore, any vertex cover of  $\mathcal{H}$  corresponds to a covering of the vertices of  $F$  with monochromatic trees. In fact, if  $A$  is a vertex cover of  $\mathcal{H}$ , then consider the monochromatic components corresponding to each vertex in  $A$ . If any vertex  $v$  of  $F$  is not covered by those components, then the

vertices in  $\mathcal{H}$  corresponding to the red, green and blue components in  $F$  containing  $v$  do not belong to  $A$  and they form an hyperedge. But this contradicts the fact that  $A$  is a vertex cover of  $\mathcal{H}$ . Therefore,

$$\text{tc}(F, \varphi') \leq \tau(\mathcal{H}). \quad (4.1)$$

Let  $L = \bigcup_{s \in V_{\text{red}}} L_s$  the union of the link graphs  $L_s$  of all vertices  $s \in V_{\text{red}}$ . Any vertex cover of this bipartite graph  $L$  corresponds to a vertex cover of  $\mathcal{H}$  of the same size. Therefore,

$$\tau(\mathcal{H}) \leq \tau(L). \quad (4.2)$$

Furthermore, by König's theorem we know that  $\tau(L) = \nu(L)$ . Thus, if  $\nu(L) \leq 3$ , then by (4.1) and (4.2), we have

$$\text{tc}(F, \varphi') \leq \tau(\mathcal{H}) \leq \tau(L) = \nu(L) \leq 3.$$

Therefore, we may assume that

$$\nu(L) \geq 4,$$

and fix a matching  $M_L$  of size at least 4 in  $L$ . Let us say that  $M_L$  consists of the edges  $G_1B_1$ ,  $G_2B_2$ ,  $G_3B_3$ , and  $G_4B_4$ , where  $\{G_1, G_2, G_3, G_4\} \subseteq V_{\text{green}}$  and  $\{B_1, B_2, B_3, B_4\} \subseteq V_{\text{blue}}$ .

Now we give an upper bound for  $\nu(\mathcal{H})$ . Note that any matching  $M_{\mathcal{H}}$  in  $\mathcal{H}$  gives us an independent set  $I$  in  $F$ . Indeed, for each hyperedge  $e \in M_{\mathcal{H}}$ , let  $v_e \in V(F)$  be any vertex in the intersection of those monochromatic components associated to the vertices in  $e$  and let  $I = \{v_e : e \in M_{\mathcal{H}}\}$ . We claim that  $I$  is an independent set in  $F$ . In fact, if  $v_e$  and  $v_f$  are adjacent vertices in  $I$ , then  $e$  and  $f$  intersect, as the edge connecting  $v_e$  to  $v_f$  in  $F$  will connect the monochromatic components containing  $v_e$  and  $v_f$  of that colour that is given to the edge  $v_e v_f$ . Therefore, since  $\alpha(F) = 2$ , we have

$$\nu(\mathcal{H}) \leq \alpha(F) = 2. \quad (4.3)$$

Now, if there are three different edges in  $M_L$  that are edges in the link graphs of three different vertices of  $V_{\text{red}}$ , then there would be a matching of size 3 in  $\mathcal{H}$ , contradicting (4.3). Therefore, we may assume that  $M_L$  is contained in the union of at most two link graphs, say  $L_{R_1}$  and  $L_{R_2}$ , of vertices  $R_1, R_2 \in V_{\text{red}}$ . Now we are left with three cases: (Case 1) two edges of  $M_L$  belong to  $L_{R_1}$  and two belong to  $L_{R_2}$ ; (Case 2) three edges of  $M_L$  belong to  $L_{R_1}$  and one to  $L_{R_2}$ ; (Case 3) the four edges of  $M_L$  belong to  $L_{R_1}$ . Without loss of generality, we can describe each of those three cases as follows (see Figures 4.3, 4.4 and 4.5):

Case 1: The edges  $G_1B_1$  and  $G_2B_2$  belong to  $L_{R_1}$  and the edges  $G_3B_3$  and  $G_4B_4$  belong to  $L_{R_2}$ . That means that all the following four sets are non-empty:

$$J_1 := R_1 \cap G_1 \cap B_1,$$

$$J_2 := R_1 \cap G_2 \cap B_2,$$

$$J_3 := R_2 \cap G_3 \cap B_3,$$

$$J_4 := R_2 \cap G_4 \cap B_4.$$

Case 2: The edges  $G_1B_1$ ,  $G_2B_2$  and  $G_3B_3$  belong to  $L_{R_1}$  and the edge  $G_4B_4$  belongs to  $L_{R_2}$ . That means that all the following four sets are non-empty:

$$J_1 := R_1 \cap G_1 \cap B_1,$$

$$J_2 := R_1 \cap G_2 \cap B_2,$$

$$J_3 := R_1 \cap G_3 \cap B_3,$$

$$J_4 := R_2 \cap G_4 \cap B_4.$$

Case 3: The edges  $G_1B_1$ ,  $G_2B_2$ ,  $G_3B_3$  and  $G_4B_4$  belong to  $L_{R_1}$ . That means that all the following four sets are non-empty:

$$J_1 := R_1 \cap G_1 \cap B_1,$$

$$J_2 := R_1 \cap G_2 \cap B_2,$$

$$J_3 := R_1 \cap G_3 \cap B_3,$$

$$J_4 := R_1 \cap G_4 \cap B_4.$$

In addition, note that in this case we may also assume that there exists a red component  $R_2$  distinct from  $R_1$  such that the set

$$J_5 := R_2 \cap G_4 \cap B_3 \tag{4.4}$$

is non-empty. To see this, let  $R_2$  be any other red component different from  $R_1$ . Then let  $B$  and  $G$  be, respectively, a blue and a green component with  $R_2 \cap B \cap G \neq \emptyset$ . Without loss of generality, let us say that  $G \notin \{G_1, G_2, G_3\}$ . If  $B \notin \{B_1, B_2, B_3\}$ , then the edges  $G_1B_1$ ,  $G_2B_2$  and  $G_3B_3$  belong to  $L_{R_1}$  and the edge  $GB$  belongs to  $L_{R_2}$ , just as in Case 2. So let us assume that  $B \in \{B_1, B_2, B_3\}$ . Let us say, without loss of generality, that  $B = B_3$ . Now if  $G \neq G_4$ , then the edges  $G_1B_1$ ,  $G_2B_2$  and  $G_4B_4$  belong to  $L_{R_1}$  and the edge  $GB$  belongs to  $L_{R_2}$ , just as in Case 2 again. Therefore we may assume that  $G = G_4$  and then the set  $J_5$  defined by 4.4, which corresponds to  $R_2 \cap G \cap B$ , is non-empty.

Let us further remark that, since  $\nu(\mathcal{H}) \leq 2$ , in each case above, we have

$$V(F) = R_1 \cup R_2 \cup G_1 \cup G_2 \cup G_3 \cup G_4 \cup B_1 \cup B_2 \cup B_3 \cup B_4.$$

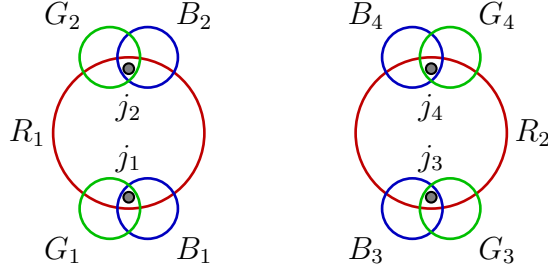


FIGURE 4.3. Case 1

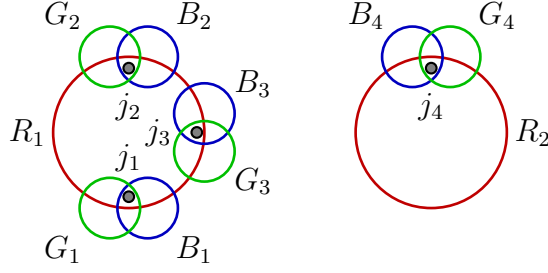


FIGURE 4.4. Case 2

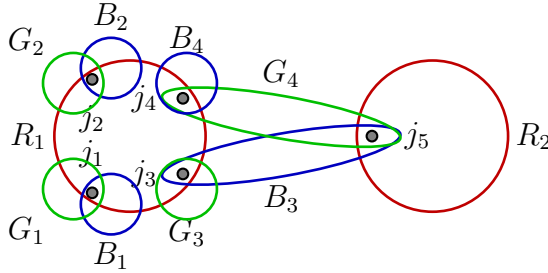


FIGURE 4.5. Case 3

Otherwise, for any uncovered vertex  $v \in V(F)$ , the hyperedge given by the red, blue and green component containing  $v$  together with the hyperedges  $R_1 B_1 G_1$  and  $R_2 B_3 G_3$  (in Cases 1 and 2) or  $R_2 B_3 G_4$  (in Case 3) give a matching of size 3 in  $\mathcal{H}$ .

Let us start with Case 1.

*Proof of Case 1:* We will prove that  $R_1$  and  $R_2$  together with possibly one further monochromatic component cover  $V(F)$ . For each  $i \in [4]$ , let  $\tilde{B}_i = B_i \setminus (R_1 \cup R_2)$  and  $\tilde{G}_i = G_i \setminus (R_1 \cup R_2)$ .

Pick vertices  $j_i \in J_i$ , with  $i \in [4]$ , arbitrarily. Consider a vertex  $o \in \tilde{B}_1$  (if such a vertex exists). Since  $\alpha(F) = 2$ , there is an edge connecting two of  $o, j_2, j_3$ . Because  $j_2$  and  $j_3$  belong to different components of each colour, such an edge must be incident to  $o$ . So let us say that such edge is  $oj_i$ , for some  $i \in \{2, 3\}$ . Since  $o \notin R_1 \cup R_2$ , the edge  $oj_i$  cannot be red. And since  $o \in B_1$ ,  $oj_i$  cannot be blue either, otherwise we would connect the blue components  $B_1$  and  $B_i$ . Now assume that  $o$  and  $j_2$  are not adjacent. Then  $oj_3$  is a green edge in  $F$ . By analogously analysing the edge between  $o, j_2$  and  $j_4$  together with the

supposition that  $oj_2$  is not an edge in  $F$ , we get that  $oj_4$  must be a green edge in  $F$ . But then we have a green path  $j_3oj_4$  connecting  $j_3$  to  $j_4$ , a contradiction. Therefore  $oj_2$  is an edge in  $F$  and it is green. That implies that  $o \in G_2$ . Therefore  $\tilde{B}_1 \subseteq G_2$ . Analogously, we can conclude the following:

$$\begin{aligned} \tilde{B}_1 &\subseteq G_2, & \tilde{G}_1 &\subseteq B_2, \\ \tilde{B}_2 &\subseteq G_1, & \tilde{G}_2 &\subseteq B_1, \\ \tilde{B}_3 &\subseteq G_4, & \tilde{G}_3 &\subseteq B_4, \\ \tilde{B}_4 &\subseteq G_3, & \tilde{G}_4 &\subseteq B_3. \end{aligned} \tag{4.5}$$

**Claim 4.3.** *We have  $\tilde{B}_1 \cup \tilde{G}_1 \cup \tilde{B}_2 \cup \tilde{G}_2 = \emptyset$  or  $\tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4 = \emptyset$ .*

*Proof.* Suppose for a contradiction that there exist  $o_1 \in \tilde{B}_1 \cup \tilde{G}_1 \cup \tilde{B}_2 \cup \tilde{G}_2$  and  $o_2 \in \tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4$ . Recall that from our choice of  $p$ , there is some  $z \in N(j_1, j_2, j_3, j_4, o_1, o_2)$ . Two of the edges  $zj_i$ , for  $i \in [4]$ , have the same colour. Since each  $j_i$  belongs to different green and blue components, those two edges are red. Since  $\{j_1, j_2\} \in R_1$  and  $\{j_3, j_4\} \in R_2$ , those two red edges are either  $zj_1$  and  $zj_2$  or  $zj_3$  and  $zj_4$ . Let us say that  $zj_1$  and  $zj_2$  are red (the other case is similar). Then one of the edges  $zj_3$  and  $zj_4$  has to be green and the other blue. Now, since  $o_1 \notin R_1$ , the edge  $zo_1$  is either green or blue. Then one of the paths  $o_1zj_3$  or  $o_1zj_4$  is green or blue. This implies that  $o_1 \in B_3 \cup G_3 \cup B_4 \cup G_4$ . On the other hand, (4.5) implies that  $o_1 \in (B_1 \cup B_2) \cap (G_1 \cup G_2)$ . But then we reached a contradiction, since that would mean that  $o_1$  belongs to two different components of the same colour.  $\square$

We may assume w.l.o.g. that  $\tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4$  is empty. Then, recalling that  $\nu(\mathcal{H}) \leq 2$  and in view of (4.5), the union of the components  $R_1, B_1, G_1$  and  $R_2$  covers every vertex of  $F$ . If we show that  $B_1 \subseteq G_1 \cup R_1 \cup R_2$  or that  $G_1 \subseteq B_1 \cup R_1 \cup R_2$ , then we get three monochromatic components covering the vertices of  $F$ . Our next claim states precisely that.

**Claim 4.4.** *We have  $\tilde{B}_1 \setminus G_1 = \emptyset$  or  $\tilde{G}_1 \setminus B_1 = \emptyset$ .*

*Proof.* Suppose that there exist two distinct vertices  $b \in \tilde{B}_1 \setminus G_1$  and  $g \in \tilde{G}_1 \setminus B_1$ . Let  $z \in N(j_1, j_2, j_3, j_4, b, g)$ . As before, either  $zj_1$  and  $zj_2$  or  $zj_3$  and  $zj_4$  are red edges. First assume that  $zj_1$  and  $zj_2$  are red. Then one of the edges  $zj_3$  and  $zj_4$  has to be green and the other blue. Now, since  $b \notin R_1$ , the edge  $zb$  is either green or blue. Then one of the paths  $bzj_3$  or  $bzj_4$  is green or blue. This implies that  $b \in B_3 \cup G_3 \cup B_4 \cup G_4$ . On the other hand, (4.5) implies that  $b \in B_1 \cap G_2$ . Then we reached a contradiction, since that would mean that  $b$  belongs to two different components of the same colour.

Therefore, the edges  $zj_3$  and  $zj_4$  are red and one of the edges  $zj_1$  and  $zj_2$  is green and the other is blue. First let us say that  $zj_1$  is green and  $zj_2$  is blue. Since  $b \notin (R_1 \cup R_2)$ , the edge  $zb$  cannot be red. Also the edge  $zb$  cannot be blue otherwise the path  $bzj_2$

would connect the components  $B_1$  and  $B_2$ . Finally,  $zb$  cannot be green, otherwise the path  $bzj_1$  would give us that  $b \in G_1$ . Therefore  $zj_1$  is blue and  $zj_2$  is green. But this case analogously leads to a contradiction (with  $g$  and  $G_i$  instead of  $b$  and  $B_i$  and green and blue switched).  $\square$

$\square$

We proceed to the proof of Case 2.

*Proof of Case 2:* As in Case 1, pick vertices  $j_i \in J_i$ , with  $i \in [4]$  arbitrarily. We claim that  $V(F) \subseteq R_1 \cup R_2 \cup B_4 \cup G_4$ . In fact, let  $o \in V(F) \setminus (R_1 \cup R_2)$ . Notice that since  $\alpha(F) = 2$ , there is an edge in each of the following sets of three vertices:  $\{o, j_4, j_1\}$ ,  $\{o, j_4, j_2\}$ , and  $\{o, j_4, j_3\}$ . We claim that  $oj_4$  is an edge of  $F$ . Indeed, if this was not the case, then since there cannot be an edge between  $j_4$  and  $j_i$  for  $i = 1, 2, 3$ , we would have the edges  $oj_1$ ,  $oj_2$  and  $oj_3$  and all of them would be coloured green or blue. Thus, two of them would be coloured the same, connecting two distinct components of one colour in this colour, a contradiction. So  $oj_4 \in E(F)$  and since  $oj_4$  cannot be red, we conclude that  $o \in (B_4 \cup G_4)$ . Therefore,  $R_1$ ,  $R_2$ ,  $B_4$  and  $G_4$  cover all vertices of  $F$ .

If  $B_4 \setminus (R_1 \cup R_2 \cup G_4) = \emptyset$  or  $G_4 \setminus (R_1 \cup R_2 \cup B_4) = \emptyset$ , then we get three monochromatic components covering  $V(F)$ . So let us assume that there exist  $b \in B_4 \setminus (R_1 \cup R_2 \cup G_4)$  and  $g \in G_4 \setminus (R_1 \cup R_2 \cup B_4)$ . If  $b$  and  $g$  are not adjacent, then since each of the sets  $\{b, g, j_i\}$ , for  $i = 1, 2, 3$ , has to induce at least one edge, there are two edges between  $b$  and  $\{j_1, j_2, j_3\}$  or two edges between  $g$  and  $\{j_1, j_2, j_3\}$ . However, from the choice of  $b$ , we know that all the edges between  $b$  and  $\{j_1, j_2, j_3\}$  are green, and therefore two of such edges would give us a green connection between two different green components, a contradiction. Similarly, from the choice of  $g$ , we know that all the edges between  $g$  and  $\{j_1, j_2, j_3\}$  are blue, and two of such edges would give us a blue connection between two different blue components, again a contradiction.

Hence, we conclude that  $bg \in F$  for any  $b \in B_4 \setminus (R_1 \cup R_2 \cup G_4)$  and any  $g \in G_4 \setminus (R_1 \cup R_2 \cup B_4)$  and any such edge  $bg$  is red. Therefore, there is a red component  $R_3$  covering  $(B_4 \triangle G_4) \setminus (R_1 \cup R_2)$ , where  $B_4 \triangle G_4$  denotes the symmetric difference. If  $B_4 \cap G_4 = \emptyset$ , then  $R_1$ ,  $R_2$  and  $R_3$  cover  $V(F)$  and we are done. Otherwise, let  $x \in B_4 \cap G_4$ . If  $R_2 \setminus (R_1 \cup B_4 \cup G_4) = \emptyset$ , then  $R_1$ ,  $B_4$ ,  $G_4$  cover  $V(F)$  and we are done. If not, let  $y \in R_2 \setminus (R_1 \cup B_4 \cup G_4)$ . Note that  $xy, xj_i \notin E(F)$ , for  $i \in \{1, 2, 3\}$ , since otherwise two different components of one colour would be connected in that colour. Now  $\alpha(F) = 2$  implies that  $yj_i \in E(F)$ , for  $i \in \{1, 2, 3\}$ . But these edges must all be green or blue, hence two of them are of the same colour, connecting two different components of one colour in that colour, a contradiction.  $\square$

We arrived at the last case, Case 3.

*Proof of Case 3:* Similarly to the previous cases, let us pick vertices  $j_i \in J_i$ , with  $i \in [5]$  arbitrarily. We will show first that we can cover all vertices of  $F$  with 4 monochromatic components. Let  $o_1, o_2 \in V(F) \setminus (R_1 \cup B_3 \cup G_4)$  and let  $z \in N(j_1, j_2, j_3, o_1, o_2, j_5)$ . At least one of the edges  $zj_1, zj_2$  and  $zj_3$  is red, as otherwise we would connect two distinct components of one colour in that colour. Therefore  $z \in R_1$ . Since  $o_1, o_2, j_5 \notin R_1$ , the edges  $zo_1, zo_2$  and  $zj_5$  cannot be red. Furthermore,  $o_1z$  and  $o_2z$  are coloured with a colour different from the colour of the edge  $j_5z$ , as otherwise they would belong to  $B_3$  or  $G_4$ . Thus,  $o_1$  and  $o_2$  are connected by a monochromatic path in green or blue. Hence, we showed that any two vertices of  $V(F) \setminus (R_1 \cup B_3 \cup G_4)$  are connected by a monochromatic path in green or blue. We infer that there is a green or blue component covering  $V(F) \setminus (R_1 \cup B_3 \cup G_4)$ . Therefore,  $R_1, B_3, G_4$  and one further blue or green component  $C$  cover all vertices of  $G$ . Let us assume that  $C$  is a green component; the case where  $C$  is a blue component is analogous.

We claim that  $R_1 \cup B_3 \cup C$ , or  $R_1 \cup G_4 \cup C$ , or  $R_1 \cup B_3 \cup G_4$  covers  $V(F)$ . Indeed, suppose for the sake of contradiction that there exist vertices  $g \in G_4 \setminus (R_1 \cup B_3 \cup C)$ ,  $b \in B_3 \setminus (R_1 \cup G_4 \cup C)$  and  $c \in C \setminus (R_1 \cup B_3 \cup G_4)$ . Let  $z \in N(j_1, j_2, j_3, g, b, c)$  and note that one of  $zj_1, zj_2$  and  $zj_3$  is red. Consequently  $gz, cz$  and  $bz$  are not red. Notice, however, that  $gz$  and  $bz$  can not be both green and neither both blue. Now let us say  $cz$  is green. Since  $c \notin G_4$  and  $g \in G_4$ , we would have  $gz$  blue in this case. But then  $bz$  must be green and since  $c \in C$  and  $C$  is a green component, we have  $b \in C$ , which is a contradiction. Therefore  $cz$  must be blue. Then, since  $c \notin B_3$  and  $b \in B_3$ , the edge  $bz$  should be green. Thus the edge  $gz$  is blue. Since this argument holds for any  $g \in G_4 \setminus (R_1 \cup B_3 \cup C)$  and  $c \in C \setminus (R_1 \cup B_3 \cup G_4)$ , we conclude that  $V(F) \setminus (R_1 \cup B_3)$  can be covered by one blue tree. Hence,  $G$  can be covered by the three monochromatic trees.  $\square$

This finishes the last case and thereby the proof of Lemma 4.2.  $\square$

## REFERENCES

- [1] R. Aharoni, *Ryser's conjecture for tripartite 3-graphs*, Combinatorica **21** (2001), no. 1, 1–4. [↑1, 3](#)
- [2] D. Bal and L. DeBiasio, *Partitioning random graphs into monochromatic components*, arXiv preprint arXiv:1509.09168 (2015). [↑1](#)
- [3] B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York. [MR1633290](#) [↑3](#)
- [4] ———, *Random graphs*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge. [MR1864966](#) [↑3](#)
- [5] J. A. Bondy, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York. [MR2368647](#) [↑3](#)
- [6] M. Bucić, D. Korándi, and B. Sudakov, *Covering random graphs by monochromatic trees and Helly-type results for hypergraphs*, arXiv preprint arXiv:1902.05055 (2019). [↑1](#)
- [7] R. Diestel, *Graph theory*, 4th ed., Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg. [MR2744811](#) [↑3](#)



- [8] P. Erdős, A. Gyárfás, and L. Pyber, *Vertex coverings by monochromatic cycles and trees*, Journal of Combinatorial Theory, Series B **51** (1991), no. 1, 90–95. [↑1](#), [3](#)
- [9] A. Gyárfás, *Partíciófedések és lefogóhalmazok hipergráfokban: Kandidátusi értekezés*, MTA Számítástechnikai és Automatizálási Kutató Intézet, 1977. [↑1](#), [2](#), [4.2](#)
- [10] P. E Haxell and Y. Kohayakawa, *Partitioning by monochromatic trees*, journal of combinatorial theory, Series B **68** (1996), no. 2, 218–222. [↑1](#)
- [11] P. E Haxell and A. D Scott, *On Ryser’s conjecture*, the electronic journal of combinatorics **19** (2012), no. 1, 23. [↑1](#)
- [12] J. R. Henderson, *Permutation decomposition of  $(0, 1)$ -matrices and decomposition transversals*, Ph.D. Thesis, 1971. [↑3](#)
- [13] S. Janson, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York. [MR1782847](#) [↑3](#)
- [14] Y. Kohayakawa, G. O. Mota, and M. Schacht, *Monochromatic trees in random graphs*, Mathematical proceedings of the cambridge philosophical society, 2019, pp. 191–208. [↑1](#), [1](#), [1](#)
- [15] D. Korándi, F. Mousset, R. Nenadov, N. Škorić, and B. Sudakov, *Monochromatic cycle covers in random graphs*, Random Structures Algorithms **53** (2018), no. 4, 667–691. [MR3875280](#) [↑3](#)

DEPARTAMENTO DE CIÊNCIA DA COMPUTAÇÃO, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA,  
UNIVERSIDADE DE SÃO PAULO, BRAZIL

*Email address:* `yoshi@ime.usp.br`

INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA, RIO DE JANEIRO, BRAZIL

*Email address:* `walner@impa.br`

CENTRO DE MATEMÁTICA, COMPUTAÇÃO E COGNIÇÃO, UNIVERSIDADE FEDERAL DO ABC, BRAZIL

*Email address:* `g.mota@ufabc.edu.br`

FACHBEREICH MATHEMATIK, UNIVERSITÄT HAMBURG, HAMBURG, GERMANY

*Email address:* `bjarne.schuelke@uni-hamburg.de`