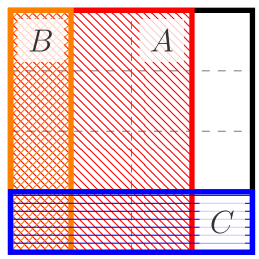
In-class Activity

ME 570 - Prof. Tron

2023-11-14

Problem 1: Review of probability

Consider a probability space $(\Omega, \mathcal{F}, P)$ where $P(A) = \frac{\mu(A)}{\mu(\Omega)}$, μ is a measure such that one square has unit area, that is, $\mu(\square) = 1$, and consider also the events (sets of outcomes) $A, B, C \in \mathcal{F}$ as represented in the figure (note that B is a subset of A).



In the following, when asked to "verify" a relation, first state the formula, then check with the numerical values.

1) Compute the probabilities of the events A, A^c, B, B^c, C, as well as the conditional probabilities $P(B|A), P(A|B), P(A|C), P(C|A)$ using the areas of the sets (keep the results as fractions). These values will be used for the other steps below.

Event	P(Event)	Event	P(Event)	Event	P(Event)	Event	P(Event)
A		B		$P(B \mid A)$		$P(A \mid C)$	
		B^c					
A^c		C		$P(A \mid B)$		$P(C \mid A)$	

2) Verify the relation between $P(A)$ and $P(A^c)$.

3) Given that $B \subset A$, verify the relative orderings of $P(A)$ versus $P(B)$ and $P(A^c)$ versus $P(B^c)$.

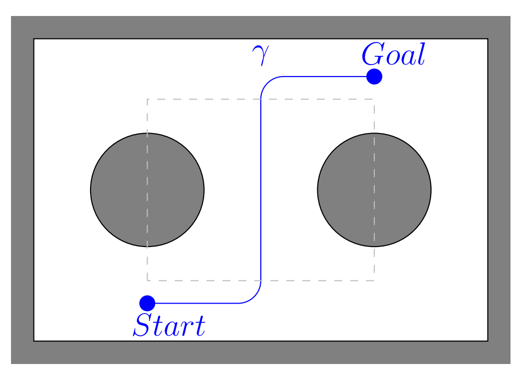
4) Verify that A and C are statistically independent ($A \perp C$).

5) Verify that $P(A|C) = P(A)$ and $P(C|A) = P(C)$ (which are implied by the fact that $A \perp C$).

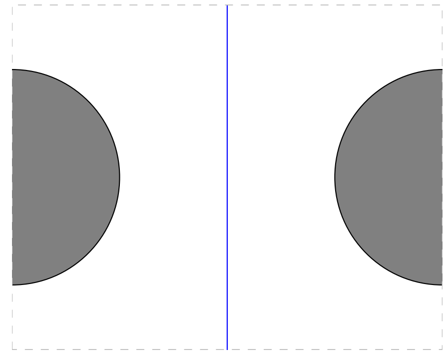
6) Verify the union bound $P(A \cup C) \leq P(A) + P(C)$.

Problem 2: Proof of probabilistic completeness of `PRM`

Question 2.1. In the figure on the right, indicate the path clearance ρ, and draw a covering of the reference path γ with a minimum number m of balls $\mathcal{B}(\cdot, \frac{\rho}{2})$ such that centers x_i, x_{i+1} of consecutive balls are contained in each other's balls. Given that the length of the path γ is L, give a formula for m:



Question 2.2. In the zoomed-in figure on the right, draw two centers x_i, x_{i+1}, the the corresponding balls $\mathcal{B}(x_i, \frac{\rho}{2})$, $\mathcal{B}(x_i, \frac{\rho}{2})$, and two samples y_i, y_{i+1}. Then use the triangular inequality to show that $d(y_i, x_{i+1}) < \rho$.



Question 2.3. Let $\mu(\mathcal{S})$ be the *volume* of a region $\mathcal{S}$. Complete the following

$$\mu(\mathcal{B}(\cdot, \frac{\rho}{2})) = \boxed{} \mu(\mathcal{B}(\cdot, 1)) \tag{1}$$

Question 2.4. Let the event I_i^c be the event where n samples $\{y_j\}_{j=1}^n$ all fall outside of $\mathcal{B}(x_i, \frac{\rho}{2})$. Using the previous answer and the bound $(1-\beta)^n \leq \exp(-\beta n)$, write a bound on the probability $P(I_i^c)$.

Question 2.5. Using the events I_i^c, give a bound on the probability that `PRM` fails when using n uniform samples $\{y_i\}_{i=1}^n$.

Question 2.6. Explain why `PRM` is probabilistically complete.