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In-class Activity

ME 570 - Prof. Tron

2023-09-19

For questions tagged with the label **scan**, answer all the questions in the space provided; to submit, scan your work and submit to Gradescope. To scan your work, you can use the flatbed scanner in the ME department office, or an app on your phone such as Adobe Scan or Dropbox, available for both iOS and Android. Please make sure to upload the pages in the correct order, and that the scans have good contrast. If needed, you can download and print this document from Blackboard. For the question tagged with the label **image** require you need to submit an image on Gradescope for evaluation.

Note that questions tagged differently will appear in separate assignments on Gradescope.

Problem 1: Reference frames and coordinates

Consider the 2-D world in Fig. 1 with reference frames $\mathcal{W}$ (e.g., world) and $\mathcal{B}$ (e.g., the body of a robot); note that $\mathcal{B}$ is rotated by exactly 45 degrees, the grid spacing is exactly one unit, and the axes of the reference frames are orthonormal. For some of the answers, you will need a little bit of trigonometry; write your answer using $\sqrt{2}$ when appropriate (i.e., do not expand to decimal numbers).

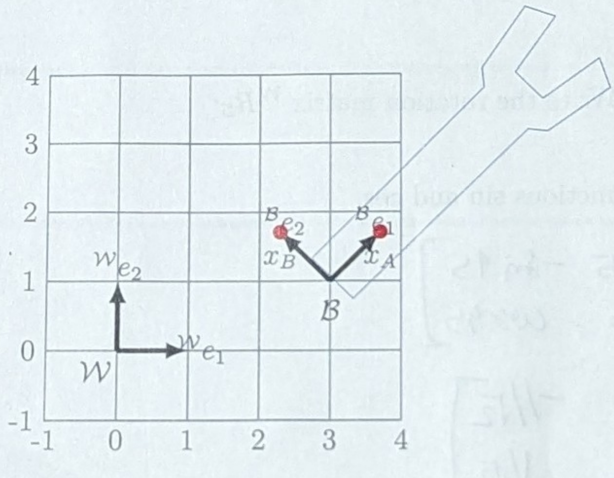

Figure 1: Points and rigid body for rigid body transformation example

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Question scan **1.1.** Write the coordinates of point x_A expressed in $\mathcal{W}$ and then $\mathcal{B}$.

$$^W x_A = \begin{bmatrix} 3+\sin 45 \\ 1+\sin 45 \end{bmatrix} = \begin{bmatrix} 3+\frac{1}{\sqrt{2}} \\ 1+\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$^B x_A = O_B + B_{e_1} + B_{e_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Question scan **1.2.** Write the coordinates of point x_B expressed in $\mathcal{W}$ and then $\mathcal{B}$.

$$^W x_B = \begin{bmatrix} 3-\frac{1}{\sqrt{2}} \\ 1+\frac{1}{\sqrt{2}} \end{bmatrix} \qquad ^B x_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Question scan **1.3.** Write the coordinates of point $O_\mathcal{B}$ (the origin of the frame $\mathcal{B}$) expressed in $\mathcal{W}$ and then $\mathcal{B}$.

$$^W x_{O_B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad ^B x_{O_B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Question scan **1.4.** Write the translation vector $^{\mathcal{W}}T_{\mathcal{B}}$.

$$^W T_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Question scan **1.5.** Write the rotation matrix $^{\mathcal{W}}R_{\mathcal{B}}$:

- In terms of $\sqrt{2}$.
- In terms of the functions sin and cos.

$$^W R_B = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

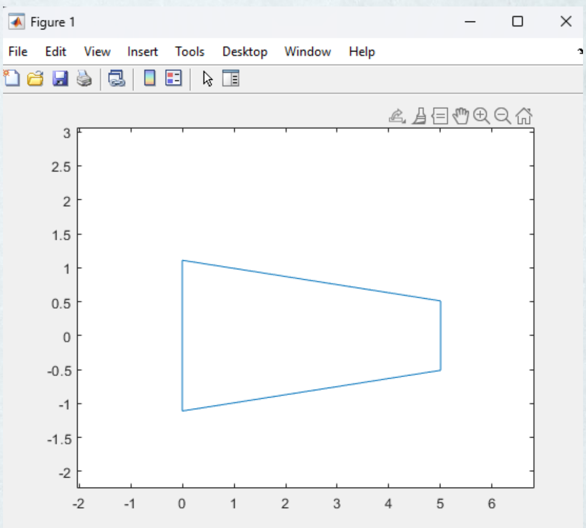
Question scan 1.6. Compute the determinant of ${}^W R_B$ (starting from the answer of Question scan 1.5)

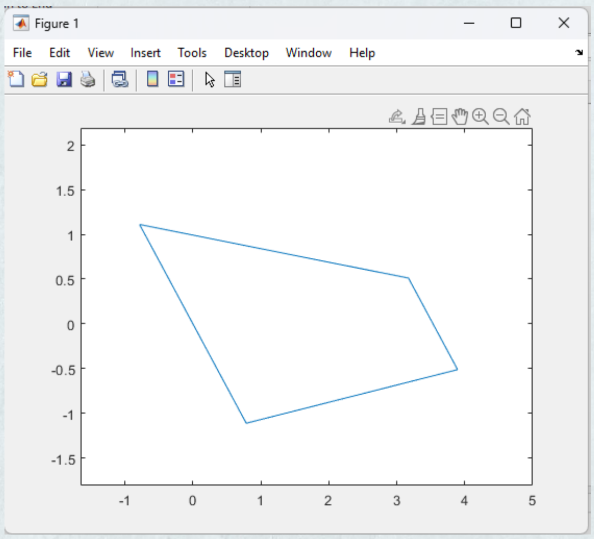
$$\det {}^W R_B = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

Question scan 1.7. Compute ${}^W R_B \, {}^B x_A + {}^W T_B$ (using the answers to Questions 1, scan 1.4 and scan 1.5), and compare it with your answer to Question 1 (hint: they should be the same).

$${}^W R_B \, {}^B x_A + {}^W T_B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 + 1/\sqrt{2} \\ 1 + 1/\sqrt{2} \end{bmatrix} = {}^W x_A$$

Question image 1.1. Using the function `rot2d( )` and the second polygon from `twolink_polygons( )` (in Matlab) or `me570_geometry.rot2d( )` and `me570_robot.polygons` (in Python) from Homework 2, reproduce the blue shape shown in Fig. 1. Specifically, starting from the polygon expressed in the frame $\mathcal{B}$ (i.e., the provided coordinates), transform them in the frame $\mathcal{W}$, and plot them.





Problem 2: Rigid body transformations composition and inversion

Consider the "Tea Cup Ride" in Figure 2a, modeled as in Figure 2b with three reference frames affixed to the world ($\mathcal{W}$), to the plate ($\mathcal{B}_1$), and to the cup ($\mathcal{B}_2$). Assume you know the rigid body transformation from cup to plate, $({}^{\mathcal{B}_1}R_{\mathcal{B}_2}, {}^{\mathcal{B}_1}T_{\mathcal{B}_2})$, and from the plate to the world, ${}^{\mathcal{W}}R_{\mathcal{B}_1}$. Let x be an arbitrary point on the cup, having known coordinates ${}^{\mathcal{B}_2}x$.

(a) Real system

(b) Model

Figure 2: Tea Cup Ride and its model

Question **scan** **2.1.** Write the expression for x:

- Expressed in $\mathcal{B}_1$, i.e., ${}^{\mathcal{B}_1}x$, as a function of ${}^{\mathcal{B}_2}x$.
- Expressed in $\mathcal{W}$, i.e., ${}^{\mathcal{W}}x$, as a function of ${}^{\mathcal{B}_1}x$.

$$^{W}R_{B_2} = {}^{W}R_{B_1}\,{}^{B_1}R_{B_2}$$

$$^{W}T_{B_2} = {}^{W}R_{B_1}\,{}^{B_1}T_{B_2} + \cancel{{}^{W}T_{B}} \to 0$$

If 2 reference frame do coincide then we can say $B_1 = B_2$

$$^{B_1}R_{B_1},\ {}^{B_1}T_{B_1} = (I, 0)$$

Question **scan** **2.2.** Write the expression for x expressed in $\mathcal{W}$, i.e., ${}^{\mathcal{W}}x$, as a function of ${}^{\mathcal{B}_2}x$. Make sure that, in the final expression you obtain, every quantity is actually know, according to the assumptions presented above. Then, rearrange the expression and highlight the parts of the formula that correspond to ${}^{\mathcal{W}}R_{\mathcal{B}_2}$ and ${}^{\mathcal{W}}T_{\mathcal{B}_2}$, i.e., the composition of rigid body motions $({}^{\mathcal{W}}R_{\mathcal{B}_2}, {}^{\mathcal{W}}T_{\mathcal{B}_2}) = ({}^{\mathcal{W}}R_{\mathcal{B}_1}, {}^{\mathcal{W}}T_{\mathcal{B}_1}) \circ ({}^{\mathcal{B}_1}R_{\mathcal{B}_2}, {}^{\mathcal{B}_1}T_{\mathcal{B}_2})$.

$$^{W}x = {}^{W}R_{B_1}\,{}^{B_1}R_{B_2}\,{}^{B_2}x + {}^{W}R_{B_1}\,{}^{B_1}T_{B_2}$$

Question scan **2.3.** Let the identity transformation be $(I, 0)$. Using the formula in the previous question, find the inverse of a rigid body transformation $({}^{W}R_{B}, {}^{W}T_{B})$, i.e., $({}^{W}R_{B}, {}^{W}T_{B})^{-1} = ({}^{B}R_{W}, {}^{B}T_{W})$ such that $({}^{W}R_{B}, {}^{W}T_{B})^{-1} \circ ({}^{W}R_{B}, {}^{W}T_{B}) = (I, 0)$.

$$ {}^{B_1}R^{W} = ({}^{W}R^{B_1})^T \quad ; \quad {}^{W}R_{B_1} \, {}^{B_1}T_W + {}^{W}T_{B_2} = 0 $$

$$ ({}^{W}R_{B_1})^T \, {}^{W}R_{B_1} \, {}^{B_1}T_W = -({}^{W}R_{B_1})^T \, {}^{W}T_{B_1} $$

$$ \therefore \; ({}^{W}R_{B_1}, {}^{W}T_{B})^{+} \cdot ({}^{W}R_{B}, {}^{W}T_{B}) \; \cancel{(I,0)} $$

$$ = \; {}^{B_1}T^{W} = -({}^{W}R_{B_1})^T \, {}^{W}T_{B_1} $$

Problem 3: Velocities for 2-D rotations and Jacobians

Preparation. Consider the parametrized curve in the space of 2-D rotations given by $R(t) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}$.

Question scan **3.1.** Compute $\dot{R}(t)$ by directly differentiating each entry.

$$ R = \begin{bmatrix} \cos wt & -\sin wt \\ \sin wt & \cos wt \end{bmatrix} ; \quad \dot{R}(t) = \frac{d}{dt}(R) = \begin{bmatrix} -w \sin wt & -w \cos wt \\ w \cos wt & -w \sin wt \end{bmatrix} $$

Question scan **3.2.** Compute $\hat{\omega}R(t)$, where $\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$

$$ \hat{w} R(t) = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \begin{bmatrix} \cos wt & -\sin wt \\ \sin wt & \cos wt \end{bmatrix} = \begin{bmatrix} -w \sin wt & -w \cos wt \\ w \cos wt & -w \sin wt \end{bmatrix} $$

Question scan **3.3.** Consider an end effector with position ${}^{\mathcal{W}}p_{\text{eff}}(t) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = R(t) \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. Compute the Jacobian of the map from t to ${}^{\mathcal{W}}p_{\text{eff}}$.

$$ J = \frac{d w_{eff}}{dt} = \frac{d}{dt}\left(R(t) \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right) ; \quad \frac{dR}{dt} = \cancel{-w \sin wt} \begin{bmatrix} -w \sin wt & -w \cos wt \\ w \cos wt & -w \sin wt \end{bmatrix} $$

$$ J = \begin{bmatrix} -w \sin wt & -w \cos wt \\ w \cos wt & -w \sin wt \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5w \sin wt \; \cancel{-5w \cos wt} \\ 5w \cos wt \end{bmatrix} $$

$$ \cancel{-5w \cos wt} \quad \cancel{-5w \cos wt} $$

$$ \therefore \; J = \begin{bmatrix} -5w \sin wt \\ 5w \cos wt \end{bmatrix} $$