In-class Activity

ME 570 - Prof. Tron

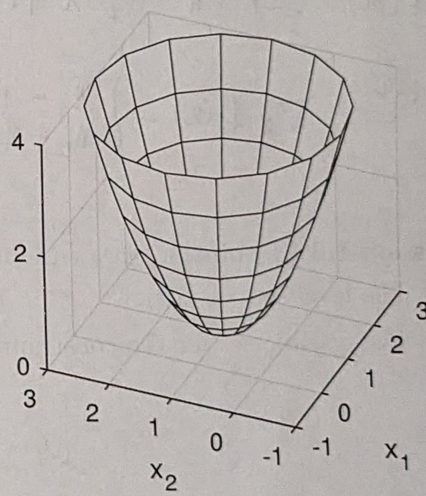
2023-10-12

For questions tagged with the label **scan**, answer all the questions in the space provided; to submit, scan your work and submit to Gradescope. To scan your work, you can use the flatbed scanner in the ME department office, or an app on your phone such as Adobe Scan or Dropbox, available for both iOS and Android. Please make sure to upload the pages in the correct order, and that the scans have good contrast. If needed, you can download and print this document from Blackboard. For the question tagged with the label **image** require you need to submit an image on Gradescope for evaluation.

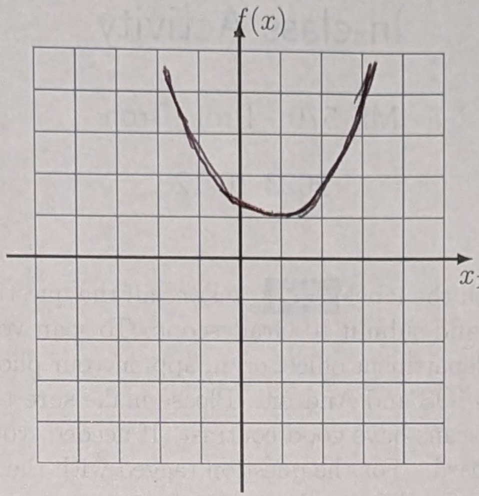
Problem 1: Constrained optimization

Consider the following quadratic cost.

$$f(x) = \frac{1}{2}(x - \begin{bmatrix}1\\1\end{bmatrix})^{\mathrm{T}}(x - \begin{bmatrix}1\\1\end{bmatrix}) = \frac{1}{2}\left\|x - \begin{bmatrix}1\\1\end{bmatrix}\right\|^2$$



Question 1.1. Sketch the cost function restricted to $x_2 = 0$ as a function of x_1 (do not worry about the scale on the y axis, just make sure that the location of the minimum is accurate).



Question 1.2. Write the function in the form of $f = \frac{1}{2}x^{\mathrm{T}}Px + q^{\mathrm{T}}x + c$ (i.e., find the values for P, q, c). Compute the gradient $\nabla_x f(x)$.

$f(x) = \frac{1}{2} x^T P x + q^T x + c$, Here $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$; $q = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$; $c = 5/2$

Now, $\nabla_x f(x) = \begin{bmatrix} x_1 - 1 + x_2 \\ x_2 + x_1 - 3 \end{bmatrix}$

Question 1.3 (3 points). Draw the following:

1) The level sets of $f(x)$.
2) The feasible set for the constraint $A_{\text{barrier}}x + b_{\text{barrier}} \leq 0$, where

$$A_{\text{barrier}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad b_{\text{barrier}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}$$

3) The locations of the unconstrained minimum argmin $f(x)$, and of the constrained minimum argmin $f(x)$subject to $A_{\text{barrier}}x + b_{\text{barrier}} \leq 0$.
4) Arrows for the gradient at those solutions.

- The Black Point represents the unconstrained minimum at (1,1).
- The arrow originating from the point (1,1) indicates the actual gradient at the unconstrained minimum
- The arrow originating from the point (0,0) indicates the gradient ~~of~~ at the potential contrained minimum
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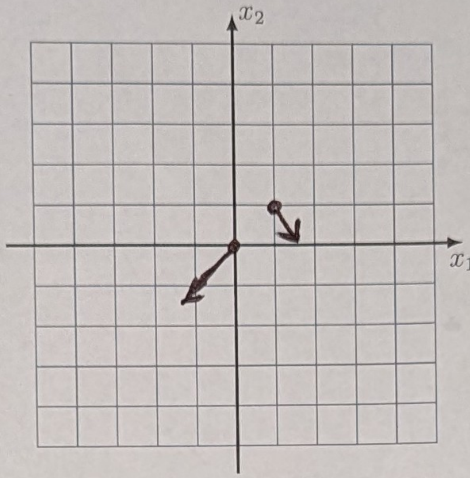

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Question image **1.1.** Use the provided function `qp_supervisor( )` to compute the solution to the constrained problem. Submit a screenshot of the code you used and the output, showing that the solution you obtained is consistent with your drawing above.

Problem 2: Lie derivative

Consider a filled-in sphere obstacle with center at the origin, $x_{\text{center}} = 0$ and radius $r = 2$. Let $h(x) = \|x - x_{\text{center}}\| - r$ be the signed distance, and $C = \{x \in \mathbb{R}^2 : h(x) \geq 0\}$ be the free configuration space.

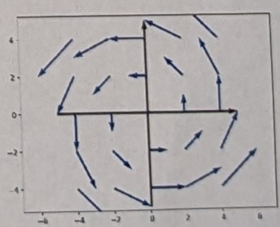
$$\nabla h = \left\{ \frac{x_1}{\sqrt{x_1^2+x_2^2}} ; \frac{x_2}{\sqrt{x_1^2+x_2^2}} \right\}$$

Question 2.1. Compute Lie derivative $\mathcal{L}_u h$ for a generic field $u(x)$.

The lie Derivative $L_u h = u \cdot \nabla h$; $h(x) = \|x - x_{\text{center}}\| - r$

and $u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}$ $\quad L_u h = u \cdot \nabla h = u_1(x) \dfrac{x_1}{\sqrt{x_1^2+x_2^2}} + u_2(x) \dfrac{x_2}{\sqrt{x_1^2+x_2^2}}$

Question optional **2.1.** Compute Lie derivative along $u(x) = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$ (shown in the figure for reference), assuming $x \in C$ (hint: it has the same value for any $x \in C$).



$u(x) = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; Lie Derivatives $L_u h = u \cdot \nabla h$

Here, $\nabla h = \left[\dfrac{x_1}{\sqrt{x_1^2+x_2^2}} , \dfrac{x_2}{\sqrt{x_1^2+x_2^2}} \right]$

$\therefore L_u h = \dfrac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left[\dfrac{x_1}{\sqrt{x_1^2+x_2^2}} , \dfrac{x_2}{\sqrt{x_1^2+x_2^2}} \right]$