In-class Activity

ME 570 - Prof. Tron

2023-09-26

For questions tagged with the label **scan**, answer all the questions in the space provided; to submit, scan your work and submit to Gradescope. To scan your work, you can use the flatbed scanner in the ME department office, or an app on your phone such as Adobe Scan or Dropbox, available for both iOS and Android. Please make sure to upload the pages in the correct order, and that the scans have good contrast. If needed, you can download and print this document from Blackboard.

Problem 1: Vector fields and solutions of ODEs

For each of the questions below:

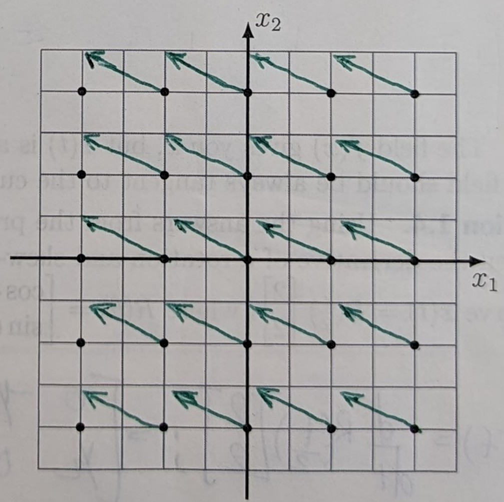
1) Draw the vector field $g\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$ as arrows at the marked points

2) Draw the image of the solution to the ODE $\dot{x} = g(x)$, $x(0) = \begin{bmatrix}2\\2\end{bmatrix}$ for $t \in [0, \infty)$.

3) Mark with an arrow or a circle any stationary point, or write *"No stationary points"* if there are none.

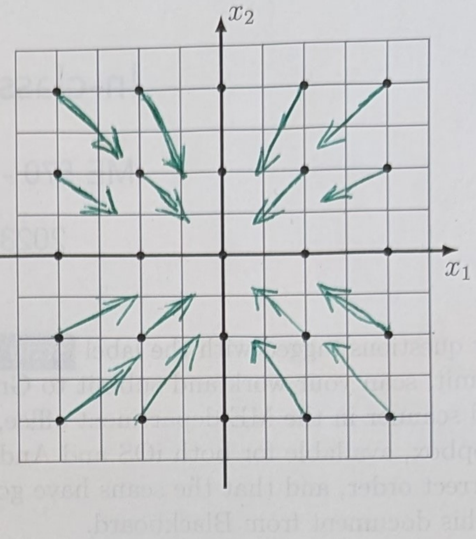
Question 1.1 (2 points).

$$g(x) = \begin{bmatrix}-2\\1\end{bmatrix}$$



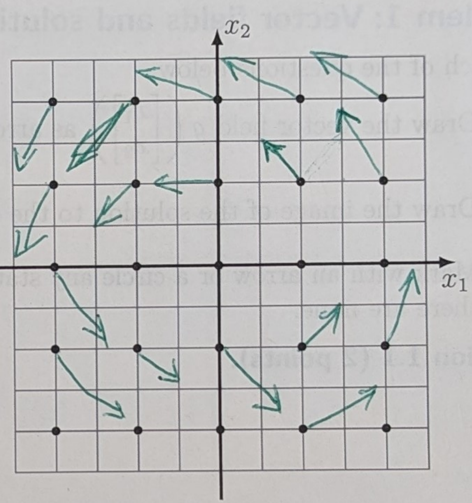
Question 1.2 (2 points).

$$g(x) = -\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(x - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + 1 \\ -\frac{1}{2} x_2 \end{bmatrix}$$



Question 1.3 (2 points).

$$g(x) = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$$



Hint: The field $f(x)$ gives you $\dot{x}$, but $\dot{x}(t)$ is also always tangent to $x(t)$. Hence, the arrows of the field should be always tangent to the curves you draw.

Question 1.4. Using the answers from the previous in-class activity regarding the relation between the derivative of a rotation and skew-symmetric matrices, compute the tangent to the curve $x(t) = R(\frac{t}{2}) \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, where $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\frac{d}{dt} x(t) = \frac{d}{dt} R(\tfrac{t}{2}) \begin{bmatrix} 2 \\ 2 \end{bmatrix}; = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} R(\tfrac{t}{2}) \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t)$$

$$\frac{d}{dt} x(t) = \begin{bmatrix} -2\sin\theta \\ \cos\theta - 2\sin\theta \end{bmatrix} \dot{\theta} \quad \& \quad x'(t) \text{ at } t = 0 \text{ i.e. } t = 2\theta$$

$\therefore$ Tangent vector = $x(t) = R(t/2) \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ at $t = 0$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Question 1.5. Comment how the answer to Question 1.4 might be related to the answer to Question 1.3.

$g(x)$ in answer for Question 1.3 is the tangent value at $x(t) = R(t)\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Problem 2: Gradients

Consider the following function $f : \mathbb{R}^d \to \mathbb{R}$.

$$f(x) = \frac{1}{2}\|x - x_0\|_A^2 = \frac{1}{2}(x - x_0)^{\mathrm{T}} A (x - x_0), \tag{1}$$

where $x \in \mathbb{R}^d$ is the variable, $A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, and $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Question 2.1. Compute the gradient of f **using the definition with the Lie derivative.**

$$\frac{d}{dt} f(x(t)) = \frac{d}{dt}[x(t) - x_0]^T A (x(t) - x_0) + [x(t) - (x_0)]^T A \frac{d}{dt}(x - x_0)$$

$$= \frac{1}{2} \dot{x}^T A (x - x_0) + \frac{1}{2}(x - x_0)^T A \dot{x}$$

$$= \nabla f^T \dot{x} \quad ; \quad a \in R \Rightarrow a^T = a$$

$$v^T w = w^T v$$

$$= \frac{1}{2}(x - x_0)^T (A^T + A) \dot{x}$$

$$\therefore \nabla f = \frac{1}{2}(2A(x - x_0)) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\left[x - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

Question 2.2. The Lie derivative along the gradient field is always positive, negative, non-positive, or non-negative? Why? What does it mean?

The lie derivative along the Gradient field is always non-positive because the Gradient feild is designed to guide the function towards higher values and a positive lie derivative would imply an increase in the functions value. contarary to the nature of gradient feild.

Question 2.3. Draw the *level sets* of f on the axes in Question $\boxed{1.2}$.

Question 2.4 (1.5 points). Let $d^2(x, x_{center}) = \|x - x_{center}\|^2$.

- Compute $\nabla_x d^2$.
- Write $\nabla_x d^2$ as a function of $\nabla_x d$, where $d(x, x_{center}) = \|x - x_{center}\|$.
- Write the expression of $\nabla_x d$.

$$\nabla_x d^2 = \begin{bmatrix} 2(x_1 - x_{center,1}) \\ 2(x_2 - x_{center\,2}) \end{bmatrix} \; ; \; \nabla_x d^2 = 2\nabla_x d.$$

$$\nabla_x d = \frac{x - x_{center}}{\|x - x_{center}\|}$$

Question 2.5. Let $d_{sphere}(x) = \|x - x_{center}\| - r$ be the *signed distance* of a point from the surface of a circular solid (filled-in) 2-D obstacle sphere having center at the point $x_{center} \in \mathbb{R}^2$ and radius $r > 0$. Compute $\nabla_x d_{sphere}$. Explain what happens for $x = x_{center}$.

$$\nabla_x d_{sphere} \text{ at } x = x_{centre} = \frac{x_{center_1} - x_{center_1}}{\|x_{center} - x_{center_1}\|}, \frac{x_{center} - x_{center_2}}{\|x_{center} - x_{center_2}\|}$$

$$\nabla_x d_{sphere} \text{ at } x = x_{center} = [0, 0].$$

It means at the center of sphere, there is no directional change in the signed distance.