U25712111

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In-class Activity

ME 570 - Prof. Tron

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Problem 1: Control Barrier Functions

Consider a filled-in sphere obstacle with center at the origin, $x_{\text{center}} = 0$ and radius $r = 2$. Let $h(x) = \|x - x_{\text{center}}\| - r$ be the signed distance, and $\mathcal{C} = \{x \in \mathbb{R}^2 : h(x) \geq 0\}$ be the free configuration space.

Question 1.1. Write the CBF constraint for h and the dynamics $\dot{x} = u$ (u is going to be the variable in the constraint), using $c_h = \frac{1}{4}$.

$$\nabla h^T f + \nabla h^T g u + c_h h \geq 0$$

$$\frac{x - x_0}{\|x - x_0\|}^T u \geq -\frac{1}{4}\left[\|x - x_{center}\| - 2\right]$$

$$\frac{x^T}{\|x\|} u \geq -\frac{1}{4}\left[\|x\| - 2\right]$$

Question 1.2 (0.5 points). Compute the constraints on u, and draw the feasible set given by the CBF condition and the condition $\|u\|^2 \leq 1$ for the points x_A, x_B, x_C shown below.

$$\frac{x - x_0}{\|x - x_0\|}^T u \geq -\frac{1}{4}\left[\|x - x_0\| - 2\right]$$

x_A

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} u \end{bmatrix} \geq 0$$

x_B

$$\begin{bmatrix} 0 & 1 \end{bmatrix} u \geq -1/2$$

x_C

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix} u \geq -\frac{1}{4}\left[4\sqrt{2} - 2\right]$$

$$\geq -\sqrt{2} + \frac{1}{2}$$

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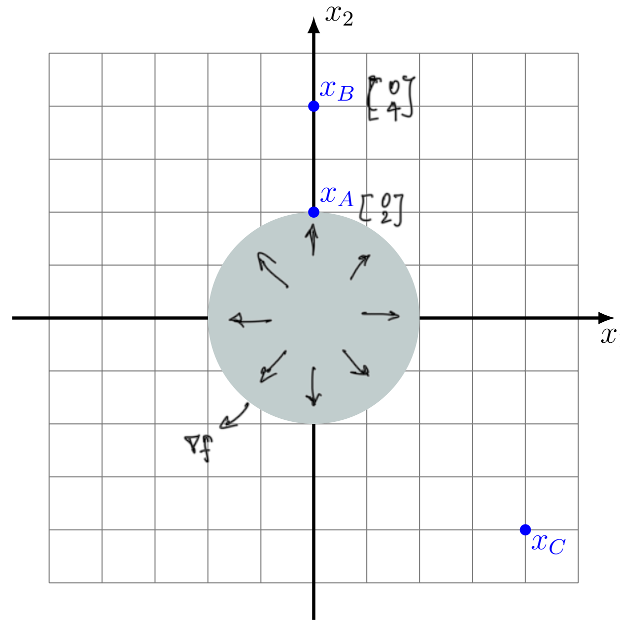

Note that the case for x_B coincides with the example of constrained optimization seen in the previous in-class activity.

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Problem 2: Consensus protocol

Consider a multi-agent systems with modeled with the graph $G = (V, E)$, with $V = \{1, 2, 3\}$, $E = \{(1, 2), (2, 3)\}$. Consider the inter-agent potential $U_{ij} = \frac{1}{2}\|x_i - x_j\|^2$, where x_i is the state of the i-th agent.

Question 2.1. Write the total potential U of the system.

$$U = \frac{1}{2}\|x_1 - x_2\|^2 + \frac{1}{2}\|x_2 - x_3\|^2$$

Question 2.2. Write the gradient of U for each node, that is $\nabla_{x_1}U$, $\nabla_{x_2}U$, $\nabla_{x_3}U$.

$\nabla_{x_1} U = \nabla_{x_1} \frac{1}{2}\|x_1 - x_2\|^2 = x_1 - x_2$ → Each gradient depends only on the state at the neighbour.

$\nabla x_2 U = 2x_2 - x_1 - x_3$

$\nabla x_3 U = x_3 - x_2$

Question 2.3. What are the stationary point of U? Why?

$\nabla x_1 \; U = 0$
$\nabla x_2 \; U = 0$
$\nabla x_3 \; U = 0$

Question 2.4. Explain why $\sum_i \nabla_{x_i}U = 0$ independently from the specific values of x_1, x_2, x_3.

$U_{ij} = \frac{1}{2}\|x_i - x_j\|^2$ | $\frac{\partial U_{ij}}{x_1} = x_i - x_j$ | $\sum \frac{\partial U}{\partial x_i} = 0$ —— (i)

$U = \sum_{(ij)\in E} U_{ij}$ | $\frac{\partial U_{ji}}{\partial x_i} = x_i - x_j$ |

Question 2.5. Using the above, and assuming that each node follows the update $\dot{x}_i = -\nabla_{x_i}U$, show that the average of the states is time-invariant, i.e., $\sum_i x_i(t) = \sum_i x_i(0)$ for all $t \geq 0$.

$\frac{d}{dt}\sum_i x_i = \sum_i \dot{x}_i = -\sum \frac{du}{dx_i}$ | So from 2.4
$x_i \gtrsim \nabla x_i U = 0$

Question 2.6. Using the answers to Questions 2.3, 2.5, what is the asymptotic value for the states x_i, i.e., what is the value of $\lim_t x_i(t)$?

$$\lim_{t \to \infty} x_i(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$$