Chapter 1

Congruency Theorems in Neutral Geometry

Note on notation: For a vertex A in $\triangle ABC$, we may refer to the interior angle $\angle CAB$ as $\angle a$. We denote the length of a segment $\overline{\mathrm{AB}}$ as $|\overline{\mathrm{AB}}|$. We do not rely on any definition of length except to say that $|\overline{\mathrm{AB}}| = |\overline{\mathrm{CD}}| \iff \overline{\mathrm{AB}} \cong \overline{\mathrm{CD}}$. Everywhere a statement about equality of lengths is made, it is equivalent to a statement about congruence of line segments or radii of circles, but we have chosen this notation for clarity and consistency with modern geometry. We assume here that angles have a measurement such that they can be related in equality and added. In particular, we assume the cancellation law for measurements of angles under addition, namely $\angle a + \angle b = \angle a + \angle c \iff \angle b = \angle c$.

Theorems are indexed $X.Y$ and constructions are indexed $X.Y.Z$.

1.1 Side-Angle-Side Congruence

Two triangles are congruent if two sides and the included angle of one are congruent respectively to two sides and the included angle of the other.

Proof. Let $\triangle ABC$ and $\triangle A'B'C'$ be constructed st. $\overline{\mathrm{AB}} \cong \overline{\mathrm{A'B'}}$, $\overline{\mathrm{BC}} \cong \overline{\mathrm{B'C'}}$, and $\angle b \cong \angle b'$. Move $\triangle A'B'C'$ so that $\overline{\mathrm{A'B'}}$ coincides with $\overline{\mathrm{AB}}$, and so that C and C' are on the same side of $\overline{\mathrm{AB}}$. By the definition of congruent angles, $\overline{\mathrm{BC}}$ and $\overline{\mathrm{B'C'}}$, starting from the same point B and extending out along $\angle b$ and $\angle b'$ respectively, coincide with each other. Since $\overline{\mathrm{BC}} \cong \overline{\mathrm{B'C'}}$, we know they have a common endpoint at C. Then, $\overline{\mathrm{AC}}$ and $\overline{\mathrm{A'C'}}$ are line segments connecting coinciding points A, A' to coinciding points C, C', so they must also coincide. Since we have now that each of the segments in $\triangle ABC$ coincides with a segment in $\triangle A'B'C'$, the two triangles are congruent.

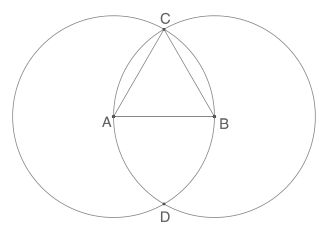
1.2 Angle-Side-Angle Congruence

Two triangles are congruent if two angles and the included side of one are congruent respectively to two angles and the included side of the other.

Proof. Let $\triangle ABC$ and $\triangle A'B'C'$ be constructed st. $\angle a \cong \angle a'$, $\overline{\mathrm{AB}} \cong \overline{\mathrm{A'B'}}$, and $\angle b \cong \angle b'$. Move $\triangle A'B'C'$ so that $\overline{\mathrm{A'B'}}$ coincides with $\overline{\mathrm{AB}}$, and so that C and C' are on the same side of $\overline{\mathrm{AB}}$. By the definition of congruent angles, $\overline{\mathrm{BC}}$ and $\overline{\mathrm{B'C'}}$, starting from the same point B and extending out along $\angle b$ and $\angle b'$ respectively, coincide with each other. Similarly, by the definition of congruent angles, $\overline{\mathrm{AC}}$ and $\overline{\mathrm{A'C'}}$, starting from the same point A and extending out along $\angle a$ and $\angle a'$ respectively, coincide with each other. By the definition of triangle, we know that $\overline{\mathrm{AC}}$ and $\overline{\mathrm{BC}}$ intersect at point C. Since $\overline{\mathrm{A'C'}}$ and $\overline{\mathrm{B'C'}}$ coincide with $\overline{\mathrm{AC}}$,$\overline{\mathrm{BC}}$ respectively, they must intersect at the point C' which coincides with C. Since A, B, C all coincide with A', B', C', the two triangles are congruent.

1.2.1 Construction of an Equilateral Triangle

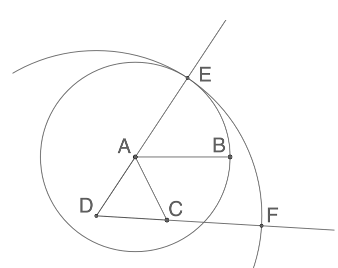
We will construct a triangle with three equal sides on a given base $\overline{\mathrm{AB}}$.



Draw a circle α with radius $|\overline{\mathrm{AB}}|$ centered at A, and a circle β with radius $|\overline{\mathrm{BA}}|$ centered at B [Euclid Post. 3]. Label a point of intersection of these circles C. Then $\triangle ABC$ is equilateral, since all points on α, including C, have distance $|\overline{\mathrm{AB}}|$ from A, so $|\overline{\mathrm{AC}}| = |\overline{\mathrm{AB}}|$. We can conclude with an identical argument $|\overline{\mathrm{BC}}| = |\overline{\mathrm{AB}}|$, thus showing $\triangle ABC$ is equilateral.
Note that this method of construction allows for two different equilateral triangles, depending on the choice of which intersection of α and β to use.

1.2.2 Construction of a Given Length

Given a line segment $\overline{\mathrm{AB}}$, and an arbitrary point C, we will construct a circle of radius $|\overline{\mathrm{AB}}|$ at C.

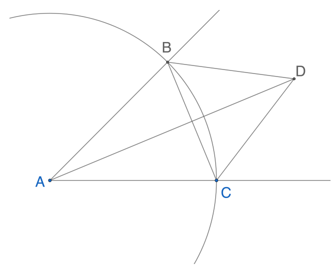


Connect A and C with a line, and with base $\overline{\mathrm{AC}}$ construct the equilateral triangle $\triangle ACD$ (the choice of intersection for D does not matter). Draw a circle, α, with a center at A with a radius of $\overline{\mathrm{AB}}$. Continue the line segments $\overline{\mathrm{DA}}$ and $\overline{\mathrm{DC}}$ indefinitely. Label the point of intersection between the line $\overrightarrow{\mathrm{DA}}$ and α E. Then draw a cir-

cle, β with center D and radius $\overline{\mathrm{DE}}$. Label the intersection of $\overrightarrow{\mathrm{DC}}$ and β F. Then, we claim $|\overline{\mathrm{AE}}| = |\overline{\mathrm{CF}}|$.
$|\overline{\mathrm{DE}}| = |\overline{\mathrm{DF}}|$, as they are both radii of β. We then have $|\overline{\mathrm{DE}}| = |\overline{\mathrm{DA}}| + |\overline{\mathrm{AE}}| = |\overline{\mathrm{DC}}| + |\overline{\mathrm{CF}}| = |\overline{\mathrm{DF}}|$. Since $|\overline{\mathrm{DA}}| = |\overline{\mathrm{DC}}|$, as they are both sides of the same equilateral triangle, we get $|\overline{\mathrm{AE}}| = |\overline{\mathrm{CF}}|$ from our former equality.

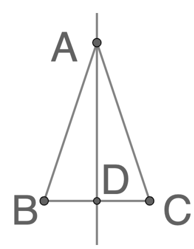
1.2.3 Angle Bisection

This will illustrate how to construct a line bisecting a given angle.



Given an angle formed by two rays extending from a point A, we draw a circle of arbitrary radius [Euclid Post. 3] and label its points of intersection on the two rays B and C. Now, with $\overline{\mathrm{BC}}$ as its base, we construct the equilateral triangle $\triangle BCD$, choosing point D as the further one from A. We then consider $\triangle ABD$ and $\triangle ACD$. They share the side $\overline{\mathrm{AD}}$. We have that $\overline{\mathrm{BD}} \cong \overline{\mathrm{CD}}$ as they are both part of the same equilateral triangle. Finally, $\overline{\mathrm{BC}} \cong \overline{\mathrm{AC}}$ as they are radii of the same circle centered at A. By [1.3], $\triangle ABD \cong \triangle ACD$. From this we have $\angle BAD \cong \angle DAC$, meaning $\overline{\mathrm{AD}}$ bisects $\angle BAC$.

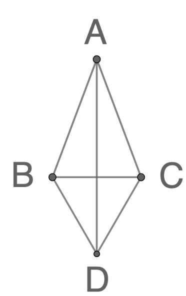
1.3 Isosceles Angle Congruence



The angles opposite the congruent sides of an isosceles triangle are congruent.

Proof. Let $\triangle ABC$ be isosceles, and let $\overline{\mathrm{AB}} \cong \overline{\mathrm{AC}}$. Construct the angle bisector of $\angle a$ [1.2.3] and extend it to intersect with $\overline{\mathrm{BC}}$, labeling the point of intersection D. Consider $\triangle ADC$ and $\triangle BDC$. We know since $\overline{\mathrm{AD}}$ is the angle bisector of $\angle a$, so $\angle BAD \cong \angle CAD$. The triangles have $\overline{\mathrm{AD}}$ in common, and have $\overline{\mathrm{AB}} \cong \overline{\mathrm{AC}}$. By [1.1, $\triangle ADC \cong \triangle BDC$, giving that $\angle ABC \cong \angle ACB$.

1.4 Isosceles Side Congruence

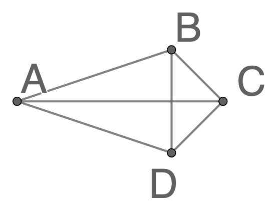


If two angles of a triangle are congruent, the sides opposite those angles are congruent (and so the triangle is isosceles). Note that this is the converse of [1.3].

Proof. Let $\triangle ABC$ have $\angle b \cong \angle c$. With base $\overline{\mathrm{BC}}$ we construct the equilateral triangle $\triangle BCD$, choosing D as the further point from A (see the note to [1.2.1]). Draw the line segment $\overline{\mathrm{AD}}$. Consider the triangles $\triangle ABD$ and $\triangle ACD$. These have $\overline{\mathrm{AD}}$ in common, have $\angle ADB \cong \angle ADC$ since they are the angles to an equilateral triangle, and have $\overline{\mathrm{BD}} \cong \overline{\mathrm{DC}}$, since these are the sides of an equilateral triangle. By [1.1] we have $\triangle ABD \cong \triangle ACD$, which gives $\overline{\mathrm{AB}} \cong \overline{\mathrm{AC}}$.

1.5 Side-Side-Side Congruence

Two triangles are congruent if the three sides of one are congruent respectively to the three sides of the other.



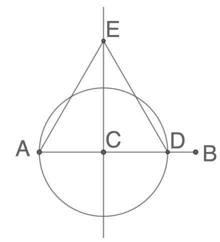
Proof. Let $\triangle ABC$ and $\triangle A'C'D$ have $|\overline{\mathrm{AC}}| = |\overline{\mathrm{A'C'}}|$, $|\overline{\mathrm{AB}}| = |\overline{\mathrm{A'D}}|$, and $|\overline{\mathrm{BC}}| = |\overline{\mathrm{DC'}}|$. Move $\triangle A'C'D$ so that $\overline{\mathrm{A'C'}}$ coincides with $\overline{\mathrm{AC}}$, and so that B and D are on different sides of $\overline{\mathrm{AC}}$. Draw $\overline{\mathrm{BD}}$. Now, $\triangle ABD$ is isosceles, and so $|\angle ABD| = |\angle ADB|$. $\triangle CBD$ is also isosceles, and so $|\angle CBD| = |\angle CDB|$. Since $|\angle ABC| = |\angle ABD| + |\angle CBD|$, and $|\angle ADC| = |\angle ADB| + |\angle CDB|$, substituting our previous angle equalities gives $\angle ABC \cong \angle ADC$. Then we have $\triangle ABC \cong \triangle A'C'D$ by [1.1].

Equilateral triangles have equal angles

Proof. This is a corollary of [1.5], but it is worth noting as it will be used later.

1.5.1 Construction of a Perpendicular Line off a Given Point

Given a line segment $\overline{\mathrm{AB}}$, and a point C on $\overline{\mathrm{AB}}$, we will construct a line extending from C that is perpendicular to $\overline{\mathrm{AB}}$ (makes right angles with $\overline{\mathrm{AB}}$).

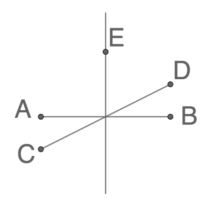


Centered at C, draw the circle α with the smaller of $|\overline{\mathrm{AC}}|$, $|\overline{\mathrm{CB}}|$ as its radius. WLOG, we will assume it is of radius $|\overline{\mathrm{AC}}|$. Let D be the point of intersection of α with $|\overline{\mathrm{AB}}|$ that is not A. Then, draw an equilateral triangle $\triangle ADE$ with base $|\overline{\mathrm{AD}}|$. Construct the line $|\overleftrightarrow{\mathrm{EC}}|$. We claim that

$|\overrightarrow{\mathrm{EC}}|$ is perpendicular to $\overline{\mathrm{AB}}$.
Clearly $\angle ACE$ and $\angle DCE$ are supplementary, and [1.5] shows that $\triangle ACE$ and $\triangle DCE$ are congruent. Since $\angle ACE$ and $\angle DCE$ are supplementary angles congruent to each other, they are right by definition.

1.6 Supplementary Angle Congruence

Supplementary angles are congruent to two right angles. Recall that supplementary angles are the angles formed by the intersection of two lines which share a common line segment.



Proof. Consider the supplementary angles $\angle AFD$ and $\angle DFB$ formed by the intersection of $\overline{\mathrm{AB}}$ with $\overline{\mathrm{CD}}$, labeling their point of intersection F. From F construct a line perpendicular to $\overline{\mathrm{AB}}$ [1.5.1]. Label a point on this line E. If $\angle AFD \cong \angle AFE$, the proof is trivial, so we will assume that $\overline{\mathrm{FE}}$ and $\overline{\mathrm{FD}}$ do not coincide, WLOG having $\angle AFE$ within $\angle AFD$. Then, $\angle AFD = \angle AFE + \angle EFD$, and $\angle EFB = \angle EFD + \angle DFB$. Adding these equalities gives:

$$\angle AFD + \angle EFD + \angle DFB = \angle AFE + \angle EFD + \angle EFB$$

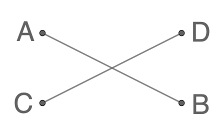
Which may simplify to

$$\angle AFD + \angle DFB = \angle AFE + \angle EFB$$

Which is exactly our theorem, as $\angle AFE + \angle EFB$ are two right angles.

1.7 Vertical Angle Congruence

Vertical angles are congruent. Recall that vertical angles (also called opposite angles) are the angles formed by the intersection of two lines which do not share a common line segment.



Proof. Consider the vertical angles $\angle AEC$ and $\angle DEB$ formed by the intersection of $\overline{\mathrm{AB}}$ with $\overline{\mathrm{CD}}$, labeling their point of intersection E. Let α represent the measure of two right angles. By [1.6], we have $\angle AEC + \angle AED = \alpha$, and by [1.6] we have $\angle DEB + \angle AED = \alpha$. From this we have $\angle AEC + \angle AED = \angle DEB + \angle AED$, and applying the cancellation property gives $\angle AEC = \angle DEB$.