- **Preparation**: See "What should I do in Week 2?" on SUNLearn.
- **Instructions**: This practical has to be done individually – no group work. Write your own practical report and hand it in on SUNLearn by Friday at 17:00. See SUNLearn for instructions.
- **Note**: Partial answers are given at the back of this assignment so that you can check your work. Study the Matlab/Simulink documentation by clicking on the *Help* menu if you do not know how to use a certain function/block.

- **Voorbereiding**: Sien "What should I do in Week 2?" op SUNLearn.
- **Instruksies**: Hierdie prakties moet individueel gedoen word – geen groepwerk. Skryf jou eie praktiese verslag en handig dit in op SUNLearn teen Vrydag 17:00. Sien SUNLearn vir instruksies.
- **Neem kennis**: Gedeeltelike antwoorde is agteraan hierdie voorskrif om jou te help om jou werk na te gaan. Bestudeer die Matlab/Simulink-dokumentasie deur na die *Help*-kieslys te gaan indien jy nie weet hoe om 'n sekere funksie/blok te gebruik nie.

Background / *Agtergrond*

So far, we have looked at modelling *linear* systems using SV equations. However, most practical systems are nonlinear and – if possible – we would like to apply the powerful linear SV modelling, analysis and control techniques to these systems. In order to do this, we first have to *linearise* the nonlinear systems. This is the topic of today's practical.

The idea of linearisation is to choose an equilibrium point (states $\mathbf{x}_0$, input u_0, and output y_0) to linearise around, and then describe the deviation from the equilibrium point (states $\delta\mathbf{x}$, input δu, and output δy) as having approximately linear dynamics, using the normal SV equations:

Ons het sover gekyk na modellering van *lineêre* stelsels met TV vergelykings. Meeste praktiese stelsels is egter nie-lineêr en – indien moontlik – sou ons graag die kragtige lineêre TV tegnieke vir modellering, analise en beheer wil toepas op hierdie stelsels. Om dit reg te kry moet ons eers die nie-lineêre stelsels *lineariseer*. Dit is die fokus van vandag se prakties.

Die idee van linearisering is om 'n ekwilibrium-punt te kies (toestande $\mathbf{x}_0$, intree u_0, en uittree y_0) om by te lineariseer, en dan die afwyking vanaf die ekwilibrium-punt (toestande $\delta\mathbf{x}$, intree δu, uittree δy) se dinamika te benader as lineêr, deur die normale TV vergelykings te gebruik:

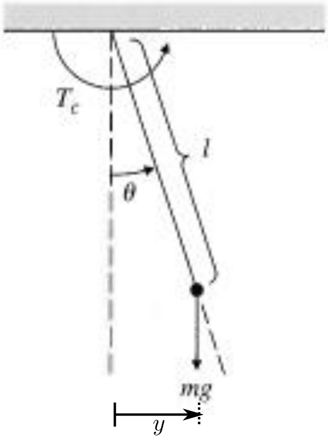
$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x} + \mathbf{b}\delta u$$
$$\delta y = \mathbf{c}\delta\mathbf{x} + d\delta u.$$

For today's practical, we will take the (nonlinear) model of a pendulum and linearise at 3 equilibria: the pendulum hanging straight down (the hanging pendulum model), the pendulum standing straight up (the inverted pendulum model), and the pendulum hanging at an angle of approximately 45°. For each of these equilibria, we will compare the behaviour of the exact, nonlinear model with that of the linearise model, using Simulink.

Vir vandag se prakties gaan ons die (nie-lineêre) model van 'n pendulum neem en lineariseer by 3 ekwilibriums: die pendulum wat ondertoe hang (die hangende pendulum model), die pendulum wat regop staan (die geïnverteerde pendulum model), en die pendulum wat hang teen 'n hoek van ongeveer 45°. Vir elkeen van hierdie ekwilibriums gaan ons die gedrag van die regte, nie-lineêre model vergelyk met die van die gelineariseerde model, deur Simulink te gebruik.

Assignment / *Voorskrif*

Consider the following diagram of a pendulum: Beskou die volgende diagram van 'n pendulum:



The pendulum angle is θ, the mass of the pendulum is m, the pendulum length is l, and a torque $u = T_c$ (the input) is applied around the pivot. The output is the horizontal position of the tip of the pendulum, y. The nonlinear differential equation describing the dynamics this system is given by

Die pendulum-hoek is θ, the massa van die pendulum is m, die pendulum-lengte is l, en 'n draaimoment $u = T_c$ (die intree) word om die draaipunt aangewend. The uittree is die horisontale posisie van die onderpunt van die pendulum, y. Die nie-lineêre differensiaalvergelyking wat die dinamika van hierdie stelsel beskryf word gegee deur

$$T_c - mgl\sin\theta = ml^2\ddot{\theta}. \tag{1}$$

The output is given by

Die uittree word gegee deur

$$y = l\sin\theta. \tag{2}$$

1 General Pendulum Model / *Algemene Pendulummodel*

Problem statement: / *Probleemstelling:*

Linearise the nonlinear pendulum model symbolically.

Lineariseer die nie-lineêre pendulummodel simbolies.

Solution development: / *Ontwikkeling van oplossing:*

(a) Choose the state vector as

Kies die toestandsvektor as

$$\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \tag{3}$$

and write down the nonlinear SV equations, i.e.

en skryf die nie-lineêre TV vergelykings neer, m.a.w.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, u) \\ y &= g(\mathbf{x}, u). \end{aligned} \tag{4}$$

(b) Draw the block diagram of this system using only integrator, gain, summation and trigonometric function blocks. Label the states, input and output.

Teken die blokdiagram van hierdie stelsel deur slegs van integreerder-, aanwins-, sommeerder- en trigonometriese funksieblokke gebruik te maak. Benoem die toestande, intree en uittree.

(c) Calculate the equilibrium point corresponding to a certain constant input u_0 *symbolically*, i.e. calculate the constant state vector $\mathbf{x}_0$ such that

Bereken die ekwilibrium-punt wat ooreenstem met 'n sekere konstante intree u_0 *simbolies*, d.w.s. bereken die konstante toestandsvektor $\mathbf{x}_0$ sodat

$$\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, u_0) = \mathbf{0}. \tag{5}$$

(d) Now linearise the system around the equilibrium point ($\mathbf{x}_0$ and u_0) and write the linearised SV equations in the form

Lineariseer nou die stelsel rondom the ekwilibrium-punt ($\mathbf{x}_0$ en u_0) en skryf die gelineariseerde TV vergelykings in die vorm

$$\begin{aligned} \delta\dot{\mathbf{x}} &= \mathbf{A}\delta\mathbf{x} + \mathbf{b}\delta u \\ \delta y &= \mathbf{c}\delta\mathbf{x} + d\delta u. \end{aligned} \tag{6}$$

2 Nonlinear Simulation / *Nie-lineêre Simulasie*

Problem statement: / *Probleemstelling:*

Simulate the nonlinear pendulum model in Simulink.

Simuleer die nie-lineêre pendulummodel in Simulink.

Solution development: / *Ontwikkeling van oplossing:*

Assume that $l = 1$ m, $m = 1$ kg, and $g = 9.81$ m.s^{-2}. Build a Simulink model of the nonlinear system[1]. Use the *Sum*, *Gain*, and *Trigonometric Function* blocks (in the *Math Operator* folder), as well as the *Integrator* block (*Continuous* folder).

Aanvaar $l = 1$ m, $m = 1$ kg, en $g = 9.81$ m.s^{-2}. Bou 'n Simulink-model van die nie-lineêre stelsel[2]. Gebruik *Sum*-, *Gain*-, en *Trigonometric Function*-blokke (in die *Math Operator*-afdeling), asook die *Integrator*-blok (*Continuous*-afdeling).

Experiments and results: / *Eksperimente en resultate:*

Use the *Step* block (*Sources* folder) for the input and apply a 1 N·m step. Display the output using a *Scope* block (*Sinks* folder). Simulate the model for a 10 s period and observe the output[3].

Gebruik die *Step*-blok (*Sources*-afdeling) vir die intree en wend 'n 1 N·m trap aan. Vertoon die uittree met 'n *Scope*-blok (*Sinks*-afdeling). Simuleer die model vir 'n 10 s periode en beskou die uittree[4].

[1] Hint for building Simulink models: to rotate a block, select the block and press Ctrl-R.

[2] Wenk vir die bou van Simulink-modelle: om 'n blok te roteer, kies die blok en druk Ctrl-R.

[3] Hint for simulation: to increase the resolution of the simulation, go to the Simulink model's *Simulation* menu, choose *Model Configuration Parameters*, and set *Max step size* to a small value.

[4] Wenk vir simulasie: om die resolusie van die simulasie te verbeter, gaan na die Simulink-model se *Simulation*-kieslys, kies *Model Configuration Parameters*, en stel *Max step size* na 'n klein waarde.

Conclusions: / *Gevolgtrekkings:*

Does the pendulum model behave as expected?

Is die gedrag van die pendulummodel wat jy verwag het?

3 Linearisation of the hanging pendulum / *Linearisering van die hangende pendulum*

Problem statement: / *Probleemstelling:*

Simulate the linear pendulum model that has been linearised at the hanging position.

Simuleer die lineêre pendulummodel wat by die hangende posisie gelineariseer is.

Solution development: / *Ontwikkeling van oplossing:*

Calculate the value of $\mathbf{x}_0$ at the equilibrium corresponding to $u_0 = 0$ N·m for the *hanging pendulum* (i.e. $-90° < \theta_0 < 90°$). Linearise the system around this equilibrium. Build the "vector" block diagram of this system in Simulink and connect its input to the same input applied to the nonlinear system. Note that the *Gain* block in Simulink can accept a matrix or vector as parameter[5], and connectors between blocks can carry vectors of signals. Connect the output to the same *Scope* block used for the nonlinear system; use the *Mux* block (*Signal Routing* folder) to combine the two output lines before feeding them into the *Scope* block.

Bereken die waarde van $\mathbf{x}_0$ by die ekwilibrium wat ooreenstem met $u_0 = 0$ N·m vir die *hangende pendulum* (d.w.s. $-90° < \theta_0 < 90°$). Lineariseer die stelsel rondom hierdie ekwilibrium. Bou die "vektor"-blokdiagram van hierdie stelsel in Simulink en verbind die intree met dié van die nie-lineêre stelsel. Neem kennis dat die *Gain*-blok in Simulink 'n matriks of vektor kan neem as parameter[6], en verbindings tussen blokke kan vektore van seine oordra. Verbind die uittree aan dieselfde *Scope*-blok wat gebruik word vir die nie-lineêre stelsel; gebruik die *Mux*-blok om die twee uittree-lyne te kombineer voor hulle by die *Scope*-blok ingevoer word.

Experiments and results: / *Eksperimente en resultate:*

Run the simulation and compare the outputs of the nonlinear and linearised systems. Increase the magnitude of the step applied to the input until the outputs differ significantly.

Simuleer en vergelyk die uittrees van die lineêre en nie-lineêre stelsels. Verhoog die grootte van die trap wat op die intree aangewend word totdat die uittrees beduidend verskil.

Conclusions: / *Gevolgtrekkings:*

Under which conditions is the linearised model respectively a good and bad approximation of the nonlinear model? In which aspects do the outputs of the nonlinear and linearised models differ?

Onder watter omstandighede is die gelineariseerde model onderskeidelik 'n goeie en 'n slegte benadering van die nie-lineêre model? In watter opsigte verskil die uittrees van die nie-lineêre en gelineariseerde model van mekaar?

[5] I suggest you define the matrices in a script (e.g. `A=[1,2;3,4]` and `b=[1;2]`) and then use the variables (e.g. `A` and `b`) in the Simulink model. You also have to change the *Multiplication* field in the *Gain* block parameters to `Matrix(K*u)` in order to do matrix/vector multiplication.

[6] Ek stel voor jy definieer die matrikse in a "script" (bv. `A=[1,2;3,4]` en `b=[1;2]`) en gebruik dan die veranderlikes (bv. `A` en `b`) in die Simulink-model. Jy moet ook die *Multiplication*-veld in die *Gain*-blok verander na `Matrix(K*u)` om matriks-/vektor-vermenigvuldiging te doen

4 Linearisation at $\theta \approx 45^\circ$ / ***Linearisering by $\theta \approx 45^\circ$***

Problem statement: / ***Probleemstelling:***

Simulate the linear pendulum model that has been linearised at an angle of approximately 45°.

Simuleer die lineêre pendulummodel wat gelineariseer is by 'n hoek van ongeveer 45°.

Solution development: / ***Ontwikkeling van oplossing:***

(a) Calculate the value of $\mathbf{x}_0$ at the equilibrium corresponding to $u_0 = 7$ N·m with $-90^\circ < \theta_0 < 90^\circ$ and change your *nonlinear* simulation as follows: Change the initial value of the *Step* block to be $u_0 = 7$ N·m and the initial value of the states to be your calculated $\mathbf{x}_0$ (you can change the intial value in the appropriate *Integrator* block). First apply a step with magnitude zero to verify that you have calculated the equilibrium correctly (the system should be in equilibrium, i.e. the states should not change), and then apply a step size of 1 N·m.

Bereken die waarde van $\mathbf{x}_0$ by die ekwilibrium wat ooreenstem met $u_0 = 7$ N·m met $-90^\circ < \theta_0 < 90^\circ$ en verander jou *nie-lineêre* simulasie soos volg: Verander die beginwaarde van die *Step*-blok tot $u_0 = 7$ N·m en die beginwaarde van die toestande volgens jou berekende $\mathbf{x}_0$ (jy kan die beginwaarde in die toepaslike *Integrator*-blok stel). Wend eers 'n nul-trapgrootte aan om te bevestig dat jy die ekwilibrium reg bereken het (die stelsel behoort in ekwilibrium te wees, d.w.s. die toestande moenie verander nie), en wend dan 'n trapgrootte van 1 N·m aan.

(b) Linearise the system around the equilibrium corresponding to $u_0 = 7$ N·m and change the "vector" block diagram of your linearised system in Simulink accordingly. Remember that the simulation of the linearised system takes the *deviation from the equilibrium input*, $\delta u \triangleq u - u_0$, as input, and produces the *deviation from the equilibrium output*, $\delta y \triangleq y - y_0$ (see Equation 6). You therefore have to add or subtract the equilibrium values at the appropriate points to calculate the full output y – use the *Constant* block (*Sources* folder) and a *Sum* block to do this.

Lineariseer die stelsel rondom die ekwilibrium wat ooreenstem met $u_0 = 7$ N·m en verander die "vektor"-blokdiagram van jou gelineariseerde stelsel in Simulink daarvolgens. Onthou dat die simulasie van die gelineariseerde stelsel die *verandering vanaf die ekwilibrium intree*, $\delta u \triangleq u - u_0$, as intree, en lewer die *verandering vanaf die ekwilibrium uittree*, $\delta y \triangleq y - y_0$ (sien Vergelyking 6). Jy moet dus die ekwilibrium-waardes op die toepaslike punte bytel of aftrek om die volledige uittree y te bereken – gebruik die *Constant*-blok (*Sources*-afdeling) en 'n *Sum*-blok om dit te doen.

Experiments and results: / ***Eksperimente en resultate:***

Run the simulation and compare the outputs of the nonlinear and linearised systems. Increase the magnitude of the step applied to the input until the outputs differ significantly.

Simuleer en vergelyk die uittrees van die lineêre en nie-lineêre stelsels. Verhoog die grootte van die trap wat op die intree aangewend word totdat die uittrees beduidend verskil.

Conclusions: / ***Gevolgtrekkings:***

Under which conditions is the linearised model a good and bad approximation respectively of the nonlinear model? How does this differ from the linearised model in Question 3, and why?

Onder watter omstandighede is die gelineariseerde model onderskeidelik 'n goeie en 'n slegte benadering van die nie-lineêre model? Hoe verskil dit van die gelineariseerde model in Vraag 3, en hoekom?

5 Linearisation at $\theta = 180^\circ$ / *Linearisering by* $\theta = 180^\circ$

Problem statement: / ***Probleemstelling:***

Simulate the linear pendulum model that has been linearised at the upright position (inverted pendulum).

Simuleer die lineêre pendulummodel wat by die regop posisie gelineariseer is (geïnverteerde pendulum).

Solution development: / ***Ontwikkeling van oplossing:***

Calculate the value of $\mathbf{x}_0$ at the equilibrium corresponding to $u_0 = 0$ N·m for the *inverted pendulum* (i.e. $90^\circ < \theta_0 < 270^\circ$) and set up a simulation to compare the nonlinear and linearised models similarly to Questions 3 and 4.

Bereken die waarde van $\mathbf{x}_0$ by die ekwilibrium wat ooreenstem met $u_0 = 0$ N·m vir die *geïnverteerde pendulum* (d.w.s. $90^\circ < \theta_0 < 270^\circ$) en stel 'n simulasie op wat die nie-lineêre and gelineariseerde modelle vergelyk soortgelyk aan Vrae 3 en 4.

Experiments and results: / ***Eksperimente en resultate:***

Do experiments similarly to Questions 3 and 4.

Doen eksperimente soortgelyk aan Vrae 3 en 4.

Conclusions: / ***Gevolgtrekkings:***

Can you make sense of the outputs? Is the system stable or unstable at this equilibrium?

Kan jy sin maak van die uittrees? Is die stelsel stabiel of onstabiel by die ekwilibrium?

Answers / *Antwoorde*

1. (a)

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}u \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = \mathbf{f}(\mathbf{x}, u)$$
$$y = l \sin x_1 = g(x_1, x_2, u) = g(\mathbf{x}, u)$$

(b)

(c)

$$\mathbf{x}_0 = \begin{bmatrix} \arcsin\left(\frac{u_0}{mgl}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta_0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u$$
$$\delta y = \begin{bmatrix} l\cos\theta_0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + 0 \cdot u$$

2. See diagram below

3. See diagram below

4. See diagram below

5. See diagram below