Numerical Project Pt. 3:
Numerical Algorithms for First Order PDEs.

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Introduction.

For part 3 of the numerical project we turn our attention to solving first order PDEs, namely, the elementary transport equation. We experiment with explicit (forward difference), upwind (backwards difference), and Lax-Wendroff schemes.

Throughout, we take advantage of the opportunities presented by these schemes to study their stability in response to changes in key parameters.

5.3.1

For this exercise, we numerically solve $u_t = 3u_x, u(0,x) = 1/(1+x^2)$, on the interval $[-10, 10]$ using an upwind scheme with $\Delta x = 0.1$, and suitably chosen Δt.

The CFL condition described in this chapter in the textbook requires that for negative wave speeds, the following be respected:

$$0 \geq \sigma = \frac{c\Delta t}{\Delta x} \geq -1, \text{ where } c \text{ denotes wave speed.} \tag{1}$$

Thus, for a given Δx of 0.1, we require that the time step Δt do not exceed 0.033333. Below, we plot the solutions at $t = 0.5, 1.0, 1.5$, for several values of Δt according to our criteria.

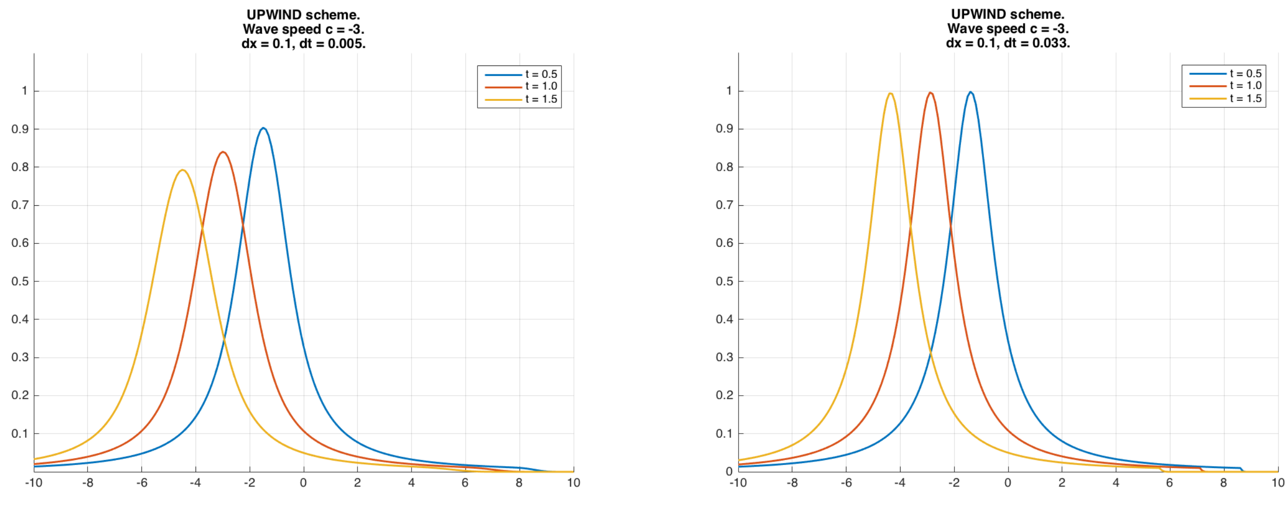

Figure 1: Upwind scheme for the wave equation. Left: $\Delta t = 0.005$. Right: $\Delta t = 0.033$

Please note that the Δt values chosen for the plots in figure 1 were strategically chosen to illustrate shortcomings of the upwind scheme, namely, its overall inaccuracy over longer time intervals.

The closer Δt gets to its upper limit of 0.033333 (for this particular exercise), the more accurate the scheme seems to get. In order to understand this, we need to look at the CFL constant σ as presented in (1) and its role in the numerics.

The upwind scheme is given as:

$$u_{j+1,m} = (\sigma+1)u_{j,m} - \sigma u_{j,m+1}, \ c \le 0, \tag{2}$$
$$u_{j+1,m} = \sigma u_{j,m-1} - (\sigma-1)u_{j,m}, \ c > 1, \tag{3}$$

which can be easily implemented by employing sparse matrices such as:

$$u_{j+1,m} = \begin{bmatrix} (\sigma+1) & -\sigma & & & & & \\ & (\sigma+1) & -\sigma & & & & \\ & & {\color{blue}(\sigma+1)} & {\color{green}-\sigma} & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & (\sigma+1) & -\sigma \\ & & & & & & (\sigma+1) \end{bmatrix} \begin{bmatrix} u(0,-10) \\ \vdots \\ \vdots \\ {\color{blue}u_{j,m}} \\ {\color{green}u_{j,m+1}} \\ \vdots \\ u(0,10) \end{bmatrix}, \tag{4}$$

for negative wave speeds, and

$$u_{j+1,m} = \begin{bmatrix} -(\sigma-1) & & & & & & \\ \sigma & -(\sigma-1) & & & & & \\ & {\color{red}\sigma} & {\color{blue}-(\sigma-1)} & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \sigma & -(\sigma-1) & \\ & & & & & \sigma & -(\sigma-1) \end{bmatrix} \begin{bmatrix} u(0,-10) \\ \vdots \\ {\color{red}u_{j,m-1}} \\ {\color{blue}u_{j,m}} \\ \vdots \\ \vdots \\ u(0,10) \end{bmatrix}, \tag{5}$$

for positive wave speeds.

Now, for $\Delta t = 0.005$, $|\sigma|$ as given in (1) has a value of .15, while for $\Delta t = 0.033$, $|\sigma| = 0.99$. Plugging these values into the (4) we get:

$$u_{j+1,m} = \begin{bmatrix} 0.01 & 0.99 & & & & & \\ & 0.01 & 0.99 & & & & \\ & & {\color{blue}(0.01)} & {\color{green}0.99} & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & 0.01 & 0.99 \\ & & & & & & 0.01 \end{bmatrix} \begin{bmatrix} u(0,-10) \\ \vdots \\ \vdots \\ {\color{blue}u_{j,m}} \\ {\color{green}u_{j,m+1}} \\ \vdots \\ u(0,10) \end{bmatrix}, \tag{6}$$

for $\Delta t = 0.033$, and

$$u_{j+1,m} = \begin{bmatrix} 0.85 & 0.15 & & & & & \\ & 0.85 & 0.15 & & & & \\ & & {\color{blue}(0.85)} & {\color{green}0.15} & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & 0.85 & 0.15 \\ & & & & & & 0.85 \end{bmatrix} \begin{bmatrix} u(0,-10) \\ \vdots \\ \vdots \\ {\color{blue}u_{j,m}} \\ {\color{green}u_{j,m+1}} \\ \vdots \\ u(0,10) \end{bmatrix}, \tag{7}$$

for $\Delta t = 0.005$.

Now, keeping in mind that the scheme relies on $u_{j,m}, u_{j,m+1}$ in order to acquire the next time step $u_{j+1,m}$, we can see that the closer the absolute value of the number on the off-diagonal is to 1, as in (6), the more accurate the scheme will be in capturing the true value of the node that lies "upwind", without the exaggerated damping seen in figure 1 (left), as a result of (7) . In other words, the closer $|\sigma|$ is to 1, the more we rely on $u_{j,m+1}$ to move forward in time, and less on $u_{j,m}$, whose coefficients equal zero when $|\sigma| = 1$. Analytically, this means we are traveling along the characteristic lines, which is what we're trying to approximate.

5.3.5

For this exercise, we again solve $u_t = 3u_x, u(0,x) = 1/(1+x^2)$, on the interval $[-10, 10]$ but this time we employ the Lax-Wendroff scheme, given by:

$$u_{j+1,m} = \frac{1}{2}\sigma(\sigma+1)u_{j,m-1} - (\sigma^2 - 1)u_{j,m} + \frac{1}{2}\sigma(\sigma-1)u_{j,m+1}, \tag{8}$$

which we again set up using a sparse matrix.

Choosing the same values for $\Delta x,\ \Delta t$ as in 5.3.1, we display the results below:

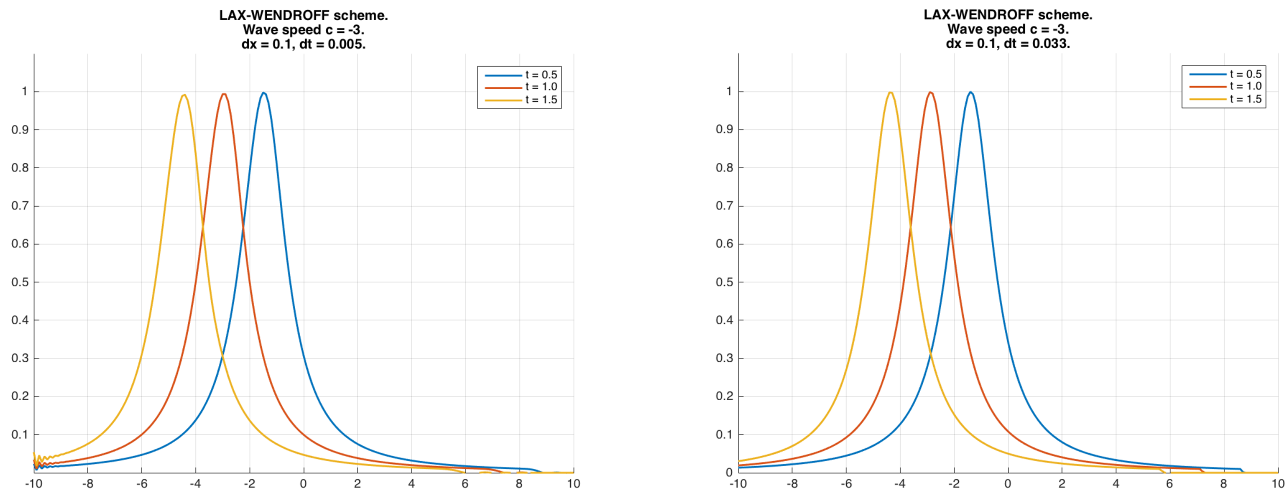

Figure 2: Lax-Wendroff scheme for the wave equation. Left: $\Delta t = 0.005$. Right: $\Delta t = 0.033$

From figure 2, it is immediately apparent that the Lax-Wendroff scheme does not suffer from the same shortcomings as the upwind scheme in terms of accuracy, as it is less sensitive to changes in Δt. Also, the Lax-Wendroff scheme is considerably better at resisting damping over longer time intervals. Figure 3 below, presents the results of the comparison.