Numerical Project Pt. 4:
Numerical Algorithms for the Wave Equation.

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Introduction.

For Part 4 of the numerical project, we employ numerical algorithms for solving the wave equation $u_{tt} = 64u_{xx}$ on the interval $0 \leq x \leq 3$, with initial boundary value problem:

$$u(t,0) = u(t,3) = 0,$$

$$u(0,x) = \begin{cases} 1 - 2|x-1| & 1/2 \leq x \leq 3/2 \\ 0, & \text{otherwise} \end{cases},$$

$$u_t(0,x) = 0.$$

5.4.1(a)

The range of values Δt allowed are determined by the CFL condition for this algorithm, which is given by $\sigma = c\frac{\Delta t}{\Delta x}$, with $\sigma \leq 1$. Once Δx is determined, the optimal value for Δt will be given by $\Delta t = \frac{\Delta x}{c}$, which in the case of a $\Delta x = 0.1$ yields $\Delta t = 0.0125$.

5.4.1(b)

The plots below illustrate the behavior of the solution for values of Δt both in the stable and unstable range. We can see that for $\Delta t = 0.0125$, i.e. the optimal value, the solution behaves quite nicely. This is due to the fact that when $\sigma = 1$, the scheme as given by the difference equation is relying on the "upwind" and "downwind" nodes only in order to obtain the next time step. The analysis is similar to the one presented in Part 3, when discussing the "upwind vs. Lax Wendroff" schemes.

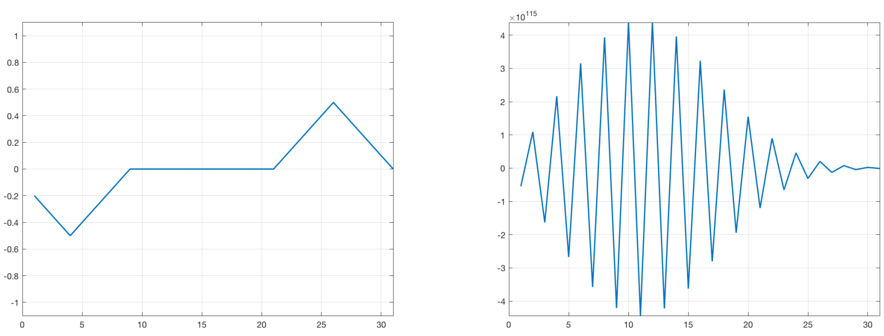

Figure 1: Numerical solutions for the wave equation. Left: $\Delta t = 0.0125$. Right: $\Delta t = 0.1$. Both solutions were captured at $t = 0.625$

However, as shown in the plot above, if $\sigma > 1$ the solution blows up. This is again due to the structure of the difference equation.

5.4.1(c)

Choosing a smaller spatial step size $\Delta x = 0.01$ has no noticeable effect on the numerical solution when we are close to the optimal CFL value. However, as we move away from said value using larger step sizes, the solution loses accuracy. This is due to the fact that a smaller spatial step size provides us with finer mesh over which to compute our solution.

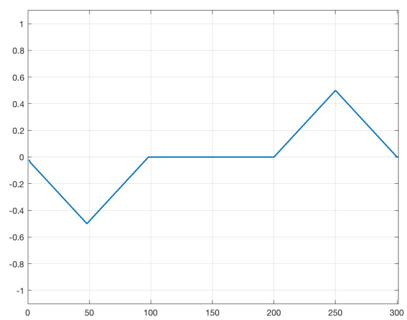

Figure 2: Numerical solution for the wave equation. $\Delta t = 0.0013, \Delta x = 0.01$. Solutions captured at $t = 0.195$