Homework 6

1. $f_{j+b} = f(x_j + bh)$

$$= f(x_j) + (bh) f'(x_j) + \frac{(bh)^2}{2} f''(x_j) + \frac{(bh)^3}{6} f'''(x_j) + \frac{(bh)^4}{24} f''''(x_j)$$

$b = -2, -1, 0, 1, 2.$

$$f''(x_j) = \alpha_{-2} f_{j-2} + \alpha_{-1} f_{j-1} + \alpha_0 f_j + \alpha_1 f_{j+1} + \alpha_2 f_{j+2} + O(h^5)$$

$$= (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2) f(x_j)$$

$$+ (-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2) h f'(x_j)$$

$$+ (4\alpha_{-2} + \alpha_{-1} + \alpha_1 + 4\alpha_2) \frac{h^2}{2} f''(x_j)$$

$$+ (-8\alpha_{-3} - \alpha_{-1} + \alpha_1 + 8\alpha_3) \frac{h^3}{6} f'''(x_j)$$

$$+ (16\alpha_{-2} + \alpha_{-1} + \alpha_1 + 16\alpha_2) \frac{1}{24} h^4 f''''(x_j)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 2h^2 & \frac{h^2}{2} & 0 & \frac{h^2}{2} & 2h^2 \\ -\frac{4}{3}h^3 & -\frac{1}{6}h^3 & 0 & \frac{h^3}{6} & \frac{4}{3}h^3 \\ \frac{2}{3}h^4 & \frac{1}{24}h^4 & 0 & \frac{1}{24}h^4 & \frac{2}{3}h^4 \end{bmatrix} \begin{bmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \alpha_{-2} = -\dfrac{1}{12h^2} \\ \alpha_{-1} = \dfrac{4}{3h^2} \\ \alpha_0 = -\dfrac{5}{2h^2} \\ \alpha_1 = \dfrac{4}{3h^2} \\ \alpha_2 = -\dfrac{1}{12h^2} \end{cases}$$

2. (a)

$$LHS = \frac{\delta}{\delta x}(u_j v_j) = \frac{u_{j+1}v_{j+1} - u_{j-1}v_{j-1}}{2h}$$

$$RHS = u_j \frac{\delta}{\delta x}(v_j) + v_j \frac{\delta}{\delta x}(u_j)$$

$$= u_j \frac{v_{j+1} - v_{j-1}}{2h} + v_j \frac{u_{j+1} - u_{j-1}}{2h}$$

$$= \frac{u_j v_{j+1} - u_j v_{j-1} + v_j u_{j+1} - v_j u_{j-1}}{2h}$$

$LHS = RHS \Leftrightarrow$

$u_{j+1}v_{j+1} - u_{j-1}v_{j-1} = u_j v_{j+1} - u_j v_{j-1} + v_j u_{j+1} - v_j u_{j-1}$

which is not guaranteed.

So, the equation is not always true.

(b) $LHS = \dfrac{u_{j+1}v_{j+1} - u_{j-1}v_{j-1}}{2h}$

$$RHS = \frac{u_{j+1} + u_{j-1}}{2} \cdot \frac{v_{j+1} - v_{j-1}}{2h} + \frac{v_{j+1} + v_{j-1}}{2} \cdot \frac{u_{j+1} - u_{j-1}}{2h}.$$

$$= \frac{1}{4h}\big(u_{j+1}v_{j+1} - u_{j+1}v_{j-1} + u_{j-1}v_{j+1} - u_{j-1}v_{j-1}$$
$$+ u_{j+1}v_{j+1} + u_{j+1}v_{j-1} - u_{j-1}v_{j+1} - u_{j-1}v_{j-1}\big)$$

$$= \frac{1}{2h}(u_{j+1}v_{j+1} - u_{j-1}v_{j-1})$$

$= LHS.$

$$\therefore \frac{\delta}{\delta x}(u_j v_j) = \overline{u_j}\frac{\delta}{\delta x}(v_j) + \overline{v_j}\frac{\delta}{\delta x}(u_j).$$

3.

$$\int_a^b f(x)dx = \sum_{i=0}^{n}\left(\int_a^b L_{n,i}(x)dx\right) f_i$$

$$= \sum_i w_i f_i$$

$$w_i = \int_a^b L_{n,i}(x)\,dx = \int_a^b \prod_{j\neq i}^{n} \frac{x-x_j}{x_i-x_j}\,dx$$

$$w_0 = \int_a^b \frac{(x-a-h)(x-a-2h)(x-b)}{-h\cdot 2h\cdot 3h}\,dx \qquad (\text{let } x-a=t)$$

$$= -\frac{1}{6h^3}\int_0^{3h} (t-h)(t-2h)(t-3h)\,dt$$

$$= -\frac{1}{6h^3}\int_0^{3h} (t^2-3ht+2h^2)(t-3h)\,dt$$

$$= -\frac{1}{6h^3}\int_0^{3h} (t^3-6ht^2+11h^2t-6h^3)\,dt$$

$$= -\frac{1}{6h^3}\left(\frac{t^4}{4}-2ht^3+\frac{11}{2}h^2t^2-6h^3t \Big|_0^{3h}\right)$$

$$= -\frac{1}{6h^3}\cdot\left(\frac{81h^4}{4}-54h^4+\frac{99}{2}h^4-18h^4\right)$$

$$= \frac{3}{8}h.$$

$$w_1 = \int_0^{3h} \frac{t(t-2h)(t-3h)}{h\cdot(-h)\cdot(-2h)}\,dt = \frac{1}{2h^3}\int_0^{3h} t^3-5ht^2+6h^2t\,dt$$

$$= \frac{1}{2h^3}\left(\frac{1}{4}t^4-\frac{5}{3}ht^3+3h^2t^2\Big|_0^{3h}\right)$$

$$= \frac{9}{8}h.$$

$$w_2 = \int_0^{3h} \frac{t(t-h)(t-3h)}{2h\cdot h\cdot(-h)}\,dh = \frac{9}{8}h.$$

$$w_3 = \int_0^{3h} \frac{t(t-h)(t-2h)}{3h\cdot 2h\cdot h}\,dh = \frac{3}{8}h$$

$$\Rightarrow \int_a^b f(x)\,dx \simeq \frac{b-a}{8}\left[f(a)+3f(a+h)+3f(a+2h)+f(b)\right]$$

where $h=\frac{b-a}{3}$.

4 Convergence of Simpson's rule

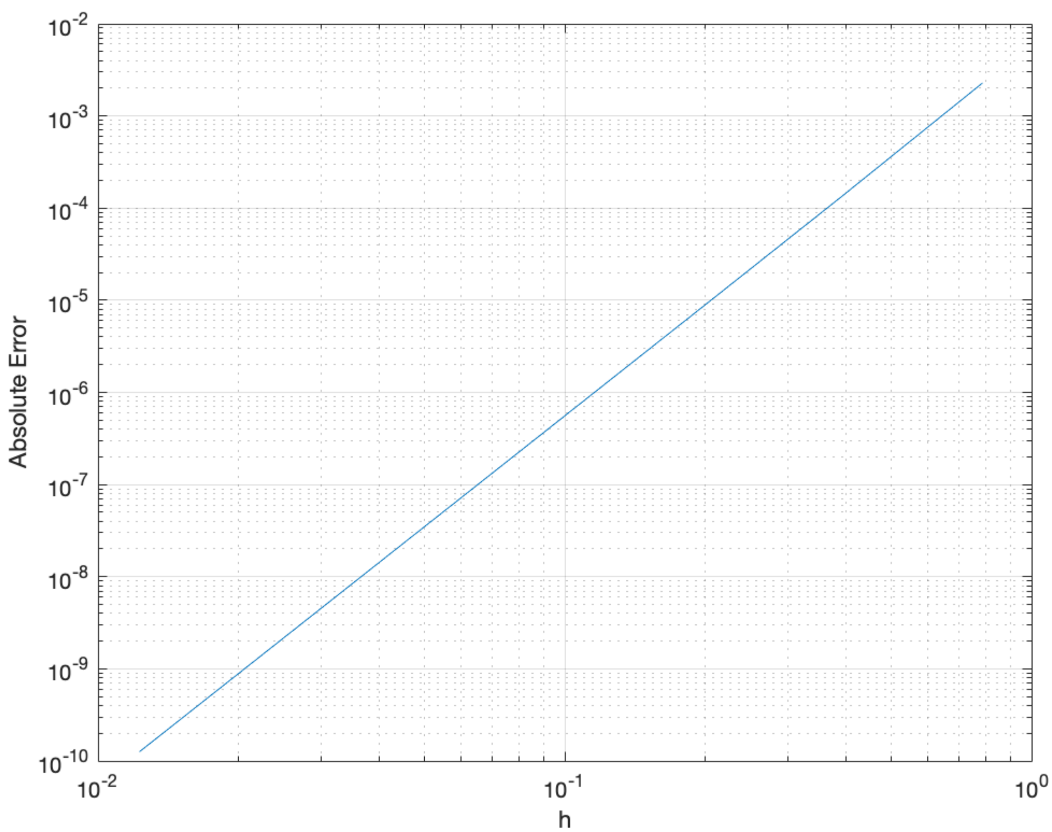
```
err = zeros(7,1);
h4 = zeros(7,1);
h = zeros(7,1);
ratio = zeros(7,1);
for i = 1:7
    n = 2^i
    err(i) = abs(compsimps(0,pi/2,n) - 1);
    h4(i) = (pi/(2*n))^4;
    h(i) = (pi/(2*n));
    ratio(i) = err(i)/h4(i);
end
ratio
loglog(h,err)
grid on;
xlabel('h')
ylabel('Absolute Error')


function ans = compsimps(a, b, n)
    h = (b-a)/n
    sum_even = 0;
    for i = 1:n/2-1
        x(i) = a + 2*i*h;
        sum_even = sum_even + f(x(i));
    end
    sum_odd = 0;
    for i = 1:n/2
       x(i) = a + (2*i-1)*h;
       sum_odd = sum_odd + f(x(i));
    end
    ans = h*(f(a)+ 2*sum_even + 4*sum_odd +f(b))/3;
end

function y = f(x)
    y = cos(x);
end
```

```
ratio =

    0.0060
    0.0057
    0.0056
    0.0056
    0.0056
    0.0056
    0.0056
```



5. $\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right)dt$

So we only need to discuss $\int_{-1}^{1} g(t)\,dt$. Note $x \to t$ is linear transform.

$$\int_{-1}^{1} t^i\,dt = I_3(t^i) \quad \text{for } t = 0,1,2,3,4,5.$$

$\int_{-1}^{1} 1\,dt = 2 = w_1 + w_2 + w_3$.

$\int_{-1}^{1} t\,dt = 0 = w_1 t_1 + w_2 t_2 + w_3 t_3$.

$\int_{-1}^{1} t^2\,dt = \frac{2}{3} = w_1 t_1^2 + w_2 t_2^2 + w_3 t_3^2$,

$\int_{-1}^{1} t^3\,dt = 0 = w_1 t_1^3 + w_2 t_2^3 + w_3 t_3^3$.

$\int_{-1}^{1} t^4\,dt = \frac{2}{5} = w_1 t_1^4 + w_2 t_2^4 + w_3 t_3^4$.

$\int_{-1}^{1} t^5\,dt = 0 = w_1 t_1^5 + w_2 t_2^5 + w_3 t_3^5$. Suppose $t_1 < t_2 < t_3$.

From symmetry, $t_2 = 0$. $t_1 + t_3 = 0$. $w_1 = w_3$.

$w_1 t_1^2 = \frac{1}{3}$.

$w_1 t_1^4 = \frac{1}{5}$.

$t_1^2 = \frac{3}{5}$, $t_1 = -\sqrt{\frac{3}{5}}$. $t_3 = \sqrt{\frac{3}{5}}$

$w_1 = \frac{5}{9} = w_3$. $w_2 = \frac{8}{9}$.

$x_i = \frac{b-a}{2} t_i + \frac{a+b}{2}$.

$$\Rightarrow \begin{cases} w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9}, \quad w_3 = \frac{5}{9}. \\ x_1 = \frac{a-b}{2} \cdot \sqrt{\frac{3}{5}} + \frac{a+b}{2} = \left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)a + \left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)b \\ x_2 = \frac{b-a}{2}\sqrt{\frac{3}{5}} + \frac{a+b}{2} = \left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)a + \left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)b. \end{cases}$$

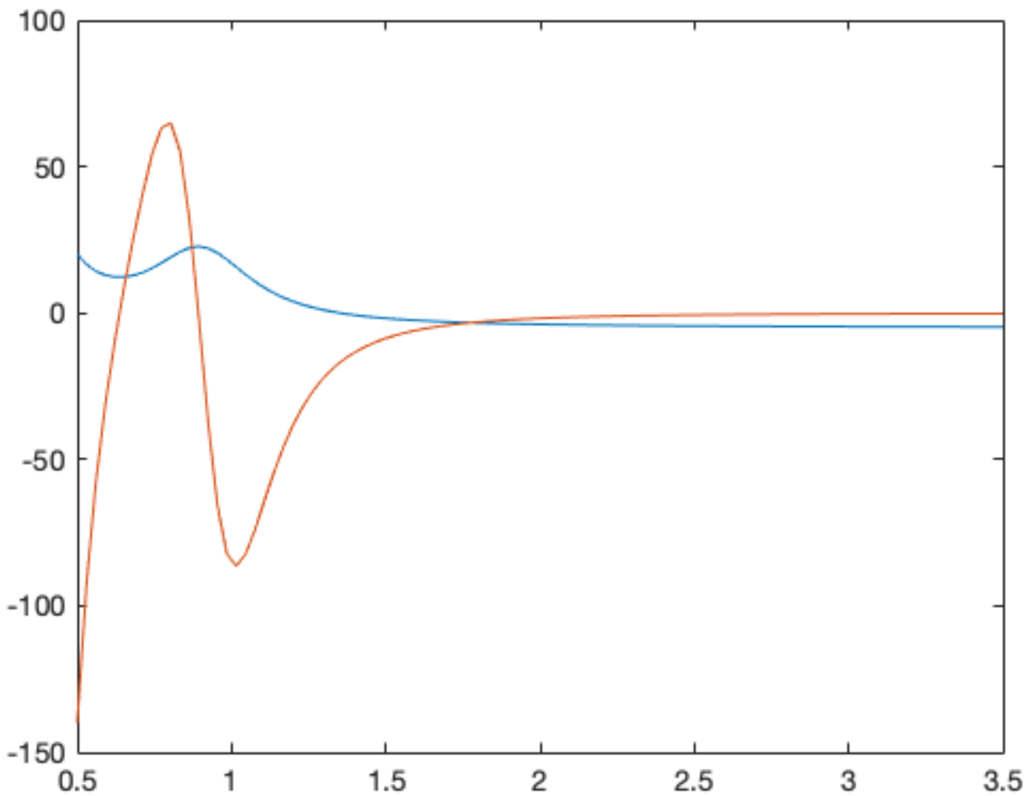
6 Newton minimization

```
syms x;
f(x) = 1/((x-0.3)^2 + 0.01) + 1/((x-0.9)^2 + 0.04) - 5
fp(x) = diff(f);
fpp(x) = diff(fp);
```

```
f(x) =

1/((x - 9/10)^2 + 1/25) + 1/((x - 3/10)^2 + 1/100) - 5
```

```
x = linspace (0.5,3.5);
plot(x,f(x))
hold on;
plot(x,fp(x))
```



Blue: $f(x)$

Orange: $f'(x)$

```
epsilon = 1e-6;
q = 0;
p = 0.5;
while (abs(p-q)>epsilon)
    q = p;
    p = vpa(p - fp(p)/fpp(p),50);
end
vpa(p,5)
```

```
ans =

0.63701
```

7 Summer correlation

```
x = [0 2 3.2 5 6.2 10];
y = [3.5 11 12.7 19.4 22 34];
```

```
p = polyfit(x,y,1)
```

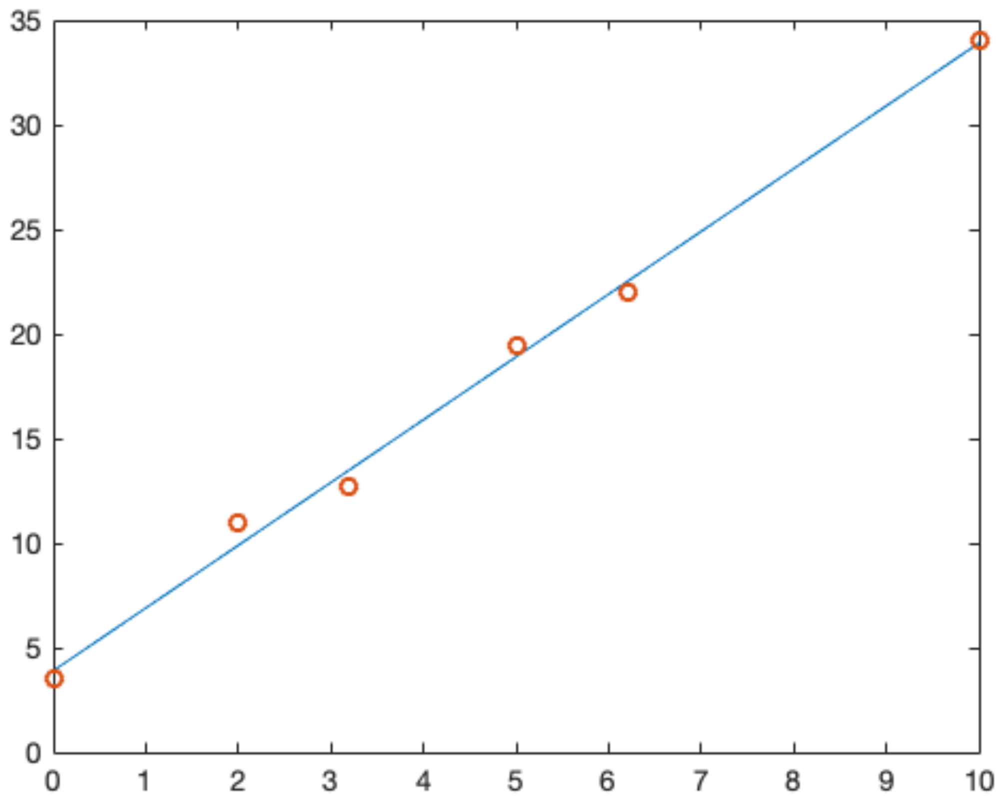
```
p =

    3.0007    3.8971
```

```
f =

    3.8971    9.8984   13.4992   18.9004   22.5012   33.9036
```

```
xx = linspace (0,10);
f = polyval(p,xx);
plot(xx,f)
hold on;
scatter(x,y);
```



The blue line is the best linear fit:

$$Y = 3.0007X + 3.8971$$

Correlation Analysis:

```
corrcoef(x,y)
```

```
ans =

    1.0000    0.9977
    0.9977    1.0000
```

The correlation coefficient = 0.9977.