Homework 2

1.

Suppose $AB = C$.

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$ By definition, $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} b_{ij} = 1$.

$$\sum_{j=1}^{n} C_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} b_{kj}\right) a_{ik}$$

$$= \sum_{k=1}^{n} a_{ik}$$

$$= 1 \quad \text{for each } 1 \le i \le n.$$

therefore, AB is right stochastic

2. (a) According to the statements,

$$\begin{cases} r_1 + r_2 + r_3 = 1240 \\ 0.1 r_1 + 0.2 r_2 + 0.3 r_3 = 276 \\ 2r_1 = r_3 \end{cases}$$

Matrix Form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.1 & 0.2 & 0.3 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1240 \\ 276 \\ 0 \end{bmatrix}$$

(b)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1240 \\ 0.1 & 0.2 & 0.3 & 276 \\ 2 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1240 \\ 0 & 1 & 2 & 1520 \\ 0 & -2 & -3 & -2480 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1240 \\ 0 & 1 & 2 & 1520 \\ 0 & 0 & 1 & 560 \end{array}\right]$$

$$\Rightarrow \begin{cases} r_1 = 280 \\ r_2 = 400 \\ r_3 = 560 \end{cases}$$

3. $L_k = I_n + g e_k^T$

where $g^T = [x_1 \ x_2 \ \cdots \ x_n]$. $\quad x_i = \begin{cases} m_i, & \text{if } i > k \\ 0, & \text{otherwise} \end{cases}$

$e_k^T = [y_1 \ y_2 \ \cdots \ y_n] \quad y_i = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases}$

Suppose $(B_k)_{ij} = \begin{cases} -m_i & \text{if } i > k \text{ and } j = k \\ 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Then, $B_k = I_n + h e_k^T$

where $h = [w_1 \ w_2 \ \cdots \ w_n]$ $w_i = \begin{cases} -m_i & \text{if } i > k \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} L_k B_k &= (I_n + g e_k^T)(I_n + h e_k^T) \\ &= I_n + (g+h) e_k^T + g e_k^T h e_k^T \\ &= I_n. \end{aligned}$$

$P = g e_k^T$. $Q = h e_k^T$

$P_{ij} = \begin{cases} m_i & j = k, i > k \\ 0 & \text{else} \end{cases}$ $\quad Q_{ij} = \begin{cases} -m_i & i > k, j = k \\ 0 & \text{else} \end{cases}$

$C = PQ$.

$C_{ij} = \sum_{t=1}^{n} P_{it} Q_{tj} = 0 \quad \therefore g e_k^T h e_k^T = 0$

L_k^{-1} is unique, therefore L_k^{-1} is given by

$$(L_k^{-1})_{ij} = \begin{cases} -m_i & \text{if } i > k \text{ and } j = k \\ 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

4. (a) Suppose $A = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (*)

$$\Rightarrow \begin{cases} l_{11} u_{11} = 0 & ① \\ l_{11} u_{12} = 1 & ② \\ l_{21} u_{11} = 1 & ③ \\ l_{21} u_{12} + l_{22} u_{22} = 1 & ④ \end{cases}$$

②.③ $\Rightarrow$ $(l_{11} u_{11}) l_{21} u_{12} = 1$.

but $l_{11} u_{11} = 0$.

Therefore, (*) is not possible. A has no LU decomposition.

(b) $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ B has an LU decomposition.

5. (a) $0.1036 x_1 + 0.2122 x_2 = -74.9028 + 75.5432 = 0.6404$
$0.2081 x_1 + 0.4247 x_2 = -150.4563 + 151.1932 = 0.7369$.

$\therefore \begin{cases} x_1 = -723 \\ x_2 = 356 \end{cases}$ is the solution.

(b)

$0.2081 \div 0.1036 = 2.009$

$2.009 \times 0.2122 = 0.4263$ $\quad$ $2.009 \times 0.6404 = 1.287$

$0.4247 - 0.4263 = -0.0016$ $\quad$ $0.7369 - 1.287 = -0.5501$

$x_2 = \dfrac{-0.5501}{-0.0016} = 343.8$ $\quad$ $343.8 \times 0.2122 = 72.95$

$0.6404 - 72.95 = -72.31$ $\quad$ $x_1 = \dfrac{-72.31}{0.1036} = -698.0$

$\begin{cases} x_1 = -698.0 \\ x_2 = 343.8 \end{cases}$

(c)

$0.2081x_1 + 0.4247x_2 = 0.7369$

$0.1036x_1 + 0.2122x_2 = 0.6404$

$0.1036 \div 0.2081 = 0.4978$. $\quad 0.4978 \times 0.4247 = 0.2114$

$0.2122 - 0.2114 = 0.0008$ $\quad 0.7369 \times 0.4978 = 0.3668$

$0.6404 - 0.3668 = 0.2736$

$$\begin{bmatrix} 0.2081 & 0.4247 & 0.7369 \\ 0 & 0.0008 & 0.2736 \end{bmatrix}$$

$$x_2 = \frac{0.2736}{0.0008} = 342.0$$

$342.0 \times 0.4247 = 145.2$ $\quad 0.7369 - 145.2 = -144.5$

$$x_1 = \frac{-144.5}{0.2081} = -694.4$$

$$\begin{cases} x_1 = -694.4 \\ x_2 = 342.0 \end{cases}$$

(d) ① In gaussian elimination, lots of basic arithmetic operations are performed. Since Limited precision cast error on almost every operation, the discrepancy can be very big.

② The coefficients of the first row is almost proportional to those of the second row. When performing gaussian elimination, we have the situation of "catastrophic cancellation", which reduced the precision significantly.