Permuted LU

Initialization & find solution:

```
A = [2 5 -9 3; 5 6 -4 2; 3 -4 2 7; 11 7 4 -8];
b = [151 103 16 -32]';
b1 = b;
ans = A\b
n = size(A,1);
P = eye(n);
```

```
ans =


    3.0000
    5.0000
  -11.0000
    7.0000
```

LU factorization with partial pivoting:

```
for k = 1:n-1
    [v,pos]=max(A(k:n,k));
    % swap rows
    A([k;pos+k-1],:) = A([pos+k-1;k],:);
    b([k;pos+k-1],:) = b([pos+k-1;k],:);
    P([k;pos+k-1],:) = P([pos+k-1;k],:);
    for i = k+1:n
        r = A(i,k) / A(k,k);
        b(i,1) = b(i,1) - r*b(k,1);
        A(i,k+1:n) = A(i,k+1:n) - r*A(k,k+1:n);
        A(i,k) = r;
    end
end
% Print L
fprintf('L:')
eye(n) + tril(A,-1)
% Print U
fprintf('U:')
triu(A)
```

```
% Print P
P
% Print LU
fprintf('Check LU:')
(eye(n) + tril(A,-1))*triu(A)
```

```
L:
ans =

    1.0000         0         0         0
    0.1818    1.0000         0         0
    0.4545    0.7561    1.0000         0
    0.2727   -1.5854   -9.4444    1.0000

U:
ans =

   11.0000    7.0000    4.0000   -8.0000
         0    3.7273   -9.7273    4.4545
         0         0    1.5366    2.2683
         0         0         0   37.6667
```

```
P =

     0     0     0     1
     1     0     0     0
     0     1     0     0
     0     0     1     0

Check LU:
ans =

   11.0000    7.0000    4.0000   -8.0000
    2.0000    5.0000   -9.0000    3.0000
    5.0000    6.0000   -4.0000    2.0000
    3.0000   -4.0000    2.0000    7.0000
```

Solve `x`:

```
% Print x
x = zeros(n,1);
A = triu(A);
for i = n: -1: 1
    x(i,1) = (b(i,1) - A(i,:)*x)/A(i,i);
end
x
```

```
x =


    3.0000
    5.0000
  -11.0000
    7.0000
```

Tridiagonal LU

a

Using Gaussian Elimination,

1. `row 2` += `row 1` * 1/2
2. `row 3` += `row 2` * 2/3
3. `row 4` += `row 3` * 3/4

$$L = \begin{bmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ & -\frac{2}{3} & 1 & \\ & & -\frac{3}{4} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -1 & & \\ & \frac{3}{2} & -1 & \\ & & \frac{4}{3} & -1 \\ & & & \frac{5}{4} \end{bmatrix}$$

$$Ux = L^{-1}b = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 2 \\ \frac{5}{2} \end{bmatrix}$$

By backward substitution,

$$x = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

b

```
A = [2 -1 0 0; -1 2 -1 0; 0 -1 2 -1; 0 0 -1 2];
b = [1 1 1 1]';
n = size(A,1);

for k = 1:n-1
    bound = min(k+2,n);
    r = A(k+1,k) / A(k,k);
    b(k+1,1) = b(k+1,1) - r*b(k,1);
    A(k+1,k+1:bound) = A(k+1,k+1:bound) - r*A(k,k+1:bound);
    A(k+1,k) = r;
end
L = eye(n) + tril(A,-1)
U = triu(A)
x = zeros(n,1);
A = triu(A);
for i = n: -1: 1
    bound = min(k+2,n);
    x(i,1) = (b(i,1) - U(i,i:bound)*x(i:bound,1))/U(i,i);
end
x
```

```
L =


    1.0000         0         0         0
   -0.5000    1.0000         0         0
         0   -0.6667    1.0000         0
         0         0   -0.7500    1.0000
```

```
U =

    2.0000   -1.0000         0         0
         0    1.5000   -1.0000         0
         0         0    1.3333   -1.0000
         0         0         0    1.2500
```

```
x =

    2.0000
    3.0000
    3.0000
    2.0000
```

Define -, *, \ operations of matrix elements as basic operations:

Computing rounds and basic operations per loop,

$$Number of computations = 7(n-2) + 5 + 3(n-1) + 1 = 10n - 11$$

and

$$N = 3n - 2$$

Therefore, number of computations roughly equals to $\frac{10}{3}N$

c

As demonstrated, the results are same.

Facts about LTs

a

$n \times n$ matrix L is invertible $\Leftrightarrow det(L) \neq 0$

$$
\begin{aligned}
det(L) &= \sum_{i=1}^{n} l_{i,n}(-1)^{i+n} det(L_{n-1}) \\
&= l_{n,n} \cdot det(L_{n-1}) \\
&= l_{n,n} \cdot l_{n-1,n-1} \cdot det(L_{n-2}) \\
&= \cdots \\
&= \prod_{i=1}^{n} l_{i,i}
\end{aligned}
$$

where

$$L_n = L(1:n, 1:n)$$

Therefore, if L is any $n \times n$ lower triangular invertible matrix, then $l_{jj} \neq 0$ for every $1 \leq j \leq n$.

b

Define

$$L(i,j) = l_{i,j}, L^{-1}(i,j) = p_{i,j}$$

For $1 < k \leq n$,

$$
\begin{aligned}
\sum_{j=1}^{n} l_{1,j} p_{j,k} &= I_{1,k} \\
&= l_{1,1} \cdot p_{1,k} \\
&= 0
\end{aligned}
$$

From *(a)*, $l_{1,1} \neq 0$. So $p_{1,k} = 0$.

Suppose for $i \leq i_0, i < k \leq n$, We have $p_{i,k} = 0$. Now consider $i = i_0 + 1, k > i$

$$\sum_{j=1}^{n} l_{i+1,j} p_{j,k} = I_{i+1,k} = 0$$

When $j < i - 1 = i_0$, $p_{j,k} = 0$; When $j > i$, $l_{i+1,j} = 0$.

Therefore,

$$\sum_{j=1}^{n} l_{i+1,j} p_{j,k} = l_{i,i} \cdot p_{i,k} = 0$$

From *(a)*, $l_{i,i} \neq 0$. So $p_{i,k} = 0$.

Then for $i \leq i_0 + 1, k > i$, We have $p_{i,k} = 0$.

Using *complete induction*, We have $p_{i,k} = 0$ for $1 \leq i < k \leq n$.

So, if L is an $n \times n$ invertible lower triangular matrix, then L^{-1} is also lower triangular.

c

Following the conventions in Problem 3 of HW2, we have

$$e_k^T g_j = 0 + 0 + \cdots + 0 = 0$$

where

$$1 \leq k < j \leq n-1$$

So,

$$\begin{aligned}
L_1 L_2 \cdots L_{n-1} &= (I + g_1 e_1^T)(I + g_2 e_2^T) \cdots (I + g_{n-1} e_{n-1}^T) \\
&= [I + (g_1 e_1^T + g_2 e_2^T) + g_1 e_1^T g_2 e_2^T](I + g_3 e_3^T) \cdots (I + g_{n-1} e_{n-1}^T) \\
&= [I + (g_1 e_1^T + g_2 e_2^T)](I + g_3 e_k^T) \cdots (I + g_{n-1} e_k^T) \\
&= [I + (g_1 e_1^T + g_2 e_2^T + g_3 e_3^T)](I + g_4 e_k^T) \cdots (I + g_{n-1} e_k^T) \\
&= \cdots \\
&= I + \sum_{k=1}^{n-1} g_k e_k^T
\end{aligned}$$