CME 108/MATH 114
Introduction to Scientific Computing
Summer 2018

Problem Set 2

Due: Monday, July 16 at 11:59pm

Warm-up Questions

The "warm-up" questions do not need to be submitted (and won't be graded). However, I *highly* encourage you to work out their solutions! I'll post answers along with the solutions to the assigned problems.

Define

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -2 \\ 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 9 \\ 11 & 3 \\ 1 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

1. Compute $A + B$ and $\frac{1}{2}A - 3B$.

2. Let
$$x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$
Compute AC, BC, Ay, xB, Cy, and $y^T C$. Also compute $x^T x$ and yy^T. (First write down the size of each product.)

3. Write a MATLAB script to verify the above computations. (You can verify all products not involving the unknowns x and y.)

4. Let D be an $n \times n$ diagonal matrix, and consider any $n \times n$ matrix E. Show that multiplying E by D on the left (DE) scales the i-th row of E by d_{ii}. Similarly, show that multiplying E by D on the right (ED) scales the j-th column of E by d_{jj}.

5. *(Practice Gaussian elimination, Bradie* 3.1.1, 3.1.2.*)* For the following systems of linear equations, write an equivalent matrix equation and solve it using Gaussian elimination.

(a)

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 4x_1 + 2x_2 + x_3 &= 4 \\ 6x_1 - 4x_2 + 2x_3 &= -2 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

Assignment problems

Solutions to the following problems should be submitted.

1. (10 points) *(Stochastic monoid.)* A square matrix A is called *right stochastic* if the elements in each row have unit sum. That is, a given $n \times n$ matrix A is right stochastic if

$$\sum_{j=1}^{n} a_{ij} = 1,$$

for each $1 \leq i \leq n$.

Suppose A and B are $n \times n$ right stochastic matrices. Show that AB is right stochastic.

2. (10 points) *(Feasible rent.)* Mary has three apartments for rent: a one-, a two- and a three-bedroom. Below are some facts about Mary's monthly rentals.

- The total rent amongst the three apartments is $\$1,240$.
- The maintenance cost for the one-bedroom apartment is 10% of the apartment's rent. Similarly, the maintenance cost for the two- and three-bedroom apartments amount to 20% and 30% of their respective rents. The total monthly maintenance cost is \$276.
- Mary rents the three-bedroom apartment for twice as much as the one-bedroom.

(a) Let r_1, r_2 and r_3 respectively denote the rent of each apartment. Using the information above, set up a system of linear equations satisfied by the unknowns r_1, r_2, and r_3.

(b) Use Gaussian elimination to find the rent of each apartment.

3. (15 points) *(Inverse Gauss transform.)* Consider the lower triangular matrices used to compute the LU factorization of a given $n \times n$ matrix A (assuming the factorization exists). In particular, for $1 \leq k \leq n-1$, let L_k be the lower triangular matrix such that

$$(L_k)_{ij} = \begin{cases} m_i & \text{if } i > k \text{ and } j = k, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Show that L_k^{-1} is given by

$$(L_k^{-1})_{ij} = \begin{cases} -m_i & \text{if } i > k \text{ and } j = k, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: Write $L_k = I_n + g e_k^T$ for an appropriate vector g and with e_k denoting the n-vector with 1 in the k-th component and 0's elswhere.

4. (10 points) *(Pivoting saves the day, Bradie 3.5.7.)*

(a) Show that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

has no LU decomposition by writing out the equations corresponding to

$$A = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix},$$

and showing that the system has no solution.

(b) Reverse the order of the rows of A and show that the resulting matrix does have an LU decomposition.

5. (25 points) *(Finite precision LU.)* A *pivoting strategy* for Gaussian elimination provides a means of determining which rows to swap and when. Sometimes, it is necessary (like when we encounter zeroes on the diagonal) to swap rows of the coefficient matrix (as in Problem 4 above) in order to continue the elimination process.

Other times, it is useful (like when the diagonal entries are *nearly* zero) to swap rows in order to avoid unnecessarily large entries in the triangular factors (which may lead to severe numerical imprecision).

Pivoting strategies are known to improve the accuracy in the solution provided by Gaussian elimination, in many cases.

Now consider the following system of simultaneous equations:

$$\begin{aligned} 0.1036x_1 + 0.2122x_2 &= 0.6404, \\ 0.2081x_1 + 0.4247x_2 &= 0.7369. \end{aligned}$$

(a) First **verify** (by hand) that the exact solution of the system is given by $x_1 = -723, x_2 = 356$.

(b) Use Gaussian elimination without pivoting (no row swapping) to solve the above system. Work in the FPS $\mathbf{F}(10, 4, -3, 3)$ and use rounding to find the floating point equivalent of the result of each intermediate calculation. Make sure that you round the result of each arithmetic operation!

(c) Now use Gaussian elimination with partial pivoting (before applying each elimination step, swap rows so that the diagonal element has the largest magnitude in its column) to solve the system. (Note that in this case $|0.2081| > |0.1036|$.)

Again work in the FPS $\mathbf{F}(10, 4, -3, 3)$ and use rounding to find the floating point equivalent of the result of each intermediate calculation. Again, make sure that you round the result of each arithmetic operation!

(d) Compare your results. Why is there so much discrepancy between the exact answer and the result obtained using Gaussian elimination?

Note that pivoting does not significantly improve the accuracy in the solution. Why is that?