CME 108/MATH 114
Introduction to Scientific Computing
Summer 2018

Problem Set 3

Due: Friday, July 20 at 11:59pm

Warm-up Questions

The "warm-up" questions do not need to be submitted (and won't be graded). However, I *highly* encourage you to work out their solutions! I'll post answers along with the solutions to the assigned problems.

Let A be an $n \times n$ invertible matrix. Note that the product AA is defined, and so is AAA. In general, for any positive integer k, set $A^k = A \cdots A$, the product on the right containig k factors (e.g. $A^4 = AAAA$).

1. Show that if $A^2 = I$ then $A^{-1} = A$.

2. If $6A^3 - 2A^2 + 4A - 2I = 0$, what is A^{-1}?

3. Show that $(A^{-1})^{-1} = A$.

4. *(LU verification, Bradie* 3.5.5.*)* Let

$$B = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}.$$

Verify that each of the following pairs forms an LU decomposition of B, and then use the decomposition to solve the system $Bx = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$.

(a)

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{bmatrix}.$$

(b)

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & -11 & 45 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 2 & 7 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c)

$$L_3 = \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 1 & 7/2 & 5/2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

In general, the LU factorization is not unique! The algorithm we learned in lecture produces the so-called "Doolittle factorization," which forces L to have a unit diagonal ($l_{ii} = 1$ for $1 \leq i \leq n$).

5. *(LU factorization, Bradie* 3.5.10.*)* Let

$$C = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix}.$$

Find a lower triangular L with ones along its diagonal and an upper triangular matrix U such that $A = LU$.

6. With B as defined in 4 above, use partial pivoting to find matrices L, U, and P so that $PB = LU$.

7. *(Block matrices.)* Let A and B be $n \times n$ matrices. For $1 \leq k \leq n$, set

$$\begin{aligned} S_{11} &= A(1:k, 1:k), \\ S_{12} &= A(1:k, k+1:n), \\ S_{21} &= A(k+1:n, 1:k), \\ S_{22} &= A(k+1:n, k+1:n). \end{aligned}$$

Similarly, let

$$\begin{aligned} T_{11} &= B(1:k, 1:k), \\ T_{12} &= B(1:k, k+1:n), \\ T_{21} &= B(k+1:n, 1:k), \\ T_{22} &= B(k+1:n, k+1:n). \end{aligned}$$

Note that A can be partitioned as

$$A = \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right],$$

and that a similar partition applies for B.

Show that

$$AB = \left[\begin{array}{c|c} S_{11}T_{11} + S_{12}T_{21} & S_{11}T_{12} + S_{12}T_{22} \\ \hline S_{21}T_{11} + S_{22}T_{21} & S_{21}T_{12} + S_{22}T_{22} \end{array}\right].$$

Note the sizes and order of the factors!

Assignment problems

Solutions to the following problems should be submitted.

1. (30 points) *(Permuted LU).* Consider the linear system $Ax = b$. In MATLAB, implement the LU factorization algorithm with partial pivoting.

To reduce the storage needs of our algorithm, we require that the algorithm overwrites A with its LU factors. That is, you may not at any point create new arrays to store the L and U factors: their entries should be stored in place of the entries of A.

Next, implement triangular solvers (backward and forward substitution) which can be used to solve the systems $Ly = b$ and $Ux = y$.

Define

$$A = \begin{bmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 151 \\ 103 \\ 16 \\ -32 \end{bmatrix},$$

and use your triangular solvers to find x. Print out L, U, and the permutation matrix P. Verify that $x =$ `A\b`.

Hint: Explicitly swapping rows of A as needed and creating a permutation matrix will make your life easier. (You should opt for implicit swaps and a permutation vector if you want to write high-perfomance code but explicit swaps and a full permutation matrix suffice here.)

2. (30 points) *(Tridiagonal LU.)*

(a) Compute an LU decomposition of the tridiagonal matrix A by hand, with

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

Let $b = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. Use the computed LU factors to solve the system $Ax = b$ (by hand). You will encounter this matrix later in the course, in the context of PDEs.

(b) Specialize the solver you wrote in Problem 1 to the case where A is tridiagonal.

Note that tridiagonal matrices are *sparse*, which means that most of their entries are 0. As you discovered in Part (a), the LU factors also turn out to be sparse in that case. In the context of solving the system $Ax = b$, sparsity of A implies that much less work is required to obtain a solution. **Your specialized solver must reflect this.** You may assume no pivoting is required. Comment on how the number of computations performed by your specialized solver depends on N, the number of non-zero entries in A.

(c) Use the specialized solver you implemented in Part (b) to verify the calculations you performed in Part (a).

3. *(Facts about LTs.)* In this problem we'll prove some facts about lower triangular matrices, which we assumed during lecture.

(a) First show that if L is any $n \times n$ lower triangular *invertible* matrix, then $l_{jj} \neq 0$ for every $1 \leq j \leq n$. You can do this by induction, starting with the case $n = 1$ and using the definition of invertibility.

Note that in part (b) you will use this result to prove that the inverse of a lower triangular matrix is lower triangular, so you may **not** assume that L^{-1} is lower triangular. (You don't need to anyways!)

Hint: For the inductive step, partition

$$L = \left[\begin{array}{c|c} L_n & 0_{n\times 1} \\ \hline l_{n+1} & l_{n+1,n+1} \end{array}\right],$$

with $L_n = L(1:n, 1:n)$ and $l_{n+1} = L(n+1, 1:n)$, and consider the equation $LL^{-1} = I$. (Refer to warm-up problem on block matrices.)

(b) Using part (a) and the definition of invertibility, show that if L is an $n \times n$ invertible lower triangular matrix, then L^{-1} is also lower triangular.

Hint: Consider l_{ij} for $j > i$ row by row, starting with $i = 1$.

(c) Recall the so-called "Gauss transforms" defined in Problem 3 of HW2. If $L_k = I + g_k e_k^T$, show that

$$L_1 L_2 \cdots L_{n-1} = I + \sum_{k=1}^{n-1} g_k e_k^T.$$