CME 108/MATH 114
Introduction to Scientific Computing
Summer 2018

Problem Set 4

Due: Monday, July 30 at 11:59pm

Warm-up Questions

1. (*Inverse and transpose commute.*) Show that if A is invertible, then

$$(A^T)^{-1} = (A^{-1})^T.$$

2. (*Projection matrices.*) A matrix P is said to be a *projection* matrix if $P^T = P$ and $P^2 = P$.

   (a) Show that if P is a projection matrix then so is $I - P$.

   (b) Show that for any $n \times k$ matrix U such that $U^T U = I$, the matrix UU^T is a projection matrix.

   (c) Show that for any $n \times k$ matrix A such that $A^T A$ is invertible, $A(A^T A)^{-1} A^T$ is a projection matrix.

3. (*Frobenius norm.*) Given an $n \times n$ matrix A, the *Frobenius norm* $\|A\|_F$ of A is defined by

$$\|A\|_F = \sqrt{\operatorname{tr}(A^T A)}.$$

   Show that the equality

$$\|A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$$

   holds for any $n \times n$ matrix A.

4. (*Determinant of diagonal matrices.*) Using induction, show that

$$\det(D) = d_{11} \cdots d_{nn} = \prod_{i=1}^{n} d_{ii},$$

for any $n \times n$ diagonal matrix D.

5. (*Linear transformations as matrices.*) For each of the following linear transformations, write down a corresponding matrix representation in the bases provided.

   (a) Let x_1, x_2 denote a basis for X and let y_1, y_2, y_3, y_4 denote a basis for Y. Consider $T : X \to Y$ such that

   $$T(x_1) = 3y_1 - y_2 + 9y_4, \text{ and } T(x_2) = 5y_2 - 7y_3 + y_4.$$

   (b) Let v_1, v_2, v_2, v_4, v_5 denote a basis for V and consider $T : V \to \mathbb{R}$ such that

   $$T(v_1) = -14, \ T(v_2) = 7, \ T(v_3) = 5, \ T(v_4) = -3, \text{ and } T(v_5) = 9.$$

   (c) Let e_1, e_2, e_3 denote a basis for $\mathbb{R}^3$ and consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

   $$T(e_1) = 2e_2 + 3e_3, \ T(e_2) = e_1 - e_2 + 4e_3, \text{ and } T(e_3) = 4e_3.$$

   Note: When the input and output spaces correspond, we take the input basis as a basis for the output space as well.

   (d) Let $v = e_1 - 2e_2 - 5e_3$. Using the matrix representation from (c), find the coordinate representation $T(v)$.

6. (*Matrices as linear transformations.*) Suppose v_1, v_2, v_3 is a basis of V and let w_1, w_2 denote a basis of W, and consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Since

$$A(\alpha x + \beta y) = \alpha(Ax) + \beta(Ay),$$

for any real numbers α, β and any length 3 vectors x, y, A corresponds to a linear transformation $T : V \to W$. Determine the action of T on the basis v_1, v_2, v_3.

Does A^T correspond to a map $S : W \to V$? If so, determine the action of that transformtion on the basis w_1, w_2.

We note that the last two problems illustrate the equivalence of matrices and linear transformations.

7. (*Lagrange polynomials.*) Let $x_0 = -1$, $x_1 = 1$, and $x_2 = 2$. Determine formulas for the Lagrange polynomials $L_{2,0}(x)$, $L_{2,1}(x)$, and $L_{2,2}(x)$ associated with the given interpolating points.

8. (*Interpolation via matrices.*) Consider the following data

t	1	2	3	4
y	11	29	65	125

Construct a third-order interpolating polynomial of the form

$$P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Use the interpolating points to form a matrix equation $Ax = b$ where x is the unknown vector $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$.

Use the MATLAB command `x = A\b` to solve for x. Use the interpolating polynomial to approximate the value $y(3.3)$.

Assignment problems

1. (20 points) (*Gauss-Seidel.*) Recall that the Gauss-Seidel method is an iterative method based on the splitting

$$A = M - N,$$

where $M = D + L$ and $N = -U$. Here D denotes the diagonal part of A, L the strict lower triangular part, and U the strict upper triangular part, as in lecture.

Explicitly, D is diagonal with $d_{ii} = a_{ii}$, L is lower triangular with $l_{ii} = 0$ and $l_{ij} = a_{ij}$ for $i > j$, and U is upper triangular with $u_{ii} = 0$ and $u_{ij} = a_{ij}$ for $j > i$.

Given an $n \times n$ matrix A and b, we would like to solve the equation $Ax = b$. The recurrence relation proposed by the Gauss-Seidel method is given by

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right),$$

for $i = 1, \ldots, n$.

Implement the Gauss-Seidel method in MATLAB and use it to solve the equation $Ax = b$, with

$$A = \begin{bmatrix} 6 & 2 & 3 & 4 & 1 \\ 2 & 6 & 2 & 3 & 4 \\ 3 & 2 & 6 & 2 & 3 \\ 4 & 3 & 2 & 6 & 2 \\ 1 & 4 & 3 & 2 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 24 \\ 8 \\ 5 \\ 24 \end{bmatrix}.$$

Start with $x_0 = 0$ and stop your iteration when the largest entry (in absolute value) of $r^{(k)} = b - Ax^{(k)}$ is less than $\epsilon = 10^{-6}$.

Hint: The following MATLAB commands might be helpful: `toeplitz`, `tril`, and `triu`. Remember to test your code against MATLAB's built-in linear equation solver!

2. (20 points) *(Polynomial differentiation.)* Let $V = \mathcal{P}_3$ denote the vector space of all polynomials of degree at most 3. That is, let V denote the set

$$V = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_i \in \mathbb{R}, i = 0, \ldots, 3\}$$

along with the regular addition and scalar multiplication of polynomials.

Note that $1, t, t^2, t^3$ forms a basis for V.

Recall that for any two differentiable functions f, g and any two real numbers α, β, we have

$$\frac{d}{dt}(\alpha f + \beta g) = \alpha\left(\frac{d}{dt}f\right) + \beta\left(\frac{d}{dt}g\right).$$

Combined with the fact that the derivative of a polynomial of degree n is a polynomial of degree $n-1$, the last equation shows that the transformation $\frac{d}{dt} : V \to V$, which maps a polynomial of degree at most 3 to its derivative, is linear. Therefore, the map has a corresponding matrix representation in the basis $1, t, t^2, t^3$.

(a) By considering the action of $\frac{d}{dt}$ on each basis element, provide a matrix representation for $\frac{d}{dt}$ in the basis $1, t, t^2, t^3$.

(b) Use your matrix from part (a) and the coordinate representation of $p(t) = 5t^2 - 12t^3$ to compute

$$\frac{d^2}{dt^2}p(t) = \frac{d}{dt}\left[\frac{d}{dt}p(t)\right].$$

3. (25 points) (*Matrix multiplication via composition.*) Let $u_1, \dots u_n$ denote a basis for U, let $v_1, \dots, v_m$ denote a basis for V, and let $w_1, \dots, w_p$ denote a basis for W. Let $S : U \to V$ and $T : V \to W$ be such that

$$S(u_j) = a_{1j}v_1 + \cdots + a_{mj}v_m, \text{ and } T(v_k) = b_{1k}w_1 + \cdots + b_{pk}w_p,$$

for $j = 1, \dots, n$ and $k = 1, \dots, m$.

Show that the matrix corresponding to the linear transformation $T \circ S : U \to W$ in the given bases is BA. For completeness, we note that $T \circ S$ denotes the composition of S and T:

$$(T \circ S)(u) = T(S(u)),$$

for any $u \in U$.

Hint: Recall that *every* linear transformation is determined by its action on a basis.

4. (25 points) (*Interpolating log.*) Consider the function $f(x) = \ln(x)$. In this problem, you will construct the polynomial which interpolates the points

$$(1, \ln 1), (2, \ln 2), (3, \ln 3)$$

in two ways.

(a) We want to find the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ such that $p(x_i) = f(x_i)$ for $x_i = i$, $i = 1, 2, 3$. Express the system of equations as a single matrix equation in the variable

$$c = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix},$$

and solve it using MATLAB. You may use the built-in solver or code that you have previously written.

(b) Now construct the Lagrange form of the interpolating polynomial p.

Note that the polynomial obtained here is equivalent to the one obtained in (a) above, by uniqueness of the interpolating polynomial.

(c) Using MATLAB, plot f and the interpolating polynomial on the same set of axes, over the interval $[1/2, 7/2]$.

5. (10 points) (*Rational interpolation.*) Consider a function of the form

$$f(x) = \frac{a_0 + a_1 x + \ldots + a_m x^m}{1 + b_1 x + \ldots + b_m x^m}.$$

Such a function is called a *rational function of degree m.*

Suppose we are given data points $(x_1, y_1), \ldots, (x_{2m+1}, y_{2m+1})$. We would like to find coefficients $a_0, \ldots, a_m$ and $b_1, \ldots, b_m$ such that f interpolates the data points; such that $y_i = f(x_i)$, for $i = 1, \ldots, 2m+1$.

Express the system of equations $y_i = f(x_i)$ as a single matrix equation in the variable c, with

$$c = \begin{bmatrix} a_0 \\ \vdots \\ a_m \\ b_1 \\ \vdots \\ b_m \end{bmatrix}.$$