Socio-Econ. Plan. Sci. Vol. 16, No. 5, pp. 223–224, 1982
Printed in Great Britain

0038-0121/82/050223-02$03.00/0
Pergamon Press Ltd.

A MULTIPLICATIVE MODEL FOR EFFICIENCY ANALYSIS†

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(*Received* 24 *November* 1981)

Abstract—This paper develops theory and algorithms for a "multiplicative" Data Envelopment Analysis (DEA) model employing virtual outputs and inputs as does the CCR ratio method for efficiency analysis. The frontier production function results here are of piecewise log-linear rather than piecewise linear form.

INTRODUCTION

In [1–3] Charnes, Cooper and Rhodes presented a method, the CCR ratio, for measuring the relative efficiency of DMU's (Decision Making units) having the same multiple inputs and multiple outputs. The essential characteristic of the CCR construction was the reduction of the multiple output-multiple input DMU situation to that of single virtual outputs and virtual inputs. For these, the ratios of virtual outputs to virtual inputs could be used to define relative efficiency in a manner similar to that in engineering practice.

Although it was understood from the first that the "additive" combinations of outputs or inputs to achieve virtual outputs and inputs was not the only way of obtaining virtual single outputs and inputs, to date no other way has been studied. It is the purpose of this paper to present a "multiplicative" combinational method. As will be seen, it has a theory similar to that of the CCR ratio and its Data Envelopment Analysis but one which is simpler in certain respects, which can prove advantageous in applications, interpretations and extensions of other theories.

A MULTIPLICATIVE EFFICIENCY MODEL

As in [1], suppose there are n DMU's and the the jth DMU has output vector Y_j with s outputs and input vector X_j with m inputs. Suppose that our measures of inputs and outputs are such that all in our "sample" on n DMU's have values greater than unity.

To determine the relative efficiency of DMU_0 with input-output pair (X_0, Y_0), we form the "multiplicative" virtual outputs and inputs for the DMU's as follows:

$$\prod_{r=1}^{s} Y_{rj}^{\mu_r} \text{ and } \prod_{i=1}^{m} X_{ij}^{\nu_i},\ j = 1, \ldots, n. \tag{1}$$

This construction is analogous to a geometric mean construction in that we take the exponents μ_r and ν_i to be non-negative, and in order that they all be positive (to avoid Pareto inefficiencies) we require them to be not less than unity.

The relative efficiency of DMU_0 will then be given as the maximum of the ratio of its virtual output to virtual input subject to the condition that the virtual output to virtual input ratio of *all* DMU's be less than or equal to unity i.e. we solve the problem

$$\begin{aligned} &\max_{\mu,\ \nu} \prod_{r=1}^{s} Y_{ro}^{\mu_r} \Big/ \prod_{i=1}^{m} X_{io}^{\nu_i} \\ &\text{s.t.} \prod_{r=1}^{s} Y_{rj}^{\mu_r} \Big/ \prod_{i=1}^{m} X_{rj}^{\nu_i} \le 1,\ j = 1, \ldots, n \\ &\mu_r \ge 1,\ r = 1, \ldots, s \\ &\nu_i \ge 1,\ 0 = 1, \ldots, m. \end{aligned} \tag{2}$$

Taking logarithms (to any base), this may be written as the linear programming problem

$$\begin{aligned} &\max \sum_{r=1}^{s} \mu_r \hat{Y}_{ro} - \sum_{i=1}^{m} \nu_i \hat{X}_{io} \\ &\text{s.t.} \sum_{r=1}^{s} \mu_r \hat{Y}_{rf} - \sum_{i=1}^{m} \nu_i \hat{X}_{ij} \le 0,\ j = 1, \ldots, n \\ &-\mu_r \qquad \le -1,\ r = 1, \ldots, s \\ &-\nu_i \qquad \le -1,\ i = 1, \ldots, m \end{aligned} \tag{3}$$

where the caret sign $(\hat{\ })$ denotes logarithms. More com pactly this can be written as

$$\begin{aligned} &\max \mu^T \hat{Y}_0 - \nu^T \hat{X}_0 \\ &\text{s.t. } \mu^T \hat{Y} - \nu^T \hat{X} \le 0 \\ &-\mu^T \qquad \le -e^T \\ &-\nu^T \qquad \le -e^T \end{aligned} \tag{4.1}$$

†This research is partly supported by ONR contract N00014-81-C-0236 and USARI contract MD--903-81-C-0365 with the Center for Cybernetic Studies, The University of Texas at Austin.
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with dual

$$\begin{aligned} \min\ & -e^T s^+ - e^T s^- \\ \text{s.t. } & \hat{Y}\lambda - s^+ = \hat{Y}_0 \\ & -\hat{X}\lambda - s^- = -\hat{X}_0 \\ & \lambda,\ s^+,\ s^- \geq 0 \end{aligned} \tag{4.2}$$

where $\hat{Y} \equiv [\hat{Y}_1, \ldots, \hat{Y}_n]$, $\hat{X} \equiv [\hat{X}_1, \ldots, \hat{X}_n]$ and the e^T are row vectors of ones of the appropriate sizes.

Evidently (4.2) is a DEA (Data Envelopment Analysis) problem with the cone on the DMU vectors enveloping $\hat{Y}_0$ from above, e.g. $\hat{Y}\lambda \geq \hat{Y}_0$ and enveloping $\hat{X}_0$ from below e.g. $\hat{X}\lambda \leq \hat{X}_0$. It is a simpler envelopment than that for the CCR ratio model. Further, its log-efficiency is given solely in terms of the optimal slacks in (4.2), whereas the CCR efficiency value also involves a change in intensity of the X_0 vector.

CHARACTERIZING EFFICENT DMU'S

Since the optimal functional value in (4.2) is the log-efficiency of DMU_0 and a DMU is efficient if and only if it has a log-efficiency of zero, we have the result,

Theorem 1. DMU_0 is efficient in an optimal solution of (4.2) iff all slacks are zero.

Again, as in the CCR ratio model, the efficiency rating of a DMU is characterized by a certain subset of efficient DMU's. For,

Theorem 2. If DMU_k appears in an optimal basic solution to (4.2), then DMU_k is efficient.

Proof. The log-efficiency problem of DMU_k replaces the right hand side in (4.2) by $(\hat{Y}_k, -\hat{X}_k)$.^{T} An optimal basic solution for DMU_0 which contains the basis vector $(\hat{Y}_k, -\hat{X}_k)^T$ is feasible for the DMU_k problem with $\lambda_k = 1$, $\lambda_j = 0$, $j \neq k$ and s^+, $s^- = 0$. It is also optimal since the dual evaluators do not involve the right hand side. Thus, DMU_k has log-efficiency zero.

Also, as in the CCR ratio model, we can project a DMU onto the efficiency facet which determines its efficiency, for

Theorem 3. If the coefficient vectors of s_B^{*+}, s_B^{*-} are basic slack vectors in an optimal basic solution of (4.2) for DMU_0, then replacing $(\hat{Y}_0, -\hat{X}_0)^T$ by $(\hat{Y}_0 + s_B^{*+}, -\hat{X}_0 + s_B^{*-})^T$ yields an efficient DMU.

Proof. If s^{*+}, $s^{*-} = 0$, DMU_0 is efficient. If $e^T s^{*+} + e^T s^{*-} > 0$, then DMU_0 is not efficient. By Theorem 2, its output-input vector cannot be part of this optimal basis for it. Evidently the vector $(\hat{Y}_0 + s_B^{*+}, -\hat{X}_0 + s_B^{*-})^T$ as a right hand side vector is feasible for this basis and with zero slacks. Thus, a fortiori, this basic solution is also a basic solution with both the right hand side and left hand side $(\hat{Y}_0, -\hat{X}_0)^T$ replaced by the new vector.

To confirm that this basic solution is still optimal for the new problem, we need only verify that the dual evaluators (μ^*, ν^*) satisfy the new constraint in the dual problem. Now our old optimal solution is the (unique) solution to

$$\begin{aligned} \hat{Y}_B \lambda_B^* - s_B^{*+} &= \hat{Y}_0 \\ -\hat{X}_B \lambda_B^* - s_B^* &= -\hat{X}_0 \end{aligned} \tag{5}$$

where the "B" subscript designates the column vectors corresponding to this basic solution. Thereby

$$\begin{aligned} \mu^{*T}(\hat{Y}_0 + s_B^{*+}) &= \mu^{*T}(\hat{Y}_B \lambda_B^*) \\ \nu^{*T}(-\hat{X}_0 + s_B^{*-}) &= \nu^{*T}(-\hat{X}_B \lambda_B^*) \end{aligned} \tag{6}$$

and

$$\mu^{*T}(\hat{Y}_0 + s_B^{*+}) - \nu^{*T}(\hat{X}_0 - s_B^{*-}) = (\mu^{*T}\hat{Y}_B - \nu^{*T}\hat{X}_B)\lambda_B^* \leq 0$$

as required to verify basis optimality in he new problem. Thus, the "projection" of the $(-\hat{X}_0, \hat{Y}_0)$ of DMU_0 onto $(-\hat{X}_0 + s_B^{*-}, \hat{Y}_0 + s_B^{*+})$ is a projection onto the efficiency facet spanned by the efficient DMU's which determined DMU_0's efficiency.

COMPUTATION OF EFFICIENCY

To compute the relative efficiency of the DMU's, we suggest using the DEA side of (4.2) for the n linear programs to be computed. Notice that they differ from one another only in the right hand side vector, a simplication from the computational situation for the CCR ratio model.

As before, however, we can immediately specify a feasible basis for DMU_0. Here it differs from a diagonal matrix only in one column,

$$\begin{bmatrix} \hat{Y}_0 & -\tilde{I} & 0 \\ -\hat{X}_0 & 0 & -I \end{bmatrix} \tag{8}$$

where $\tilde{I}$ designates an $(s-1)\times(s-1)$ identity matrix with an additional first row of zeroes. We note, parenthetically, that the inverse of (8) can be written explicitly as

$$\begin{bmatrix} \hat{\bar{Y}}_0 y_{10}^{-1} & \tilde{I} & 0 \\ -\hat{X}_0 y_{10}^{-1} & 0 & -I \end{bmatrix} \tag{9}$$

where $\bar{Y}_0$ denotes Y_0 with y_{10} replaced by unity.

To speed up computation while making accessible the efficient facet information corresponding to DMU_0, we point out that by Theorem 2 on reaching an optimal basis for DMU_0, any DMU_g in this basis is efficient and its dual evaluators are those corresponding to DMU_0! Thus we do not need to separately compute an optimal tableau for DMU_g. We need only record the slacks for DMU_0, its dual evaluator vector and what its subset of efficient DMU_g's is and proceed to compute log-efficiency for the remaining unrecorded DMU's. Further, any DMU_p *not* in the basis, but which has zero "$(z_p - c_p)$" is also in the same efficient facet. For, it is an alternate optimum which can be pivoted into the basis. But, since the old dual evaluators are still dual evaluators for it, no additional computation need be made. Thus, such DMU_p's can also be recorded as corresponding to DMU_0.

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