THE COEFFICIENT OF RESOURCE UTILIZATION[1]

By Gerard Debreu

A numerical evaluation of the "dead loss" associated with a nonoptimal situation (in the Pareto sense) of an economic system is sought. Use is made of the intrinsic price systems associated with optimal situations of whose existence a noncalculus proof is given. A coefficient of resource utilization yielding measures of the efficiency of the economy is introduced. The treatment is based on vector-set properties in the commodity space.

1. INTRODUCTION

The activity of the economic system we study can be viewed as the transformation by n production units and the consumption by m consumption units of l commodities (the quantities of which may or may not be perfectly divisible). Each consumption unit, say the ith one, is assumed to have a preference ordering of its possible consumptions, and therefore an index of its satisfaction, $\mathcal{S}_i$. Each production unit has a set of possibilities (depending, for example, on technological knowledge) defined independently of the limitation of physical resources and of conditions in the consumption sector. Finally, the total *net* consumption of all consumption units and all production units for each commodity must be at most equal to the available quantity of this commodity.

If we impose on the economic system the constraints defined by (1) the set of possibilities of each production unit and (2) the limitation of physical resources, we cannot indefinitely increase the m satisfactions. In trying to do so we would find situations where it is impossible to increase any satisfaction without making at least one other one decrease. In any one of these situations all the resources are fully exploited, and it can be considered optimal. When a situation is nonoptimal is it possible to find some measure of the loss involved, indicating how far it is from being optimal? The basic difficulty comes from the fact that no meaningful metrics exists in the satisfaction space.

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For this reason we take up the following dual problem. We impose on the economic system the constraints defined by (1) the set of possibilities of each production unit and (2) the condition that for each consumption unit the satisfaction $\mathcal{S}_i$ is at least equal to a given value $\mathcal{S}_i^0$. We cannot decrease indefinitely the l quantities of available physical resources. In trying to do so we would find situations where it is impossible to decrease one of them without making at least one other one increase. In any one of these situations the prescribed levels of satisfaction have been attained with as small an amount of physical resources as possible, and it can be considered optimal. The loss associated with a nonoptimal situation is now a measure of the distance from the actually available complex of resources to the set of optimal complexes; this concept is far simpler than the former one because we are dealing now with *quantities of commodities*. The two definitions of optimality are equivalent if the saturation cases are excluded.

Using the second definition of optimality we proceed to a noncalculus proof of the intrinsic existence of price systems associated with the optimal complexes of physical resources—the basic theorem of the new welfare economics. This proof is more general than the usual ones since it does not require the existence of derivatives which, indeed, do not exist in simple and realistic cases; more complete, since it deals with global instead of local properties of maxima or minima; more concise, as the synthetic nature of the problem requires it to be; it gives a deeper explanation of the intrinsic existence of prices by its geometric interpretation in the commodity space. These reasons seem to justify the higher level of abstraction on which it is placed.

This proof is based on convexity properties which imply continuity of quantities of commodities; if this assumption of continuity is dropped, the same technique shows that to achieve an optimal situation the use of a (real or virtual) price system is still sufficient but no longer necessary.

We are now prepared to measure the distance from the actually given complex of physical resources to the set of optimal complexes, i.e., the minimum of the distance from the given complex to a varying optimal complex. To evaluate such a distance we multiply, for each commodity, the difference between the available quantity and the optimal quantity by the price derived from the intrinsic price system whose existence has been previously proved. We take the sum of all such expressions for all commodities, and we divide by a price index in order to eliminate the arbitrary multiplicative factor affecting all the prices. It is then proved that the distance function so defined reaches its minimum for an optimal complex resulting from a reduction of all quantities of the nonoptimal complex by a ratio ρ, the coefficient of resource utilization of the economic system. This number, equal to 1 if the situation is optimal, smaller than 1 if it is nonoptimal, measures the efficiency of the economy and summarizes (1) the underemployment of physical resources, (2) the technical inefficiency of production units, and (3) the inefficiency of economic organization (due, for example, to monopolies or a system of indirect taxes or tariffs).

The money value of the "dead loss" associated with a nonoptimal situation can be derived from ρ, and the inefficiency of the economy is now described by a certain number of dollars representing the value of the physical resources which could be thrown away without preventing the achievement of the prescribed levels of satisfaction. This definition seems to obviate the shortcomings of the older ones.

The theory which led to the introduction of ρ can be imbedded in a more general one. Let us consider the ratio of the money value of any complex of resources that allows one to achieve for each consumption unit at least $\mathcal{S}_\iota^0$ to the money value of actually available resources, the price system being arbitrary. The antagonistic activities of a central agency, which chooses the prices so as to make this ratio as large as possible, and of the economic units, which behave in such a way as to make it as small as possible, eventually give the value ρ to it.

This minimax interpretation of ρ points out a rather striking isomorphism with the theory of statistical decision functions.

The end of Section 9 might be useful as a supplement to this introduction by its more detailed exposition of the significance of the coefficient of resource utilization. The two most important sections are 6, where the noncalculus proof of the basic theorem of the new welfare economics is given, and 9, where ρ is introduced. Section 11, which gives the minimax interpretation of ρ, is a natural complement of 9. Section 12, which brings out the isomorphism with the theory of statistical decision functions, includes an elementary and self-contained exposition of the latter.

2. BASIC MATHEMATICAL CONCEPTS

Vectors are denoted by bold face lower case roman or Greek types; their components, by corresponding ordinary lower case types with a subscript characterizing the coordinate axis. We use the following notations for inequalities among vectors:

$$\mathbf{u} \geqq \mathbf{v} \text{ if } u_\iota \geqq v_\iota \text{ for every } \iota,$$

$$\mathbf{u} > \mathbf{v} \text{ if } u_\iota > v_\iota \text{ for every } \iota,$$

$$\mathbf{u} \geq \mathbf{v} \text{ if } \mathbf{u} \geqq \mathbf{v} \text{ and } \mathbf{u} \neq \mathbf{v}.$$

A function $\mathbf{w}(\mathbf{u})$ is increasing (resp. nondecreasing) if "$\mathbf{u}^2 \geqslant \mathbf{u}^1$" implies "$\mathbf{w}(\mathbf{u}^2) \geqslant \mathbf{w}(\mathbf{u}^1)$ [resp. $\mathbf{w}(\mathbf{u}^2) \geqq \mathbf{w}(\mathbf{u}^1)$]."

Sets are denoted by German letters. According to the usual terminology, $\mathbf{v}$ is a *maximal* (resp. *minimal*) element of $\mathfrak{U}$ if (1) $\mathbf{v} \in \mathfrak{U}$ and (2) there is no $\mathbf{u}$ such that $\mathbf{u} \in \mathfrak{U}$ and $\mathbf{v} \leqslant \mathbf{u}$ (resp. $\mathbf{v} \geqslant \mathbf{u}$). The set of maximal (resp. minimal) elements of $\mathfrak{U}$ is denoted by $\mathfrak{U}^{\max}$ (resp. $\mathfrak{U}^{\min}$).

The vector sum of a finite number of sets $\mathfrak{U}_\iota$, $\mathfrak{V} = \sum_\iota \mathfrak{U}_\iota$, is the set of $\mathbf{v} = \sum_\iota \mathbf{u}_\iota$, $\mathbf{u}_\iota \in \mathfrak{U}_\iota$.

A set $\mathfrak{U}$ is convex if "$\mathbf{u} \in \mathfrak{U}$, $\mathbf{v} \in \mathfrak{U}$, $0 \leqq t \leqq 1$" implies "$t\mathbf{u} + (1 - t)\mathbf{v} \in \mathfrak{U}$."

A set $\mathfrak{U}$ is closed if it contains every point at a zero distance from $\mathfrak{U}$.

$\mathbf{u}$ is an interior point of $\mathfrak{U}$ if there exists a sphere of nonzero radius, centered at $\mathbf{u}$ and entirely contained in $\mathfrak{U}$.

A set $\mathfrak{U}^2$ is greater than (more strictly speaking, at least as great as) a set $\mathfrak{U}^1$ if it includes $\mathfrak{U}^1$; i.e., $\mathfrak{U}^2 \supset \mathfrak{U}^1$.

3. DESCRIPTION OF THE ECONOMIC SYSTEM

A *commodity* of the economic system is characterized by a subscript h ($h = 1, \cdots, l$). This concept can be given various contents: it can be a good or a service, direct or indirect, playing a role in any production or consumption process (for example, the training of pilots by some Air Force agency). The quantity of the hth commodity can either vary continuously or be an integral multiple of a given unit. The discontinuous case, which is indeed very widespread, can easily be included in the frame we present, as will be shown.

A *consumption unit* is characterized by a subscript i ($i = 1, \cdots, m$); its activity is represented by a *consumption vector* $\mathbf{x}_i$ of the l-dimensional Euclidean commodity space $\mathfrak{R}_l$; the components x_{hi} are quantities of commodities actually consumed or negatives of quantities of commodities produced (for a consumer of the classical type the only negative components correspond to the different kinds of labor he can produce). We assume that, if $\mathbf{x}_i^1$ and $\mathbf{x}_i^2$ are two arbitrary consumption vectors of the ith consumption unit, it either "prefers $\mathbf{x}_i^1$ to $\mathbf{x}_i^2$," "thinks $\mathbf{x}_i^1$ equivalent to $\mathbf{x}_i^2$," or "prefers $\mathbf{x}_i^2$ to $\mathbf{x}_i^1$" ($\mathbf{x}_i^2 \geqslant \mathbf{x}_i^1$ excluding $\mathbf{x}_i^1$ preferred to $\mathbf{x}_i^2$), with the usual transitivity property. One can therefore construct equivalence classes (an equivalence class may happen to contain only one vector), which will be denoted by $\mathfrak{S}_i$; a given $\mathbf{x}_i$ belongs to one and only one such class, $\mathfrak{S}_i(\mathbf{x}_i)$. The preference ordering on the $\mathbf{x}_i$ induces a *complete* ordering on the $\mathfrak{S}_i$ which will be denoted by $\mathfrak{S}_i^2 \geqq \mathfrak{S}_i^1$ (an element of $\mathfrak{S}_i^1$ is *not* preferred to an element of $\mathfrak{S}_i^2$). The usual procedure is to assume that a one-to-one, order preserving correspondence can be established between the set of $\mathfrak{s}_i$ and the set of real numbers so that a satisfaction function $s_i(\mathbf{x}_i)$ is obtained. First of all, such a correspondence need not exist, but even more important is the fact that the numerical value of this function has never any role to play, that only the ordering itself matters. The advisability of introducing such a function (always accompanied by the mention "defined but for an arbitrary monotonically increasing transformation"), which is useless and which moreover might not exist at all, may be questionable. However, one's intuition is likely to be helped if one views the ordering of the $\mathfrak{s}_i$ as the ordering of real numbers; we will draw Figures 1a and 3a in that spirit. The m $\mathfrak{s}_i$ are considered as the components of the element $\mathfrak{s} = (\mathfrak{s}_1, \cdots, \mathfrak{s}_m)$ of the product space $\mathfrak{E}$. $\mathfrak{s}_i$ could conveniently be called the satisfaction or standard of living of the ith consumption unit and $\mathfrak{s}$ the satisfaction or standard of living of the economic system. A partial ordering on the satisfaction space $\mathfrak{E}$ is defined in the following way: $\mathfrak{s}^2 > \mathfrak{s}^1$ if $\mathfrak{s}_i^2 \geqq \mathfrak{s}_i^1$ for every i and $\mathfrak{s}^2 \neq \mathfrak{s}^1$. The basic features of this reasoning are well known; our purpose is only to reformulate it in a language applicable to more general cases, including the discontinuous case. Here again the content of the concept of the *consumption unit* is left indeterminate: it can be a consumer, a household unit, a governmental agency, etc. In an economy provided with a central planning board incarnating a social welfare function there is only one consumption unit. The whole economic system can be divided into nations among which consumption units are distributed. The theory to be developed applies to all such cases.

The production activity of the system is represented by the *total input vector* $\mathbf{y} \in \mathfrak{R}_l$; the components of $\mathbf{y}$ are inputs (*net* quantities of commodities consumed by the *whole* production sector during the period considered) or negatives of outputs (defined in a symmetrical way). Constraints such as the limitation of technological knowledge determine the set $\mathfrak{Y}$ of possible $\mathbf{y}$. $\mathfrak{Y}$ is defined independently of the limitation of physical resources (which will be dealt with later) and of conditions in the consumption sector. The set of efficient vectors in production is $\mathfrak{Y}^{\min}$. (This concept is studied in great detail in [**14**] for the case where $\mathfrak{Y}$ is a convex polyhedral cone.)

A family of sets $\mathfrak{Y}_j$ $(j = 1, \cdots, n)$ is a *decomposition*[2] of $\mathfrak{Y}$ if $\mathfrak{Y} = \sum_j \mathfrak{Y}_j$; in other words $\mathbf{y} = \sum_j \mathbf{y}_j$, $\mathbf{y}_j \in \mathfrak{Y}_j$. The *input vector* $\mathbf{y}_j$ characterizes the activity of the jth *production unit*. The concept of *production unit* may coincide with that of industry, firm, plant, etc. This formulation allows for production and consumption of intermediate commod-

[2] This decomposition is not meant to be unique. If $\mathbf{0} \in \mathfrak{Y}_j$ for every j, $\mathfrak{Y}_j \subset \mathfrak{Y}$. A study of decentralization of economic decisions is concerned with the extent to which decisions can be made with respect to $\mathfrak{Y}_j$ instead of with respect to $\mathfrak{Y}$.

ities, even in a circular way, with as many intermediate steps as one wants. It allows, of course, for discontinuities of variables, or, if they are continuous, for nonsmooth surfaces $\mathfrak{Y}_j^{\min}$, for the existence of fixed ratios between some variables, etc. The more usual exposition, which amounts to starting from the $\mathfrak{Y}_j$ to obtain $\mathfrak{Y}$, is valid only if the assumption that $\mathfrak{Y}$ is nothing more than $\sum_j \mathfrak{Y}_j$ is explicitly made. In order that $\mathbf{y} \in \mathfrak{Y}^{\min}$ it is necessary but not sufficient that $\mathbf{y}_j \in \mathfrak{Y}_j^{\min}$ $(j = 1, \cdots, n)$.

The vector $\mathbf{x} = \sum_i \mathbf{x}_i$ is the total *consumption vector*, and $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is the total *net* consumption of the *whole* economy (all consumption units and all production units) which can come only from the available physical resources: we call it the *utilized physical resources vector* as opposed to $\mathbf{z}^0$, a vector of $\mathfrak{R}_l$, whose components are the available quantities of each commodity (natural resources and services of existing capital, for example; the different kinds of labor would give rise to zero components[3]). We call $\mathbf{z}^0$ the *utilizable physical resources vector*. We thus have $\mathbf{z} \leqq \mathbf{z}^0$.

4. OPTIMUM AND LOSS DEFINED IN THE SATISFACTION SPACE

The constraints imposed on the economic system are[4]

$$\mathbf{y} \in \mathfrak{Y}, \quad \mathbf{z} \leqq \mathbf{z}^0;$$

this determines in $\mathfrak{E}$ the set $\mathfrak{S}$ of attainable $\mathfrak{s}$. According to the Paretian criterion, if the goal of the economic system is to make the $\mathfrak{s}_i$, which cannot be compared to each other, as great as possible, $\mathfrak{s}^2$ is better than $\mathfrak{s}^1$ if, and only if, $\mathfrak{s}^2 \geqslant \mathfrak{s}^1$, and $\mathfrak{s}$ is *optimal* if, and only if, it is *maximal*: $\mathfrak{s} \in \mathfrak{S}^{\max}$. Any economic system, anxious to satisfy the needs of the consumption units as well as possible, and confronted with the problem of selecting one $\mathfrak{s}$ in $\mathfrak{S}$, would in fact restrict its choice to $\mathfrak{S}^{\max}$.

If $\mathfrak{s}^0 \notin \mathfrak{S}^{\max}$ (Figure 3a)[5] a *dead loss* is associated with $\mathfrak{s}^0$; its magnitude is, intuitively, the distance from $\mathfrak{s}^0$ to the set $\mathfrak{S}^{\max}$ (i.e., the minimum of the distance from $\mathfrak{s}^0$ to a variable $\mathfrak{s}$ belonging to $\mathfrak{S}^{\max}$). The very nature of the space $\mathfrak{E}$ prevents us from finding a meaningful content for that definition.

[3] The quantity of a certain kind of labor can be treated in two different ways: either as a component of the $\mathbf{x}_i$, if one wishes to emphasize the possibility of varying it, or as a nonzero component of $\mathbf{z}^0$ if the opposite assumption is made.

[4] Supplementary constraints such as the existence of some minimum standard of living, $\mathfrak{s} \geqq \mathfrak{s}^0$, would involve no essential change in the following analysis.

[5] Occasional references will be made to figures. They are all drawn in the two-dimensional case and, with one exception, contain only smooth curves, while the reasoning deals with a greater number of dimensions, nonsmooth surfaces, and even discrete sets of points. They are therefore mere illustrations, loosely connected with the text but likely to be found useful.

5. OPTIMUM AND LOSS DEFINED IN THE COMMODITY SPACE

Let us therefore study the following *dual* problem and consider in $\mathfrak{R}_l$ the set $\mathfrak{Z}(\mathfrak{s}^0)$ of vectors $\mathbf{z}$ defined by the constraints

$$\mathbf{y} \in \mathfrak{Y}, \quad \mathfrak{s} \geqq \mathfrak{s}^0.$$

$\mathfrak{Z}$ is the set of utilized physical resources vectors which, taking into account the production possibilities $\mathfrak{Y}$, enable the economy to achieve at least $\mathfrak{s}^0$. Let $\mathfrak{X}_i(\mathfrak{s}_i^0)$ be the set of $\mathbf{x}_i$ defined by $\mathfrak{s}_i(\mathbf{x}_i) \geqq \mathfrak{s}_i^0$, and $\mathfrak{X}(\mathfrak{s}^0) = \sum_i \mathfrak{X}_i$; then, since $\mathbf{z} = \mathbf{x} + \mathbf{y}$, $\mathfrak{Z}$ is nothing else than $\mathfrak{Z} = \mathfrak{X} + \mathfrak{Y}$.

$\mathfrak{Z}^{\min}(\mathfrak{s}^0)$ is a natural concept: it describes the minimal physical resources required to achieve at least $\mathfrak{s}^0$. One sees that $\mathbf{z}^0$ can be defined as optimal with respect to $\mathfrak{Y}$ and $\mathfrak{s}^0$ if, and only if, $\mathbf{z}^0 \in \mathfrak{Z}^{\min}$, and that, if $\mathbf{z}^0 \notin \mathfrak{Z}^{\min}$ (Figure 3b), the dead loss can be defined as the distance from $\mathbf{z}^0$ to the set $\mathfrak{Z}^{\min}$. This distance can now be meaningful since the coordinates of the commodity space $\mathfrak{R}_l$ are *quantities of commodities*.

The definitions of optimum and loss given in Sections 4 and 5 are not necessarily equivalent but, under conditions which amount essentially to excluding the saturation cases,[6] "$\mathfrak{s}^0 \in \mathfrak{S}^{\max}$" is equivalent to "$\mathbf{z}^0 \in \mathfrak{Z}^{\min}$." (See Figures 1a and 1d.)

6. THE OPTIMUM THEOREM

Let us now assume that the sets $\mathfrak{X}_i$, $\mathfrak{Y}_j$ are all *convex* and closed (convexity implies, of course, that the quantities of all commodities can be varied continuously); it follows that $\mathfrak{Z} = \sum_i \mathfrak{X}_i + \sum_j \mathfrak{Y}_j$ is convex.[7] As for the $\mathfrak{X}_i$, the assumption is a classical one and needs no particular comment; as for the $\mathfrak{Y}_j$, it may be worth noticing that if one added the two postulates,

1. $\mathbf{y}_1 \in \mathfrak{Y}_j$, $\mathbf{y}_2 \in \mathfrak{Y}_j$ *implies* $\mathbf{y}_1 + \mathbf{y}_2 \in \mathfrak{Y}_j$ (*additivity postulate*),

2. $\mathbf{0} \in \mathfrak{Y}_j$,

then $\mathfrak{Y}_j$ would be a cone.

[6] This can be done by a few additional simple postulates which we do not discuss in detail for they would lead to very formal developments without throwing more light on the heart of our problem. The saturation case has been considered by K. J. Arrow in the paper, "A Generalization of the Basic Theorem of Classical Welfare Economics" (to be published in the *Proceedings of the Second Berkeley Symposium*), given in the summer of 1950 at the Berkeley meeting of the Econometric Society independently of the present paper, which was given at the same time at the Harvard meeting of the Society.

[7] The sum of a finite number of closed sets is not necessarily closed when those sets are unbounded. However, it is sufficient that they are all contained in some closed, convex, pointed cone for their sum to be closed. We will assume that such is the case so that $\mathfrak{Z}$ is closed.

We now concentrate our attention on a vector $\mathbf{z}^0 = \sum_i \mathbf{x}_i^0 + \sum_j \mathbf{y}_j^0$ ($\mathbf{x}_i^0 \in \mathfrak{X}_i$, $\mathbf{y}_j^0 \in \mathfrak{Y}_j$) (Figure 1). If $\mathfrak{P}$ denotes the positive orthant of $\mathfrak{R}_l$ (the set of vectors of $\mathfrak{R}_l \geqq \mathbf{0}$), we have the following chain of *equivalent* propositions:

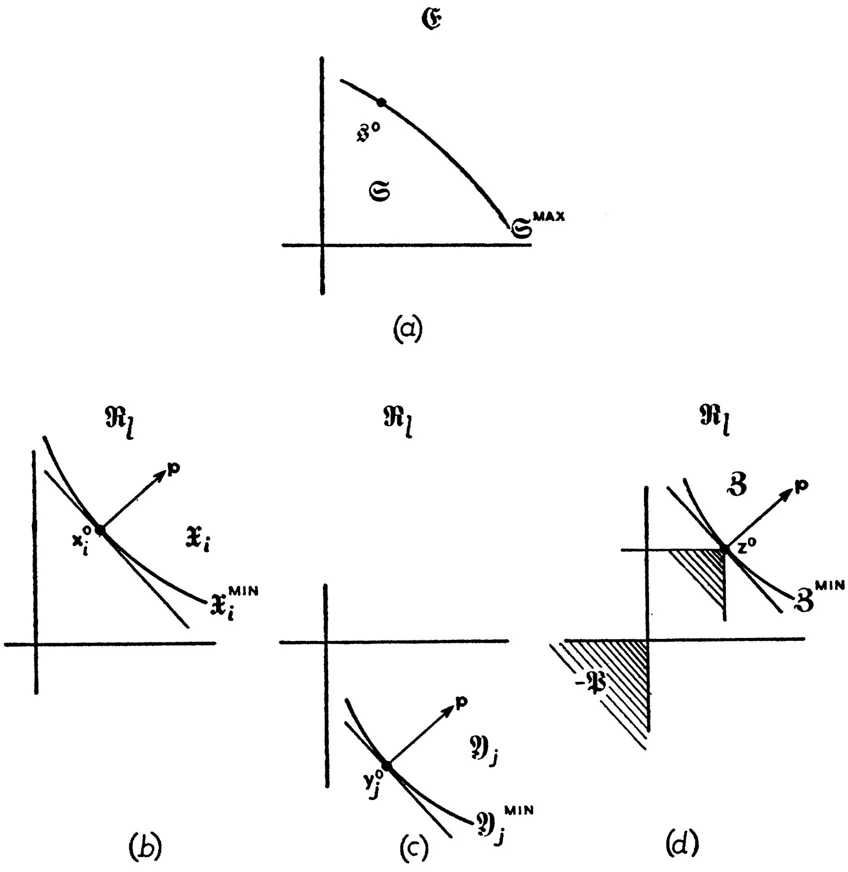

FIGURE 1

1. $\mathbf{z}^0 \in \mathfrak{Z}^{\min}$.
2. The convex sets $\mathfrak{Z}$ and $\mathbf{z}^0 - \mathfrak{P}$ have no other point in common than $\mathbf{z}^0$ (this is just another way of expressing proposition 1).[8]
3. There is a plane through $\mathbf{z}^0$ separating these two sets. (When two closed convex sets with interior points have only one point in common there is at least one plane through that point separating them [**6**].)
4. There is a vector $\mathbf{p} > \mathbf{0}$ (normal to the separating plane) such that $\mathbf{z} \in \mathfrak{Z}$ implies $\mathbf{p} \cdot (\mathbf{z} - \mathbf{z}^0) \geqq 0$.

[8] No distinction is made between a vector such as $\mathbf{z}^0$ and the set containing only this vector. $\mathbf{z}^0 - \mathfrak{P}$ is written in short for $\mathbf{z}^0 + (-\mathfrak{P})$.

5. There is a vector $\mathbf{p} > \mathbf{0}$ such that

(1) $\mathbf{x}_i \in \mathfrak{X}_i$ implies $\mathbf{p}\cdot(\mathbf{x}_i - \mathbf{x}_i^0) \geqq 0$ for every i,

(2) $\mathbf{y}_j \in \mathfrak{Y}_j$ implies $\mathbf{p}\cdot(\mathbf{y}_j - \mathbf{y}_j^0) \geqq 0$ for every j.[9]

Proof of the equivalence of propositions 4 and 5: $\mathbf{p}\cdot(\mathbf{z} - \mathbf{z}^0) = \sum_i \mathbf{p}\cdot(\mathbf{x}_i - \mathbf{x}_i^0) + \sum_j \mathbf{p}\cdot(\mathbf{y}_j - \mathbf{y}_j^0)$; the condition is therefore sufficient. It is necessary, for if one term of the right-hand member could be made < 0, as all the others can be made $= 0$, the left-hand member could be made < 0.[10]

Finally, (1) is equivalent to

(1′) $\mathbf{p}\cdot(\mathbf{x}_i - \mathbf{x}_i^0) < 0$ implies $\mathbf{x}_i \notin \mathfrak{X}_i$ [i.e., $\mathfrak{s}_i(\mathbf{x}_i) < \mathfrak{s}_i(\mathbf{x}_i^0)$ for every i]

or

(1″) $\mathbf{p}\cdot(\mathbf{x}_i - \mathbf{x}_i^0) \leqq 0$ implies $\mathfrak{s}_i(\mathbf{x}_i) \leqq \mathfrak{s}_i(\mathbf{x}_i^0)$ for every i.

Interpreting $\mathbf{p}$ as a price vector and, defining $a_i = \mathbf{p}\cdot\mathbf{x}_i^0$ $(i = 1, \cdots, m)$, we have the statement:

The necessary and sufficient condition for $\mathfrak{s}^0$ *to be maximal, or for* $\mathbf{z}^0 = \sum_i \mathbf{x}_i^0 + \sum_j \mathbf{y}_j^0$ *to be minimal, is the existence of a price vector* $\mathbf{p} > \mathbf{0}$ *and of a set of numbers* a_i $(i = 1, \cdots, m)$ *such that*

(α) $\mathbf{x}_i$ *being constrained by* $\mathbf{p}\cdot\mathbf{x}_i \leqq a_i$, $\mathfrak{s}_i(\mathbf{x}_i)$ *reaches its maximum at* $\mathbf{x}_i^0$, *for every* i,

(β) $\mathbf{y}_j$ *being constrained by* $\mathbf{y}_j \in \mathfrak{Y}_j$, $\mathbf{p}\cdot\mathbf{y}_j$ *reaches its minimum at* $\mathbf{y}_j^0$ *for every* j.

This is a formalization of well-known rules of behavior for consumption units and production units: each consumption unit, subject to a budgetary constraint, maximizes its satisfaction and each production unit, subject to technological constraints, maximizes its profit.

Given $\mathbf{z}^0$, $\mathfrak{Y}$, and $\mathfrak{s}^0 \in \mathfrak{S}^{\max}$, the direction of $\mathbf{p}$ is not always uniquely determined.[11] It is only constrained to belong to the set of directions normal to supporting planes for $\mathfrak{Z}$ through $\mathbf{z}^0$, which we call briefly the cone of normals. Even if its direction is known, $\mathbf{p}$ is determined only up to a multiplication by a positive scalar. Once $\mathbf{p}$ is known, the set of m numbers (a_i) is determined.

[9] Therefore $\mathbf{x}_i^0 \in \mathfrak{X}_i^{\min}$, $\mathbf{y}_j^0 \in \mathfrak{Y}_j^{\min}$ (this could be seen directly).

[10] The geometric interpretation of this is the following: The necessary and sufficient condition for the existence of a supporting plane through $\mathbf{z}^0$ for $\mathfrak{Z}$ is the existence of a family of parallel supporting planes through the $\mathbf{x}_i^0$ (resp. $\mathbf{y}_j^0$) for the $\mathfrak{X}_i$ (resp. $\mathfrak{Y}_j$). Such is the deeper meaning of the optimum theorem to be enunciated in a moment.

[11] Unless, of course, $\mathfrak{Z}^{\min}$ is a smooth surface having only one normal direction at each point.

Given $\mathbf{z}^0$ and $\mathfrak{Y}$, the different $\mathfrak{s}^0$ belonging to $\mathfrak{S}^{\max}$ determine all possible pairs $\mathbf{p}$, (a_i). To attain an arbitrary maximal $\mathfrak{s}^0$ one can imagine the following procedure: choose (a_i) among its possible values; then find a $\mathbf{p} > \mathbf{0}$ such that, when

(α') *every consumption unit maximizes* $\mathfrak{s}_i(\mathbf{x}_i)$ *subject to* $\mathbf{p}\cdot\mathbf{x}_i \leqq a_i$,

(β') *every production unit minimizes* $\mathbf{p}\cdot\mathbf{y}_j$ *subject to* $\mathbf{y}_j \in \mathfrak{Y}_j$,

the $\mathbf{x}_i^0$ *and* $\mathbf{y}_j^0$ *thus determined satisfy* $\sum_i \mathbf{x}_i^0 + \sum_j \mathbf{y}_j^0 = \mathbf{z}^0$.

A proper choice of (a_i) can lead to any given point $\mathfrak{s}^0 \in \mathfrak{S}^{\max}$ that one wishes to attain.[12]

If the activity of the economic system extends over t successive time intervals of equal length, the subscript h can be made to characterize the time interval as well. Nothing is changed in the preceding analysis; $\mathbf{p}$ now need only be interpreted as a set of actual prices for present and future commodities.

7. HISTORICAL NOTE

Proofs of this basic theorem of the new welfare economics published so far were based on the use of the calculus.[13] They required unnecessary restrictive assumptions on the existence of derivatives, assumptions which cannot be made, for example, in the very simple and realistic case of linear programming in which $\mathfrak{Y}$ is a polyhedral cone [**14**]; moreover, they could at best establish the existence of a *local* maximal. Indeed, they generally limited themselves to the study of first-order conditions.

Pareto himself, who defined [**18, 19**] an optimal $\mathfrak{s}$ as a maximal $\mathfrak{s}$ and conceived the set[14] $\mathfrak{S}^{\max}$ [**20**], did not establish those conditions satisfactorily in spite of lengthy developments [**19**]. The gradual improvements brought by Barone [**3**], Bergson [**4, 5**], Hotelling [**12, 13**], Hicks [**8**], Lange [**15**], Lerner [**16**], Allais [**1, 2**], Samuelson [**21**], and Tintner [**25**] clarified, made more rigorous, and extended the content of his writings.

The long and piecemeal treatment, which consisted of proving that the rates of substitution between any two commodities are independent of the individual, of the industry, etc., failed to comply with the *synthetic* nature of the problem; moreover, it put the emphasis on the

[12] If the conditions of differentiability are fulfilled, (β') coincides with the well-known rule that "every production unit produces its output at the smallest possible total cost and sells it at marginal cost."

If $\mathfrak{Y}_j$ is a cone, the minimum of $\mathbf{p}\cdot\mathbf{y}_j$ is zero and (β') coincides with the rule of perfect competition within the jth industry.

[13] K. J. Arrow's paper quoted in footnote 6 contains a noncalculus proof of the basic theorem. Unfortunately, I had his manuscript in my hands for too short a time to appraise it fully here.

[14] Here, as in similar cases, the author quoted did not use the language we use; however, the translations should not raise any difficulty.

equality of rates of substitution, which disappear in the simplest cases (polyhedral cones), instead of putting it on the necessary and sufficient existence of a price system (real or virtual), which is the actually meaningful operational concept. For these reasons the proofs given independently by O. Lange [**15**] and M. Allais [**1, 2**] were of particular interest: they were essentially synthetic and some of their Lagrange multipliers could be interpreted immediately as prices, which was done forcefully by M. Allais. However, they used an asymmetrical exposition (one individual or one commodity played a particular role) to obtain symmetrical results from symmetrical assumptions. Their Lagrange multipliers were a mathematical trick obscuring the more fundamental facts; they had the weaknesses of calculus proofs already mentioned.

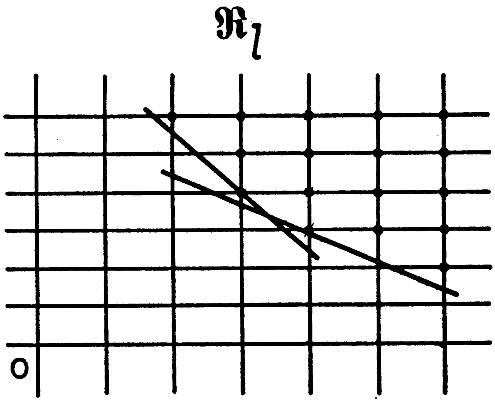

FIGURE 2

8. THE DISCONTINUOUS CASE

If the quantities of some commodities vary discontinuously, we can, in an attempt to preserve certain properties of convexity, define a *quasi-convex* set (Figure 2) as a set which has at least one supporting plane through each minimal point. But the assumption that all the $\mathfrak{X}_i$ and $\mathfrak{Y}_j$ are quasi-convex does not imply that $\mathfrak{Z} = \sum_i \mathfrak{X}_i + \sum_j \mathfrak{Y}_j$ is quasi-convex.[15] In other words, the theorem proved in Section 6 was based on the additivity property of convexity. Quasi-convexity is not additive and the theorem cannot be extended; that is, the existence of a price vector $\mathbf{p} > \mathbf{0}$ used according to the rules (α) and (β) is still sufficient but no longer necessary for $\mathfrak{z}^0$ to be maximal.

9. THE COEFFICIENT OF RESOURCE UTILIZATION

We therefore go back to the convexity case and concern ourselves with the measurement of the dead loss associated with a vector $\mathbf{z}^0 \notin$

[15] It is easy to build a two-dimensional counterexample.

$\mathfrak{Z}^{\min}$ ($\mathfrak{z}^0$) (Figure 3). This loss is depicted entirely by, and only by, the relative positions of $\mathbf{z}^0$ and the set $\mathfrak{Z}^{\min}$. However, if we want, instead of this complex picture, a simple representation by a number, we define the magnitude of the loss as the distance from $\mathbf{z}^0$ to $\mathfrak{Z}^{\min}$, i.e., the minimum of the distance from the fixed point $\mathbf{z}^0$ to the point $\mathbf{z}$ varying in $\mathfrak{Z}^{\min}$. To have a distance with an economic meaning we evaluate the vector $\mathbf{z}^0 - \mathbf{z}$, which represents the nonutilized resources, by the *intrinsic* price vector $\mathbf{p}$ associated with $\mathbf{z}$, whose existence we proved in Section 6. We thus obtain $\mathbf{p}\cdot(\mathbf{z}^0 - \mathbf{z})$. In fact, there can be several $\mathbf{p}$ associated with $\mathbf{z}$; it is easy to see that, whatever the $\mathbf{p}$ chosen in the cone of normals, the result to be obtained below is the *same*. No other price vector can be taken for this evaluation, for (1) it is quite possible that no price vector exists at all in the concrete economic situation observed (if, for example, there is no uniqueness of price for one commodity) and (2) even if there were one, let us say $\mathbf{p}^0$, it would have no intrinsic significance.

Before we engage in a minimization process we must not forget that $\mathbf{p}$ is affected by an arbitrary positive multiplicative scalar whose influence we eliminate by dividing by a price index for which we may take either $\mathbf{p}\cdot\mathbf{z}$ or $\mathbf{p}\cdot\mathbf{z}^0$. It must be pointed out that the result to be obtained is again independent of this choice. Indeed, the use of $\mathbf{p}\cdot\mathbf{z}$ has a very intuitive justification: all the points $\mathbf{z}$ of $\mathfrak{Z}^{\min}$ then have the same "value."

We are thus led to look for

$$\operatorname*{Min}_{\mathbf{z}\in\mathfrak{Z}^{\min}} \frac{\mathbf{p}\cdot(\mathbf{z}^0 - \mathbf{z})}{\mathbf{p}\cdot\mathbf{z}}, \quad \text{i.e., for} \quad \operatorname*{Max}_{\mathbf{z}\in\mathfrak{Z}^{\min}} \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0}.$$

Let $\mathbf{z}^*$ be the vector collinear with $\mathbf{z}^0$ and belonging to $\mathfrak{Z}^{\min}$:

$$\mathbf{z}^* = \rho\mathbf{z}^0, \quad \mathbf{z}^* \in \mathfrak{Z}^{\min},$$

$$\operatorname*{Max}_{\mathbf{z}\in\mathfrak{Z}^{\min}} \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0} = \rho \operatorname*{Max}_{\mathbf{z}\in\mathfrak{Z}^{\min}} \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^*}.$$

But the convexity of $\mathfrak{Z}$ insures that

$$\mathbf{p}\cdot(\mathbf{z}^* - \mathbf{z}) \geqq 0, \quad \text{i.e.,} \quad \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^*} \leqq 1;$$

since this ratio is equal to 1 when $\mathbf{z} = \mathbf{z}^*$, we have

$$\operatorname*{Max}_{\mathbf{z}\in\mathfrak{Z}^{\min}} \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0} = \rho;$$

the maximum is reached at every point $\mathbf{z} \in \mathfrak{Z}^{\min}$ such that a supporting plane through $\mathbf{z}$ contains $\mathbf{z}^*$ (i.e., at every point $\mathbf{z}$ contained in a supporting plane through $\mathbf{z}^*$).

We call ρ defined in the preceding way *the coefficient of resource utilization* of the economic system; it is a function $\rho(\mathfrak{s}^0, \mathbf{z}^0, \mathfrak{Y})$ describing the efficiency of the economy. To be precise, it is the smallest fraction of the actually available physical resources that would permit the achievement of $\mathfrak{s}^0$. $\rho = 1$ if and only if $\mathfrak{s}^0$ is maximal (i.e., $\mathbf{z}^0$ minimal). $\rho < 1$ if and only if $\mathfrak{s}^0$ is attainable but not maximal. ρ decreases if $\mathfrak{s}^0$ decreases or if $\mathfrak{Y}$ increases, for in both cases $\mathfrak{Z}$ increases; it is hardly more difficult to see that ρ decreases if $\mathbf{z}^0$ increases ("decreases" has been used for short to mean "does not increase" and vice versa).

The appellation suggested for ρ has a general content which must be clearly brought out. An economic system has three kinds of *resources*:

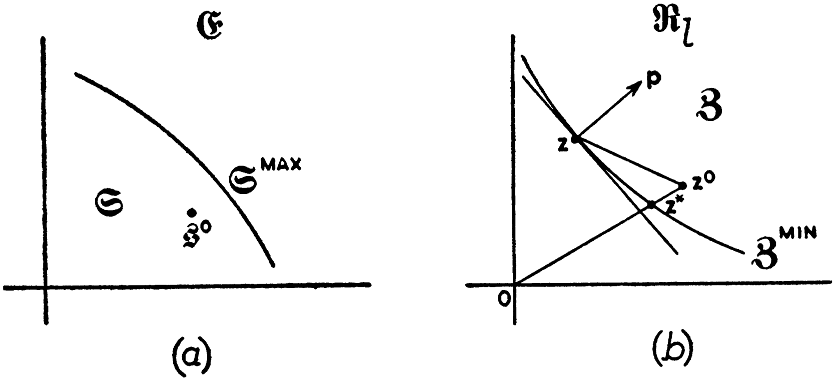

FIGURE 3

(1) its physical resources $\mathbf{z}^0$, (2) its production possibilities $\mathfrak{Y}$, and (3) its economic organization possibilities. If $\mathfrak{s}^0$ is not maximal there is a loss originating from one or several of the following sources:

1. underemployment of physical resources, such as unemployment of labor, idle machinery, lands uncultivated by agreement, etc. This is the most obvious source of loss. In a very narrow sense a coefficient of resource utilization would describe only this phenomenon; this is by no means our only purpose.

2. inefficiency in production; $\mathbf{y}_j \in \mathfrak{Y}_j^{\min}$. This kind of loss is already much less obvious but is not, by its very nature, the main concern of the economist.

3. imperfection of economic organization; if the physical resources are fully utilized, if the production is perfectly efficient, it is still possible that $\mathfrak{s}^0$ is not maximal if the conditions of the basic theorem are not

satisfied. As is well known, such a case arises, for example, with monopolies or indirect taxation or a system of tariffs. This kind of loss is the most subtle (in fact, perhaps hardly conceivable to the layman) and therefore the one for which a numerical evaluation is the most necessary.

The coefficient ρ takes into account the three kinds of loss.

The definition of the coefficient of resource utilization as the ratio of a vector collinear with $\mathbf{z}^0$ to $\mathbf{z}^0$ can be legitimized in cases more general than the convexity case, and defined in still further general cases, but this would lead to a certain amount of undesirable sophistication.

10. DEFINITIONS OF THE ECONOMIC LOSS

In summary, the loss is $\mathbf{z}^0 - \mathbf{z}^* = \mathbf{z}^0(1 - \rho)$; its *value* is $\mathbf{p}^* \cdot \mathbf{z}^0(1 - \rho)$, $\mathbf{p}^*$ being the price vector associated with $\mathbf{z}^*$. However, $\mathbf{p}^*$ has no immediate concrete significance and, if a price vector $\mathbf{p}^0$ exists in the economic situation actually observed, a more interesting evaluation is probably $\mathbf{p}^0 \cdot \mathbf{z}^0(1 - \rho)$. $\mathbf{p}^0$, which was inacceptable in the minimization process leading to $\mathbf{z}^0$, is, of course, acceptable now that an approximate numerical evaluation is sought. Whether one wants the magnitude of the loss due to monopolies (in the absence of other distortions, the total degree of monopoly could be taken as $1 - \rho$), or to a taxation system, or to tariffs,[16] the above expression gives the answer under the form of a certain number of billions of dollars. ρ itself, a percentage describing the degree of efficiency of the economy, can be found more useful in some cases.

Since J. Dupuit [**7**] several definitions of the loss described have been more or less explicitly suggested. A very simple one is the variation of real national income[17] [**9, 22**] [according to our notation,[18] $\mathbf{p} \cdot (\mathbf{x}^2 - \mathbf{x}^1)$]; other definitions were directly based on the various notions of consumers' surplus as presented in their modern forms by J. R. Hicks [**10, 11**]. All of them derived the value of the loss from the comparison of two sets of individual consumptions $(\mathbf{x}_1^1, \cdots, \mathbf{x}_m^1)$ and $(\mathbf{x}_1^2, \cdots, \mathbf{x}_m^2)$, and, if those two sets varied in such a way that $\mathcal{S}^1$ and $\mathcal{S}^2$ did not change, the value of the loss *did vary*. This was inconsistent with the Paretian philosophy which considers two situations which yield the same point $\mathcal{S}$ to be equivalent.[19] Even if this were overcome by the construction of a plausible numerical index of comparison of $\mathcal{S}^1$ and $\mathcal{S}^2$, it would still be

[16] The loss thus measured is the loss for the set of nations trading with each other as a whole.

[17] But it can be defined only if a price vector with some intrinsic meaning exists at all.

[18] The superscripts 1 and 2 will denote the two economic situations compared.

[19] Moreover, the roles played by situations (1) and (2) were generally asymmetrical in such a way that inconsistencies pointed out by T. de Scitovsky [**23, 24**] arose.

unsatisfactory for finding the loss associated with $\mathfrak{s}^1$ to compare it with an $\mathfrak{s}^2$ *arbitrarily* selected in $\mathfrak{S}^{\max}$ instead of comparing it with the set $\mathfrak{S}^{\max}$.

The treatment of this question by M. Allais [**1**] overcomes this difficulty, but its exposition *and its results* rely entirely on the asymmetrical role played by a particular commodity.

11. MINIMAX INTERPRETATION OF ρ

In Section 9 we were led to consider the expression $(\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$, where $\mathbf{z} \in \mathfrak{Z}^{\min}(\mathfrak{s}^0)$ and $\mathbf{p}$ is one of the normals to $\mathfrak{Z}^{\min}$ at $\mathbf{z}$. It was proved that its maximum is reached at $\mathbf{z}^*$ (and possibly at other points) collinear with $\mathbf{z}^0$, and that its value is ρ, the ratio of $\mathbf{z}^*$ to $\mathbf{z}^0$. This is but a part of a more complete theory that we present now.

We still assume that the quantity of every commodity varies continuously, but we drop for a moment the convexity hypothesis and look for

$$\operatorname*{Min}_{\mathbf{z}\in\mathfrak{Z}} \operatorname*{Max}_{\mathbf{p}\in\mathfrak{P}'} \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0},$$

where $\mathfrak{P}'$ is the closed positive orthant, origin excluded. $\mathbf{z}$ being given,

$$\operatorname*{Max}_{\mathbf{p}\in\mathfrak{P}'} \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0} = \operatorname*{Max}_{h} \frac{z_h}{z_h^0},$$

which may be infinite.[20]

If $\mathbf{z} \in \mathfrak{Z}$, $(z_h/z_h^0) \geqq \rho$ for at least one h; otherwise one would have $\mathbf{z} < \mathbf{z}^*$, contradicting the fact that $\mathbf{z}^* \in \mathfrak{Z}^{\min}$. Therefore, $\operatorname{Max}_{\mathbf{p}\in\mathfrak{P}'} (\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0) \geqq \rho$ whatever be $\mathbf{z}$ in $\mathfrak{Z}$; it is equal to ρ if, and only if, $\mathbf{z} = \mathbf{z}^*$ (again an immediate consequence of "$\mathbf{z}^* \in \mathfrak{Z}^{\min}$"). In other words, $\operatorname{Min}_{\mathbf{z}\in\mathfrak{Z}} \operatorname{Max}_{\mathbf{p}\in\mathfrak{P}'}(\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0) = \rho$; it is reached for $\mathbf{z}^*$ and $\mathbf{p}$ arbitrary in $\mathfrak{P}'$. If $\mathbf{p}$ is chosen (say by some central agency) in $\mathfrak{P}'$ so as to make $(\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$ as great as possible, and if $\mathbf{z}$ is chosen in $\mathfrak{Z}$ so as to make this expression as small as possible [this amounts to choosing $\mathbf{y}_j$ in $\mathfrak{Y}_j$ (resp. $\mathbf{x}_i$ in $\mathfrak{X}_i$) so as to make $\mathbf{p}\cdot\mathbf{y}_j$ (resp. $\mathbf{p}\cdot\mathbf{x}_i$) as small as possible for every j (resp. i)], the economic system is led to $\mathbf{z}^*$ and the final value of the expression is ρ. The order in which the operations Max and Min are carried out is of utmost importance.

If the set $\mathfrak{Z}$ is convex (this property of $\mathfrak{Z}$ has been studied in Sections 6 and 9), this order becomes *indifferent*. In effect, let us look for

$$\operatorname*{Max}_{\mathbf{p}\in\mathfrak{P}'} \operatorname*{Min}_{\mathbf{z}\in\mathfrak{Z}} \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0}.$$

[20] Some ratios (z_h/z_h^0) might be of the form $(0/0)$; they would be disregarded in the operation described by the right-hand member.

$\mathbf{p}$ being given, $\mathrm{Min}_{\mathbf{z}\in\mathfrak{Z}}\ (\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$ is reached for a point of $\mathfrak{Z}^{\min}$ (and possibly other points of $\mathfrak{Z}$). We can therefore restrict ourselves to the case where $\mathbf{z} \in \mathfrak{Z}^{\min}$ and $\mathbf{p}$ is a normal to $\mathfrak{Z}^{\min}$ at $\mathbf{z}$. The problem of finding the maximum of $(\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$ under these conditions is precisely the problem we solved in Section 9. The maximum ρ is reached at $\mathbf{z}^*$ (and possibly at other points of $\mathfrak{Z}^{\min}$), the corresponding $\mathbf{p}^*$ being any one of the normals to $\mathfrak{Z}^{\min}$ at $\mathbf{z}^*$.

To sum up,

$$\rho = \underset{\mathbf{z}\in\mathfrak{Z}}{\mathrm{Min}}\ \underset{\mathbf{p}\in\mathfrak{P}'}{\mathrm{Max}}\ \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0} = \underset{\mathbf{p}\in\mathfrak{P}'}{\mathrm{Max}}\ \underset{\mathbf{z}\in\mathfrak{Z}}{\mathrm{Min}}\ \frac{\mathbf{p}\cdot\mathbf{z}}{\mathbf{p}\cdot\mathbf{z}^0},$$

the set of saddle points of the function $(\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$ is the product [**17**, Section 13] of:

the set of $\mathbf{z}$ where $\mathrm{Min}_{\mathbf{z}}\ \mathrm{Max}_{\mathbf{p}}\ (\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$ is reached; it is composed of $\mathbf{z}^*$ only;

the set of $\mathbf{p}$ where $\mathrm{Max}_{\mathbf{p}}\ \mathrm{Min}_{\mathbf{z}}\ (\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$ is reached; it is composed of the normals $\mathbf{p}^*$ to $\mathfrak{Z}^{\min}$ at $\mathbf{z}^*$.[21]

If $\mathfrak{s}^0$ is maximal, the value of the minimax is, of course, 1.

12. ISOMORPHISM WITH THE THEORY OF STATISTICAL DECISION FUNCTIONS

If none of the components of $\mathbf{z}^0$ is null, we can, by an appropriate choice of the units, make them all equal to 1. The expression $(\mathbf{p}\cdot\mathbf{z}/\mathbf{p}\cdot\mathbf{z}^0)$ then takes the form $(\mathbf{p}/\sum_h p_h)\cdot\mathbf{z}$; we put $\bar{\mathbf{p}} = (\mathbf{p}/\sum_h p_h)$, normalizing the price vector in such a way that the sum of its components is 1, and we have, finally, the very simple form $\bar{\mathbf{p}}\cdot\mathbf{z}$, where $\mathbf{z} \in \mathfrak{Z}$, $\bar{\mathbf{p}} \in \bar{\mathfrak{P}}$, the simplex defined by $\sum_h \bar{p}_h = 1$ and $\bar{p}_h \geqq 0$.

We have proved that, $\mathfrak{Z}$ being convex,

$$\rho = \underset{\mathbf{z}\in\mathfrak{Z}}{\mathrm{Min}}\ \underset{\bar{\mathbf{p}}\in\bar{\mathfrak{P}}}{\mathrm{Max}}\ \bar{\mathbf{p}}\cdot\mathbf{z} = \underset{\bar{\mathbf{p}}\in\bar{\mathfrak{P}}}{\mathrm{Max}}\ \underset{\mathbf{z}\in\mathfrak{Z}}{\mathrm{Min}}\ \bar{\mathbf{p}}\cdot\mathbf{z}.$$

The saddle points are $\mathbf{z}^*$, all of whose components are equal to ρ, associated with any normal $\bar{\mathbf{p}}^*$ to $\mathfrak{Z}^{\min}$ at $\mathbf{z}^*$. They appear to be the result of the antagonistic activities of a central agency which chooses $\bar{\mathbf{p}}$ in $\bar{\mathfrak{P}}$ so as to *maximize* $\bar{\mathbf{p}}\cdot\mathbf{z}$ and of production units (resp. consumption units) which choose $\mathbf{y}_j$ (resp. $\mathbf{x}_i$) in $\mathfrak{Y}_j$ (resp. $\mathfrak{X}_i$) so as to *minimize* $\bar{\mathbf{p}}\cdot\mathbf{z}$.[22]

[21] If we were interested only in the fact that the operations Min and Max can be inverted, we could give the very short following proof: choose $\mathbf{z}^*$ and one of the $\mathbf{p}^*$ and show that this is a saddle point [**17**, Section 13]. This is indeed immediate but hardly enlightening.

[22] The structure of the set $\mathfrak{Z}$ makes these antagonistic activities formally different from a zero-sum two-person game in the von Neumann-Morgenstern [**17**, Section 17] sense.

On the other hand, a simple case of the theory of statistical decision functions can be presented in the following way.[23] Let $F(\mathbf{x})$ be the cumulative distribution function of a random variable $\boldsymbol{\xi}$, a vector with possibly a denumerable infinity of components (probability that $\boldsymbol{\xi} < \mathbf{x}$); F is merely known to be an element of a finite set $(F_1, \cdots, F_\iota, \cdots, F_\nu)$. The statistician is faced with the choice of a decision d in a set $\mathfrak{D}$. With every pair F_ι, d is associated a number $r \geqq 0$ called risk, expressing what it costs to use d when F_ι is true.

The expression $r(F_\iota, d)$ can be more conveniently written $r_\iota(d)$; it is thus clear that to each d corresponds a risk vector $\mathbf{r}(d)$ $(r_1, \cdots, r_\iota, \cdots, r_\nu)$ of the space $\mathfrak{R}_\nu$. The image of the set $\mathfrak{D}$ by the function $\mathbf{r}(d)$ is a set $\mathfrak{R}$ of $\mathfrak{R}_\nu$, and the initial problem of choice of d in

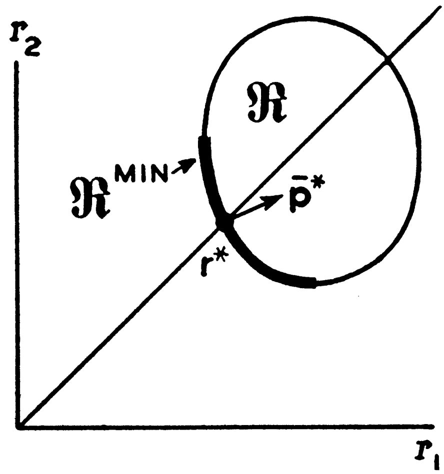

FIGURE 4

$\mathfrak{D}$ can be replaced by the problem of choosing a point $\mathbf{r}$ in $\mathfrak{R}$. In the usual framework of the theory (including the use of randomized decisions; i.e., d_1 and d_2 being two decisions, one can choose d_1 with the probability α and d_2 with the probability $1 - \alpha$), $\mathfrak{R}$ is closed and convex. If $\mathbf{r}^1 \leqslant \mathbf{r}^2$, $\mathbf{r}^1$ is better than $\mathbf{r}^2$ [whatever be the true F_ι, $r(F_\iota, d_1) \leqq r(F_\iota, d_2)$, the strict inequality holding for at least one ι], and the choice of $\mathbf{r}$ is therefore restricted to $\mathfrak{R}^{\min}$. Let us make the further assumption that the straight line whose equations are $r_1 = r_2 = \cdots = r_\nu$ meets

[23] The theory of which this paragraph and the three following ones give a summary is developed in greater detail and generality in the basic work of A. Wald [**26**]. Its geometric interpretation was pointed out by J. Wolfowitz at the Chicago meeting of the Econometric Society in December, 1950, in a paper, "Some Recent Advances in the Theory of Decision Functions," which is unfortunately not available in printed form.

$\mathfrak{R}$ and meets it for the first time (when one is moving away from the origin) at a point $\mathbf{r}^*$ of $\mathfrak{R}^{\min}$ (Figure 4); this assumption is not significant for the theory of statistical decision functions, but the isomorphism can be brought out in this case.

The principle of minimizing the maximum risk amounts to taking $\text{Min}_{\mathbf{r}}\ \text{Max}_{\nu}\ (r_1, \cdots, r_\nu)$; it leads to the selection of $\mathbf{r}^*$. Indeed, if $\bar{\mathbf{p}}$ is a vector whose ν components p_ι satisfy $\bar{p}_\iota \geqq 0$ and $\sum \bar{p}_\iota = 1$, the Min Max operation mentioned is equivalent to $\text{Min}_{\mathbf{r}}\ \text{Max}_{\bar{\mathbf{p}}}\ \bar{\mathbf{p}}\cdot\mathbf{r}$. The formal analogy with our former concepts is obvious: $\mathbf{z}$ and $\mathfrak{Z}$ have been replaced by $\mathbf{r}$ and $\mathfrak{R}$; the normalized price vector $\bar{\mathbf{p}}$, by the vector $\bar{\mathbf{p}}$ whose interpretation will be given in a moment. We proved in Section 11 that the operation $\text{Min}_{\mathbf{r}}\ \text{Max}_{\bar{\mathbf{p}}}\ \bar{\mathbf{p}}\cdot\mathbf{r}$ leads to $\mathbf{r}^*$.

We proved also that the operations Min and Max can be inverted. $\text{Max}_{\bar{\mathbf{p}}}\ \text{Min}_{\mathbf{r}}\ \bar{\mathbf{p}}\cdot\mathbf{r}$ now has the following interpretation: $\bar{\mathbf{p}}$ is a probability vector, $\bar{p}_\iota$ being the a priori probability that F_ι is true. The statistician minimizes the expected risk and $\text{Min}_{\mathbf{r}}\ \bar{\mathbf{p}}\cdot\mathbf{r}$ gives the Bayes solution relative to the a priori distribution $\bar{\mathbf{p}}$. It is a point of $\mathfrak{R}^{\min}$. (Conversely, every point of $\mathfrak{R}^{\min}$ is a Bayes solution for a properly chosen $\bar{\mathbf{p}}$.) Therefore $\mathbf{r}^*$ is the Bayes solution relative to $\bar{\mathbf{p}}^*$ (one of the normals to $\mathfrak{R}^{\min}$ at $\mathbf{r}^*$); $\bar{\mathbf{p}}^*$ is the a priori distribution which gives the greatest value to the minimum expected risk, i.e., the least favorable a priori distribution.

In the same way that prices were historically first considered as primary data and later only as an indirect theoretical construction with optimal properties, the controversial concept of an a priori distribution, at first taken at its face value, is here considered as an indirect construction with intrinsic optimal properties.

The formal analogies between the theories of zero-sum two-person games, statistical decision functions, and resource allocation are valuable since a result obtained in any one of them can have an interesting counterpart in the two others; the differences between their philosophies should, however, by no means be overlooked. In a game we have a clear-cut case of naturally antagonistic interests: one player tries to make his gain as great as possible, the other tries to make his loss as small as possible. In a statistical decision problem, according to A. Wald's words [**26**, Section 1.6], "Whereas the experimenter wishes to minimize the risk $r(F, d)$, we can hardly say that Nature wishes to maximize $r(F, d)$. Nevertheless, since Nature's choice is unknown to the experimenter, it is perhaps not unreasonable for the experimenter to behave as if Nature wanted to maximize the risk. But, even if one is not willing to take this attitude, the theory of games remains of fundamental importance for the problem of statistical decisions, since ... it leads to basic results concerning admissible decision functions and complete classes of decision functions." In the resource allocation problem the central agency determining $\bar{\mathbf{p}}$ is not inert and its behavior can be chosen precisely to conflict fully with the behavior of the various economic units.

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