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THE MEASUREMENT OF PRODUCTIVE EFFICIENCY*

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1. INTRODUCTION

THE problem of measuring the productive efficiency of an industry is important to both the economic theorist and the economic policy maker. If the theoretical arguments as to the relative efficiency of different economic systems are to be subjected to empirical testing, it is essential to be able to make some actual measurements of efficiency. Equally, if economic planning is to concern itself with particular industries, it is important to know how far a given industry can be expected to increase its output by simply increasing its efficiency, without absorbing further resources.

A number of attempts have been made to solve this problem, but, although they usually produced careful measurements of some or all of the inputs and outputs of the industry, they failed to combine these measurements into any satisfactory measure of efficiency. This failure was partly due to a pure neglect of the theoretical side of the problem. Indeed, for a long time it was considered adequate to measure the average productivity of labour, and to use this as a measure of efficiency. This is a patently unsatisfactory measure, as it ignores all inputs save labour, but it was so widely used by economic statisticians that it is now enjoying an extensive popular vogue, which may indeed have unfortunate effects on economic policy. More recently, attempts have been made to construct "indices of efficiency", in which a weighted average of inputs is compared with output. These attempts have naturally run into all the usual index number problems.

It is the purpose of this paper to provide a satisfactory measure of productive efficiency—one which takes account of all inputs and yet avoids index number problems—and to show how it can be computed in practice. In doing so, an estimate of the relevant production function is obtained. The method is illustrated by an application to agricultural production in the United States.

It is hoped that the paper will be of interest to a wide range of economic statisticians, business men and civil servants, many of whom have little knowledge of economic theory or mathematics. For their benefit, the main exposition is in the sort of terms used by elementary economic text-books, and such elementary mathematics as is necessary for discussion of the general case or of computing problems is confined to sections 2.3 and 5. Similarly, although the treatment of the efficient production function is largely inspired by activity analysis,† no reference is made to this in the exposition. The professional economist can easily draw the necessary parallels for himself,

* I should like to express my gratitude to Sir Dennis Robertson, Dr. T. E. Easterfield and Dr. M. R. Fisher for their helpful comments on this paper, and to Mr. P. Fisk and Mr. E. Osborn for their assistance with the calculations. I am also grateful to the Director of the Cambridge University Mathematical Laboratory for permitting the larger-scale calculations to be performed on EDSAC, and to Dr. L. J. Slater for her help in this. I am, however, most deeply indebted to Mr. J. A. C. Brown, who not only offered many helpful comments on the paper, but devoted an immense amount of time and energy to acting as interpreter between EDSAC and myself.

† See, for example, Koopmans (1951).

as indeed, he can note the similarity of the measure of "technical efficiency" and Debreu's "coefficient of resource utilization" (Debreu, 1951).

The measures developed are intended to be quite general, applicable to any productive organization from a workshop to a whole economy. For ease of exposition, they are developed primarily in terms of the efficiency of a firm, but, as will be seen later, the application to industries is in many ways more interesting and, indeed, the illustrative example is of this kind.

2. Statement of the Method

2.1. *Efficiency Measures in a Simple Case*

When one talks about the efficiency of a firm one usually means its success in producing as large as possible an output from a given set of inputs. Provided that all inputs and outputs were correctly measured, this usage would probably be generally accepted. At any rate, the measure of *technical efficiency* defined below conforms to this usage.

Consider, for the sake of simplicity, a firm employing two factors of production to produce a single product, under conditions of constant returns to scale (These restrictions will all be relaxed later.) Suppose the *efficient production function* is known; that is, the output that a perfectly efficient firm could obtain from any given combination of inputs. This concept, too, will be discussed later.

The assumption of constant returns permits all the relevant information to be presented in a simple "isoquant" diagram, of the kind familiar in elementary text-books. In Diagram 1, the point P represents the inputs of the two factors, per unit of output, that the firm is observed to use. The isoquant SS' represents the various combinations of the two factors that a perfectly efficient firm might use to produce unit output.

Now the point Q represents an efficient firm using the two factors in the same ratio as P. It can be seen that it produces the same output as P using only a fraction OQ/OP as much of each factor. It could also be thought of as producing OP/OQ times as much output from the same inputs. It thus seems natural to define OQ/OP as the *technical efficiency* of the firm P.

This ratio has the properties that a measure of efficiency obviously needs. It takes the value unity (or 100 per cent.) for a perfectly efficient firm, and will become indefinitely small if the amounts of input per unit output become indefinitely large. Moreover, so long as SS' has a negative slope, an increase in the input per unit output of one factor will, *ceteris paribus*, imply lower technical efficiency.

However, one also needs a measure of the extent to which a firm uses the various factors of production in the best proportions, in view of their prices. Thus, in Diagram 1, if AA' has a slope equal to the ratio of the prices of the two factors, Q' and not Q is the optimal method of production; for although both points represent 100 per cent. technical efficiency, the costs of

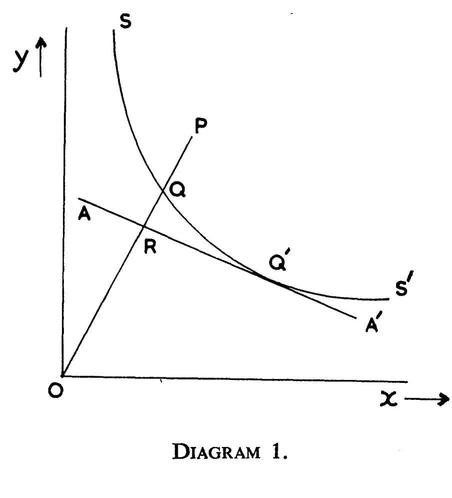


DIAGRAM 1.

production at Q' will only be a fraction OR/OQ of those at Q. It is natural to define this ratio as the *price efficiency* of Q.

Further, if the observed firm were to change the proportions of its inputs until they were the same as those represented by Q', while keeping its technical efficiency constant, its costs would be reduced by a factor OR/OQ, so long as factor prices did not change. It is therefore reasonable to let this ratio measure the price efficiency of the observed firm P, too. This argument is not entirely conclusive, as it is impossible to say what will happen to the technical efficiency of a firm as it changes the proportions of its inputs, but, with this qualification, it seems the best measure available. It also has the desirable property of giving the same price efficiency to firms using the factors in the same proportions.

If the observed firm were perfectly efficient, both technically and in respect of prices, its costs would be a fraction OR/OP of what they in fact are. It is convenient to call this ratio the *overall efficiency* of the firm, and one may note that it is equal to the product of the technical and price efficiencies.

2.2. *The Efficient Production Function in the Simple Case*

These measures of efficiency have been defined on the assumption that the efficient production function is known. In other words, they are methods of comparing the observed performance of a firm with some postulated standard of perfect efficiency, so that each of the measures has, in general, corresponding to each postulated standard, a different value and a different significance. It is therefore necessary to consider the definition of the efficient production function before discussing the significance of the efficiency measures.

Although there are many possibilities, two at once suggest themselves—a theoretical function specified by engineers and an empirical function based on the best results observed in practice. The former would be a very natural concept to choose—after all, should not a postulated standard of perfect efficiency represent the best that is theoretically attainable? Certainly it is the concept used by engineers themselves when they discuss the efficiency of a machine or a process. However, although it is a reasonable and perhaps the best concept for the efficiency of a single production process, there are considerable objections to its application to anything so complex as a typical manufacturing firm, let alone an industry.

In the first place, it is very difficult to specify a theoretical efficient function for a very complex process. Even the best engineer is likely to overlook some problems, and it must be very difficult indeed to estimate, *a priori*, a plant's need of, say, indirect labour. Thus, the more complex the process, the less accurate is the theoretical function likely to be. Also, partly because of this, and partly because the more complex a process, the more scope it allows to human frailty, the theoretical function is likely to be wildly optimistic. If the measures are to be used as some sort of yardstick for judging the success of individual plants, firms, or industries, this is likely to have unfortunate psychological effects; it is far better to compare performances with the best actually achieved than with some unattainable ideal.

Thus, although the theoretical standard is perfectly valid and has its own uses, this paper will be concerned with the observed standard. The next problem is thus to estimate an efficient production function from observations of the inputs and outputs of a number of firms. On the same assumptions as before, each firm can be represented by a point on an isoquant diagram, so that a number of firms will yield a scatter of points like that on Diagram 2. The efficient production function will be represented by an isoquant, and the problem is to estimate such an efficient isoquant from the scatter diagram.

If it can be assumed that the isoquant is convex to the origin and has nowhere a positive slope, then the curve SS' is the most conservative (or pessimistic) estimate of it. That is to say, SS' is the least exacting standard of efficiency that is consistent with the observed points and satisfies these two assumptions.

Of these two assumptions, that of convexity is almost always made in economic theory. It amounts to assuming that if two points are attainable in practice, then so is any point representing a weighted average of them. Since constant returns to scale have already been assumed, this merely requires that the processes represented by the two points can be carried on without interfering with each other. The assumption that the slope of the isoquant is nowhere positive is

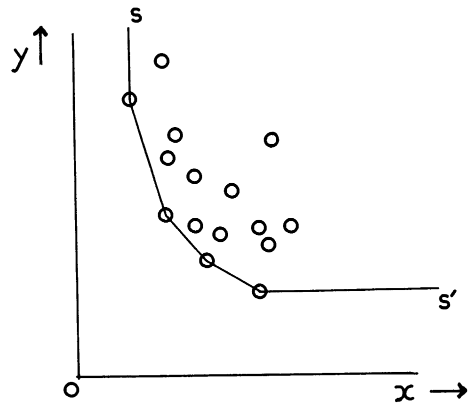


DIAGRAM 2.

not usually made in the text-books, but it ought to be. If it did not hold, increased applications of both factors would result in reduced output!

The curve SS', then, will be taken as the estimate of the efficient isoquant. It will be seen that this method of measuring the technical efficiency of a firm consists in comparing it with a hypothetical firm which uses the factors in the same proportions. This hypothetical firm is constructed as a weighted average of two observed firms, in the sense that each of its inputs and outputs is the same weighted average of those of the observed firms, the weights being chosen so as to give the desired factor proportions.

It is worth emphasizing that it is this that is the essence of the method, and not the representation in an isoquant diagram, which is purely an expository device. In the generalization in the next section to allow many inputs and outputs the isoquant diagram is perforce abandoned, but the basic principle of forming a hypothetical firm as a weighted average of the appropriate number of observed firms remains unchanged.

2.3. *Generalization to the Case of Many Inputs and Outputs*

The curve SS' may be defined geometrically as follows. It is composed of the line-segments joining certain pairs of points, chosen from a set A of points consisting of the observed points plus the two points $(0,\infty)$ and $(\infty, 0)$. (The two points at infinity are added to give the parts of SS' parallel to the axes.) The pairs of points chosen are those for which the line joining them satisfies the two conditions:

(i) that its slope is not positive;
(ii) that no observed point lies between it and the origin.

These two conditions can conveniently be expressed as the single condition that no point of A lies on the same side of the line as the origin.

This is equivalent to the following algebraic definition. Write any point in the form $P_i = (x_{i1}, x_{i2})$, and let λ_{ijk}, μ_{ijk} be the solution of the equations

$$\left.\begin{aligned} \lambda x_{i1} + \mu x_{j1} &= x_{k1} \\ \lambda x_{i2} + \mu x_{j2} &= x_{k2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P_i, P_j and P_k are points in A. Then the line-segment joining P_i and P_j is part of SS' if and only if

$$\lambda_{ijk} + \mu_{ijk} \geqslant 1 \text{ for all } P_k \text{ in } A \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is perhaps worth while to elaborate a little. Any point on the line P_iP_j can be written as $\lambda x_{i1} + \mu x_{j1}, \lambda x_{i2} + \mu x_{j2})$ where $\lambda + \mu = 1$; and for points between P_i and P_j, $\lambda, \mu \geqslant 0$. Hence, if P_iP_j lies between P_k and the origin, $\lambda_{ijk} + \mu_{ijk} > 1$; and if OP_k cuts P_iP_j internally, $\lambda_{ijk}, \mu_{ijk} \geqslant 0$.

Thus the equations (1) may be used to determine the technical efficiency of any point P_k. It is first necessary to find which segment of SS' is intersected by OP_k—that is, to find the segment P_iP_j of SS' for which λ_{ijk}, $\mu_{ijk} \geqslant 0$. Then the technical efficiency of

$$P_k = \frac{1}{\lambda_{ijk} + \mu_{ijk}}.$$

An equivalent but more elegant definition (and one which is useful in actual computations) is that the technical efficiency of P_k is the maximum of

$$\frac{1}{\lambda_{ijk} + \mu_{ijk}}$$

for all segments P_iP_j of SS'. The convexity of SS' ensures that this expression reaches its maximum where λ, $\mu \geqslant 0$.

The generalization to permit n inputs, while retaining the assumptions of a single product and constant returns, is straightforward. Each observed firm is now represented by a point in n-dimensional space, written typically as a column vector x_i. The set A is constructed by adding to the observed points the n points

$$(\infty, 0, \quad . \; . \; . \quad , 0)\ (0, \infty, \quad . \; . \; . \quad , 0) \quad . \; . \; . \quad , (0, 0, \quad . \; . \; . \quad , \infty).$$

Just as in two dimensions pairs of points in A defined lines and line-segments, so now sets of n points in A define hyperplanes and "facets". Here "facet" is used to describe that part of a hyperplane whose points can be expressed as weighted averages, with non-negative weights, of the n defining points. The efficient isoquant is now a surface S in n dimensions, composed of such facets.

To the equations (1) there corresponds the matrix equation

$$[x_i, x_{i+1}, \quad . \; . \; . \quad , x_{i+n-1}]\lambda = x_k \qquad . \quad . \quad . \quad . \quad (3)$$

whose solution is the column vector λ, and the facet defined by the n points P_i, P_{i+1}, . . . , P_{i+n-1} is part of S if and only if

$$\lambda' u \geqslant 1 \text{ for all } P_k \text{ in } A \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(Here u is a column vector, all whose elements are unity.) As before, the technical efficiency of P_k may be defined either as $1/(\lambda' u)$ for that facet intersected by OP_k, or as the maximum of $1/(\lambda' u)$ for all facets of S.

The relaxation of the single-product assumption is slightly more complex. Since output is no longer a scalar quantity, it is no longer possible to reduce the observations to points on our isoquant diagram by dividing inputs by output. Instead, each observed firm has a vector X_i of outputs and a vector x_i of inputs (note the change in notation) and must be represented by a point in $n + m$-dimensional space. The efficient surface S is now composed of facets determined by sets of $n + m$ points, chosen from a set A including, besides the observed points and the points at infinity, the origin. As constant returns are still assumed, the definition of a facet is changed to permit the origin a negative weight. It follows that the origin appears in every efficient facet.

The counterpart of equations (1) and (3) is now the matrix equations

$$\left.\begin{aligned} [X_i, X_{i+1}, \quad . \; . \; . \quad , X_{i+m+n-2}, 0]\lambda &= (\lambda' u)\, X_k \\ [x_i, x_{i+1}, \quad . \; . \; . \quad , x_{i+m+n-2}, 0]\lambda &= x_k \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (5)$$

equivalent to $n + m$ linear equations. The counterpart of conditions (2) and (4) is that

$$\lambda' u \geqslant 1 \text{ for all } P_k \text{ in } A \qquad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the efficiency of P_k is defined in terms of $\lambda' u$ precisely as before.

It may not be immediately obvious that this criterion is simply a generalization of the previous ones. It can, however, easily be shown that in the case where $m = 1$ (so that X_i is a scalar) the two procedures are equivalent. Let $\lambda = (\lambda_1, \lambda_2, \quad . \; . \; . \quad , \lambda_{n+1})$ be the solution of (5) and define $\mu = (\mu_1, \mu_2, \quad . \; . \; . \quad , \mu_n)$ by

$$X_{i+j-1}\,\lambda_j = X_k\,\mu_j \qquad j = 1, 2, \quad . \; . \; . \quad , n.$$

Then the equations (5) may be written

$$\left.\begin{aligned} \lambda' u &= \mu' u \\ \left[\frac{1}{X_i}\,x_i, \frac{1}{X_{i+1}}\,x_{i+1}, \quad . \; . \; . \quad , \frac{1}{X_{i+n-1}}\,x_{i+n-1}\right]\mu &= \frac{1}{X_k}\,x_k \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

It will be seen that (7) consists of (3) re-written in the present notation, plus the statement that $\lambda'\,u = \mu'\,u$, i.e. that the two criteria are equivalent.

The interpretation of the technical efficiency of P_k defined in this way is precisely the same as before. If an efficient firm were to produce outputs X_k, it could do so from inputs $1/(\lambda' u)\;x_k$; or, using inputs x_k, it could produce outputs $\lambda'\,u\;X_k$. Thus, except for the need to change the geometrical picture, this generalization, too, is quite straightforward.

Where there is more than one output, it would be natural to extend the measure of price efficiency to cover the extent to which the firm's choice of outputs is adapted to the prices of these outputs. This has not been done, largely because, as will be argued later, price efficiency is in any case a measure with rather limited usefulness, so that further elaboration would be of rather academic interest. A measure of price efficiency defined as before remains a valid measure of the firm's adaptation to factor prices.

2.4. *Increasing and Diminishing Returns to Scale*

It is, unfortunately, more difficult to relax the assumption of constant returns to scale. It is quite simple to allow for diseconomies of scale. All that is necessary is to alter the definition of a facet in the last section so as not to permit a negative weight to the origin, and equations (3) so as to permit any set of $n + m$ points from A (not necessarily including the origin) to determine a facet. But there is no entirely satisfactory way of allowing for economies of scale.

The difference between the two cases is illustrated in Diagram 3 for the simple case of one input and one output. Whereas under diseconomies of scale, the efficient production function S is convex, so that the average of two points on S is attainable (indeed, probably inefficient), in the case of economies of scale this is not so. This is a serious matter, as the whole method is based on the assumption of convexity. It means that the method will give an optimistic instead of a conservative estimate of S—some straight line like OR—and, of course a pessimistic estimate of the efficiency of any point. Indeed, none of the justifications of the method advanced in previous

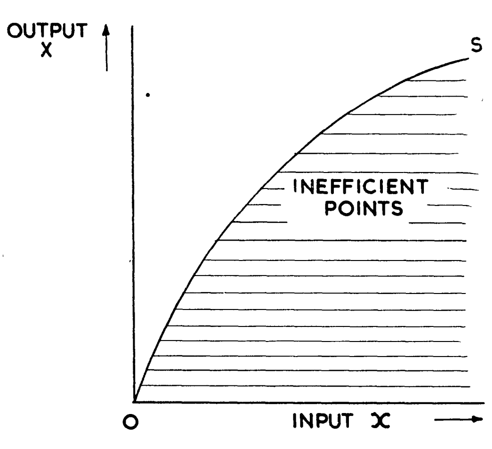


DIAGRAM 3*a*.—Diseconomies of scale.

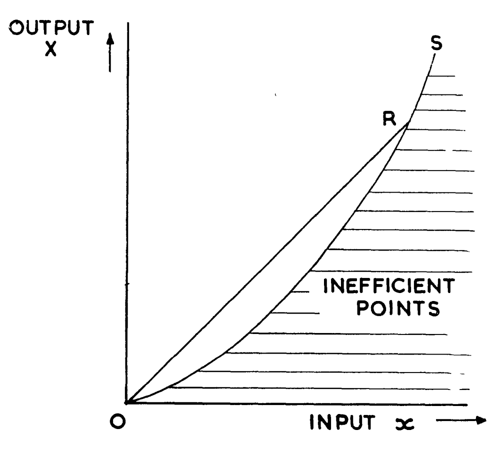


DIAGRAM 3*b*.—Economies of scale.

sections will hold, and furthermore, the estimated S will always show apparently constant returns.

The only practical method of dealing with this problem seems to be that of dividing the observations into groups of roughly equal output, and applying the method to each of these groups separately, the assumption being that returns are constant within a group to a sufficient degree of approximation. This will yield a different efficient isoquant for each level of output, and comparison of these isoquants will show the extent and nature of the economies of scale.

This device should work satisfactorily if there are a sufficient number of observations, but it clearly needs many more observations than the original method. Fortunately, where economies of scale are likely to be important, as in measuring the efficiency of firms, or plants, there are likely to be relatively many observations; while, in those cases where there are likely to be only a few observations, as where the efficiency of an industry is measured, economies of scale are likely to be unimportant. It is worth noting that the relevant economies of scale are those where a larger industry permits greater specialization by firms; for those "economies of large-scale industry" that take the form of cheaper inputs are irrelevant, while it is unlikely that the method will be applied to industries that are too small to permit firms of optimum size.

Of course, where economies or diseconomies of scale are present there are two measures of technical efficiency. The given output would have been efficiently produced with only a fraction e_1 of the given inputs, and $1/e_2$ times as great an output could have been produced from the given inputs, where e_1 and e_2 are not in general equal except under constant returns to scale. The ambiguity is, of course, systematic, and the choice of measure must depend on whether it is desired to produce the given output from minimal inputs, or maximum output from the given inputs.

If in practice there are several inputs but only a single output, it will be sensible to choose e_1 as the measure of efficiency, as this involves measuring the scale of production in terms of a single variable, output. Otherwise, the scale of production is a vector and not a scalar quantity. The practice of using a single input as a measure of scale—as labour in manufacturing, or land in agriculture—is clearly indefensible, and leads to obvious anomalies.

2.5. *Quasi-factors*

It is well known that production depends on certain factors besides those generally regarded as factors of production—air and water, for instance, and climate and location. The treatment of such *quasi-factors* must, of course, be determined *ad hoc*, but certain general observations may be made. Air and water, for example, are really perfectly ordinary factors of production, save that their price is (or may be) zero. They should therefore be treated accordingly, and the method has the advantage that any deviation of the price from zero can readily be dealt with.

Other quasi-factors, like the thickness of seam in coal-mining, are more difficult to deal with in that, although they undoubtedly influence the output obtained from a given set of inputs, diminishing returns to the other factors will not set in if the input of the quasi-factor is held constant. The simplest and most obvious solution to this problem is analogous to that of economies of scale, that is, to divide the observations into groups homogeneous (to a desired degree of approximation) in the quasi-factor. Alternatively, if the quasi-factor lends itself to representation by a continuous variable a method analogous to that for diseconomies of scale can be used.

3. INTERPRETATION

3.1. *The Technical Efficiency of a Firm*

One of the most important features of the method outlined above is the distinction between price and technical efficiency. The former measures a firm's success in choosing an optimal set of inputs, the latter its success in producing maximum output* from a given set of inputs. This distinction is quite a natural one, but it has also the merit that most of the difficulties are associated with price efficiency, leaving technical efficiency as a relatively uncomplicated measure. For example, even a firm's operating above or below its optimum output will affect only its price efficiency. Even so, there are a number of qualifications that must be made in interpreting technical efficiency.

* It will be convenient in this discussion to talk as though there were a single product. It is easy to see how the arguments should be extended to cover several.

In the first place, the definition chosen of the efficient production function means that a firm's technical efficiency is relative to the set of firms from which the function is estimated. If additional firms are introduced into the analysis, they may reduce, but cannot increase, the technical efficiency of a given firm. This, of course, is perfectly natural—a firm may be highly efficient by British standards, but not by world standards.

A more important qualification arises in respect of the measurement of inputs. It is not proposed to discuss here the practical difficulties of measuring particular inputs, or the old riddles about the measurement of capital and the length of the working week. These problems remain, and are often formidable, but there is also a problem of more general interest. No one would dispute that the output produced from *given inputs* is a genuine measure of efficiency, but there is room for doubt whether, in a particular application, the inputs of a given firm are really "the same" as those represented by the corresponding point on the efficient isoquant. Of course, if, as has been implicitly assumed, there are a small number of homogeneous factors of production, each of which can be measured in physical units, there is no problem. It is perhaps trivial to point out that the omission of one of these factors would give a firm that used relatively much of this factor a relatively high technical efficiency. But so would quality differences in a factor favour a firm using a relatively high grade, and, in practice, factors of production are notoriously heterogeneous.

It is important to note that mere heterogeneity of factors will not matter, so long as it is spread evenly over firms. It is when there are differences between firms in the average quality (or, more strictly, in the distribution of qualities) of a factor, that a firm's technical efficiency will reflect the quality of its inputs as well as the efficiency of its management. If these differences in quality are physically measurable, it may be possible to reduce this effect by defining a larger number of relatively homogeneous factors of production, but in practice it is never likely to be possible completely to eliminate it.*

Thus, the technical efficiency of a firm must always, to some extent, reflect the quality of its inputs; it is impossible to measure the efficiency of its management entirely separately from this factor. This is, indeed, as it should be, for it is never possible to decide precisely how far the fertility of a particular farmer's land is due to nature and how far to good husbandry, how far the laziness and intractability of a particular firm's labour force is ingrained and how far the product of bad management.

Technical efficiency, then, is defined in relation to a given set of firms, in respect of a given set of factors measured in a specific way, and any change in these specifications will affect the measure. This is inevitable in any such measure. But with these qualifications it functions in a natural and satisfactory way as a measure of efficiency.

3.2. *The Price Efficiency of a Firm*

The price efficiency of a firm also depends on the measurement of inputs, but in a rather complex way, so that such problems are best discussed *ad hoc*. It depends, too, on the set of firms in the analysis, and is indeed much more sensitive to the introduction of new firms than is technical efficiency. It can be seen from Diagram 2 that OR/OQ, the price efficiency of P, depends on the slope of AA', the slope of SS' at Q, and its curvature between Q and Q'. The introduction of fresh observations is likely, if it affects this part of SS' at all, to affect both the slope and the curvature, in either direction and perhaps quite substantially. This would have a large effect on the price efficiency of P, as indeed would a change in the slope of AA'. Thus price efficiency is very sensitive to the introduction of new observations and to errors in estimating factor prices,

* Capital presents a particularly difficult problem. The relative productiveness of two different sets of equipment may vary considerably according to the amounts of other inputs. That is to say, each must be represented by a production function of a different shape. In this case, it would be impossible to measure capital by any scalar quantity, such as a money value. It is only when the functions are of roughly the same shape, so that relative productiveness does not vary much with changes in other inputs, that measurement in money terms will be satisfactory.

The treatment of capital must, then, be to define a number of different sorts of capital, each homogeneous in this sense, and to measure each in either physical units or money terms (say, replacement cost). It is to be hoped that in practice a small number of sorts of capital will suffice!

so that it is likely to be rather unstable. Of course, in a particular case where many observations and accurate price information were available, it might be quite reliable.

The use of the straight line AA' (see Diagram 2) in defining price efficiency, implicitly assumes a perfectly elastic supply of each factor. In fact, the elasticities of supply will usually be positive, so that changing the proportions of the inputs will change their price ratios, those of which relatively more is used becoming relatively dearer. This means that OR/OQ will tend to underestimate the true price efficiency of the firm.

However, over and above these problems there remains the question of whether a high price efficiency is necessarily desirable. It may be that a firm's best policy is to operate for a time well above optimum output (in a period of expansion, say, or one of temporarily high demand) even though this gives it a low price efficiency.* Similarly, a firm whose inputs are adjusted to past or expected future prices may not be inefficient. In short, its price efficiency measures the extent of a firm's adaptation to a particular set of prices, and will therefore provide a good measure of its efficiency in adapting to factor prices only in a completely static situation.

Thus, price efficiency is a measure that is both unstable and dubious of interpretation; its virtue lies in leaving technical efficiency free of these faults, rather than in any intrinsic usefulness. From the point of view of the planner, the technical efficiency of a firm or plant indicates the undisputed gain that can be achieved by simply "gingering up" the management, while its price efficiency indicates the gain that, on certain assumptions about the future price structure, can be obtained by varying the input ratios.

3.3. *Price and Technical Efficiency of an Industry*

For the purpose of this exposition, an industry may be regarded as any group of firms making the same product. It might be thought, then, that the technical efficiency of an industry with respect to a given efficient isoquant would be simply a weighted average† of the technical efficiencies with respect to the same isoquant of its constituent firms. This is basically true, but it needs to be qualified, for in so far as its constituent firms use their inputs in different proportions, this dispersion will reduce the technical efficiency of the industry. This can easily be seen from the fact that the average of two points on SS' will in general lie beyond SS'—that is, that two firms which, taken individually, are perfectly efficient (technically, but not, of course, price-wise) are not perfectly technically efficient when taken together. By the same token, this dispersion will mean that the price efficiency of the industry will be greater than the weighted average of the price efficiencies of the constituent firms. (The over-all efficiencies must, of course, be the same.) Thus horizontal aggregation leads to the transfer of some elements of inefficiency from price to technical efficiency. It can easily be seen that the same is true of vertical aggregation.

Thus, with respect to a given efficient isoquant, the technical efficiency of an industry will tend to be somewhat less than the weighted average of the technical efficiencies of its constituent firms. But in an application of the method to industries, it is to be expected that the efficient isoquant will itself be estimated from observations of industries. What will be the effect of aggregation on the efficient isoquant? The answer is simply that it will lead to a less optimistic estimate. Except in very special circumstances, the combination of two or more efficient points will shift SS' away from the origin; so will, *a fortiori*, the combination of efficient and inefficient points. Conversely, of course, disaggregation will push the efficient isoquant closer to the origin.

With these qualifications, the method can be applied to industries in the same way as to firms. The set of observations may be either international, say the steel industries of a number of different countries, or inter-regional, say the agricultural industries of each of the 48 American states (the illustrative example given below). The latter is easier, in that many of the problems of comparability of data are avoided, but the former is in many ways more interesting.

It may be worth noting that whereas in the case of firms, if economies or diseconomies of scale are present, it will usually be desirable to allow for them in the ways mentioned in section 2.4, where industries are concerned this will not be so. The extent to which its firms are of

* A period of temporarily *low* demand might lead to short-time working. If labour input were measured in man-hours, as is conceptually correct, this too would affect price efficiency, but if in number of men employed, it would affect the firm's technical efficiency.

† Weighted by output.

optimum size may legitimately be considered part of an industry's efficiency. Of course, if an industry were too small to permit its firms to achieve optimum size or the optimum degree of specialization, it might be desirable to allow for this.

3.4. *The Structural Efficiency of an Industry*

In principle, an industry analysis can be carried out quite satisfactorily as outlined above, but in practice the scarcity of data is a serious limitation. It is notoriously difficult to obtain comparable input measurements in an international analysis, while the number of cases in which an inter-regional analysis will provide as many observations as United States agriculture is small. (There are few countries with so many regions as the United States, and few industries so ubiquitous as agriculture.) In any case, international comparisons are likely, where they are possible, to be the more interesting, and here it is likely to be difficult to obtain sufficient observations (on a comparable basis) to provide a satisfactory estimate of the efficient isoquant.

There is, however, a very satisfactory way of getting round this problem: that is, by comparing an industry's performance with the efficient production function derived from its own constituent firms. The "technical efficiency" of an industry, measured in this way, will be called its *structural efficiency*, and is a very interesting concept. It measures the extent to which an industry keeps up with the performance of its own best firms. It is a measure of what it is natural to call the structural efficiency of an industry—of the extent to which its firms are of optimum size, to which its high-cost firms are squeezed out or reformed, to which production is optimally allocated between firms in the short run.

It is open to the criticism that it does not reflect the extent to which the best practice in an industry compares with the best practice elsewhere. Thus an industry whose firms were uniformly inefficient would have a higher structural efficiency than one where there were both efficient and inefficient firms. This means that, in an international analysis, a comparison of structural efficiencies should be supplemented by a comparison of efficient isoquants, as described in section 3.5 below. But the introduction of the concept of structural efficiency confers two advantages. First, it permits a comparison of some of the most important aspects of efficiency to be made, even where the incomparability of input measurements precludes a comparison of efficient isoquants. Secondly, where some sort of comparison can be made of efficient isoquants, it separates this (probably rather shaky) comparison from the more reliable one of structural efficiencies.

However, perhaps the greatest importance of the concept is due to the fact that it is possible to compare the structural efficiencies of industries producing different commodities—where, of course, any comparison of efficient isoquants is out of the question. It is thus a measure that could well be used in empirical work on the comparative efficiency of economic systems. It may not measure *precisely* what the economic theorist wishes to measure, but it does measure a surprisingly large proportion of the relevant factors, and it may well be the best measure available in practice. It is hoped to develop this argument further elsewhere.

3.5. *The Efficient Production Function*

The method provides, as a sort of by-product, an estimate of the efficient production function which can be put to various uses, some of which have already been indicated. This section will be devoted to seeing what those uses are and how well the method of estimation is suited to them. It will not be surprising if the method of estimation is not the best for any particular use, for it was chosen simply as providing the best measure of technical efficiency.

One virtue of the estimate is that it involves no assumptions about the shape of the isoquant, other than negative slope and convexity. This is a considerable advantage for, as in so many econometric problems, the data are likely to be such that quite a variety of specific functions might yield a "good fit". (Indeed, in the absence of *a priori* reasoning about the shape of the function and the empirical testing of a wide variety of possible functions, little significance can usually be attached to a "good fit" in an econometric study.) Thus the present estimates can, by virtue of their freedom from special assumptions, cast useful light of the shape of efficient isoquants in practice.

An obvious use of an efficient production function is to provide estimates of the marginal rates of substitution of pairs of factors at various points. These estimates, like the related measure of price efficiency, are likely to be somewhat unstable, and it might be argued that some estimate based (by ordinary regression methods) on all the observations might be preferable. It would certainly be more stable, but as it would be a marginal rate of substitution in average rather than in best practice, its preferability would depend on which of these concepts it was desired to measure.

There remain a group of uses that are formally similar—they all involve comparing two or more efficient isoquants derived in different circumstances. As has been said, in an international comparison it is desirable, if possible, to compare the best practice of the various countries as well as their structural efficiencies. This is done by comparing their efficient isoquants. Again, investigating economies of scale is done by deriving an efficient isoquant for each size-group of arms and comparing them. Finally, to investigate technological progress, efficient isoquants are derived for an industry at different points in time and then compared.

In any of these cases the isoquants may cross, possibly at more than one point. In the first two cases this is obvious, but in the third case it requires some comment, as it would seem to imply forgetfulness on the part of our technologists! In fact, it would reflect a decline in the quality of some input—for example, land in the American Dust Bowl, or labour in Britain. Whether they cross or not the isoquants may well be of very different shapes, and thus contain a great deal of information that would be lost in any point-comparison; indeed, the latter might be wildly misleading.

3.6. *The Method of Estimation from a Statistical Viewpoint*

As the efficient production function is defined in section 2.2, there can be no doubt that the method chosen is the appropriate way of estimating it. It does give the most conservative function consistent with the observations and with the assumptions of convexity and non-positive slope. However, if one looks at it from a statistical point of view, one may wish to reformulate the problem as follows. There exists some efficient function, from which all the observed points deviate randomly but in the same direction. What, then, is the best estimate of the function?

It might seem at first sight as though the method chosen involved a great waste of information—only a handful of "efficient" points contribute directly to the estimate, the rest being "ignored". However, in the problem of estimating the extremes of the rectangular distribution, the relevant information is all contained in the extreme observations, and this seems to be an analogous case. On the other hand, the estimate chosen will almost certainly have a pessimistic bias (that is, a bias away from the origin) but in the absence of *a priori* knowledge of the distribution of the deviations, it is impossible to remove this bias.

Similarly, errors of observation will introduce an optimistic bias, which can only be eliminated if the distributions of both errors and efficiencies are known. This is an interesting problem for any theoretical statistician but for practical purposes the important fact is that if the errors are small compared with the variation in efficiencies, this bias will be negligible. In fact, the problem is more important in the illustrative example than it is ever likely to be in an industrial application. The variation in harvests in agriculture is much larger than genuine errors of observation are likely to be, although it is not strictly variation from year to year in harvests that matters, but differential variation. That is to say, it would not matter that all harvests were bad, but it would matter if some States suffered more than others.

4. Previous Methods

4.1. *The Average Productivity of Labour*

Of the efficiency measures previously used, perhaps the most popular, and certainly the least satisfactory, was the average productivity of labour.* Ignoring as it does all inputs save labour, this measure is so obviously unsatisfactory that one would not waste space discussing it, were it not for the danger that its popularity with the general public may do the economy serious harm through over-capitalization. Very often, the easiest way to increase the average productivity of labour in an industry is to use more machinery, and this, pushed beyond a certain point, will lower the price efficiency of the industry. Indeed, it is quite possible for a country to increase "productivity" in every industry and yet to lower its standard of living: the reduction in the

* See, for example, Rostas (1948).

numbers employed in the consumer goods industries outweighs the increase in their average productivity.

Economists are well aware of this possibility, and in using the concept must have at least implicit reservations on this point. However, the concept has been widely adopted by the public, and one finds it in frequent use amongst politicians, trades unionists, journalists and the like. It would be surprising if all these people made the appropriate qualifications when using the concept, and indeed it seems that there is a large body of public opinion that believes any increase in "productivity" to be desirable. It may be argued that this does not matter, as the price mechanism would prevent any over-capitalization; but public opinion has nowadays little respect for the price mechanism, and might easily urge that it be over-ridden in the interests of "increased productivity". It is perhaps not too much to say that the eradication of "productivity" as a criterion of efficiency could make a considerable difference to our future standard of living.

4.2. *Indices of Efficiency*

The unsatisfactory nature of the "productivity" criterion led naturally to attempts to produce measures of efficiency that took account of all factors of production by adding up (in some sense) a firm's inputs of the different factors. They sought to represent the vector of inputs by a scalar—to add up land, labour, materials, capital and so on into a single quantity. They were naturally doomed to failure—one simply cannot add men and acres, pounds and horsepower.

To be more precise, one cannot add them without (explicitly or implicitly) pricing them, and the choice of a set of prices introduces an arbitrary element into the measure. The indices all included a weighted average of inputs, and whatever it is intended to represent, such a weighted average is always equivalent to a valuation of the inputs at prices proportional to the weights. Thus, any attempt to compare efficiency by such an index may be regarded as making a cost comparison, with a set of factor prices proportional to the weights in the index. The difficulty is simply that of choosing a suitable set of prices as a basis for the comparison. Of course, if all the firms in the analysis face the same set of prices, it is natural to choose that set; but then the "index" boils down to a simple cost comparison. Otherwise, the choice of a set of prices must be arbitrary, and leave at least one of the firms facing prices different from those chosen. In this case, the index will reflect not only the technical efficiency of the firm but also the extent of its adaptation to a set of factor prices different from those facing it.

4.3. *Cost Comparisons*

The most obvious measure of a firm's efficiency is its costs. Comparisons of costs must clearly be limited to situations where all the firms compared face the same factor prices, but in such cases they constitute a much better criterion than "productivity" and are equivalent to the best "efficiency index". That they have drawbacks even in such cases can be seen from the fact that a firm's costs are proportional to its over-all efficiency, so that cost comparisons are open to the same objections as price efficiency—principally, objections arising from the possible divergence of past or future factor price ratios from their present values. It is the virtue of the present method that it separates price from technical efficiency, and leaves the latter free from these criticisms. It is perhaps worth noting that, in analysing any particular group of firms, the lowest cost firm must be technically perfectly efficient, but so, in general, will a number of other firms.

5. THE COMPUTING PROBLEM

5.1. *The Methods Used*

This section will discuss the computing problem actually met in the illustrative example, that is, where there are n inputs, one output and constant returns to scale.* The problem is essentially very simple, as a glance at section 2.3 will show. (4) is a necessary and sufficient condition for the n vectors on the left-hand side of (3) to define an efficient facet. An obvious solution is thus to solve (3) for all possible combinations of n points in A, and to determine in each case whether (4) is satisfied. If there are m observed points, this involves $N = {}_{n+m}C_n$ matrix inversions and

* The methods discussed below can easily be adapted to cases of many outputs or decreasing returns to scale, so that computing problems introduce no restrictions additional to those discussed in section 2.4.

an equal number of matrix multiplications. Thus number would usually be prohibitively large for desk computers, but not for EDSAC for the moderate values of n and m met in the illustrative example. Here there were 48 observed points, but a large proportion of these could usually be eliminated from the calculation by inspection as clearly inefficient. Thus a typical case might give $n = 4$, $m = 12$ so that $N = 1{,}820$—a calculation that would take EDSAC about two hours. However, quite small increases in n and m could make the task large even for EDSAC. If $n = 5$, $m = 20$, $N = 55{,}130$ and the computation would take about 60 hours.* Thus the obvious method is practicable only for a very limited range of values of n and m.

There are also difficulties associated with the points at infinity in A. The most attractive method of dealing with them is simply to treat them on a par with the other vectors, substituting a suitably large number for the infinite element. Unfortunately, the capacity of the existing matrix inversion sub-routine for EDSAC is not large enough to permit an adequate approximation to infinity, so that this method could not be used. The alternative is to obtain inverses for matrices with infinite elements by inverting lower order matrices and bordering them appropriately. This may well be the best method in the long run, but it was felt that the complexity of the EDSAC programme involved would be greater than was justified by the present exercise.

Accordingly, the following method of avoiding the difficulty was adopted. Those facets defined by n observed points were christened "interesting" and those involving points at infinity "uninteresting", and the two sets of facets were computed separately. The interesting facets were computed by a simple adaptation of the above method. The exclusion of the points at infinity has two advantages. It avoids the problem mentioned above, and somewhat reduces N. Thus ${}_{12}C_4 = 495$ representing only half-an-hour's work, and ${}_{20}C_5 = 18{,}504$, representing about 20 hours' work. It also has a disadvantage—a separate criterion for non-positive slope must be introduced. Write B for the $n \times n$ matrix corresponding to a facet and C for the $n \times m$ matrix whose columns are the observed vectors. Then B corresponds to an efficient facet if and only if the column sums of B^{-1} are $\geqslant 0$ and those of $B^{-1}C \geqslant 1$. This procedure, which can easily be programmed for EDSAC, was used to find the interesting facets.

If n is small enough, the uninteresting facets can be filled in by elementary geometry, but as n increases the intuition sooner or later fails (in the author's case, at $n = 4$). However, it is possible to "build up" the uninteresting part of the efficient surface in n dimensions from a knowledge of the efficient surfaces in $n-1$ dimensions. An efficient surface in $n-1$ dimensions can be regarded mathematically as the efficient surface obtained when the observed points are projected orthogonally on to one of the co-ordinate hyper-planes, or from an economic point of view as that obtained when one of the inputs is ignored. It is clear that there is a one-to-one correspondence between the efficient facets in n dimensions that are parallel to a particular co-ordinate axis and the efficient facets in the hyper-plane orthogonal to that axis. Thus, if the n efficient surfaces in $n-1$ dimensions are known, it is a simple matter to fill in the uninteresting facets in n dimensions.

It might seem that the calculation of efficient surfaces in lower-dimensional space is a considerable and unnecessary addition to the computing burden, but in fact it is not large enough to offset the reduction produced by omitting the points at infinity, and, in addition, is helpful in interpreting the analyses (see section 6.3).

5.2. *More Sophisticated Methods*

Although these methods proved adequate to this particular problem, it is clear that problems could easily arise for which the simple testing of all possible hyper-planes would be prohibitive even on EDSAC. Some more selective procedure is clearly necessary. One method is to start with an efficient facet (there are always some that can be written down from inspection) and then to determine all the neighbouring efficient facets, and so on, systematically.

It can be shown that, given an efficient facet passing through $x_1, x_2 \; . \; . \; . \; x_n$, its neighbouring efficient facet on the side away from x_n passes through $x_1, x_2 \; . \; . \; . \; x_{n-1}, x_k$ where x_k is that one of the remaining observed points for which $\lambda_n < 0$ and for which

$$\frac{\sum_i \lambda_i - 1}{|\lambda_n|}$$

* The new machine EDSAC II (at present under construction) could complete this calculation in about an hour; but the point, of course, remains that the number of inversions rises very rapidly with n and m.

is least. (This condition must be modified in an obvious way of x_n is a dummy vector.) In this way, the number of inversions can be reduced to the number of efficient facets, with a very considerable saving in computing.

However, Mr. P. Swinnerton-Dyer has suggested the following very elegant method, which has the great advantage of side-stepping the matrix inversions. His method consists in starting with the efficient surface corresponding to a very small number of observed points (that for one observed point may be written down from inspection) and successively introducing the remaining points, amending the surface as necessary.

Suppose the equations of the facets of the efficient surface corresponding to a given set of observed points are written in the form $L_i(x) = 0$ (with $L_i(0) < 0$) and the further point x_k is introduced. Then if $L_i(x_k) > 0$ for all i, x_k is not an efficient point and the surface is unchanged. If however $L_i(x_k) < 0$ for some facets, these facets must be replaced by facets passing through x_k and the edges of the neighbouring facets.

That is to say, suppose that $L_i(x_k) < 0$ and for a neighbouring facet $L_j(x_k) > 0$. Then the amended surface will contain a facet passing through the intersection of these facets and through x_k. The equation of this facet may be written

$$L_j(x_k)\, L_i(x) - L_i(x_k)\, L_j(x) = 0$$

a form which is simple to calculate given $L_i(x)$ and $L_j(x)$. The method has to be modified in an obvious way where a facet parallel to one of the axes is discarded.

6. The Illustrative Example

6.1. *Choice of Illustrative Example*

The example chosen to illustrate the method is that of agriculture in the United States. The reasons for this choice are two. The first, and perhaps the more important, is the availability of information. American agricultural statistics give considerable information as to the agricultural inputs and outputs of each state. One is thus provided with measures of inputs and outputs, on a comparable basis, for 48 productive units. It is difficult to think of any other application where such adequate information is so readily available. Secondly, in so far as land plays a more important role in agriculture than in manufacturing, agricultural production has a more genuinely multi-variate nature than most industries. It might also be argued that agricultural outputs are more homogeneous than those of other industries, but against this must be set the heterogeneity of land inputs.

Thus, American agriculture provides an interesting application of the method, and one which is possible with relatively little preliminary work on the measurement of inputs and outputs. It should be emphasized, however, that it is intended purely as an *illustrative* application. The handy measures of inputs are not always the best, and a full-scale study of American agriculture by the method would require vastly more work on the measurement of several of the inputs than was worth while for the present purpose. At the least, one would need to make a detailed attempt to allow for the heterogeneity of land inputs, before one could draw more than the roughest inferences about American agricultural efficiency. This heterogeneity reflects differences in location and climate, as well as in fertility.

DIAGRAM 4 (key).

1. Australia.
2. Austria.
3. Belgium.
4. Canada.
5. Chile.
6. Columbia.
7. Cuba.
8. Denmark.
9. Finland.
10. France.
11. Germany.
12. Greece.
13. Hungary.
14. Ireland.
15. Italy.
16. Japan.
17. Netherlands.
18. New Zealand.
19. Poland.
20. Spain.
21. Sweden.
22. Switzerland.
23. South Africa.
25. United Kingdom.
26. United States.
28. Jugoslavia.

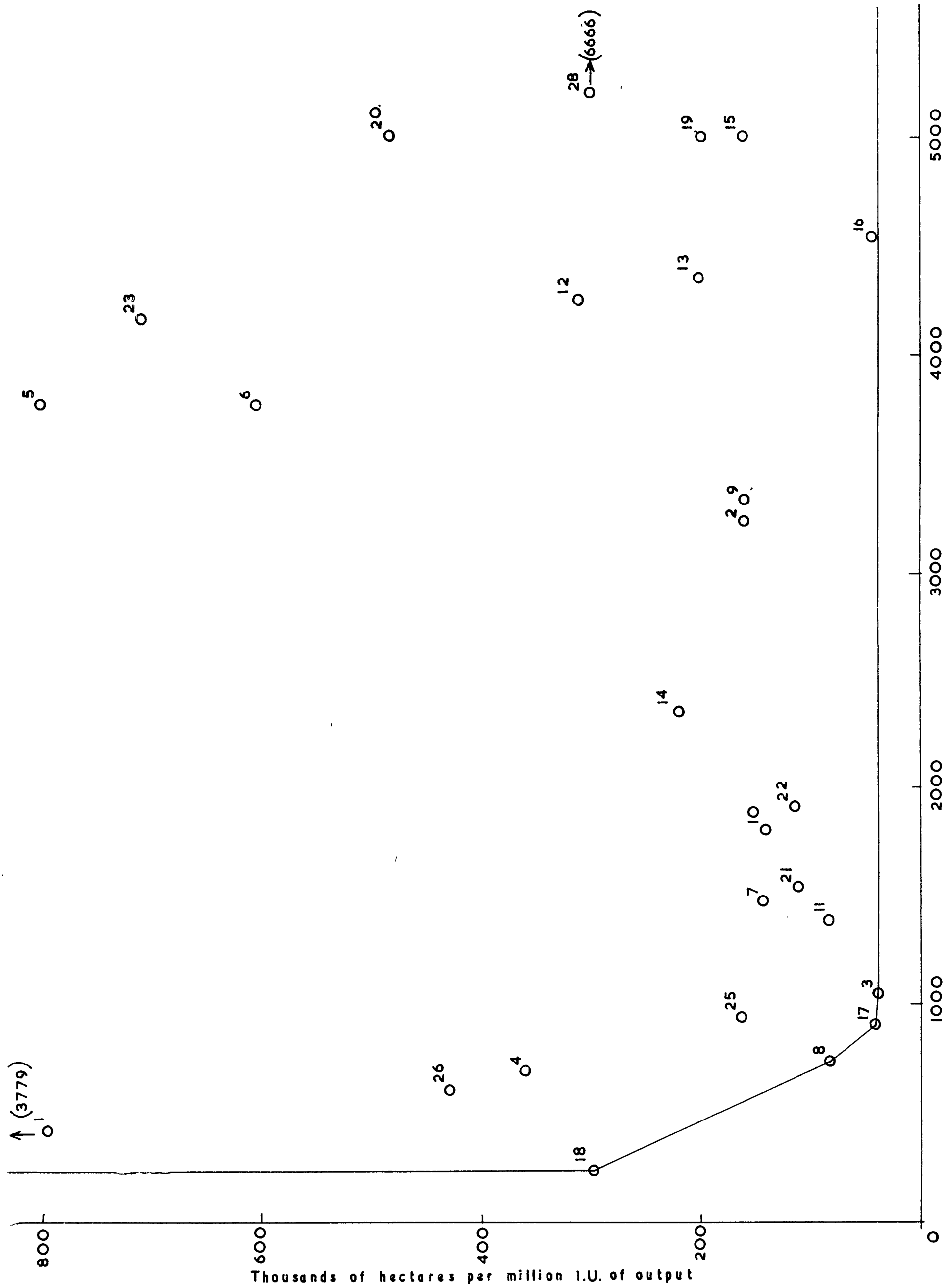

Thousands of hectares per million I.U. of output
Men per million I.U. of output
(3779)
(6666)

DIAGRAM 4.

These considerations represent a justification of the illustrative example, but the actual history of the choice may also be of interest to society. In 1954 Mr. Colin Clark read a paper to the Society in which, *inter alia*, he compared the average productivity of agricultural labour in a number of countries. In the discussion it was suggested that the average productivity of land was

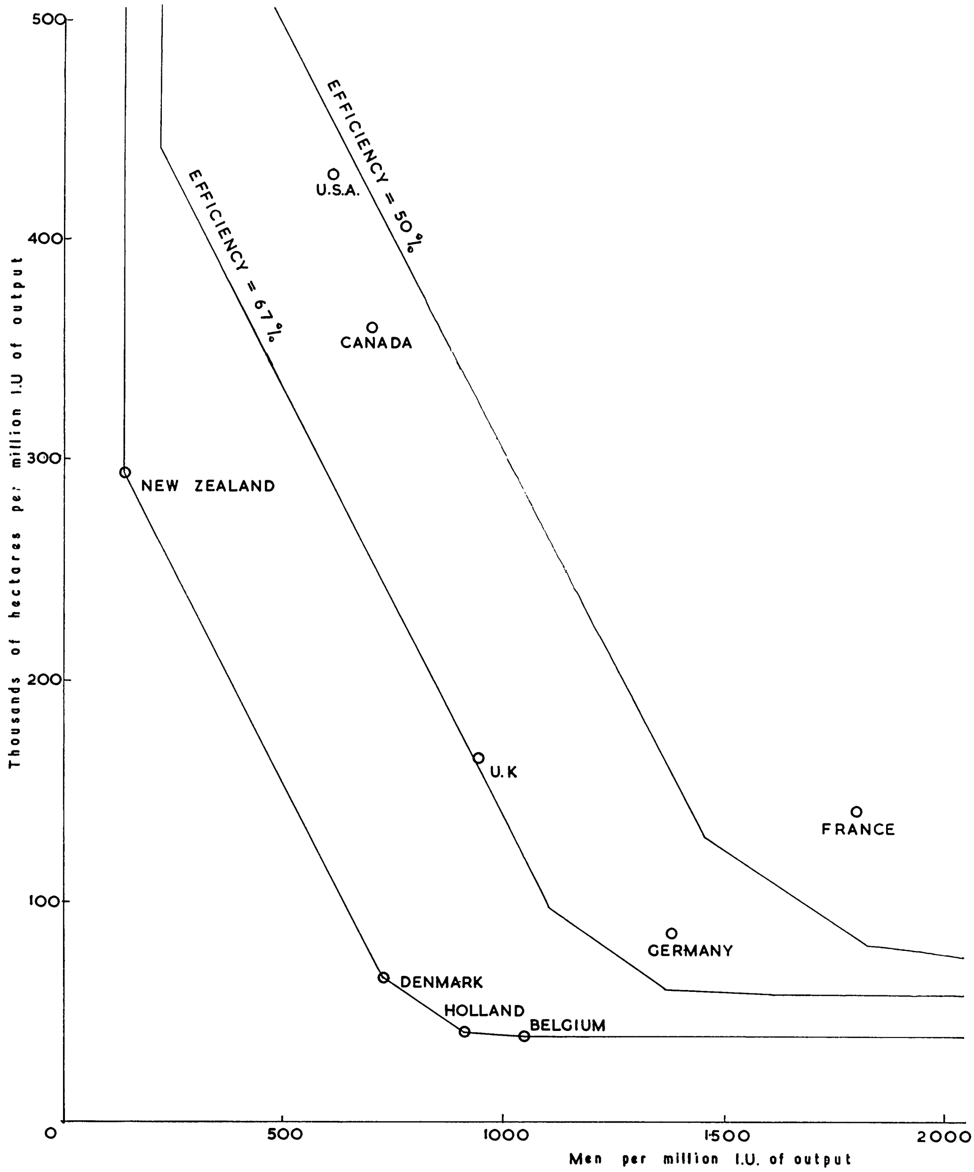


DIAGRAM 5.

at least as important, and this provoked the present author to draw Diagram 4. The figures for labour per unit of output are reciprocals of Mr. Clark's productivity figures, while those for land per unit of output were obtained from them by applying estimates of the number of workers per hectare, derived in most cases from the *F.A.O. Year Book*, 1952. Diagram 5 gives an enlargement of part of Diagram 4, with constant-efficiency contours drawn in. The positions of various countries on this diagram may be suggestive, but it must be emphasized that no firm conclusions may be drawn; the two diagrams are included only to indicate how Mr. Clark's international comparisons change when a second factor is introduced, and it is certainly not suggested that they themselves form an adequate basis for such comparisons.

The greatest objection is, of course, that they consider only two factors. It was felt that the difficulty of getting internationally comparable measurements made it preferable to apply the more elaborate analysis to American data, rather than to attempt it on the international figures.

6.2. *Definition of Variables*

(a) *Output*—Cash receipts from farming *plus* value of home consumption in millions of dollars.

(*Agricultural Statistics*, 1952, Table 700.) The figures include government payments, a mixed bag of subsidies and payments for restriction. It would ideally have been desirable to include the former and exclude the latter, but as government payments are never more than 3 per cent., and seldom more than 1 per cent., of a state's total cash receipts, this particular inadequacy is trivial. The figures were not corrected for "duplication" as it was felt that this problem is best handled by treating purchases of agricultural produce as an input.

A more serious problem is the variation in prices between states. This is a matter partly of heterogeneity of product and partly of distance from the main markets. The former is properly reflected in output, while the latter should, ideally, have been eliminated. This would be done either by revaluing all the outputs at a constant set of prices, which would have involved a disproportionate amount of labour, or by introducing location as a factor of production. An unsuccessful attempt was made to do the latter.

(b) *Land*—Land in farms *minus* woodland and other land not pastured, in thousands of acres.

(*Agricultural Statistics*, 1952, Table 637.) There seemed to be no obvious alternative to this definition. Of course, the land included is highly heterogeneous, and in a full-scale study one would have to make careful allowance for the different natural characteristics of the land. These include not merely geological factors, but also climate and location, although perhaps the latter are best dealt with by introducing special inputs or quasi-factors.

(c) *Labour*—Men on farms, including farmers, farm managers, and unpaid family workers.

(*Statistical Abstract of the U.S.*, 1953, Table 225.) The choice of definition here was somewhat arbitrary, but the analysis did not seem very sensitive to changes in definition.

(d) *Materials*—Expenditure on feed, livestock and seed purchased in thousands of dollars.

(*U.S. Census of Agriculture*, 1950.) It should be noted that whereas the previous figures refer to 1950, these are for 1949. This choice was largely dictated by the availability of information, but the implicit assumptions about time-lags in production seem as plausible as any.

(e) *Capital*—Value of implements and machinery on farms, 1950, in thousands of dollars.

These figures, which are a breakdown of his published results (Tostlebee, 1954) were very kindly made available to me by Dr. A. S. Tostlebee. From these figures the variables shown in Table 1 (p. 279) were constructed, thus:—

$$x_1 = \frac{b}{a}, x_2 = \frac{c}{a}, x_3 = \frac{d}{a}, x_4 = \frac{e}{a}$$

Thus, each of these variables represents the input of a particular factor per unit of output. Each observation represents a state, and the agricultural industry of a state must be regarded as an industry in the sense of section 3.4. Perhaps the most serious objection to the data is that agricultural output is notoriously subject to year-to-year variations (see section 3.6, above).

6.3. *The Estimates of Technical Efficiencies*

The estimates of the technical efficiencies of the states are given in Table 2 (p. 280). For each state there is a different estimate of efficiency, according to the factors of production included in the analysis. It will be seen that, in accordance with the theory developed in section 3, the introduction of a new factor of production into the analysis cannot lower, and in general, raises the technical efficiency of any particular state. Thus, the more factors that are considered, the more are apparent differences in efficiency explained as being due to differing inputs of these factors.

Of course, there is no doubt that the process could (and in principle should) be pushed beyond the four factors considered in this paper. That is to say, the apparent differences in efficiency in the final column of Table 2 reflect factors like climate, location and fertility that have not been included in the analysis, as well as "genuine" differences in efficiency. However, from drawing scatter diagrams, it was found that there was little correlation between the efficiencies of the final column and variables representing location, temperature and rainfall. It may well be objected that such a test depends too much on the selection of variables, and indeed on the judgment of whether or not the scatter diagram shows "appreciable" correlation. An analysis of variance was therefore applied to these efficiencies, the groups used being those shown in Table 2, which are the regional groupings normally used by the United States Department of Agriculture. It was found that the variance between groups was less than that within groups, so that it was unlikely that much of the difference in efficiencies could be explained in terms of variables whose variation was mainly between regions—as is the case in general with climatic and locational factors.

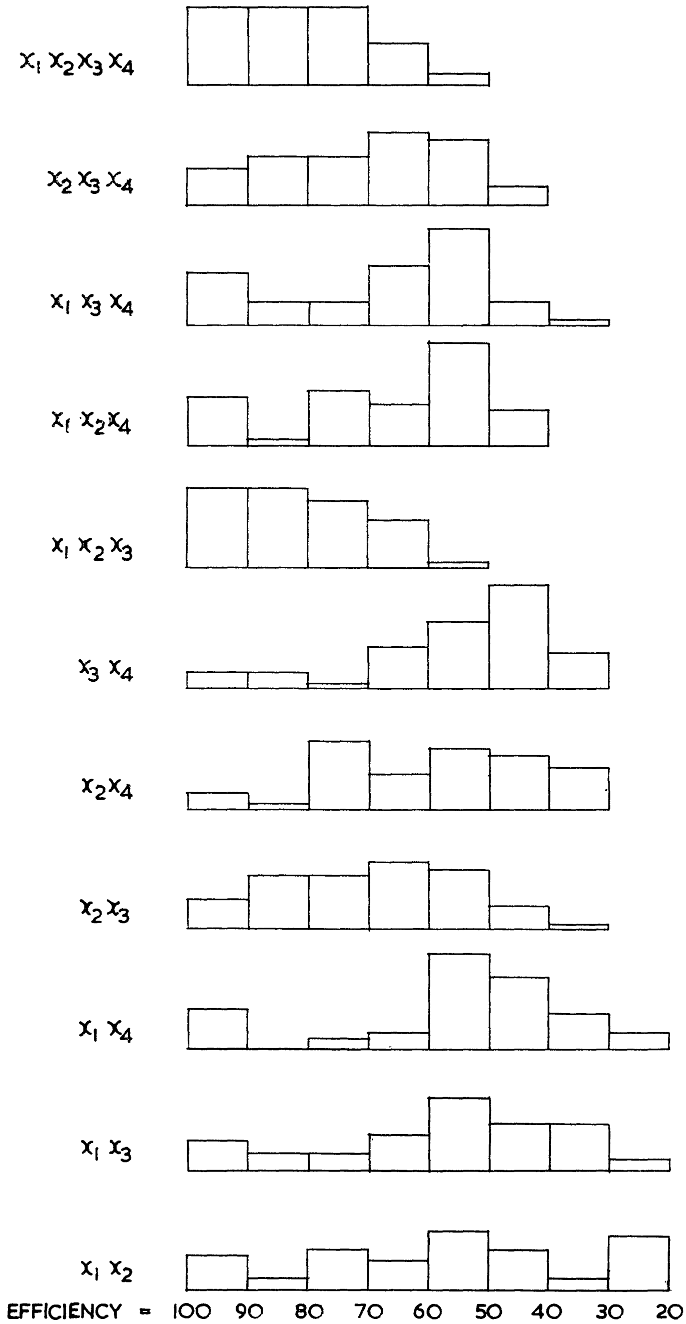
This does not mean, of course, that one must conclude that climate and location have no influence on agricultural production. A more likely explanation is that some of the factors already included are correlated with climate and location, and have already "explained" a good deal of the relevant variation in efficiency. The situation is thus analogous to the familiar one of a partial regression analysis where the pre-determined variables are highly inter-correlated, and the solution, too, must be analogous. That is, the analysis must, regretfully, be stopped at this point. It might be thought that some climatic and locational variables should be introduced while omitting some of the other variables, but, on the one hand, it is not clear how these factors should be represented, and on the other hand, it would be hard to justify an analysis from which any one of land, labour, capital and materials was missing.

It is perhaps also instructive to look at the frequency distribution of efficiencies, and Diagram 6 shows the distribution corresponding to each column of Table 2 in histogram form. It is to such frequency distributions that one must look for a measure of the success of the analysis, corresponding to the multiple correlation coefficient in regression analysis. It might seem at first sight that the obvious counterpart of the correlation coefficient was simply the average level of efficiency, being a measure of the extent to which differences have been explained in causal terms. But whereas in regression analysis the object is to explain as much of the variation as possible, here one does not wish to explain away all differences. In other words, one regards as ideal not an analysis that makes all states perfectly efficient, but one that leaves only "genuine" differences in efficiency, as discussed in section 3.1.

As any direct test of whether the differences in efficiency are genuine or not is impossible (except, of course, the sort of test described above) one is reduced to a consideration of the plausibility of the distribution of efficiencies. This is necessarily a highly subjective matter, and one hesitates to attempt to lay down any objective criteria of plausibility. Indeed, if one were to approach the matter in general terms it is doubtful whether one could secure agreement. Fortunately, in particular cases agreement seems much easier to obtain. Thus, in Diagram 6 people seem to agree that the distributions corresponding to $x_1x_2x_3x_4$ and $x_1x_2x_3$ are quite plausible while most of the others are not; and that the distribution for x_1x_4 is a good example of an implausible one.

In the latter case, there is a substantial group of states with very high efficiencies, and then a virtual gap until the 60 per cent. point; three-quarters of the states are less than 60 per cent. efficient. The reason why this is regarded as implausible is perhaps that one feels that there must be some neglected input accounting for the apparently very low efficiency of such a large group of states. The same sort of argument would apply to the other bimodal distributions. It is difficult to see any rational basis for regarding rectangular, J-shaped, or unimodal distributions as more plausible. Thus, it might be felt that the distribution for x_2x_3 was as plausible as those for $x_1x_2x_3$ and $x_1\ x_2\ x_3\ x_4$.

However, in comparing these three distributions (which seem to be the most plausible of those in Diagram 6) other criteria than the mere shape of the distribution may be introduced. Thus, in going from x_2x_3 to $x_1x_2x_3$, by introducing land (obviously an important factor of production in agriculture) one has clearly explained a good deal of the apparent variation in efficiency, so



$X_1 X_2 X_3 X_4$
$X_2 X_3 X_4$
$X_1 X_3 X_4$
$X_1 X_2 X_4$
$X_1 X_2 X_3$
$X_3 X_4$
$X_2 X_4$
$X_2 X_3$
$X_1 X_4$
$X_1 X_3$
$X_1 X_2$
EFFICIENCY = 100 90 80 70 60 50 40 30 20

DIAGRAM 6.

that $x_1x_2x_3$ is clearly a better analysis. In going from $x_1x_2x_3$ to $x_1x_2x_3x_4$, by introducing capital (another relevant factor) only a little of the apparent variation in efficiency is explained. Thus although in principle $x_1x_2x_3x_4$ is the better analysis, it seems that in fact relatively little difference is made to the efficiency estimates by omitting x_4. (This may perhaps be thought of as analogous to a regression analysis where the introduction of a variable does not significantly improve the fit; although here there is no objective measure of significance.) That this is not due to inter-correlation is shown by the fact that none of the other three-factor analyses gives a very plausible distribution of efficiencies. One must therefore conclude that either capital does not play an important role in agricultural production, or, more likely, that the measure of capital used is not a satisfactory one. It is well known that depreciated value is not the ideal measure of capital input in a production process. Nor was the choice of implements and machinery entirely appropriate, as it includes motor cars but excludes buildings, whereas one would have wished to exclude most of the former and include part of the latter.

The conclusion must be that $x_1x_2x_3$ and $x_1x_2x_3x_4$ are the two best analyses, but that it is difficult to decide between them. Accordingly, both efficient production functions are given below.

6.4. *The Efficient Production Functions*

The efficient production function in four dimensions is specified in Table 3 (p. 281) and that in three dimensions in Table 4 (p. 281). For each facet the points determining it are given* together with the parameters of the equation of the hyper-plane of which it is part, the equation being written in the form

$$\Sigma a_i x_i = 1{,}000 \qquad (8)$$

However, although the tables completely specify the efficient production functions, they do not lend themselves to analysis by inspection, and it is desirable to have a more graphic method

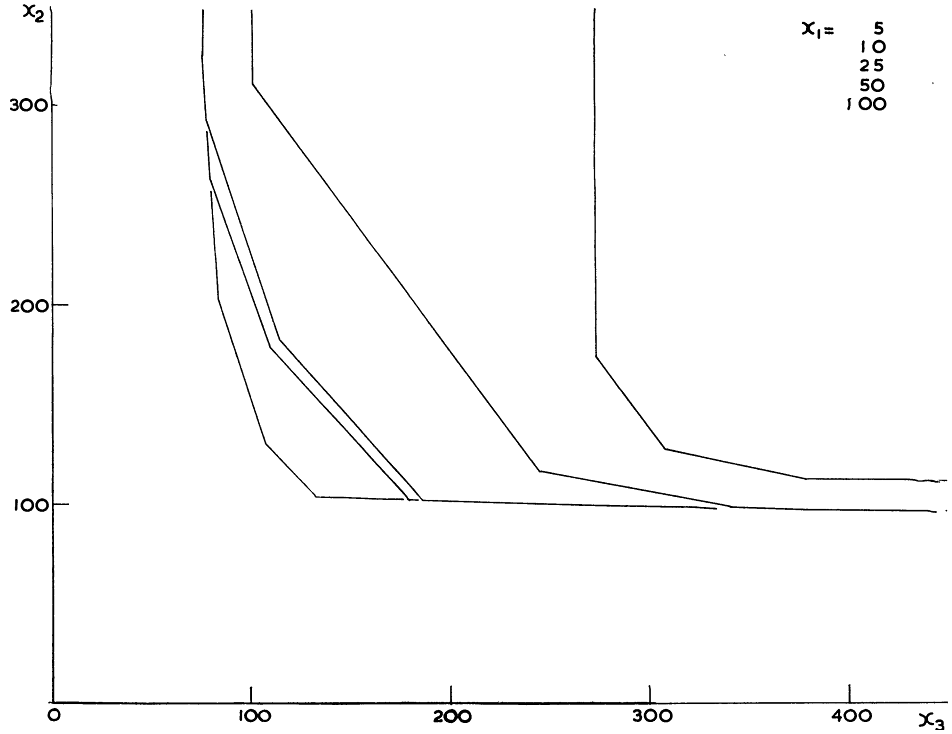


DIAGRAM 7*a*.

* The observed points are numbered as in Table 1, while $D_1D_2D_3$ and D_4 are the dummy vectors described in section 2.3. Thus, in four dimensions, D_1 =(' 80 , 0, 0), and so on.

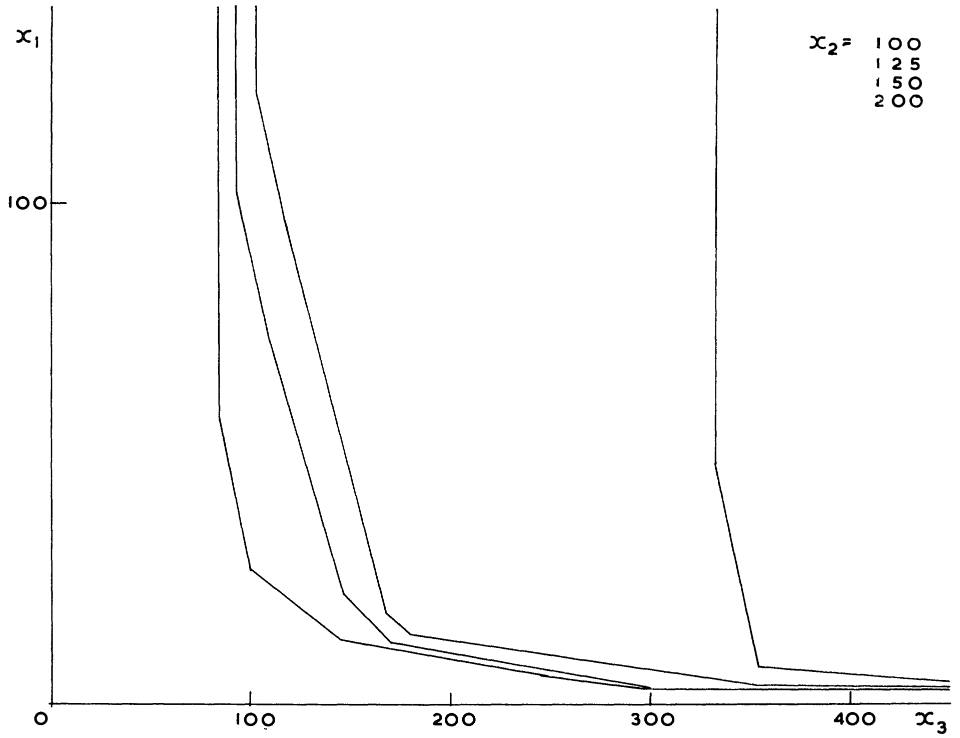


DIAGRAM 7*b*.

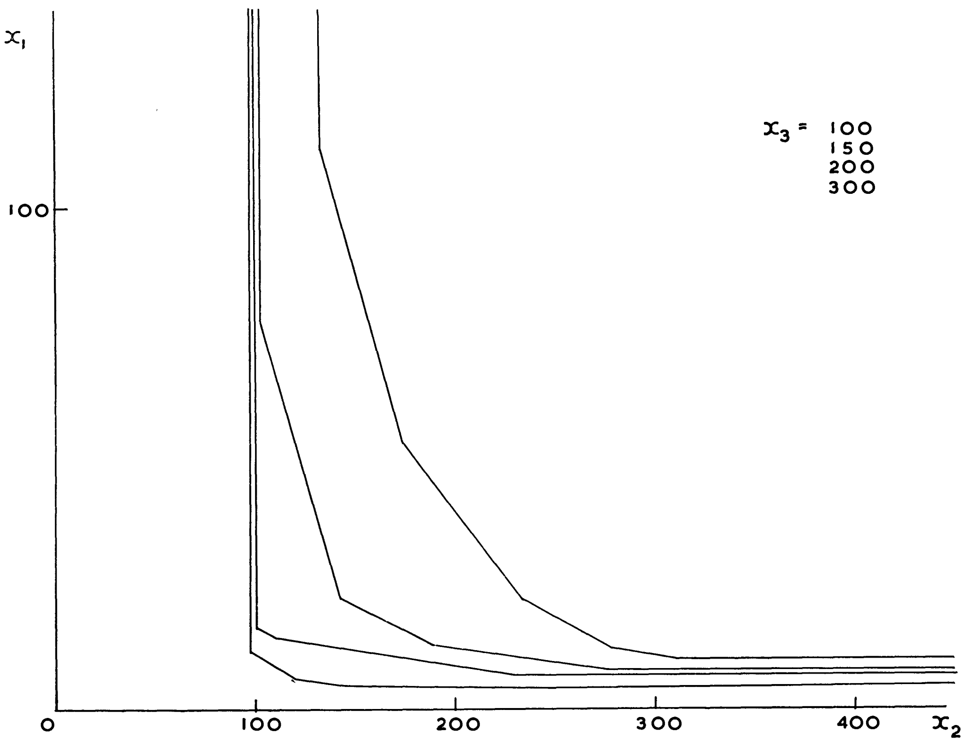


DIAGRAM 7*c*.

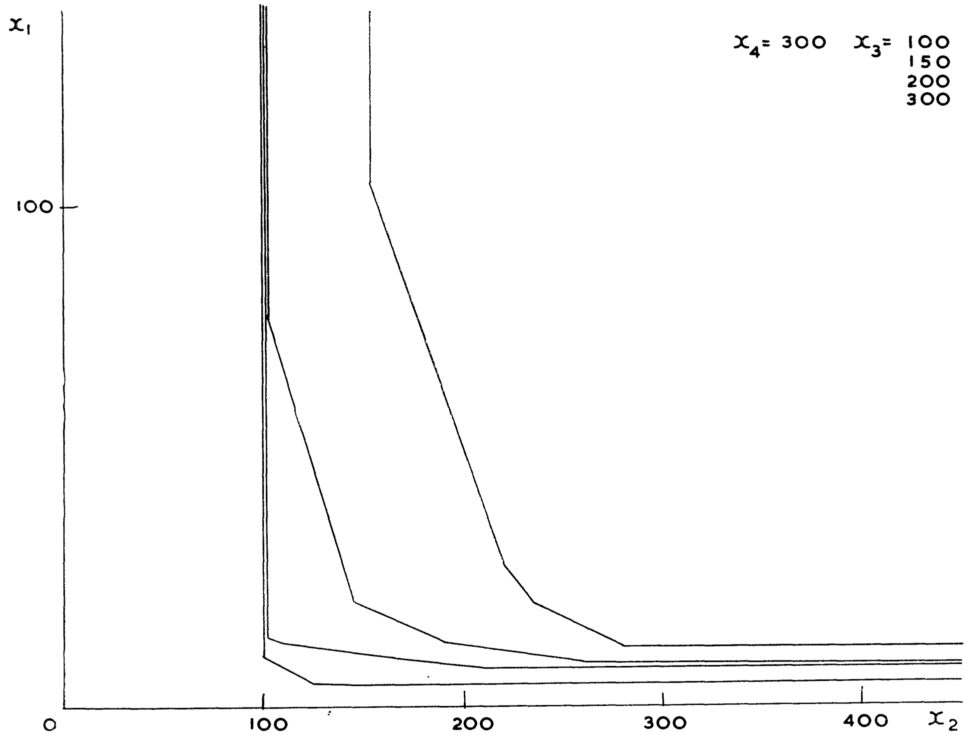


DIAGRAM 8*a*.

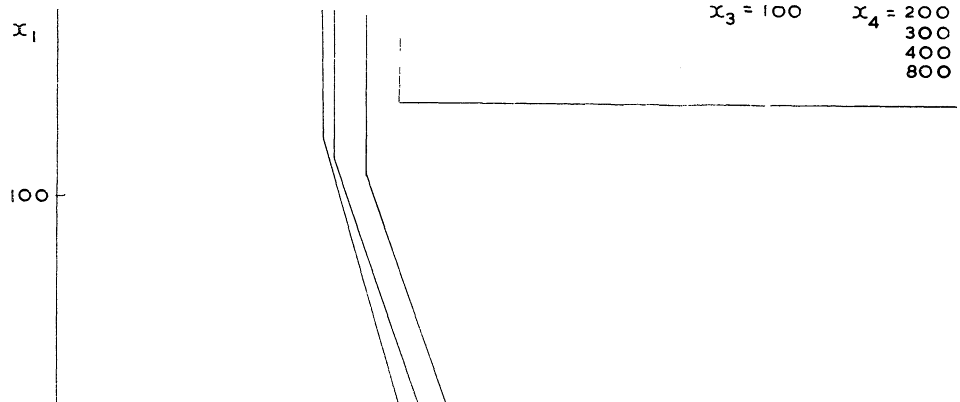


DIAGRAM 8B.

of showing the functions. This can be done by drawing contour diagrams like those of Diagrams 7 and 8. In three dimensions, if one of the variables is held constant, the relationship between the other two variables can be represented as a contour line in a two-dimensional diagram. If such contours are plotted for a number of different values of the third variable, the resultant contour map gives a picture of the relationships between all three variables, as is done in Diagram 7.

In four dimensions, the procedure must be pushed a stage further. Holding one variable constant, a contour map can be drawn to show the relationship between the remaining three variables when the fourth variable has that particular value. Thus Diagram 8*a* shows the relationship between x_1, x_2 and x_3 for $x_4 = 300$. In order to give a complete picture of the function a very large number of such diagrams would have to be drawn—reasons of space preclude giving here more than a small sample. Of course, the contour maps of Diagram 7 also help to complete the picture for the four-dimensional function, as they are identical with the contour maps for the latter corresponding to large values of x_4.

Perhaps the most striking impression given by these diagrams is that the Law of Diminishing Returns operates very sharply. Indeed, this has led to the contours being drawn at unequal intervals—using equally-spaced contours would give an even more spectacular result. This is at least in part due to the method, which assumes that the elasticity of substitution falls to zero outside the range of observed factor proportions. Various rather obvious conclusions may be drawn, but perhaps the most interesting diagram is 8*b*. Here one notes that for a process to be economical in both capital and materials, it must use a great deal of land—i.e. grazing. One also notes how sharply returns to capital appear to fall off, but this is probably a reflection of the factors discussed in the previous section.

6.5. *The Cobb-Douglas Approximation*

It will be appreciated from the previous section that it is a laborious business both to specify and to analyse the efficient production function as it stands. This does not constitute a criticism

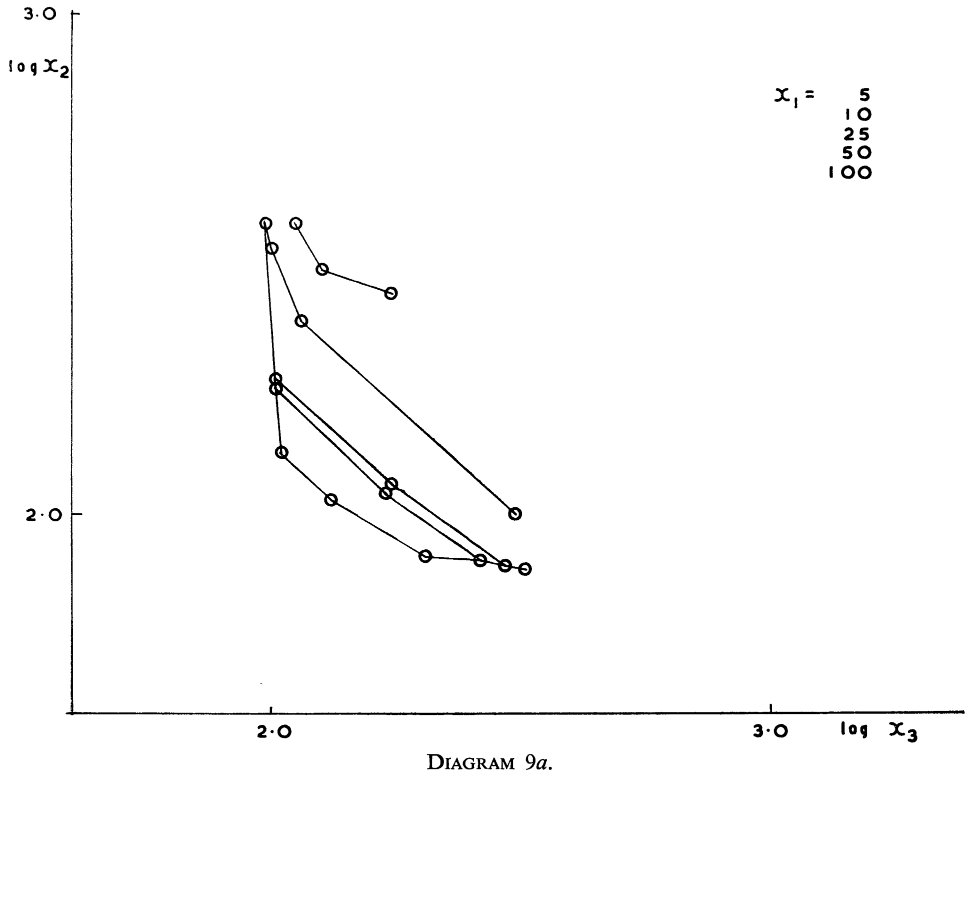


DIAGRAM 9*a*.

of the method—if the world is complex, its analysis must needs be laborious. However, it would be a great convenience to approximate the efficient production function by a simple mathematical function. It would then be specified by a simple formula, together with the values of a few parameters, and its properties could be analysed algebraically.

The best known and perhaps the most plausible of such approximations is the Cobb-Douglas function (Cobb and Douglas, 1928) which may be written in our notation

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\alpha_1 + \alpha_2 + \alpha_3 = 1.$$

This defines a surface, convex to the origin, very similar in principle to the efficient production function. If it were to turn out to be a good approximation to the observed function, it would have the great advantage that it could be specified in terms of four or five parameters. Its disadvantage is that it makes quite strong assumptions about the shape of the efficient function, and for this reason it would not have been desirable to fit such a function at the beginning of the analysis.

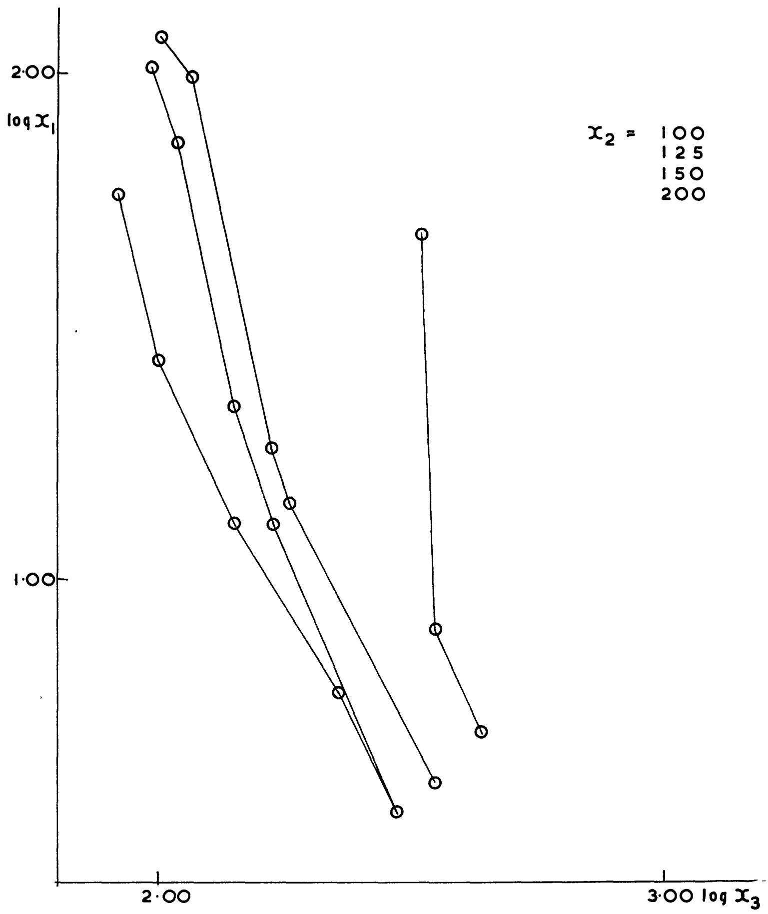


DIAGRAM 9*b*.

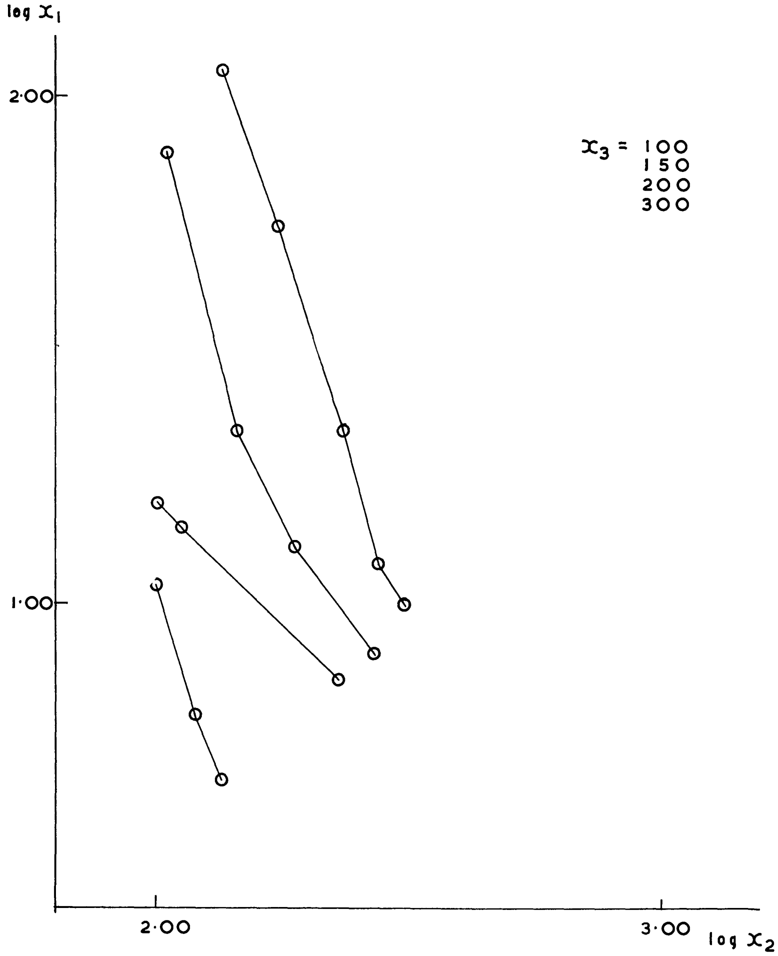


DIAGRAM 9*c*.

As three (independent) parameters have to be estimated from (effectively) seven observations (or four from nine) the paraphernalia of regression analysis is clearly unjustified. However, a rough test by inspection can be obtained from Diagram 9. Equation (9) may be written

$$\alpha_1 \log x_1 + \alpha_2 \log x_2 + \alpha_3 \log x_3 = \log k \quad . \quad . \quad . \quad (10)$$

so that, if the logarithms of the x's are plotted, the contour diagrams corresponding to the Cobb-Douglas function are sets of parallel straight lines. In Diagram 9 the corner points of the contours of Diagram 7 are plotted on logarithmic scales, and it can be seen that, apart from the extreme points, they lie roughly on parallel straight lines.

As the extreme points reflect in part the implicit assumption of zero elasticity of substitution outside the range of observations, one may conclude that the Cobb-Douglas approximation gives a fair fit within the range of observations. By inspection, the values of the parameters seem to be (very roughly)

$$\alpha_1 = \cdot 15$$

$$\alpha_2 = \cdot 45$$

$$\alpha_3 = \cdot 40$$

If x_4 is introduced, the parameters become (again roughly)

$$\alpha_1 = \cdot 12$$

$$\alpha_2 = \cdot 36$$

$$\alpha_3 = \cdot 32$$

$$\alpha_4 = \cdot 20$$

The low values of α_1 reflect the very great variation in x_1 as compared with the other variables. (The efficient points in four dimensions show a range of variation of about 3,000 per cent. for x_1, 350 per cent. for x_2, 600 per cent. for x_3 and 480 per cent. for x_4.)

It would be very interesting to compare these results with other agricultural production functions, for example those that Tintner (1944) and J. O. Jones (1956) derived from data for individual farms. Unfortunately, differences both in method and in definition of variables are so large that such comparisons cannot at all easily be made.

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TABLE 1

State	Land x_1	Labour x_2	Materials x_3	Capital x_4
1. Maine	11·79	153·9	222·7	341·8
2. New Hampshire	13·58	175·4	411·2	346·0
3. Vermont	23·46	207·1	328·0	454·9
4. Massachusetts	5·06	138·2	309·6	286·9
5. Rhode Island	4·57	151·5	331·0	274·5
6. Connecticut	5·29	125·7	317·3	258·3
7. New York	15·46	179·7	277·5	535·7
8. New Jersey	4·17	130·9	317·4	307·6
9. Pennsylvania	13·96	189·2	303·6	581·4
10. Ohio	18·97	199·8	209·0	525·0
11. Indiana	16·56	157·4	199·4	428·3
12. Illinois	15·66	130·1	197·4	434·8
13. Michigan	20·73	211·4	155·2	655·5
14. Wisconsin	19·96	217·6	189·7	625·3
15. Minnesota	22·65	185·4	187·7	528·7
16. Iowa	14·37	118·8	251·5	378·0
17. Missouri	27·87	220·7	226·7	398·8
18. North Dakota	74·18	176·4	87·6	761·8
19. South Dakota	81·58	172·2	207·3	574·0
20. Nebraska	44·21	134·0	246·2	407·3
21. Kansas	43·01	137·3	213·2	428·6
22. Delaware	5·62	97·2	452·0	252·0
23. Maryland	11·42	198·8	296·4	425·4
24. Virginia	18·19	299·9	189·5	342·3
25. West Virginia	37·13	367·6	243·9	342·6
26. North Carolina	10·66	333·3	76·8	324·6
27. South Carolina	19·56	441·4	88·3	386·3
28. Georgia	26·74	383·1	144·3	420·1
29. Florida	27·02	176·8	113·9	182·9
30. Kentucky	24·58	391·1	173·6	397·1
31. Tennessee	25·34	440·0	153·1	494·2
32. Alabama	29·80	462·8	119·1	413·2
33. Mississippi	26·87	442·3	91·5	406·1
34. Arkansas	25·09	336·8	132·7	449·6
35. Louisiana	22·52	344·4	97·2	473·9
36. Oklahoma	54·84	229·9	193·5	475·5
37. Texas	63·04	177·4	161·8	341·0
38. Montana	155·69	135·1	140·4	534·0
39. Idaho	38·14	161·3	200·7	584·6
40. Wyoming	233·93	140·8	282·3	428·3
41. Colorado	69·00	122·3	254·5	407·8
42. New Mexico	207·94	158·1	186·0	288·3
43. Arizona	137·89	105·3	109·4	158·9
44. Utah	64·87	167·6	309·3	421·5
45. Nevada	144·65	122·4	186·4	338·1
46. Washington	30·08	127·7	161·2	398·4
47. Oregon	45·53	143·2	180·6	441·2
48. California	14·53	104·8	191·1	244·4

TABLE 2

	Efficiency when Variables Considered are:—										
	x_1x_2	x_1x_3	x_1x_4	x_2x_3	x_2x_4	x_3x_4	$x_1x_2x_3$	$x_1x_2x_4$	$x_1x_3x_4$	$x_2x_3x_4$	$x_1x_2x_3x_4$
1. Maine	63	72	71	68	65	49	94	71	83	68	94
2. New Hampshire	55	52	69	58	58	46	69	69	70	58	71
3. Vermont	47	39	51	50	48	35	58	53	56	50	58
4. Massachusetts	89	95	94	74	73	55	98	95	99	74	99
5. Rhode Island	91	94	100	68	68	58	94	100	100	68	100
6. Connecticut	92	92	100	81	80	62	98	100	100	81	100
7. New York	54	55	46	58	54	37	75	54	57	58	75
8. New Jersey	100	100	100	78	76	52	100	100	100	78	100
9. Pennsylvania	51	57	43	55	51	34	74	51	57	55	74
10. Ohio	49	52	46	53	49	45	78	49	58	53	78
11. Indiana	62	58	56	67	62	50	87	62	68	67	87
12. Illinois	75	61	56	80	75	50	93	75	69	80	93
13. Michigan	46	51	37	64	46	49	88	46	51	64	88
14. Wisconsin	45	51	39	55	45	45	79	45	51	55	79
15. Minnesota	52	46	45	58	52	48	82	52	56	58	82
16. Iowa	82	60	64	86	82	43	88	82	73	86	89
17. Missouri	44	37	55	48	47	46	68	57	62	48	68
18. N. Dakota	55	88	27	100	55	88	100	55	88	100	100
19. S. Dakota	56	37	32	61	56	44	71	56	45	61	71
20. Nebraska	73	31	49	77	73	43	78	73	51	77	78
21. Kansas	71	36	48	76	71	47	81	71	52	76	81
22. Delaware	100	74	100	100	100	63	100	100	100	100	100
23. Maryland	49	66	58	52	51	37	78	58	68	52	78
24. Virginia	32	54	67	50	46	55	71	67	78	55	78
25. West Virginia	26	31	58	40	46	46	52	58	60	46	60
26. North Carolina	39	100	75	100	49	100	100	75	100	100	100
27. South Carolina	25	87	60	87	41	87	87	60	87	87	87
28. Georgia	25	53	53	58	38	62	67	53	64	62	67
29. Florida	55	67	100	84	87	94	100	100	100	94	100
30. Kentucky	25	44	57	50	40	56	63	57	66	56	66
31. Tennessee	22	50	47	54	32	56	62	47	58	57	62
32. Alabama	21	64	53	66	38	70	66	53	71	70	71
33. Mississippi	22	84	55	84	39	84	84	55	84	84	84
34. Arkansas	29	58	51	64	35	64	74	51	65	68	74
35. Louisiana	28	79	49	83	34	79	86	49	79	84	86
36. Oklahoma	42	40	41	54	44	49	67	48	51	54	67
37. Texas	55	47	53	66	58	61	81	64	64	66	81
38. Montana	72	55	33	78	72	57	81	72	57	78	81
39. Idaho	60	38	38	65	60	45	79	60	46	65	79
40. Wyoming	69	27	39	73	69	38	73	69	39	73	73
41. Colorado	79	30	45	84	79	42	85	79	45	84	85
42. New Mexico	61	41	57	66	65	58	66	65	58	66	66
43. Arizona	92	70	100	100	100	100	100	100	100	100	100
44. Utah	58	25	43	62	58	38	62	58	43	62	62
45. Nevada	79	41	51	85	79	56	85	79	57	85	85
46. Washington	76	48	55	82	76	59	99	76	63	82	99
47. Oregon	68	43	46	73	68	53	87	68	54	73	87
48. California	93	65	93	99	95	65	100	99	100	99	100

TABLE 3

Facet				a_1	a_2	a_3	a_4
5	6	8	22	77·056	1·062	·113	1·638
6	8	22	48	22·991	4·597	·918	·036
6	8	26	48	28·024	1·196	1·798	·507
6	22	29	48	10·491	·926	·303	2·834
6	26	29	48	17·352	·012	1·020	2·258
18	26	29	43	1·184	1·395	6·046	·179
22	29	43	48	2·084	2·231	·050	2·972
D_1	18	26	43	0	1·011	7·924	·167
D_2	5	6	8	51·771	0	1·333	1·173
D_2	6	8	26	49·480	0	1·414	1·121
D_2	6	22	29	13·155	0	·129	3·444
D_2	6	26	29	17·367	0	1·014	2·271
D_2	26	29	43	·599	0	6·003	1·641
D_3	5	6	22	62·514	·169	0	2·509
D_3	5	8	22	86·511	·986	0	1·659
D_3	22	29	43	2·045	2·138	0	3·097
D_4	8	22	48	23·006	4·671	·922	0
D_4	8	26	48	29·730	1·562	2·116	0
D_4	22	43	48	·141	9·034	·268	0
D_4	29	43	48	2·187	3·182	3·322	0
D_4	26	29	48	10·270	1·884	3·418	0
D_4	18	26	29	3·090	1·631	5·516	0
D_4	18	29	43	2·069	2·978	3·666	0
D_1	D_2	26	43	0	0	7·109	1·399
D_1	D_3	22	43	0	8·395	0	·730
D_1	D_4	18	26	0	·690	10·025	0
D_1	D_4	18	43	0	2·164	7·057	0
D_1	D_4	22	43	0	9·269	·219	0
D_2	D_3	5	6	59·630	0	0	2·650
D_2	D_3	5	8	126·787	0	0	1·532
D_2	D_3	6	22	53·134	0	0	2·783
D_2	D_3	22	29	11·952	0	0	3·701
D_2	D_3	29	43	1·146	0	0	5·298
D_2	D_4	8	26	78·545	0	2·119	0
D_3	D_4	8	22	102·018	4·390	0	0
D_1	D_2	D_3	43	0	0	0	6·293
D_1	D_2	D_4	26	0	0	13·021	0
D_1	D_3	D_4	22	0	10·288	0	0
D_2	D_3	D_4	8	239·808	0	0	0

TABLE 4

Facet			a_1	a_2	a_3
8	22	48	23·006	4·671	·922
8	26	48	29·730	1·562	2·116
22	43	48	·141	9·034	·268
29	43	48	2·187	3·182	3·322
26	29	48	10·270	1·884	3·418
18	26	29	3·090	1·631	5·516
18	29	43	2·069	2·978	3·666
D_1	18	26	0	·690	10·025
D_1	18	43	0	2·164	7·057
D_1	22	43	0	9·269	·219
D_2	8	26	78·545	0	2·119
D_3	8	22	102·018	4·390	0
D_1	D_2	26	0	0	13·021
D_2	D_3	8	239·808	0	0
D_3	D_1	22	0	10·288	0

Discussion on Mr. Farrell's Paper

Mr. Colin Clark: It is a great pleasure to propose the Society's thanks to Mr. Farrell. Later this evening I am leaving for Japan, where I really do hope to learn something about agricultural productivity.

It would be wrong to imply that I am *au fait* with every aspect of Mr. Farrell's method. It does appear to me however that Mr. Farrell has designed a most ingenious procedure which is a very great advance on anything we have had before, and which still remains within the limits of practicability. Some of us first became acquainted with Mr. Farrell's work through his study of the demand for motor cars, in which he showed an ingenuity which put his work well ahead of that of previous workers in this field.

I think that the first point on which comment is required is on the choice of data. One generally thinks that intricate methods of this sort can be better applied to industrial than to agricultural data. Mr. Farrell, however, showed an inspired intuition, which has succeeded, in taking agricultural data as matter for his experiment.

The data are more precise than might appear at first sight. No agricultural data are perfect, but the United States has spent a remarkable amount of money on improving its agricultural statistics.

In other countries besides the United States, agriculture is of great political importance, and therefore may receive more than its share of the funds available for statistical work, and many industrial statistics for that reason still leave much to be desired.

I think we will agree, having heard this paper, that Mr. Farrell has already reached some interesting and successful results and has come nearer than any previous investigator to a true measure of agricultural efficiency, which figures in their turn will send the economists further down the road hunting for the social and other factors which lie behind them.

We all share Mr. Farrell's concern about his inability to introduce climatic factors into his analysis. I would suggest that in the present state of our knowledge we would probably do best by introducing climatic factors *a priori*. Some plant physiologists have reached the stage at which they can quantify their knowledge of plant requirements of moisture and temperature. I hope in later discussions with Mr. Farrell to be able to put before him some evidence about some approximate methods of quantifying the climatic element for purposes of international comparison. I certainly found his treatment of my international comparative data most valuable and I look forward to many interesting results in future. I have been puzzled for years about getting some three-cornered relationship between land, labour and agricultural product, and Mr. Farrell has opened up the hope of securing a valuable relationship.

The Cobb-Douglas approximation, which is still being ferociously debated by American statisticians, still gives a useful result. Many will be relieved to hear this, because it is certainly one of the simplest functions which we can hope to handle.

Mr. Farrell has warned us, on purely mathematical grounds, that we cannot use any functions which imply general decreasing or increasing returns to scale (as apart from diminishing returns to individual factors). It was that condition, I think, which prevented the fruitful application of the Cobb-Douglas approximation to United States manufacture, except for a very limited period. It will continue to limit the scope of this work.

As to the very crude methods of using weighted averages of costs, pioneered by G. T. Jones in the 'twenties, to which Mr. Farrell refers, I think every one of Mr. Farrell's criticisms is justified; but, nevertheless, Jones was searching for, and managed to find to some extent, a measure of the extent of increasing returns, in a field in which it is still impossible to use more refined functions. I think that we all regret that industrial statisticians have done so little over the last 30 years to improve on Jones's crude results, and in industrial as well as agricultural statistics we should be able to look forward to a lot of fruitful work in the future, started by Mr. Farrell's paper.

Mr. C. B. Winsten: Two aspects of Mr. Farrell's paper I found especially useful. The first is his emphasis on the importance of measuring more than one input when facing a problem of efficiency. Diagrams like diagram 2 of his paper would be very helpful in many such situations. The second aspect is the introduction of what he calls the "efficient production function" as a function to be fitted from empirical data. Most production functions in the past have either been fitted by regression methods, or by extensions of these methods by a simultaneous equation approach (for example, Marschak's and Andrew's work, "Random simultaneous equations and the theory of production" in *Econometrica*, 1944). All such regression methods estimate what are, in a sense, average production functions, or perhaps the production function of the average firm. It seems an interesting further question to ask what the production function of an efficient firm might be, and Mr. Farrell's method of estimating this function is ingenious. It would also be interesting to know whether in practice this efficient production function turned out to be parallel to the average production function, and whether it might not be possible to fit a line to the averages, and then shift it parallel to itself to estimate the efficient production function.

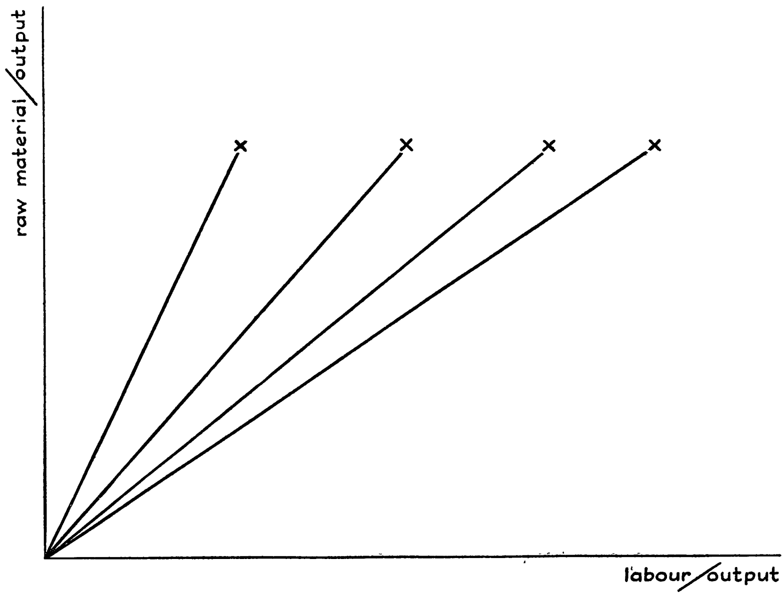
I do not, however, find Mr. Farrell's explanation of his index of technical efficiency convincing. He states that "when one talks about the efficiency of the firm one usually means its success in producing as large as possible an output from a given set of inputs". Mr. Farrell has suggested that he is dealing with an immensely wide field "from a workshop to a whole economy", and it is difficult to summarize all the sorts of questions of efficiency which may arise and difficult, therefore, to judge whether a particular index is "usually" useful. But all of them have this in common: it is necessary to specify the range of variation (of choice perhaps) open to the firm before one can set a standard of efficiency. The most familiar example is the distinction between short and long run problems. In the short run a factory management may have to work with the plant it has, and a short run standard of efficiency would be set accordingly. In the longer run it can change its plant, but whether it does so will depend on the conditions of its present plant. The standard of efficiency will be changed. Thus, any standard, or index, of efficiency depends on the possibilities which are considered open, and these should enter in an obvious way into the definition of indices.

Whether one can interpret the definition Mr. Farrell gives within this framework depends partly on how he defines his inputs, and what are to be included. All of us, but especially the journalists, trade unionists, politicians and the like, will have learnt that they must not take labour as the only input, but it is difficult to find in the paper what should be taken. Yet the fixing of such inputs help to define the constraints under which the factory manager is to operate, and the standards by which he is to be judged.

From the examples given the list of inputs to be used is wide. It includes raw materials and various forms of capital. It also includes quasi inputs such as position and seam thickness and weather. Even air and water are mentioned. But any plant manager would be astonished if his technical efficiency (his "need for gingering up") were judged by the amount of air he used. A usable definition of the inputs corresponding to actual standards to be set is an essential of any index, technical or otherwise.

Some of the inputs used are composite commodities; for example, capital might consist of different sorts of equipment. A method of determining how capital should be aggregated is given in the footnote on 3.1. The method suggested depends on the production characteristics of the equipment. But the way one aggregates equipment in this way affects the type of question one is asking. If, in diagram 2, two sorts of capital are aggregated to give one input, this means that while this aggregate input remains constant, the types of capital used may be varied. Thus the combining of two forms of capital in this way relaxes the restrictions under which the firm is supposed to produce its output. But such restrictions should depend on possibilities of change, not on the production characteristics of the equipment. Another difficulty is that what seem to be used as inputs in the paper can occur in actual problems in two roles. Sometimes they are in the control of the manager, and so might justifiably be called inputs, and sometimes they are, in the terms of the problem, fixed environmental factors which the manager cannot change. When they occur in the latter role they may affect the production function, and may be useful in helping to determine the production function empirically. But we should expect an index of efficiency to distinguish these two roles.

The following is an example, extreme but not far fetched, which suggests that Mr. Farrell's index does not answer one kind of question concerning efficiency in which we may be interested. Suppose we are studying a set of plants, each producting one output and, for the sake of simplification, using just two inputs to do this. We might call these inputs labour and raw material. Now suppose it happens that, whatever the plant, it requires a fixed amount of the raw material to give a given level of output. For example, the plant might be cutting up strip, and different plants have little variation in scrap or waste material. But suppose that these plants use very different amounts of labour. Then on a diagram like diagram 2, points might look like those shown on page 284; and the "technical efficiencies" of the firms in Mr. Farrell's sense will all be 100 per cent. even though some are far more wasteful in the use of their workers than others. Notice that the actual technique which they use may be the same. But in this case one would be interested in the waste of labour (labour productivity), a different comparison to the one implied by Mr. Farrell's index. The point will still arise if there are many inputs one or some of them being closely related to output. Then these will dominate the index. So Mr. Farrell's index does not answer the questions we are likely to ask in this case. But that is not surprising. To attempt to answer such a wide range of questions by an index is a very ambitious thing.



Mr. Farrell has contributed a lively paper which I think will stimulate further work on his subject. I have much pleasure in seconding the vote of thanks.

The vote of thanks was put to the meeting and carried unanimously.

Dr. A. J. HOFFMAN: I shall make a few comments about the proposed methods of computation described in 5.2. The principal problem to be solved is: given points $P_1, \ldots, P_n$ (including the infinite points) in m-space, to find the equations of the facets of the convex hull of the P's. This is demanding a great deal of information, and m and n do not have to be very large for the problem to become hopeless.

The second method discussed in 5.2 which, under the name of the "double description method", was proposed by Motzkin, Raiffa, Thompson and Thrall in 1950, is probably the best, but computational experience indicates that the number of potential facets that have to be considered, although they are eventually discarded, may become terribly large.

If one's ambition is more modest, and it is only desired to compute the efficiencies of some of the points $\{P_n\}$, then the following method seems best. It is essentially the first idea proposed in 5.2, combined with an intelligent way of proceeding to neighbouring facets. Indeed, there is no difficulty in principle of continuing until the efficiencies of all points and all efficient facets have been found.

Let $P_1, P_2, \ldots$ stand for the technologies of each firm and for the infinite points as well; then the problem is, for each P_k, to find the smallest value of λ such that λP_k is in the convex hull of all the P's. This number is the reciprocal of the maximum value of $x_1 + \ldots + x_n$, where $x_j \geqq 0$ and $P_k = x_1P_1 + \ldots + x_nP_n$. The latter is a problem of linear programming, and the dual simplex method of C. E. Lemke (*Naval Research Logistics Quarterly*, 1, 48–54) is particularly adapted to solve this problem. Without going into details of the method let me nevertheless illustrate some of its virtues.

It first solves the problem for P_1, and then permits an easy determination of all points P which have the same efficient facet as P_1 and computes their efficiency as well. Turning to the next point P whose status is still undetermined it proceeds "efficiently" through efficient facets to determine that facet which is efficient for this P, and so on. At each stage, all the points for whom the associated efficient facet is the one currently under examination are readily found and (if they are not on the efficiency boundary) can be easily discarded in subsequent computations.

Mr. STURROCK: I am a practical economist, not a statistician. One of the duties of my department is to provide measures of efficiency that can be used by the advisory services in agriculture. These measures may not be very elegant from the theoretical point of view but they must work in practice. It is, therefore, of particular interest to listen to Mr. Farrell's paper which deals with the statistical basis of such measures.

There are, first of all, three points that Mr. Farrell might like to take into account in improving his technique:

Firstly, I am a little sceptical of a system which depends on joining up the *best* results. If a large sample is used, individual results may be freakish. Not only are there errors but there are chance variations that have nothing to do with efficiency. Every business has good years and bad years depending on the state of the market, chance variation in prices and other factors over which the manager has no possible control. To call only these freakishly good results "100 per cent. efficient" would result in hanging the carrot too high and the donkey would be discouraged. It would be better to have either "average results" or 'premium results" (say the average of the upper 10 or 20 per cent.).

Secondly, the factors quoted by Mr. Farrell are not always homogeneous, and if care is not taken this may give rise to spurious rates of substitution. To give an example: Suppose we were quoting two farms in South Dakota and Iowa respectively. Suppose the first of these had 600 acres and a yield of 15 bushels an acre and the second had 150 acres and a yield of 60 bushels an acre, both, therefore, having the same total output. Suppose also that the first farmer uses 150 dollars worth of fertilizer and the second 600 dollars worth. If one takes acres of land as an input, one might conclude that 450 dollars worth of fertilizer was a substitute for 450 acres of land. In practice, the South Dakota farmer would be wasting his time if he put on more fertilizer because there is not enough moisture to give a bigger yield. It would be more accurate in a case like this to take the real estate value of land instead of acres. In this case, the output of these two farms is about the same and the real "input" of land in estate value might be about the same—not four times as great on one as the other as a consideration of acreage alone might lead one to suppose.

Thirdly, in moving from one State to another along Mr. Farrell's curves, the nature of the function is changing completely. For example, at the top of the curve, one may have a negro farmer in the deep South who is thinking of substituting a tractor for a mule in the production of cotton. In the middle of the curve, one may have a mid-West farmer who is wondering if he should buy a cornpicker instead of hiring one from a neighbour. This is part of hog production. At the bottom, we may be considering a Wisconsin dairy farmer who is wondering if he should buy a mechanical cleaner for his cowhouse. This is milk production. These three points, however, prepresent three different industries with no product in common. Does a curve joining these three points have any particular meaning? The same fault is apt to arise in the use of Cobb-Douglas functions. The input-output ratios are apt to be very mixed, and for that reason I am not sure that there is not more hope in the use of linear programming.

There is another more theoretical point. As every economist knows, the average cost curve, as output increases in a business, is normally U-shaped. Average cost per unit first falls owing to economies of scale, then rises as one reaches the full capacity of the existing equipment. The optimum level of production, however, is not the point of lowest average cost. In practice, it pays to increase production until marginal cost has risen to marginal revenue. In Mr. Farrell's first diagram, a firm that was expanding output might migrate from point P to point Q, then back again to point P as output reached the full economic level. Point Q might, therefore, represent a firm operating below optimum capacity. A business that appears highly efficient in diagram 1 may, therefore, be operating below the optimum output.

I must confess that this last criticism also applies to many of the efficiency factors that we use ourselves.

Finally, I should like to make a plea to Mr. Farrell to use his methods on data from British farms. There is ample information available. I mention these points not as a fundamental criticism of Mr. Farrell's paper but merely in the hope that he may use them in improving the techniques described. We are always very interested in research of this kind and would be among the first to adopt any technique that would give practical results.

Dr. EASTERFIELD: I have been quite fascinated by Mr. Farrell's line of approach. I was privileged to see some of this work while under development, and it struck me as an enormously useful addition to the equipment of tools for comparing the efficiency of firms and for seeking out the causes of efficiency variations in order to find ways of raising efficiency.

There is, however, one point of his approach which I should like to criticize to some extent. It seems to me to suggest that one can get a measure of efficiency which will answer all the questions and show which firms ought to get the medals and which need gingering up. But I do not believe it. It seems to me that some of the measures which Mr. Farrell has dismissed—for example measures of the productivity of labour—may be very useful for particular purposes. It may be well worth while for firms to know whether they are using much the same labour per unit of output as others. That they are using much more labour per unit of output is not necessarily a criticism, provided they know they are doing it and are doing it deliberately.

Again, the engineering type of efficiency, the comparison with what would theoretically be possible if everything went well, was dismissed as being psychologically depressing. But it may be useful to be given an idea of what is possible in an imperfect world. There is obviously a difference between knowing that you are running an engine at something like 50 per cent. of its capacity and knowing that in the cutting of a bar of steel you are using something like 10 to quite a high power times the work theoretically needed to make two surfaces. In the latter case the efficiency is quite irrelevant, but in the former it may be useful to know you are getting only 50 per cent. of the efficiency you might get, even if you know that no one has yet managed to get to 60 per cent.

I should like to take up a point implicit in Mr. Winsten's comments, which is that an important way of looking at efficiency is to assess it in terms of what one is trying to do. One may, for example, reasonably be required to consider what sort of fluctuations one may expect to be up against. If one plans a firm to be able to cope with changing factors of demand, and so on, it is extremely likely that one will pay insurance policies in some form, in labour, stocks or reserves, and to that extent incur costs and in a sense lower efficiency. But if in so doing the firm is enabled to survive then it seems to me that in another sense one is more efficient. I would, therefore, come back to the point that one must not use this method of efficiency, or any other method, as the sole means of knowing which is the best firm. In fact, the idea of looking for the best firm ought to be cut out. It has bedevilled all the ideas of productivity.

I should like finally to make a suggestion. Is it possible to get the collaboration, if not of the Census of Production then maybe of a trade association, to try to apply Mr. Farrell's methods in an industry where one could work with something a little more homogeneous than the land which the farmer uses? And, if so, might we not consider a factor which one or two writers recently have placed great stress on, namely, indirect costs of production? There have been suggestions that a great deal of indirect costs in the way of research and so on relate to ways of increasing efficiency or productivity (or something that ought to be increased, anyway). There have also been suggestions that this is quite wrong and that by having people on such work you are carrying a lot of passengers who are no doubt very busy but on quite irrelevant activities. An analysis on Mr. Farrell's lines that covered raw materials, labour, indirect costs of the type just referred to, and possibly other input factors, might give very interesting results.

Professor M. G. KENDALL (read by Mr. Quenouille): The method employed by Mr. Farrell is admittedly a rather elaborate one, and, as he points out, would strain the resources even of an electronic computer if many variables were involved. I have therefore considered whether a simpler approach might not be adequate. The following is based on one of the methods for measuring agricultural productivity which I discussed in a paper read to the Society in 1939.

If we take the figures in the first column "Land" of Table 1, and rank the 48 States according to them, we obtain a ranking of the numbers 1 to 48. The same procedure can be followed for the other three columns, and the rank numbers summed for the four variables, giving 48 rank sums. These numbers may then be used to arrange the 48 States in a descending order of "productivity" and the results compared with the ranking obtained from the final column of Table 2.

In the rankings obtained from Mr. Farrell's figures there are a number of ties but this does not seriously affect the comparison. I have carried it out, and find that with a few exceptions the ordering given by the ranking method is very similar to the one given by Mr. Farrell's. The Spearman correlation co-efficient between my ranking and Mr. Farrell's is about 0·76, and, if we exclude six anomalous values, is 0·92.

The anomalous values in question concern New Mexico, New Hampshire, Kentucky, Michigan, North Dakota, and Idaho, and I have looked at these individually. It seems to me that the ranking method gives results which on the face of it are at least as acceptable as those derived by Mr. Farrell. For example, in the case of North Dakota, my ranking makes it the 37th, whereas Mr. Farrell gives it an efficiency of 100 per cent.

On the individual variables, North Dakota is 42nd for land, 25th for labour, second for materials, and 48th for capital. The discrepancy seems to arise from an extraordinarily low figure on materials, and on the whole it seems to me that my ranking is a fairer reflection of the position than Mr. Farrell's, since he gives it an efficiency of 100 per cent., notwithstanding that it is the least efficient in the use of capital and nearly the least efficient in the use of land.

The ranking method, of course, purports to arrange the States of the Union in order and not to quantify the measurement of efficiency. If it is no more, however, it is an easy check on the more elaborate method, and from the examples I have given I think it may well prove to be more than that.

All this is without prejudice to the question whether productivity in the agricultural sense can be measured in this way. In my own work on the English crop counties—which, incidentally, also were 48 in number—I worked on the crop yields without reference to the factors of production. The same method has more recently been used by Professor Dudley Stamp in his book on *Land Today and Tomorrow* in arranging the countries of the world according to agricultural productivity.

Mr. QUENOUILLE: The point that is in part the basis of Professor Kendall's remarks involves the choice of scale for the input factors. Obviously, the measurement of technical efficiency for the more extreme points may be greatly affected by choice of scale.

For instance, suppose in the analysis of diagram 1 the logarithms of x and y are used. The lines through the origin are then parallel diagonal lines (with the slope of one) and the logarithm of the efficiency is the diagonal distance of the point P from the curve SS'. The whole approach of diagram 1 transforms simply.

The estimation of diagram 2 gives rise to SS' as a series of exponential curves joining the extreme points, and, at the ends, to exponential curves approaching the axes. This approach, probably, will not greatly affect the estimated efficiencies for the central point, but points near the axes may have completely different estimates of efficiency.

The following contributions were received in writing after the meeting:

Mr. J. A. C. BROWN: I should like to comment briefly on Mr. Farrell's remarks on estimation in section 3.6. Mr. Farrell's emphasis is placed mainly on assuring us of the efficiency of his procedure for estimating the efficient production function and he rather minimizes the importance of errors of observation. But in many applications interest will centre on the estimates of the relative efficiency of individual firms, and most economic statisticians would wish, I think, to attach some measure of error to these and to correct any possible bias. It is important at the outset to distinguish between the approach based on a hypothetical distribution of efficiencies from that based on errors of observation, particularly in relation to the size of sample considered. For on the first approach the *pessimistic* bias in the estimate of the efficient function will diminish as the sample size increases, whereas on the second approach the *optimistic* bias will increase with the sample size. I do not myself think that the assumption of a hypothetical parent distribution of efficiencies is appropriate here, since it involves a departure from Mr. Farrell's principle of comparison with an observed rather than a theoretical standard (section 2.2). This particularly applies if the sample consists of all the firms in an industry, or all the farms in a region. It is therefore probably better to consider the true efficiencies as given variables, but their measurement is subject to error and random disturbance because of accidental variations in the inputs of factors. It is true, as Mr. Farrell says, that these random disturbances are likely to be most important in agriculture, but it will be very difficult even in industrial applications for the statistician to impute inputs correctly to a particular product during a particular period.

I hope, therefore, that Mr. Farrell can be encouraged to develop this aspect of his method further, for it is not only of theoretical interest.

Lady HALL: In the course of making a comparative analysis of the structure of United States and British retail trade, based on the 48 American States, and Washington D.C. and also on the regional data for Britain, I and my colleagues* have sought for a measure of technical efficiency for use in appraising actual performance. Mr. Farrell's most provocative paper is an invitation to raise some of the conceptual difficulties involved which his paper sets aside.

Mr. Farrell's diagram 2 is based on the assumption that a unique measure of each output can be found. In the analysis of retail trade, the conceptual difficulties involved in measuring output soon become apparent. A choice can be made between including the goods sold as part of the output or regarding the output simply as a series of services to the consumer which will include "transformations" like breaking bulk, wrapping etc.; these transformations are analogous to the transformations of goods from place to place, the service output of the transport industry. The choice of a particular output measure commits us to a choice of inputs appropriate to the

* For a discussion of conceptual difficulties involved see Margaret Hall and John Knapp, *Productivity Measurement Review*, No. 8, February, 1957, pp. 24 and 25 (English edition).

supposed production function. If one includes the merchandise as an output then one must include the merchandise as an input but if one restricts oneself to the concept of output as services, this would be inappropriate. As far as I can see, there is, in the case of retail trade, no conclusive reason for preferring one of these output measures to the other, yet the use of different conceptions would seem to lead to different plottings on Mr. Farrell's diagram 2. He assumes, then, that a unique set of outputs can be derived but when there is a choice, as in the case just quoted, different measures of output yield different diagrams and therefore different results, according to the choice. This problem is present, surely in Mr. Farrell's own example of American agriculture: there are different ways of regarding agricultural production: for example, as a transformation involving feeding stuffs into milk and meat or as the production of milk and meat with feeding stuffs as an input. As an example, a farm may buy cattle and eventually sell them for meat. This could be treated as a transformation, or the cattle sold could be treated as on output in which case cattle bought would be an input. The lack of a uniquely defensible measure of output means that there will be not one but more than one point at which output is maximized for a given input.

Mr. Farrell's forcible general case for considering all inputs in measuring efficiency is associated with what are, I think, too general structures on the use of partial input measures, in particular labour productivity. The use of this measure has sometimes been dictated by lack of data in contexts where it is unsuitable. This should not blind us to the fact that in other contexts, it may well be the most appropriate and interesting measure. In retail trade, for example, the capital involved is mostly in the form of buildings and fixtures, forms of capital which are inflexible in the short or medium run. We should therefore only wish to bring in capital in thinking how to improve efficiency in retail trade in the long run. Moreover, wages cost is such a large proportion of the total expenses of retailing that, at the moment, the principal increases in efficiency often result not from substitution of capital for labour but from the redeployment of labour.

Finally, heterogeneity of environmental circumstances is inherent in retail trade. Neo-classical economics referred to this feature as the imperfectly competitive character of the retail market. In his treatment of economies of scale, relevant for example to questions of the optimum size of shop, Mr. Farrell assumes that the simple choice is between maximizing output for a given input and minimizing input for a given output. The presentation of these alternatives seems to be on purely formal mathematical grounds. In practice, the most efficient size of productive unit will be influenced by a number of exogenous environmental factors, for example, the size of shop which is suitable in Piccadilly would be too big in a small village. It is not clear how these environmental factors can be reduced to "quasi-inputs" as Mr. Farrell suggests. If, for example, higher *per capita* incomes mean that the average size of transaction is larger in one area than in another, this might mean that the amount of retail (service) output involved in selling one pound's worth of merchandise was less and sales per person engaged and sales per unit of total cost were higher. Before we can regard this factor as any sort of input, we have to be assured that it is a variable factor and not a constraint and this assurance is lacking. But even if we go along with Mr. Farrell in supposing it, it is not clear whether this factor is a quasi-input or a quasi-output. It reduces the cost of selling a given quantity of merchandise but it also reduces the work involved in selling a given quantity of merchandise. In any case, it is not clear how Mr. Farrell would suggest incorporating his ingenious quasi-inputs in his generalized efficiency measure. Consider, for example, a given increase in *per capita* income. The extra trade it brings about can easily be accommodated by shop A, costs are reduced, efficiency increased. Shop B, however, is on a smaller scale, cannot easily accommodate the increased custom, and incurs an increase in costs through overcrowding. There has been an identical increase in a quasi-input yet its effect on different production units has been different*; how is this quasi-input to be fitted in to the analysis?

The above examples give some idea of practical difficulties of a particular exercise.

Mr. Farrell (in reply): I should like to thank those concerned for the extremely helpful discussion. Mr. Clark has suggested that I cannot deal with increasing and decreasing returns to scale. This is not so—decreasing returns are quite easy to handle, while increasing returns, although more difficult, are by no means impossible.

The only point on which I disagree with Dr. Easterfield is his suggestion that he is disagreeing with me—I wholeheartedly endorse his positive comments. I fully agree with Mr. Sturrock about the importance of the heterogeneity of land inputs and, to a lesser extent, of agricultural output. As I said in my paper, I regard some solution of the former problem as essential to a definitive study of agricultural efficiency, and the use of real estate values in place of acreage, while, it has obvious drawbacks, may well be the best solution. I am grateful to Dr. Hoffman for his erudite contribution, but I hope the prospect is not quite so black as he paints it. In particular, I cannot see why the first method of section 5.2 must be restricted to "some" of the observed points.

* This idea was suggested to me by Mr. Michael Posner of the Oxford Institute of Statistics.

Mr. Winsten has raised a great many points. I shall attempt to deal with only two to-night. First, it is quite true (as indeed I emphasized in section 3.1) that the measure of efficiency depends on the inputs considered, and that the specification of the relevant set of inputs is therefore a necessary first step in any application of the method. But I cannot see that this constitutes any sort of objection to the validity of the measure itself. Secondly, while I agree that the measurement of capital is a difficult matter, I feel that Mr. Winsten's problem is a bogus one. So long as two types of capital equipment have the same production characteristics (in the sense of my footnote to section 3.1) it is irrelevant in what proportions they are used or whether these proportions can be varied.

Mr. FARRELL subsequently replied in writing as follows:

Mr. Winsten's example, where material inputs are (almost) the same for all plants, but labour inputs vary, is certainly spectacular and seems at first glance to raise an important objection to the method. However, on closer inspection it appears to be no more than a suggestion that, if certain additional technical information is available, a different way of splitting over-all efficiency into price and technical efficiency is possible and may have certain advantages. In diagram 1, the comparison of the observed point P with the optimal point Q yields the measure of over-all efficiency, OR/OP. This comparison I split into two parts, first a comparison between P and the point Q on the efficient isoquant that uses the factor in the same proportions, yielding the measure of technical efficiency, OQ/OP; and secondly, the comparison between Q and Q', yielding the measure of price efficiency, OR/OQ. Now suppose a line through P and parallel to Oy is drawn to cut Ox in U, AA' in V and SS' in W. Mr. Winsten suggests that, in his example, it would be better first to compare P with W, yielding technical efficiency UW/UP, and the W with Q', yielding price efficiency UV/UW. The final comparison is still between P and Q', and the over-all efficiency UV/UP is naturally equal to OR/OP. Thus, he splits the same measure of over-all efficiency so as to give what can easily be shown to be a smaller technical efficiency and a proportionately larger price efficiency.

Now my measure of technical efficiency can be interpreted as measuring how much more output could be obtained from the same inputs; that is to say, it was deliberately designed to avoid all the questions Mr. Winsten raises as to what variations in inputs are "possible" or "desirable". It is the measure of efficiency based on the most conservative assumption that no inputs may be varied, and is consequently the most charitable measure of technical efficiency. If additional technical information such as Mr. Winsten postulates, is available, one can of course construct a more stringent measure of technical efficiency, by asserting that certain variation of inputs may properly be demanded of a manager. In the extreme case, where all inputs are regarded as variable, technical efficiency will fall to over-all efficiency and price efficiency will rise to unity. It will be seen that such an extension of the concept of technical efficiency is rather a tricky business, and where possible one would prefer to stick to the original and quite unequivocal concept.

However, as Mr. Winsten suggests, cases may arise where the ratio of one input to output is so nearly constant that my measure of technical efficiency is unduly charitable, so that it is necessary to bring in extra information and define a more stringent measure. For the most part, this is likely to happen in cases like that cited by Mr. Winsten, where the process is a very simple one and all the plants use the same technique. Fortunately, the necessary technical information is likely to be easily obtained here, while in more complex cases, where the information is harder to come by, it is unlikely to be needed. A rather interesting case in point is Lady Hall's query as to whether the goods purchased and sold by a distributor should be counted as an input. As the correlation between purchases and sales is likely to be high (except in the case of perishables) this seems to be a good instance of the desirability of using this information to provide a more stringent measure; which can be done by simply omitting purchases from the list of inputs.

I found Lady Hall's note most refreshing, and the application of my method to distribution certainly raises some interesting problems. I think, however, her remarks on capital betray a misunderstanding of the problem. It is essential to bring capital into the computation of technical efficiency (which we have seen to be relevant in the very shortest run) because otherwise we shall make the mistake of regarding the most capital-intensive firms as the most efficient. (I was surprised to learn that in retail trade the capital is mostly in the form of fixtures, as I had always supposed stocks to be very important). The correct specification of quasi-inputs for an analysis of efficiency in distribution requires, of course, a good deal of thought, but I would suggest tentatively that differences of location could be treated by regarding the rental value of the premises as an ordinary input. I think it would be a mistake to treat the incomes of customers as a quasi-input. If they are regarded as simply producing an increase in demand, the paradox suggested by Mr. Posner is easily resolved—it is merely an illustration of the law of diminishing returns.

Before the increase in demand, B has its inputs in the optimal proportions but A is using too much space; after the increase, both increase their inputs of all factors save space, which is assumed fixed. This improves A's proportions but worsens B's.

I agree with Mr. Brown and Mr. Sturrock that the problem of random errors (discussed in section 3.6) is both difficult and of some importance. Indeed, I still hope that some theoretical statistician will take up the challenge offered by this knotty little problem. It is interesting that Professor Kendall's ranking method gives a similar ranking of states to my efficiency measure, but I fear he has missed the point of my paper. For I have attempted to produce a measure with a clear economic interpretation, not a method (as Dr. Easterfield would say) of handing out medals. I think the same point applies to Mr. Quenouille's ingenious proposal for a logarithmic transformation of the variables. I am most grateful for the interest these theoretical statisticians have shown in an economist's problems, but I must emphasize that the possibility of economic interpretation of any measure is vitally important. The field of economic index number theory is a veritable graveyard, so full is it of mathematically ingenious indices that are useless because they have no economic significance.

As a result of the ballot taken during the meeting the candidates named below were elected Fellows of the Society:

David Elliott Barton.
William James Chambers.
David Henry Davies.
Robert Joseph Deam.
Alexander David Threipland Deans.
Robert Dow.
George Wallace Henning.
Edward Francis John.
Kironde Kabunda.
Brendan Moore.
George Oyakhire.