ON ESTIMATING THE INDUSTRY PRODUCTION FUNCTION

By D. J. AIGNER AND S. F. CHU*

The purpose of this paper is to present an estimation technique which allows the economist to make a traditional interpretation of an empirically estimated microproduction function where the underlying production process is assumed to be deterministic. To be specific, we may estimate with mathematical programming a production function for the firm which can be "so defined that it expresses the *maximum product* obtainable from the (input) combination at the existing state of technical knowledge" [2a, pp. 14–15].

We shall argue here that the distinguishing features of firm production for a given industry may be embodied in attained values for certain technical parameters in an "industry" production function, differences in them reflecting relative scales of operation, varying organizational structures, etc. In the spirit of M. J. Farrell [5] [6] who constructs an envelope isoquant for the industry, the "industry" production function is conceptually a frontier of potential attainment for given input combinations. The production function for any particular firm may conceptually be obtained from the industry function in terms of the firm's ability to implement optimal values of parameters in the industry.

Of course the frontier concept is not new, but heretofore our available quantitative tools have forced an unusual amount of effort to be directed toward the interpretation of fitted functions—their meaning in light of the accepted theory, questions of identification of production functions versus factor demand equations and/or output supply equations, and the like. A justification for this effort is that regression analysis and the estimation techniques available for multi-equation systems depend upon the specification of stochastic terms with zero means. For the goal of fitting a function through a series of observations on firms for output and several inputs, this implies that an "average" function is obtained.

From the initial attempts of Reder [14] and Bronfenbrenner [2] through that of Marschak and Andrews [12] to the most recent literature (Hildebrand and Liu [9]), our theoretical notion of the "average" function and its meaning for the industry and its component firms has not matured appreciably.

* University of Wisconsin and University of Nevada. This paper was written while the authors were associate professor of economics and graduate student in economics, University of Illinois. We wish to thank G. Judge and A. Goldberger for helpful comments on an earlier draft of the manuscript.

Some econometricians, for instance, seemingly adopt the "average" function as the correct *conceptual* construct with persuasive arguments about "sustained" versus "unusual" output. While such an interpretation has obvious appeal for measurement purposes, the necessary theoretical development which derives from this construct has not been forthcoming. The fact remains that the frontier production function presently forms the core of microeconomic theory, and, in the sense that our marginalism is based on instantaneous, single adjustments rather than persistent ones, appears to be a reasonable goal to pursue empirically.

By applying the techniques of mathematical programming to cross-section data on firms we may produce the envelope function by controlling the disturbance term (in either a single or simultaneous equation setting) to be of one sign only. For a linear programming formulation the objective appears as the sum of such disturbances (a linear loss function), while for quadratic programming the criterion is a minimum for the sum of squared residuals (a quadratic loss function). No presumptions need be made about returns to scale for the industry function, as was necessary in the Farrell work. The empirical assumption required for a programming application is that disturbances are of one sign, i.e., that observed points in the production space lie on or below the frontier only. So long as measurement error may be neglected such a specification can be justified.

However, the estimation potential of these techniques is reduced to some extent by a lack of available statistical inference procedures for use in discriminating between functional forms, among variables, etc. Indeed, our old standbys in classical single-equation regression—R^2, t-ratios, and the like—are of little if any use to the model-builder who builds frontiers. Moreover, it is easy to argue that such estimation must be biased in every dimension of the production space, analogous to the sample maximum as an estimator of a population maximum.

This paper, therefore, aims toward a primarily provocative goal. We begin by outlining an empirical framework within which the frontier production function is observable. The empirical problem is then cast into a mathematical programming mold for both single and multiequational settings. Finally, an application—to compare the programming estimates with some work of Hildebrand and Liu [9]—is presented. The concluding remarks focus on implications of the paper for further extensions of the frontier concept, both empirical and theoretical.

I. *The Industry Production Function*

In microeconomic theory a firm's production function is usually defined in conjunction with a given state of the arts and expresses the *maximum product* obtainable by the firm from a given combination of factors during the (assumed) short period of time required to produce this output. It is assumed that inputs are applied at some point in time and output appears at some advanced point in time. The notion of continuous production is not considered within the traditional framework, except as it may be approximated by a discrete-time process. In this context, the production function sets the highest possible limit on the output which a firm can hope to obtain with a certain combination of factors at the given state of technical knowledge during the production period. This maximum output applies not only to the particular firm of interest; conceptually it holds for all other firms in the same industry. We might call the function so defined the *industry production function*, to be distinguished from the industry's aggregate production function, which expressed the relationship between aggregate output and the aggregate inputs of that particular industry.

Possibly all and certainly many constituent firm outputs lie below the frontier for a variety of reasons:

1. *Due to pure random shocks in the production process*. For example, some parts of a product may be damaged through careless handling; or, some products are defective, etc.

2. *Due to differences in technical efficiency*. One reason for such differences follows from varied holdings of capital equipment, both in quantity and vintage. Large firms are usually in a better position to replace old equipment than smaller firms, either because of superior self-financing systems or from advantages in obtaining credit terms. And new equipment generally reflects technical improvement. As the composition of capital equipment differs from firm to firm and large firms tend to possess more new equipment, the large firms are generally more efficient. Similarly, there exist efficiency differences in the labor input; for instance, large firms may afford to divert resources to the improvement of labor efficiency while small firms cannot.

3. *Due to differences in economic efficiency*. Given a production function and the market situation, the firm should produce a certain level of output so as to maximize its profits. Such a maximization procedure simultaneously determines the level of output produced and the levels of inputs used. Whenever there is a change in the market situation, the levels of output and inputs must adjust accordingly to assure profit maximization. However, the ability to make such adjustments can hardly be expected to be equal for all firms. Presumably the higher a firm's economic efficiency the higher the level of output that can be achieved for a given input combination.

The technical differences in (2) above are assumed to be manifested in the attained values of technical parameters in the industry production function. Hence, although the individual production functions of firms are of like form, their technical parameters differ. This realized production function we call the *firm production function.*

In the early interfirm and intrafirm production function discussion, Reder [14] seems to have had these ideas in mind. He first assumed that the production function "is known with complete accuracy, i.e., the values which its parameters take are not affected by random fluctuations" [14, p. 259] and then began to talk about the so-called interfirm production function, which obviously presupposed the existence of differences in the individual firm functions even though the quotation implies that all firms have the same production function. This latter idea is analogous to what we have termed the *industry production function.* The different production functions implied by the concept of the interfirm function are equivalent to what we have called *firm production functions.* Reder's main position seems to have been a justifiable one. However, presumably because of the lag in methodological development which prevented Reder from estimating the industry and firm functions empirically, he remained silent to the criticism of inconsistency leveled against him by Marschak and Andrews [12]. Yet from the above analysis it is clear that actual observations on output should conceptually lie below the true production surface, since had there been any level of output obtained with a given combination of inputs higher than that allowed by the production function, a new state of arts should have been defined.

To date, work in the area of microproduction function estimation assumes that the firm's actual output may either be greater or smaller than the industry production function permits. A group of economists did notice the obvious conflict with theory, however, and some rationalization of this position was attempted.[1] What they did was to assume that the function to be estimated, i.e., the conceptual construct, is an "average" production function for the industry. Some firms could therefore produce more than the average; some, less. But the meaning of such an "average" function is not necessarily clear. Average in the sense of what? a conditional median? a mean? or, a mode?[2] More importantly, average *about* what? about output? about some input? about technology? or about something else? Some economists refer to it as the function for a "firm of average size." This interpretation cannot be correct unless it is assumed that the parameters of the function are random variables and have their expectations equal to those of the firm of "average size." Others seem to refer to the average function as reflecting

[1] See, e.g., Bronfenbrenner [2], Marschak and Andrews [12], and, more recently, Nerlove[13].

[2] For example, in a recent paper, Arthur Goldberger [7] has pointed out the common negligence in interpreting the fitted Cobb-Douglas form as a conditional mean output, when, in fact, it provides the conditional median.

some sort of "average technology." But it would be infeasible to assume that a firm which possesses "average technology" with respect to capital also has an "average technology" with respect to labor. Such a coincidence is even less likely when factors are treated in more definitive categories.

Another appealing interpretation is that "true" productive capacity only makes sense as an output level which can be sustained. The "average" production function, devoid of random fluctuations due to "lucky" coincidences of good weather, sunspots, etc., and "unlucky" ones as well, is thus promoted as the function of sustained output. But the notion of sustained output has long-run overtones which tend to invalidate this argument for the average function in a short-run context. Production processes which depend on uncontrollables like weather as explicit inputs (or catalysts) may be of a different conceptual nature than those of industry, for example, and to the extent that they exist (as in some forms of agricultural production) the frontier concept is unacceptable, but *only* because it assumes inputs to be "variable" in the short run in the sense that the firm may control them.

From a more practical standpoint, if, for example, we wish to estimate how much output *on the average*, could be obtained for a firm in the industry with a certain set of inputs, then the "average" concept would obviously be the correct one to employ. Another very important use of the average function which previously has been overlooked is that in some cases we can approximate the industry's aggregate production function when aggregate data cannot be obtained but data at the firm level are available; or, we can approximate an "average" firm production function when we have data only on industry aggregates. The latter point is especially important because in practice data at the firm level are usually not available. Hildebrand and Liu's [9] belief that observations for some "representative establishments" could be used to estimate U.S. manufacturing production functions is consistent with the first part of the above statement.

Despite these uses for the "average" production function, use of the frontier function is appropriate in order to ascertain the maximum productive capacity of an industry (for purposes of planning, for example), of measuring the potential output of an economy, etc. Along these same lines, M. J. Farrell [5] [6] defined an "efficient production function" which resembles our industry function, and devised an ingenious way of estimating this efficient production function through constructing isoquants. The difficulty with Farrell's method, however, is that it is not general enough. Many types of production cannot be characterized within his models. For example, it is not possible to estimate a production function with his method which conforms to the Law of Variable Proportions.

In the following section we attempt to indicate how familiar mathematical programming methods may be used in some cases where Farrell's method would fail to obtain the required production surface. In terms of the generality of approach, while we can alleviate the inherent difficulties of assuming something in advance about the shape of the expansion path, our methods apply only to single output situations.

II. *Programming Methodology*

To begin, let us consider only the effects of random shocks on the production process; all differences in technical efficiency are subsumed within the disturbance term. Errors of measurement in all variables are also assumed negligible. For simplicity we take a one-output, two-input Cobb-Douglas model over firms.

$$x_0 = A x_1^{\alpha} x_2^{\beta} u, \tag{3.1}$$

with

$x_0 =$ output

$x_1, x_2 =$ inputs

$u =$ random shock

$A, \alpha, \beta =$ parameters

Our problem is to obtain an estimated function

$$A x_1^{\hat{\alpha}} x_2^{\hat{\beta}} = \hat{x}_0 \tag{3.2}$$

such that

$$A x_1^{\hat{\alpha}} x_2^{\hat{\beta}} \geq x_0 \tag{3.3}$$

If we take logarithms of both sides of (3.1) and rewrite equations (3.2) and (3.3) compactly in matrix notation, we have:

$$X_0 = XC + e \tag{3.4}$$

$$XC \geq X_0, \tag{3.5}$$

where

$X_0 = \log x_0,\ X_1 = \log x_1,\ X_2 = \log x_2,$

$X = [1\ X_1 X_2]$

$C = [A \quad \alpha \quad \beta]'$

$e =$ vector of measured residuals

One way to approach the estimation problems imposed by equations (3.4) and (3.5) is to minimize the sum of squared residuals ($e'e$) subject to (3 5), in which case we formulate the problem as follows, in its form for the general linear production model (positive coefficients on inputs):

To find the positive vector C[3] that minimizes the quadratic function

$$(3.6)\quad e'e = (X\hat{C} - X_0)'(X\hat{C} - X_0) = \hat{C}'X'X\hat{C} - 2\hat{C}'X'X_0 + X_0'X_0$$

subject to:

$$X\hat{C} \geq X_0,$$

(3.6) poses a typical quadratic programming problem and may be solved by Wolfe's algorithm [16].

Since the shocks lie only on one side of the industry production frontier, we may also treat the estimation problem easily within the framework of linear programming. In other words, we may minimize the sum of residuals, as a linear loss function, rather than the sum of squared residuals. The problem may then be written as follows:

To minimize

$$(3.7)\qquad l'e = l'(X\hat{C} - X_0), \qquad \text{where } l' = [1\ 1 \cdot \cdot \cdot 1]$$

subject to:

$$(3.8)\qquad \begin{gathered} X\hat{C} \geq X_0 \\ \hat{C} \geq 0 \end{gathered}$$

It is to be noted that for some purposes the linear programming estimator may be superior to the quadratic programming estimator since the criterion function is less influenced by extreme values. However, the quadratic programming method provides a convenient treatment when the simultaneous equation model is considered, as we shall see below.

Next, let us attempt to introduce technical efficiency into the model in such a fashion as to make the identification of individual firm functions possible within the context of the industry production function. One way to treat interfirm differences in technical efficiency is to regard them as a part of the disturbance term, as we have above, following Marschak and Andrews [12]. Another approach is adopted by Hildebrand and Liu [9] (hereafter referred to as H-L). These authors attempt to separate the effects of technical efficiency from shocks on theoretical grounds, in an effort to allow only the shocks to influence estimates. The advantage of their method is that (1) if the theoretical specification is correct, then unbiased estimates may be obtained,[4] and (2) individual

[3] Here we would allow the constant term to be of either sign by redefining X as $X = [-1\ 1\ X_1\ X_2]$, and C as $C = [A_1\ A_2\ \alpha\ \beta]'$. Then the algorithm is free to select A_1 or A_2 (both positive). If A_1 is chosen ($A_2 = 0$), the constant term is negative.

[4] Under the assumption E (log u) = 0. In a review of the H-L work McFadden [11] demonstrates that their empirical specification is *incorrect*, but that corrections to recover estimates of model structural parameters are available.

firm functions may be derived because technical differences have been specified explicitly.

If we let R be the ratio of value of equipment to plant, and r be the ratio of number of technical personnel to production workers, we may use R and r as proxy variables for describing the technical level of firms in utilizing the capital and labor inputs, respectively. One Cobb-Douglas production function used by Hildebrand and Liu is

$$x_0 = A x_1^{\alpha \log r} x_2^{\beta \log R} u, \tag{3.9}$$

where x_1 and x_2 represent the labor and capital inputs, respectively.[5] If data on R and r can be obtained, estimation may be carried out as described in the previous model. Also, since r and R vary over firms, the firm production function can be derived from the industry function by appropriately adjusting these proxy variables.

However, actual output is not only determined by a technical relationship with inputs, it also is affected by an economic adjustment process given some market situation for the procurement of inputs and sale of output. If we assume perfect competition, profit maximization for the firm would yield the following system of simultaneous equations, containing the production function and two derived demand equations for factors, again using the Hildebrand-Liu formulation as a basis for the exposition:

$$x_0 = A x_1^{\alpha \log r} x_2^{\beta \log R} u \tag{3.10}$$

$$p_1 x_1 = \alpha(\log r) p_0 x_0 + v_1 \tag{3.11}$$

$$p_2 x_2 = \beta(\log R) p_0 x_0 + v_2 \tag{3.12}$$

where

p_0 = price of output

p_1, p_2 = prices of first and second input, respectively

$E(v_1) = E(v_2) = 0$

$E(v_1 v_1') = \sigma_2^2 I,\ E(v_2 v_2') = \sigma_3^2 I$

If we let

$x_0' = \log x_0$

$x_1' = \log r \log x_1$

[5] McFadden also points out that these proxies for technological differences between firms, *as they are used in the Cobb-Douglas function,* may not be adequate over the range of input levels on a priori grounds: for instance, $x_1^{\alpha \log r}$ is not monotonic in $\log r$ over $x_1 \geq 0$, but only over $x_1 \geq 1$.

$$x_2' = \log R \log x_2$$
$$z_1 = (\log r)p_0x_0$$
$$z_2 = (\log R)p_0x_0$$
$$y_1 = p_1x_1$$
$$A' = \log A$$
$$u' = \log u, \text{ with } E(uu') = \sigma_1^2 I$$
$$T = \text{number of observations}$$

then equations (3.10)–(3.12) can be rewritten as:

$$W = ZC + U, \tag{3.13}$$

where

$$W = \begin{bmatrix} x_0' \\ y_1 \\ y_2 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & x_1' & x_2' \\ 0 & z_1 & 0 \\ 0 & 0 & z_2 \end{bmatrix} \quad C = \begin{bmatrix} A' \\ \alpha \\ \beta \end{bmatrix} \quad U = \begin{bmatrix} u' \\ v_1 \\ v_2 \end{bmatrix}$$

For our purposes the simultaneous setting presents the additional problem of specifying $\Sigma = EUU'$, the variance-covariance matrix of disturbances, for use in constructing the appropriate quadratic loss function. In Zellner's work [17], for example, the Aitken generalized estimator is used for efficiency purposes. This specification, of a full Σ matrix, has no known efficiency properties under the restrictions we impose. Considering the nature of the disturbance u',$E[(u')(u')'] = \sigma_1^2 I$ has no particular significance for the loss function and also cannot, apparently, be estimated from sample data without specifying $E(u')$. Moreover, one can quite reasonably argue for independence of the inputs with output (or value added) in production models of this sort, so that the production function may be considered alone for purposes of estimation.[6]

In light of this discussion, if a simultaneous determination of coefficients is pursued Σ must be specified in order to produce the quadratic loss function. Two alternatives seem plausible, but on intuitive grounds only: $\Sigma = \mathrm{I}$ (where I is $3T \times 3T$), and

$$\Sigma = \begin{bmatrix} kI & 0 & 0 \\ 0 & \sigma_2^2 I & 0 \\ 0 & 0 & \sigma_3^2 I \end{bmatrix}$$

[6] Cf. Zellner, *et al.* [18, p. 787]. The Zellner paper considers production as stochastic a priori (as contrasted to our deterministic specification), but the key point is specification independent: that the "effect of the disturbance on output cannot be known until after the preselected quantities of inputs have been employed in production."

where k is an arbitrary weighting factor. With S a sample estimate of Σ (or $S=\mathrm{I}$), our estimation problem is:

To minimize

$$U'U = (W - Z\hat{C})'S^{-1}(W - Z\hat{C}) \tag{3.14}$$

subject to

$$X\hat{C} \geq X_0$$
$$\hat{C} \geq 0.$$

As with (3.6), (3.14) should be viewed as being specified for a general linear system of equations.

III. *An Example of Use*

To illustrate the use of the programming methods, some empirical results are given here. The production function is version III in Hildebrand and Liu [9; cf., p. 65], as applied to the primary metals industry.[7] The complete model in their notation is as follows:

$$\log V = \log A + b_0 \log L + e_0 \log R_{-1} \log K_{-1} + \log u, \tag{4.1}$$

the production function, and

$$\log L = \lambda \log B + (1 - \lambda) \log L_{-1} + (1 - m)\lambda \log V - (1 - g)\lambda \log W - \lambda \log z \tag{4.2}$$

the labor demand function.[8]

Three versions of the programming method are used to produce estimates of the coefficients in (4.1): linear programming (LP) on (4.1), quadratic programming-single equation (QP1) on (4.1), and quadratic programming-simultaneous equation (QP2) on the two equations.[9] The results from such application, together with estimates derived from single equation least squares (1SLS) and two-stage least squares (2SLS) are tabulated below. The data used were for 1957–58, as it was not possible to reproduce perfectly the H-L data for 1956–57. The original H-L results are included in parentheses.

The table is presented in terms of H-L's original empirical work. However, McFadden's criticism that H-L ignored the effects of intermediate good input and the output demand structure on b_0 and e_0 directs us to modify our procedures for computing the last four columns

[7] H-L's data are state aggregates. We use this example for illustrative purposes primarily and ignore the obvious problems of specification and interpretation of the aggregation process on a framework developed for microdata.

[8] L is labor, K_{-1} is the lagged (one year) value of capital, V is value added for output, and W is the wage rate; z is a random term and λ is specified as describing the speed of adjustment of the actual demand for labor to its optimal (profit maximizing) value.

[9] In the QP2 results S was taken to be an identity matrix.

from b_0 and e_0. We will not, for purposes of this example, recompute all the relevant quantities. However, as an indication of changes in them, corrected estimates for the 2SLS row are: $h=4.13$, $MPP_L=1.005$, capital-output elasticity $=.021$, and technology-output elasticity $=.084$. In general McFadden's results suggest that H-L's estimates of the scale factor for the two-digit industries considered are high. The conclusion which follows (as contrasted to H-L's) is that no evidence is present which would indicate other than constant returns to scale in these manufacturing industries.

Reviewing the tabled results, the labor-output elasticity (b_0) is highest for QP1 (1.071) and lowest for QP2 (.822). It seems that the estimates from all five versions are plausible. The marginal physical product of labor per dollar of wage costs (MPP_L) is again highest for QP1 (1.822) and lowest for QP2 (1.399). Following Hildebrand and Liu, the marginal revenue product of labor (ϕ) is specified at 0.8, which leads to derived estimates of the demand elasticity for output (h): (h) is

Empirical Results for Five Estimation Methods

	b_0 (Labor-Output Elasticity)	e_0	MPP_L	h[a] (Output Demand Elasticity)	Capital-Output Elasticity[b]	Technology-Output Elasticity[b]
1SLS	.908	.0333	1.546	2.072	.1278	.5115
(III.A.1)[c]	(.988)	(.0343)[d]	(1.703)	(1.886)	(.1318)	(.3351)
2SLS	.917	.0321	1.560	2.053	.1232	.4931
(III.A.2)[c]	(1.000)	(.0326)[d]	(1.724)	(1.865)	(.1251)	(.3143)
LP	.873	.0031	1.485	2.168	.0132	.0165
QP1	1.071	.0269	1.822	1.783	.1148	.1441
QP2	.822	.0219	1.399	2.336	.0935	.1173

[a] Marginal revenue product of labor per dollar of wage cost (ϕ) assumed at 0.8.
[b] $\ln R_{(\max)}$ used for LP, QP1, QP2; $\ln \overline{R}$ for 1SLS, 2SLS.
[c] H-L's designation, cf. [9, p. 93].
[d] H-L's results are reported in terms of common log transformations, while we have used the natural system. The base used is immaterial (so long as it is consistently used) in obtaining the coefficient b_0, but the capital coefficient is affected by the base of the logarithms used. The translation is simple: for the equation fitted in common logarithms, the counterpart of the coefficient e_0 if data is transformed by natural logs is $e_0' = e_0 \cdot \log_{10}(2.718282) = .434294\, e_0$. Hildebrand and Liu are apparently consistent in their use of common logarithms for the empirical results, but overlook the above scaling problem elsewhere (although it does not seem to cause any invalidation of results). For example [9, p. 50, footnote 8] there is the following:

$$\frac{d}{dR}(e_0 \log R) = \frac{e_0}{R}$$

If log R is a common logarithm, as they use in the empirical work, then

$$\frac{d}{dR}(e_0 \log R) = \frac{d}{dR}(e_0 \log_{10}(2.718) \ln R) = \frac{e_0 \log_{10}(2.718)}{R} = \frac{.434294\, e_0}{R}.$$

largest for QP2 (2.336) and smallest for QP1 (1.783). In view of the nature of this industry, these (indeed, all) values appear to be high, which suggests a smaller value of ϕ should be used.[10]

The significant difference between programming and regression results appears in the capital coefficient and derivatives of it. The capital-output elasticities for LP, QP1, and QP2 are respectively .0132, .1148, and 0.935, while those for one- and two-stage least squares are .1278 and .1232. The technology-output elasticity for LP is .0165; for QP1 and QP2, .1441 and .1173, respectively; for 1SLS and 2SLS, .5115 and .4931.

Using the proxy variable R as an indicator of technological efficiency within the firms of the primary metals industry, our industry production function (from QP2) would be given by

$$
\begin{aligned}
\text{(4.3)} \qquad \ln V &= .8221 \ln L + .0219 \cdot 4.2673 \ln K_{-1} + \text{constant} \\
&= .8221 \ln L + .0935 \ln K_{-1} + \text{constant}
\end{aligned}
$$

where the maximum value for $\ln R$ (for the state of Louisiana) is inserted into the general form (4.1). This function would provide the maximum possible output (value added) from the labor and capital inputs given the state of the arts in the industry. Clearly, any individual function (in this case, for a state) may be obtained by substituting the attained value of $\ln R$ into (4.3).

Without an attempt to identify technological differences among firms explicitly, as with the proxy variables r and R, we still may, of course, obtain the industry production function in a simultaneous equation setting. However, individual firm (micro) functions may not be concomitantly estimated.

IV. *Final Remarks*

A viable distinction between the average and frontier functions as predictors of capacity (and this is not at all intended to give ground in the theoretical conflict) derives from a probability interpretation of alternative forecasts. On the one hand, the frontier function forecasts an "unlikely" event (output level), whereas the probability attached to the output level given by the average function is higher. A fitted Cobb-Douglas function provides the conditional median output. So 50 per cent of firm outputs for a selected input combination should lie above that output predicted by the fitted Cobb-Douglas function. Of course, these remarks hold in the short run only: The frontier function under a fixed technology gives that output which, in the short run, only a few firms at most can produce for any given input combination (neglecting

[10] The McFadden work suggests that even a wider discrepancy exists in adjusting ϕ so the demand elasticity becomes more "reasonable."

scale differences). Allowing a longer run, as implied by the sustained output idea, but still assuming a fixed technology, more firms presumably become able to produce at the maximum level; under this circumstance, therefore, the frontier function should be the relatively better capacity predictor.

Finally, we have attempted to interpret the traditional theory strictly, in that the frontier we construct is truly a surface of maximum points. Under a different goal one may pursue less than 100 per cent frontiers using the chance-constrained programming ideas of Charnes-Cooper [3], where (3.3) would be translated into a probability statement, $Pr(\hat{A}x_1^{\hat{\alpha}}x_2^{\hat{\beta}} \geq x_0) \geq \Upsilon$, with Υ a specified minimum probability with which the statement is to hold.

References

1. V. Ashar and T. D. Wallace, "A Sampling Study of the Minimum Absolute Deviations Estimator," *Operations Research*, Sept.–Oct. 1963, *11*, 747–58.
2. M. Bronfenbrenner, "Production Functions: Cobb-Douglas, Inter-Firm, Intra-Firm," *Econometrica*, Jan. 1944, *12*, 35–44.

2a. Sune Carlson, *A Study on the Pure Theory of Production*. New York 1956.

3. A. Charnes and W. W. Cooper, "Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints," *Operations Research*, Jan.–Feb. 1963, 18–39.
4. G. Dantzig and A. Orden, "Duality Theorems," RAND Report R. M. 1265.
5. M. J. Farrell, "The Measurement of Productive Efficiency," *Jour. Royal Stat. Soc.*, Ser. A, 1957, *120*, Pt. 3, 253–81.
6. ———, and M. Fieldhouse, "Estimating Efficient Production Under Increasing Returns to Scale," *Jour. Royal Stat. Soc.*, Ser. A, 1962, *125*, Pt. 2, 252–67.
7. A. Goldberger, "On the Interpretation and Estimation of Cobb-Douglas Functions," *Econometrica*, forthcoming.
8. J. M. Henderson and R. Quandt, *Microeconomic Theory*. New York 1958.
9. G. Hildebrand and T. Liu, *Manufacturing Production Functions in the United States, 1957*. Ithaca 1965.
10. S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Vol. I, Reading, Mass., 1959.
11. D. McFadden, "Review of Hildebrand and Liu," *Jour. Am. Stat. Assoc.*, June 1967, *62*, 295–300.
12. J. Marschak, and W. Andrews, "Random Simultaneous Equations and the Theory of the Production Function," *Econometrica*, July–Oct. 1944, *12*, 143–205.
13. Marc Nerlove, *Estimation and Identification of Cobb-Douglas Production Functions*. Chicago 1965.

14. REDER, M. W., "An Alternative Interpretation of the Cobb-Douglas Production Function," *Econometrica*, Oct. 1953, *11*, 259–64.
15. T. TAKAYAMA, AND G. JUDGE, "A Method for Handling Inequality Restrictions in Regression Analysis," *Jour. Am. Stat. Assoc.*, March 1966, *61*, 166–81.
16. P. WOLFE, "The Simplex Method for Quadratic Programming," *Econometrica*, July 1959, *27*, 383–98.
17. A. ZELLNER, "An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias," *Jour. Am. Stat. Assoc.*, June 1962, *57*, 348–68.
18. ———, J. KMENTA AND J. DRÈZE, "Specification and Estimation of Cobb-Douglas Production Function Models," *Econometrica*, Oct. 1966, *34*, 784–95.