UNIVERSITÀ
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DEPARTMENT OF ECONOMIC AND BUSINESS SCIENCES

MASTER'S DEGREE IN FINANCE AND RISK MANAGEMENT

DATA ENVELOPMENT ANALYSIS
AND ITALIAN MUTUAL FUNDS

Author:
Walter Giannattasio

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1 - INTRODUCTION

1.1 - Evaluation of mutual funds

I Mutual funds are usually evaluated based on past returns, which, however, do not in any way guarantee future performance.
In academic literature, several indices have been developed to assess performance without relying solely on fund returns. These tools inherently take other variables into account to achieve a more comprehensive and balanced evaluation.
The most widely used indices are:
-Sharpe Ratio(Sharpe, 1966);
-Treynor Ratio(Treynor, 1964);
-Jensen's Alpha(Jensen, 1968)

These financial performance evaluation tools measure the portfolio's expected excess return relative to a benchmark while accounting for the associated risk. However, each adopts different methodologies and approaches.
The Sharpe and Treynor ratios represent, respectively, the relationship between the portfolio's expected excess return and its volatility (measured by standard deviation) or systematic risk (beta). In this way, they condense both expected return and risk into a single indicator.
Jensen's alpha, on the other hand, is not a ratio but a difference: it measures the actual excess return achieved by the portfolio compared to the expected return, calculated based on systematic risk using the CAPM model. It is distinguished by evaluating the portfolio's overperformance or underperformance without directly relating expected return to risk but rather isolating the excess return relative to a theoretical benchmark.

The limitations in evaluating portfolio performance through these indices are:

1. The appropriate choice of benchmark for comparison;
2. The role of market timing;
3. The exclusion of transaction costs consideration;

Furthermore, although they represent an improvement over analysis based solely on the returns achieved, they continue to ignore the subscription and redemption fees associated with the investment. While these fees are indirectly related to the fund's share value, they contribute to determining the overall return.
To include the entry fees of the investment, one can use a performance measurement methodology that allows for the evaluation of the efficiency of a decision-making unit in the presence of multiple inputs and outputs: Data Envelopment Analysis (DEA), proposed by Charnes, Cooper, and Rhodes in 1978. Initially developed to measure the relative efficiency of public sector activities and nonprofit organizations, it was later applied to many profit-oriented companies, such as bank branches. Various applications of this technique are presented in Charnes et al.(1994)

The DEA methodology measures relative performance without requiring the specification of a benchmark and incorporates transaction costs. It will thus be used to define the efficiency of mutual funds, considering multiple inputs, particularly various risk measures and investment costs.
Pioneers in applying DEA to the financial sector include Murthi et al.(1997), who, through the DPEI (DEA Portfolio Efficiency Index), incorporated investment costs into the definition of a mutual fund performance index. The DEA technique also allows for identifying, for each inefficient fund, a corresponding set of efficient funds (the peer group), which represents a composite portfolio viewed as a specific benchmark and characterizes the portfolio's style.
Finally, an empirical application was carried out using data from the Italian financial market to test the applicability and properties of the proposed DEA indices and to compare the results with those obtained using traditional performance indices.

1.2 - Introduction to the concept of efficiency

The concept of efficiency refers to a company's ability to optimally use the available resources, i.e., inputs, to generate outputs. According to Farrell(1957), overall efficiency can be broken down into two fundamental components: technical efficiency and allocative efficiency.

Technical efficiency focuses on a company's ability to maximize the level of output from a given set of inputs, considering the substitution ratio between those resources. In the economic literature, two main definitions of technical efficiency emerge:

-The first definition, proposed by Koopmans(1951), considers a producer to be technically efficient when it is not possible to improve the production of one output without compromising another output or increasing the use of at least one input. Similarly, a producer is efficient if it is impossible to reduce the use of one input without increasing the use of another input or decreasing the production of an output;

-The second definition, introduced by Debreu(1951) and later expanded by Farrell(1957), is known as the Debreu-Farrell measure. In this perspective, input-oriented technical efficiency is defined as the complement to one of the maximum proportional reduction of all inputs needed to produce a given level of output. Conversely, output-oriented technical efficiency is based on the proportional maximization of all outputs while keeping the level of inputs constant.

Allocative efficiency refers to a company's ability to combine inputs and outputs in optimal proportions, considering the prices or weights associated with the available production factors. It reflects the company's ability to maximize the outcome of the production process by making the best use of the opportunities offered by the market, such as savings from purchasing inputs or selling outputs, which allow the company to obtain more resources for the same monetary expenditure.

Also, according to Farrell(1957), technical efficiency can be calculated by assuming some simplifying assumptions:

- The company uses exclusively two factors of production, labeled as x and y, to obtain a single unit of output;
- The returns to scale are considered constant;
- The efficient production function is known and defined.

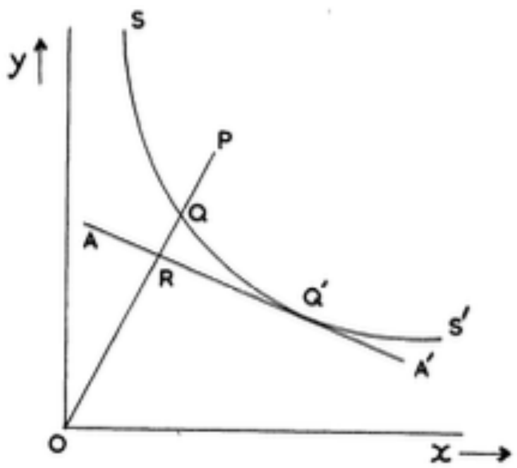

Figure 1: The Measurement of Productive Efficiency, Farrell(1957) – Technical and Allocative efficiency

The isoquant SS' represents the various combinations of the two production factors that a perfectly efficient company should use to produce units of output. Point P represents the combination of the two factors used by the company, while point Q identifies an efficient company that uses the two factors in the same proportions as P. It can be noted that Q produces the same output as P but using only a fraction OQ/OP of each factor. Therefore, OQ/OP is defined as the technical efficiency (T) of company P. As previously mentioned, it indicates the company's ability to achieve the maximum output from a given set of inputs. Moreover, if AA' has a slope equal to the ratio between the prices of the two production factors, Q' and not Q is the optimal production method. Indeed, although both points are technically efficient, the production costs at Q' are only a fraction OR/OQ of those at Q.

The ratio OR/OQ is defined as the allocative efficiency (A) of Q; it indicates the optimal proportion of input usage given their respective prices.
If the observed company were perfectly efficient, its costs would be a fraction OR/OP of what they actually are. Therefore, the ratio OR/OP is defined as overall efficiency (E). It is equal to the product of technical efficiency and allocative efficiency.
Analytically, this is obtained as:

$$E = \frac{OR}{OP} = \frac{OQ}{OP} * \frac{OR}{OQ} = T * A$$

The tangent point between the production isoquant SS' and the line AA', that is, point Q', is efficient both from a technical and allocative perspective.
Efficiency measures have been defined assuming that the efficient production function is known.

1.3 - Frontier approach

Frontier approaches define a theoretical limit of efficiency that represents the maximum output achievable for a given level of input, or the minimum input required for a specified level of output. Productive units are evaluated against this frontier, identifying those that are efficient, which lie on the frontier, and those that are inefficient, which are located below the frontier.

1.4 - Comparison between parametric and non-parametric methods

In parametric approaches, the production frontier is expressed through a predefined mathematical function that contains a fixed number of parameters to be estimated. These methods assume that the form of the production function is known a priori or can be estimated statistically. The main advantage of this approach is that assumptions about the form of the function can be tested

through statistical analyses, and the relationships between inputs and outputs follow well-defined functional models. The Cobb-Douglas function is one of the most commonly used functional forms, often estimated using regression techniques(see Cobb & Douglas, 1928).
In contrast, in non-parametric approaches, the production frontier is directly determined from the observed data without making assumptions about its form. In this case, the frontier is constructed as the envelope of the data, satisfying certain efficiency properties. Methods such as Data Envelopment Analysis (DEA) do not require the specification of an explicit functional form but instead seek maximum efficiency based on the available dataset, making the approach more flexible than parametric methods. The primary strength of these approaches lies in their ability to handle complex relationships without being constrained by a predefined functional model (see Charnes et al., 1978).

1.4.1 - Deterministic and stochastic parametric methods

Parametric frontiers can be distinguished based on the presence of stochastic elements in the underlying mathematical function of the model.
In deterministic parametric frontiers, no assumptions are made about the data generation mechanism, and the absence of stochastic error is assumed. The difference between theoretical production and observed production represents the inefficiency of the production unit. Therefore, deterministic frontier functions attribute the deviation of an observation from its theoretical maximum exclusively to inefficiency within the firm, i.e., the producer’s choices. This specification does not consider potential uncontrollable random shocks specific to the production process.
In stochastic parametric frontiers, this difference is split into two components: stochastic error and inefficiency measurement. Hence, it acknowledges that deviations from the frontier can be justified by factors outside the producer’s control.
In the literature, we can cite two relevant studies regarding parametric frontiers, specifically deterministic and stochastic:

- Deterministic Frontiers: These frontiers do not consider any random error. All deviations from the frontier are solely attributed to technical inefficiency. A classic example is the Deterministic Frontier Model (DFM), introduced by Aigner & Chu(1968), which assumes a predefined production function. The measure of inefficiency is given by the residuals from the estimation, with a corrective procedure known as Corrected Ordinary Least Squares (COLS) used to eliminate any deviations inconsistent with economic theory. However, the assumption of no random noise can be restrictive, as it does not account for exogenous shocks or measurement errors.

- Stochastic Frontiers: Introduced later by D. Aigner et al.(1977) with the Stochastic Frontier Model (SFM), these frontiers decompose the deviation from the frontier into two components: a symmetric part attributable to measurement errors or exogenous shocks, and an asymmetric part related to inefficiency. This decomposition allows for the consideration of random factors that influence the performance of production units, providing greater flexibility compared to deterministic frontiers. However, a limitation of this approach lies in the difficulty of precisely distinguishing between the two types of deviation.

1.4.2 - Non-parametric method: Data Envelopment Analysis

Non-parametric methods are based on constructing a reference system against which the individual observations concerning the performance of the production units, referred to as Decision Making Units (DMUs), are evaluated. These methods do not require a priori assumptions about the characteristics of the reference parameter, a characteristic assumption of parametric statistics; specifically, it is not assumed that the data come from a normal or Gaussian population.

The non-parametric approach is flexible in the construction of the efficient frontier, as it does not rely on any theoretical model that specifies the functional form of the frontier by predetermining a fixed number of parameters explaining the production set. Non-parametric methods make few assumptions and do not hypothesize the form of the function generating the production frontier or the position of observations relative to it. No probabilistic assumption is made regarding the data collection methodology, treating it as descriptive rather than inferential(Coelli et al., 2005).

Additionally, since the production functions of many firms are not known, no assumption about the form of the production function can be made a priori. Therefore, non-parametric methods are more effective for estimating the efficiency of DMUs and modeling the sample under analysis.

An efficient production function can be constructed empirically using input and output observations from a sample of firms. This innovation was originally proposed by Farrell(1957) and later embraced by the theorists of Data Envelopment Analysis (DEA), notably Charnes et al.(1978).

1.5 - Fundamental concepts of DEA

Data Envelopment Analysis (DEA) is a non-parametric methodology used to assess the relative performance of decision-making units (DMUs), such as companies, schools, or hospitals, which utilize multiple resources (inputs) to produce outcomes (outputs). DEA involves solving a linear programming problem, derived from a fractional programming model, to determine the technical efficiency of the DMUs in the analyzed sample.

This approach extends the measure of technical efficiency introduced by Farrell(1957), which was originally limited to the case of a single input and a single output. With DEA, it is possible to include more complex scenarios with multiple inputs and outputs, making it a flexible methodology that aligns more closely with real-world production processes. In fact, in the economic world, production processes typically use a variety of inputs to generate one or more outputs.

The complexity of this extension lies in the weighting of the quantitative variables, as they are often dimensionally different.
In DEA models, which will be presented below, n production units (DMUs) are considered, indexed by j = 1, 2, …, n, each of which uses m inputs (v_i, where i = 1, 2, …, m) to produce t outputs (u_r, where r = 1, 2, …, t). The DEA method allows for variable weights to be assigned to both inputs and outputs, represented by v_i and u_r, to calculate an efficiency index for each DMU.
This approach was originally developed by Charnes et al.(1978), who introduced the CCR model to evaluate the relative efficiency of DMUs. The weights are not fixed but are derived directly from the observations and chosen to maximize the "output/input" efficiency indices for each unit relative to others in the sample. To ensure the validity of the model, the weights must be positive or at least non-negative, and the resulting relative efficiency index must be between zero and one.
Thanks to these characteristics, the DEA method is easily adaptable to different situations and application areas. Unlike parametric techniques, it does not require a detailed description of the production process, nor does it make probabilistic assumptions about the data collection methodology or the functional form of the production function. This descriptive nature of DEA, as a non-parametric model, has been discussed by Coelli et al.(2005), who emphasize its usefulness in measuring the empirical efficiency of DMUs.

1.5.1 - DEA models over time

Data Envelopment Analysis (DEA) has evolved over time, leading to the development of various models, each characterized by specific assumptions and distinctive approaches. Below, the main features of the most significant models are highlighted:

1. CCR Model(Charnes et al., 1978): it is based on the assumption of constant returns to scale (CRS), evaluating the overall efficiency of DMUs; it uses virtual inputs and outputs, i.e., the weighted aggregations of inputs and outputs, to estimate the level of efficiency; it allows for the

identification of sources of inefficiency, providing a comprehensive overview of the performance of the analyzed units.

2. BCC Model(Banker et al., 1984): it introduces the assumption of variable returns to scale (VRS), distinguishing "pure" technical efficiency from scale efficiency; it is particularly useful for analyses where the size of DMUs may influence overall efficiency, allowing for a more detailed estimation of performance. In fact, a DMU might be inefficient not due to poor resource utilization but because it operates at a scale that is too large (diseconomies of scale) or too small (economies of scale).
3. Additive Model (Charnes et al., 1985): it focuses on the analysis of inefficiency, highlighting how much and in what way each DMU deviates from optimal efficiency in terms of inputs and outputs; it links efficiency results to the economic concept of Pareto-optimality, as interpreted in the works of Koopmans(1951), offering a broader economic perspective on the analysis.

1.5.2 - Main variables of a DEA model

The main alternative characteristics of the theorized DEA models concern the type of returns to scale and the orientation used to evaluate DMUs.

Regarding the type of returns to scale, DEA models can assume constant returns to scale (CRS) or variable returns to scale (VRS):

- Constant returns to scale (CRS), introduced by Charnes et al.(1978), assume that a proportional increase in all inputs leads to a proportional increase in outputs.

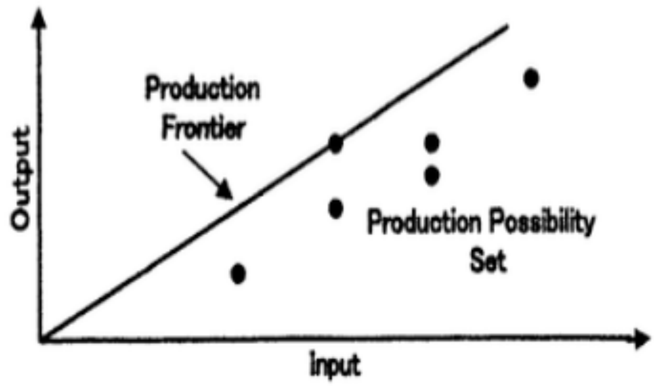

Figure 2: Measuring the efficiency of decision making units, Charnes et al., 1978 – Constant returns to scale

- Variable returns to scale (VRS), introduced by Banker et al.(1984), describe a situation where the proportionality between inputs and outputs of a DMU is not constant as the production scale changes. When all inputs are increased proportionally, the outputs do not increase in the same proportion. Economies of scale, or increasing returns to scale, occur when increasing inputs by a certain percentage leads to a greater increase in outputs, thanks to higher efficiency. Conversely, decreasing returns to scale, or diseconomies of scale, occur when an increase in inputs results in a smaller increase in outputs, often due to inefficiencies related to excessive growth. Finally, local constant returns to scale occur when the relationship between inputs and outputs remains proportional in some operational phases but may shift to increasing or decreasing returns depending on the scale achieved.

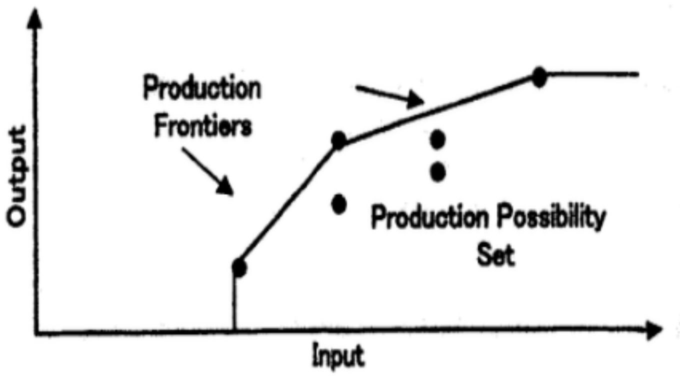

Figure 3: Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis, Banker et al., 1984 – Variable returns to scale

We refer to pure technical efficiency, which evaluates the ability to transform inputs into outputs without considering scale; and scale efficiency, which assesses whether the DMU is operating at the most appropriate size to maximize productivity.

Through the use of DEA models, it is possible to identify the causes and calculate the amount of inefficiency for DMUs that do not lie on the frontier, indicating the remedies needed to become efficient. After calculating the efficient frontier formed by the decision-making units (DMUs) with the best performance, an efficiency score is assigned to each unit not on the efficient frontier based on their distances from the efficient frontier itself.

Regarding the type of orientation, a model can be input-oriented or output-oriented, depending on the objective of the analysis:

- In the case of an input-oriented approach, the goal is to minimize the inputs required to produce a given level of output, thus keeping production constant. This approach is useful when the aim is to evaluate efficiency in terms of resource savings.

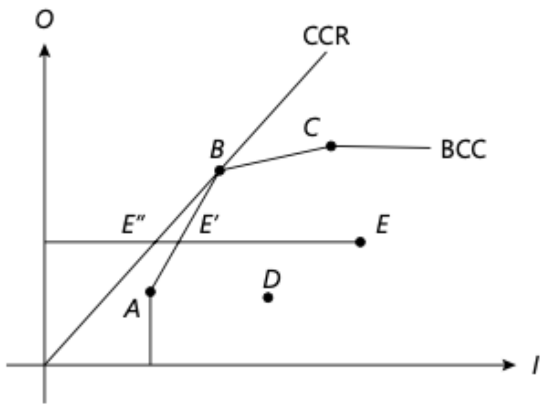

Figure 4: input-oriented case for the CCR and BCC models

- An output-oriented model aims to maximize the output achievable from a given level of input. This orientation is useful in contexts where the focus is on increasing process productivity while keeping resources constant.

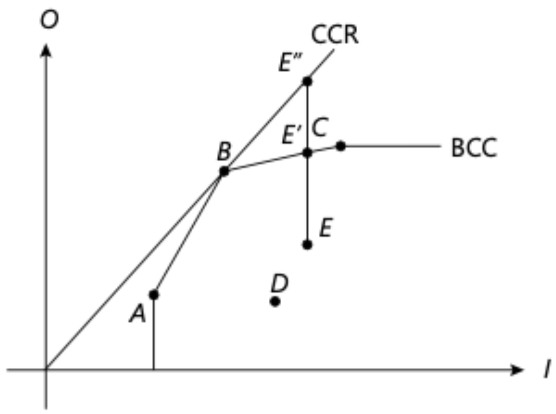

Figure 5: output-oriented case for the CCR and BCC models

The measure of efficiency is given by the ratio between output and input:

$$\frac{\text{Output}}{\text{Input}}$$

An economic unit can be technically inefficient, Farrell(1957), both in the case of input-oriented and output-oriented approaches. It is referred to as input-oriented when the DMU wastes inputs during production, and output-oriented when, given the inputs, it does not maximize outputs.

1.6 - CRR Model

The CCR model is the first of the basic DEA models, proposed by Charnes, Cooper, and Rhodes in 1978.
In DEA, the production unit is referred to by the abbreviation DMU, which stands for Decision Making Unit. A DMU is considered the entity responsible for converting inputs into outputs, whose performance needs to be evaluated.

Suppose there are a number n of DMUs: DMU1, DMU2, ..., DMUn.
The input and output elements for each of these j = 1, ..., n DMUs are selected based on the following assumptions:

1. Numerical data are available for each input and output, provided that the data are positive for all DMUs;
2. Returns to scale are constant (CRS);
3. All inputs and all outputs are reduced to a single virtual input and a single virtual output, expressed as a weighted sum of inputs and outputs;
4. The weights must be non-negative, and the ratio for all DMUs must be greater than 1.

It is assumed that each DMU uses an amount m of inputs to produce an amount t of outputs.
Therefore, the series of inputs is i = 1, ..., m;
while the series of outputs is r = 1, ..., t.

Consequently, the following input (X) and output (Y) matrices will be obtained:

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{t1} & \cdots & y_{tn} \end{pmatrix}$$

where x1j, x2j, ..., xmj are the inputs and y1j, y2j, ..., ytj are the outputs of the j-th DMU.

For each DMU, we denote the weights for both inputs vi and outputs ur, necessary to reduce each DMUj to a single virtual input and output, calculated through the weighted sum of inputs and outputs, respectively:

$$virtual\ input = v_1 x_{1j} + \ldots + v_m x_{mj}$$

$$virtual\ output = u_1 y_{1j} + \ldots + u_t y_{tj}$$

Therefore, the weights are determined using linear programming (LP) in order to maximize their ratio:

$$\frac{virtual\ output}{virtual\ input}$$

Considering the inputs and outputs, the efficiency of each DMU is calculated through n optimizations, one for each DMU to be evaluated. In each optimization, the efficiency of a DMU is determined as:

$$\theta = \frac{\sum_{r=1}^{t} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}$$

Each DMU is assigned a set of optimal weights, ur*ur* for outputs and vi*vi* for inputs, which, instead of being predefined, are determined directly from the data. These values are calculated by the model to maximize the efficiency ratio and can vary from one DMU to another, as the goal is to optimize their own

efficiency index.
If an input is abundant, the model tends to reduce the weight associated with that factor, as the excessive availability of input diminishes its impact on the DMU's efficiency. Conversely, if an output is difficult to achieve or scarce, the model assigns a higher weight to that result to highlight its value.
The weights cannot be negative, as it would not make sense to assign negative values to inputs or outputs. Additionally, the selection of weights must ensure that, when applied to other DMUs, their efficiency indices never exceed 1. This is referred to as relative efficiency, where a DMU is considered efficient only if its performance cannot be improved without compromising other factors, compared to the other DMUs.

1.7 - Extensions and variants of DEA

DEA, while being a robust model for efficiency analysis, has various extensions and variants that expand its applicability and improve its ability to represent complex scenarios. Among the main variants, the BCC method (Banker, Charnes, and Cooper) introduces a modification to the CCR model, allowing for the analysis of efficiency under conditions of variable returns to scale, a feature particularly useful when DMUs operate at different production scales.
Additionally, there are additive and multiplicative approaches that propose alternative formulations for calculating efficiency, with specific features in managing deviations and defining optimization objectives. These models offer new perspectives in DMU analysis, adapting to situations where the CCR model might be limited. In the following paragraphs, these variants will be examined in detail, highlighting their distinctive characteristics and potential applications.

1.7.1 - BCC Model

The BCC model, developed by Banker, Charnes, and Cooper (1984), is one of the main extensions of DEA and differs from the CCR model in its treatment of variable returns to scale. Unlike the CCR model, which assumes that DMUs

operate under constant returns to scale, the BCC model considers the possibility that DMUs may operate in an environment where returns to scale are not necessarily constant but can vary depending on the size and productive efficiency of each unit.
This distinction is crucial for analyzing organizations operating at different production scales, as the BCC model provides more realistic efficiency estimates when the relationship between inputs and outputs is non-linear(Banker et al., 1984). The introduction of variable returns to scale allows for a more precise efficiency assessment of DMUs operating in dynamic and diverse environments.
Algebraically, the BCC model introduces an additional constraint compared to the CCR model, enabling the analysis of DMUs in relation to their returns to scale. This modification allows for an efficiency evaluation that takes into account the possibility that some DMUs may benefit from increasing economies of scale, while others may experience decreasing economies of scale.

1.7.2 - Other approaches: additive and moltiplicative methods

Beyond the CCR and BCC models, other approaches exist for evaluating efficiency using DEA, including the additive and multiplicative models. These methods differ in their mathematical formulation and the approach they take in handling the relationships between inputs and outputs.
In the additive model(Charnes et al., 1985), efficiency is assessed through the sum of the input and output slacks. Unlike the CCR and BCC models, it imposes the condition that the sum of the slacks must be minimized. This approach is particularly useful when a more direct evaluation of inefficiencies at the input and output levels is desired, avoiding the introduction of relative weights as in the primal CCR model. The additive approach is considered advantageous in situations where slacks need to be analyzed separately, without affecting the efficiency ratio.
The multiplicative model(Charnes et al., 1982), on the other hand, is based on the multiplication of efficiency factors rather than their sum. In this case, the

objective function seeks to maximize a product of efficiency, allowing for an evaluation in terms of relative results among DMUs. This model is useful for applications where the relationships between inputs and outputs can be better represented by a product rather than a sum.

1.8 - Advantages and limitations of the DEA approach

1.8.1 - Advantages of the DEA approach

DEA offers several advantages over traditional methods:

1. Non-parametric nature, it does not require theoretical models as benchmarks but measures performance relative to the best-performing funds within the same category.
2. Inclusion of transaction costs, it simultaneously analyzes returns, management costs, and risk.
3. Multi-input/output analysis, it allows the consideration of multiple inputs and outputs in the evaluation process.
4. Detailed efficiency measurement, DEA provides a specific efficiency index for each fund, identifying inefficiencies and suggesting improvements.

1.8.2 - Limitations of the DEA Approach

The limitations that persist in the DEA methodology are:

1. Sensitivity to sample size; in a small sample with few decision-making units, many DMUs may appear efficient simply because there is insufficient discrimination. This makes the method less useful in contexts with small samples;
2. The measured efficiency is relative, meaning it is based on the data observed in the dataset, not absolute efficiency. Therefore, if all the analyzed DMUs are relatively inefficient, DEA may still designate some

as efficient, but this consideration is strictly related to the examined sample;

3. The analysis remains volatile depending on the selected period for the sample analysis and the performance of the investment funds.

2 - MATERIALS AND METHODS

2.1- Calculations and considerations

2.1.1 - From the fractional programming problem to the linear programming problem

Given that DMU_0 is the unit for which efficiency is to be evaluated relative to all other DMU_j, the following fractional programming (FP) problem is solved. Through this process, the optimal values of the input weights vi, where i = 1, 2, ..., m, and the output weights ur, where r = 1, 2, ..., t, are obtained.

(FP_0)

$$\max_{v,u} \theta = \frac{\sum_{r=1}^{t} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \quad (1.0)$$

s.v.

$$\frac{\sum_{r=1}^{t} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \leq 1 \text{ , j = 1, …, n (1.1)}$$

$$v_r \geq \varepsilon \text{ (1.2)}$$

$$u_i \geq \varepsilon \text{ (1.3)}$$

where ε is a small positive number that prevents the weights from becoming zero (ε should be understood as a non-Archimedean constant; see Charnes et al.(1994).

The goal is to maximize the efficiency of DMU_0, θ, given by the ratio of the sum of the weighted output weights to the sum of the weighted input weights. Constraint 1.1 indicates that the ratio of the virtual output to the virtual input has an upper limit of 1 for each DMU. Constraints 1.2 and 1.3 establish that the input and output weights must be non-negative.

Through some algebraic steps, it is possible to transform the fractional programming (FP) problem into an equivalent linear programming (LP) problem, without altering the result.

Consider the input-oriented case, where the goal is to minimize the inputs required to achieve a given level of output. In the fractional problem, the denominator of the FP constraint 0, specified in formula (1.1), represents the weighted sum of the inputs of the DMU_0 being analyzed, calculated using the respective weights vi. Under the assumption that the inputs xij and the weights vi are non-negative, this denominator is also non-negative for every j. This property allows multiplying both sides of inequality 1.1 by the denominator.

$$\sum_{i=1}^{m} v_i x_{i0}$$

without altering its sign, and obtaining the following inequality:

$$\sum_{i=1}^{m} v_i x_{i0} * \frac{\sum_{r=1}^{t} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \leq 1 * \sum_{i=1}^{m} v_i x_{i0}$$

Simplifying, we then arrive at the inequality:

$$\sum_{r=1}^{t} v_r x_{r0} \leq \sum_{i=1}^{m} v_i x_{i0} \text{ (1.4)}$$

and thus, the fractional programming problem (FP0) is equivalent to the linear programming problem (LP0).
Then, setting the denominator for the DMU_0 under examination equal to 1 as follows:

$$\sum_{i=1}^{m} v_i x_{i0} = 1$$

This leads to the transformation of the fractional programming problem into a linear programming problem. In fact, the denominator of the objective function

(1.0), set equal to 1, is imposed as a constraint in the linear programming problem.
The LP0 solution will be obtained by maximizing the numerator, subject to certain conditions:

(LP_0)

$$\max_{u,v} \theta = \sum_{r=1}^{t} u_r y_{r0} \text{ (1.5)}$$

s.v.

$$\sum_{i=1}^{m} v_i x_{i0} = 1 \text{ (1.6)}$$

$$\sum_{r=1}^{t} u_r y_{r0} \leq \sum_{i=1}^{m} v_i x_{i0}, \text{ j = 1, …, n (1.7)}$$

$$v_r \geq \varepsilon \text{ (1.8)}$$

$$u_i \geq \varepsilon \text{ (1.9)}$$

Compared to the fractional programming problem, the objective now is to maximize the sum of the weighted output weights. As for the constraints, 1.6 imposes that the weighted sum of inputs must be equal to 1; 1.7 ensures that, for each decision-making unit (DMUj), the weighted sum of outputs does not exceed the weighted sum of inputs. In other words, the efficient frontier must dominate or coincide with each DMUj, and no unit can exceed the relative efficiency limit established by this frontier. Constraints 1.8 and 1.9 are analogous to 1.2 and 1.3.
Given that the optimal solution of LP0 is v = v*, u = u*, and the optimal value is θ*, the solution v = v* is also optimal for FP0. Therefore, both problems have the same optimal solution θ*.
As explained by Basso & Funari(2001), "*this problem has t + m variables (the weights ur and vi, which must be found to maximize the efficiency of the target unit j = 0) and n + t + m + 1 constraints. The dual problem will be used subsequently:*

$$\min z_0 - \varepsilon \sum_{r=1}^{t} s_r^{+} - \varepsilon \sum_{i=1}^{m} s_i^{-} \text{ (2.0)}$$

s.v.

$$x_{\mathrm{ij}0} z_0 - s_i^- - \sum_{j=1}^{n} x_{ij}\lambda_j = 0 \quad (2.1)$$

$$-s_r^+ + \sum_{j=1}^{n} y_{\mathrm{rj}}\,\lambda_j = y_{rj_0} \quad (2.2)$$

$$\lambda_j \geq 0 \quad (2.3)$$

$$s_i^- \geq 0 \quad (2.4)$$

$$s_r^+ \geq 0 \quad (2.5)$$

It can be observed that the CCR model provides a piecewise linear production surface that, in economic terms, represents a production frontier. In fact, it provides the maximum output that can be empirically obtained by a decision-making unit given its level of input; from another perspective, it provides the minimum amount of input required to achieve the given levels of output.
The relative efficiency measures of the decision-making units represent only one type of information derived from the DEA methodology. In fact, the DEA approach can also suggest to inefficient units a "virtual unit" that they could emulate to improve their efficiency.

The solution of the dual model allows us to identify, for each inefficient unit, a corresponding group of efficient units, known as peer units (see Boussofiane et al.(1991)). Peer units are associated with strictly positive multipliers in the optimal solution λ*j, i.e., the non-zero optimal dual variables. Therefore, for each inefficient unit j0, it is possible to construct a composite unit with output given by the weighted sum,* $\sum_{j=1}^{n} y_{\mathrm{rj}}\,\lambda_j^*$, r = 1, …, t, and input given by the weighted sum $\sum_{j=1}^{n} x_{ij}\lambda_j^*$, i = 1, ..., m, which surpasses the performance of unit j0 and lies on the efficient frontier.

The optimization problem for the calculation of λ is formulated as follows:

$$\min \theta$$

s.v.

$$\sum_{j=1}^{n} x_{ij}\lambda_j^* \leq \theta x_{i0}$$

$$\sum_{j=1}^{n} y_{\mathrm{rj}}\,\lambda_j^* \geq y_{r0}$$

$$\lambda_j \geq 0$$

where:

- θ: represents the efficiency index of the fund under consideration (DMU_0). A value of θ equal to 1 indicates that the fund is efficient, while a value less than 1 indicates inefficiency.
- λ_j: these are the weights associated with each j-th fund. These weights indicate how much each fund contributes to the evaluation of the efficiency of the fund under consideration.
- x_{ij}: represents the i-th input of the j-th fund. Inputs are the resources or costs used by the fund to produce outputs.
- y_{rj}: represents the r-th output of the j-th fund. Outputs are the results or benefits produced by the fund.
- x_{i0} and y_{r0}: these are the inputs and outputs, respectively, of the inefficient fund under consideration (DMU_0).

The optimization problem aims to minimize the efficiency index θ subject to constraints that ensure a proportional reduction of inputs and the maintenance of outputs. The constraints ensure that the fund under consideration can be compared with a linear combination of the other funds, using non-negative weights λ_j.

From a financial perspective, this composite unit can be seen as a benchmark for the inefficient fund j0. The latter could improve its performance by trying to emulate the behavior of the efficient composite unit. This optimal (i.e., efficient) portfolio exhibits an input/output orientation similar to that of the inefficient fund j0. Therefore, knowing the corresponding composition can be useful for

analyzing the portfolio management style. For further insights on the importance of studying a portfolio's management style, see Sharpe(1992).
Finally, DEA efficiency indices are calculated using the rate of return of mutual funds, which implies that decision-making units are scaled based on the same amount of invested capital. For this reason, it is appropriate to normalize the multipliers using the following formula:

$$l_j = \frac{\lambda_j^*}{\sum_{k=1}^{n} \lambda_k^*} \quad (2.6)$$

It indicates the relative composition of the reference portfolio for each inefficient fund.

2.1.2 - Invariance of units in the model

The calculated relative efficiency value is not affected by the scale or units of measurement used for inputs and outputs, as long as there is consistency in the measurements. This principle gives the DEA methodology great flexibility, making it applicable in contexts where inputs and outputs can be expressed in heterogeneous units, such as physical quantities, monetary values, or percentages, without the need for preprocessing or normalization techniques to standardize the analyzed input and output variables. The optimal efficiency value θ*, calculated in the fractional programming problem and its corresponding linear programming problem, remains unchanged regardless of the units of measurement for inputs and outputs. This invariance is due to the fractional nature of the objective function, where both the numerator and denominator are proportionally affected by changes in the units of measurement. This fractional ratio allows obtaining an efficiency value that depends solely on the relative ratio between inputs and outputs for each DMU, regardless of the units of measurement used.

2.1.3 - Efficiency of the Decision Making Units

In the input-oriented approach, a DMU is efficient if the value θ^* equals 1, and if there exists at least one optimal solution with positive input and output weights $v^* > 0$ and $u^* > 0$. A DMU is efficient if it maximizes the available resources in inputs, achieving the best possible result in outputs.
When a DMU does not reach this level of efficiency, it is defined as inefficient. This can occur in two scenarios:

1. When $\theta < 1$, meaning the DMU is not using its inputs optimally to produce the desired outputs, achieving less than what would be possible with the same inputs.
2. When $\theta = 1$, the j-th DMU is on the efficient frontier.

2.1.4 - Production Possibility Set of the model

The Production Possibility Set (P) represents the set of all possible production combinations in an efficient context. It is bounded by the space of DMUs operating with maximum efficiency, i.e., those that use the minimum amount of inputs to produce a given amount of output. The set includes all possible combinations of inputs and outputs that can be obtained from the observed DMUs, where each pair of values xj and yj represents the activity of a specific DMU. To formally define this set, the data are organized into matrices X = xj and Y = yj, which represent the inputs and outputs of each DMU, respectively. The production possibility set P can be expressed through a formula involving a vector λ, which represents the weights associated with the DMUs in the set:

$$P = \{(x, y) \mid x \geq X\lambda, y \leq Y\lambda, \lambda \geq 0\}$$

The vector λ represents a set of weights assigned to each DMU. These weights are used to construct a linear combination of the DMUs, which is a kind of "weighted average" of the performance of other DMUs. This allows comparing the efficiency of the DMU under examination with those that are more efficient.

The value of λj indicates how much each DMU contributes to this combination. Additionally, λ is constrained to be non-negative, ensuring that it is not possible to nullify a DMU or assign it a negative weight, thus guaranteeing that the linear combination always represents a positive weighted sum of the performance of the DMUs.

If a DMU is inefficient, the model will attempt to represent it as a combination of more efficient DMUs, assigning positive values to the DMUs that are used to "replicate" the inefficient DMU. This implies that the inefficient DMU could reduce its inputs to achieve an equivalent level of output, leveraging the resources of the efficient DMUs. On the other hand, if a DMU is already efficient, the weights associated with other DMUs in the λ vector will also be positive, but this will indicate that the DMU under examination is already using its inputs optimally, without the need to combine other DMUs to improve its efficiency.

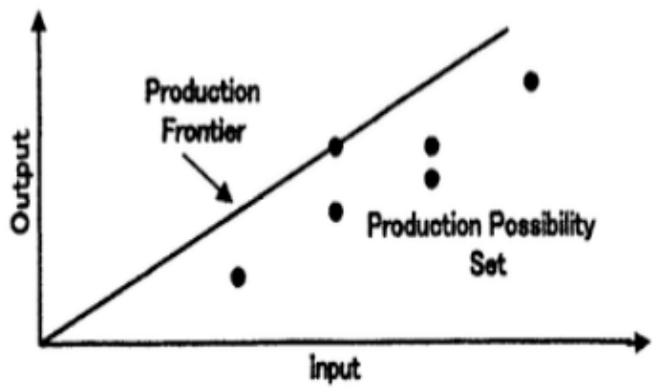

Figure 6: Measuring the efficiency of decision making units, Charnes et al., 1978 – Constant Returns to Scale

2.2 - Dataset

The dataset used in the analysis includes a total of 52 mutual funds, divided into three categories according to the classification provided by Morningstar:

- 22 mutual funds investing in Italian equity securities, classified as 'Italian Equity';

- 15 funds investing in European government bonds, classified as 'European Government Bonds EUR';
- 15 funds in the 'Moderate Balanced EUR' category.

The choice of these three categories responds to the objective of continuing the study "A Data Envelopment Analysis Approach to Measure Mutual Fund Performance" by Basso & Funari(2001). The approach used in this study guided both the selection of the funds and the methodological configuration of the DEA model.

To determine the membership of each fund in a specific category, Morningstar was used as a reference. This ensured uniformity and consistency in the classification, avoiding discrepancies due to subjective interpretative criteria.

All funds considered are actively managed and belong to recognized issuing companies, including: AcomeA, Schroders, Symphonia, Algebris, Allianz, Amundi, Anima, Arca, Azimut, BNL, Eurizon, Fidelity, Fideuram, Mediolanum, Zenit, Schroder, Investimenti, Interfund, Fonditalia, Pictet, JPMorgan, BlueBay, DWS, BNP Paribas, Degroof Petercam, Epsilon, Euromobiliare, Generali, HSBC, iShares, Carmignac, Groupama, UBS, Xtrackers.

From a geographical perspective, the Italian equity funds include only instruments from the Italian market, while the European government bond funds consist of securities issued by European governments. Similarly, the funds in the 'Moderate Balanced EUR' category hold both equity and bond securities exclusively from this geographical area. The choice of the last two categories is dictated by the absence in the market of funds that allow investment in bonds issued solely by the Italian government or in a balanced mix of Italian equity and bond assets.

Table 1: fund, related ISIN, and category of belonging

Name	ISIN	Asset Class
AcomeA PMITALIA ESG A1	IT0000390044	Azionario Italia
Schroder International Selection Fund Italian Equity C Accumulation EUR	LU0106239527	Azionario Italia

Symphonia Azionario Small Cap Italia	IT0004464233	Azionario Italia
Algebris Core Italy R EUR Acc	IE00BF4RGB4	Azionario Italia
Allianz Azioni Italia All Stars A	IT0004287840	Azionario Italia
Amundi Impegno Italia B	IT0004253800	Azionario Italia
Amundi Sviluppo Attivo Italia A	IT0005245243	Azionario Italia
Anima Iniziativa Italia A	IT0005186041	Azionario Italia
Anima Italia A	IT0001040051	Azionario Italia
Arca Azioni Italia P	IT0000388907	Azionario Italia
Azimut Trend Italia	IT0001055158	Azionario Italia
Bnl Azioni Italia	IT0000382561	Azionario Italia
Eurizon Azioni Italia R	IT0001021192	Azionario Italia
Fidelity Funds - Italy Fund A-acc-eur	LU0922333322	Azionario Italia
Fideuram Italia R	IT0000388147	Azionario Italia
Mediolanum Challenge Italian Equity Fund L Acc	IE0004905604	Azionario Italia
Mediolanum Flessibile Futuro Italia La	IT0001019329	Azionario Italia
Zenit Pianeta Italia R	IT0001070645	Azionario Italia
Schroder International Selection Fund Italian Equity A Accumulation Eur	LU0106238719	Azionario Italia
Investimenti Azionari Italia A	IT0001023628	Azionario Italia
Interfund Equity Italy	LU0074298604	Azionario Italia
Fonditalia Equity Italy R	LU0058495788	Azionario Italia
Pictet-eur Government Bonds I	LU0241467157	Obbligazionario Governativi EUR
AMUNDI EURO GOVERNMENT BOND - AE ©	LU1050470373	Obbligazionario Governativi EUR
Anima Tricolore A	IT0005186082	Obbligazionario Governativi EUR
JPMorgan Funds - EU Government Bond Fund D (acc)	LU0355584037	Obbligazionario Governativi EUR
BlueBay Funds - BlueBay Investment Grade Euro Government Bond Fund R EUR Acc	LU0549537040	Obbligazionario Governativi EUR
Dws Invest Euro-gov Bonds Nc	LU0145652649	Obbligazionario Governativi EUR
Bnp Paribas Funds Euro Government Bond Privl Capitalisation	LU0111549217	Obbligazionario Governativi EUR
DPAM B Bonds Eur Govt W Cap	BE6246046229	Obbligazionario Governativi EUR

Epsilon Fund - Euro Bond Class Unit R Eur Accumulation	LU0367640660	Obbligazionario Governativi EUR
Euromobiliare Reddito A	IT0000382405	Obbligazionario Governativi EUR
Generali Investments Sicav - Euro Bond Fund Dx	LU0145476817	Obbligazionario Governativi EUR
Hsbc Euro Gvt Bond Fund Hc	FR0000971293	Obbligazionario Governativi EUR
Ishares Euro Government Bond Index Fund (lu) F2 Eur	LU0836516103	Obbligazionario Governativi EUR
Jpmorgan Funds - Eu Government Bond Fund A (acc) - Eur	LU0363447680	Obbligazionario Governativi EUR
Generali Investments Sicav - Euro Bond Fund Ex	LU0169250635	Obbligazionario Governativi EUR
Investimenti Bilanciati Internazionali C	IT0004941628	Bilanciati Moderati EUR
Anima Visconteo Plus A	IT0005158966	Bilanciati Moderati EUR
Anima Visconteo Plus AD	IT0005159006	Bilanciati Moderati EUR
Arca Economia Reale Bilan Italia 55 PIR	IT0005252686	Bilanciati Moderati EUR
Azimut Dinamico	IT0000384567	Bilanciati Moderati EUR
Carmignac Portfolio Patrimoine Europe A EUR Acc	LU1744628287	Bilanciati Moderati EUR
Groupama Bilanciato - NC	FR0000995128	Bilanciati Moderati EUR
Eurizon Bilanciato Euro Multimanager	IT0000380300	Bilanciati Moderati EUR
Fideuram Bilanciato	IT0000382389	Bilanciati Moderati EUR
UBS (Lux) KSS Eur GrInc € P-acc	LU1038902331	Bilanciati Moderati EUR
Fidelity Funds - European Multi Asset Income Fund A-Acc-EUR	LU0261950553	Bilanciati Moderati EUR
Dpam B - Balanced Flexible B	BE0940785794	Bilanciati Moderati EUR
Groupama Bilanciato - Ic	FR0010270314	Bilanciati Moderati EUR
Investimenti Bilanciati Internazionali A	IT0000382181	Bilanciati Moderati EUR
Ubs (lux) Key Selection Sicav - European Growth And Income (eur) Q-acc	LU1240794898	Bilanciati Moderati EUR

2.2.1 - Data sources

To collect the data necessary for the analysis, platforms that represent the standard for reliability and accuracy in the financial sector were used:

- Morningstar: used to classify mutual funds according to their categories. For each fund, the platform provides key documents that fund managers are required to produce, such as the Key Investor Information Document (KIID). This document contains transparent information regarding entry, management, and exit fees, which are essential for selecting the input variable 'Entry Fee'.
- Investing.com: used to gather weekly historical time series of the funds during the analysis period. The platform provided access to the necessary data for calculating indicators such as expected return (used as an output) and risk indices, including standard deviation and Beta (used as inputs).

Only funds for which all required information was available were selected, ensuring a complete dataset.

2.2.2.1 - Selection of Inputs

For the DEA analysis of mutual funds, three inputs were chosen: standard deviation, beta, and entry fees:

- Standard Deviation: used to measure the risk of the funds, as suggested by Markowitz(1952), representing the volatility of their returns. It was calculated using weekly logarithmic returns, which were then annualized for each fund over the analysis period. The use of logarithmic returns, rather than percentage returns, eliminates potential issues arising from compounding effects and asymmetric percentage variations, ensuring a more consistent and stable risk measurement over time.
- Beta: selected as a measure of the fund's systematic risk, as discussed by Sharpe(1964), reflecting the fund's sensitivity to market movements. It was calculated through linear regression between the fund's weekly logarithmic returns and those of the FTSE Italia All Share index, considered as the market benchmark, over the same period. Beta serves

as an indicator to distinguish the fund-specific risk from the overall market risk. A beta value greater than 1 indicates that the fund tends to move more aggressively than the market, while a value below 1 suggests lower sensitivity.

- Entry Fees: used to measure the impact of costs on the investor's net return. This data was sourced from the KIID documents of each fund, which provide a detailed breakdown of applicable fees. Unlike management fees, which are deducted directly from the fund's net asset value, entry fees are an upfront cost for the investor at the time of purchase. While they do not reduce the invested capital directly, they represent an additional expense necessary for subscription. It is important to note that exit fees were not considered, as none of the sampled funds charged such a fee—contrary to the research period of Basso & Funari(2001), when exit fees were more common.

2.2.2.2 - Selection of output

The output used for the mutual fund analysis is expected return, calculated as the annualized logarithmic return, obtained by multiplying the weekly logarithmic return by 52, following the methodology of Basso & Funari(2001).
The choice of logarithmic return over percentage return—similar to the calculation of input variables—ensures continuous return measurement and avoids compounding errors that arise with percentage returns.
Expected return was chosen as the output since it represents the primary objective for an investor, who typically focuses on a fund's future returns in relation to the risk undertaken and costs incurred.

2.2.2.3 - Calculation of inputs and output

Below are the algebraic steps required for the computation of each input and output, based on the methodology previously described.

- Entry fees were obtained directly from the KIID document, which is periodically issued by the fund provider.

Table 2: entry fee percentage of each mutual fund

Fund	**Entry fee**
AcomeA PMITALIA ESG A1	4,00%
Schroder International Selection Fund Italian Equity C Accumulation EUR	1,00%
Symphonia Azionario Small Cap Italia	4,00%
Algebris Core Italy R EUR Acc	3,00%
Allianz Azioni Italia All Stars A	2,00%
Amundi Impegno Italia B	3,00%
Amundi Sviluppo Attivo Italia A	2,00%
Anima Iniziativa Italia A	4,00%
Anima Italia A	4,00%
Arca Azioni Italia P	4,00%
Azimut Trend Italia	2,00%
Bnl Azioni Italia	2,00%
Eurizon Azioni Italia R	1,50%
Fidelity Funds - Italy Fund A-acc-eur	5,25%
Fideuram Italia R	3,00%
Mediolanum Challenge Italian Equity Fund L Acc	3,00%
Mediolanum Flessibile Futuro Italia La	3,00%
Zenit Pianeta Italia R	2,00%
Schroder International Selection Fund Italian Equity A Accumulation Eur	5,00%
Investimenti Azionari Italia A	2,00%
Interfund Equity Italy	6,50%
Fonditalia Equity Italy R	3,80%
Pictet-eur Government Bonds I	5,00%
AMUNDI EURO GOVERNMENT BOND - AE ©	4,50%
Anima Tricolore A	1,00%
JPMorgan Funds - EU Government Bond Fund D (acc)	3,00%
BlueBay Funds - BlueBay Investment Grade Euro Government Bond Fund R EUR Acc	5,00%
Dws Invest Euro-gov Bonds Nc	1,50%
Bnp Paribas Funds Euro Government Bond Privl Capitalisation	3,00%

DPAM B Bonds Eur Govt W Cap	2,00%
Epsilon Fund - Euro Bond Class Unit R Eur Accumulation	1,50%
Euromobiliare Reddito A	2,00%
Generali Investments Sicav - Euro Bond Fund Dx	3,00%
Hsbc Euro Gvt Bond Fund Hc	2,00%
Ishares Euro Government Bond Index Fund (lu) F2 Eur	5,00%
Jpmorgan Funds - Eu Government Bond Fund A (acc) - Eur	3,00%
Generali Investments Sicav - Euro Bond Fund Ex	2,00%
Investimenti Bilanciati Internazionali C	2,00%
Anima Visconteo Plus A	3,00%
Anima Visconteo Plus AD	3,00%
Arca Economia Reale Bilan Italia 55 PIR	3,00%
Azimut Dinamico	2,00%
Carmignac Portfolio Patrimoine Europe A EUR Acc	4,00%
Groupama Bilanciato - NC	3,00%
Eurizon Bilanciato Euro Multimanager	1,50%
Fideuram Bilanciato	1,50%
UBS (Lux) KSS Eur GrInc € P-acc	4,00%
Fidelity Funds - European Multi Asset Income Fund A-Acc-EUR	5,25%
Dpam B - Balanced Flexible B	2,00%
Groupama Bilanciato - Ic	3,00%
Investimenti Bilanciati Internazionali A	2,00%
Ubs (lux) Key Selection Sicav - European Growth And Income (eur) Q-acc	4,00%

The standard deviation, as proposed by Markowitz(1952), was calculated using the weekly logarithmic returns, Rt, of each fund, following the formula:

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \quad (2.6)$$

where Pt is the fund price at week t, and P(t-1) is the fund price in the previous week.

Next, the average of the logarithmic returns is calculated:

$$\bar{R} = \frac{1}{n}\sum_{t=1}^{n} R_t \quad (2.7)$$

where n is the total number of weeks in the considered period.
Finally, the weekly standard deviation is computed using the formula:

$$\sigma_{weekly} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(R_t - \bar{R})^2} \quad (2.8)$$

and is then annualized as follows:

$$\sigma_{ann} = \sigma_{weekly} * \sqrt{52} \quad (2.9)$$

Table 3: standard deviation of annualized logarithmic returns for each fund

Fund	Standard deviation
AcomeA PMITALIA ESG A1	0,123
Schroder International Selection Fund Italian Equity C Accumulation EUR	0,108
Symphonia Azionario Small Cap Italia	0,124
Algebris Core Italy R EUR Acc	0,086
Allianz Azioni Italia All Stars A	0,128
Amundi Impegno Italia B	0,117
Amundi Sviluppo Attivo Italia A	0,099
Anima Iniziativa Italia A	0,122
Anima Italia A	0,129
Arca Azioni Italia P	0,131
Azimut Trend Italia	0,134
Bnl Azioni Italia	0,138
Eurizon Azioni Italia R	0,130
Fidelity Funds - Italy Fund A-acc-eur	0,103
Fideuram Italia R	0,133
Mediolanum Challenge Italian Equity Fund L Acc	0,124
Mediolanum Flessibile Futuro Italia La	0,114
Zenit Pianeta Italia R	0,103

Schroder International Selection Fund Italian Equity A Accumulation Eur	0,100
Investimenti Azionari Italia A	0,130
Interfund Equity Italy	0,129
Fonditalia Equity Italy R	0,131
Pictet-eur Government Bonds I	0,035
AMUNDI EURO GOVERNMENT BOND - AE ©	0,033
Anima Tricolore A	0,053
JPMorgan Funds - EU Government Bond Fund D (acc)	0,019
BlueBay Funds - BlueBay Investment Grade Euro Government Bond Fund R EUR Acc	0,024
Dws Invest Euro-gov Bonds Nc	0,020
Bnp Paribas Funds Euro Government Bond Privl Capitalisation	0,018
DPAM B Bonds Eur Govt W Cap	0,021
Epsilon Fund - Euro Bond Class Unit R Eur Accumulation	0,017
Euromobiliare Reddito A	0,015
Generali Investments Sicav - Euro Bond Fund Dx	0,020
Hsbc Euro Gvt Bond Fund Hc	0,019
Ishares Euro Government Bond Index Fund (lu) F2 Eur	0,034
Jpmorgan Funds - Eu Government Bond Fund A (acc) - Eur	0,019
Generali Investments Sicav - Euro Bond Fund Ex	0,020
Investimenti Bilanciati Internazionali C	0,084
Anima Visconteo Plus A	0,048
Anima Visconteo Plus AD	0,049
Arca Economia Reale Bilan Italia 55 PIR	0,054
Azimut Dinamico	0,042
Carmignac Portfolio Patrimoine Europe A EUR Acc	0,037
Groupama Bilanciato - NC	0,046
Eurizon Bilanciato Euro Multimanager	0,052
Fideuram Bilanciato	0,054
UBS (Lux) KSS Eur GrInc € P-acc	0,057
Fidelity Funds - European Multi Asset Income Fund A-Acc-EUR	0,049
Dpam B - Balanced Flexible B	0,035
Groupama Bilanciato - Ic	0,046
Investimenti Bilanciati Internazionali A	0,071

Ubs (lux) Key Selection Sicav - European Growth And Income (eur) Q-acc	0,057

The beta, as proposed by Sharpe(1964), was calculated using the linear regression formula, which relates the fund's weekly returns Rt to the market index's weekly returns Rm.

The process begins with the calculation of logarithmic returns for both the fund and the market index. Then, the average weekly returns for the fund, R̅_f, and for the index, R̅_m, are determined using the same formulas as for the standard deviation (2.6 and 2.7).

Next, the covariance between the fund returns and the market returns is calculated:

$$Cov\left(R_f, R_m\right) = \frac{1}{n-1}\sum_{t=1}^{n}\left(R_{f,t} - \bar{R}_f\right)\left(R_{m,t} - \bar{R}_m\right) \text{ (3.0)}$$

And the variance of market returns:

$$Var(R_m) = \frac{1}{n-1}\sum_{t=1}^{n}\left(R_{m,t} - \bar{R}_m\right)^2 \text{ (3.1)}$$

Finally, the beta can be calculated using the formula:

$$\beta = \frac{Cov\left(R_f, R_m\right)}{Var(R_m)} \text{ (3.2)}$$

Table 4: Beta of each mutual fund

Fund	**Beta**
AcomeA PMITALIA ESG A1	0,123
Schroder International Selection Fund Italian Equity C Accumulation EUR	0,108
Symphonia Azionario Small Cap Italia	0,124
Algebris Core Italy R EUR Acc	0,086
Allianz Azioni Italia All Stars A	0,128

Amundi Impegno Italia B	0,117
Amundi Sviluppo Attivo Italia A	0,099
Anima Iniziativa Italia A	0,122
Anima Italia A	0,129
Arca Azioni Italia P	0,131
Azimut Trend Italia	0,134
Bnl Azioni Italia	0,138
Eurizon Azioni Italia R	0,130
Fidelity Funds - Italy Fund A-acc-eur	0,103
Fideuram Italia R	0,133
Mediolanum Challenge Italian Equity Fund L Acc	0,124
Mediolanum Flessibile Futuro Italia La	0,114
Zenit Pianeta Italia R	0,103
Schroder International Selection Fund Italian Equity A Accumulation Eur	0,100
Investimenti Azionari Italia A	0,130
Interfund Equity Italy	0,129
Fonditalia Equity Italy R	0,131
Pictet-eur Government Bonds I	0,035
AMUNDI EURO GOVERNMENT BOND - AE ©	0,033
Anima Tricolore A	0,053
JPMorgan Funds - EU Government Bond Fund D (acc)	0,019
BlueBay Funds - BlueBay Investment Grade Euro Government Bond Fund R EUR Acc	0,024
Dws Invest Euro-gov Bonds Nc	0,020
Bnp Paribas Funds Euro Government Bond Privl Capitalisation	0,018
DPAM B Bonds Eur Govt W Cap	0,021
Epsilon Fund - Euro Bond Class Unit R Eur Accumulation	0,017
Euromobiliare Reddito A	0,015
Generali Investments Sicav - Euro Bond Fund Dx	0,020
Hsbc Euro Gvt Bond Fund Hc	0,019
Ishares Euro Government Bond Index Fund (lu) F2 Eur	0,034
Jpmorgan Funds - Eu Government Bond Fund A (acc) - Eur	0,019
Generali Investments Sicav - Euro Bond Fund Ex	0,020
Investimenti Bilanciati Internazionali C	0,084
Anima Visconteo Plus A	0,048
Anima Visconteo Plus AD	0,049

Arca Economia Reale Bilan Italia 55 PIR	0,054
Azimut Dinamico	0,042
Carmignac Portfolio Patrimoine Europe A EUR Acc	0,037
Groupama Bilanciato - NC	0,046
Eurizon Bilanciato Euro Multimanager	0,052
Fideuram Bilanciato	0,054
UBS (Lux) KSS Eur GrInc € P-acc	0,057
Fidelity Funds - European Multi Asset Income Fund A-Acc-EUR	0,049
Dpam B - Balanced Flexible B	0,035
Groupama Bilanciato - Ic	0,046
Investimenti Bilanciati Internazionali A	0,071
Ubs (lux) Key Selection Sicav - European Growth And Income (eur) Q-acc	0,057

The expected return was calculated based on the weekly logarithmic returns RtR_tRt of each fund. In this case, formulas 2.6 and 2.7 were applied to each fund.
Finally, to annualize the weekly returns, the weekly average was multiplied by the number of weeks in a year:

$$R_{ann} = \bar{R}_{weekly} * 52 \quad (3.3)$$

Table 5: Expected annual logarithmic return of each mutual fund

Fund	Expected return
AcomeA PMITALIA ESG A1	0,0694
Schroder International Selection Fund Italian Equity C Accumulation EUR	0,0664
Symphonia Azionario Small Cap Italia	0,0789
Algebris Core Italy R EUR Acc	0,0878
Allianz Azioni Italia All Stars A	0,0487
Amundi Impegno Italia B	0,0489
Amundi Sviluppo Attivo Italia A	0,0539
Anima Iniziativa Italia A	0,0579
Anima Italia A	0,0485

Arca Azioni Italia P	0,0480
Azimut Trend Italia	0,0479
Bnl Azioni Italia	0,0434
Eurizon Azioni Italia R	0,0579
Fidelity Funds - Italy Fund A-acc-eur	0,0658
Fideuram Italia R	0,0708
Mediolanum Challenge Italian Equity Fund L Acc	0,0461
Mediolanum Flessibile Futuro Italia La	0,0468
Zenit Pianeta Italia R	0,0454
Schroder International Selection Fund Italian Equity A Accumulation Eur	0,0640
Investimenti Azionari Italia A	0,0491
Interfund Equity Italy	0,0595
Fonditalia Equity Italy R	0,0539
Pictet-eur Government Bonds I	0,0146
AMUNDI EURO GOVERNMENT BOND - AE ©	0,0140
Anima Tricolore A	0,0236
JPMorgan Funds - EU Government Bond Fund D (acc)	0,0152
BlueBay Funds - BlueBay Investment Grade Euro Government Bond Fund R EUR Acc	0,0180
Dws Invest Euro-gov Bonds Nc	0,0117
Bnp Paribas Funds Euro Government Bond Privl Capitalisation	0,0137
DPAM B Bonds Eur Govt W Cap	0,0193
Epsilon Fund - Euro Bond Class Unit R Eur Accumulation	0,0166
Euromobiliare Reddito A	0,0104
Generali Investments Sicav - Euro Bond Fund Dx	0,0149
Hsbc Euro Gvt Bond Fund Hc	0,0130
Ishares Euro Government Bond Index Fund (lu) F2 Eur	0,0141
Jpmorgan Funds - Eu Government Bond Fund A (acc) - Eur	0,0160
Generali Investments Sicav - Euro Bond Fund Ex	0,0140
Investimenti Bilanciati Internazionali C	0,0399
Anima Visconteo Plus A	0,0253
Anima Visconteo Plus AD	0,0192
Arca Economia Reale Bilan Italia 55 PIR	0,0399
Azimut Dinamico	0,0168
Carmignac Portfolio Patrimoine Europe A EUR Acc	0,0635
Groupama Bilanciato - NC	0,0399

Eurizon Bilanciato Euro Multimanager	0,0354
Fideuram Bilanciato	0,0270
UBS (Lux) KSS Eur GrInc € P-acc	0,0368
Fidelity Funds - European Multi Asset Income Fund A-Acc-EUR	0,0298
Dpam B - Balanced Flexible B	0,0286
Groupama Bilanciato - Ic	0,0432
Investimenti Bilanciati Internazionali A	0,0348
Ubs (lux) Key Selection Sicav - European Growth And Income (eur) Q-acc	0,0400

2.2.3 - Descriptive analysis of inputs and outputs

Let us now analyze the input and output variables used in the DEA model, using the boxplot to describe the distribution of each variable.

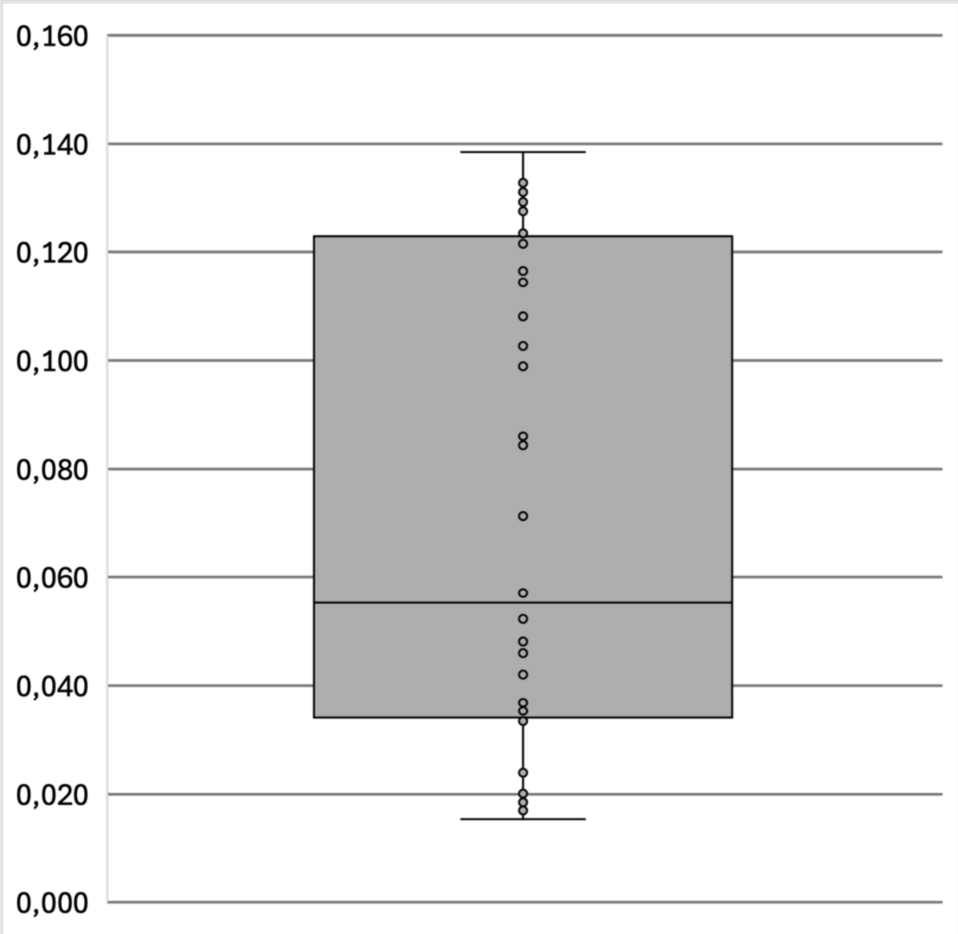

Figure 7: standard deviation boxplot

The boxplot of the standard deviation shows the distribution of the weekly volatility of the funds. The values indicate the dispersion of the annualized logarithmic returns, with the minimum at 0.015, the first quartile at 0.034, the median at 0.055, the third quartile at 0.122, and the maximum at 0.138. Funds with a lower standard deviation can be considered less risky for the investor, while those with a higher standard deviation may be more suitable for investors willing to tolerate higher risk in search of higher returns. Additionally, it is useful to note the differences based on the category of the funds. Funds in the 'Azionario Italia' category have an average standard deviation of 0.120, compared to an average of 0.052 and 0.024 for funds in the 'Bilanciati Moderati EUR' and 'Obbligazionario Governativo EUR' categories, respectively.

As documented by Fama & French(1989); Merton(1974), the equity asset has proven to be more volatile than the bond asset.

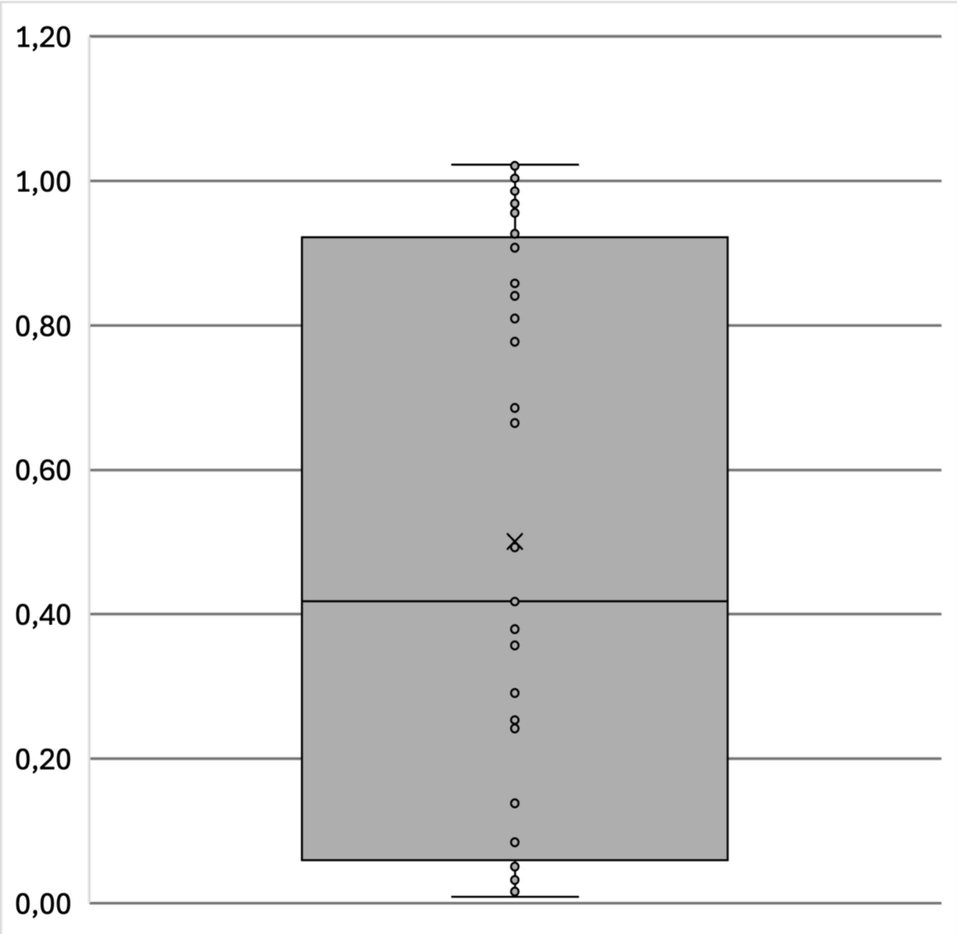

Figure 8: Beta boxplot

The boxplot of the fund beta shows the sensitivity of each fund to market movements compared to the reference index FTSE Italia All Share. The values shown in the graph are: minimum of 0.009, first quartile of 0.076, median of 0.418, third quartile of 0.912, and maximum of 1.022.

Funds with a lower beta indicate less intense fluctuations compared to market movements, while funds with a higher beta are considered riskier but also with higher potential returns, as they tend to move more with wider variations in response to market changes.

Based on the category, it is possible to highlight that fully equity-based funds had higher beta values, closer to the market, with an average of 0.912. In contrast, balanced and purely bond funds had more modest values for this input, indicating less intense fluctuations compared to market changes, with averages of 0.358 and 0.040, respectively. This finding in the sample aligns with literature (see Elton & Gruber(1973); Fama & French(1993); Sharpe(1964)).

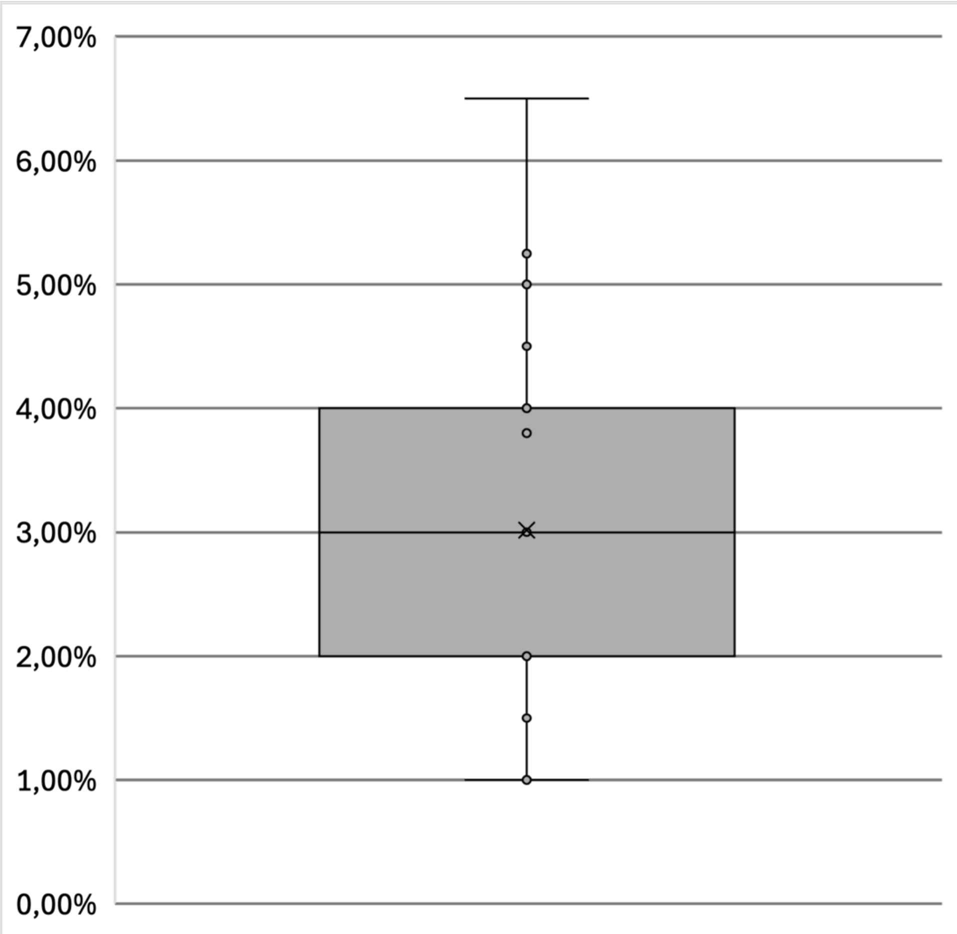

Figure 9: entry fee boxplot

The calculated values are: minimum of 1%, first quartile of 2%, median of 3%, third quartile of 4%, and maximum of 6.5%.
Funds with higher entry fees should be justified by more sophisticated investment strategies and rewarded with higher returns.
For this input, there is no strong evidence to suggest that the category of the fund requires higher or lower fees. Fund managers apply an average entry fee of 3.2% for equity funds and 2.9% for balanced and bond funds.

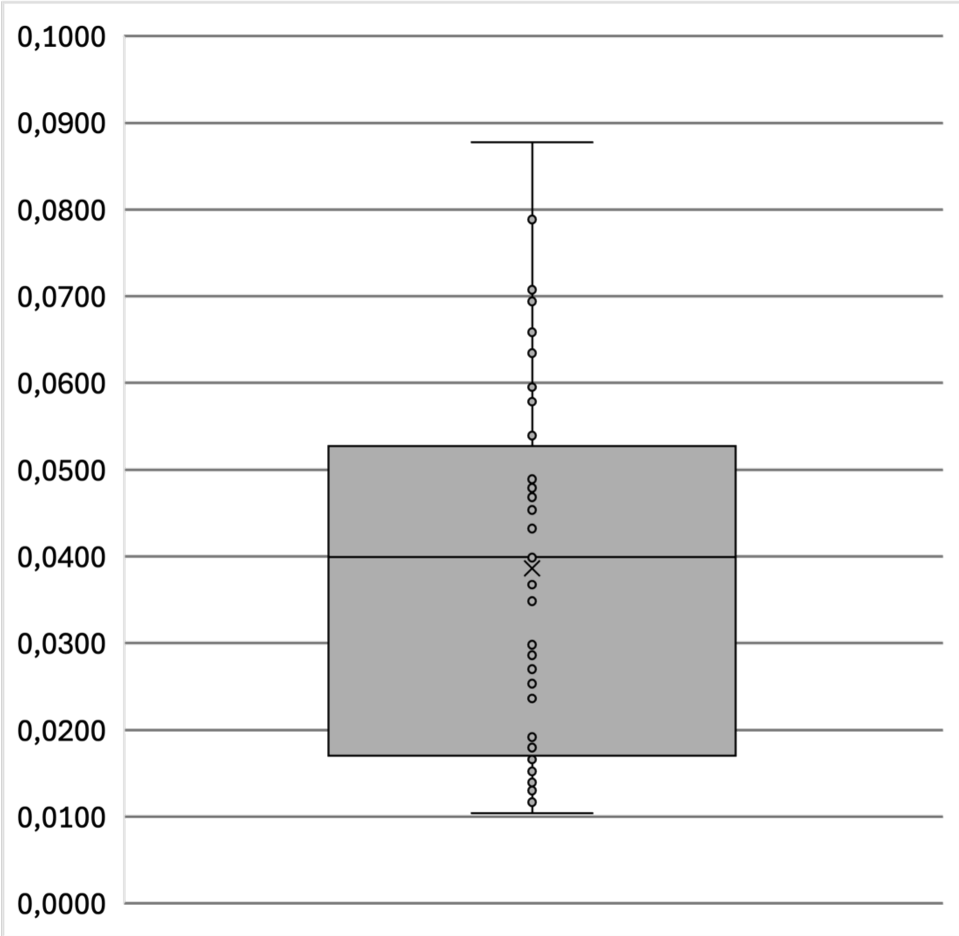

Figure 10: expected return boxplot

The values in the figure above are: minimum of 0.010 (1.05% annual), first quartile of 0.018 (1.78% annual), median of 0.040 (4.07% annual), third quartile of 0.050 (5.16% annual), and maximum of 0.088 (9.07% annual).
Within the categories, evidence can be found in favor of equity assets for the expected return. Indeed, the average return for this category is 0.057 (5.89% annual), compared to 0.035 (3.53% annual) and 0.015 (1.54% annual) for balanced and bond funds, respectively, for the same value. This finding aligns with the literature over time (see Fama & French(1993); Sharpe(1964)).
In conclusion, the funds in the 'Azionario Italia' category showed a greater output in exchange for higher risk inputs, namely deviation standard and Beta.
The efficiency analysis using the DEA methodology serves as a tool to evaluate whether assuming greater risk, measured by the model inputs 'deviation

standard' and 'Beta,' has been justified compared to the 'expected return' output obtained.

2.2.4 - Analysis period and macroeconomic context

The data for the analysis were collected weekly for the period from 01/01/2019 to 30/06/2021, a time frame that saw global macroeconomic events, characterized by a context of great uncertainty and economic instability.
In 2019, global markets experienced a relatively stable period, although the trade tensions between the United States and China generated volatility. Despite this, expansive monetary policies implemented by the Federal Reserve and the European Central Bank helped support economic growth, limiting the effects of geopolitical uncertainties.

In contrast, 2020 was marked by the economic crisis caused by the COVID-19 pandemic, which had a severely negative impact on the global economy. The restrictions imposed to contain the virus spread caused a global recession, with a sharp contraction in demand and disruption of economic activities. To face this emergency situation, central banks implemented extraordinary monetary policies, such as interest rate cuts and asset purchase programs. Despite a significant drop in stock markets during the first months of 2020, the second half of the year saw a partial recovery, thanks to interventions from fiscal and monetary authorities.

In 2021, the analyzed period closed with a partial recovery, although in a context of uncertainties related to the evolution of the pandemic and the difficulty of managing its potential repercussions. Fears of inflation grew, with rising commodity prices and supply chain problems negatively impacting many sectors. Moreover, while accommodative monetary policies continued to support the economy, concerns arose about their potential side effects, particularly on the bond market.

In Italy, the economy experienced a sharp decline in 2020, with one of the most significant GDP contractions in Europe. Key sectors such as tourism and manufacturing were severely impacted by the economic crisis, although fiscal stimulus policies and European aid began to support the recovery in 2021.
The Italian stock market, represented by the FTSE Italia All Share and FTSE MIB, saw significant declines in 2020, followed by a recovery in 2021.
EUR government bond funds were strongly influenced by the ECB's expansive monetary policies, with historically low yields and increased volatility due to inflation concerns.

EUR balanced funds, which combine stocks and bonds, showed some resilience thanks to diversification, but their overall performance was affected by the weakness of the bond markets and the moderate recovery of the stock market.

3 - RESULTS

3.1 - Fund efficiency

The analysis of the efficiency of mutual funds was conducted using the CCR input-oriented model. In such a model, the efficiency of a fund is assessed based on its ability to reduce inputs (volatility, beta, and entry fees) while keeping the output, i.e., the expected return, unchanged. Reducing the inputs indicates greater efficiency, as the fund is able to achieve the same return with lower risk or cost.

Table 6: Efficiency score for each fund

DMU	Efficientcy score
AcomeA_PMITALIA_ESG_A1	0,581
Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	1,000
Symphonia_Azionario_Small_Cap_Italia	0,804
Algebris_Core_Italy_R_EUR_Acc	1,000
Allianz_Azioni_Italia_All_Stars_A	0,536
Amundi_Impegno_Italia_B	0,483
Amundi_Sviluppo_Attivo_Italia_A	0,691
Anima_Iniziativa_Italia_A	0,492
Anima_Italia_A	0,404
Arca_Azioni_Italia_P	0,390
Azimut_Trend_Italia	0,521
Bnl_Azioni_Italia	0,484
Eurizon_Azioni_Italia_R	0,629
Fidelity_Funds_-_Italy_Fund_A-acc-eur	0,555
Fideuram_Italia_R	0,668
Mediolanum_Challenge_Italian_Equity_Fund_L_Acc	0,437
Mediolanum_Flessibile_Futuro_Italia_La	0,480
Zenit_Pianeta_Italia_R	0,582
Schroder_International_Selection_Fund_Italian_Equity_A_Accumulation_Eur	0,550
Investimenti_Azionari_Italia_A	0,541
Interfund_Equity_Italy	0,383
Fonditalia_Equity_Italy_R	0,438
Pictet-eur_Government_Bonds_I	1,000
AMUNDI_EURO_GOVERNMENT_BOND_-_AE	0,735

Anima_Tricolore_A	1,000
JPMorgan_Funds_-_EU_Government_Bond_Fund_D_acc	1,000
BlueBay_Funds_-_BlueBay_Investment_Grade_Euro_Government_Bond_Fund_R_EUR_Acc	0,770
Dws_Invest_Euro-gov_Bonds_Nc	0,699
Bnp_Paribas_Funds_Euro_Government_Bond_Privl_Capitalisation	0,940
DPAM_B_Bonds_Eur_Govt_W_Cap	0,796
Epsilon_Fund_-_Euro_Bond_Class_Unit_R_Eur_Accumulation	1,000
Euromobiliare_Reddito_A	0,741
Generali_Investments_Sicav_-_Euro_Bond_Fund_Dx	0,776
Hsbc_Euro_Gvt_Bond_Fund_Hc	1,000
Ishares_Euro_Government_Bond_Index_Fund_(lu)_F2_Eur	0,592
Jpmorgan_Funds_-_Eu_Government_Bond_Fund_A_acc_-_Eur	0,948
Generali_Investments_Sicav_-_Euro_Bond_Fund_Ex	0,836
Investimenti_Bilanciati_Internazionali_C	0,661
Anima_Visconteo_Plus_A	0,411
Anima_Visconteo_Plus_AD	0,311
Arca_Economia_Reale_Bilan_Italia_55_PIR	0,607
Azimut_Dinamico	0,371
Carmignac_Portfolio_Patrimoine_Europe_A_EUR_Acc	1,000
Groupama_Bilanciato_-_NC	0,745
Eurizon_Bilanciato_Euro_Multimanager	0,695
Fideuram_Bilanciato	0,515
UBS_(Lux)_KSS_Eur_GrInc_P-acc	0,476
Fidelity_Funds_-_European_Multi_Asset_Income_Fund_A-Acc-EUR	0,367
Dpam_B_-_Balanced_Flexible_B	0,680
Groupama_Bilanciato_-_Ic	0,807
Investimenti_Bilanciati_Internazionali_A	0,571
Ubs_(lux)_Key_Selection_Sicav_-_European_Growth_And_Income_EUR_Q-acc	0,518

From the analysis of the efficiency scores of the funds in the sample, 8 funds were found to be efficient, with a score of 1. These funds lie on the efficiency frontier. The other 44 funds in the sample had an efficiency score lower than 1, indicating that they could achieve the same return with a lower level of input, meaning that there are combinations of funds that would improve these funds in order to become more efficient.

The analysis of the results shows that the funds belonging to the "Government Bonds EUR" category were on average more efficient, with an average score of 0.856. This indicates that, despite the lower returns of these funds compared to other categories, the risk they were exposed to was significantly lower, thus justifying their higher efficiency. This is followed by balanced funds, with an average score of 0.582, which showed intermediate performance in terms of efficiency, balancing risk and return more evenly. Finally, the "Italian Equity" funds, with a score of 0.575, despite achieving higher returns, did not adequately compensate for the higher risks taken, thus being less efficient. This highlights how, despite higher returns, equity funds did not show an adequate level of efficiency compared to bond funds, which, despite lower returns (5.89% vs 1.54% annually on average), exhibited significantly lower risk.

3.2 - Efficient combinations on the virtual frontier

The table below presents the efficient combinations obtained through the DEA analysis of the funds. Specifically, it shows the combinations of funds that, according to the CCR input-oriented model, are efficient, based on the value of the vector λ. The vector λ represents the weights associated with each fund in the linear combination used to evaluate the efficiency, where λ is positive for the funds that contribute to the efficient combination for the examined DMU (Decision Making Unit) in the j-th case. The process then proceeds with the application of formula 2.6, yielding:

Table 7: Combination on the efficient frontier for each fund

Index	DMU			
1	AcomeA_PMITALIA_ESG_A1	$l_2 = 0,048$	$l_4 = 0,952$	
2	Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	$l_2 = 1,000$		
3	Symphonia_Azionario_Small_Cap_Italia	$l_4 = 0,322$	$l_{25} = 0,562$	$l_{43} = 0,116$
4	Algebris_Core_Italy_R_EUR_Acc	$l_4 = 1,000$		
5	Allianz_Azioni_Italia_All_Stars_A	$l_2 = 0,697$	$l_4 = 0,303$	
6	Amundi_Impegno_Italia_B	$l_2 = 0,291$	$l_4 = 0,709$	

7	Amundi_Sviluppo_Attivo_Italia_A	l_2 = 0,517	l_4 = 0,483	
8	Anima_Iniziativa_Italia_A	l_2 = 0,012	l_4 = 0,988	
9	Anima_Italia_A	l_2 = 0,059	l_4 = 0,941	
10	Arca_Azioni_Italia_P	l_2 = 0,114	l_4 = 0,886	
11	Azimut_Trend_Italia	l_2 = 0,711	l_4 = 0,289	
12	Bnl_Azioni_Italia	l_2 = 0,683	l_4 = 0,317	
13	Eurizon_Azioni_Italia_R	l_2 = 0,711	l_4 = 0,289	
14	Fidelity_Funds_-_Italy_Fund_A-acc-eur	l_4 = 0,652	l_{43} = 0,348	
15	Fideuram_Italia_R	l_2 = 0,370	l_4 = 0,630	
16	Mediolanum_Challenge_Italian_Equity_Fund_L_Acc	l_2 = 0,363	l_4 = 0,637	
17	Mediolanum_Flessibile_Futuro_Italia_La	l_2 = 0,223	l_4 = 0,777	
18	Zenit_Pianeta_Italia_R	l_2 = 0,516	l_4 = 0,484	
19	Schroder_International_Selection_Fund_Italian_Equity_A_Accumulation_Eur	l_4 = 0,623	l_{43} = 0,377	
20	Investimenti_Azionari_Italia_A	l_2 = 0,697	l_4 = 0,303	
21	Interfund_Equity_Italy	l_4 = 0,544	l_{43} = 0,456	
22	Fonditalia_Equity_Italy_R	l_2 = 0,114	l_4 = 0,886	
23	Pictet-eur_Government_Bonds_I	l_{23} = 1,000		
24	AMUNDI_EURO_GOVERNMENT_BOND_-_AE	l_{23} = 0,431	l_{26} = 0,569	
25	Anima_Tricolore_A	l_{25} = 1,000		
26	JPMorgan_Funds_-_EU_Government_Bond_Fund_D_acc	l_{25} = 1,000		
27	BlueBay_Funds_- _BlueBay_Investment_Grade_Euro_Government_Bond_Fund_R_EUR_Acc	l_{26} = 0,893	l_{43} = 0,107	
28	Dws_Invest_Euro-gov_Bonds_Nc	l_{34} = 0,889	l_{43} = 0,111	
29	Bnp_Paribas_Funds_Euro_Government_Bond_Privl_Capitalisation	l_{26} = 0,994	l_{43} = 0,006	
30	DPAM_B_Bonds_Eur_Govt_W_Cap	l_{34} = 0,571	l_{43} = 0,429	
31	Epsilon_Fund_-_Euro_Bond_Class_Unit_R_Eur_Accumulation	l_{31} = 1,000		
32	Euromobiliare_Reddito_A	l_{26} = 0,382	l_{31} = 0,599	l_{43} = 0,020
33	Generali_Investments_Sicav_-_Euro_Bond_Fund_Dx	l_{26} = 0,902	l_{43} = 0,098	
34	Hsbc_Euro_Gvt_Bond_Fund_Hc	l_{34} = 1,000		
35	Ishares_Euro_Government_Bond_Index_Fund_(lu)_F2_Eur	l_{23} = 0,091	l_{26} = 0,909	
36	Jpmorgan_Funds_-_Eu_Government_Bond_Fund_A_acc_-_Eur	l_{26} = 0,948	l_{43} = 0,052	
37	Generali_Investments_Sicav_-_Euro_Bond_Fund_Ex	l_{34} = 0,889	l_{43} = 0,111	

38	**Investimenti_Bilanciati_Internazionali_C**	l_{2} = 0,070	l_{4} = 0,930	
39	**Anima_Visconteo_Plus_A**	l_{4} = 0,415	l_{43} = 0,585	
40	**Anima_Visconteo_Plus_AD**	l_{4} = 0,410	l_{25} = 0,023	l_{43} = 0,566
41	**Arca_Economia_Reale_Bilan_Italia_55_PIR**	l_{4} = 0,522	l_{43} = 0,578	
42	**Azimut_Dinamico**	l_{4} = 0,472	l_{25} = 0,177	l_{43} = 0,351
43	**Carmignac_Portfolio_Patrimoine_Europe_A_EUR_Acc**	l_{43} = 1,000		
44	**Groupama_Bilanciato_-_NC**	l_{4} = 0,091	l_{25} = 0,298	l_{43} = 0,611
45	**Eurizon_Bilanciato_Euro_Multimanager**	l_{4} = 0,596	l_{25} = 0,237	l_{43} = 0,167
46	**Fideuram_Bilanciato**	l_{4} = 0,648	l_{26} = 0,225	l_{43} = 0,128
47	**UBS_(Lux)_KSS_Eur_GrInc_P-acc**	l_{4} = 0,316	l_{43} = 0,684	
48	**Fidelity_Funds_-_European_Multi_Asset_Income_Fund_A-Acc-EUR**	l_{4} = 0,037	l_{43} = 0,963	
49	**Dpam_B_-_Balanced_Flexible_B**	l_{4} = 0,414	l_{25} = 0,080	l_{43} = 0,506
50	**Groupama_Bilanciato_-_Ic**	l_{4} = 0,091	l_{26} = 0,298	l_{43} = 0,611
51	**Investimenti_Bilanciati_Internazionali_A**	l_{2} = 0,092	l_{4} = 0,908	
52	**Ubs_(lux)_Key_Selection_Sicav_-_European_Growth_And_Income_EUR_Q-acc**	l_{4} = 0,316		l_{43} = 0,684

Below is a table showing the weighted inputs and outputs based on the efficient combinations. The inputs and outputs are calculated using the weights assigned to each fund, and therefore reflect the values of standard deviation, Beta, entry fees, and expected return of the respective efficient combination, which lies on the frontier. These values represent the conditions that would have allowed the investor to achieve the highest level of efficiency.

Table 8: Combination on the efficient frontier for each fund

DMU	**Deviazione standard**	**Beta**	**Commissione d'ingresso**	**Rendimento atteso**
AcomeA_PMITALIA_ESG_A1	0,087	0,70	0,03	0,087
Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	0,108	0,84	0,01	0,066
Symphonia_Azionario_Small_Cap_Italia	0,062	0,33	0,02	0,049
Algebris_Core_Italy_R_EUR_Acc	0,086	0,69	0,03	0,088
Allianz_Azioni_Italia_All_Stars_A	0,101	0,79	0,02	0,073
Amundi_Impegno_Italia_B	0,092	0,73	0,02	0,082
Amundi_Sviluppo_Attivo_Italia_A	0,097	0,77	0,02	0,077
Anima_Iniziativa_Italia_A	0,086	0,69	0,03	0,088

Anima_Italia_A	0,087	0,70	0,03	0,087
Arca_Azioni_Italia_P	0,088	0,71	0,03	0,085
Azimut_Trend_Italia	0,102	0,80	0,02	0,073
Bnl_Azioni_Italia	0,101	0,79	0,02	0,073
Eurizon_Azioni_Italia_R	0,102	0,80	0,02	0,073
Fidelity_Funds_-_Italy_Fund_A-acc-eur	0,069	0,53	0,03	0,079
Fideuram_Italia_R	0,094	0,75	0,02	0,080
Mediolanum_Challenge_Italian_Equity_Fund_L_Acc	0,094	0,74	0,02	0,080
Mediolanum_Flessibile_Futuro_Italia_La	0,091	0,72	0,03	0,083
Zenit_Pianeta_Italia_R	0,097	0,77	0,02	0,077
Schroder_International_Selection_Fund_Italian_Equity_A_Accumulation_Eur	0,068	0,52	0,03	0,079
Investimenti_Azionari_Italia_A	0,101	0,79	0,02	0,073
Interfund_Equity_Italy	0,064	0,49	0,03	0,077
Fonditalia_Equity_Italy_R	0,088	0,71	0,03	0,085
Pictet-eur_Government_Bonds_I	0,035	0,01	0,05	0,015
AMUNDI_EURO_GOVERNMENT_BOND_-_AE	0,026	0,02	0,04	0,015
Anima_Tricolore_A	0,053	0,14	0,01	0,024
JPMorgan_Funds_-_EU_Government_Bond_Fund_D_acc	0,019	0,02	0,03	0,015
BlueBay_Funds_-_BlueBay_Investment_Grade_Euro_Government_Bond_Fund_R_EUR_Acc	0,021	0,04	0,03	0,020
Dws_Invest_Euro-gov_Bonds_Nc	0,021	0,04	0,02	0,019
Bnp_Paribas_Funds_Euro_Government_Bond_Privl_Capitalisation	0,019	0,02	0,03	0,015
DPAM_B_Bonds_Eur_Govt_W_Cap	0,027	0,11	0,03	0,035
Epsilon_Fund_-_Euro_Bond_Class_Unit_R_Eur_Accumulation	0,017	0,04	0,02	0,017
Euromobiliare_Reddito_A	0,018	0,04	0,02	0,017
Generali_Investments_Sicav_-_Euro_Bond_Fund_Dx	0,021	0,04	0,03	0,020
Hsbc_Euro_Gvt_Bond_Fund_Hc	0,019	0,02	0,02	0,013
Ishares_Euro_Government_Bond_Index_Fund_(lu)_F2_Eur	0,020	0,02	0,03	0,015
Jpmorgan_Funds_-_Eu_Government_Bond_Fund_A_acc_-_Eur	0,020	0,03	0,03	0,018
Generali_Investments_Sicav_-_Euro_Bond_Fund_Ex	0,021	0,04	0,02	0,019
Investimenti_Bilanciati_Internazionali_C	0,088	0,70	0,03	0,086
Anima_Visconteo_Plus_A	0,057	0,43	0,04	0,074
Anima_Visconteo_Plus_AD	0,057	0,42	0,04	0,073
Arca_Economia_Reale_Bilan_Italia_55_PIR	0,063	0,48	0,03	0,076

Azimut_Dinamico	0,063	0,43	0,03	0,068
Carmignac_Portfolio_Patrimoine_Europe_A_EUR_Acc	0,037	0,24	0,04	0,064
Groupama_Bilanciato_-_NC	0,046	0,25	0,03	0,054
Eurizon_Bilanciato_Euro_Multimanager	0,070	0,48	0,03	0,069
Fideuram_Bilanciato	0,072	0,51	0,03	0,070
UBS_(Lux)_KSS_Eur_GrInc_P-acc	0,052	0,38	0,04	0,071
Fidelity_Funds_-_European_Multi_Asset_Income_Fund_A-Acc-EUR	0,039	0,26	0,04	0,064
Dpam_B_-_Balanced_Flexible_B	0,059	0,42	0,03	0,070
Groupama_Bilanciato_-_Ic	0,046	0,25	0,03	0,054
Investimenti_Bilanciati_Internazionali_A	0,088	0,70	0,03	0,086
Ubs_(lux)_Key_Selection_Sicav_-_European_Growth_And_Income_EUR_Q-acc	0,052	0,38	0,04	0,071

4 - DISCUSSION

This chapter will analyze the results obtained from the application of Data Envelopment Analysis (DEA), comparing them with those obtained through more traditional approaches. Additionally, some key issues in the current financial context will be explored.

In the first section, a comparison will be made between the fund classification obtained using the DEA model and the one derived from the use of traditional indicators such as the Sharpe and Treynor ratios, as mentioned in previous chapters.

Next, the role of ETFs, investment instruments whose popularity is steadily increasing due to their cost-efficiency, will be examined. The integration of ETFs into the DEA model represents an innovation compared to the reference study by Basso & Funari(2001), enabling an assessment of how the passive nature of these instruments influences efficiency analysis compared to actively managed funds.

Finally, the topic of ESG funds will be explored, a field of growing importance in the European financial landscape. In particular, it will be examined whether funds with a high sustainability rating can translate this characteristic into a competitive advantage in terms of efficiency, compared to funds without a specific ESG designation.

4.1 - Comparison between DEA results and traditional indices

To compare the classification of funds obtained through the DEA approach with traditional methodologies, the Sharpe ratio and the Treynor ratio are calculated for each analyzed fund. Both indicators provide a measure of risk-adjusted performance, although they adopt different methodologies.

The Sharpe ratio evaluates the excess return over the risk-free asset per unit of total volatility, thus providing a performance measure that considers the overall risk of the investment. The Treynor ratio, on the other hand, is based on the ratio of excess return to systematic risk, represented by the beta coefficient, making it particularly suitable for contexts where diversifiable risk is negligible.

The joint analysis of these indicators provides a more comprehensive perspective on the efficiency of the funds, highlighting any discrepancies between the classification based on DEA and that derived from traditional approaches.

The calculation of the Sharpe ratio follows the formula:

$$S_i = \frac{R_i - R_f}{\sigma_i} \quad (3.4)$$

where:

- S_i is the Sharpe ratio of the i-th fund
- R_i is the average annual return of the fund over the analyzed period, used as the output of the DEA model
- R_f is the risk-free rate, calculated as the average return of the 12-month Italian BOT
- σ_i is the standard deviation of the fund’s returns, already used as input in the DEA model

The Sharpe ratio quantifies the excess return relative to the risk-free rate per unit of total risk assumed by the fund, measured through the standard deviation of returns. A high value of the ratio indicates efficient risk management, as it shows that the fund has been able to generate a higher return than a risk-free investment, considering the volatility assumed.

Similarly, the Treynor ratio is calculated as:

$$T_i = \frac{R_i - R_f}{\beta_i} \quad (3.5)$$

where:

- T_i is the Treynor ratio of the i-th fund
- R_i is the average annual return of the fund over the analyzed period, used as the output of the DEA model
- R_f is the risk-free rate, calculated as the average return of the 12-month Italian BOT
- β_i is the Beta of the fund, which measures the sensitivity of the fund's returns to the market benchmark and is the same as the input used in the DEA model.

The Treynor ratio measures the return adjusted for systematic risk, represented by the fund's Beta relative to the market. Unlike the Sharpe ratio, which uses total volatility as a measure of risk, the Treynor ratio focuses only on market risk, excluding the specific risk of the individual fund.

A high value of the Treynor ratio suggests that the fund has achieved a superior return for each unit of systematic risk assumed. This implies greater efficiency in the fund's ability to compensate investors for exposure to market risk.

Below are the values of the two traditional indices for each fund, which will subsequently be compared with the efficiency score estimated through the DEA approach. This comparison will help identify any discrepancies between the methodologies and explore their respective strengths and weaknesses.

Table 9: DEA efficiency index, Sharpe ratio, and Treynor index for each fund

Fund	DEA	Sharpe	Treynor
AcomeA_PMITALIA_ESG_A1	0,581	0,595	0,077
Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	1,000	0,649	0,083

Symphonia_Azionario_Small_Cap_Italia	0,804	0,677	0,126
Algebris_Core_Italy_R_EUR_Acc	1,000	1,084	0,136
Allianz_Azioni_Italia_All_Stars_A	0,536	0,404	0,052
Amundi_Impegno_Italia_B	0,483	0,444	0,057
Amundi_Sviluppo_Attivo_Italia_A	0,691	0,576	0,070
Anima_Iniziativa_Italia_A	0,492	0,503	0,066
Anima_Italia_A	0,404	0,396	0,053
Arca_Azioni_Italia_P	0,390	0,387	0,050
Azimut_Trend_Italia	0,521	0,379	0,050
Bnl_Azioni_Italia	0,484	0,332	0,047
Eurizon_Azioni_Italia_R	0,629	0,470	0,060
Fidelity_Funds_-_Italy_Fund_A-acc-eur	0,555	0,678	0,081
Fideuram_Italia_R	0,668	0,564	0,076
Mediolanum_Challenge_Italian_Equity_Fund_L_Acc	0,437	0,392	0,049
Mediolanum_Flessibile_Futuro_Italia_La	0,480	0,433	0,058
Zenit_Pianeta_Italia_R	0,582	0,467	0,062
Schroder_International_Selection_Fund_Italian_Equity_A_Accumulation_Eur	0,550	0,675	0,084
Investimenti_Azionari_Italia_A	0,541	0,398	0,052
Interfund_Equity_Italy	0,383	0,489	0,063
Fonditalia_Equity_Italy_R	0,438	0,434	0,056
Pictet-eur_Government_Bonds_I	1,000	0,473	1,849
AMUNDI_EURO_GOVERNMENT_BOND_-_AE	0,735	0,468	0,939
Anima_Tricolore_A	1,000	0,482	0,184
JPMorgan_Funds_-_EU_Government_Bond_Fund_D_acc	1,000	0,892	1,062
BlueBay_Funds_-_BlueBay_Investment_Grade_Euro_Government_Bond_Fund_R_EUR_Acc	0,770	0,825	0,388
Dws_Invest_Euro-gov_Bonds_Nc	0,699	0,679	0,332
Bnp_Paribas_Funds_Euro_Government_Bond_Privl_Capitalisation	0,940	0,830	0,814
DPAM_B_Bonds_Eur_Govt_W_Cap	0,796	0,981	0,250
Epsilon_Fund_-_Euro_Bond_Class_Unit_R_Eur_Accumulation	1,000	1,080	0,451
Euromobiliare_Reddito_A	0,741	0,786	0,377
Generali_Investments_Sicav_-_Euro_Bond_Fund_Dx	0,776	0,823	0,427
Hsbc_Euro_Gvt_Bond_Fund_Hc	1,000	0,787	0,658
Ishares_Euro_Government_Bond_Index_Fund_(lu)_F2_Eur	0,592	0,465	0,601
Jpmorgan_Funds_-_Eu_Government_Bond_Fund_A_acc_-_Eur	0,948	0,950	0,530

Generali_Investments_Sicav_-_Euro_Bond_Fund_Ex	0,836	0,781	0,404
Investimenti_Bilanciati_Internazionali_C	0,661	0,501	0,086
Anima_Visconteo_Plus_A	0,411	0,565	0,076
Anima_Visconteo_Plus_AD	0,311	0,424	0,058
Arca_Economia_Reale_Bilan_Italia_55_PIR	0,607	0,791	0,099
Azimut_Dinamico	0,371	0,439	0,063
Carmignac_Portfolio_Patrimoine_Europe_A_EUR_Acc	1,000	1,822	0,277
Groupama_Bilanciato_-_NC	0,745	0,919	0,169
Eurizon_Bilanciato_Euro_Multimanager	0,695	0,719	0,104
Fideuram_Bilanciato	0,515	0,539	0,076
UBS_(Lux)_KSS_Eur_GrInc_P-acc	0,476	0,683	0,093
Fidelity_Funds_-_European_Multi_Asset_Income_Fund_A-Acc-EUR	0,367	0,645	0,087
Dpam_B_-_Balanced_Flexible_B	0,680	0,866	0,121
Groupama_Bilanciato_-_Ic	0,807	0,991	0,182
Investimenti_Bilanciati_Internazionali_A	0,571	0,520	0,074
Ubs_(lux)_Key_Selection_Sicav_-_European_Growth_And_Income_EUR_Q-acc	0,518	0,742	0,101

Table 10: Ranking of each fund relative to each index (DEA efficiency index, Sharpe ratio, Treynor index)

Fund	**Ranking DEA**	**Ranking Sharpe**	**Ranking Treynor**
AcomeA_PMITALIA_ESG_A1	30	26	31
Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	1	24	29
Symphonia_Azionario_Small_Cap_Italia	13	22	20
Algebris_Core_Italy_R_EUR_Acc	2	2	19
Allianz_Azioni_Italia_All_Stars_A	35	46	48
Amundi_Impegno_Italia_B	41	41	44
Amundi_Sviluppo_Attivo_Italia_A	22	27	36
Anima_Iniziativa_Italia_A	39	32	37
Anima_Italia_A	47	48	46
Arca_Azioni_Italia_P	48	50	50
Azimut_Trend_Italia	36	51	49
Bnl_Azioni_Italia	40	52	52

Eurizon_Azioni_Italia_R	26	37	41
Fidelity_Funds_-_Italy_Fund_A-acc-eur	32	21	30
Fideuram_Italia_R	24	29	34
Mediolanum_Challenge_Italian_Equity_Fund_L_Acc	45	49	51
Mediolanum_Flessibile_Futuro_Italia_La	42	44	42
Zenit_Pianeta_Italia_R	29	39	40
Schroder_International_Selection_Fund_Italian_Equity_A_Accumulation_Eur	33	23	28
Investimenti_Azionari_Italia_A	34	47	47
Interfund_Equity_Italy	49	34	39
Fonditalia_Equity_Italy_R	44	43	45
Pictet-eur_Government_Bonds_I	3	36	1
AMUNDI_EURO_GOVERNMENT_BOND_-_AE	19	38	3
Anima_Tricolore_A	4	35	16
JPMorgan_Funds_-_EU_Government_Bond_Fund_D_acc	5	8	2
BlueBay_Funds_-_BlueBay_Investment_Grade_Euro_Government_Bond_Fund_R_EUR_Acc	16	11	11
Dws_Invest_Euro-gov_Bonds_Nc	20	20	13
Bnp_Paribas_Funds_Euro_Government_Bond_Privl_Capitalisation	10	10	4
DPAM_B_Bonds_Eur_Govt_W_Cap	14	5	15
Epsilon_Fund_-_Euro_Bond_Class_Unit_R_Eur_Accumulation	6	3	8
Euromobiliare_Reddito_A	18	15	12
Generali_Investments_Sicav_-_Euro_Bond_Fund_Dx	15	12	9
Hsbc_Euro_Gvt_Bond_Fund_Hc	7	14	5
Ishares_Euro_Government_Bond_Index_Fund_(lu)_F2_Eur	28	40	6
Jpmorgan_Funds_-_Eu_Government_Bond_Fund_A_acc_-_Eur	9	6	7
Generali_Investments_Sicav_-_Euro_Bond_Fund_Ex	11	16	10
Investimenti_Bilanciati_Internazionali_C	25	33	27
Anima_Visconteo_Plus_A	46	28	32
Anima_Visconteo_Plus_AD	52	45	43
Arca_Economia_Reale_Bilan_Italia_55_PIR	27	13	24
Azimut_Dinamico	50	42	38
Carmignac_Portfolio_Patrimoine_Europe_A_EUR_Acc	8	1	14
Groupama_Bilanciato_-_NC	17	7	18
Eurizon_Bilanciato_Euro_Multimanager	21	18	22
Fideuram_Bilanciato	38	30	33

UBS_(Lux)_KSS_Eur_GrInc_P-acc	43	19	25
Fidelity_Funds_-_European_Multi_Asset_Income_Fund_A-Acc-EUR	51	25	26
Dpam_B_-_Balanced_Flexible_B	23	9	21
Groupama_Bilanciato_-_Ic	12	4	17
Investimenti_Bilanciati_Internazionali_A	31	31	35
Ubs_(lux)_Key_Selection_Sicav_-_European_Growth_And_Income_EUR_Q-acc	37	17	23

Below is the list of funds showing the most discordant values for ranking based on the three indices considered (DEA efficiency index, Sharpe ratio, and Treynor ratio):

- Schroder International Selection Fund Italian Equity C Accumulation EUR: DEA 1st, Sharpe 24th, Treynor 29th
- Anima Tricolore A: DEA 4th, Sharpe 35th, Treynor 16th
- DPAM B - Balanced Flexible B: DEA 23rd, Sharpe 9th, Treynor 21st
- Fidelity Funds - Italy Fund A-acc-EUR: DEA 32nd, Sharpe 21st, Treynor 30th
- Groupama Bilanciato - IC: DEA 12th, Sharpe 4th, Treynor 17th
- JPMorgan Funds - EU Government Bond Fund D acc: DEA 5th, Sharpe 8th, Treynor 2nd
- Carmignac Portfolio Patrimoine Europe A EUR Acc: DEA 8th, Sharpe 1st, Treynor 14th
- Pictet-eur Government Bonds I: DEA 3rd, Sharpe 36th, Treynor 1st
- BNP Paribas Funds Euro Government Bond Privl Capitalisation: DEA 10th, Sharpe 10th, Treynor 4th
- Anima Visconteo Plus AD: DEA 52nd, Sharpe 45th, Treynor 43rd

The difference between the results obtained through the DEA approach and those derived from traditional indices can be attributed to the ability of the DEA model to provide a more comprehensive and realistic evaluation of funds, incorporating additional variables, such as entry costs, which are not considered by traditional indicators. This aspect allows for a more accurate estimate of the net return received by the investor.

Specifically, a fund might register a high value in the Sharpe or Treynor indices, indicating good performance in relation to the risk taken. However, since these indices do not account for entry costs, the actual return may be lower than what the traditional measures suggest, thus providing a potentially distorted view of the fund's true efficiency.

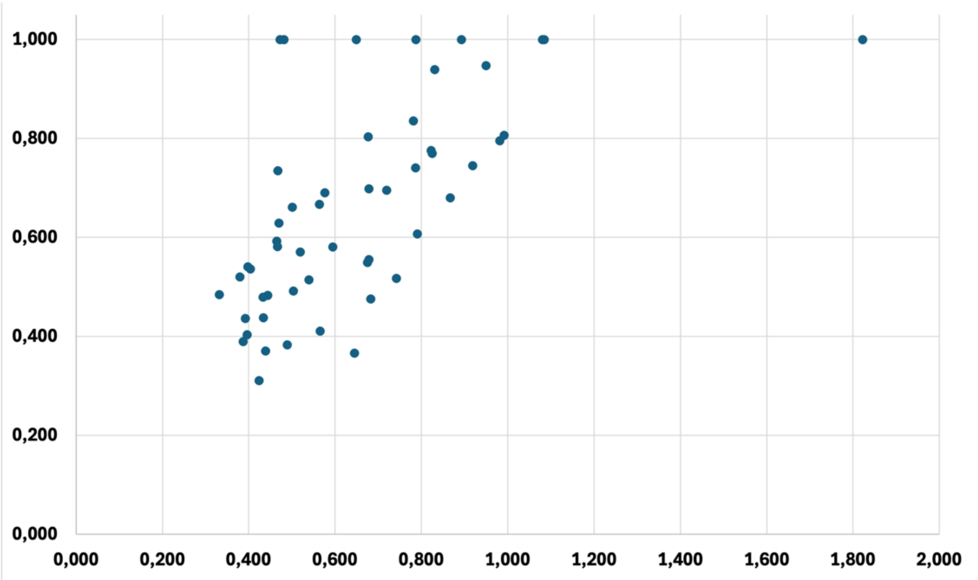

Figure 11: scatter plot of the Sharpe ratio (X-axis) versus the DEA efficiency index (Y-axis) for each fund in the sample.

As highlighted in the scatter plot, the analyzed funds can be classified into three main categories based on the relationship between the Sharpe ratio and the efficiency score estimated through DEA:

- Funds with a proportional Sharpe ratio and DEA efficiency score: these funds exhibit both a high risk-adjusted return and efficient operations, characterized by low entry costs or reduced sensitivity to market fluctuations. For example, Algebris Core Italy R EUR Acc records a DEA score of 1.000 and a Sharpe ratio of 1.084, while Groupama Bilanciato-Ic has a DEA of 0.807 and a Sharpe of 0.991. In these cases, the alignment between the two indicators suggests that the additional costs captured by the DEA analysis are compensated by efficient fund management.
- Funds with a low Sharpe ratio and a high DEA efficiency score: this category includes funds that, although not showing particularly high

returns relative to the risk taken (low Sharpe ratio), are efficient due to competitive entry costs or management that results in a low Beta, thus less exposure to market risk. An example is Pictet-EUR Government Bonds I, which records a DEA score of 1.000 but a Sharpe ratio of 0.473, having reported a Beta of 0.01 during the analyzed period. Similarly, Anima Tricolore A has a DEA efficiency score of 1.000 and a Sharpe ratio of 0.482. In this case, the high DEA efficiency is due to particularly low entry fees (1.00% compared to an average of 3.00%), which help offset the lower risk-return performance.

- Funds with a high Sharpe ratio and a low DEA score: this type includes funds that, despite having a good risk-adjusted return (high Sharpe ratio), receive a lower DEA score due to high entry costs or greater exposure to systematic risk. An example is Fidelity Funds - European Multi Asset Income Fund A-Acc-EUR, which has a Sharpe ratio of 0.645 but a DEA score of 0.367, penalized by entry costs of 5.25%, significantly higher than the average of 3.00%. In these cases, although the fund shows good performance in terms of risk-return, the high fees reduce operational efficiency, limiting the net return actually received by the investor.

The linear correlation index between the Sharpe ratio and the DEA efficiency score is 0.639. This value indicates a moderate positive correlation between the two indicators, suggesting that, in general, funds with better risk-adjusted returns also tend to show greater operational efficiency. However, the correlation is not perfect, highlighting that the two indices capture different aspects of performance: while the Sharpe ratio focuses on risk-return, DEA also evaluates the impact of costs and risk management. The discrepancy between the two approaches underscores the importance of considering both indicators for a comprehensive assessment of fund performance.

These results highlight the interpretative differences between the two indicators: while the Sharpe ratio provides a measure of return relative to total risk, DEA allows for a more nuanced evaluation, incorporating not only the impact of entry costs but also the risk taken relative to the benchmark market, measured through Beta. The discrepancies between the two approaches help identify those funds that, despite showing good performance in terms of risk-return, are penalized by a high-cost structure or inefficient risk management, and those that, while having a lower return, stand out for their greater operational efficiency due to reduced costs and prudent risk management.

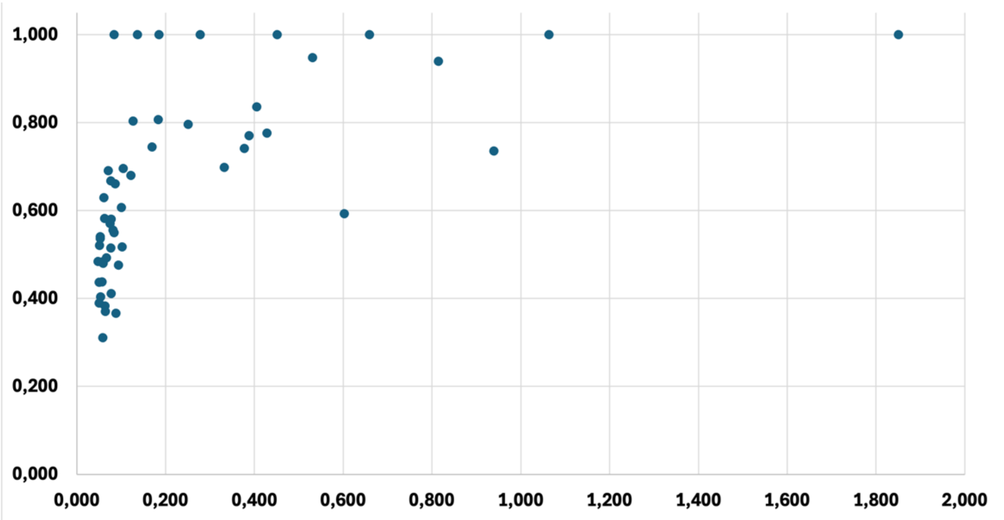

Figure 12: scatter plot of the Treynor index (X-axis) versus the DEA efficiency index (Y-axis) for each fund in the sample.

As highlighted in the scatter plot, the funds can exhibit the following characteristics:

- Proportional Treynor index and DEA efficiency score: these funds show a good return per unit of systematic risk, as well as efficient operational management, with low entry costs or a well-managed risk structure. In other words, these funds combine high returns relative to systematic risk with effective resource and cost management.
- Low Treynor index and high DEA efficiency score: in this case, the fund does not show particularly high returns per unit of systematic risk but manages to be efficient due to competitive entry costs or effective management of systematic risk. An example of such behavior is represented by the Algebris Core Italy R EUR Acc fund, which has a

DEA score of 1.000 and a Treynor index of 0.136. The high DEA efficiency indicates that the fund benefits from very low entry fees and effective management of volatility, thus compensating for the relatively lower systematic return.

- High Treynor index and low DEA score: in this scenario, the funds show good returns per unit of systematic risk (indicated by a high Treynor index) but a low DEA score, generally due to high entry costs or less efficient management of systematic risk. An example is the Ishares Euro Government Bond Index Fund (LU) F2 EUR, which has a Treynor index of 0.601 and a DEA score of 0.592. This indicates that, despite the good return per unit of systematic risk (a relatively high Treynor index compared to other funds in the sample, many of which have values below 0.1-0.2), the fund is less efficient operationally (with a relatively low DEA efficiency score compared to the maximum of 1.000). This discrepancy suggests that, while generating a return adjusted for systematic risk, the fund's overall efficiency is penalized by high entry costs or higher volatility, factors that DEA can capture but that do not directly influence the Treynor index.

The linear correlation index between the Treynor ratio and the DEA efficiency score is 0.585. This value indicates a moderate positive correlation between the two indicators, suggesting that, in general, funds with better risk-adjusted returns for systematic risk (measured by Beta) also tend to show greater operational efficiency. However, the correlation is not perfect, highlighting that the two indices capture different aspects of performance: while the Treynor ratio focuses on return relative to systematic risk, DEA also evaluates the impact of costs and operational management. The discrepancy between the two approaches underscores the importance of considering both indicators for a more comprehensive assessment of fund performance.

These examples highlight the difference in the interpretation of the indicators: while the Treynor index measures return per unit of systematic risk (beta), DEA provides a more comprehensive evaluation, which also includes entry costs and risk management in relation to the benchmark market. The divergences between the two indices can, therefore, reveal whether a fund, despite appearing high-performing in terms of systematic risk, is penalized by a high-cost structure or inefficient management, or if a fund with a less impressive return can still be efficient due to low costs and optimal risk management.

4.2 - Analysis of the results' implications in relation to ETFs

In the context of investment management, Exchange-Traded Funds (ETFs) are generally considered a more efficient form of investment compared to active funds due to their lower management costs. ETFs are investment instruments that replicate a market index, adopting a passive management strategy. In contrast, active funds are managed by professionals who make strategic decisions based on analysis and forecasts, which results in a higher cost structure. However, the efficiency of a fund is not solely dependent on the type of management strategy used but is also influenced by other factors, such as entry costs, ongoing expenses, long-term performance, and associated risk.

The advantage of ETFs, in terms of management and entry fees, is made possible by passive management, which eliminates the need for dedicated managers to select and manage portfolios. This reduction in costs could translate into greater efficiency, but this would only occur if the activities of the managers in active funds do not generate returns proportional to the risk, which would be higher than the additional cost incurred by investors for active management.

Next, the same DEA analysis developed so far will be conducted, introducing two ETFs: the first replicates the Italian equity market, specifically the FTSE MIB; the second replicates the European bond market, tracking the IBOXX® € SOVEREIGNS EUROZONE INDEX. No additional ETFs were included, as, due to the passive nature of this instrument, the performance of ETFs that track the same index is virtually identical.

Below, the inputs and outputs related to the two ETFs added to the previously analyzed sample are presented.

Table 11: Inputs and outputs of the ETFs added to the DEA analysis

DMU	Standard deviation	Beta	Entry fee	Expected return
AcomeA_PMITALIA_ESG_A1	0,121	1,03	0,00	0,0587
Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	0,019	0,03	0,00	0,0146

Table 12: DEA analysis with the introduction of ETFs

DMU	Efficiency score ETF
AcomeA_PMITALIA_ESG_A1	0,580
Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	1,000
Symphonia_Azionario_Small_Cap_Italia	0,789
Algebris_Core_Italy_R_EUR_Acc	1,000
Allianz_Azioni_Italia_All_Stars_A	0,535
Amundi_Impegno_Italia_B	0,483
Amundi_Sviluppo_Attivo_Italia_A	0,684
Anima_Iniziativa_Italia_A	0,490
Anima_Italia_A	0,401
Arca_Azioni_Italia_P	0,389
Azimut_Trend_Italia	0,516
Bnl_Azioni_Italia	0,478
Eurizon_Azioni_Italia_R	0,629
Fidelity_Funds_-_Italy_Fund_A-acc-eur	0,562
Fideuram_Italia_R	0,665
Mediolanum_Challenge_Italian_Equity_Fund_L_Acc	0,440
Mediolanum_Flessibile_Futuro_Italia_La	0,476
Zenit_Pianeta_Italia_R	0,581
Schroder_International_Selection_Fund_Italian_Equity_A_Accumulation_Eur	0,546

Investimenti_Azionari_Italia_A	0,539
Interfund_Equity_Italy	0,381
Fonditalia_Equity_Italy_R	0,437
Pictet-eur_Government_Bonds_I	1,000
AMUNDI_EURO_GOVERNMENT_BOND_-_AE	0,729
Anima_Tricolore_A	0,939
JPMorgan_Funds_-_EU_Government_Bond_Fund_D_acc	1,000
BlueBay_Funds_-_BlueBay_Investment_Grade_Euro_Government_Bond_Fund_R_EUR_Acc	0,801
Dws_Invest_Euro-gov_Bonds_Nc	0,686
Bnp_Paribas_Funds_Euro_Government_Bond_Privl_Capitalisation	0,901
DPAM_B_Bonds_Eur_Govt_W_Cap	0,787
Epsilon_Fund_-_Euro_Bond_Class_Unit_R_Eur_Accumulation	1,000
Euromobiliare_Reddito_A	0,734
Generali_Investments_Sicav_-_Euro_Bond_Fund_Dx	0,743
Hsbc_Euro_Gvt_Bond_Fund_Hc	1,000
Ishares_Euro_Government_Bond_Index_Fund_(lu)_F2_Eur	0,592
Jpmorgan_Funds_-_Eu_Government_Bond_Fund_A_acc_-_Eur	0,908
Generali_Investments_Sicav_-_Euro_Bond_Fund_Ex	0,821
Investimenti_Bilanciati_Internazionali_C	0,636
Anima_Visconteo_Plus_A	0,408
Anima_Visconteo_Plus_AD	0,310
Arca_Economia_Reale_Bilan_Italia_55_PIR	0,629
Azimut_Dinamico	0,368
Carmignac_Portfolio_Patrimoine_Europe_A_EUR_Acc	1,000
Groupama_Bilanciato_-_NC	0,741
Eurizon_Bilanciato_Euro_Multimanager	0,693
Fideuram_Bilanciato	0,515
UBS_(Lux)_KSS_Eur_GrInc_P-acc	0,473
Fidelity_Funds_-_European_Multi_Asset_Income_Fund_A-Acc-EUR	0,359
Dpam_B_-_Balanced_Flexible_B	0,677
Groupama_Bilanciato_-_Ic	0,802
Investimenti_Bilanciati_Internazionali_A	0,563
Ubs_(lux)_Key_Selection_Sicav_-_European_Growth_And_Income_EUR_Q-acc	0,515
iShares FTSE MIB UCITS ETF (Acc)	**0,631**

Xtrackers II Eurozone Government Bond UCITS ETF 1C	**1,000**

The iShares FTSE MIB UCITS ETF, with an efficiency score of 0.631, falls below the efficiency threshold of 1, indicating that the ETF does not optimally utilize the available resources. This score suggests that the fund is not on the efficient frontier, as there exists a combination of other DMUs that could generate the same expected return with a lower resource input.
In contrast, the Xtrackers II Eurozone Government Bond UCITS ETF 1C, with an efficiency score of 1.000, achieves the highest level of efficiency. This implies that the ETF has ideally optimized the available resources and is located on the efficient frontier. It is noteworthy that the maximum efficiency was achieved despite the expected return not being particularly high—indeed, it is significantly lower than the sample average (1.47% versus 3.96%). However, this lower return was achieved with substantially lower input usage compared to the average selection, as demonstrated by the following data: a standard deviation of 0.019, compared to an average of 0.073; a Beta of 0.03, compared to an average of 0.501; and zero entry fees, compared to an average of 3.00%.

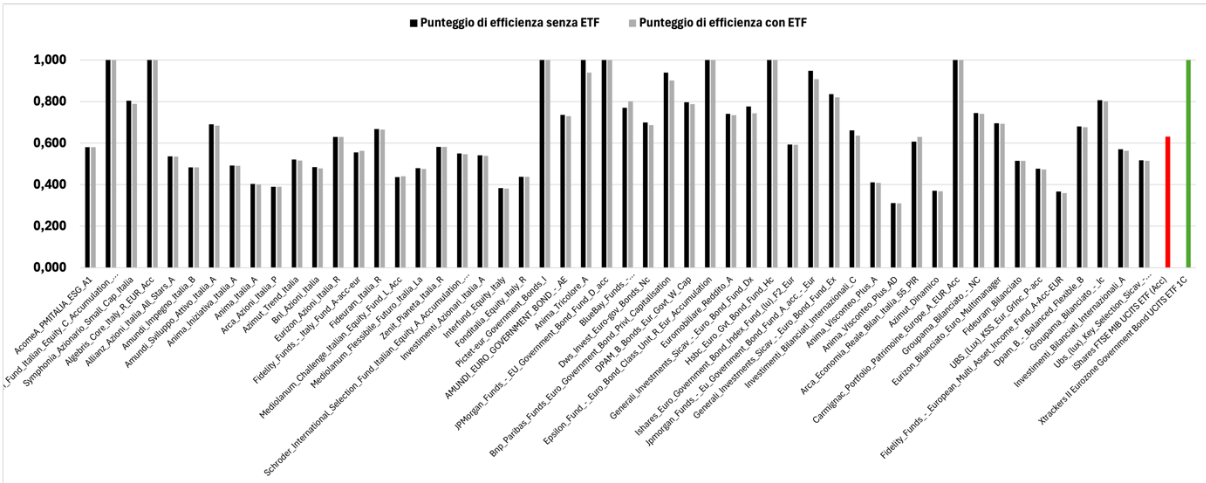

Figure 13: histogram of the efficiency scores of each fund, introducing the two passive funds (ETFs).

The figure illustrates the distribution of efficiency scores obtained from the DEA analysis for each fund in the sample, before the inclusion of the ETFs (black bars) and with the addition of the two ETFs in the analysis (gray bars). The addition of the two ETFs to the sample resulted in a minimal change in the efficiency scores. This outcome can be partially attributed to the inherent nature of DEA: as the sample size increases, the inclusion of additional Decision Making Units (DMUs) tends to have progressively less influence on the efficiency frontier, as the contribution of each new fund becomes marginal.

Furthermore, the inclusion of additional ETFs that replicate the same index is redundant, as it equates to introducing the same index multiple times, given the passive replication mechanism of these instruments, with the same input and output metrics, without adding value to the analysis.

The ETF “Xtrackers II Eurozone Government Bond UCITS ETF 1C” reached the maximum score of 1.000, while the “iShares FTSE MIB UCITS ETF (Acc)” scored 0.631.

These results suggest that, although ETFs are generally considered more efficient than active funds in terms of cost structure, it is also crucial to consider the potential advantages of active management. In fact, while active funds involve higher entry and management fees, these costs can be justified by activities that, through targeted strategic decisions, can generate superior returns or reduce risk exposure, thus offsetting the additional expenses.

4.3 - Difference in efficiency between ESG funds and neutral funds

An ESG (Environmental, Social, Governance) fund integrates environmental, social, and governance criteria into its investment decisions. These funds select and include stocks of companies that meet specific criteria, in addition to traditional financial parameters, such as the environmental impact of the companies, social resource management, and the adoption of best governance practices. The adoption of these criteria implies an investment strategy focused not only on generating financial returns but also on promoting responsible and sustainable business practices.

From an investor's perspective, ESG funds should, theoretically, offer benefits such as reduced long-term risk, derived from, for example, reduced exposure to harmful or risky environmental and social practices. In this sense, such funds might perform better during periods of increasing global attention to sustainability. Furthermore, some studies suggest that companies with solid ESG policies may be better positioned to face future challenges, such as the introduction of increasingly stringent environmental regulations, especially in the European context, potentially leading to higher returns for investors.

To identify ESG funds with a positive profile, this analysis considers funds that have been assigned a rating of at least 4 stars out of 5 by Morningstar for this parameter.

Table 13: ESG rating for each fund in the sample

DMU	ESG
AcomeA_PMITALIA_ESG_A1	1
Schroder_International_Selection_Fund_Italian_Equity_C_Accumulation_EUR	0
Symphonia_Azionario_Small_Cap_Italia	1
Algebris_Core_Italy_R_EUR_Acc	1
Allianz_Azioni_Italia_All_Stars_A	0
Amundi_Impegno_Italia_B	0
Amundi_Sviluppo_Attivo_Italia_A	1
Anima_Iniziativa_Italia_A	0
Anima_Italia_A	0
Arca_Azioni_Italia_P	0
Azimut_Trend_Italia	1
Bnl_Azioni_Italia	0
Eurizon_Azioni_Italia_R	0
Fidelity_Funds_-_Italy_Fund_A-acc-eur	0
Fideuram_Italia_R	0
Mediolanum_Challenge_Italian_Equity_Fund_L_Acc	0

Mediolanum_Flessibile_Futuro_Italia_La	0
Zenit_Pianeta_Italia_R	1
Schroder_International_Selection_Fund_Italian_Equity_A_Accumulation_Eur	0
Investimenti_Azionari_Italia_A	0
Interfund_Equity_Italy	0
Fonditalia_Equity_Italy_R	0
Pictet-eur_Government_Bonds_I	0
AMUNDI_EURO_GOVERNMENT_BOND_-_AE	0
Anima_Tricolore_A	0
JPMorgan_Funds_-_EU_Government_Bond_Fund_D_acc	0
BlueBay_Funds_-_BlueBay_Investment_Grade_Euro_Government_Bond_Fund_R_EUR_Acc	0
Dws_Invest_Euro-gov_Bonds_Nc	0
Bnp_Paribas_Funds_Euro_Government_Bond_Privl_Capitalisation	0
DPAM_B_Bonds_Eur_Govt_W_Cap	0
Epsilon_Fund_-_Euro_Bond_Class_Unit_R_Eur_Accumulation	0
Euromobiliare_Reddito_A	1
Generali_Investments_Sicav_-_Euro_Bond_Fund_Dx	0
Hsbc_Euro_Gvt_Bond_Fund_Hc	0
Ishares_Euro_Government_Bond_Index_Fund_(lu)_F2_Eur	0
Jpmorgan_Funds_-_Eu_Government_Bond_Fund_A_acc_-_Eur	0
Generali_Investments_Sicav_-_Euro_Bond_Fund_Ex	0
Investimenti_Bilanciati_Internazionali_C	1
Anima_Visconteo_Plus_A	1
Anima_Visconteo_Plus_AD	1
Arca_Economia_Reale_Bilan_Italia_55_PIR	0
Azimut_Dinamico	1
Carmignac_Portfolio_Patrimoine_Europe_A_EUR_Acc	0
Groupama_Bilanciato_-_NC	1
Eurizon_Bilanciato_Euro_Multimanager	1
Fideuram_Bilanciato	0
UBS_(Lux)_KSS_Eur_GrInc_P-acc	1
Fidelity_Funds_-_European_Multi_Asset_Income_Fund_A-Acc-EUR	1
Dpam_B_-_Balanced_Flexible_B	1
Groupama_Bilanciato_-_Ic	1

Investimenti_Bilanciati_Internazionali_A	1
Ubs_(lux)_Key_Selection_Sicav_-_European_Growth_And_Income_EUR_Q-acc	1

The analysis of efficiency scores calculated through the DEA methodology revealed a moderate difference between traditional funds and those classified as ESG. Non-ESG funds recorded an average efficiency score of 0.687, while ESG funds reported an average value of 0.607. In general, ESG funds appear to be slightly less efficient compared to traditional funds, although this difference is not particularly significant. The distribution of efficiency scores shows considerable overlap between the two groups. Considering the sample size, which includes 33 non-ESG funds and 19 ESG funds, this variation may not be statistically significant, suggesting that, overall, ESG factors do not result in substantial differences in efficiency levels within the analyzed sample.

5 - CONCLUSION

This research applies the DEA (Data Envelopment Analysis) methodology to evaluate the efficiency of mutual funds, extending and comparing the results with the pioneering study by Basso & Funari (2001). Although both studies confirm the usefulness of DEA in overcoming the limitations of traditional indices, significant differences emerge in the implications, related to both the temporal context and the innovations introduced in this analysis.

A key aspect that distinguishes this research from the study lies in the different objective pursued. While the publication "A Data Envelopment Analysis Approach to Measure Mutual Fund Performance" primarily focused on demonstrating the application of the DEA technique, concentrating on an intra-category comparison (analyzing the efficiency of funds within each class: equity, bond, and balanced funds), this thesis introduces an inter-category approach, using a unique sample that groups all three categories. This allows for evaluating the efficiency within each category, as well as directly comparing the different types, identifying which one was the most efficient in the analyzed period.

For a direct comparison with the study by Basso & Funari (2001), intra-class efficiency coefficients were also calculated. Specifically, they recorded an average efficiency index of 0.907 for equity funds, 0.439 for bond funds, and 0.900 for balanced funds, while this research yielded results of 0.595, 0.863, and 0.687, respectively. These discrepancies can be attributed to various factors, including the temporal context and the macroeconomic conditions of the two periods analyzed.

Table 14: historical comparison of DEA application compared to the results of Basso & Funari

Period	Equity Efficiency*	Bond Efficiency*	Balanced Efficiency*	DEA-Sharpe Correlation	DEA-Treynor Correlation

01/01/1997 - 30/06/1999**	0.907	0.439	0.900	0.727	0.624
01/01/2019 - 30/06/2021	0.595	0.863	0.687	0.639	0.585

*Intra-category efficiency measurement

**Basso, A., & Funari, S. (2001). *A data envelopment analysis approach to measure the mutual fund performance*

Basso & Funari's study focused on the period 01/01/1997-30/06/1999, an era characterized by relative stability in financial markets and sustained economic growth, during which equity and balanced funds benefited from high returns. In contrast, the period analyzed in this thesis, 01/01/2019-30/06/2021, was marked by extraordinary events such as the COVID-19 pandemic, which introduced a high degree of uncertainty and volatility in the markets. In this context, bond funds, due to their lower risk exposure and intrinsic stability, achieved higher efficiency scores compared to equity and balanced funds, which instead suffered from higher volatility and higher costs.

While in Basso & Funari's study equity and balanced funds demonstrated higher efficiency, in the more recent period, bond funds proved to be more efficient, reflecting investors' preference for low-risk instruments in times of high uncertainty. This suggests that fund efficiency is not a static attribute but varies based on market conditions and investor preferences.

Additionally, the greater efficiency of bond funds in the 2019-2021 period may also be linked to the expansive monetary policy of central banks, which kept interest rates at historically low but stable levels. In contrast, equity funds were impacted by greater market volatility, which amplified risk and reduced overall efficiency.

At the inter-category level, the results further confirm the trends observed in the intra-class analysis. Government EUR bond funds stand out as the most efficient, with an average score of 0.856, surpassing both equity and balanced funds. This performance is attributed to their low risk (average standard deviation of 0.024 and Beta of 0.040), despite lower returns compared to equity funds. In contrast, equity funds, despite recording higher returns, are less efficient (average score: 0.575) due to greater risk exposure (average standard deviation of 0.024 and Beta of 0.912).

This result, not explored in Basso & Funari's publication, emphasizes how, in macroeconomic contexts marked by high uncertainty (such as the COVID-19 pandemic), lower volatility is rewarded by DEA. The greater efficiency of bond funds reflects not only their intrinsic stability but also investors' preference for low-risk instruments in turbulent periods, thus confirming the importance of the temporal context in analyzing fund efficiency.

Another innovative aspect of this research compared to Basso & Funari's study is the introduction of ETFs (Exchange-Traded Funds) into the analyzed sample. Unlike actively managed funds, ETFs passively replicate a market index, which translates into generally lower management and entry costs. However, the results show that passive management does not automatically guarantee maximum efficiency.

The Xtrackers Eurozone Government Bond UCITS ETF 1C achieves a DEA efficiency score of 1.000, positioning it on the efficient frontier. This result is attributed to its combination of low risk (standard deviation: 0.019) and absence of entry costs, despite a relatively modest expected return (1.47%). In contrast, the iShares FTSE MIB UCITS ETF (Acc), despite replicating a benchmark index, achieves a lower efficiency score (0.631) due to higher volatility (standard deviation: 0.121) and a lower expected return compared to actively managed equity funds.

These results highlight that, while ETFs are often considered efficient instruments due to their low-cost structure, their performance heavily depends on the type of underlying asset. Bond ETFs, due to their intrinsic stability, are more efficient, while equity ETFs may perform less well in highly volatile contexts.

Another original contribution of this research is the analysis of the efficiency of ESG (Environmental, Social, Governance) funds compared to traditional funds. ESG funds integrate sustainability criteria into investment decisions, aiming to generate positive environmental, social, and governance impacts. However, contrary to expectations, the results show that ESG funds do not present an efficiency advantage over traditional funds.
In the analyzed sample, ESG funds recorded an average efficiency score of 0.607, lower than traditional funds, which scored 0.687. This suggests that sustainability criteria, while representing an added value in terms of social and environmental responsibility, do not necessarily translate into higher financial efficiency.

These results, in line with Basso & Funari's expectations for future in-depth analysis, raise questions about the ability of ESG funds to reconcile financial goals and sustainability. While ESG criteria may help reduce long-term risk, they do not seem to provide a competitive advantage in terms of operational efficiency in the short term.

As also highlighted in Basso & Funari's study, DEA provides a more detailed assessment compared to traditional indices like Sharpe and Treynor. For example, the Pictet-EUR Government Bonds I fund achieves a DEA score of 1.000, due to its low volatility and contained costs, despite a relatively modest Sharpe ratio (0.473) and a high Treynor ratio (1.849), reflecting its low sensitivity to systematic risk (Beta: 0.01). In contrast, funds with a high Sharpe ratio, such as Carmignac Portfolio Patrimoine Europe A EUR Acc (Sharpe: 1.822), may prove less efficient in DEA due to higher entry costs or greater exposure to systematic risk.

These examples confirm that DEA complements, but does not replace, traditional approaches, enriching the analysis with multi-dimensional metrics that include not only risk-return but also costs and resource management. While Sharpe and Treynor ratios focus respectively on total risk and systematic risk, DEA provides a more comprehensive view, considering the impact of costs and operational efficiency.

To summarize, the results of this research have significant implications for investors and fund managers. Firstly, the greater efficiency of bond funds in the analyzed period suggests that, in high-volatility contexts, these instruments can offer a better balance between risk and return. Secondly, the analysis of ETFs highlights that passive management is not always the optimal choice, even if it involves no entry costs. Finally, the results regarding ESG funds raise questions about their ability to reconcile financial goals and sustainability, suggesting the need for further research in this area. These contributions open the way for future research, both to deepen the role of ETFs and ESG funds and to explore the application of DEA in different geographical and temporal contexts. In conclusion, this thesis not only confirms the usefulness of DEA as an evaluation tool but also extends its application to new areas of interest, offering a more complete and nuanced perspective for the analysis of mutual funds.

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