A FIRST COURSE

IN

TOPOLOGY

A FIRST COURSE

IN

TOPOLOGY

MAT4002 Notebook

Lecturer

Prof. Daniel Wong

The Chinese University of Hongkong, Shenzhen

Tex Written By

Mr. Jie Wang

The Chinese University of Hongkong, Shenzhen

香港中文大學(深圳)
The Chinese University of Hong Kong, Shenzhen

Contents

Acknowledgments xv

Notations xvii

1 Week1 1

1.1 Monday for MAT3040 1

1.1.1 Introduction to Advanced Linear Algebra 1

1.1.2 Vector Spaces 2

1.2 Monday for MAT3006 5

1.2.1 Overview on uniform convergence 5

1.2.2 Introduction to MAT3006 6

1.2.3 Metric Spaces 7

1.3 Monday for MAT4002 11

1.3.1 Introduction to Topology 11

1.3.2 Metric Spaces 12

1.4 Wednesday for MAT3040 14

1.4.1 Review 14

1.4.2 Spanning Set 14

1.4.3 Linear Independence and Basis 16

1.5 Wednesday for MAT3006 20

1.5.1 Convergence of Sequences 20

1.5.2 Continuity 24

1.5.3 Open and Closed Sets 25

1.6 Wednesday for MAT4002 27

1.6.1 Forget about metric 27

1.6.2 Topological Spaces 30

1.6.3 Closed Subsets . . . 31

2 Week2 . . . 33

2.1 Monday for MAT3040 33

2.1.1 Basis and Dimension . . . 33

2.1.2 Operations on a vector space . . . 36

2.2 Monday for MAT3006 38

2.2.1 Remark on Open and Closed Set . . . 38

2.2.2 Boundary, Closure, and Interior . . . 41

2.3 Monday for MAT4002 44

2.3.1 Convergence in topological space . . . 44

2.3.2 Interior, Closure, Boundary . . . 46

2.4 Wednesday for MAT3040 51

2.4.1 Remark on Direct Sum . . . 51

2.4.2 Linear Transformation . . . 52

2.5 Wednesday for MAT3006 59

2.5.1 Compactness . . . 59

2.5.2 Completeness . . . 64

2.6 Wednesday for MAT4002 66

2.6.1 Remark on Closure . . . 66

2.6.2 Functions on Topological Space . . . 68

2.6.3 Subspace Topology . . . 70

2.6.4 Basis (Base) of a topology . . . 72

3 Week3 . . . 73

3.1 Monday for MAT3040 73

3.1.1 Remarks on Isomorphism . . . 73

3.1.2 Change of Basis and Matrix Representation . . . 74

3.2 Monday for MAT3006 81

3.2.1 Remarks on Completeness . . . 81

3.2.2 Contraction Mapping Theorem . . . 82

3.2.3 Picard Lindelof Theorem . . . 86

3.3 Monday for MAT4002 87

3.3.1 Remarks on Basis and Homeomorphism . . . 87

3.3.2 Product Space . . . 90

3.4 Wednesday for MAT3040 92

3.4.1 Remarks for the Change of Basis . . . 92

3.5 Wednesday for MAT3006 98

3.5.1 Remarks on Contraction . . . 98

3.5.2 Picard-Lindelof Theorem . . . 98

3.6 Wednesday for MAT4002 103

3.6.1 Remarks on product space . . . 103

3.6.2 Properties of Topological Spaces . . . 106

4 Week4 . . . 109

4.1 Monday for MAT3040 109

4.1.1 Quotient Spaces . . . 109

4.1.2 First Isomorphism Theorem . . . 112

4.2 Monday for MAT3006 115

4.2.1 Generalization into System of ODEs . . . 115

4.2.2 Stone-Weierstrass Theorem . . . 117

4.3 Monday for MAT4002 121

4.3.1 Hausdorffness . . . 121

4.3.2 Connectedness . . . 122

4.4 Wednesday for MAT3040 126

4.4.1 Dual Space . . . 131

4.5 Wednesday for MAT3006 **134**

4.5.1 Stone-Weierstrass Theorem . . . 135

4.6 Wednesday for MAT4002 **140**

4.6.1 Remark on Connectedness . . . 140

4.6.2 Compactness . . . 142

5 Week5 . . . 145

5.1 Monday for MAT3040 **145**

5.1.1 Remarks on Dual Space . . . 146

5.1.2 Annihilators . . . 148

5.2 Monday for MAT3006 **152**

5.2.1 Stone-Weierstrass Theorem in $\mathbb{C}$ . . . 153

5.2.2 Baire Category Theorem . . . 155

5.3 Monday for MAT4002 **157**

5.3.1 Continuous Functions on Compact Space . . . 157

5.4 Wednesday for MAT3040 **160**

5.4.1 Adjoint Map . . . 161

5.4.2 Relationship between Annihilator and dual of quotient spaces . . . 163

5.5 Wednesday for MAT3006 **165**

5.5.1 Remarks on Baire Category Theorem . . . 165

5.5.2 Compact subsets of $C[a,b]$ . . . 167

5.6 Wednesday for MAT4002 **169**

5.6.1 Remarks on Compactness . . . 169

5.6.2 Quotient Spaces . . . 170

6 Week6 . . . 175

6.1 Monday for MAT3040 **175**

6.1.1 Polynomials . . . 175

6.2 Monday for MAT3006 177

6.2.1 Compactness in Functional Space 177

6.2.2 An Application of Ascoli-Arzela Theorem 178

6.3 Monday for MAT4002 179

6.3.1 Quotient Topology 179

6.3.2 Properties in quotient spaces 181

6.4 Wednesday for MAT3040 185

6.4.1 Eigenvalues & Eigenvectors 188

6.5 Wednesday for MAT4002 191

6.5.1 Remarks on Compactness 191

7 Week7 193

7.1 Monday for MAT3040 193

7.1.1 Minimal Polynomial 193

7.1.2 Minimal Polynomial of a vector 198

7.2 Monday for MAT3006 200

7.2.1 Remarks on the outer measure 200

7.3 Monday for MAT4002 205

7.3.1 Quotient Map 205

7.3.2 Simplicial Complex 206

7.4 Wednesday for MAT3040 211

7.4.1 Cayley-Hamiton Theorem 211

7.5 Wednesday for MAT4002 218

7.5.1 Remarks on Triangulation 218

7.5.2 Simplicial Subcomplex 220

7.5.3 Some properties of simplicial complex 222

8 Week8 225

8.1 Monday for MAT3040 225

8.1.1 Cayley-Hamiton Theorem 227

8.1.2 Primary Decomposition Theorem 230

8.2 Monday for MAT3006 232

8.2.1 Remarks for Outer Measure 232

8.2.2 Lebesgue Measurable 233

8.3 Monday for MAT4002 238

8.3.1 Quotient Map 238

8.3.2 Simplicial Complex 239

8.4 Wednesday for MAT3006 242

8.4.1 Remarks on Lebesgue Measurability 242

8.4.2 Measures In Probability Theory 243

8.5 Wednesday for MAT4002 247

8.5.1 Homotopy 249

9 Week9 251

9.1 Monday for MAT3040 251

9.1.1 Remarks on Primary Decomposition Theorem 251

9.2 Monday for MAT3006 257

9.2.1 Measurable Functions 257

9.3 Monday for MAT4002 262

9.3.1 Remarks on Homotopy 262

9.4 Wednesday for MAT3040 268

9.4.1 Jordan Normal Form 268

9.4.2 Inner Product Spaces 273

9.5 Wednesday for MAT3006 275

9.5.1 Remarks on Measurable function 275

9.5.2 Lebesgue Integration 276

9.6 Wednesday for MAT4002 280

9.6.1 Simplicial Approximation Theorem 280

10 Week10 285

10.1 Monday for MAT3040 285

10.1.1 Inner Product Space 285

10.1.2 Dual spaces 288

10.2 Monday for MAT3006 291

10.2.1 Remarks on Markov Inequality 291

10.2.2 Properties of Lebesgue Integration 291

10.3 Monday for MAT4002 294

10.3.1 Group Presentations 296

10.4 Wednesday for MAT3040 297

10.4.1 Orthogonal Complement 297

10.4.2 Adjoint Map 300

10.5 Wednesday for MAT3006 303

10.5.1 Consequences of MCT 306

10.6 Wednesday for MAT4002 309

10.6.1 Reviewing On Groups 309

10.6.2 Free Groups 311

10.6.3 Relations on Free Groups 313

11 Week11 315

11.1 Monday for MAT3040 315

11.1.1 Self-Adjoint Operator 315

11.1.2 Orthononal/Unitary Operators 318

11.2 Monday for MAT3006 320

11.2.1 Consequences of MCT I 320

11.2.2 MCT II . . . 323

11.3 Monday for MAT4002 325

11.3.1 Cayley Graph for finitely presented groups . . . 327

11.3.2 Fundamental Group . . . 329

11.4 Wednesday for MAT3040 332

11.4.1 Unitary Operator . . . 332

11.4.2 Normal Operators . . . 335

11.5 Wednesday for MAT3006 337

11.5.1 Properties of Lebesgue Integrable Functions . . . 338

11.6 Wednesday for MAT4002 342

11.6.1 The fundamental group . . . 342

12 Week12 . . . 349

12.1 Monday for MAT3040 349

12.1.1 Remarks on Normal Operator . . . 349

12.1.2 Tensor Product . . . 353

12.2 Monday for MAT3006 355

12.2.1 Remarks on MCT . . . 355

12.2.2 Dominated Convergence Theorem . . . 358

12.3 Monday for MAT4002 362

12.3.1 Some basic results on $\pi_1(X,b)$ . . . 365

12.4 Wednesday for MAT3040 367

12.4.1 Introduction to Tensor Product . . . 367

12.5 Wednesday for MAT3006 371

12.5.1 Riemann Integration & Lebesgue Integration . . . 371

12.5.2 Continuous Parameter DCT . . . 373

12.6 Wednesday for MAT4002 376

12.6.1 Groups & Simplicial Complices . . . 376

13 Week13 381
13.1 Monday for MAT3040 381
13.1.1 Basis of $V \otimes W$ 383
13.1.2 Tensor Product of Linear Transformation 387
13.2 Monday for MAT3006 388
13.2.1 Double Integral 389
13.3 Monday for MAT4002 393
13.3.1 Isomorphsim between Edge Loop Group and the Fundamental Group 393
13.4 Wednesday for MAT3040 399
13.4.1 Tensor Product for Linear Transformations 399
13.5 Wednesday for MAT3006 405
13.5.1 Fubini's and Tonell's Theorem 405
13.6 Wednesday for MAT4002 410
13.6.1 Applications on the isomorphism of fundamental group 410
14 Week14 417
14.1 Monday for MAT3040 417
14.1.1 Multilinear Tensor Product 417
14.1.2 Exterior Power 420
14.2 Monday for MAT3006 422
14.2.1 Tonelli's and Fubini's Theorem 422
14.3 Monday for MAT4002 426
14.3.1 Fundamental group of a Graph 426
15 Week15 431
15.1 Monday for MAT3040 431
15.1.1 More on Exterior Power 431
15.1.2 Determinant 433

15.2 Monday for MAT3006 **438**

15.2.1 Applications on the Tonell's and Fubini's Theorem 438

15.3 Monday for MAT4002 **443**

15.3.1 The Selfert-Van Kampen Theorem 444

Acknowledgments

This book is taken notes from the MAT4002 in spring semester, 2019. These lecture notes were taken and compiled in LaTeX by Jie Wang, an undergraduate student in spring 2019. The tex writter would like to thank Prof. Daniel Wong and some students for their detailed and valuable comments and suggestions, which significantly improved the quality of this notebook. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

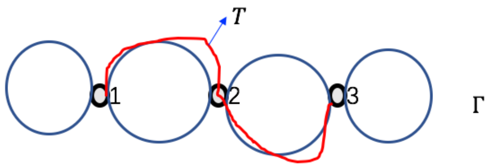
Notations and Conventions

$(X,\mathcal{T})$	Topological space
$X \cong Y$	The space X is homeomorphic to space Y
$G \cong H$	The group G is isomorphic to group H
p_X	Project mapping
$X \times Y$	Product Topology
$X/\sim$	Quotient Topology related to the topologcial space X and the equivalence class $\sim$
S^n	The n-sphere $\{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \lVert\boldsymbol{x}\rVert = 1\}$
D^n	The n-disk $\{\boldsymbol{x} \in \mathbb{R}^{n} \mid \lVert\boldsymbol{x}\rVert \leq 1\}$
$E^\circ, \partial E, \overline{E}$	The interior, boundary, closure of E
$\mathbb{T}^2$	The torus in $\mathbb{R}^3$
Δ^n	The n-simplex
$i : A \hookrightarrow X$	Inclusion mapping from $A \subseteq X$ to X
$K = (V, \Sigma)$	(Abstract) Simplicial Complex
$\lvert K\rvert$	Topological realization of the simplicial complex K
$\langle X \mid R\rangle$	The presentation of a group
$H : f \overset{H}{\simeq} g$	f and g are homotopic, where H denotes the homotopy
$X \simeq Y$	The space X and Y are homotopy equivalent
$\pi_1(X,x)$	The fundamental group of X w.r.t. the basepoint $x \in X$
$E(K,b)$	The edge loop group of the space K w.r.t. the basepoint b
f_*	The induced homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ for $f : X \to Y$

15.3. Monday for MAT4002

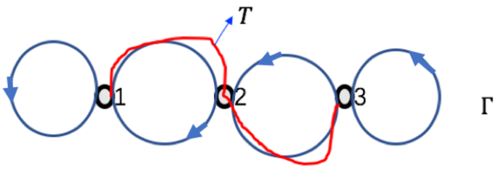
Theorem 15.4 Let Γ be a connected graph. Then $\pi(\Gamma)$ is isomorphic to the free group generated by $\#\{E(\Gamma) \setminus E(T)\}$ elements, for any maximal tree of Γ.

Now we give a proof for this theorem on one special case of Γ:

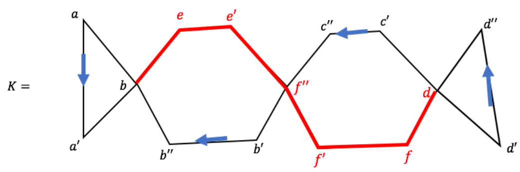


Proof.

- Fix an orientation for each $e \in E(\Gamma) \setminus E(T)$:



- Now let K be a simplicial complex with $|K| \cong \Gamma$:



As a result, $E(K,b) \cong \pi_1(\Gamma)$

- Now we construct the group homomorphism

$$
\begin{aligned}
\phi: \quad & \langle \alpha, \beta, \gamma, \delta \rangle \to E(K,b) \\
\text{with} \quad & \phi(\alpha) = [ba'a''b] \\
& \phi(\beta) = [bee'f''b'b''b] \\
& \phi(\gamma) = [bee'f''f'fdc'c''f''e'eb] \\
& \phi(\delta) = [bee'f''f'fdd''d'dff'f''e'eb]
\end{aligned}
$$

- We can show the group homomorphism ϕ is bijective. In particular, the inverse of ϕ is given by:

$$\Psi: \quad E(K,b) \to \langle \alpha,\beta,\gamma,\delta\rangle$$

where for any $[\ell] := [bv_1 \cdots v_n] \in E(K,b)$, the mapping $\Psi[\ell]$ is constructed by

  (a) Remove all other letters appearing in ℓ except $b, a', a'', b', b'', c', c'', d', d''$

  (b) Assign

$$\alpha,\ \ \alpha^{-1},\ \ \beta,\ \ \beta^{-1},\ \ \gamma,\ \ \gamma^{-1},\ \ \delta,\ \ \delta^{-1}$$

  for each appearance of

$$a'a'',\ \ a''a',\ \ b'b'',\ \ b''b',\ \ c'c'',\ \ c''c',\ \ d'd'',\ \ d''d',$$

  respectively.

■

15.3.1. The Selfert-Van Kampen Theorem

Theorem 15.5 Let $K = K_1 \cup K_2$ be the union of two **path-connected open** sets, where $K_1 \cap K_2$ is also path-connected. Take $b \in K_1 \cap K_2$, and suppose the group presentations for $\pi_1(K_1,b), \pi_1(K_2,b)$ are

$$\pi_1(K_1,b) \cong \langle X_1 \mid R_1\rangle, \quad \pi_1(K_2,b) \cong \langle X_2 \mid R_2\rangle.$$

Let the inclusions be

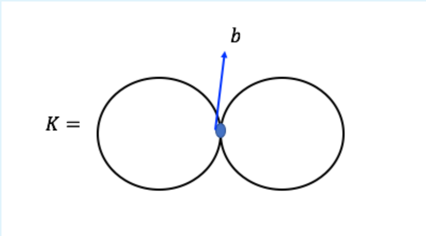
$$i_1 : K_1 \cap K_2 \hookrightarrow K_1, \quad i_2 : K_1 \cap K_2 \hookrightarrow K_2,$$

then a presentation of $\pi_1(K,b)$ is given by:

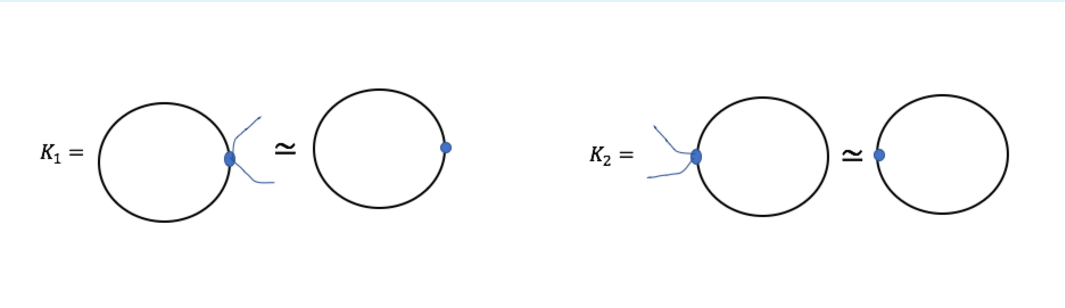
$$\pi_1(K,b) \cong \langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{(i_1)_*(g) = (i_2)_*(g) : \forall g \in \pi_1(K_1 \cap K_2, b)\}\rangle.$$

(Here $(i_1)_*: \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_1, b)$ and $(i_2)_*: \pi_1(K_1 \cap K_2, b) \hookrightarrow \pi_1(K_2, b)$.)

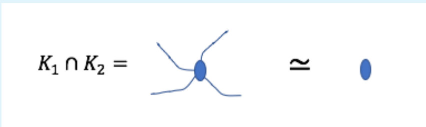
■ **Example 15.4** 1. Let $K = S^1 \wedge S^1$ given by



(a) Then construct b as the intersection between two circles, and construct K_1, K_2 as shown below:



We can see that $K_1 \cap K_2$ is contractible:



(b) As we have shown before, $\pi_1(S^1) \cong \mathbb{Z}$, which follows that

$$\pi_1(K_1, b) \cong \langle \alpha \rangle, \quad \pi_1(K_2, b) \cong \langle \beta \rangle$$

Also, $\pi_1(K_1 \cap K_2, b) \cong \pi_1(\{b\}, b) \cong \{e\}$.

(c) It's easy to compute $(i_1)_*$ and $(i_2)_*$:

$$\begin{array}{ll} (i_1)_*: & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} & e \mapsto e \end{array}, \qquad \begin{array}{ll} (i_2)_*: & \pi_1(K_1 \cap K_2) \to \pi_1(K_2) \\ \text{with} & e \mapsto e \end{array},$$

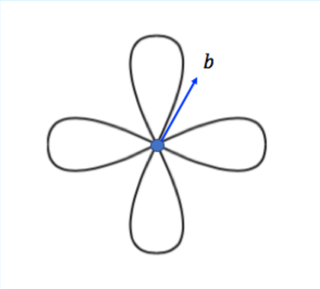
(d) Therefore, by Seifert-Van Kampen Theorem,

$$\pi_1(K, b) \cong \langle \alpha, \beta \mid e = e \rangle \cong \langle \alpha, \beta \rangle$$

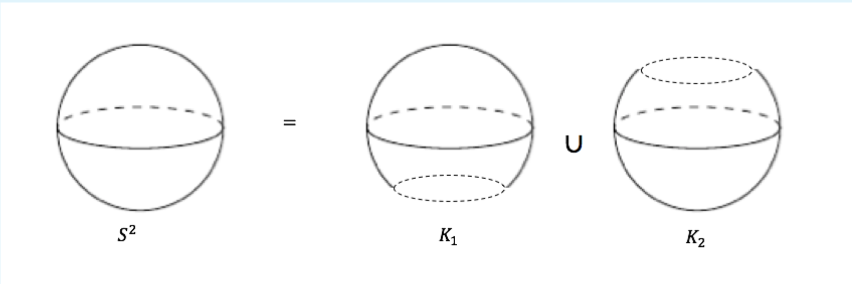
2. By induction,

$$\pi_1(\wedge^n S^1, b) \cong \langle a_1, \dots, a_n \rangle$$

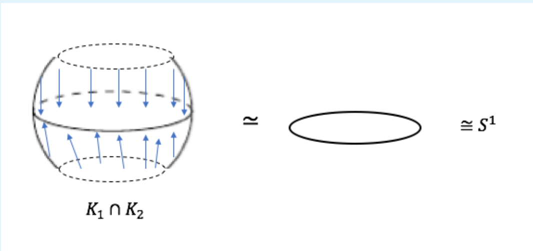
For instance, the figure illustration for $\wedge^4 S^1$ and the basepoint b is given below:



3. (a) Construct $S^2 = K_1 \cup K_2$, which is shown below:



Therefore, we see that $K_1 \cap K_2 \simeq S^1$:



(b) It's clear that K_1 and K_2 are contractible, and therefore

$$\pi_1(K_1) \cong \langle \beta \mid \beta \rangle, \quad \pi_1(K_2) \cong \langle \gamma \mid \gamma \rangle$$

and $\pi_1(K_1 \cap K_2) \cong \pi_1(S^1) \cong \langle \alpha \rangle$.

(c) Then we compute $(i_1)_*$ and $(i_2)_*$. In particular, the mapping $(i_1)_*$ is defined as

$$\begin{aligned} (i_1)_* : \quad & \pi_1(K_1 \cap K_2) \longrightarrow \pi_1(K_1) \\ \text{with} \quad & [\alpha] \mapsto [i_1(\alpha)] \end{aligned}$$

where α is any loop based at b. Since K_1 is contractible, we imply α in K_1 is homotopic to c_b, i.e.,

$$(i_1)_*([\alpha]) = [i_1(\alpha)] = e, \forall \alpha \in \pi_1(K_1 \cap K_2).$$

Similarly, $(i_2)_*([\alpha]) = e$.

(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(S^2) \cong \langle \beta, \gamma \mid \beta, \gamma, e = e \rangle \cong \{e\}$$

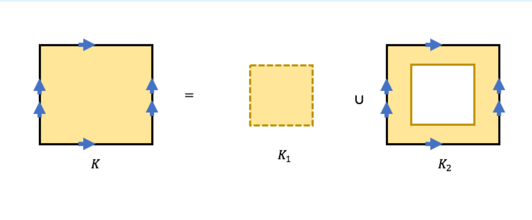
4. Homework: Use the same trick to check that $\pi_1(S^n) = \{e\}$ for all $n \geq 2$. Hint: for S^3, construct

$$K_1 = \{(x_1, \ldots, x_4) \in S^3 \mid x_4 > -1/2\}$$

and

$$K_1 = \{(x_1, \ldots, x_4) \in S^3 \mid x_4 < 1/2\}$$

5. (a) Consider the quotient space $K \cong \mathbb{T}^2$, and we construct $K = K_1 \cup K_2$ as follows:



Therefore, we can see that K_1 is contractible, and K_2 is homotopy equivalent to $S^1 \wedge S^1$:

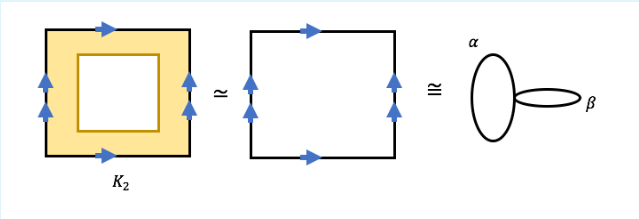
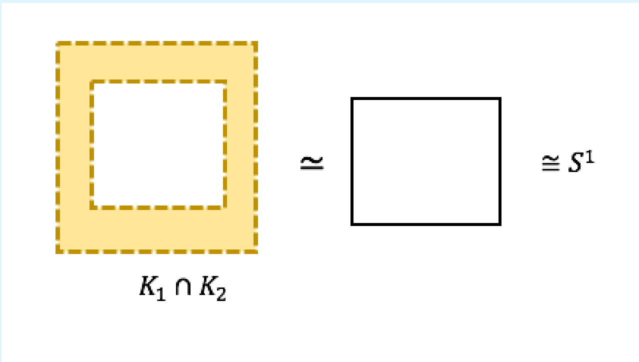


Figure 15.2: Illustration for $K_2 \simeq S^1 \wedge S^1$

and $K_1 \cap K_2$ is homotopic equivalent to the circle:



(b) It follows that

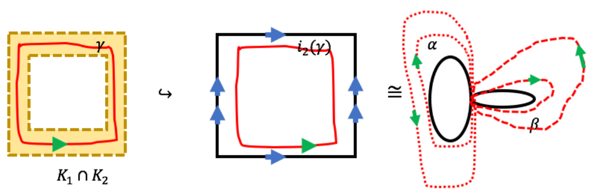
$$\pi_1(K_1) \cong \{e\}, \quad \pi_1(K_2) \cong \langle \alpha, \beta \rangle,$$

and $\pi_1(K_1 \cap K_2) \cong \langle \gamma \rangle$.

(c) Then we compute $(i_1)_*$ and $(i_2)_*$. In particular, $(i_1)_*$ is trivial:

$$\begin{aligned} (i_1)_* : \quad & \pi_1(K_1 \cap K_2) \to \pi_1(K_1) \\ \text{with} \quad & [\alpha] \mapsto e \end{aligned}$$

Then compute $(i_2)_*$. In particular, for any loop γ, we draw the graph for $i_2(\gamma)$:



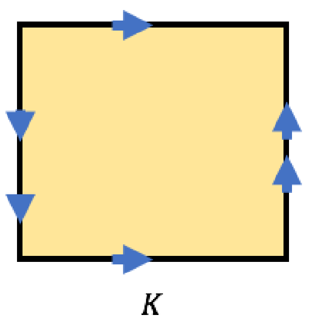
Therefore,

$$(i_2)_*[\gamma] = [i_2(\gamma)] = [\alpha\beta\alpha^{-1}\beta^{-1}]$$

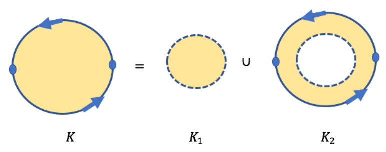
(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha, \beta \mid \beta, \alpha\beta\alpha^{-1}\beta^{-1} = e \rangle \cong \langle \alpha, \beta \mid, \alpha\beta = \beta\alpha \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

6. Exerise: The Klein bottle K shown in graph below satisfies $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$.



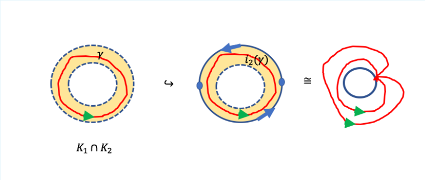
7. Consider the quotient space $K = \mathbb{R}P^2$. We construct $K = K_2 \cup K_2$, which is shown below:



(a) It's clear that K_1 is contractible. In hw3, question 1, we can see that $K_2 \simeq S^1$. Moreover, similar as in (5), $K_1 \cap K_2 \simeq S^1$.

(b) Therefore, $\pi_1(K_1) = \{e\}$ and $\pi_1(K_2) = \langle \alpha \rangle$, $\pi_1(K_1 \cap K_2) = \langle \gamma \rangle$.

(c) It's easy to see that $(i_1)_*([\gamma]) = e$ for any loop γ. For any loop γ, we draw the graph for $i_2(\gamma)$:



Therefore, $(i_2)_*([\gamma]) = [i_2(\gamma)] = [\alpha^2]$.

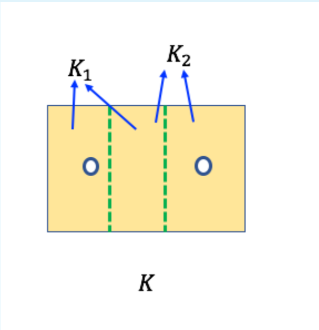
(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha \mid \alpha^2 = e \rangle \cong \mathbb{Z}/2\mathbb{Z} \cong \{0,1\}_{\text{mod }(2)}$$

8. Let $K = \mathbb{R}^2 \setminus \{2 \text{ points } \alpha, \beta\}$. As have shown in hw3, $K \simeq S^1 \wedge S^1$, which implies

$$\pi_1(K) \cong \pi_1(S^1 \wedge S^1) \cong \langle \alpha, \beta \rangle.$$

We can compute the fundamental group for K directly. Construct $K = K_1 \cup K_2$ as follows:



(a) It's clear that $K_1 \cong \mathbb{R}^2 \setminus \{\text{one point}\} \simeq S^1$ and similarly $K_2 \simeq S^1$. Moreover, $K_1 \cap K_2$ is contractible

(b) Therefore,

$$\pi_1(K_1) \cong \langle \alpha \rangle, \quad \pi_1(K_2) \cong \langle \beta \rangle, \quad \pi_1(K_1 \cap K_2) \cong \{e\}$$

(c) Therefore, $(i_1)_*$ and $(i_2)_*$ is trivial since $\pi_1(K_1 \cap K_2) \cong \{e\}$.

(d) By Seifert-Van Kampen Theorem, we conclude that

$$\pi_1(K) \cong \langle \alpha, \beta \mid e = e \rangle \cong \langle \alpha, \beta \rangle$$

■