15.2. Monday for MAT3006

Theorem 15.2 — Tonell. Let $f:\mathbb{R}^2\to[0,\infty]$ be a measurable function (i.e., $f^{-1}((a,\infty])\in\mathcal{M}\otimes\mathcal{M}$), then

$$\int f\,\mathrm{d}x=\int\left(\int f(x,y)\,\mathrm{d}x\right)\mathrm{d}y=\int\left(\int f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$$

Theorem 15.3 — Fubini. Let $f:\mathbb{R}^2\to[-\infty,\infty]$ be integrable (i.e., $f=f^+-f^-$ with $f^\pm:\mathbb{R}^2\to[0,\infty]$ measurable and $\int f^\pm\,\mathrm{d}x<\infty$), then

$$\int f\,\mathrm{d}\pi=\int\left(\int f(x,y)\,\mathrm{d}x\right)\mathrm{d}y=\int\left(\int f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$$

Corollary 15.2 Suppose that $f:\mathbb{R}^2\to[-\infty,\infty]$ is measurable, and either

$$\int(\int|f(x,y)|\,\mathrm{d}x)\,\mathrm{d}y\qquad(*)$$

or

$$\int(\int|f(x,y)|\,\mathrm{d}y)\,\mathrm{d}x\qquad(**)$$

exists, then f is integrable, and the result of Fubini follows. (i.e., one can switch the order of integration as long as the integral of $|f|$ exists)

Proof. By Tonell's theorem if (*) or (**) is finite, then $\int|f|\,\mathrm{d}\pi<\infty$, which implies $|f|$ is integrable, i.e., f is integrable.

Then apply the Fubini's theorem. ∎

▪ **Example 15.2** Compute $I=\int_0^1\int_0^x\sqrt{\frac{1-y}{x-y}}\,\mathrm{d}y\,\mathrm{d}x$.

Consider $f(x,y):=\sqrt{\frac{1-y}{x-y}}\mathcal{X}_E(x,y)$, with E shown in the figure. Therefore,

$$I=\int f(x,y)\,\mathrm{d}\pi$$

Consider

$$\begin{aligned}
\int\left(\int f(x,y)\,\mathrm{d}x\right)\mathrm{d}y &= \int_0^1\left(\int_y^1\sqrt{\frac{1-y}{x-y}}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^1\sqrt{1-y}\left(\int_y^1\frac{1}{\sqrt{x-y}}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^1\sqrt{1-y}\left(\int_0^{1-y}\frac{1}{\sqrt{t}}\,\mathrm{d}t\right)\mathrm{d}y \\
&= \int_0^1\sqrt{1-y}2(\sqrt{1-y})\,\mathrm{d}y \\
&= 2\int_0^1(1-y)\,\mathrm{d}y \\
&= 1
\end{aligned}$$

Since $\int(\int|f|\,\mathrm{d}x)\,\mathrm{d}y<\infty$, we know that

$$\int(\int f\,\mathrm{d}y)\,\mathrm{d}x=\int(\int f\,\mathrm{d}x)\,\mathrm{d}y$$

by corollary, i.e., $I=1$. ▪

R The function $f(x,y)$ is continuous on E and hence measurable. When it approaches the line $y\to x$, $f(x,y)\to\infty$.

We have two measures on $\mathbb{R}^2$:

- $\mathcal{M}\otimes\mathcal{M}$, and
- $\mathcal{M}_{\mathbb{R}^2}$, given by

$$\mathcal{M}_{\mathbb{R}^2}=\{E\subseteq\mathbb{R}^2\mid m^*(A)=m^*(A\cap E)+m^*(A\cap E^c)\text{ for all subsets }A\subseteq\mathbb{R}^2\}$$

We have $\mathcal{M}_{\mathbb{R}^2}$ equals the completion of $\mathcal{M}\otimes\mathcal{M}$ (analogy of $\mathcal{B}$ and $\mathcal{M}$), i.e., all $E\subseteq\mathcal{M}_{\mathbb{R}^2}$ can be decomposed as

$$E=B\cup(E\setminus B),$$

where $B\in\mathcal{M}\otimes\mathcal{M}$ and $E\setminus B\in\mathcal{M}_{\mathbb{R}^2}$ with $\pi(E\setminus B)=0$.

Question: does Tonell's theorem holds for (Lebesgue) measurable functions $f : \mathbb{R}^2 \to [0,\infty]$ (i.e., $f^{-1}((a,\infty]) \in \mathcal{M}_{\mathbb{R}^2}$ for any $a \in [0,\infty)$?)

Answer: Yes.

To see so, we need the following:

Proposition 15.4 Let $(\mathbb{R}^2, \mathcal{M}_{\mathbb{R}^2}, \pi)$ be the Lebesgue measure on $\mathbb{R}^2$, and $N \in \mathcal{M}_{\mathbb{R}^2}$ be such that $\pi(N) = 0$. Then for almost all values of $x \in \mathbb{R}$, $N_x \in \mathcal{M}$ and $m_Y(N_x) = 0$.

Proof. For $N \in \mathcal{M}_{\mathbb{R}^2}$. By hw3, there exists $B' \in \mathcal{M} \otimes \mathcal{M}$ such that $N \subseteq B'$, with

$$\pi(B') = \pi(N)$$

If N is null, then $\pi(B') = 0$. By Tonell's theorem on $M \otimes \mathcal{M}$, we imply

$$\pi(B') = \int m_Y(B'_x)\,\mathrm{d}x = \int m_X(B'_y)\,\mathrm{d}y = 0$$

Therefore, $m_Y(B'_x) = 0$ for almost all $x \in \mathbb{R}$. Since $N \subseteq B'$, we imply $N_x \subseteq B'_x$, i.e., N_x is also a null set.

Therefore, $N_x \in \mathcal{M}$ and $m_Y(N_x) = 0$. ■

■ **Example 15.3** Consider the integral

$$\int_0^\infty \int_0^\infty y e^{-y^2(1+x^2)}\,\mathrm{d}y\,\mathrm{d}x$$

Let $f(x,y) = ye^{-y^2(1+x^2)}$, which is continous on $(0,\infty) \times (0,\infty)$, and therefore measurable.

Then

$$\begin{aligned}
\int_0^\infty \int_0^\infty y e^{-y^2(1+x^2)}\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\infty \left(\lim_{n\to\infty} \int_0^n y e^{-y^2(1+x^2)}\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^\infty \left(\frac{1}{1+x^2} \frac{1}{2} \right) \mathrm{d}x \\
&= \lim_{n\to\infty} \int_0^n \frac{1}{2} \frac{1}{1+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{4}
\end{aligned}$$

By the corollary of Fubini's theorem,

$$\int_0^\infty \int_0^\infty y e^{-y^2(1+x^2)} \, \mathrm{d}x \, \mathrm{d}y = \frac{\pi}{4}$$

$$\int_0^\infty y e^{-y^2} \int_0^\infty e^{-x^2 y^2} \, \mathrm{d}x \, \mathrm{d}y = \frac{\pi}{4}$$

$$\int_0^\infty y e^{-y^2} \lim_{n\to\infty} \int_0^n e^{-x^2 y^2} \, \mathrm{d}x \, \mathrm{d}y = \frac{\pi}{4}$$

$$\int_0^\infty y e^{-y^2} \lim_{n\to\infty} \frac{1}{y} \int_0^{ny} e^{-t^2} \, \mathrm{d}t \, \mathrm{d}y = \frac{\pi}{4}, \quad t = xy$$

$$\int_0^\infty e^{-y^2} \int_0^\infty e^{-t^2} \, \mathrm{d}t \, \mathrm{d}y = \frac{\pi}{4}$$

$$\int_0^\infty e^{-x^2} \, \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

∎