A GRADUATE COURSE

IN

CONVEX OPTIMIZATION

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DDA6110 Notebook

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Contents

Acknowledgments

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Notations and Conventions

sup	least upper bound
inf	greatest lower bound
$\overline{E}$	closure of E
$f \circ g$	composition
$\limsup(\liminf)$	upper (lower) limit
$L(\mathcal{P}, f), U(\mathcal{P}, f)$	Riemann sums
$\mathcal{R}[a, b]$	classes of Riemann integrable functions on $[a, b]$
$\underline{\int_a^b} f(x)\,\mathrm{d}x, \overline{\int_a^b} f(x)\,\mathrm{d}x$	Riemann integrals
$\langle \boldsymbol{x}, \boldsymbol{y} \rangle$	inner product
$\omega(f; E)$	oscillation of f over set E
$\lVert \cdot \rVert$	norm
∇f	gradient
$\frac{\partial f}{\partial x_i}, f_{x_i}, f_i, \partial_i f, D_i f$	partial derivatives
$D_{\boldsymbol{v}} f$	directional derivative at direction $\boldsymbol{v}$
$\frac{\partial(y_1, \ldots, y_m)}{\partial(x_1, \ldots, x_n)}$	Jacobian
$\mathbb{S}^n$	set of real symmetric $n \times n$ matrices
$\succ\ (\succeq)$	positive (semi)-definite
$\mathcal{C}^m$	classes of m-th order continuously differentiable functions
$\mathcal{C}(E; \mathbb{R}^m)$	set of $\mathcal{C}^1$ mapping from E to $\mathbb{R}^m$
$(\mathcal{H}, d)$	metric space

Chapter 1

Week1

1.1. Monday

1.1.1. Introduction to Convex Optimization

The basic optimization model is as follows:

$$\begin{aligned} \min \quad & f(x) : \mathbb{R}^n \to \mathbb{R} \\ \text{s.t.} \quad & x \in X \subseteq \mathbb{R}^n \end{aligned} \tag{1.1}$$

For instance, the constraint X can be union of some inequality constraints:

$$X = \left\{ x \in \mathbb{R}^n \middle| f_i(x) \le b_i, \quad i = 1, \dots, m \right\}$$

We only consider **convex** problems in this course, which means that in (1.1):

- the objective function f is convex;
- the constraint set X is a convex set.

The goal of optimization is to find $x^* \in X$ such that

$$f(x^*) \le f(x), \quad \forall x \in X.$$

or determine whether such an x^* exists or not.

■ **Example 1.1** The least squares problem is a convex problem:

$$\begin{aligned} \min \quad & \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|_2^2 \\ \text{s.t.} \quad & x \in \mathbb{R}^n \end{aligned}$$

The optimal solution can be compactly written as

$$\boldsymbol{x}^* = \boldsymbol{A}^* \boldsymbol{b}$$

where $\boldsymbol{A}^*$ denotes the pseudo inverse of $\boldsymbol{A}$. ■

■ **Example 1.2** The linear programming is also a convex problem:

$$\begin{aligned} \min \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \le b_i, \quad i = 1, 2, \dots, m. \end{aligned}$$

The linear programming is a special case of the conic programming. We will cover conic programming in this course. ■

Definition 1.1 [Line Segments] The line segment between $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ is defined as

$$\{\theta\boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \mid \theta \in [0,1]\} = \{\boldsymbol{x}_2 + \theta(\boldsymbol{x}_1 - \boldsymbol{x}_2) \mid \theta \in [0,1]\}$$

■

Definition 1.2 [Convex Set in $\mathbb{R}^n$] The set $X \subseteq \mathbb{R}^n$ is convex if for any $x_1, x_2 \in X$,

$$\theta x_1 + (1-\theta)x_2 \in X, \qquad \forall \theta \in [0,1].$$

■

Here are two examples for convex sets over matrices:

$$\mathcal{S}^n = \{\boldsymbol{W} \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{W} = \boldsymbol{W}^{\mathrm{T}}\}, \quad \mathcal{S}_+^n = \{\boldsymbol{W} \in \mathcal{S}^n \mid \boldsymbol{W} \succeq 0\},$$

R Sometimes the convex set could be a collection of matrices, but we can vectorize a matrix into a vector.

Definition 1.3 [Convex Function] A function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if

- dorm(f) is a convex set, and
- $\forall x, y \in$ dorm(f), it holds that for $\theta \in [0,1]$,

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y),$$

which means that the secant line betweeen any two points is above the function.

Proposition 1.1 $f(x) = \max_i f_i(x)$ is convex if $f_i(x)$ is convex for each i.

Proof. For any $\theta \in [0,1]$, the following inequality holds:

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \max_i \ f_i(\theta x + (1-\theta)y) \\ &\le \max_i \ \theta f_i(x) + (1-\theta) f_i(y) \\ &\le \max_i \ \theta f_i(x) + \max_i \ (1-\theta) f_i(y) \\ &= \theta f(x) + (1-\theta) f(y) \end{aligned}$$

■

■ **Example 1.3** Define the function

$$\begin{aligned} & f : \mathbb{S}^n \to \mathbb{R} \\ \text{with} \quad & f(\boldsymbol{X}) = \lambda_{\max}(\boldsymbol{X}) \triangleq \max_{\|\boldsymbol{v}\|=1} \ \boldsymbol{v}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{v} \end{aligned}$$

This function is convex since f can be written as the maximization of a collection of affines (in terms of $\boldsymbol{X}$):

$$f(\boldsymbol{X}) = \max_{\|\boldsymbol{v}\|=1} f_{\boldsymbol{v}}(\boldsymbol{X}), \qquad \text{with } f_{\boldsymbol{v}}(\boldsymbol{X}) = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{v}.$$

1.2. Wednesday

Convex Programming (CP):

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & x \in X \subseteq \mathbb{R}^n \end{aligned}$$

with f and X being convex. The standard constraint set X can be written as:

$$X = \left\{ x \middle| f_i(x) \le b_i, \;\; i = 1,2,3,\dots,m, \quad Ax = c \right\}$$

with $A \in \mathbb{R}^{p \times n}$, $p < n$, and f_i's are convex.

Proposition 1.2 The constraint set

$$X = \left\{ x \middle| f_i(x) \le b_i, \;\; i = 1,\dots,m \right\}$$

is **convex** if all f_i's are convex.

Proof. For any $x_1, x_2 \in X$ and $\theta \in [0,1]$, verify that

$$\begin{aligned} f_i(\theta x_1 + (1-\theta)x_2) &\le \theta f_i(x_1) + (1-\theta) f_i(x_2) \\ &\le \theta b_i + (1-\theta) b_i = b_i \end{aligned}$$

where the first inequality is because of the convexity, and the second inequality is because that $x_i \in X, i = 1,2$. As a result, $\theta x_1 + (1-\theta)x_2 \in X$, i.e., X is a convex set. ∎

Proposition 1.3 If $\{X_i\}_{i \in \mathcal{I}}$ is a collection of convex sets, then the union $X = \bigcap_{i \in \mathcal{I}}$ is convex.

Proof. Consider any $x_1, x_2 \in X$, i.e., $x_1, x_2 \in X_i, \forall i \in \mathcal{I}$, then by the convexity of X_is, the line segment

$$\theta x_1 + (1-\theta)x_2 \in X_i, \forall i \in \mathcal{I},$$

i.e., $\theta x_1 + (1-\theta)x_2 \in X$. ∎

Definition 1.4 [Epigraph] Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, the epigraph of f is a set defined as

$$\text{epi}(f) = \left\{ (x,t) \middle| f(x) \le t, x \in \text{dorm}(f) \right\} \in \mathbb{R}^{n+1}.$$

Proposition 1.4 The function f is convex if and only if the epigraph epi(f) is convex.

Proof. We only talk about the proof on the forward direction. Consider any $(x_1, t_1), (x_2, t_2) \in$ epi(f), then we show that $\theta(x_1, t_1) + (1-\theta)(x_2, t_2) \in$ epi(f) for any $\theta \in [0,1]$:

$$\begin{aligned} f(\theta x_1 + (1-\theta)x_2) &\le \theta f(x_1) + (1-\theta) f(x_2) \\ &\le \theta t_1 + (1-\theta) t_2, \end{aligned}$$

which proves the desired result. ■

Let's start to introduce some useful convex sets.

Definition 1.5 [Combination of two points] Consider the combination of two points x_1, x_2:

$$\theta_1 x_1 + \theta_2 x_2,$$

- When $\theta_1, \theta_2 \in \mathbb{R}$, it is called an affine combination; the resulted space is an **affine space**.
- When $\theta_1, \theta_2 \in \mathbb{R}_+$, it is called a non-negative combination; the resulted space is a **convex cone**.
- When $\theta_2 = 1 - \theta_1, \theta_1 \in [0,1]$, it is called a convex combination; the resulted space is a **convex set**.

Definition 1.6 [Affine Set] The standard form of an affine space is $\{ \boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \}$.

Definition 1.7 [Cone] A set is called a cone with a vertex at the origin if for any $\boldsymbol{x} \in X$, $a\boldsymbol{x} \in X$ for any $a \geq 0$.

The standard form of a convex conic programming is the following:

$$\begin{aligned} \min \quad & \langle \boldsymbol{C}, \boldsymbol{X} \rangle \\ \text{s.t.} \quad & \langle \boldsymbol{A}_i, \boldsymbol{X} \rangle = \boldsymbol{B}_i, \quad i = 1, 2, \ldots, m \\ & \boldsymbol{X} \in \mathcal{K} \end{aligned}$$

where $\mathcal{K}$ is a convex cone.

One special form of conic programming is the linear programming:

$$\begin{aligned} \min \quad & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Another is the second-order cone programming:

$$\mathcal{K} = \{(\boldsymbol{x}, t) \mid \|\boldsymbol{x}\|_2 \leq t\}$$

The semidefinite programming also belongs to the case of conic programming:

$$\mathcal{K} = \mathcal{S}_+^n \triangleq \{\boldsymbol{X} \in \mathbb{S}^n \mid \boldsymbol{X} \succeq 0, \forall \boldsymbol{v} \in \mathbb{R}^n\}$$

Convex hull of a set S:

It is the smallest convex set containing S, called $\operatorname{conv}(S)$.

Definition 1.8 [Polyhedron]

- A hyperplane in $\mathbb{R}^n$ can be written as the form $\{\boldsymbol{x} \mid \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = b\}$.
- Half space: $\{\boldsymbol{x} \mid \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{b}\}$.
- Polyhedron: a intersection of finite hyperplanes and half spaces.

- Ellipsoid:

$$\left\{\boldsymbol{x} \,\middle|\, (\boldsymbol{x}-\boldsymbol{x}_c)^{\mathrm{T}}\boldsymbol{P}^{-1}(\boldsymbol{x}-\boldsymbol{x}_c) \le 1,\ \boldsymbol{P} \succ 0\right\} = \{\boldsymbol{x} \mid \|\boldsymbol{P}^{-1/2}(\boldsymbol{x}-\boldsymbol{x}_c)\|_2^2 \le 1\}$$

$$= \{\boldsymbol{x} \mid \boldsymbol{x} = \boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u}, \|\boldsymbol{u}\|_2 \le 1\}$$

Chapter 2

Week2

2.1. Monday

Operations that preserve convexity:

- Intersection:

$$B = \bigcap_{\|a\|=1} \{\boldsymbol{x} \mid \boldsymbol{a}^{\mathrm{T}} x \le 1\} = \{\boldsymbol{x} \mid \|\boldsymbol{x}\|_2 \le 1\}.$$

- Affine function: $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^n$. As long as $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$,

$$f(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha f(\boldsymbol{x}) + \beta f(\boldsymbol{y}).$$

  - When $S \subseteq \mathbb{R}^n$ is convex, $f(S) \subseteq \mathbb{R}^n$ is convex.
  - When $C \subseteq \mathbb{R}^m$ is convex, then $f^{-1}(C) \subseteq \mathbb{R}^n$ is convex. Consider $x_1, x_2 \in f^{-1}(C)$, then $f(x_1), f(x_2) \in C$, and $\theta f(x_1) + (1-\theta) f(x_2) \in C$, i.e., $f(\theta x_1 + (1-\theta)x_2) \in C$. As a result, $\theta x_1 + (1-\theta)x_2 \in f^{-1}(C)$.

Example 2.1 Consider the set

$$L = \left\{\boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n x_i \boldsymbol{A}_i \preceq \boldsymbol{B}\right\}, \quad \text{with } \boldsymbol{A}_i, \boldsymbol{B} \in \mathbb{S}_+^n$$

Write $f(\boldsymbol{x}) = \boldsymbol{B} - \sum_{i=1}^n x_i \boldsymbol{A}_i$, which is an affine function. It follows that

$$L = \left\{\boldsymbol{x} \in \mathbb{R}^n \,\middle|\, f(\boldsymbol{x}) \preceq 0\right\} = f^{-1}(\mathbb{S}_+^n)$$

Since $\mathbb{S}_+^n$ is convex, L is convex as well. ■

■ **Example 2.2** Consider the set

$$\mathcal{H} = \left\{ \boldsymbol{x} \middle| \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x} \le (\boldsymbol{c}^{\mathrm{T}} \boldsymbol{x})^2, \qquad \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \ge 0 \right\}, \qquad \boldsymbol{P} \succeq 0.$$

Define

$$f(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{P}^{1/2} \\ \boldsymbol{c}^{\mathrm{T}} \end{pmatrix} \boldsymbol{x} : \mathbb{R}^n \to \mathbb{R}^{n+1}$$

Therefore, $\mathcal{H} = f^{-1}(\text{second order cone})$, with

$$\text{second order cone} = \left\{ \begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix} \middle| \|\boldsymbol{x}\| \le t \right\}$$

■

Perspective transformation:

$$P(\boldsymbol{x}, t) = \frac{\boldsymbol{x}}{t}, \quad t > 0 : \mathbb{R}^{n+1} \to \mathbb{R}^n.$$

Consider $S \subseteq \mathrm{dorm}(P)$ and $P(S) \subseteq \mathbb{R}^n$.

We show that line segments map to line segments:

Proof. Let $(x_1, t_1), (x_2, t_2) \in S$ with $t_1, t_2 > 0$. Consider

$$\begin{aligned}
\begin{pmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta t_1 + (1-\theta) t_2 \end{pmatrix} &\to \frac{\theta x_1 + (1-\theta) x_2}{\theta t_1 + (1-\theta) t_2} \\
&= \frac{\theta t_1}{\theta t_1 + (1-\theta) t_2} \frac{x_1}{t_1} + \frac{(1-\theta) t_2}{\theta t_1 + (1-\theta) t_2} \frac{x_1}{t_2} \\
&= \mu \frac{x_1}{t} + (1-\mu) \frac{x_2}{t}
\end{aligned}$$

■

Linear fractional:

$$f(\boldsymbol{x}) = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{d}}, \qquad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{d} > 0.$$

Proper cone $\mathcal{K}$:

- convex;
- closed;
- solid;
- pointed, i.e., contain no line.

Generalized inequality:

$$\boldsymbol{x} \preceq_{\mathcal{K}} \boldsymbol{y} \implies \boldsymbol{y} - \boldsymbol{x} \in \mathcal{K}$$

Define minimum element v.s. minimal element.

2.2. Wednesday

K proper cone: convex, closed, solid, pointed.

Definition 2.1 [Generalized Inequality] Given a set K, we define $x \leq_K y$ if $y - x \in K$.

Pointed cone:

$$x \leq_K y, y \leq_K x \implies x = y.$$

Definition 2.2 [Dual Cone] Given a cone $K \subseteq \mathbb{R}^n$, the dual cone is defined as

$$K^* = \{y \in \mathbb{R}^n \mid y^{\mathrm{T}} x \geq 0, \forall x \in K\}.$$

R The dual cone K^* is convex, closed, and pointed. The convexity is because that it is an intersection of many half-spaces.

Self-dual cones:

$$(\mathbb{R}^n_+)^* = \mathbb{R}^n_+, \qquad (\mathbb{S}^n_+)^* = \mathbb{S}^n_+.$$

Proof. Let $X, Y \in \mathbb{S}^n_+$, then

$$\langle X, Y \rangle = \sum_i \lambda_i q_i^{\mathrm{T}} Y q_i \geq 0.$$

Therefore, $\mathbb{S}^n_+ \subseteq (\mathbb{S}^n_+)^*$. When $Y \notin (\mathbb{S}^n_+)$, there exists u such that

$$\boldsymbol{u}^{\mathrm{T}} Y \boldsymbol{u} < 0 \implies \langle Y, \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \rangle < 0,$$

which implies $Y \notin (\mathbb{S}^n_+)^*$. ∎

2.2.1. Separation Theorems

If C and D are disjoint convex sets, there exists affine function such that

$$\boldsymbol{a}^{\mathrm{T}} \boldsymbol{x} \leq \boldsymbol{b}, \forall \boldsymbol{x} \in C, \qquad \boldsymbol{a}^{\mathrm{T}} \boldsymbol{x} \geq \boldsymbol{b}, \forall \boldsymbol{x} \in D.$$

Theorem 2.1 If C is convex, then for any $x_0 \in \partial C$, there exists a hyper-plane $\{\boldsymbol{x} \mid \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{b}\}$ with $\boldsymbol{a} \neq \mathbf{0}$ such that

$$a^{\mathrm{T}}\boldsymbol{x}_0,$$

and $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{b}, \forall \boldsymbol{x} \in C$.

The converse is true if C is closed.

Chapter 3

Week3

3.1. Monday

3.1.1. Convex Functions