A BRIEF ON
FUNCTIONAL ANALYSIS

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MAT4010 Notebook

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Acknowledgments

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Jie, 2021-01-11

Notations and Conventions

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{R}^{m \times n}$	set of all $m \times n$ real-valued matrices
$\mathbb{C}^{m \times n}$	set of all $m \times n$ complex-valued matrices
x_i	ith entry of column vector $\boldsymbol{x}$
a_{ij}	(i, j)th entry of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i$	ith column of matrix $\boldsymbol{A}$
$\boldsymbol{a}_i^{\mathrm{T}}$	ith row of matrix $\boldsymbol{A}$
$\mathbb{S}^n$	set of all $n \times n$ real symmetric matrices, i.e., $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $a_{ij} = a_{ji}$ for all i, j
$\mathbb{H}^n$	set of all $n \times n$ complex Hermitian matrices, i.e., $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ and $\bar{a}_{ij} = a_{ji}$ for all i, j
$\boldsymbol{A}^{\mathrm{T}}$	transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$ means $b_{ji} = a_{ij}$ for all i, j
$\boldsymbol{A}^{\mathrm{H}}$	Hermitian transpose of $\boldsymbol{A}$, i.e, $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{H}}$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\mathrm{trace}(\boldsymbol{A})$	sum of diagonal entries of square matrix $\boldsymbol{A}$
$\mathbf{1}$	A vector with all 1 entries
$\mathbf{0}$	either a vector of all zeros, or a matrix of all zeros
$\boldsymbol{e}_i$	a unit vector with the nonzero element at the ith entry
$\mathcal{C}(\boldsymbol{A})$	the column space of $\boldsymbol{A}$
$\mathcal{R}(\boldsymbol{A})$	the row space of $\boldsymbol{A}$
$\mathcal{N}(\boldsymbol{A})$	the null space of $\boldsymbol{A}$
$\mathrm{Proj}_{\mathcal{M}}(\boldsymbol{A})$	the projection of $\boldsymbol{A}$ onto the set $\mathcal{M}$

Chapter 1

Week1

1.1. Monday

1.1.1. Metric Space

Definition 1.1 [Metric Space] A metric space is a pair (X, d), where X is a set and d is a metric on X, i.e., a function defined on $X \times X$ such that for any $x, y, z \in X$,

- d is real-valued, non-negative, and finite;
- $\mathsf{d}(x, y) = 0$ if and only if $x = y$;
- $\mathsf{d}(x, y) = \mathsf{d}(y, x)$;
- $\mathsf{d}(x, y) \leq \mathsf{d}(x, z) + \mathsf{d}(z, y)$.

Definition 1.2 [Subspace] A subspace $(Y, \tilde{\mathsf{d}})$ of (X, d) is obtained if we take $Y \subseteq X$ and restrict d into $Y \times Y$, denoted as

$$\tilde{\mathsf{d}} = \mathsf{d}\mid_{Y \times Y}.$$

The metric $\tilde{\mathsf{d}}$ is called the metric induced on Y by d.

■ **Example 1.1** Examples about metric space:

- Real line $\mathbb{R}$: a set of all real numbers with the usual metric $\mathsf{d}(x, y) = |x - y|$;

- Euclidean space $\mathbb{R}^2$: the metric space $\mathbb{R}^2$ is obtained by defining the metric

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad x = (x_1, x_2), y = (y_1, y_2).$$

- More generally, the Euclidean space $\mathbb{R}^n$, unitary space $\mathbb{C}^n$, and complex plane $\mathbb{C}$ are metric spaces.

- Sequence space ℓ^∞: As a set X, we take the set of all bounded sequences of complex numbers, i.e.,

$$X \ni x = (x_1, x_2, \ldots) := (x_i)$$

such that for all i, $|\xi_i| \leq C_x$, where C_x is a real number which may be dependent on x but does not depend on i. We choose the metric

$$\mathrm{d}(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i|,$$

with $x = (x_i), y = (y_i)$.

- Function space $\mathcal{C}[a,b]$: The set of all real-valued functions $x, y, \ldots$, which are functions of an independent variable t, and are defined and continuous on $\mathcal{J} = [a,b]$. We choose the metric

$$\mathrm{d}(x,y) = \max_{t \in \mathcal{J}} |x(t) - y(t)|.$$

- Space $\mathcal{B}(A)$ of bounded functions: each element $x \in \mathcal{B}(A)$ is a function defined and bounded on a set A. The metric is defined by

$$\mathrm{d}(x,y) = \sup_{A} |x(t) - y(t)|.$$

Proof. It's eash to see that $\mathrm{d}(x,y)$ is real-valued, finite, non-negative,and $\mathrm{d}(x,y) = \mathrm{d}(y,x)$. As for the second condition, when $d(x,y) = 0$,

$$x(t) - y(t) = 0, \quad \forall t \in A,$$

so that $x = y$. As for the last condition, for all $t \in A$,

$$\begin{aligned}|x(t) - y(t)| &\le |x(t) - z(t)| + |z(t) - y(t)| \\ &\le \sup_{t \in A} |x(t) - z(t)| + \sup_{t \in A} |z(t) - y(t)|\end{aligned}$$

This indicates that $|x(t) - y(t)| \le \mathsf{d}(x,z) + \mathsf{d}(z,y)$ for all $t \in A$. Taking the supremum on A both sides gives the desired result. ■

- Space ℓ^p: Let $p \ge 1$ be a fixed number. By definition, each element in ℓ^p is a sequence $x = (x_i) = (x_1, x_2, \dots)$ of numbers such that

$$\sum_{i \ge 1} |x_i|^p < \infty.$$

The metric is defined by

$$\mathsf{d}(x,y) = \left(\sum_i |x_i - y_i|^p \right)^{1/p}.$$

When $p = 2$, d reduces into the 2-norm. Then ℓ^2 is called the Hilbert space.

Proposition 1.1 The space ℓ^p is a metric space.

We proof this result by four intermediate steps:

- An auxiliary inequality.
- the Holder inequality.
- the Minkowski inequality.
- the triangle inequality.

Proof.

- Let $p > 1$ and define q by $1/p + 1/q = 1$. Let α and β be any non-negative numbers, then we have

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

The condition $1/p + 1/q = 1$ implies $1/(p-1) = q - 1$, then

$$u = t^{p-1} \implies t = u^{q-1}.$$

It follows that

$$\begin{aligned} \alpha\beta &\leq \int_0^\alpha t^{p-1}\,\mathrm{d}t + \int_0^\beta u^{q-1}\,\mathrm{d}u \\ &= \frac{\alpha^p}{p} + \frac{\beta^q}{q}. \end{aligned}$$

- Take any non-zero $x = (x_i) \in \ell^p$ and $y = (y_i) \in \ell^q$, we have the Holder inequality:

$$\sum_{i\geq 1} |x_i y_i| \leq \left(\sum_{i\geq 1} |x_i|^p\right)^{1/p} \left(\sum_{i\geq 1} |y_i|^q\right)^{1/q}.$$

Let $(\tilde{x}_i)$ and $(\tilde{y})$ be such that

$$\sum |\tilde{x}_i|^p = 1, \sum |\tilde{y}_i|^q = 1.$$

By applying the Auxiliary inequality,

$$|\tilde{x}_i \tilde{y}_i| \leq \frac{1}{p}|\tilde{x}_i|^p + \frac{1}{q}|\tilde{y}_i|^q.$$

Therefore,

$$\sum_i |\tilde{x}_i \tilde{y}_i| \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Now take any non-zero $x = (x_i) \in \ell^p$ and $y = (y_i) \in \ell^q$. Set

$$\tilde{x}_i = \frac{x_i}{\left(\sum_i |x_i|^p\right)^{1/p}}, \quad \tilde{y}_i = \frac{y_i}{\left(\sum_i |y_i|^q\right)^{1/q}}$$

It follows that

$$\sum_i \left| \frac{x_i}{\left(\sum_i |x_i|^p\right)^{1/p}} \frac{y_i}{\left(\sum_i |y_i|^q\right)^{1/q}} \right| \leq 1.$$

The desired result follows.

■