12.2. Monday for MAT3006

▪ **Example 12.2**

$$\lim_{n\to\infty}\int_0^{n\pi}\cos\left(\frac{x}{2n}\right)xe^{-x^2}\,\mathrm{d}x$$

Let $f_n(x)=\cos\left(\frac{x}{2n}\right)xe^{-x^2}\mathcal{X}_{[0,n\pi]}$.

- Since $\cos(x/2n)<\cos(x/2(n+1))$ for any $x\in[0,n\pi]$,we imply $f_n(x)\le f_{n+1}(x)$ for any x.
- $f_n(x)$ are integrable for all n.
- Note that
$$\int f\,\mathrm{d}m\le\int_0^{\infty}xe^{-x^2}\,\mathrm{d}x=\frac{1}{2}(1-e^{-n^2a^2})\le\frac{1}{2}<\infty,$$
i.e., $\sup\int f_n\,\mathrm{d}m\le\frac{1}{2}<\infty$

Then MCT I applies and

$$\lim_{n\to\infty}\int_0^{n\pi}\cos\left(\frac{x}{2n}\right)xe^{-x^2}\,\mathrm{d}x=\int\left(\lim_{n\to\infty}f_n\right)\mathrm{d}m$$

with

$$\lim_{n\to\infty}f_n=xe^{-x^2}\mathcal{X}_{[0,\infty)}.$$

As a result,

$$\begin{aligned}\int\left(\lim_{n\to\infty}f_n\right)\mathrm{d}m&=\lim_{m\to\infty}\int_0^m xe^{-x^2}\,\mathrm{d}x\\&=\int_0^{\infty}xe^{-x^2}\,\mathrm{d}x\\&=\frac{1}{2}\end{aligned}$$

where the first equality is by applying MCT I with $g_m(x)=xe^{-x^2}\mathcal{X}_{[0,m]}$. ▪

Corollary 12.3 [Lebesgue Series Theorem] Let $\{f_n\}$ be a series of measurable functions such that

$$\sum_{n=1}^{\infty}\int |f_n|\,\mathrm{d}m < \infty.$$

Then $\sum_{n=1}^{k} f_n$ converges to an integrable function $f = \sum_{n=1}^{\infty} f_n$ a.e., with

$$\int f\,\mathrm{d}m = \sum_{n=1}^{\infty}\int f_n\,\mathrm{d}m$$

Proof. For each f_n, consider

$$f_n = f_n^+ - f_n^-, \text{ where } f_n^+, f_n^- \text{ are nonnegative}$$

By consequence of MCT (I),

$$\int \sum_{n=1}^{\infty} f_n^+\,\mathrm{d}m = \sum_{n=1}^{\infty}\int f_n^+\,\mathrm{d}m \le \sum_{n=1}^{\infty}\int |f_n|\,\mathrm{d}m < \infty.$$

The same by replacing $+$ with $-$. Therefore, $f^+ := \sum_{n=1}^{\infty} f_n^+ = \lim_{k\to\infty}\sum_{n=1}^{k} f_n^+$ is integrable.

As a result, $f^+(x), f^-(x) < \infty, \forall x \in U$, where U^c is null.

Construct $f := f^+ - f^-$ on U, and $f(x) = 0$ on U^c.

As a result, on U,

$$f(x) = \left(\lim_{k\to\infty}\sum_{n=1}^{k} f_n^+(x)\right) - \left(\lim_{k\to\infty}\sum_{n=1}^{k} f_n^-(x)\right) = \lim_{k\to\infty}\left[\sum_{n=1}^{k}(f_n^+(x) - f_n^-(x))\right] = \sum_{n=1}^{\infty} f_n(x)$$

Moreover,

$$\begin{aligned}
\int f \,\mathrm{d}m &= \int f^+ \,\mathrm{d}m - \int f^- \,\mathrm{d}m \\
&= \int \sum_{n=1}^{\infty} f_n^+ \,\mathrm{d}m - \int \sum_{n=1}^{\infty} f_n^- \,\mathrm{d}m \\
&= \left(\sum_{n=1}^{\infty} \int f_n^+ \,\mathrm{d}m\right) - \left(\sum_{n=1}^{\infty} \int f_n^- \,\mathrm{d}m\right) \\
&= \sum_{n=1}^{\infty} \left(\int f_n^+ \,\mathrm{d}m - \int f_n^- \,\mathrm{d}m\right) \\
&= \sum_{n=1}^{\infty} \int f_n \,\mathrm{d}m
\end{aligned}$$

■

▪ **Example 12.3** Compute

$$\int_0^1 e^{-x} x^{\alpha-1} \,\mathrm{d}x, \ \alpha > 0$$

Consider

$$e^{-x} x^{\alpha-1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{\alpha+n-1}}{n!}, \text{ pointwise converge}$$

Define $f_n(x) = (-1)^n \frac{x^{\alpha+n-1}}{n!} \mathcal{X}_{(0,1]}$, $n \geq 0$.

Then each $f_n(x)$ is integrable, with

$$\int |f_n| \,\mathrm{d}m = \frac{1}{(\alpha+n)n!} \text{ (MCT I)}$$

and $\int |f_n| \,\mathrm{d}m \leq \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)n!} < \infty$ converges.

Therefore, $\int (\sum_{n=0}^{\infty} f_n) \,\mathrm{d}m = \sum_{n=0}^{\infty} \int f_n \,\mathrm{d}m$, and

$$\int_0^1 e^{-x} x^{\alpha-1} \,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\alpha+n)n!}$$

▪

R It's essential to have $\sum \int |f| \,\mathrm{d}m < \infty$ rather than $\int f_n \,\mathrm{d}m < \infty$ in the theorem.

For example, let

$$f_n = \frac{(-1)^{n+1}}{(n+1)} \mathcal{X}_{[n,n+1)} \implies \sum_{n=1}^{\infty} \int f_n \, \mathrm{d}m = \log(2) < \infty$$

but $\sum f_n := f$ is not integrable.

12.2.1. Dominated Convergence Theorem

Theorem 12.2 Let $\{f_n\}$ be a sequence of measruable functions such that $|f_n| \leq g$ a.e., and g is integrable. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e., then

1. f is integrable,
2. $$\int f \, \mathrm{d}m = \lim_{n\to\infty} \int f_n \, \mathrm{d}m$$

Proof.

$$|f_n| \leq g \implies \lim_{n\to\infty} |f_n| \leq g \implies |f| \leq g$$

By comparison test, g is integrable, g is integrable, we imply $|f|$ is integrable, and f is integrable.

Now consider the sequence of non-negative functions $\{g - f_n\}_{n\in\mathbb{N}}$ and $\{g + f_n\}_{n\in\mathbb{N}}$

By Fatou's Lemma,

$$\begin{aligned}
\liminf_{n\to\infty} \int (g - f_n) \mathrm{d}m &\geq \int \liminf_{n\to\infty} (g - f_n) \mathrm{d}m \\
&= \int (g - f) \mathrm{d}m \\
&= \int g \, \mathrm{d}m - \int f \, \mathrm{d}m
\end{aligned}$$

which follows that

$$\int g \, \mathrm{d}m - \limsup_{n\to\infty} \int f_n \, \mathrm{d}m \geq \int g \, \mathrm{d}m - \int f \, \mathrm{d}m$$

i.e.,

$$\int f \, dm \geq \limsup_{n\to\infty} \int f_n \, dm$$

Similarly,

$$\liminf_n (g + f_n) \, dm \geq \int \liminf_n (g + f_n) \, dm$$

which implies

$$\liminf_{n\to\infty} \int f_n \, dm \geq \int f \, dm$$

As a result,

$$\limsup_{n\to\infty} \int f_n \, dm \leq \int f \, dm \leq \liminf_{n\to\infty} \int f_n \, dm,$$

which implies

$$\int f \, dm = \lim_n \int f_n \, dm$$

■

Corollary 12.4 [Bounded Convergence Theorem] Suppose that $E \in \mathcal{M}$ be such that $m(E) < \infty$. If

- $|f_n(x)| \leq K$ for any $x \in E, n \in \mathbb{N}$
- $f_n \to f$ a.e. in E

Then f is integrable in E with

$$\int_E f \, dm = \lim_{n\to\infty} \int f_n \, dm$$

Proof. Take $g = K\mathcal{X}_E$ in DCT. ■

■ **Example 12.4** Let f be a proper Riemann integrable function on $[a, b]$, then we will see $f\mathcal{X}_{[a,b]}$ is Lebesgue integrable with

$$\int f \, dm = \int_a^b f(x) \, dx$$

If the Riemann integral is improper on $[a,b]$, then there is still a chance that f is not Lebesgue integrable:

$$f(x)=\sum_{n=1}^{\infty}(-1)^n n\cdot\mathcal{X}_{(1/(n+1),1/n]},\ x\in[0,1]$$

12.5. Wednesday for MAT3006

■ **Example 12.6** Find

$$L = \lim_{n\to\infty} \int_0^1 \frac{nx\log(x)}{1+n^2x^2}\,\mathrm{d}x$$

Let $f_n(x) = v\mathcal{X}_{(0,1]}$, which is continuous on $[0,1]$, i.e., integrable on $[0,1]$.

At the same time, $f_n(x) \to 0, \forall x \in [0,1]$ pointwisely, as $n \to \infty$. The goal is to show $L = 0$.

Note that $t/(1+t^2) \le \frac{1}{2}, \forall t \ge 0$. Take $t = nx$, we imply

$$|f_n(x)| \le \frac{1}{2}|\log(x)|\mathcal{X}_{(0,1]}$$

We claim that $\frac{1}{2}|\log(x)|\mathcal{X}_{(0,1]} := -\frac{1}{2}\log(x)\mathcal{X}_{(0,1]}$ is integrable.

Indeed, by MCT I,

$$\int -\frac{1}{2}\log(x)\mathcal{X}_{(0,1]}\,\mathrm{d}m = \lim_{n\to\infty}\int_{1/n}^1 -\frac{1}{2}\log(x)\,\mathrm{d}x = \frac{1}{2} < \infty.$$

Therefore, the DCT applies, and

$$\lim_{n\to\infty}\int_0^1 \frac{nx\log(x)}{1+n^2x^2}\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty}\frac{nx\log(x)}{1+n^2x^2}\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0$$

However, $f_n(x)$ does not converge to $f(x) \equiv 0$ uniformly on $[0,1]$:

$$\sup_{0\le x\le 1}|f_n(x) - 0| \ge |f_n(1/n) - 0| = \frac{1}{2}\log(n) \to \infty, \text{as } n \to \infty$$

Therefore, we cannot switch integral symbol and limit by using the tools in MAT2006. ■

■ **Example 12.7** Suppose that $f(x)$ is a proper Riemann integrable function on $[a,b]$. Then $f(x)$ is Lebesgue integrable on $[a,b]$ with $\int_{[a,b]} f\,\mathrm{d}m = \int_a^b f(x)\,\mathrm{d}x$.

First of all, note that f is Riemann inregrable implies $f(x)$ is bounded on $[a,b]$, i.e., $|f(x)| \le K, \forall x \in [a,b]$. Let ϕ_n, ψ_n be the Riemann lower and upper function with 2^n equal subintervals.

- $\phi_n(x) \le f(x) \le \psi_n(x), \forall n$
- $\phi_n(x)$ is monotone increasing
- $\psi_n(x)$ is monotone decreasing

Therefore, there exists $\phi(x), \psi(x)$ such that

$$\phi_n(x) \to \phi(x), \quad \psi_n(x) \to \psi(x)$$

Now apply bounded convergence theorem on $\psi_n - \phi_n$:

- $|\psi_n(x) - \phi_n(x)| \le 2K$ on $[a, b]$
- $\psi_n - \phi_n \to \psi - \phi$

Therefore,

$$\begin{aligned}\int |\psi - \phi| \, \mathrm{d}m &= \int \psi - \phi \, \mathrm{d}m \\ &= \lim_{n\to\infty} \int \psi_n - \phi_n \, \mathrm{d}m \\ &= \text{Riemann Upper Sum} - \text{Riemann Lower Sum} \\ &= 0\end{aligned}$$

Therefore, $\int |\psi - \phi| \, \mathrm{d}m = 0$ implies $\psi(x) = \phi(x)$ a.e. By sandwich theorem,

$$\psi(x) = f(x) = \phi(x) \text{ a.e.}$$

Therefore,

$$\int f \, \mathrm{d}m = \int \phi \, \mathrm{d}m = \lim_{n\to\infty} \int \phi_n \, \mathrm{d}m = \int_a^b f(x) \, \mathrm{d}x$$

where the second equality is by MCT II. ∎

R As we saw before, the example does not necessarily hold for improper Riemann integrable functions $f(x)$.

However, if we assume $f(x) \ge 0$, then $f(x)$ is improper Riemann integrable implies $f(x)$ is Lebesgue integrable, with the same integral value.

Apply the previous example and MCT I.

Theorem 12.4 — Continous parameter DCT. Let $I, J \subseteq \mathbb{R}$ be intervals, and $f : I \times J \to \mathbb{R}$ be such that

1. For any $y \in J$, then $x \mapsto f(x, y)$ is an integrable function over I.
2. Fix any $y \in J$, then
$$\lim_{y' \to y} f(x, y') = f(x, y)$$
for almost all $x \in I$
3. For all $y \in J$, there exists integrable $g(x)$ such that
$$|f(x, y)| \le g(x)$$
for almost all $x \in I$.

As a result,
$$F(y) = \int_I f(x, y)\, \mathrm{d}x$$
is a continuous function on J.

Proof. Let $\{y_n\}$ be a sequence on J such that $y_n \to y$. It suffices to show $F(y_n) \to F(y)$.

Let $f_n(x) = f(x, y_n)$, it follows that

- $f_n(x)$ is integrable (by (1))
- $|f_n(x)| \le g(x)$ a.e. (by (3))
- $$\lim_{n \to \infty} f_n(x) = f(x, y) \text{ a.e. by (2)}$$

Therefore, the DCT applies, and

$$\lim \int_I f_n\, \mathrm{d}m = \int \lim f_n\, \mathrm{d}m$$

$$\lim_{n\to\infty}\int_I f(x,y_n)\,\mathrm{d}x = \int_I f(x,y)\,\mathrm{d}x$$

$$\lim_{n\to\infty} F(y_n) = F(y)$$

■

■ **Example 12.8** Consider $f(x,y) = e^{-x}x^{y-1}$ with $I\times J = (0,\infty)\times[m,M]$, where $0 < m < M < \infty$. We will study

$$\Gamma(y) = \int_0^\infty e^{-x}x^{y-1}\,\mathrm{d}x$$

1. Fix $k\in[m,M]$, $e^{-x}x^{k-1}$ is integrable on $(0,\infty)$

$$\left(e^{-x}x^{k-1}\right)\mathcal{X}_{(0,\infty)} \le 1\cdot x^{k-1}\mathcal{X}_{(0,K]} + 10e^{-x/2}\mathcal{X}_{[K,\infty)}$$

2. (2) follows directly from the contiuity of $f(x,y)$
3.

$$\begin{aligned}|f(x,y)| &\le e^{-x}x^{m-1}\mathcal{X}_{[0,1]} + e^{-x}x^{M-1}\mathcal{X}_{(1,\infty)}\\ &\le x^{m-1}\mathcal{X}_{[0,1]} + \text{an integrable function by the argument in (1)}\end{aligned}$$

Therefore, $T(y)$ is continuous for any $m\le y\le M$.

Note that the choice of $0<m<M<\infty$ is arbitrary, and therefore $T(y)$ is continous for all $y>0$.

We also hope that

$$F'(y) = \int_I \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$$

■