Chapter 1

Week1

1.1. Tuesday

1.1.1. Difference between ODE and SDE

We first discuss the difference between deterministic differential equations and stochastic ones by considering several real-life problems.

Problem 1: Population Growth Model. Consider the first-order ODE

$$\begin{cases} \dfrac{\mathrm{d}N(t)}{\mathrm{d}t} = a(t)N(t) \\ N(0) = N_0 \end{cases}$$

where $N(t)$ denotes the **size** of the population at time t; $a(t)$ is the given (deterministic) function describing the **rate** of growth of population at time t; and N_0 is a given constant.

If $a(t)$ is not completely known, e.g.,

$$a(t) = r(t) \cdot \text{noise, or } r(t) + \text{noise},$$

with $r(t)$ being a deterministic function of t, and the "noise" term models something random. The question arises: How to *rigorously* describe the "noise" term and solve it?

Problem 2: Electric Circuit. Let $Q(t)$ denote the charge at time t in an electrical circuit, which admits the following ODE:

$$\begin{cases} LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = F(t), \\ Q(0) = Q_0, \quad Q'(0) = Q'_0 \end{cases}$$

where L denotes the inductance, R denotes the resistance, C denotes the capacity, and $F(t)$ denotes the potential source.

Now consider the scenario where $F(t)$ is not completely known, e.g.,

$$F(t) = G(t) + \text{noise}$$

where $G(t)$ is deterministic. The question is how to solve the problem.

R The differential equations above involving non-deterministic coefficients are called the **stochastic differential equations** (SDEs). Clearly, the solution to an SDE should involve the randomness.

1.1.2. Applications of SDE

Now we discuss some applications of SDE shown in the finance area.

Problem 3: Optimal Stopping Problem. Suppose someone holds an asset (e.g., stock, house). He plans to sell it at some future time. Denote $X(t)$ as the price of the asset at time t, satisfying the following dynamics:

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = rX(t) + \alpha X(t) \cdot \text{noise}$$

where r, α are given constants. The goal of this person is to maximize the expected selling price:

$$\sup_{\tau \geq 0} \quad \mathbb{E}[X(\tau)]$$

where the optimal solution τ^* is the optimal stopping time.

Problem 4: Portfolio Selection Problem. Suppose a person is interested in two types of assets:

- A risk-free asset which generates a deterministic return ρ, whose price $X_1(t)$ follows a deterministic dynamics

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = \rho X_1(t),$$

- A risky asset whose price $X_2(t)$ satisfies the following SDE:

$$\frac{\mathrm{d}X_2(t)}{\mathrm{d}t} = \mu X_2(t) + \sigma X_2(t) \cdot \text{noise}$$

  where $\mu, \sigma > 0$ are given constants.

The policy of the investment is as follows. The wealth at time t is denoted as $v(t)$. This person decides to invest the fraction $u(t)$ of his wealth into the risky asset, with the remaining $1 - u(t)$ part to be invested into the safe asset. Suppose that the utility function for this person is $U(\cdot)$, and his goal is to maximize the expected total wealth at the terminal time T:

$$\max_{u(t), 0 \leq t \leq T} \mathbb{E}[U(v^u(T))]$$

where the decision variable is the portfolio function $u(t)$ along whole horizon $[0, T]$.

Problem 5: Option Pricing Problem. The financial derivates are products in the market whose value depends on the underlying asset. The European call option is a typical financial derivative. Suppose that the underlying asset is stock A, whose price at time t is $X(t)$. Then the call option gives the option holder the right (not the obligation) to buy one unit of stock A at a specified price (strike price) K at maturity date T. The task is to inference the fair price of the option at the current time. The formula for the price of the option is the following:

$$c_0 = \mathbb{E}[(X(T) - K)^+]$$

which is the famous Black-Sholes-Merton Formula.

1.1.3. Reviewing for Probability Space

Firstly, we review some basic concepts in real analysis.

Definition 1.1 [σ-Algebra] A set $\mathcal{F}$ containing subsets of Ω is called a σ-algebra if:

1. $\Omega \in \mathcal{F}$;
2. $\mathcal{F}$ is closed under complement, i.e., $A \in \mathcal{F}$ implies $\Omega \setminus A \in \mathcal{F}$;
3. $\mathcal{F}$ is closed under countably union operation, i.e., $A_i \in \mathcal{F}, i \geq 1$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition 1.2 [Probability Measure] A function $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is called a **probability measure** on $(\Omega, \mathcal{F})$ if

- $\mathbb{P}(\Omega) = 1$;
- $\mathbb{P}(A) \geq 0, \forall A \in \mathcal{F}$;
- $\mathbb{P}$ is σ-additive, i.e., when $A_i \in \mathcal{F}, i \geq 1$ and $A_i \cap A_j = \emptyset, \forall i \neq j$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

where $\mathbb{P}(A)$ is called the **probability of the event** A.

Definition 1.3 [Probability Space] A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ defined as follows:

1. Ω denotes the **sample space**, and a point $\omega \in \Omega$ is called a sample point;
2. $\mathcal{F}$ is a σ-algebra of Ω, which is a collection of subsets in Ω. The element $A \in \mathcal{F}$ is called an "event"; and
3. $\mathbb{P}$ is a probability measure defined in the space $(\Omega, \mathcal{F})$.

Definition 1.4 [Almost Surely True] A statement S is said to be **almost surely (a.s.) true** or **true with probability 1**, if

- $\mathfrak{B} := \{w : S(w) \text{ is true}\} \in \mathcal{F}$
- $\mathbb{P}(F) = 1.$

Definition 1.5 [Topological Space] A **topological space** $(X, \mathcal{T})$ consists of a (non-empty) set X, and a family of subsets of X ("open sets" $\mathcal{T}$) such that

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T}$ implies $U \bigcap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$.

When $A \in \mathcal{T}$, A is called the open subset of X. The $\mathcal{T}$ is called a **topology** on X.

Definition 1.6 [Borel σ-Algebra] Consider a topological space Ω, with $\mathcal{U}$ being the topology of Ω. The **Borel σ-Algebra** $\mathcal{B}(\Omega)$ on Ω is defined to be the *minimal* σ-algebra containing $\mathcal{U}$:

$$\mathcal{B}(\Omega) \triangleq \sigma(\mathcal{U}).$$

Any element $B \in \mathcal{B}(\Omega)$ is called the **Borel set**.

Definition 1.7 [$\mathcal{F}$-Measurable / Random Variable]

1. A function $f : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called $\mathcal{F}$-**measurable** if

   $$f^{-1}(\boldsymbol{B}) = \{w \mid f(w) \in \mathcal{B}\} \in \mathcal{F},$$

   for any $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)$.
2. A random variable X is a function $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and is $\mathcal{F}$-measurable.

Definition 1.8 [Generated σ-Algebra] Suppose X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the σ-algebra generated by X, say $\mathcal{H}_X$ is defined to be the **minimal σ-algebra** on Ω to make X measurable.

Proposition 1.1 $\mathcal{H}_X = \{X^{-1}(\boldsymbol{B}) : \boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)\}$.

Proof. Since X is $\mathcal{H}_X$-measurable, for any $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)$, $X^{-1}(\boldsymbol{B}) \in \mathcal{H}_X$. Thus $\mathcal{H}_X \supseteq \{X^{-1}(\boldsymbol{B}) : \boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)\}$. It suffices to show that $\{X^{-1}(\boldsymbol{B}) : \boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)\}$ is a σ-algebra to finish the proof, which is true since $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{U})$, with $\mathcal{U}$ being the topology of X. ∎

1.2. Thursday

Reviewing for Probability Space.

- $(\Omega, \mathcal{F}, \mathbb{P})$;
- Random variable;
- Generated σ-algebra;

1.2.1. More on Probability Theory

Definition 1.9 [Distribution] A probability measure μ_X on $\mathbb{R}^n$ induced by the random variabe X isdefined as

$$\mu_X(\boldsymbol{B}) = \mathbb{P}(X^{-1}(\boldsymbol{B})),$$

where $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)$. The μ_X is called the **distribution** of X.

Definition 1.10 [Expectation] The expectation of X is given by

$$\mathbb{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

When $\Omega = \mathbb{R}^n$, the expectation can be written in terms of distribution function:

$$\mathbb{E}[X] = \int_{\mathbb{R}^n} y\,\mathrm{d}\mu_X(y)$$

Note that the expectation of the random variable X is well-defined when X is integrable:

Definition 1.11 [Integrable] The random variable X is **integrable**, if

$$\int_\Omega |X(w)|\,\mathrm{d}\mathbb{P}(w) < \infty.$$

In other words, X is said to be $\mathcal{L}^1$-integrable, denoted as $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

■ **Example 1.1** If $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable, and $\int_\Omega |f(X(\omega))| \, \mathrm{d}\mathbb{P}(\omega) < \infty$, then

$$\mathbb{E}[f(x)] = \int_\Omega f(X(\omega)) \, \mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^n} f(y) \, \mathrm{d}\mu_X(y).$$

■

Definition 1.12 [L^p space] Suppose $X: \ \Omega \to \mathbb{R}$ is a random variable and $p \ge 1$.

- Define L^p-norm of X as

$$\|X\|_p = \left(\int_\Omega |X(\omega)|^p \, \mathrm{d}\mathbb{P} \right)^{1/p}$$

  If $p = \infty$, define

$$\|X\|_\infty = \inf\{N \in \mathbb{R} \mid |X(w)| \le N, \text{ a.s.}\}$$

- A random variable X is said to be in the L^p space (p-th integrable) if

$$\int_\Omega |X(\omega)|^p \, \mathrm{d}\mathbb{P}(\omega) < \infty,$$

  denoted as $X \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$.

■

Proposition 1.2 If $p \ge q$, then $\|X\|_p \ge \|X\|_q$. Thus $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{L}^q(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. The inequality is shown by using Holder's inequality:

$$\|X\|_q^q = \int_\Omega |X|^q \, \mathrm{d}\mathbb{P} \le \left(\int_\Omega (|X|^q)^{p/q} \, \mathrm{d}\mathbb{P} \right)^{q/p} = \left(\int_\Omega |X|^p \, \mathrm{d}\mathbb{P} \right)^{\frac{1}{p} \cdot q} = \|X\|_p^q.$$

■

Then we discuss how to define independence between two random variables, by the following three steps:

Definition 1.13 [Independence]

1. Two events $A_1, A_2 \in \mathcal{F}$ are said to be **independent** if $\mathbb{P}(A_1 \bigcap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$.
2. Two σ-algebras $\mathcal{F}_1, \mathcal{F}_2$ are said to be **independent** if F_1, F_2 are independent events for $\forall F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2$
3. Two random variables X, Y are said to be **independent** if $\mathcal{H}_X, \mathcal{H}_Y$, the σ-algebra generated by X and Y, respectively, are independent.

R The independence defined above can be generalized from two events into finite number of events.

Proposition 1.3 If X and Y are two independent random variables, and $\mathbb{E}[|X|] < \infty, \mathbb{E}[|Y|] < \infty$, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] < \infty.$$

Proof. The first step is to simplify the probability distribution for the product random variable (X, Y), i.e., $\mu_{X,Y}$.

R From now on, we also write the event $\{X^{-1}(\boldsymbol{B})\}$ as $\{X \in \boldsymbol{B}\}$ for $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)$.

By the definition of independence, we have the following:

$$\begin{aligned}\mu_{X,Y}(A_1 \times A_2) &\triangleq \mathbb{P}(\{(X,Y) \in (A_1 \times A_2)\}) = \mathbb{P}(\{X \in A_1, Y \in A_2\}) \\ &= \mathbb{P}(\{X \in A_1\})\mathbb{P}(\{Y \in A_2\}) = \mu_X(A_1)\mu_Y(A_2).\end{aligned}$$

Now we begin to simplify the expectation of product:

$$\begin{aligned}\mathbb{E}[XY] &= \int xy \,\mathrm{d}\mu_{X,Y}(x,y) = \iint xy \,\mathrm{d}\mu_X(x)\mu_Y(y) \\ &= \int y\left[\int x \,\mathrm{d}\mu_X(x)\right]\mu_Y(y) = \int \mathbb{E}[X] y \mu_Y(y) = \mathbb{E}[X]\mathbb{E}[Y].\end{aligned}$$

■

1.2.2. Stochastic Process

Consider a set T of time index, e.g., a non-negative integer set or a time interval $[0,\infty)$. We will discuss a discrete/continuous time stochastic process.

Definition 1.14 [Stochastic Process] A collection of random variables $\{X_t\}_{t\in T}$, defined on $(\Omega,\mathcal{F},\mathbb{P})$ and taking values in $\mathbb{R}^n$, is called a **stochastic process**.

R A stochastic process $\{X_t\}_{t\in T}$ can also be viewed as a random function, since it is a mapping $\Omega\times T\to\mathbb{R}^n$. Sometimes we omit the subscript to denote a stochastic process $\{X_t\}$.

Definition 1.15 [Sample Path] Fixing $\omega\in\Omega$, then $\{X_t(\omega)\}_{t\in T}$ (denoted as $X.(\omega)$) is called a **sample path**, or **trajectory**.

Definition 1.16 [Continuous] A stochastic process $\{X_t\}$ is said to be **continuous** (right-cot, left-cot, resp.) a.s., if $t\to X_t(\omega)$ is **continuous** (right-cot, left-cot, resp.) a.s., i.e.,

$$\mathbb{P}\Big(\{\omega: t\to X_t(\omega) \text{ is continuous (right-cot, left-cot, resp.)}\}\Big)=1.$$

Example 1.2 [Poisson Process] Consider $(\xi_j, j=1,2,\dots)$ a sequence of i.i.d. random variables with Possion distribution with intensity $\lambda>0$. Let $T_0=0$, and $T_n=\sum_{j=1}^n\xi_j$. Define $X_t=n$ if $T_n\le t<T_{n+1}$. Verify that $\{X_t\}$ is a stochastic process with right-continuity and left-limit exists. Instead of giving a mathematical proof, we provide a numerical simulation of $\{X_t\}$ plotted in Figure. 1.1. [a]

[a]The corresponding matlab code can be found in

https://github.com/WalterBabyRudin/CourseWare/tree/master/MAT4500/week1

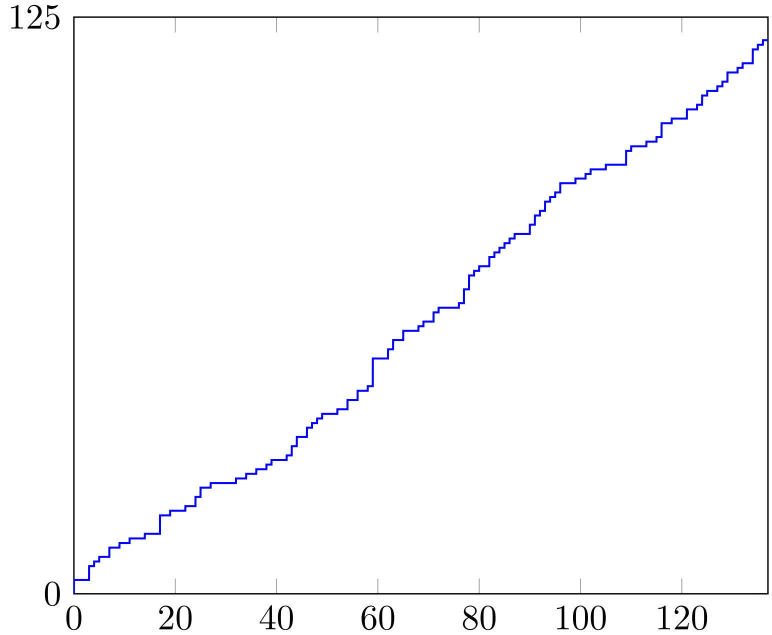

Figure 1.1: One simulation of $\{X_t\}$ with intensity $\lambda = 1.2$ and 500 samples