On Achievable Rates of Line Networks with Generalized Batched Network Coding

Jie Wang, Shenghao Yang, Yanyan Dong and Yiheng Zhang

Abstract

To better understand the wireless network design with a large number of hops, we investigate a line network formed by general discrete memoryless channels (DMCs), which may not be identical. Our focus lies on Generalized Batched Network Codes (GBNCs), a class of codes that encompasses most existing schemes as special cases and achieves the min-cut upper bounds as the parameters batch size and inner block length tend to infinity. The inner blocklength of a GBNC provides upper bounds on the required latency and buffer size at intermediate network nodes. By employing a "bottleneck status" technique, we derive new upper bounds on the achievable rates of GBNCs. These bounds surpass the cut-set bound for large network lengths when the inner blocklength and batch size of GBNCs are finite. For line networks of canonical channels, certain upper bounds hold even with relaxed inner blocklength constraints. Additionally, we employ a "channel reduction" technique to generalize the existing achievability results for line networks with identical DMCs to networks with non-identical DMCs. For line networks with packet erasure channels, we make refinement in both the upper bound and the coding scheme, and showcase their proximity through numerical evaluations.

Index Terms

multi-hop network, line network, batched network code, capacity bound, buffer size, latency

I. Introduction

We investigate multi-hop, line topology networks formed by concatenating discrete memoryless channels (DMCs), which are fundamental channel models in communication systems.

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In this line network, the first node is the source node, the last node is the destination node, and the intermediate nodes connect them. Without limitations on storage and computation at the intermediate nodes, the network capacity is determined by the min-cut from the source to the destination, achievable through hop-by-hop implementation of capacity-achieving channel codes [1]. However, as the number of hops increases, the hop-by-hop coding approach introduces significant communication latency and storage expenses at the intermediate nodes. This paper addresses the study of such networks with a large number of hops, while considering constraints on intermediate node buffer size and latency.

A. Related Work

Multi-hop wireless communication networks are utilized in diverse domains, such as wireless LAN [2], underwater acoustic networks [3], deep space communication networks [4], and cellular networks with integrated access and backhaul (IAB). The IAB enables wireless interconnection between multiple base stations, eliminating the need for wireline backhaul links for some stations [5], [6]. These networks typically involve a significant number of hops. Communication latency and rate are critical factors in multi-hop wireless networks [**?**], [**?**], [7]–[9]. With the hop-by-hop coding approach, increasing the number of hops necessitates longer code blocklength to minimize hop-by-hop error probability. Consequently, the end-to-end communication latency grows at a rate higher than linear as the number of hops increases. In general, the increase in the number of hops affects both the end-to-end latency and communication rate, warranting further investigation.

In their work [10], Niesen, Fragouli, and Tuninetti investigated the line network capacity with a fixed inner blocklength N at the intermediate nodes. This fixed blocklength affects delay and buffer size. Assuming all channels in the line network are identical (denoted as Q), and when the zero-error capacity of Q is non-zero, they demonstrated that using a constant N allows achieving any constant rate below the zero-error capacity for any given number of hops L. Conversely, when the zero-error capacity of Q is zero, a class of codes with a constant N can achieve rates on the order of $\Omega(e^{-cL})$, where c is a constant. Additionally, if N is of the order of $\ln L$, it is possible to achieve any rate below the capacity of Q.

Contrary to the achievability results discussed earlier, the min-cut remains the strongest upper bound for line networks. It is still uncertain whether the diminishing rates observed with increasing network length are essential or if there exist more efficient coding strategies that can achieve higher rates. In addition, although the previous work [10] did not explicitly examine the processing latency and buffer size at intermediate nodes, it is worth exploring the possibility of reducing the processing latency and buffer size requirements beyond the $O(N)$ complexity. With these inquiries in mind, we embark on a comprehensive investigation of line networks formed by DMCs.

B. Paper Contributions

We investigate Generalized Batched Network Codes (GBNCs) for line networks formed by general DMCs. GBNCs, defined in §II, consist of an outer code and an inner code, where the outer code encodes information messages into batches of coded symbols, and the inner code performs recoding within each batch. GBNCs are chosen for several reasons. First, GBNCs encompass a wide range of codes, including the coding schemes studied in [10] where the inner blocklength is the same as the batch size. GBNCs extend batched network codes developed for networks with packet loss to accommodate general DMCs. Moreover, when both the batch size and inner blocklength can be arbitrarily large, GBNCs can achieve the min-cut. Since it is challenging to determine non-trivial upper bounds for line networks with constraints like buffer size or inner blocklength, as suggested in the network information theory literature [11], considering GBNCs provides a valuable alternative approach. Second, GBNCs allow us to explicitly characterize latency and buffer size. Our formulation reveals that the recoding latency and buffer size at an intermediate node are bounded above by a linear order of the inner blocklength.

In this paper, we analyze the achievable rates of GBNCs in terms of the network length L. GBNCs have two parameters: the batch size M and the inner blocklength N, which also represents the buffer size required for caching received symbols and the corresponding latency. We assume that the computation costs associated with recoding, such as space and time requirements, are not considered. Consequently, the buffer size needed for caching symbols at an intermediate node is of the order $O(N)$, and the accumulated end-to-end latency for a batch is of the order $O(NL)$. This paper presents refined and extended results compared to our previous conference papers [12], [13], with main results improved or new.

1) Upper Bound: We introduce a “bottleneck status” technique to establish new upper bounds on the achievable rate of GBNCs for line networks consisting of channels with 0 zero-error capacity. We begin by proving the converses for a class of channels known as *canonical channels*, which are characterized by having an output symbol that occurs with a positive probability for all possible input symbols. Examples of canonical channels include the binary erasure channel (BEC) and binary symmetric channel (BSC).

In the case of a canonical channel, the bottleneck status occurs when all N channel uses for a batch produce the same fixed output value. By decomposing the end-to-end transition matrix induced by the inner code into two parts, one occurring under the bottleneck status and the other occurring otherwise, we can obtain an upper bound on the achievable rate of GBNCs using the convexity of mutual information (Theorem 4). Numerical evaluations demonstrate that our upper bound can outperform the cut-set bound for large network lengths for certain values of M and N.

For networks formed by channels that are not necessarily canonical, we develop a channel concatenation technique to transform the network into one with canonical channels. In Theorem 7, we derive a general upper bound on the achievable rates of GBNCs. Notably, when $N = O(1)$, our upper bound indicates that the achievable rate must exponentially decay with L, aligning with the achievable rates obtained in [10]. When $N = O(\ln L)$ and $M = O(1)$, our upper bound shows that the achievable rate is $O(1/\ln L)$, which matches the order of achievable rates obtained in [10]. For canonical channels, we obtain even stronger upper bounds scalability, demonstrating that the achievable rate is $O(1/\ln L)$ when $M = O(1)$ and N is unbounded.

2) Lower Bound: We present coding schemes that achieve rates matching the order of the upper bound in terms of the network length L (see §IV). When all links in the line networks are identical DMCs, it has been shown in [10] that GBNCs with $M, N = O(\ln L)$ achieve a constant rate, while GBNCs with $M, N = O(1)$ achieve exponentially decreasing rates. We extend these achievability results to line networks with non-identical DMCs by employing a "channel reduction" technique, transforming the network into an equivalent network with identical channels (see §IV-A and §IV-C).

Furthermore, we demonstrate that rates of $\Omega(1/\ln L)$ can be achieved using $M = O(1)$ and $N = O(\ln L)$. In a general decode-and-forward approach, a buffer size of $O(\ln L)$ is required. However, specific codes allow for a reduced buffer size of $O(\ln \ln L)$ (see §IV-B). To exemplify this result, we consider a repetition coding scheme, which prompts us to explore simpler schemes such as convolutional codes with Viterbi decoding for line networks with a large number of hops.

3) Extensions: In the context of line networks with packet erasure channels, we make advancements in both the upper bound and the coding scheme. Through extensive numerical evaluations,

TABLE I

SOME NOTATIONS USED IN THE PAPER, LISTED IN THE ALPHABETICAL ORDER.

Notation	Explanation	Notation	Explanation
$\mathcal{A}$	Batch alphabet.	M	Batch size.
$C(Q)$	Channel capacity of channel Q.	N	Inner blocklength.
$C_0(Q)$	Zero-error capacity of channel Q.	Q	Discrete memoryless channel.
$C_L(M,N)$	Maximum achievable rate of all recoding schemes with batch size M and inner blocklength N.	$Q^{\otimes N}$	Discrete memoryless channel with N channel uses.
$E_{0,\ell}$	Event that all N outputs of Q_ℓ are equal to the same value regardless of channel input.	$\mathcal{Q}_{\mathrm{i}}/\mathcal{Q}_{\mathrm{o}}$	Input/output alphabets of Q.
E_0	Event that there exists one link ℓ such that $E_{0,\ell}$ holds.	$\mathbf{U}_\ell/\mathbf{Y}_\ell$	The input/output of N uses of the ℓ-th communication link.
Er_ℓ	Coding error exponent for channel Q_ℓ	W_L	End-to-end transition matrix of the batch channel from $\mathbf{X}$ to $\mathbf{Y}_L$.
Er^*	Smallest coding error exponent among all $\ell \geq 1$.	$\mathbf{X} \in \mathcal{A}^M$	A generic batch.
ℓ	Index of node/channel.	$\mathbf{X}[k]$	The k-th entry in $\mathbf{X}$.
L	Network length.	$\mathbf{Z}_\ell$	Channel status of Q_ℓ.

we establish the close proximity between the upper bound and the achievable rates of the coding scheme (see §V). This finding serves as motivation for future research endeavors aimed at further improving the upper bound and developing more efficient coding schemes tailored to specific channel characteristics.

Furthermore, we extend our results to networks where certain channels have a positive zero-error capacity (see §VI). We argue that the achievable rate of the network scales like a line network formed by only the channels with the 0 zero-error capacity.

Throughout this paper, we use $\log$ to denote the logarithm of base 2 and $\ln$ to denote the logarithm of base e. All omitted proofs and experiment codes can be found in the supplementary material at https://github.com/WalterBabyRudin/JSAC-SCNF-22-Supplementary. Most of the notations used throughout this manuscript are defined in Table I. Others are defined following their first appearances, as needed.

II. LINE NETWORKS AND GENERALIZED BATCHED NETWORK CODING

In this section, we describe the line network model and introduce batched network coding.

A. Line Network Model

A line network of length L consists of nodes labeled as $0, 1, \ldots, L$, with directed communication links from node $\ell - 1$ to node ℓ. Each link is a discrete memoryless channel (DMC) with fixed finite input and output alphabets $\mathcal{Q}_\mathrm{i}$ and $\mathcal{Q}_\mathrm{o}$ respectively. The transition matrix for link ℓ is denoted as Q_ℓ. The line network is formed by concatenating $Q_1, Q_2, \ldots, Q_L$. This study focuses on communication between the first node, referred to as the *source node*, and the last node, known as the *destination node*. The nodes numbered $1, 2, \ldots, L-1$ are referred to as the *intermediate nodes*.

Let $C(Q)$ and $C_0(Q)$ denote the channel capacity and zero-error capacity of a DMC with transition matrix Q respectively. Without any constraints at the network nodes, the capacity of the network is given by $\min_{\ell=1}^{L} C(Q_\ell)$, which is also known as the *min-cut*. Achieving the min-cut involves using a capacity achieving code at each hop, where intermediate nodes decode the previous link's code and encode the message using the next link's code. This scheme is commonly referred to as *decode-and-forward*. However, as we will discuss later, decode-and-forward is not always the optimal solution when considering both latency and buffer size at the intermediate nodes. Next, we present a general coding scheme for the line network and examine the relationship between the coding parameters and latency as well as buffer size.

B. Generalized Batched Network Coding

A *Generalized Batched Network Code* (GBNC) comprises an outer code and an inner code. The outer code, executed at the source node, encodes a message from a finite set and generates multiple batches, each containing M symbols from a finite set $\mathcal{A}$. The parameter M is known as the *batch size*. The inner code operates on individual batches separately, employing recoding operations at nodes $0, 1, \ldots, L-1$.

Let's define the recoding process for a generic batch $\mathbf{X} \in \mathcal{A}^M$. At the source node, the recoding transforms the original M symbols of $\mathbf{X}$ into N recoded symbols $\mathbf{U}_1$ in $\mathcal{Q}_\mathrm{i}$, where N is a positive integer referred to as the *inner blocklength*. The recoding at the source node is represented by the function $\phi_0 : \mathcal{A}^M \to \mathcal{Q}_\mathrm{i}^N$, such that $\mathbf{U}_1 = \phi_0(\mathbf{X})$.

At an intermediate node ℓ, recoding is performed on the N received symbols $\mathbf{Y}_\ell \in \mathcal{Q}_\mathrm{o}^N$ to generate N recoded symbols $U_{\ell+1} \in \mathcal{Q}_\mathrm{i}^N$ for transmission on the outgoing link of node ℓ. Due

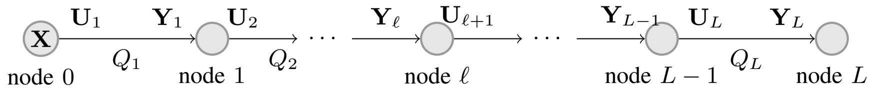

Fig. 1. A line network with the random variables involved in recoding.

to the memoryless property of Q_ℓ, the conditional probability of $\mathbf{Y}_\ell = \mathbf{y}$ given $\mathbf{U}_\ell = \mathbf{u}$ is

$$P(\mathbf{Y}_\ell = \mathbf{y}|\mathbf{U}_\ell = \mathbf{u}) = Q_\ell^{\otimes N}(\mathbf{y}|\mathbf{u}) \triangleq \prod_{i=1}^{N} Q_\ell(\mathbf{y}[i]|\mathbf{u}[i]), \tag{1}$$

where $\mathbf{y}[k]$ ($1 \leq k \leq M$) represents the kth entry in $\mathbf{y}$. The recoding at node ℓ is represented by the function $\phi_\ell : \mathcal{A}^M \to \mathcal{Q}_{\mathrm{i}}^N$, such that $\mathbf{U}_{\ell+1} = \phi_\ell(\mathbf{Y}_\ell)$. In general, the number of recoded symbols transmitted by different nodes can vary [14], [15]. However, for simplicity, we assume they are all the same for the analysis.

At the destination node, all received symbols, which may belong to different batches, are jointly decoded. The inner code's end-to-end operation, with the given recoding function ϕ_ℓ at all nodes, can be viewed as a memoryless channel referred to as a *batch channel*, which takes $\mathbf{X}$ as the input and produces $\mathbf{Y}_L$ as the output. Fig. 1 illustrates the variables involved in the recoding process, forming the Markov chain:

$$\mathbf{X} \to \mathbf{U}_1 \to \mathbf{Y}_1 \to \cdots \to \mathbf{U}_L \to \mathbf{Y}_L. \tag{2}$$

The end-to-end transition matrix W_L of the batch channel can be derived using ϕ_ℓ and Q_ℓ.

The outer code serves as a channel code for the batch channel W_L to ensure end-to-end reliability. Given a recoding scheme $\{\phi_\ell\}$, the maximum achievable rate of the outer code is $\max_{p_\mathbf{X}} I(\mathbf{X}; \mathbf{Y}_L)$ for N channel uses, where $p_\mathbf{X}$ represents the distribution of $\mathbf{X}$. The objective of designing a recoding scheme, given parameters M and N, is to maximize $\frac{1}{N} \max_{p_\mathbf{X}} I(\mathbf{X}; \mathbf{Y}_L)$. Let $C_L(M, N)$ denote the maximum achievable rate among all recoding schemes with batch size M and inner blocklength N, defined as:

$$C_L(M, N) = \frac{1}{N} \max_{\{\phi_\ell\}} \max_{p_\mathbf{X}} I(\mathbf{X}; \mathbf{Y}_L) = \frac{1}{N} \max_{\{\phi_\ell\}} \max_{p_\mathbf{X}} I(p_\mathbf{X}, W_L). \tag{3}$$

$C_L(M, N)$ is also referred to as the capacity of GBNCs with parameters M and N. We can then maximize $C_L(M, N)$ while considering constraints on M and N, which impact both the recoding latency and the buffer size.

Recoding functions $\{\phi_\ell\}$ can generally be random. However, the convexity of $I(p_\mathbf{X}, W_L)$ for a fixed $p_\mathbf{X}$ with respect to W_L implies the existence of a deterministic recoding scheme that achieves $C_L(M, N)$. In particular, the coding scheme analyzed in [10] considers the case where $M = N$. A special inner code known as decode-and-forward will be discussed in §IV. GBNCs generalize the batched network codes studied for networks with packet erasure channels in literature (see discussion in §V).

C. Buffer Size and Latency at Intermediate Nodes

Let's now delve into the buffer size requirement and latency at the intermediate nodes in GBNCs. In this discussion, we consider a sequential transmission model where symbols of a batch are transmitted consecutively. We will disregard the space and time costs associated with executing recoding ϕ_ℓ, but instead focus on the buffer size needed for caching received symbols and the corresponding latency. Specifically, we will discuss the buffer size required for caching the received symbols for recoding at an intermediate node, as well as the latency between receiving the first symbol of a batch and transmitting the first symbol of the same batch.

The key principle of GBNCs is the independent application of recoding to each batch. In the worst case scenario, an intermediate node begins transmitting the first recoded symbol of a batch only after receiving all N symbols of that batch. Consequently, the latency of a batch at an intermediate node is upper bounded by $O(N)$. Since an intermediate node can only transmit symbols of a batch after receiving at least one symbol from that batch, the lower bound on the latency at an intermediate node is 1. The accumulated end-to-end recoding latency across all intermediate nodes falls within the range of $\Omega(L)$ to $O(NL)$.

Similarly, in the worst-case scenario, an intermediate node starts transmitting the first recoded symbol of a batch only after receiving all N symbols of that batch. Additionally, these received symbols need to be cached for N more channel uses. Therefore, an intermediate node needs to cache at most $2N$ symbols: N symbols of the batch for transmitting and N symbols of the same batch for receiving. This indicates that the buffer size required for caching symbols at an intermediate node is $O(N)$.

III. Converse for Line Networks of Channels with 0 Zero-error Capacity

In this section, we study upper bounds on $C_L(M, N)$ for line networks of channels with 0 zero-error capacity, i.e., $C_0(Q_\ell) = 0$. One known upper bound of $C_L(M, N)$ is the min-cut $\min_{\ell=1}^{L} C(Q_\ell)$. However, this bound may not be sufficient for small values of M and N. In this section, we introduce a technique called a "bottleneck status" to derive a tighter bound on $C_L(M,N)$ when M and N are small.

The bottleneck status refers to an event E_0 that is associated with the channel W_L. Let

$$W_L^{(0)}(\mathbf{y} \mid \mathbf{x}) = P(\mathbf{Y}_L = \mathbf{y} \mid \mathbf{X} = \mathbf{x}, E_0), \quad W_L^{(1)}(\mathbf{y} \mid \mathbf{x}) = P(\mathbf{Y}_L = \mathbf{y} \mid \mathbf{X} = \mathbf{x}, \overline{E_0}). \tag{4}$$

The channel W_L can be expressed as $W_L = W_L^{(0)} p_0 + W_L^{(1)} p_1$, where $p_0 = P(E_0), p_1 = P(\overline{E_0})$. As mutual information $I(p_{\mathbf{X}}, W_L)$ is convex w.r.t. W_L for given $p_{\mathbf{X}}$, we can establish the upper bound as follows:

$$I(p_{\mathbf{X}}, W_L) \le p_0 I(p_{\mathbf{X}}, W_L^{(0)}) + p_1 I(p_{\mathbf{X}}, W_L^{(1)}). \tag{5}$$

The crucial step is to design the event E_0 in order to obtain the desired upper bound. We first introduce our technique for canonical channels, and then discuss the general channels.

A. *Line Network of Canonical Channels*

For $0 < \varepsilon \le 1$, we call a channel $Q : \mathcal{Q}_{\mathrm{i}} \to \mathcal{Q}_{\mathrm{o}}$ an *ε-canonical channel* if there exists $y^* \in \mathcal{Q}_{\mathrm{o}}$ such that for every $x \in \mathcal{Q}_{\mathrm{i}}$, $Q(y^*|x) \ge \varepsilon$. In other words, for a canonical channel, there exists an output symbol y^* that occurs with a positive probability for all the inputs. The binary erasure channel (BEC) and binary symmetric channel (BSC) are both canonical channels, but a typewriter channel is non-canonical. Note that a canonical Q has $C_0(Q) = 0$. In this subsection, we study a line network consisting of ε-canonical channels $Q_\ell, \ell = 1, \ldots, L$.

To design the bottleneck status E_0, we adopt a formulation of DMCs in [16, §7.1]. The relation between the input X and output Y of a DMC Q can be modeled as a function α as $Y = \alpha(X, Z)$, where Z is a random variable independent of X. In particular, we have

$$Y = \alpha(X, Z = (Z[x], x \in \mathcal{Q}_{\mathrm{i}})) = \sum_{x \in \mathcal{Q}_{\mathrm{i}}} \mathbf{1}\{X = x\} Z[x], \tag{6}$$

where $\mathbf{1}$ denotes the indicator function, and $Z[x], x \in \mathcal{Q}_{\mathrm{i}}$ are independent random variables on $\mathcal{Q}_{\mathrm{o}}$ with the distribution $P(Z[x] = y) = Q(y|x)$. Here $Z = (Z[x], x \in \mathcal{Q}_{\mathrm{i}})$ is also called *channel status variable*, and α is called the *channel function*. We denote by α_ℓ the channel function of Q_ℓ.

Consider a GBNC with inner blocklength N for the line network. With the alternative channel formulation (6), we can write for $\ell = 1, \ldots, L$, and $i = 1, \ldots, N$, $\mathbf{Y}_\ell[i] = \alpha_\ell(\mathbf{U}_\ell[i], \mathbf{Z}_\ell[i])$. Here $\mathbf{Z}_\ell[i] = (\mathbf{Z}_\ell[i, x], x \in \mathcal{Q}_{\mathrm{i}})$ is the channel status variable for the i-th use of the channel Q_ℓ, where

$$P(\mathbf{Z}_\ell[i, x] = y) = Q_\ell(y|x). \tag{7}$$

Define $\mathbf{Z}_\ell = (\mathbf{Z}_\ell[i], i = 1, \ldots, N)$. For notation simplicity, we rewrite the channel relation as

$$\mathbf{Y}_\ell = \alpha_\ell^{(N)}(\mathbf{U}_\ell, \mathbf{Z}_\ell). \tag{8}$$

Given that Q_ℓ is ε-canonical, there exists an output denoted as y_ℓ^* satisfying

$$Q_\ell(y_\ell^*|x) \geq \varepsilon \text{ for all } x \in \mathcal{Q}_\mathrm{i}. \tag{9}$$

Let's define $E_{0,\ell} = \{\mathbf{Z}_\ell[i,x] = y_\ell^*, i \in \{1,\ldots,N\}, x \in \mathcal{Q}_\mathrm{i}\}$. Under the condition $E_{0,\ell}$, all N outputs of Q_ℓ are equal to y_ℓ^* for any possible channel input, rendering the channel useless. We can quantify the probability of $E_{0,\ell}$ as follows:

$$P(E_{0,\ell}) = \prod_{i\in\{1,\ldots,N\}, x\in\mathcal{Q}_\mathrm{i}} P(\mathbf{Z}_\ell[i,x] = y_\ell^*) = \prod_{i\in\{1,\ldots,N\}, x\in\mathcal{Q}_\mathrm{i}} Q_\ell(y_\ell^*|x) \geq \varepsilon^{|\mathcal{Q}_\mathrm{i}|N}, \tag{10}$$

where the second equality follows from (7), and the inequality follows from (9). Now consider the event

$$E_0 = \vee_{\ell=1}^{L} E_{0,\ell}. \tag{11}$$

This event implies the existence of at least one link ℓ in the network that is deemed useless. To establish this rigorously, we show the following lemma.

Lemma 1. *When Q_ℓ, $\ell = 1, \ldots, L$ are all ε-canonical channels, for $W_L^{(0)}$ defined in* (4) *and E_0 defined in* (11)*, $I(p_\mathbf{X}, W_L^{(0)}) = 0$.*

Proof: Write $E_0 = \vee_{\ell=1}^{L}(E_{0,\ell} \wedge_{\ell'>\ell} \overline{E_{0,\ell'}})$, where $(E_{0,\ell} \wedge_{\ell'>\ell} \overline{E_{0,\ell'}})$, $\ell = 1, \ldots, L$ are disjoint. Hence,

$$P(\mathbf{y}_L, \mathbf{x}, E_0) = \sum_{\ell=1}^{L} P(\mathbf{y}_L, \mathbf{x}, E_{0,\ell}, \wedge_{\ell'>\ell}\overline{E_{0,\ell'}}) = \sum_{\ell,\mathbf{y}_\ell,\mathbf{u}_\ell} P(\mathbf{y}_L, \mathbf{x}, E_{0,\ell}, \wedge_{\ell'>\ell}\overline{E_{0,\ell'}}, \mathbf{y}_\ell, \mathbf{u}_\ell) \tag{12}$$

$$= \sum_{\ell,\mathbf{y}_\ell} P(\mathbf{y}_L, \wedge_{\ell'>\ell}\overline{E_{0,\ell'}} \mid \mathbf{y}_\ell) \sum_{\mathbf{u}_\ell} P(\mathbf{x}, E_{0,\ell}, \mathbf{y}_\ell, \mathbf{u}_\ell), \tag{13}$$

where the last equality follows from the Markov chain in (2). Further,

$$\sum_{\mathbf{u}_\ell} P(\mathbf{x}, E_{0,\ell}, \mathbf{y}_\ell, \mathbf{u}_\ell) = \sum_{\mathbf{u}_\ell} P(\mathbf{x}, \mathbf{u}_\ell) P(E_{0,\ell}) P(\mathbf{y}_\ell \mid \mathbf{u}_\ell, E_{0,\ell}) \tag{14}$$

$$= \sum_{\mathbf{u}_\ell} P(\mathbf{x}, \mathbf{u}_\ell) P(E_{0,\ell}) P(\mathbf{y}_\ell \mid E_{0,\ell}) = p_\mathbf{X}(\mathbf{x}) P(E_{0,\ell}) P(\mathbf{y}_\ell \mid E_{0,\ell}), \tag{15}$$

where (14) follows from $\mathbf{Y}_\ell = \alpha_\ell^{(N)}(\mathbf{U}_\ell, \mathbf{Z}_\ell | E_{0,\ell}) = y_\ell^*$ (ref. (8)). By (13) and (15),

$$P(\mathbf{y}_L, \mathbf{x}, E_0) = p_{\mathbf{X}}(\mathbf{x}) \sum_{\ell, \mathbf{y}_\ell} P(\mathbf{y}_L, \wedge_{\ell'>\ell} \overline{E_{0,\ell'}} \mid \mathbf{y}_\ell) P(\mathbf{y}_\ell, E_{0,\ell}) \tag{16}$$

$$= p_{\mathbf{X}}(\mathbf{x}) \sum_{\ell, \mathbf{y}_\ell} P(\mathbf{y}_L, \wedge_{\ell'>\ell} \overline{E_{0,\ell'}}, \mathbf{y}_\ell, E_{0,\ell}) = P(\mathbf{y}_L, \mathbf{x}, E_0), \tag{17}$$

which implies $I(p_{\mathbf{X}}, W_L^{(0)}) = 0$. ■

Lemma 2. *When Q_ℓ, $\ell = 1, \ldots, L$ are all ε-canonical channels, for $W_L^{(1)}$ defined in* (4) *and E_0 defined in* (11)*, we have 1) $P(\overline{E_0}) \leq (1 - \varepsilon^{|\mathcal{Q}_\mathrm{i}|N})^L$ and 2)*

$$I(p_{\mathbf{X}}, W_L^{(1)}) \leq \min \left\{ H(\mathbf{X}), \max_{p_{\mathbf{U}_L}} I(\mathbf{U}_\ell; \mathbf{Y}_\ell \mid \overline{E_{0,\ell}}), \ell = 1, \ldots, L \right\}. \tag{18}$$

Proof: As $\overline{E_0} = \wedge_{\ell=1}^L \overline{E_{0,\ell}}$, by (10), $P(\overline{E_0}) = \prod_{\ell=1}^L (1 - P(E_{0,\ell})) \leq (1 - \varepsilon^{|\mathcal{Q}_\mathrm{i}|N})^L$. We first show that given $\overline{E_0}$, $\mathbf{Z}_1, \ldots, \mathbf{Z}_L$ are independent. Write

$$P(\overline{E_0}) = P(\overline{E_{0,\ell}}, \ell = 1, \ldots, L) = \prod_{\ell=1}^L P(\overline{E_{0,\ell}}) \tag{19}$$

$$P(\mathbf{Z}_\ell, \ell = 1, \ldots, L, \overline{E_0}) = P(\mathbf{Z}_\ell, \overline{E_{0,\ell}}, \ell = 1, \ldots, L) = \prod_{\ell=1}^L P(\mathbf{Z}_\ell \mid \overline{E_{0,\ell}}) P(\overline{E_{0,\ell}}) \tag{20}$$

Hence,

$$P(\mathbf{Z}_\ell, \ell = 1, \ldots, L \mid \overline{E_0}) = \prod_{\ell=1}^L P(\mathbf{Z}_\ell \mid \overline{E_{0,\ell}}) = \prod_{\ell=1}^L P(\mathbf{Z}_\ell \mid \overline{E_0}). \tag{21}$$

Under the condition of $\overline{E_0}$, as $\mathbf{Z}_1, \ldots, \mathbf{Z}_L$ are independent, we have the Markov chain in (2) holds and hence $I(p_{\mathbf{X}}, W_L^{(1)}) \leq I(\mathbf{U}_\ell; \mathbf{Y}_\ell \mid \overline{E_{0,\ell}})$ and $I(p_{\mathbf{X}}, W_L^{(1)}) \leq H(\mathbf{X})$. ■

In Lemma 2, $\max_{p_{\mathbf{U}}} I(\mathbf{U}_\ell; \mathbf{Y}_\ell \mid \overline{E_{0,\ell}})$ is the capacity of the channel Q_ℓ^N under the condition $\overline{E_{0,\ell}}$. One upper bound is $\frac{1}{N} \max_{p_{\mathbf{U}_L}} I(\mathbf{U}_\ell; \mathbf{Y}_\ell \mid \overline{E_{0,\ell}}) \leq \log \min(|\mathcal{Q}_\mathrm{i}|, |\mathcal{Q}_\mathrm{o}|)$. In the following lemma, we give a better upper bound that converges $C(Q_\ell)$ when N tends to infinity.

Lemma 3. *Consider a channel Q as defined in* (6) *by (α, Z). Fix an output y^* such that $Q(y^*|x) = P(Z[x] = y^*) \geq \epsilon$ for all input x, where $\epsilon > 0$. For N uses of the channel, let $Z[i, x]$ be the channel variable of the ith uses associated with the input x. Let E_0 be the event that*

$\{Z[i,x]=y^*, i=1,\ldots,N, x\in\mathcal{Q}_\mathrm{i}\}$. Let W be the channel formed by N uses of Q under the condition of $\overline{E_0}$. Then

$$\frac{1}{N}I(p,W)\le C^*(Q,N)\triangleq\frac{1}{1-p_0}\left(C(Q)+\frac{1}{N}\left((q^*+p_0)\log\frac{q^*-p_0}{\epsilon^N-p_0}+q^*\log\frac{\epsilon^N}{q^*}\right)\right), \quad (22)$$

where $p_0=P(E_0)=[\prod_x Q(y^*|x)]^N$, and $q^*=\max_{\mathbf{x}} Q^{\otimes N}(\mathbf{y}^*|\mathbf{x})$.

Based on the relation (5) together with Lemmas 1, 2, and 3, we derive the following theorem.

Theorem 4. *Consider a length-L line network of ε-canonical channels with finite input and output alphabets $\mathcal{Q}_\mathrm{i}$ and $\mathcal{Q}_\mathrm{o}$, respectively. The capacity of GBNCs with batch size M and inner blocklength N has the following upper bound:*

$$C_L(M,N)\le(1-\varepsilon^{|\mathcal{Q}_\mathrm{i}|N})^L\min\left\{C^*(Q_\ell,N),\log|\mathcal{Q}_\mathrm{i}|,\log|\mathcal{Q}_\mathrm{o}|,\frac{M\log|\mathcal{A}|}{N}\right\}. \quad (23)$$

Moreover,

1) *when $N=O(1)$, $\max_M C_L(M,N)=O((1-\varepsilon^{|\mathcal{Q}_\mathrm{i}|N})^L)$;*
2) *when $M=O(1)$, $\max_N C_L(M,N)=O(1/\ln L)$;*
3) *when M and N are arbitrary, $\max_{M,N} C_L(M,N)=O(1)$.*

Proof: Recall that $C_L(M,N)=\frac{1}{N}\max_{\{\phi_\ell\}}\max_{p_\mathbf{X}} I(p_\mathbf{X},W_L)$, and we obtain

$$\begin{aligned}
I(p_\mathbf{X},W_L)&\le p_0 I(p_\mathbf{X},W_L^{(0)})+p_1 I(p_\mathbf{X},W_L^{(1)}) && (24)\\
&=p_1 I(p_\mathbf{X},W_L^{(1)}) && (25)\\
&\le(1-\varepsilon^{|\mathcal{Q}_\mathrm{i}|N})^L\min\left\{H(\mathbf{X}),\max_{p_{\mathbf{U}_L}} I(\mathbf{U}_\ell;\mathbf{Y}_\ell\mid\overline{E_{0,\ell}}),\ell=1,\ldots,L\right\} && (26)\\
&\le(1-\varepsilon^{|\mathcal{Q}_\mathrm{i}|N})^L\min\{NC^*(Q_\ell,N),N\log|\mathcal{Q}_\mathrm{i}|,N\log|\mathcal{Q}_\mathrm{o}|,M\log|\mathcal{A}|\}, && (27)
\end{aligned}$$

where (24) follows from (5), (25) is obtained by applying Lemma 1, (26) follows from Lemma 2, and (27) holds due to $H(\mathbf{X})\le M\log|\mathcal{A}|$, $I(\mathbf{U}_\ell;\mathbf{Y}_\ell\mid\overline{E_{0,\ell}})\le\min(\log|\mathcal{Q}_\mathrm{i}^N|,\log|\mathcal{Q}_\mathrm{o}^N|)$, and Lemma 3.

The remainder part of the theorem is proved by analyzing the upper bound in (23) for different values of M and N. In particular, Case 2) is obtained using the following Lemma 5, whose proof is provided in Appendix A.

Lemma 5. *For fixed real number $0<\epsilon<1$ and integer $L>1$, the function $F(N)=(1-\epsilon^N)^L/N$ of integer N is maximized when N is $\Theta(\ln L)$, and the optimal value of $F(N)$ is $\Theta\left(\frac{\ln(1/\epsilon)}{\ln L}\right)$.*

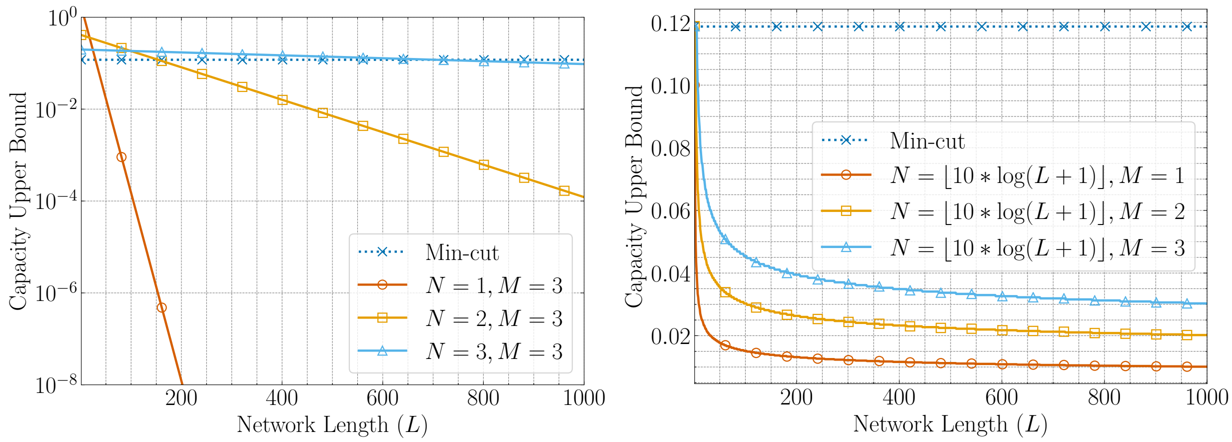

Fig. 2. Numerical illustrations of the capacity of GBNCs using BSC with crossover probability $\epsilon = 0.3, |\mathcal{A}| = 2$. The hop length L ranges from 1 to 1000. Figures from left to right correspond to (a) the capacity upper bound (23) with inner blocklength $N \in \{1, 2, 3\}$ and batch size $M = 3$; and (b) the capacity upper bound (23) with inner blocklength $N = \lfloor 10 * \log(L + 1) \rfloor$ and batch size $M \in \{1, 2, 3\}$.

■

To illustrate the capacity upper bound in Theorem 4, we evaluate it for the network formed by BSCs in Fig. 2, and use the min-cut for baseline comparison. Fig. 2(a) depicts, for each hop length L, the upper bound (23) when $M, N = O(1)$. It reveals the exponential decay of the capacity with respect to L, and the min-cut is in geneal a loose upper bound. Fig. 2(b) shows the upper bound (23) when $M = O(1), N = O(\ln L)$. In this case, the capacity decays slowly as L increases, and the min-cut is a loose upper bound as well.

B. *General Channels*

Consider a channel $Q : \mathcal{Q}_\mathrm{i} \to \mathcal{Q}_\mathrm{o}$ with $C_0(Q) = 0$, modeled as in (6). As Q may not be canonical, an output symbol that occurs with a positive probability for all the inputs may not exist. Moreover, if Q is non-canonical, $Q^{\otimes m}$ is non-canonical for any positive integer m. For example, define channel $Q_{3\times 3}$ with $\mathcal{Q}_\mathrm{i} = \mathcal{Q}_\mathrm{o} = \{0, 1, 2\}$ and $Q_{3\times 3}(0|0) = Q_{3\times 3}(0|1) = Q_{3\times 3}(1|0) = Q_{3\times 3}(1|2) = Q_{3\times 3}(2|1) = Q_{3\times 3}(2|2) = 1/2$. For a non-canonical channel, we cannot find an event of the channel status with a positive probability such that the channel outputs are the same for all inputs.

To study the converse of general channels, our technique is to use the concatenation of multiple channels by recoding, which forms a new channel that can be canonical. We use $Q_{3\times 3}$ to illustrate

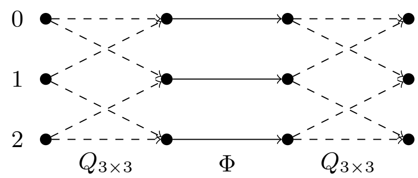

Fig. 3. Concatenation of two $Q_{3\times 3}$'s with recoding Φ. The end-to-end channel is given by $W = Q_{3\times 3}\Phi Q_{3\times 3}$. Here Φ is a deterministic transition matrix with $\Phi(i|i) = 1$. The transition from an input to an output connected by a dashed has probability $1/2$. Any output of W can occur with a positive probability for all inputs.

the idea. Consider the concatenation of two copies of $Q_{3\times 3}$ with a 3×3 *deterministic* transition matrix Φ, which yields the new channel $W = Q_{3\times 3}\Phi Q_{3\times 3}$. In other words, Φ maps an output of the first channel as an input of the second channel. See an illustration in Fig. 3. For the first channel, at least one output of $\{0, 1\}$ occurs for any channel input. If Φ maps these two outputs 0 and 1 of the first channel to the same input, say 2, of the second channel, then output 2 of W occurs with a positive probability for any input of W. If Φ maps these two outputs 0 and 1 of the first channel to two different inputs, say 1 and 2 respectively, of the second channel, then output 2 of W occurs with a positive probability for any input of W.

Now we discuss the general case. For a channel $Q : \mathcal{Q}_\mathrm{i} \to \mathcal{Q}_\mathrm{o}$, denote by ε_Q the maximum value such that for any $x, x' \in \mathcal{Q}_\mathrm{i}$, there exists $y \in \mathcal{Q}_\mathrm{o}$ so that $Q(y|x) \geq \varepsilon_Q$ and $Q(y|x') \geq \varepsilon_Q$. For $Q_{3\times 3}$, we have $\varepsilon_{Q_{3\times 3}} = 1/2$. Note that $\varepsilon_Q > 0$ if and only if $C_0(Q) = 0$ (see [17]). As $C_0(Q_\ell) = 0$, For any two channel inputs of Q_ℓ, it is possible to observe the same output. Based on this fact, we can further show that for any subset $\mathcal{S}_\mathrm{i}$ of $\mathcal{Q}_\mathrm{i}$, there exists a subset $\mathcal{S}_\mathrm{o}$ of $\mathcal{Q}_\mathrm{o}$ size less than half of $\mathcal{S}_\mathrm{i}$ such that for any input in $\mathcal{S}_\mathrm{i}$, it is possible to observe an output in $\mathcal{S}_\mathrm{o}$. Formally, we have the following lemma, proved in Appendix A:

Lemma 6. *Consider a DMC $Q : \mathcal{Q}_\mathrm{i} \to \mathcal{Q}_\mathrm{o}$ with $\varepsilon_Q > 0$, channel function α and channel status variable Z. For any non-empty set $\mathcal{S}_\mathrm{i} \subseteq \mathcal{Q}_\mathrm{i}$, there exist a subset $\mathcal{Z}$ of the range of Z and a subset $\mathcal{S}_\mathrm{o} \subseteq \mathcal{Q}_\mathrm{o}$ with $|\mathcal{S}_\mathrm{o}| \leq \lceil |\mathcal{S}_\mathrm{i}|/2 \rceil$ such that $\alpha(x, z) \in \mathcal{S}_\mathrm{o}$ for any $x \in \mathcal{S}_\mathrm{i}$ and $z \in \mathcal{Z}$, and $P(Z \in \mathcal{Z}) \geq \varepsilon_Q^{|\mathcal{S}_\mathrm{i}|}$.*

Based on the above lemma, we can concatenate a number of consecutive channels in the line network to form a canonical channel, and hence the line network becomes one formed by canonical channels. Then similar to Theorem 4, we can prove the following result, where the proof is in Appendix A.

Theorem 7. *Consider a length-L line network of channels $\{Q_\ell\}_{\ell=1}^L$ with finite input and output alphabets and $\varepsilon_{Q_\ell} \geq \varepsilon > 0$ for all ℓ. The capacity of GBNCs with batch size M and inner blocklength N has the following upper bound:*

$$C_L(M,N) \leq (1-\varepsilon^{N(2|\mathcal{Q}_\mathrm{i}|^N+K)})^{\lfloor L/K \rfloor} \min\{M/N \log|\mathcal{A}|, \log|\mathcal{Q}_\mathrm{i}|, \log|\mathcal{Q}_\mathrm{o}|\}, \tag{28}$$

where $K = \lceil N \log|\mathcal{Q}_\mathrm{i}| \rceil$. Moreover,

1) *when $N = O(1)$, $C_L^* = O((1-\varepsilon')^L)$ for certain $\varepsilon' \in (0,1)$;*
2) *when $M = O(1)$ and $N = \Omega(\ln L)$, $C_L^* = O(1/\ln L)$;*
3) *when M and N are arbitrary, $C_L^* = O(1)$.*

When applied to line networks of canonical channels, Theorem 4 gives stronger results than that of Theorem 7. The upper bound in (23) is strictly better than that in (28). For scalabilities, the first and the third cases in both theorems have the same condition and the same scalability order of L. For the second case, in Theorem 7, there is an extra condition that $N = \Omega(\ln L)$.

IV. Achievable Rates using Decode-and-Forward Recoding

In this section, we discuss the lower bounds of the achievable rates of line networks. We will first study the achievable rates when $N = O(\ln L)$ using two recoding schemes: decode-and-forward and repetition, which can achieve different scalability of the buffer size. When $N = O(1)$, for a line network of identical channels, a rate that exponentially decays with L can be achieved as proved in [10]. We will extend their results for line networks where channels may not be identical.

A. Decode-and-forward Recoding

We discuss a class of GBNC recoding called *decode-and-forward*. When there is a trivial outer code, decode-and-forward has been extensively studied and widely applied in the existing communication systems [11]. We first describe decode-and-forward recoding in the GBNC framework, and then discuss the scalability of achievable rates.

Following the notations in §II-B, we consider a GBNC with batch size M. Let (f_ℓ, g_ℓ) be an (N,M) channel code for Q_ℓ where $f_\ell : \mathcal{A}^M \to \mathcal{Q}_\mathrm{i}^N$ and $g_\ell : \mathcal{Q}_\mathrm{o}^N \to \mathcal{A}^M$ are the encoding and decoding functions, respectively. Consider the transmission of a generic batch $\mathbf{X}$. The source node transmits $\mathbf{U}_1 = f_1(\mathbf{X})$. Each intermediate node ℓ first catches the N received symbols of $\mathbf{Y}_\ell$ and then transmits $\mathbf{U}_{\ell+1} = f_{\ell+1}(g_\ell(\mathbf{Y}_\ell))$. In other words, the recoding function ϕ_ℓ behaves as follows:

- For $i = 1, \ldots, N$, the recoding just keeps the received symbols in the buffer. Therefore, the buffer size is $\Theta(N)$.
- After receiving the N symbols of $\mathbf{Y}_\ell$, the recoding generates $f_{\ell+1}(g_\ell(\mathbf{Y}_\ell))$. If the decoding is correct at nodes $1, \ldots, \ell$, then $g_\ell(\mathbf{Y}_\ell) = \mathbf{X}$ and $\mathbf{U}_{\ell+1} = f_{\ell+1}(\mathbf{X})$.

Let ϵ_ℓ denote the maximum decoding error probability of (f_ℓ, g_ℓ) for Q_ℓ. Due to the fact that if the decoding is correct at all the nodes $1, \ldots, L$, it holds that $g_L(\mathbf{Y}_L) = \mathbf{X}$, we have

$$P(g_L(\mathbf{Y}_L) = \mathbf{X}) \geq \prod_{\ell=1}^{L} (1 - \epsilon_\ell). \tag{29}$$

Let $C = \min_{\ell=1}^{L} C(Q_\ell)$ be the min-cut of the line network. When $\frac{M}{N} \log |\mathcal{A}| < C$ and N is sufficiently large, by the channel coding theorem of DMCs, there exists (f_ℓ, g_ℓ) such that ϵ_ℓ can be arbitrarily small. This gives us the well-known result that the min-cut C is achievable using decode-and-forward recoding when M, N and B are allowed to be arbitrarily large [1]. When all the channels are identical, it has been shown that if $M = \Theta(N)$ and $N = O(\ln L)$ (and hence $B = O(\ln L)$), a constant rate lower than C can be achieved by GBNC [10].

We briefly rephrase their discussion for the case where the channels of the line network are not identical. Consider a sequence of DMCs $Q_\ell, \ell = 1, 2, \ldots$ with $C = \inf\{C(Q_\ell), \ell \geq 1\} > 0$. Suppose parameters M and N are chosen to satisfy $r \triangleq \frac{M}{N} \log |\mathcal{A}| \in [0, C]$. Using random coding arguments [18], there exists (f_ℓ, g_ℓ) such that

$$\epsilon_\ell \leq \exp(-N \mathrm{Er}_\ell(r)), \tag{30}$$

where Er_ℓ is the random coding error exponent for Q_ℓ. For certain $0 < C' \leq C$, assume $\mathrm{Er}^*(r) \triangleq \inf\{\mathrm{Er}_\ell(r), \ell \geq 1\} > 0$ for all $0 \leq r < C'$. The following theorem shows the achievable rate of decode-and-forward recoding scheme.

Theorem 8. *For the line network of length L, where the ℓ-th link is Q_ℓ, the GBNC with decode-and-forward recoding scheme, batch size M, and inner blocklength N achieves rate*

$$C_L(M, N) \geq \frac{M \log |\mathcal{A}|}{N} \left(1 - e^{-N \mathrm{Er}^*(M \log |\mathcal{A}|/N)}\right)^L - \frac{1}{N}. \tag{31}$$

The lower bound (31) *is non-trivial only if*

$$\left(1 - e^{-N \mathrm{Er}^*(M \log |\mathcal{A}|/N)}\right)^L > |\mathcal{A}|^{-M}. \tag{32}$$

Moreover,

1) *when* $M = \Theta(N)$ *and* $N = O(\ln L)$*,* $C_L^* = \Omega(1)$*;*
2) *when* $M = O(1)$ *and* $N = O(\ln L)$*,* $C_L^* = \Omega(1/\ln L)$*.*

Proof: Substituting the error bound of ϵ_ℓ in (30) into (29), we obtain the end-to-end decoding error bound:

$$P(g_L(\mathbf{Y}_L) \neq \mathbf{X}) \leq 1 - \prod_{\ell=1}^{L}(1 - e^{-N\mathrm{Er}_\ell(r)}) \leq 1 - \left(1 - e^{-N\mathrm{Er}^*(r)}\right)^L. \tag{33}$$

Using a similar argument as in the proof of [10, Theorem V.3], the GBNC achieves rate $r\left(1 - e^{-N\mathrm{Er}^*(r)}\right)^L - 1/N$. When (32) does not hold, this rate is bounded by $\frac{1}{N}[\log(|\mathcal{A}|^M)/|\mathcal{A}|^M - 1] \leq 0$, which is trivial. Next, we discuss the scalability of the rate for different scalings of M and N.

- When $M = \Theta(N)$, i.e., $r_1 \leq r \leq r_2$ for some $0 < r_1 < r_2 < C'$, as long as $N = O(\ln L)$ and $\left(1 - e^{-N\mathrm{Er}^*(r_2)}\right)^L > |\mathcal{A}|^{-M}$, the achievable rate of GBNC is at least $r_2\left(1 - e^{-N\mathrm{Er}^*(r_2)}\right)^L - \frac{1}{N} = \Theta(1)$. Note that in $\left(1 - e^{-N\mathrm{Er}^*(r_2)}\right)^L > |\mathcal{A}|^{-r_1 N} > |\mathcal{A}|^{-M}$, the first inequality can be satisfied when $N = O(\ln L)$ and L is sufficiently large.
- When $M = O(1)$, i.e., $r \leq r_3$ for some $0 < r_3 < C$, as long as $N = O(\ln L)$ and $\left(1 - e^{-N\mathrm{Er}^*(r)}\right)^L > |\mathcal{A}|^{-M}$, the achievable rate of GBNC is at least $r\left(1 - e^{-N\mathrm{Er}^*(r)}\right)^L - \frac{1}{N} = \Omega(\ln L)$. Note that in $\left(1 - e^{-N\mathrm{Er}^*(r)}\right)^L \geq \left(1 - e^{-N\mathrm{Er}^*(r_3)}\right)^L > |\mathcal{A}|^{-M}$, the second inequality can be satisfied when $N = O(\ln L)$ and $M = O(1)$ are chosen properly. ■

We provide a special example showcasing the achievable rates of GBNC based on decode-and-forward in Fig. 4: we use a BSC with crossover error probability $\epsilon = 0.2$, $|\mathcal{A}| = 2$ and vary the hop length L from 1 to 2000. The solid lines correspond to the case where $M = \Theta(N), N = O(\ln L)$, from which we find the achievable rate remains to be a constant for increasing hop length L. The dash lines correspond to the case where $M = O(1), N = O(\ln L)$, whereas in this situation, the achievable rate decays slowly when L increases.

As a summary, decode-and-forward recoding can achieve the same order of rate scalability as the upper bound in Theorem 7 for case 2) and 3), where the buffer size requirement is $O(N) = O(\ln L)$. The above approach, however, cannot be used to show the scalability with $M = O(1)$ and $N = O(1)$, since (32) does not hold when $N = O(1)$ and L is large. This case will be discussed in Sec.IV-C using another approach.

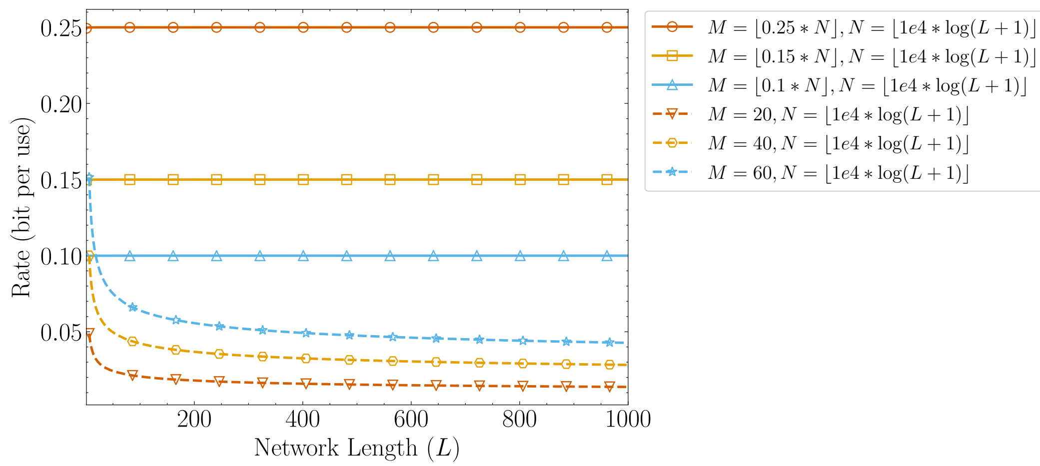

Fig. 4. Numerical illustrations of the achievable rates of GBNC based on decode-and-forward recoding (see (31)) using BSC with crossover probability $\epsilon = 0.2, |\mathcal{A}| = 2$. The hop length L ranges from 1 to 1000. The solid lines correspond to achievable rates when $M = \lfloor c_1 N \rfloor, N = \lfloor c_2 \log(L+1) \rfloor$, and the dashed lines correspond to rates when $M = c_1', N = \lfloor c_2 \log(L+1) \rfloor$. Here constants $c_1 \in \{0.1, 0.15, 0.25\}, c_2 = 1e4, c_1' \in \{20, 40, 60\}$.

B. Repetition Recoding

In this subsection, we show that it is possible to achieve $\Omega(1/\ln L)$ using $M = O(1)$ and $N = O(\ln L)$, but the buffer size requirement $O(\ln \ln L)$. Specifically, we discuss the repetition recoding scheme, which is a special decode-and-forward recoding scheme. In the following, we introduce this recoding scheme by specifying $f_\ell, g_\ell, \ell = 1, \ldots, L$ defined in §IV-A.

We first discuss the case $M = 1$. For any ℓ, let $\mathcal{Q}_{\mathrm{i}}^\ell$ be the maximal subset of $\mathcal{Q}_{\mathrm{i}}$ such that for any $x \neq x' \in \mathcal{Q}_{\mathrm{i}}^\ell$, $Q_\ell(\cdot|x) \neq Q_\ell(\cdot|x')$. For $\ell = 1, \ldots, L$, assume $|\mathcal{Q}_{\mathrm{i}}^\ell| \geq |\mathcal{A}| \geq 2$, and let u_ℓ be a one-to-one mapping from $\mathcal{A}$ to $\mathcal{Q}_{\mathrm{i}}^\ell$. For a generic batch $x \in \mathcal{A}$ with $M = 1$, node $\ell - 1$ transmits $u_\ell(x)$ for N times, i.e.,

$$f_\ell(x) = (u_\ell(x), \ldots, u_\ell(x)). \tag{34}$$

Suppose $\mathbf{Y}_\ell = \mathbf{y}_\ell$, i.e., node ℓ receives $\mathbf{y}_\ell$ for the transmission $f_\ell(x)$. The decoding function g_ℓ is defined based on the maximum likelihood (ML) criterion:

$$g_\ell(\mathbf{y}_\ell) = \arg\max_{x \in \mathcal{A}} \prod_{i=1}^{N} Q_\ell(\mathbf{y}_\ell[i] \mid u_\ell(x)), \tag{35}$$

where a tie is broken arbitrarily. Let

$$\mathcal{L}_\ell(x; \mathbf{y}_\ell) = \sum_{i=1}^{N} \ln Q_\ell(\mathbf{y}_\ell[i] \mid u_\ell(x)) = \sum_{y \in \mathcal{Q}_{\mathrm{o}}} \mathcal{N}(y|\mathbf{y}_\ell) \ln Q_\ell(y \mid u_\ell(x)), \tag{36}$$

where $\mathcal{N}(y|\mathbf{y}_\ell)$ denote the number of times that y appears in $\mathbf{y}_\ell$. Then the ML decoding problem can be equivalently written as $g_\ell(\mathbf{y}_\ell) = \arg\max_{x\in\mathcal{A}} \ \mathcal{L}_\ell(x;\mathbf{y}_\ell)$.

To perform the ML decoding, node ℓ needs to count the frequencies of symbols y for any $y \in \mathcal{Q}_\text{o}$ among N received symbols. As a result, a buffer of size $O(|\mathcal{Q}_\text{o}|\log N) = O(\ln N)$ at each intermediate is required. Additionally, the computation cost of the repetition recoding is $O(N)$ per batch. The following lemma bounds the maximum decoding error probability ϵ_ℓ of (f_ℓ, g_ℓ) for Q_ℓ, whose proof is provided in Appendix B.

Lemma 9. *For any $\ell = 1,\dots,L$, under the condition $|\mathcal{Q}_\text{i}^\ell| \geq |\mathcal{A}| \geq 2$, using the repetition encoding f_ℓ and the ML decoding g_ℓ in* (34) *and* (35)*, respectively, the maximum decoding error probability ϵ_ℓ for Q_ℓ satisfies $\epsilon_\ell \leq \big(|\mathcal{A}|-1\big)\exp\left(-NE_\ell\right)$, where $E_\ell > 0$ is a constant depends only on the channel Q_ℓ.*

Based on the above lemma, the following theorem establishes the achievable rate of repetition recoding scheme. Consider a sequence of DMCs $Q_\ell, \ell = 1, 2, \dots$ with $C(Q_\ell) > 0, \ell \geq 1$ and

$$E^* \triangleq \inf\{E_\ell, \ell \geq 1\} > 0, \quad S^* \triangleq \inf\{|\mathcal{Q}_\text{i}^\ell| : \ \ell \geq 1\} \geq 2. \tag{37}$$

We choose the alphabet $\mathcal{A}$ such that $|\mathcal{A}| \in [2, S^*]$.

Theorem 10. *Assume N is chosen such that $\left(1-e^{-NE^*}\right)^L \geq |\mathcal{A}|^{-1}$. For the line network of length L, where the ℓ-th link is Q_ℓ, the GBNC with repetition recoding scheme, batch size $M = 1$, inner blocklength N, and batch alphabet $\mathcal{A}$ achieves rate*

$$C_L(1,N) \geq \frac{1}{N}\left\{\log|\mathcal{A}| - \left(1-\left(1-e^{-NE^*}\right)^L\right)\log(|\mathcal{A}|-1) - \mathcal{H}\left(\left(1-e^{-NE^*}\right)^L\right)\right\}, \tag{38}$$

where $\mathcal{H}(\cdot)$ denotes the binary entropy function. When specifying $N = O(\ln L)$ and $B = O(\ln N) = O(\ln\ln L)$, it holds that $C_L^* = \Omega(1/\ln L)$.

We plot the rate of repetition recoding using BSC with crossover error probability $\epsilon \in \{0.05, 0.1, 0.15, 0.2\}$ and $|\mathcal{A}| = 2$ with respect to the hop length L in Fig. 5. In Fig.5(a), for each hop length L, we plot the optimal value of N maximizing the lower bound (38), which is denoted as N_L^*. This illustration highlights the observed trend of N_L^* increasing roughly in the order of $\ln L$. In Fig. 5(b), we plot the lower bound (38) for each hop length L, showcasing an approximate decrease rate in the order of $1/\ln L$.

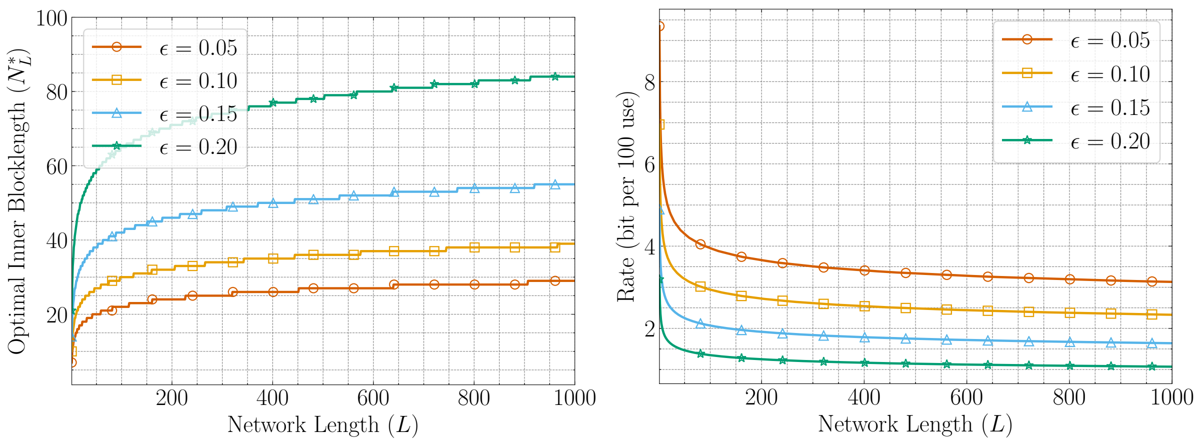

Fig. 5. Numerical illustrations of the achievable rates of GBNC based on repetition recoding (see (38)) using BSC with crossover probability $\epsilon = 0.2, |\mathcal{A}| = 2$. Figures from left to right correspond to (a) plot of the optimal value of N, denoted by N_L^*, that maximizes the lower bound of $C_L(1, N)$ in (38); and (b) plot of the achievable rates with optimized inner blocklength N and fixed batch size $M = 1$.

Proof of Theorem 10: As $C(Q_\ell) > 0$, $|\mathcal{Q}_{\mathrm{i}}^\ell| \geq 2$. We consider a GBNC with $M = 1$ and $\mathcal{A}$ such that $\inf\{|\mathcal{Q}_{\mathrm{i}}^\ell|, \ell \geq 1\} \geq |\mathcal{A}| \geq 2$. Let $E^* = \inf\{E_\ell, \ell \geq 1\}$. By the condition of the theorem, $E^* > 0$, and hence $P(g_L(\mathbf{Y}_L) = \mathbf{X}) \geq \left(1 - e^{-NE^*}\right)^L$. Applying an argument in [10, Theorem V.3], we obtain the desired lower bound (38). When $N = O(\ln L)$ is chosen properly, we have $C_L(1, N) = \Omega(1/\ln L)$. ∎

Though in the proof of the above theorem, we use GBNC with $M = 1$, the repetition coding scheme can be extended to $M > 1$. Fix integers $M = O(1)$ and $\tilde{N} = O(\ln L)$. For an integer m, let $\mathcal{Q}_{\mathrm{i}}^{m,\ell}$ be the maximum subset of $\mathcal{Q}_{\mathrm{i}}^m$ such that for any $\mathbf{x} \neq \mathbf{x}' \in \mathcal{Q}_{\mathrm{i}}^{m,\ell}$, $Q_\ell^{\otimes m}(\cdot|\mathbf{x}) \neq Q_\ell^{\otimes m}(\cdot|\mathbf{x}')$. Following the similar notations as in (37), we define

$$E^{m,*} = \inf\{E_\ell^m, \ell \geq 1\}, \quad S^{m,*} = \inf\{|\mathcal{Q}_{\mathrm{i}}^{m,\ell}| : \ \ell \geq 1\} \geq 2, \tag{39}$$

with E_ℓ^m being the random coding error exponent for $Q_\ell^{\otimes m}$. Assume that $(1 - e^{-NE^{m,*}})^L \geq |\mathcal{A}|^{-M}$. Fix $m = O(1)$ and a finite alphabet $\mathcal{A}$ such that $|\mathcal{A}|^M \in [2, S^{m,*}]$. Conseuqently, we can view the line network of channels $Q_1, \ldots, Q_L$ as one of $Q_1^{\otimes m}, \ldots, Q_L^{\otimes m}$. For the latter, we can apply the repetition recoding with batch size 1, inner blocklength $\tilde{N}$ and the batch alphabet $\mathcal{A}^M$, which for the original line network of $Q_1, \ldots, Q_L$ is a GBNC with batch size M, inner blocklength $m\tilde{N}$ and the batch alphabet $\mathcal{A}$. Based on Theorem 10, such a coding scheme achieves rate

$$\sup_N \frac{1}{N}\left\{\log|\mathcal{A}|^M - \left(1-\left(1-e^{-NE^{m,*}}\right)^L\right)\log(|\mathcal{A}|^M-1) - \mathcal{H}\left(\left(1-e^{-NE^{m,*}}\right)^L\right)\right\}. \tag{40}$$

While the repetition code may appear straightforward, it serves as an illustrative example of how to reduce the buffer size at the intermediate node. Using convolutional codes with Viterbi decoding, due to their analogous encoding and decoding nature, can achieve the same order of the buffer size. However, the corresponding achievable rate is challenging to analyze.

C. *Channel Reduction*

When all the links in the line network are identical DMCs, it has been shown in [10] that an exponentially decreasing rate can be achieved using $N = O(1)$, which corresponds to the first case in Theorem 7. Here we discuss how to generalize this scalability result to line networks where the DMCs Q_ℓ are not necessarily identical. Our approach is to perform recoding so that the line network is reduced to one with identical channels.

We introduce the reduction of an $m\times n$ stochastic matrix Q with $C(Q) > 0$. Let $r = \mathrm{rank}(Q)$. Note that $C(Q) > 0$ if and only if $r \geq 2$. Let s be an integer such that $2 \leq s \leq r$. We would like to reduce Q by multiplying an $s \times m$ matrix R and an $n \times s$ matrix S before and after Q, respectively, so that RQS becomes an $s \times s$ matrix $U_s(\varrho)$ with $(U_s(\varrho))_{i,j} = \varrho$ if $i = j$ and otherwise $(U_s(\varrho))_{i,j} = \frac{1-\varrho}{s-1}$, where ϱ is a parameter in the range $(1/s, 1]$. When $1/s < \varrho \leq 1$, among all the $s \times s$ stochastic matrices with trace $s\varrho$, $U_s(\varrho)$ is the one that has the least mutual information for the uniform input distribution (ref. [10, Theorem V.3]). The reduction described above, if exists, is called *uniform reduction*.

We give an example of uniform reduction with $s = 2$. Choose R so that RQ is an s-row matrix formed by s linearly independent rows of Q. Let a_{ij} be the (i,j) entry of RQ, where $i = 1,2$ and $1 \leq j \leq n$. Define an $n \times 2$ stochastic matrix $W = (w_{ij})$ as

$$w_{i1} = \begin{cases} \frac{a_{1i}}{a_{1i}+a_{2i}} & \text{if } a_{1i}+a_{2i} > 0, \\ 1 & \text{otherwise,} \end{cases} \tag{41}$$

and $w_{i2} = 1 - w_{i1}$, where $1 \leq i \leq n$. With the above R and W, we see that $RQW = U_2(\varrho)$, where $\varrho = \sum_{k: a_{1k}+a_{2k}>0} \frac{a_{1k}^2}{a_{1k}+a_{2k}}$. The following lemma, proved in Appendix B, states a range of ϱ such that the reduction to $U_2(\varrho)$ is feasible.

Lemma 11. *For a stochastic matrix Q such that $C(Q) > \epsilon$ for some $\epsilon > 0$, there exists a constant $B > 1/2$ depending only on ϵ such that Q has a uniform reduction to $U_2(\varrho)$ for all $1/2 < \varrho \leq B$.*

Fix any $\epsilon > 0$. Consider the line network formed by $Q_1, \ldots, Q_L$, where $C(Q_\ell) > \epsilon$ and hence $\mathrm{rank}(Q_\ell) \geq 2$. We discuss a GBNC with $|\mathcal{A}| = 2$ and $M = N = 1$. By Lemma 11, there exists $\varrho > 1/2$ such that for any ℓ, there exists stochastic matrices R_ℓ and S_ℓ such that $R_\ell Q_\ell S_\ell = U_2(\varrho)$. Define the recoding at the source node as R_1, and for $\ell = 1, \ldots, L-1$, define the recoding at node ℓ as $S_\ell R_{\ell+1}$. At the destination node, process all the received batches by R_L. The overall operation of a batch from the source node to the destination node is $W'_L \triangleq (U_2(\varrho))^L$. Applying the argument in [10, Theorem III.5], we get

$$\log\left(\frac{1}{2\varrho - 1}\right) \leq \liminf_{L\to\infty} -\frac{1}{L}\log C(W'_L) \leq \limsup_{L\to\infty} -\frac{1}{L}\log C(W'_L) \leq 2\log\left(\frac{1}{2\varrho-1}\right), \tag{42}$$

where $\frac{1}{2\varrho-1}$ is the second largest eigenvalue of $U_2(\varrho)$. Therefore, a channel code for the transition matrix W_L as the outer code can achieve the rate $\Omega(e^{-cL})$ as $L \to \infty$, where the constant c is between $\log\left(\frac{1}{2\varrho-1}\right)$ and $2\log\left(\frac{1}{2\varrho-1}\right)$. The above discussion is summarized as the following theorem:

Theorem 12. *Consider a sequence of DMCs $Q_\ell, \ell = 1, 2, \ldots$ with $\inf\{C(Q_\ell), \ell \geq 1\} > 0$. For the line network of length L, where the ℓ-th link is Q_ℓ, the GBNC with $M = O(1)$ and $N = O(1)$ achieves rate $C^*_L \geq c' \cdot e^{-cL}$, where c is a constant between $\log\left(\frac{1}{2\varrho-1}\right)$ and $2\log\left(\frac{1}{2\varrho-1}\right)$, and $c' > 0$ is a constant.*

The technique used in the proof of Theorem 12 can be generalized for $M, N \geq 1$. We first show that for an $m \times n$ stochastic matrix Q with $\mathrm{rank}(Q) \geq 2$, for any $2 \leq s \leq r$, the uniform reduction to $U_s(\varrho)$ exists if ϱ is sufficiently close to $1/s$. For an integer $2 \leq s \leq r$, let

$$\kappa_s(Q) = \max_{\substack{s\times m \text{ stochastic matrix } R \\ n\times s \text{ stochastic matrix } W}} \min \mathrm{inv}(RQW) \tag{43}$$

where $\min\mathrm{inv}(RQW)$ is the minimum value of $(RQW)^{-1}$ when RQW is invertible and is ∞ otherwise. We give an example of R and W such that RQW is invertible. Choose R so that RQ is an s-row matrix formed by s linearly independent rows of Q. Let a_{ij} be the (i,j) entry of RQ, where $1 \leq i \leq s$ and $1 \leq j \leq n$. To simplify the discussion, we assume all the columns of RQ are non-zero. Define $W = D(RQ)^\top$, where D is an $n \times n$ diagonal matrix with the (i,i) entry $1/\sum_{j'} a_{j'i}$. With the above R and W, we see that RQW is positive definite and hence invertible. Let $\rho_s(Q) = \frac{\min\{\kappa_s(Q),0\}-1}{s\min\{\kappa_s(Q),0\}-1}$. We see that $\rho_s(Q) > 1/s$. The following lemma, proved in Appendix B, states a range of ϱ such that the reduction to $U_s(\varrho)$ is feasible.

Lemma 13. *Consider an $m \times n$ stochastic matrix Q with rank $r \geq 2$. For any $2 \leq s \leq r$ and $1/s < \varrho \leq \rho_s(Q)$, there exist an $s \times m$ stochastic matrix R and an $n \times s$ stochastic matrix S such that $RQS = U_s(\varrho)$.*

Remark 1. Lemma 11 is stronger than Lemma 13 for the case $s = 2$ as the former gives a uniform bound B that does not depend on Q as long as $C(Q) > \epsilon$.

Consider a line network formed by $Q_1, \dots, Q_L$, where $C(Q_\ell) > 0$ and hence $\operatorname{rank}(Q_\ell) \geq 2$. Let $r = \min_{\ell=1}^{L} \operatorname{rank}(Q_\ell)$. Assuming $r \geq |\mathcal{A}|$, we first discuss a recoding scheme with $M = N = 1$. Let $\varrho = \min_{\ell=1}^{L} \rho_r(Q_\ell)$. By Lemma 13, there exists stochastic matrices R_ℓ and S_ℓ such that $R_\ell Q_\ell S_\ell = U_r(\varrho)$. The following argument is similar as that of the proof of Theorem 12. Now we consider recoding with $M, N = O(1)$. Fix $M, N = O(1)$ and a finite alphabet $\mathcal{A}$ such that $r^N \geq |\mathcal{A}|^M$. Regarding the line network $\mathcal{L}$ as one formed by $Q_1^{\otimes N}, \dots, Q_L^{\otimes N}$, we can apply the above GBNC with batch size 1, inner blocklength 1 and the batch alphabet $\mathcal{A}^M$, which for the original line network $\mathcal{L}$ of $Q_1, \dots, Q_L$ is a GBNC with batch size M, inner blocklength N and the batch alphabet $\mathcal{A}$.

V. Line Networks of Packet Erasure Channels

Batched network codes for line networks of *packet erasure channels* have been studied as efficient variations of random linear network coding [19]–[26]. In this section, we discuss line networks with identical packet erasure channels, for which, we demonstrate stronger converse and achievability results than the general ones.

Fix the alphabet $\mathcal{A}$ with $|\mathcal{A}| \geq 2$. Suppose that the input alphabet $\mathcal{Q}_\mathrm{i}$ and the output alphabet $\mathcal{Q}_\mathrm{o}$ are both $\mathcal{A} \cup \{e\}$ where $e \notin \mathcal{A}$ is called the erasure. For example, we may use a sequence of bits to represent a packet so that $\mathcal{A} = \{0,1\}^T$, i.e., each packet is a sequence of T bits. Henceforth, a symbol in $\mathcal{A}$ is also called a packet in this section.

A packet erasure channel with erasure probability ϵ $(0 < \epsilon < 1)$ has the transition matrix Q_era: for each $x \in \mathcal{A}$, $Q_\mathrm{era}(y|x) = 1 - \epsilon$ if $y = x$ and $Q_\mathrm{era}(y|x) = \epsilon$ if $y = e$. The input e can be used to model the input when the channel is not used for transmission and we define $Q_\mathrm{era}(e|e) = 1$. When the input e is not used for encoding information, erasure codes can achieve a rate of $1 - \epsilon$ symbols (in $\mathcal{A}$) per use. It is also clear that $C_0(Q_\mathrm{era}) = 0$.

A. Upper Bound

We follow the argument in §III-A to obtain the upper bound. It is worth noting that for this special channel, the upper bound in Lemma 2 can be tightened as $p_1 = P(\overline{E_0}) = (1-\epsilon^N)^L$. Based on this observation and following a similar procedure as in the proof of Theorem 4, we have

$$C_L(M,N) \leq \frac{(1-\epsilon^N)^L}{N} \min\{M \log|\mathcal{A}|, N \log|\mathcal{Q}_{\mathrm{o}}|\}, \tag{44}$$

which is a tighter upper bound than (23).

B. Achievability by Random Linear Recoding

We now introduce a class of inner codes with batch size $M = O(1)$, which provides the achievability counterpart for the cases 1) and 2) in Theorem 4. Let $\mathbb{F}_q$ be the finite field of q symbols, and let $T > 0$ be an integer. Suppose $\mathcal{A} = \mathbb{F}_q^T$, i.e., each packet is a sequence of T symbols from the finite field $\mathbb{F}_q$. The outer code generates batches that consist of M packets in $\mathcal{A}$, and can be represented as a $T \times M$ matrix over $\mathbb{F}_q$. In each packet generated by the outer code, the first M symbols in $\mathbb{F}_q$ are called the *coefficient vector*. A batch $\mathbf{X}$ has the first M rows, called the *coefficient matrix*, forming the identity matrix. In the following discussion, we treat the erasure e as the all-zero vector $\mathbf{0}$ in $\mathbb{F}_q^T$, which is not used as a packet in the batches. In other words, when a packet is erased, an intermediate node assumes $\mathbf{0}$ is received.

The inner code is formed by *random linear recoding*, which have been studied in [19]–[26]. A random linear combination of vectors in $\mathcal{A}$ has the linear combination coefficients chosen uniformly at random from $\mathbb{F}_q$. The inner code includes the following operations:

- The source node generates N packets for a batch using random linear combinations of the M packets of the batch generated by the outer code.
- Each intermediate node generates N packets for a batch using random linear combinations of all packets of the received packets of the batch.

Note that for each batch, only the packets with linearly independent coefficient vectors are needed for random linear recoding. Therefore, the buffer size used to store batch content is $O(MT \log q)$ bits. Also, the computational cost of the above recoding scheme for each intermediate node is $O(N^2 T \log q)$ per batch.

At each node, the rank of the coefficient matrix of a batch (i.e., the first M rows of the matrix formed by the generated/received packets of the batch) is also called the rank of the batch. At each node, the ranks of all the batches follow an identical and independent distribution. Denote by π_ℓ the rank distribution of a batch at node ℓ. As all the batches at the source node have rank M, we know that $\pi_0 = (0, 0, \dots, 0, 1)$. Moreover, the rank distributions $\pi_0, \pi_1, \dots, \pi_L$ form a Markov chain so that for $\ell = 1, \dots, L$, it holds that

$$\pi_\ell = \pi_{\ell-1}\mathbf{P} \tag{45}$$

where $\mathbf{P}$ is the transition matrix characterized in [27, Lemma 4.2] (see (80) in Appendix B for the formula).

The maximum achievable rate of this class of GBNCs is $(1 - \frac{M}{T})\frac{\mathbf{E}[\pi_L]}{N}$ packets (in $\mathcal{A}$) per use, and can be achieved by BATS codes [27], [28], where the factor $1 - \frac{M}{T}$ comes from the overhead of M symbols in a packet used to transmit the coefficient vector. Denote

$$\text{BATS}_L(M, N) = \left(1 - \frac{M}{T}\right)\frac{\mathbf{E}[\pi_L]}{N}\log|\mathcal{A}|. \tag{46}$$

In Fig. 6, we compare numerically the upper bound and the achievable rates of BATS codes by evaluating (44) and (46), respectively. Throughout the experiment, we specify parameters $\epsilon = 0.2$, $q = 256$ and $T = 1024$ following the same setup as in [24, Fig. 10], which are decided based on the following considerations: Firstly, in many practical wireless communication systems, a packet loss rate of around 10 to 20 percent is commonly observed. Secondly, a finite field of size 256 is frequently utilized in real-world implementations. Lastly, a packet of 1024 bytes is a typical choice in internet-based communication scenarios. Note that each packet has 8192 bits and the min-cut is 6553.6 bits per use.

- First, we consider fixed $M = N = 2, 3, 4$, and plot the calculation for L up to 1000 in Fig. 6(a). We see from the figure that for a fixed N, the achievable rates of BATS codes and the upper bound in (44) share the same exponential decreasing trend.
- Second, we consider fixed $M = 2, 4, 8, 16, 32$. For each value of M, we find the optimal value of N, denoted by N_L^*, that maximizes $\text{BATS}_L(M, N)$. We see from Fig. 6(b) that N_L^* demonstrates a low increasing rate with L. We further illustrate $\text{BATS}_L(M, N_L^*)$ and $\text{PEC}_L^{\text{UB}}(M, N_L^*)$ for each value of M in Fig. 6(c).

The following theorem, proved in Appendix B, justifies the scalability of $\text{BATS}_L(M, N)$ when L is large, where the $M = 1$ case was proved in [27].

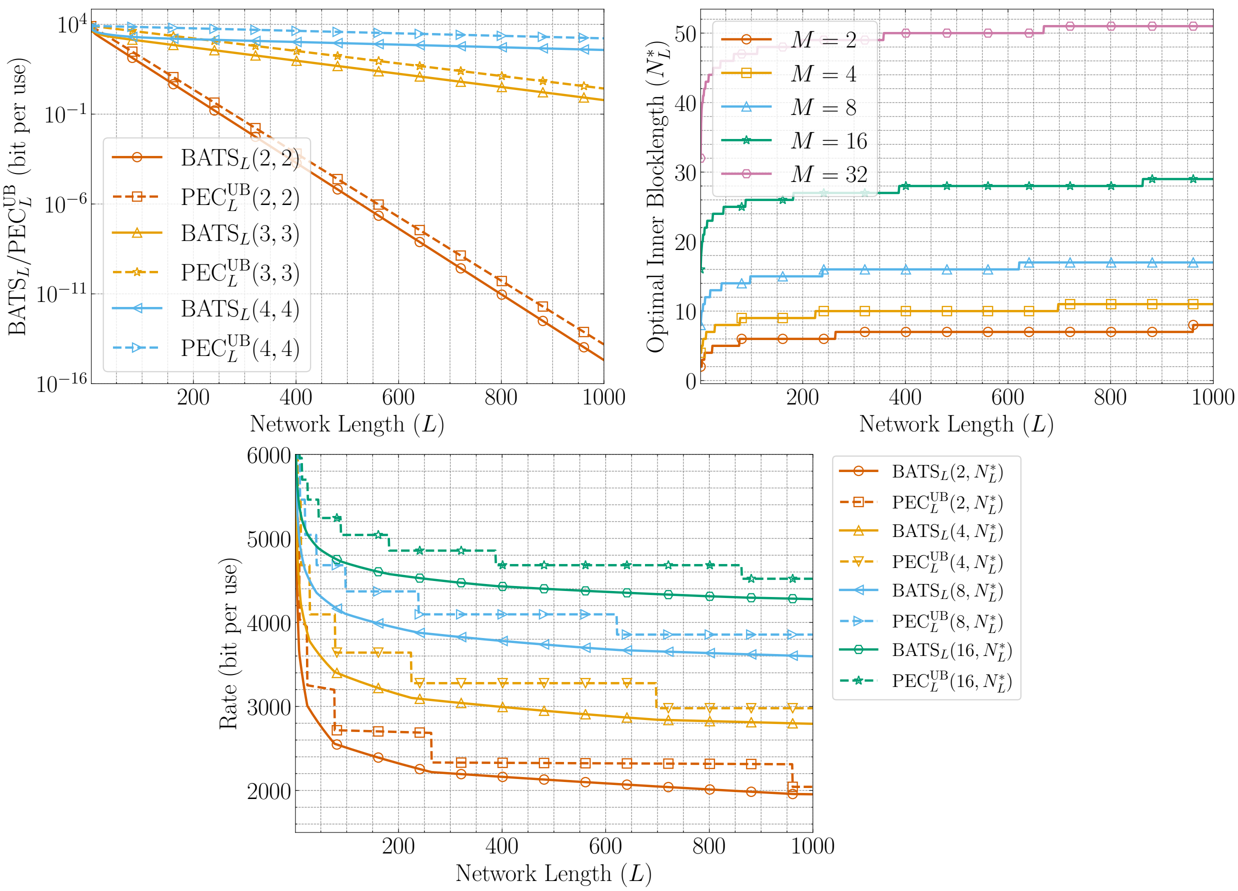

Fig. 6. Numerical illustrations of the upper bound and achievable rates of BATS codes. Figures from left to right correspond to (a) plot of $\mathrm{BATS}_L(M,N)$ and $\mathrm{PEC}_L^{\mathrm{UB}}(M,N)$ when L increases for the case $M=N$ fixed; (b) plot of the optimal value of N, denoted by N_L^*, that maximizes $\mathrm{BATS}_L(M,N)$ for a fixed value of M; and (c) plot of $\mathrm{BATS}_L(M,N)$ and $\mathrm{PEC}_L^{\mathrm{UB}}(M,N)$ when L increases for the case that M is fixed and $N=N_L^*$.

Theorem 14. *Consider a line network of L packet erasure channels with erasure probability ϵ. For GBNCs of fixed batch size $M<T$ and inner blocklength N using random linear recoding,*

$$\mathit{BATS}_L(M,N)=\Theta\left(\frac{\left(1-(\epsilon+(1-\epsilon)/q)^N\right)^L}{N}\right). \tag{47}$$

When q is relatively large, $\mathrm{BATS}_L(M,N)$ has nearly the same scalability as $\mathrm{PEC}_L^{\mathrm{UB}}(M,N)$, as illustrated by Fig. 6(c). Consider two cases of N for the scalability of $\mathrm{BATS}_L(M,N)$:

- When N is a fixed number, $\mathrm{BATS}_L(M,N)$ decreases exponentially with L.
- When M is a fixed number and N is unconstrained, based on the optimization theory (see, e.g., [12, Lemma 1]) we know that $\max_N \mathrm{BATS}_L(M,N)=\Theta(1/\ln L)$, and the maximum is achieved by $N=\Theta(\ln L)$.

VI. Line Networks with Channels of Positive Zero-Error Capacity

In this section, we study the capacity scalability of line networks of channels that have positive capacity but may also have positive zero-error capacity. Denote by $\mathcal{L}(L)$ a line network of length L formed by channels $Q_1, \ldots, Q_L$, where it is not necessary that $C_0(Q_\ell) = 0$. Recall that for a GBNC with batch size M and inner blocklength N, the end-to-end transition matrix of a batch for $\mathcal{L}(L)$ with recoding operations $\{\phi_\ell\}$ is

$$W_L = \phi_0 Q_1^{\otimes N} \phi_1 Q_2^{\otimes N} \phi_2 \cdots \phi_{L-1} Q_L^{\otimes N}. \tag{48}$$

Denote the maximum achievable rate of all recoding schemes with batch size M and inner blocklength N for $\mathcal{L}(L)$ as $C_{\mathcal{L}(L)}(M, N)$. Let $L_0(L)$ be the number of channels in $\mathcal{L}(L)$ with 0 zero-error capacity, i.e., $L_0(L) = |\{1 \leq \ell \leq L : C_0(Q_\ell) = 0\}|$. Let $\{l_1, \ldots, l_{L_0(L)}\} = \{1 \leq \ell \leq L : C_0(Q_\ell) = 0\}$ where $l_1 < l_2 < \cdots < l_{L_0(L)}$. In the following, we argue that $C_{\mathcal{L}(L)}(M, N)$ scales like a line network of length $L_0(L)$ formed by channels with 0 zero-error capacity.

A. Upper Bound

Denote by $\mathcal{L}'(L_0(L))$ the line network formed by the concatenation of $Q_{l_1}, \ldots, Q_{L_0(L)}$, each of which has 0 zero-error capacity in $\mathcal{L}(L)$. For a GBNC with batch size M and inner blocklength N, the end-to-end transition matrix of a batch for $\mathcal{L}'(L_0(L))$ with recoding operations $\{\phi'_\ell\}_{\ell=0}^{L_0(L)-1}$ is $W'_{L_0(L)} = \phi'_0 Q_{l_1}^{\otimes N} \phi'_1 Q_{l_2}^{\otimes N} \Phi'_2 \cdots \Phi'_{L_0(L)-1} Q_{l_{L_0(L)}}^{\otimes N}$. Denote the maximum achievable rate of all recoding schemes with batch size M and inner blocklength N for $\mathcal{L}'(L_0(L))$ as $C_{\mathcal{L}'(L_0(L))}(M, N)$.

Notice that $W'_{L_0(L)} = W_L$ when proper recoding operations $\{\phi'_\ell\}_{\ell=0}^{L_0(L)-1}$ are selected, and hence $C_{\mathcal{L}(L)}(M, N) \leq C_{\mathcal{L}'(L_0(L))}(M, N)$. For network $\mathcal{L}'(L_0(L))$, §III provides the upper bounds on the achievable rates as functions of length $L_0(L)$ under certain coding parameter sets, which are also upper bounds for network $\mathcal{L}(L)$.

B. Lower Bound

We derive a lower bound of achievable rates of $\mathcal{L}(L)$ using the uniform reduction approach introduced in §IV-C. Suppose $C = \inf\{C(Q_{l_i}) : \ i \geq 1\} > 0$. For Q_{l_i}, $i = 1, \ldots, L_0(L)$, we know $C_0(Q_{l_i}) = 0$. By Lemma 11, there exists a constant $B \in (1/2, 1)$ depending only on C such that there exist stochastic matrices R_{l_i} and S_{l_i} with $R_{l_i} Q_{l_i} S_{l_i} = U_2(B)$ for all i. For Q_ℓ with $C_0(Q_\ell) > 0$, we can find R_ℓ and S_ℓ so that $R_\ell Q_\ell S_\ell$ equals the identity matrix I_2. The existence of R_ℓ and S_ℓ is guaranteed by the following lemma.

Lemma 15. *For an $m \times n$ stochastic matrix Q with $C_0(Q) > 0$, there exists a $2 \times m$ stochastic matrix R and a $n \times 2$ stochastic matrix S such that $RQS = I_2$, the 2×2 identity matrix.*

Proof: For a DMC Q, two channel inputs x_1 and x_2 are said to be *adjacent* if there exists an output y such that $Q(y|x_1)Q(y|x_2) > 0$. Denote by $M_0(Q)$ the largest number of inputs in which adjacent pairs do not exist. For a DMC with $C_0 > 0$, we have that $M_0(Q) \geq 2$ and then $C_0(Q) \geq 1$, since otherwise it is easy to verify $M_0(Q^{\otimes n}) \leq 1$ for any n which leads to $C_0(Q) = 0$.

When the channel Q satisfies $C_0(Q) > 0$, we have $M_0(Q) \geq 2$. Define R as a two-row deterministic stochastic matrix that selects two rows of Q that correspond to two non-adjacent inputs. Denote by a_{ij} the (i,j) entry of RQ. We have $a_{1j}a_{2j} = 0$ for all $j = 1, \ldots, n$. Let S be defined same as the matrix W in defined in (41). ■

Denote by $\mathcal{L}''(L_0(L))$ the line network formed by the concatenation of $L_0(L)$ identical channels $U_2(B)$. For a GBNC with batch size M and inner blocklength N, the end-to-end transition matrix of a batch for $\mathcal{L}''(L_0(L))$ with recoding operations $\{\phi''_\ell\}_{i=0}^{L_0(L)-1}$ is

$$W''_{L_0(L)} = \phi''_0 U_2(B)^{\otimes N} \phi''_{l_1} U_2(B)^{\otimes N} \Phi''_{l_2} \cdots \Phi''_{l_{L_0(L)-1}} U_2(B)^{\otimes N} \tag{49}$$

$$= \phi''_0 R_{l_1}^{\otimes N} Q_{l_1}^{\otimes N} S_{l_1}^{\otimes N} \phi''_{l_1} R_{l_2}^{\otimes N} Q_{l_2}^{\otimes N} S_{l_2}^{\otimes N} \Phi''_{l_2} \cdots \Phi''_{l_{L_0(L)-1}} R_{l_{L_0(L)}}^{\otimes N} Q_{l_{L_0(L)}}^{\otimes N} S_{l_{L_0(L)}}^{\otimes N}. \tag{50}$$

Denote the maximum achievable rate of all recoding schemes with batch size M and inner blocklength N for $\mathcal{L}''(L_0(L))$ as $C_{\mathcal{L}''(L_0(L))}(M,N)$.

In addition to the recoding operations F and $\{\phi_i\}_{i=1}^{L-1}$ as used in (48), deploying an extra recoding operation ϕ_L at the destination node, the end-to-end transition matrix of a batch for $\mathcal{L}(L)$ becomes $W_L^* = W_L \phi_L$. By properly choosing $\phi_0, \phi_1, \ldots, \phi_L$, W_L^* becomes $W''_{L_0(L)}$:

- Let $\phi_0 = \phi''_0 R_1^{\otimes N}$ and $\phi_L = S_L^{\otimes N}$;
- If $L_0(L) \neq L$, let $\phi_{L_0(L)} = S_{L_0(L)}^{\otimes N} R_{L_0(L)+1}^{\otimes N}$.
- For $\ell = 1, \ldots, L-1$ and $\ell \neq L_0(L)$, if $C_0(Q_\ell) > 0$, let $\phi_\ell = S_\ell^{\otimes N} R_{\ell+1}^{\otimes N}$; if $C_0(Q_\ell) = 0$, let $\phi_\ell = S_\ell^{\otimes N} \phi''_\ell R_{\ell+1}^{\otimes N}$.

Hence, we obtain $C_{\mathcal{L}(L)}(M,N) \geq C_{\mathcal{L}''(L_0(L))}(M,N)$, where the later can be lower bounded by the techniques in §IV. In particular, the error exponent condition in Theorem 10 can be verified by checking the proof of Lemma 9 for the special case of BSCs.

VII. Concluding Remarks

This paper examines the achievable rates of generalized batched network codes (GBNCs) in line networks with general discrete memoryless channels (DMCs). The findings suggest that capacity-achieving codes for DMCs may not be the only consideration for the inner code. Simple codes like repetition and convolutional codes can achieve the same rate order while requiring lower buffer sizes. Additionally, reliable hop-by-hop communication is not always optimal when buffer size and latency constraints are present.

Feedback is useful in certain communication scenarios. Hop-by-hop feedback does not increase the network capacity (the min-cut). However, exploring its potential benefits is an intriguing area of research in the context of generalized batched network coding (GBNC). Hop-by-hop feedback within batches does not increase the upper bound since it does not increase the capacity of a discrete memoryless channel (DMC). However, when hop-by-hop feedback crosses batches, it introduces memory in the batched channel, which may increase the capacity. Additionally, feedback can also simplify coding schemes.

Future research directions also include investigating better upper bounds and recoding schemes for line networks with special channels like binary symmetric channels (BSCs), generalizing the analysis to channels with infinite alphabets and continuous channels, and exploring whether the upper bound holds for more general codes beyond GBNCs would be valuable.

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Supplementary Material for "*On Achievable Rates of Line Networks with Generalized Batched Network Coding*"

Appendix A
Proofs about Converse

Proof of Lemma 3: Denote by $\mathbf{y}^* = (y^* \cdots y^*)$. We have

$$W(\mathbf{y}|\mathbf{x}) = \begin{cases} \frac{Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) - p_0}{1-p_0} & \mathbf{y} = \mathbf{y}^*, \\ \frac{Q^{\otimes N}(\mathbf{y}|\mathbf{x})}{1-p_0} & \text{otherwise.} \end{cases} \tag{51}$$

Let $P(\mathbf{y}) = \sum_{\mathbf{x}} Q^{\otimes N}(\mathbf{y}|\mathbf{x}) p(\mathbf{x})$ and $P'(\mathbf{y}) = \sum_{\mathbf{x}} W(\mathbf{y}|\mathbf{x}) p(\mathbf{x})$. We have

$$P'(\mathbf{y}) = \begin{cases} \frac{1}{1-p_0}(P(\mathbf{y}) - p_0) & \mathbf{y} = \mathbf{y}^*, \\ \frac{1}{1-p_0} P(\mathbf{y}) & \text{otherwise.} \end{cases} \tag{52}$$

Substituting (51) and (52) into $I(p, W)$, we get

$$\begin{aligned} I(p, W) &= \sum_{\mathbf{x}} p(\mathbf{x}) \sum_{\mathbf{y}} W(\mathbf{y}|\mathbf{x}) \log \frac{W(\mathbf{y}|\mathbf{x})}{P'(\mathbf{y})} & (53) \\ &= \frac{1}{1-p_0} I(p, Q^{\otimes N}) + \frac{1}{1-p_0} U(\mathbf{y}^*), & (54) \end{aligned}$$

where

$$U(\mathbf{y}^*) \triangleq \sum_{\mathbf{x}} p(\mathbf{x}) \left((Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) - p_0) \log \frac{Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) - p_0}{P(\mathbf{y}^*) - p_0} - Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) \log \frac{Q^{\otimes N}(\mathbf{y}^*|\mathbf{x})}{P(\mathbf{y}^*)} \right). \tag{55}$$

Using $P(\mathbf{y}^*) = \sum_{\mathbf{x}} Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) p(\mathbf{x}) \geq \sum_{\mathbf{x}} \epsilon^N p(\mathbf{x}) = \epsilon^N$, we have

$$\begin{aligned} U(\mathbf{y}^*) &= -p_0 \sum_{\mathbf{x}} p(\mathbf{x}) \log(Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) - p_0) + P(\mathbf{y}^*) \log \frac{P(\mathbf{y}^*)}{P(\mathbf{y}^*) - p_0} & (56) \\ &\quad + p_0 \log(P(\mathbf{y}^*) - p_0) + \sum_{\mathbf{x}} p(\mathbf{x}) Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) \log \frac{Q^{\otimes N}(\mathbf{y}^*|\mathbf{x}) - p_0}{Q^{\otimes N}(\mathbf{y}^*|\mathbf{x})} & (57) \\ &\leq -p_0 \log(\epsilon^N - p_0) + q^* \log \frac{\epsilon^N}{\epsilon^N - p_0} + p_0 \log(q^* - p_0) + q^* \log \frac{q^* - p_0}{q^*} & (58) \\ &= (q^* + p_0) \log \frac{q^* - p_0}{\epsilon^N - p_0} + q^* \log \frac{\epsilon^N}{q^*} & (59) \end{aligned}$$

The proof is completed by combining (54) and (59). ■

Proof of Lemma 5: We relax N to a real number and solve $\frac{\mathrm{d}F(N)}{\mathrm{d}N} = 0$, i.e.,

$$1 - \epsilon^N + LN\epsilon^N \ln \epsilon = 0, \tag{60}$$

or

$$\epsilon^{-N} - 1 + LN \ln \epsilon = 0. \tag{61}$$

Let $t = -N \ln \epsilon$, and denote by $t^*(L)$ the solution of $g(t) \triangleq e^t - 1 - Lt = 0, t > 0$. Then the solution of (60) is $N^* = t^*(L)/\ln(1/\epsilon)$.

We know that $g(t) < 0$ for $0 < t < t^*(L)$; and $g(t) > 0$ for $t > t^*(L)$. Since $g(\ln L) = L - 1 - L\ln L < 0$ and $g(2\ln L) = L^2 - 1 - 2L\ln L > 0$ when $L > 1$, we have $\ln l < t^*(L) < 2\ln L$ when $L > 1$. Last, using $\epsilon^{N^*} = e^{-t^*(L)}$,

$$0.25 \le (1 - 1/L)^L \le (1 - \epsilon^{N^*})^L \le \left(1 - 1/L^2\right)^L < 1, \tag{62}$$

and hence $F(N^*) = \frac{(1-\epsilon^{N^*})^L}{N^*} = \frac{\ln\frac{1}{\epsilon}(1-\epsilon^{N^*})^L}{t^*(L)} = \Theta(\frac{\ln\frac{1}{\epsilon}}{\ln L})$. ∎

Proof of Lemma 6: We group the elements of $\mathcal{S}_\mathrm{i}$ into $\lceil |\mathcal{S}_\mathrm{i}|/2 \rceil$ pairs, denoted collectively as $\mathcal{S}_\mathrm{i}^{(2)}$, where each element of $\mathcal{S}_\mathrm{i}$ appears in exactly one pair. When $|\mathcal{S}_\mathrm{i}|$ is even, all pairs have distinct entries. When $|\mathcal{S}_\mathrm{i}|$ is odd, exactly one pair has the two entries same and the other pairs have distinct entries.

For each pair $(x, x') \in \mathcal{S}_\mathrm{i}^{(2)}$, fix $y_{x,x'}$ such that $Q(y_{x,x'}|x) \ge \varepsilon_Q$ and $Q(y_{x,x'}|x') \ge \varepsilon_Q$. Define $\mathcal{Z}$ as the collection of $z = (z_x, x \in \mathcal{Q}_\mathrm{i})$ such that $z_x = y_{x,x'}$ and $z_{x'} = y_{x,x'}$ for all pairs $(x, x') \in \mathcal{S}_\mathrm{i}^{(2)}$. Let $\mathcal{S}_\mathrm{o} = \{y_{x,x'} : (x, x') \in \mathcal{S}_\mathrm{i}^{(2)}\}$. Therefore, $|\mathcal{S}_\mathrm{o}| \le \lceil |\mathcal{S}_\mathrm{i}|/2 \rceil$. Hence for any $x \in \mathcal{S}_\mathrm{i}$ and $z \in \mathcal{Z}$, $\alpha(x, z) = z_x \in \mathcal{S}_\mathrm{o}$. When $\mathcal{A}$ is even,

$$P(Z \in \mathcal{Z}) = \prod_{(x,x') \in \mathcal{S}_\mathrm{i}^{(2)}} P(Z[x] = y_{x,x'}) P(Z_{x'} = y_{x,x'}) \tag{63}$$

$$= \prod_{(x,x') \in \mathcal{S}_\mathrm{i}^{(2)}} Q(y_{x,x'}|x) Q(y_{x,x'}|x') \ge \prod_{(x,x') \in \mathcal{S}_\mathrm{i}^{(2)}} \varepsilon_Q^2 = \varepsilon_Q^{|\mathcal{S}_\mathrm{i}|}. \tag{64}$$

When $\mathcal{A}$ is odd,

$$P(Z \in \mathcal{Z}) = \prod_{(x,x') \in \mathcal{S}_\mathrm{i}^{(2)} : x \ne x'} P(Z[x] = y_{x,x'}) P(Z_{x'} = y_{x,x'}) \prod_{(x,x) \in \mathcal{S}_\mathrm{i}^{(2)}} P(Z[x] = y_{x,x}) \tag{65}$$

$$= \prod_{(x,x') \in \mathcal{S}_\mathrm{i}^{(2)} : x \ne x'} Q(y_{x,x'}|x) Q(y_{x,x'}|x') \prod_{(x,x) \in \mathcal{S}_\mathrm{i}^{(2)}} Q(y_{x,x}|x) \ge \varepsilon_Q^{|\mathcal{S}_\mathrm{i}|}. \tag{66}$$

∎

Proof of Theorem 7: Consider a line network of length L of general DMCs Q_ℓ with $\varepsilon_{Q_\ell} \geq \epsilon > 0$ and a GBNC as described in §II. Without loss of optimality, we assume a deterministic recoding scheme, i.e., ϕ_ℓ are deterministic. Channel $Q_\ell^{\otimes N}$ can be modelled by the function α_ℓ^N with the channel status variable $Z_\ell = (Z_\ell[\mathbf{x}], \mathbf{x} \in \mathcal{Q}_{\mathrm{i}}^N)$ so that

$$\mathbf{Y}_\ell = \alpha_\ell^N(\mathbf{U}_\ell, Z_\ell). \tag{67}$$

As $\varepsilon_{Q_\ell^{\otimes N}} \geq \varepsilon_{Q_\ell}^N > 0$, the condition of applying Lemma 6 on $Q_\ell^{\otimes N}$ is satisfied.

Let $\mathcal{S}_{\mathrm{i}}^{(1)} = \mathcal{Q}_{\mathrm{i}}^N$. Applying Lemma 6 on $Q_1^{\otimes N}$ w.r.t. $\mathcal{S}_{\mathrm{i}}^{(1)}$, there exists subsets $\mathcal{Z}^{(1)}$ and $\mathcal{S}_{\mathrm{o}}^{(1)} \subseteq \mathcal{Q}_{\mathrm{o}}^N$ with $|\mathcal{S}_{\mathrm{o}}^{(1)}| \leq \lceil |\mathcal{S}_{\mathrm{i}}^{(1)}|/2 \rceil$ such that $\alpha_1^N(\mathbf{x}, z_1) \in \mathcal{S}_{\mathrm{o}}^{(1)}$ for any $\mathbf{x} \in \mathcal{S}_{\mathrm{i}}^{(1)}$ and $z_1 \in \mathcal{Z}^{(1)}$, and $P(Z_1 \in \mathcal{Z}^{(1)}) \geq \varepsilon^{N|\mathcal{Q}_{\mathrm{i}}|^N}$. Fix an integer $K = \lceil N \log |\mathcal{Q}_{\mathrm{i}}| \rceil$. For $i = 2, 3, \ldots, K$, define recursively $\mathcal{S}_{\mathrm{i}}^{(i)}$, $\mathcal{S}_{\mathrm{o}}^{(i)}$ and $\mathcal{Z}^{(i)}$ as follows: $\mathcal{S}_{\mathrm{i}}^{(i)} = \left\{ \mathbf{x} \in \mathcal{Q}_{\mathrm{i}}^N : \mathbf{x} = \phi_{i-1}(\mathbf{y}) \text{ for certain } \mathbf{y} \in \mathcal{S}_{\mathrm{o}}^{(i-1)} \right\}$, and $\mathcal{S}_{\mathrm{o}}^{(i)}$ and $\mathcal{Z}^{(i)}$ are determined as in the proof of Lemma 6 w.r.t. $Q_i^{\otimes N}$ and $\mathcal{S}_{\mathrm{i}}^{(i)}$ so that $\alpha_i^{\otimes N}(\mathbf{x}, z) \in \mathcal{S}_{\mathrm{o}}^{(i)}$ for any $\mathbf{x} \in \mathcal{S}_{\mathrm{i}}^{(i)}$ and $z \in \mathcal{Z}^{(i)}$, and $P(Z_i \in \mathcal{Z}^{(i)}) \geq \varepsilon^{N|\mathcal{S}_{\mathrm{i}}^{(i)}|}$.

According to the construction, $|\mathcal{S}_{\mathrm{i}}^{(i)}| \leq |\mathcal{S}_{\mathrm{o}}^{(i-1)}|$ and $|\mathcal{S}_{\mathrm{o}}^{(i)}| \leq \lceil |\mathcal{S}_{\mathrm{i}}^{(i)}|/2 \rceil$. Hence $|\mathcal{S}_{\mathrm{o}}^{(K)}| \leq \lceil |\mathcal{S}_{\mathrm{i}}^{(1)}|/2^K \rceil = 1$. Since the set $\mathcal{S}_{\mathrm{o}}^{(K)}$ is non-empty, we have $|\mathcal{S}_{\mathrm{o}}^{(K)}| = 1$, i.e., there exists an output of $Q_K^{\otimes N}$ that occurs with a positive probability for all inputs of $Q_1^{\otimes N}$. Define the channel $G_1 = Q_1^{\otimes N} \phi_1 Q_2^{\otimes N} \cdots \phi_{K-1} Q_K^{\otimes N}$. Under the condition $Z_i \in \mathcal{Z}^{(i)}, i = 1, \ldots, K$, the output of G_1 must be unique for all possible channel inputs, i.e., G_1 is canonical. Note that

$$P(Z_i \in \mathcal{Z}^{(i)}, i = 1, \ldots, K) \geq \varepsilon^{N \sum_{i=1}^K |\mathcal{A}_i|} \geq \varepsilon^{N(2|\mathcal{Q}_{\mathrm{i}}|^N + K)}. \tag{68}$$

Let $L' = \lfloor L/K \rfloor$. For $i = 2, \ldots, L'$, define $G_i = Q_{K(i-1)+1}^{\otimes N} \phi_{K(i-1)+1} Q_{K(i-1)+2}^{\otimes N} \cdots \phi_{Ki-1} Q_{Ki}^{\otimes N}$. Similar as G_1, we know that G_i, $i = 2, \ldots, L'$ are all canonical. We see that G_i, $i = 1, \ldots, L'$ forms a length-L' network. Let $\tilde{W}_{L'} = \phi_0 G_1 \phi_K G_2 \phi_{2K} \cdots G_{L'}$, which is the end-to-end transition matrix of a GBNC with inner block length 1 for the length-L' network of canonical channels G_i. By the data processing inequality, $I(p_{\mathbf{X}}, W_L) \leq I(p_{\mathbf{X}}, \tilde{W}_{L'})$. Based on this relation, we are ready to prove the theorem, similar to that of Theorem 4. ∎

Appendix B
Proofs about Achievability

Proof of Lemma 9: Suppose that the node $\ell - 1$ transmits $u_\ell(x)$ for N times, where $x \in \mathcal{A}$. We know that the entries of $\mathbf{y}_\ell$ are i.i.d. random variables with distribution $Q_\ell(\cdot \mid u_\ell(x))$. The error probability for ML decoding at the node ℓ satisfies

$$\epsilon_\ell(x) \le P\left(\vee_{\overline{x}\neq x}\mathcal{L}_\ell(\overline{x};\mathbf{y}_\ell) \ge \mathcal{L}_\ell(x;\mathbf{y}_\ell)\right\} \tag{69}$$

$$\le \sum_{\overline{x}\in\mathcal{A}:\ \overline{x}\neq x} P\left(\mathcal{L}_\ell(\overline{x};\mathbf{y}_\ell) \ge \mathcal{L}_\ell(x;\mathbf{y}_\ell)\right), \tag{70}$$

where the second inequality follows from the union bound. For fixed $\overline{x}\in\mathcal{A}$ so that $\overline{x}\neq x$, we bound the probability $P\left(\mathcal{L}_\ell(\overline{x};\mathbf{Y}_\ell) \ge \mathcal{L}_\ell(x;\mathbf{Y}_\ell)\right)$ by considering two cases.

If there exists a non-empty subset $\mathcal{Y}_0 \subseteq \mathcal{Q}_\mathrm{o}$ so that for any $y_0 \in \mathcal{Y}_0$, $Q_\ell(y_0 \mid u_\ell(x)) > 0$ but $Q_\ell(y_0 \mid u_\ell(\overline{x})) = 0$, as long as $\mathbf{y}_\ell[i] \in \mathcal{Y}_0$ for some i, we can assert that $\mathcal{L}_\ell(\overline{x};\mathbf{y}_\ell) < \mathcal{L}_\ell(x;\mathbf{y}_\ell)$. Therefore,

$$P\left(\mathcal{L}_\ell(\overline{x};\mathbf{y}_\ell) \ge \mathcal{L}_\ell(x;\mathbf{y}_\ell)\right) \le P\left(\mathbf{Y}_\ell[i]\notin\mathcal{Y}_0, i=1,\ldots,N\right) \tag{71}$$

$$= \left[\sum_{y\notin\mathcal{Y}_0} Q_\ell(y \mid u_\ell(x))\right]^N = \exp\left(-N\log\frac{1}{\sum_{y\notin\mathcal{Y}_0} Q_\ell(y \mid u_\ell(x))}\right), \tag{72}$$

where $\sum_{y\notin\mathcal{Y}_0} Q_\ell(y \mid u_\ell(x)) = 1 - \sum_{y\in\mathcal{Y}_0} Q_\ell(y \mid u_\ell(x)) < 1$.

Otherwise, consider that the support of $Q_\ell(\cdot \mid u_\ell(x))$ belongs to the support of $Q_\ell(\cdot \mid u_\ell(\overline{x}))$. For $i = 1,\ldots,N$, define the random variable $D_i = \log\frac{Q_\ell(\mathbf{Y}_\ell[i]|u_\ell(\overline{x}))}{Q_\ell(\mathbf{Y}_\ell[i]|u_\ell(x))}$. We see that D_i are i.i.d., and satisfy

$$\log\varrho_\ell \le D_i \le -\log\varrho_\ell, \tag{73}$$

where $\varrho_\ell = \min_{x\in\mathcal{Q}_\mathrm{i}, y\in\mathcal{Q}_\mathrm{o}: Q_\ell(y|x)>0} Q_\ell(y|x)$, and

$$\mathbb{E}[D_i] = E'_\ell \triangleq -\mathcal{D}_{\mathrm{KL}}\left(Q_\ell(\cdot \mid u_\ell(x)) \| Q_\ell(\cdot \mid u_\ell(\overline{x}))\right), \tag{74}$$

where $\mathcal{D}_{\mathrm{KL}}$ denotes the Kullback-Leibler divergence. We see that $E'_\ell > -\infty$. Moreover, as $u_\ell(x) \neq u_\ell(\bar{x}) \in \mathcal{Q}_\mathrm{i}^\ell$, $Q_\ell(\cdot \mid u_\ell(x)) \neq Q_\ell(\cdot \mid u_\ell(\overline{x}))$ and hence $E'_\ell \neq 0$. Applying Hoeffding's inequality, we obtain

$$P\left(\mathcal{L}_\ell(\overline{x};\mathbf{y}_\ell) \ge \mathcal{L}_\ell(x;\mathbf{y}_\ell)\right) = P\left(\sum_{i=1}^N D_i \ge 0\right) \tag{75}$$

$$= P\left(\sum_{i=1}^N \left(D_i - E'_\ell\right) \ge -NE'_\ell\right) \tag{76}$$

$$\le \exp\left(-\frac{NE'^2_\ell}{2\log^2\varrho_\ell}\right). \tag{77}$$

The proof is completed by combining both cases. ■

Proof of Lemma 11: Suppose Q has size $m \times n$. As $C(Q) > \epsilon > 0$, $m \geq 2$. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a row of Q, and construct a new $m \times n$ stochastic matrix $\tilde{Q}$ with all the rows $\mathbf{a}$. We have $C(\tilde{Q}) = 0$ and hence $|C(Q) - C(\tilde{Q})| > \epsilon$. Since channel capacity as a function of stochastic matrices is uniformly continuous [10, Lemma I.1], there exists a constant $\delta > 0$ depending on ϵ such that $\|\tilde{Q} - Q\|_\infty > \delta$. As a consequence, there exists another row $\mathbf{a}' = (a'_1, \ldots, a'_n)$ of Q such that $\|\mathbf{a} - \mathbf{a}'\|_\infty > \delta$. Denote by j the index such that $|a_j - a'_j| > \delta$.

Using the example of uniform reduction with $s = 2$, we can choose R so that RQ is formed by $\mathbf{a}$ and $\mathbf{a}'$. Then we can find W so that $RQW = U_2(\rho_1)$, where

$$\rho_1 = \sum_{k: a_k + a'_k > 0} \frac{a_k^2}{a_k + a'_k} = 1 - \sum_{k: a_k + a'_k > 0} \frac{a_k a'_k}{a_k + a'_k}. \tag{78}$$

Based on the relation that

$$\frac{1}{2} - \sum_{k: a_k + a'_k > 0} \frac{a_k a'_k}{a_k + a'_k} = \frac{1}{4} \sum_{k: a_k + a'_k > 0} \frac{(a_k - a'_k)^2}{a_k + a'_k} \geq \frac{1}{4} \frac{(a_j - a'_j)^2}{a_j + a'_j} \geq \frac{\delta^2}{8}, \tag{79}$$

we have the lower bound $\rho_1 \geq B$ with $B = \frac{1}{2} + \frac{\delta^2}{8} > 1/2$. For any ϱ such that $1/2 < \varrho \leq B$, we have $U_2(\varrho) = U_2(\rho_1) U_2(\frac{\rho_1 + \varrho - 1}{2\rho_1 - 1})$, and hence $RQWU_2(\frac{\rho_1 + \varrho - 1}{2\rho_1 - 1}) = U_2(\varrho)$. ■

Proof of Lemma 13: As $\operatorname{rank}(Q) = r \geq s$, we can find stochastic matrices R and W such that $\min \operatorname{inv}(RQW) = \kappa_s(Q)$. Let $B = (RQW)^{-1}$, and $K = BU_s(\varrho)$. As $RQWK = U_s(\varrho)$, we only need to show that for $1/s < \varrho \leq \rho_s(Q)$, K is a stochastic matrix. Let $\mathbf{1}$ be the all-one vector of certain length. We see that $K\mathbf{1} = BU_s(\rho)\mathbf{1} = B\mathbf{1} = \mathbf{1}$, where the last equality follows because $RQW\mathbf{1} = \mathbf{1}$ and RQW is invertible.

It remains to show that all the entries of K are nonnegative. Let b_{ij} be the (i,j) entry of B. The (i,j) entry of K is $k_{ij} = \frac{1}{s-1}\left[(1-\varrho) + b_{ij}(s\varrho - 1)\right] \geq \frac{1}{s-1}\left[(1-\varrho) + \kappa_s(Q)(s\varrho - 1)\right]$. When $\kappa_s(Q) \geq 0$, we have $k_{ij} \geq 0$ for any $\varrho \in (1/s, 1]$. When $\kappa_s(Q) < 0$, we have $k_{ij} \geq 0$ for any $\varrho \in (1/s, \frac{\kappa_s(Q) - 1}{s\kappa_s(Q) - 1}]$. ■

Proof of Theorem 14: Recall the Markov chain relation in (45), where the transition matrix $\mathbf{P}$ is an $(M+1) \times (M+1)$ matrix with the (i,j) entry ($0 \leq i, j \leq M$):

$$p_{i,j} = \begin{cases} 0 & i < j, \\ \sum_{k=j}^{N} f(k; N, \epsilon) \zeta_j^{i,k} & i \geq j, \end{cases} \tag{80}$$

where $f(k; N, \epsilon) = \binom{N}{k}(1-\epsilon)^k \epsilon^{N-k}$ is the probability mass function (PMF) of the binomial distribution with parameters N and $1 - \epsilon$, and $\zeta_j^{i,k}$ is the probability that the $i \times k$ matrix with independent entries uniformly distributed over the field $\mathbb{F}_q$ has rank j. We know that (ref. [27, (2.4)]) $\zeta_j^{i,k} = \frac{\zeta_j^i \zeta_j^k}{\zeta_j^j q^{(i-j)(k-j)}}$, where

$$\zeta_r^m = \begin{cases} 1 & r = 0, \\ (1-q^{-m})(1-q^{-m+1})\cdots(1-q^{-m+r-1}) & 1 \le r \le m. \end{cases} \tag{81}$$

As shown in [29], the matrix $\mathbf{P}$ admits the eigendecomposition $\mathbf{P} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, where $\mathbf{V} = (v_{i,j})_{0 \le i,j \le M}$ and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_M)$. Here $\lambda_j = \sum_{k=j}^N f(k; N, \epsilon)\zeta_j^k$, $v_{i,j} = \zeta_j^i$ for $i \ge j$ and otherwise $v_{i,j} = 0$. It can be checked that $\lambda_0 > \lambda_1 > \cdots > \lambda_M$. Denote the (i,j) entry $0 \le i,j \le M$ of V^{-1} by $u_{i,j}$. We know that $u_{i,j} = 0$ for $i < j$ and $u_{i,i} = 1/\zeta_i^i$. Based on the formulation above, we have

$$\begin{aligned}
\mathbf{E}[\pi_L] &= \pi_0 \mathbf{V}\mathbf{\Lambda}^L\mathbf{V}^{-1}\begin{bmatrix}0 & 1 & \cdots & M\end{bmatrix}^\top = \sum_{i=1}^M \lambda_i^L v_{M,i} \sum_{j=1}^i j u_{i,j} && (82)\\
&= \lambda_1^L v_{M,1} u_{1,1}\left(1 + \sum_{i=2}^M \frac{\lambda_i^L v_{M,i}}{\lambda_1^L v_{M,1} u_{1,1}} \sum_{j=1}^i j u_{i,j}\right) && (83)\\
&= \Theta(\lambda_1^L), && (84)
\end{aligned}$$

where (83) follows from the fact that $v_{M,1}u_{1,1} > 0$, and (84) is obtained by noting that

$$\sum_{i=2}^M \frac{\lambda_i^L v_{M,i}}{\lambda_1^L v_{M,1} u_{1,1}} \sum_{j=1}^i j u_{i,j} = o(1) \tag{85}$$

as $\lambda_i \le \lambda_1$ for $i \ge 2$. By (81), we further have

$$\begin{aligned}
\lambda_1 &= \sum_{k=1}^N f(k; N, \epsilon)(1 - q^{-k}) = \sum_{k=1}^N f(k; N, \epsilon) - \sum_{k=1}^N f(k; N, \epsilon) q^{-k} && (86)\\
&= 1 - f(0; N, \epsilon) - \sum_{k=1}^N \binom{N}{k} (1-\epsilon)^k \epsilon^{N-k} q^{-k} = 1 - (\epsilon + (1-\epsilon)/q)^N. && (87)
\end{aligned}$$

The proof is completed. ■