Achievable Rate and Latency of Line Networks with Outage Links

Yanyan Dong, Jie Wang, Shenghao Yang and Raymond W. Yeung

Abstract

This paper studies the communication rate and latency in a line topology, multi-hop wireless network when the number of hops goes large. The achievable rate of line networks formed by discrete memoryless channels (DMCs) has been studied, but real-world wireless communication channels are more complex than DMCs. We propose an outage link model to approximate the behavior of practical wireless communication devices for slow fading channels, which can better capture wireless communication properties than DMCs. We study the achievable rate and latency for line networks with outage links using batched network coding with three types of inner codes: random linear network coding (RLNC), decode-and-forward (DF) and hop-by-hop retransmission. Under a technical condition on the outage function, the DF scheme at most achieves $\Theta(\frac{1}{L})$ rate with the end-to-end latency $O(L^2)$ when the network length L goes large. When ideal feedback is assumed, the retransmission scheme has the constant achievable rate and the end-to-end latency $O(L\ln L)$. The RLNC scheme can avoid the drawbacks associated with feedback, and achieve $\Theta(\frac{1}{\ln L})$ and $\Theta(1)$ rate with the end-to-end latency $O(L\ln L)$ and $O(L(\ln L)^2)$, respectively.

Index Terms

multi-hop wireless network, slow fading channel, batched network code, rate adaption

I. Introduction

Multi-hop wireless communication networks are drawing more and more attention over the past decade. Both WLAN (WiFi) and the cellular networks enable multihop wireless networking [1], [2]. Emerging applications like underwater acoustic networks [3], Low Earth Orbit

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(LEO) satellite networks [4] and deep space communication networks [5] all need the support of multi-hop wireless networks. Moreover, due to the large scale of ocean and the relatively short communication range of sound, wireless underwater acoustic networks of hundreds of hops can be useful [6]. Wireless networks of a large number of hops are also desired by smart city, intelligent transportation [7], infrastructure-vehicle collaborative autonomous driving [8], etc.

However, compared with the comprehensive theory of the single hop wireless communications, multi-hop wireless networks are still far from well-understood, especially when the number of hops is large. In this paper, we study the scalability of the communication rate and latency in a line topology, multi-hop wireless network when the number of hops goes large. Existing works have studied the achievable rates of line networks formed by discrete memoryless channels (DMCs). The network capacity is the minimum channel capacity among all the DMCs in the network, which is also called the *min-cut* and can be achieved by applying capacity achieving channel codes for these DMCs hop-by-hop [9]. When the network length L increases, the blocklength of these channels codes needs to be $O(\ln L)$ to achieve a constant rate below the min-cut [10]–[12], where the end-to-end latency for each code block is $O(L \ln L)$. When the blocklength is a constant, the achievable rate decreases exponentially fast when L increases, and the end-to-end latency for each code block is $O(L)$ [10]–[12].

Real-world wireless communication channels are more complex than memoryless channels with known channel statistics due to interference, multipath fading, mobility, etc. We may wonder whether the rate and latency scalability results for line networks of DMCs apply to multi-hop wireless networks. One way to model a wireless channel is to use a time-varying and random capacity [13]. If the capacity value is known by the transmitter instantaneously, the hop-by-hop communication in the line network is reliable and hence end-to-end communication problem becomes a tandem queuing system [14]. In practice, however, adaptive rate control cannot perfectly track the channel state changes [15], and it is possible that the coding rate used for communication is higher than the instantaneous capacity so that the communication fails, which is called an *outage* [13].

In this paper, we study the line networks where the network links have outage, which can better help us to understand the performance of multi-hop wireless networks. As outage already takes the effect of interference into consideration, we assume that all the wireless links are separated spatially, by technologies such as scheduling, directional antenna, power control and beamforming [16].

A. Paper Contributions

We define an *outage link* to model the outage performance of a wireless channel. An outage link can be used for hop-by-hop communication at a controllable rate r in a feasible range. The outage probability of rate r is specified by the outage function $P_{\text{out}}(r)$, which is a non-decreasing function of r. If an outage occurs, the transmitted packet is erased; otherwise, the transmitted packet is correctly received. The outage link captures the behavior of the slow (quasi-static) fading channels, where the channel gain is random but remains constant for the duration of each codeword [13]. This definition of the outage function also matches the behavior of adaptive rate control: a higher rate can be achieved with a higher target outage probability [15].

A network with outage links can be regarded as an extension of a network with packet loss. The capacity of a network with packet loss can be achieved by random linear network coding (RLNC) [17]–[25]. *Batched network coding* extends RLNC by introducing an *inner code–outer code* structure so that the computation cost and the coefficient vector overhead of RLNC can be reduced while preserving the advantage of network coding [26]–[32]. A batched network code can be regarded as a general framework for network communications that includes both hop-by-hop coding and RLNC as special cases. Though batched network coding has been extensively studied for networks with packet loss [33], network links with controllable communication rate and packet loss probability have not been considered in literature.

We study batched codes with three inner codes: RLNC, decode-and-forward (DF) and hop-by-hop retransmission, and compare the maximum achievable rate and the end-to-end latency in the network for each scheme. Under certain technical constraints for the outage function, the DF scheme at most achieves $\Theta(\frac{1}{L})$ rate with the end-to-end latency $O(L^2)$ when the network length L goes large. Note that under a much looser condition on outage function, the DF scheme have the rate upper bounded by $O(\frac{1}{L^{1/s}})$ for a positive integer s, which is a little-o of $\frac{1}{\ln L}$. Specifically when the outage function is infinitely differentiable at zero, the loose condition becomes $P_{\text{out}}(r)$ is not a flat function which has all derivatives vanish at $r = 0$. In the following we present achievability rate results under the outage function that is non-degraded, which only excludes extreme cases. It turns out to be easy to break the scalability of decode-and-forward scheme when the ideal feedback is assumed since the retransmission scheme achieves the constant achievable rate with the end-to-end latency $O(L \ln L)$. In practice, however, feedback may have latency and error, and hence both the rate and latency of the feedback scheme can be harmed. In underwater acoustic communications, for example, the hop-by-hop sound propagation latency can be longer than a couple of seconds, and the communication reliability is low due to the dynamic of communication media. The RLNC scheme can avoid the drawbacks associated with feedback, which achieves $\Theta(\frac{1}{\ln L})$ and $\Theta(1)$ rate with the end-to-end latency $O(L\ln L)$ and $O(L(\ln L)^2)$, respectively.

B. Paper Organization

The remaining of this paper is organized as follows. Sec. II formulates the communication problem on a line network formed by outage links and introduce a general framework of batched network coding for the line network. Sec. III presents the capacity scalibility of batched network coding with uniformly random linear inner code as the inner code and discusses the systematic inner code as well. Sec IV introduces the capacity scalability of decode-and-forward scheme and hop-by-hop retransmission. The end-to-end latency is also studied in addition to the achievable rate in Sec V. Sec. VI provides concluding remarks. Appendix presents some technical proofs.

II. Problem Formulation

In this section, we first describe a network link model called an outage link, and then introduce batched network codes for a line network formed by outage links.

In the following discussion of this paper, we use $\Pr$ to denote the probability of events and $\mathbb{E}[X]$ to denote the expectation value of the random variable X. Denote by $\mathbb{N}$ the set of natural numbers.

A. Outage Links

An outage link is a generalized packet erasure channel with a controllable rate. Fix an alphabet $\mathcal{A}$ with $|\mathcal{A}| \geq 2$. The input and the output of an outage link are from the set $\mathcal{A}^T$ and $\mathcal{A}^T \cup \{e\}$ respectively, where $e \notin \mathcal{A}^T$ is called the *erasure*. In this paper, a symbol in $\mathcal{A}^T$ is called a packet. The sender transmits a packet from $\mathcal{A}^T$ by using n seconds of the link, where $r = \frac{T}{n}$ is a controllable *rate* such that $0 \leq r \leq r_{\max}$. When $r = 0$, the outage link transmits no packets regardless the input; when $0 < r \leq r_{\max}$, the link uses $\frac{T}{r}$ seconds to transmit a packet.

To model the communication quality, the outage link is associated with an *outage function* $P_{\text{out}}(r)$, which is a non-decreasing function in $0 \leq r \leq r_{\max}$ satisfying $P_{\text{out}}(0) = 0$ and $P_{\text{out}}(r_{\max}) \leq 1$. If a packet $x \in \mathcal{A}^T$ is transmitted with rate r, then with probability $1 - P_{\text{out}}(r)$, the output of the outage link is x; otherwise, the output is e, i.e., the input packet is erased.

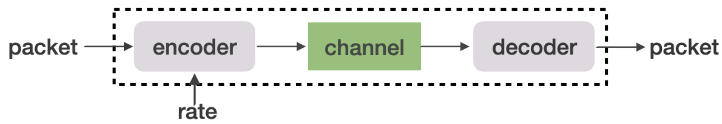

Fig. 1. The dashed box represents an outage link formed by a channel together with a pair of encoder and decoder.

Fig. 1 illustrates that an *outage link* can be regarded as a channel together with a pair of encoder and decoder with controllable rates. In the following we show connections of the outage links with some commonly used channel models and discuss the empirical approach of obtaining an outage link model.

1) Outage Link for Discrete Memoryless Channels

For a discrete memoryless channel (DMC) with finite input and output alphabets, the Shannon capacity C (bit per second of using the channel) is well defined and non-negative. The communication performance of the DMC can be approximated by an outage link Q_{DMC} with $\mathcal{A} = \{0,1\}^T$ and the outage function $P_{\mathrm{out}}(r) = 0$ for $0 \leq r < C$ and $P_{\mathrm{out}}(r) = 1$ for $r \geq C$. When $0 \leq r < C$, due to the achievability of the channel coding theorem, there exists a code of rate at least r with an arbitrarily small decoding error probability when the blocklength is sufficiently large. When $r > C$, due to the converse of Wolfowitz [34], the decoding error probability of any code of rate at least r is arbitrarily close to 1 when the blocklength is sufficiently large. Therefore, the communication performance of the DMC can be approximated by that of the outage link Q_{DMC} arbitrarily accurate for $r \neq C$ with the increase of the blocklength.

2) Outage Link for Quasi-static Channels

A *quasi-static* channel has been widely used to model wireless communications with slowly-varying fading [13]. A quasi-static channel U has a channel status (e.g., channel gain in wireless communications) that is independently chosen by sampling a random variable $\mathcal{S}$ but remains constant for transmitting each codeword. We assume that the channel status information is not available at both the sender and the receiver sides. For a fixed status s, assume that U is memoryless and has the capacity $C(s)$ (bits per second of using the channel). There are different approaches for measuring the communication performance of U [13], [35].

Using the compound channel approach, the reliable communication capacity of U is the minimum of $C(s)$ over the support $\mathcal{S}$ of s when the same input distribution achieves the channel capacity for each fixed $s \in \mathcal{S}$ and $\mathcal{S}$ is compact [35, Sec. 23.3]. The compound capacity, however, is usually 0 for practical fading models. Therefore, the outage probability [13, Sec. 5.4.1], [35, Sec. 23.3] is a more useful performance measure for a quasi-static channel. Define the outage function

$$P_{\text{out}}(r) = \Pr\{C(S) \leq r\}. \tag{1}$$

The communication at rate r can be reliable if and only if $r < C(S)$, and hence the failure probability is $P_{\text{out}}(r)$ at rate r. From (1), we see that $P_{\text{out}}(r)$ is non-decreasing, and when the compound capacity is 0, $P_{\text{out}}(r) > 0$ for any $r > 0$.

Using the similar argument as for DMCs, the communication performance of a quasi-static channel can be approximated by that of an outage link with the outage function $P_{\text{out}}(r) = \Pr\{C(S) \leq r\}$. For some fading channel models [13], the outage function is $P_{\text{out}}(r) = \frac{a^r - 1}{a-1}, r \in [0, 1]$, where $a > 1$ is a constant.

3) Outage Link from Empirical Evaluation

To obtain an outage link model close to the real performance of the wireless communication system, we can use the empirical performance to generate the outage function $P_{\text{out}}(r)$. Suppose that a sequence of channel codes of different rates are specified, for example, as in the physical-layer of IEEE 802.11 or 5G New Radio. We can test or perform simulations for each code of rate r and let $P_{\text{out}}(r)$ be the empirical decoding error rate.

B. A Line Network with Outage Links

A line network of length L is formed by a sequence of nodes labeled by $0, 1, \ldots, L$, where directed outage links exist only from node $\ell - 1$ to node ℓ for $\ell = 1, \ldots, L$. We call the link from node $\ell - 1$ to node ℓ the ℓ-th link or link ℓ. To simplify discussion, we assume that all the outage links have the same packet set $\mathcal{A}^T$ and outage function P_{out} with the maximum rate $r_{\max}$. We study the communication from the first node to the last node, which are called the *source node* and the *destination node*, respectively. Nodes $1, 2, \ldots, L-1$ are called the *intermediate nodes*.

We say that an outage function P_{out} is *degraded* if $P_{\text{out}}(r_0) = 0$ for a certain $r_0 > 0$ or $P_{\text{out}}(r) = 1$ for any $r \in (0, r_{\max}]$. For an outage link with a degraded outage function, a packet either can be transmitted with zero loss probability when the link rate is not larger than r_0 or cannot be transmitted successfully for any positive rate. For the line network discussed in this paper, if the outage function is degraded, a constant positive rate or only a zero rate can be achieved for any L. Henceforth in this paper, we focus on non-degraded outage functions (i.e.,

Fig. 2. The dashed box represents a batch channel. ENC and DEC are respectively the encoding and decoding of the outer code.

$P_{\text{out}}(r) > 0$ for $r > 0$ and $P_{\text{out}}(r_0) < 1$ for some $r_0 \in (0, r_{\max}]$), which match the real-world wireless communication scenario with fading and interference.

C. Batched Network Codes for Line Networks

We study *batched network codes (BNCs)* for the line network of outage links. BNCs that have been extensively studied for line networks of packet erasure channels [26]–[31], [36], [37] and the line networks of DMCs [11], [12], [38]. In the following of this section, we introduce batched codes for a line network with outage links, where we have to take the controllable rate into consideration.

A BNC for the line network has an outer code and an inner code. The source node encodes the message by the outer code into a sequence of batches, each of which includes M packets from $\mathcal{A}^T$, where M is a positive integer called the *batch size*. The inner code at each node *recodes* the packets belonging to the same batch.

At the source node, the recode function f is performed on the original M packets of a batch and generates N_1 packets in $\mathcal{A}^T$. At an intermediate network node ℓ ($1 \leq \ell < L$), the recode function ϕ_ℓ is performed on the received packets belonging to the same batch to generate $N_{\ell+1}$ packets. At each link ℓ, $\ell = 1, \ldots, L$, each packet in the batch is transmitted with link rate r_ℓ and outage probability $P_{\text{out}}(r_\ell)$. The inner code of a batched code consists of the functions f and ϕ_ℓ, $\ell = 1, \ldots, L-1$. We denote the inner code by Φ_f. Note that N_ℓ can be random variables, so that f and ϕ_ℓ are random.

As illustrated in Fig. 2, for the given inner code operations at all the nodes, the end-to-end operation of the network on a batch by the series of the inner code operations can be regarded as a channel with $\mathcal{A}^{T\times M}$ as the input alphabet, called a *batch channel*. The outer code acts as a channel code for this batch channel. As the inner code is independent for each batch, the batch channel is memoryless batch-wisely. Let $\mathbf{X}$ be a random variable on $\mathcal{A}^{T\times M}$. Let $\mathbf{Y}$ be the output of the channel L when the input of the batch channel is $\mathbf{X}$. For a given inner code Φ_f, the maximum achievable rate (the Shannon capacity) of the outer code is $\max_{p_{\mathbf{X}}} I(\mathbf{X};\mathbf{Y})$ using $\frac{T\mathbb{E}[N_\ell]}{r_\ell}$ seconds of link ℓ. The *maximum achievable rate* of the batched code with inner code Φ_f for the line network is the rate of the outer code normalized by $\max_{l=1}^{L} \frac{T\mathbb{E}[N_\ell]}{r_\ell}$, i.e.,

$$R_L^{\Phi_f}\Big(T, M, r_\ell, N_\ell, \ell = 1, \ldots, L)\Big) \triangleq \frac{\max_{p_{\mathbf{X}}} I(\mathbf{X};\mathbf{Y})}{\max_{l=1}^{L} \frac{T\mathbb{E}[N_\ell]}{r_\ell}}. \tag{2}$$

For a BNC, we know how to design an outer code to achieve the above rate [30], [32], [39]. In the following two sections, we will introduce several classes of inner codes, and discuss how to maximize the capacity of the batch channel for each class of inner codes by optimizaing the code parameters T, M, r_ℓ and N_ℓ, $\ell = 1, \ldots, L$.

III. Batched Code with Random Linear Inner Codes

In this section, we introduce two types of random linear inner codes called *uniformly random linear inner code* (UniRL inner code) and *systematic inner code*, which are both the random linear network coding (RLNC). We will analyze the scalability of maximum achievable rate of the batched code when two types of random linear inner codes are utilized.

A. Uniformly Random Linear Inner Code

Now we introduce the UniRL inner code. Let $\mathbb{F}_q$ be the finite field of size q, and $T \geq M$ be a positive integer. Suppose the packet set $\mathbb{F}_q^T$, i.e., a packet has T symbols in $\mathbb{F}_q$. Fix an integer $N \geq M$. We first describe the UniRL inner code at the source node. The M packets of a batch generated by the outer code are put into an $T \times M$ matrix denoted by $\mathbf{X}$, where the first M rows form the identity matrix $\mathbf{I}_M$, i.e.,

$$\mathbf{X} = \begin{bmatrix} \mathbf{I}_M \\ \tilde{\mathbf{X}} \end{bmatrix}. \tag{3}$$

The identity matrix is used to recover the end-to-end effect of the inner code on a batch. Next, a $T \times N$ matrix $\mathbf{U}_1$ is generated as

$$\mathbf{U}_1 = \mathbf{X}\mathbf{B}_0, \tag{4}$$

where $\mathbf{B}_0$ is an $M \times N$ uniformly random matrix over $\mathbb{F}_q$ (i.e, all entries are uniformly at random chosen from $\mathbb{F}_q$). Each column of $\mathbf{U}_1$ is transmitted as a packet into link 1 at rate r.

Next, we describe the operation at other nodes recursively. For $1 \leq \ell \leq L$, suppose that node $\ell - 1$ transmits N packets of a batch at rate r. Let N'_ℓ be the number of correctly received packets by node ℓ for a batch transmitted by the node $\ell - 1$, which follows the binomial distribution

$\mathrm{B}(N, 1-P_{\text{out}}(r))$. Denote by $\mathbf{Y}_\ell$ the $T \times N'_\ell$ matrix formed by juxtaposing the N'_ℓ received packets by node ℓ. When $1 \le \ell < L$, node ℓ generates a $T \times N$ matrix

$$\mathbf{U}_{\ell+1} = \mathbf{Y}_\ell \mathbf{B}_\ell \tag{5}$$

where $\mathbf{B}_\ell$ is an N-column uniformly random matrix over $\mathbb{F}_q$. Each column of $\mathbf{U}_{\ell+1}$ is transmitted as a packet into link $\ell+1$ at rate r. When no packets are received for a batch, i.e., $\mathbf{Y}_\ell$ is the empty matrix, no packets are transmitted.

At the destination node, the received matrix $\mathbf{Y}_L$ for a batch is used for the outer code decoding. For the inner code formulated above, the end-to-end batch channel is a *matrix multiplication channel* with the input $\mathbf{X}$ and output $(\mathbf{Y}_L, \mathbf{H}_L)$ such that $\mathbf{Y}_L = \mathbf{X}\mathbf{H}_L$, where $\mathbf{H}_L$ be the first M rows of $\mathbf{Y}_L$. Note that among all the q^T packets in $\mathbb{F}_q^T$, only q^{T-M} of them are used for transmitting information since the first M components are used to form the identity matrix as shown in (3). Since the capacity of the matrix multiplication channel is $T\,\mathbb{E}[\operatorname{rank}(\mathbf{H}_L)]$ by [39] and the time of transmitting a batch in each link is $\frac{TN}{r}$, the maximum achievable rate (as defined in (2)) of the batched code with UniRL inner code is

$$\frac{T-M}{T}\frac{T\,\mathbb{E}[\operatorname{rank}(\mathbf{H}_L)]}{\frac{TN}{r}} = \frac{r}{N}\left(1-\frac{M}{T}\right)\mathbb{E}[\operatorname{rank}(\mathbf{H}_L)],$$

where the expected rank $\mathbb{E}[\operatorname{rank}(\mathbf{H}_L)]$ depends on M, N and the outage probability $P_{\text{out}}(r)$ (see the formula in [33, Sec. 4.1]). Let

$$R_L^{\text{rl}}(M, N, r) = \frac{r}{N}\,\mathbb{E}[\operatorname{rank}(\mathbf{H}_L)].$$

When T is much larger than M, $R_L^{\text{rl}}(M, N, r)$ gives a close approximation of the maximum achievable rate of the batched code with UniRL inner code.

Let

$$C_L^{\text{rl}}(M, N) \triangleq \sup_{r \in [0, r_{\max}]} R_L^{\text{rl}}(M, N, r). \tag{6}$$

We study the supremum of $C_L^{\text{rl}}(M, N)$ for M, N subject to different constraints on M and N, which affect the inner code complexity in (4) and (5). In the following theorem, proved in Appendix A, we show the scalability of the supremum of $C_L^{\text{rl}}(M, N)$ when $N = O(\ln L)$ subject to different constraints on M.

Theorem 1: Consider an L-hop line network formed by identical outage links with the non-degraded outage function P_{out}. Suppose the batched code with UniRL inner code is applied.

1) When $M = O(1)$ and $N = O(\ln L)$, for any field size q, the supremum of $C_L^{\mathrm{rl}}(M, N)$ is $\Theta(\frac{1}{\ln L})$. Moreover, when r, T satisfies $r = \Theta(1)$, $T = \Theta(1)$ and $P_{\mathrm{out}}(r) \leq 1 - \epsilon$ for some constant $\epsilon \in (0, 1)$, $R_L^{\mathrm{rl}}(M, N, r)$ is $\Theta(\frac{1}{\ln L})$.
2) When $M = O(\ln L)$, $N = O(\ln L)$ and the field size $q \geq 2M$, the supremum of $C_L^{\mathrm{rl}}(M, N)$ is $\Theta(1)$. Moreover, when r, T satisfies $r = \Theta(1)$, $T = \Theta(\ln L)$ and $P_{\mathrm{out}}(r) \leq 1 - \epsilon$ for some constant $\epsilon \in (0, 1)$, $R_L^{\mathrm{rl}}(M, N, r)$ is $\Theta(1)$.

When M is fixed, the above theorem shows that batched network coding with UniRL inner code achieves rate $\Theta(\frac{1}{\ln L})$. When M is increasing with L, it is possible to achieve a constant rate. In practice, it is preferred that M is a small integer, e.g., 16 or 64, which are sufficient to achieve good performance for a practically long network.

B. Systematic Inner Code

Systematic inner code directly uses the received packets as recoded packets, and can reduce both the computation cost and the latency [33]. The operation of the systematic inner code is similar to the UniRL inner code discussed in the last subsection, except that the matrices $\mathbf{B}_\ell$, $\ell = 0, 1, \ldots, L$ are slightly different. For systematic recoding, the first M columns of $\mathbf{B}_\ell$ form the identity matrix and for $\ell = 1, \ldots, L$, the first N'_ℓ columns of $\mathbf{B}_\ell$ form the identity matrix.

It is shown in [33] that the systematic inner code can achieve better expected rank at the next hop than the UniRL inner code. The gain of the systematic recoding is larger when the field size for recoding is smaller. In addition to the lower computation cost, another advantage of systematic inner code is to have lower latency than the UniRL inner code, as we will discuss in the following Sec. V.

IV. Decode-and-Forward and Hop-by-hop Retransmission

In this section, we discuss two transmission schemes for the line network: decode-and-foward (DF) and hop-by-hop retransmission. The DF and retransmission schemes are special batched network codes. For both schemes, the outer code is a packet-wise erasure code, e.g., a fountain code, where the batch size $M = 1$. The inner code has $N_\ell = 1$ for DF and $N_\ell \leq N$ for hop-by-hop retransmission.

A. DF Scheme

For this scheme, an intermediate node only forwards the packets correctly decoded. At the source node 0, a packet in $\mathcal{A}^T$ is transmitted at link rate r. At each intermediate node ℓ, if the packet is received correctly, the node will transmit it at link rate r, and otherwise, no packet is transmitted. The average time of using the link ℓ per packet is $\frac{T(1-P_{\text{out}}(r))^{\ell-1}}{r}$ seconds, which is upper bounded by $\frac{T}{r}$. The end-to-end transmission for a packet is a *packet erasure channel* with the input alphabet $\mathcal{A}^T$ and the erasure probability $1-(1-P_{\text{out}}(r))^L$. By (2), the maximum achievable rate of the network using DF is the capacity of the packet erasure channel $T(1-P_{\text{out}}(r))^L$ normalized by $\frac{T}{r}$ seconds (the maximum average time of using each link), i.e., $r(1-P_{\text{out}}(r))^L$. Denote

$$R_L^{\text{DF}}(r) = r(1-P_{\text{out}}(r))^L. \tag{7}$$

Then the *capacity of the DF scheme* is defined as

$$C_L^{\text{DF}} \triangleq \sup_{r\in[0,r_{\max}]} R_L^{\text{DF}}(r). \tag{8}$$

Theorem 2: Consider an L-hop line network formed by identical outage links with the outage function P_{out} and the DF scheme being applied.

1) Suppose there exists a function $Q(r)$ such that $P_{\text{out}}(r) \geq Q(r)$, where $Q(0) = 0$, $Q^{(n)}(0) = 0$ for any $1 \leq n < s$ and $Q^{(s)}(0) > 0$ for some positive integer s. Then it holds that $C_L^{\text{DF}} = O(\frac{1}{(sL)^{1/s}})$.
2) Suppose there exists a function $P(r)$ such that $P_{\text{out}}(r) \leq P(r)$, where $P(0) = 0$, $(P^{-1})^{(n)}(0)$ $= 0$ for any $1 \leq n < s$ and $(P^{-1})^{(s)}(0) > 0$ for some positive integer s. Then it holds that $C_L^{\text{DF}} = \Omega(\frac{1}{L^s})$. Moreover, when $r = \Theta(\frac{1}{L^s})$, $R_L^{\text{DF}}(r) = \Omega(\frac{1}{L^s})$.

The proof of Theorem 2 is provided in Appendix B.

Remark 1: For a fixed positive integer s, $\frac{1}{(sL)^{1/s}}$ is a little-o of $\frac{1}{\ln L}$ as L goes to infinity. Therefore, when the condition 1) of Theorem 2 and condition in Theorem 1 both hold, the batched code with UniRL inner code achieves strictly better capacity scalability than DF scheme.

To better understand the conditions on P_{out} in Theorem 2, suppose $P_{\text{out}}(r)$ is infinitely differentiable around 0. Then, the condition in Theorem 2-1) hold if $\liminf_{r\to 0^+} P_{\text{out}}^{(n)}(r) = 0$ for any $1 \leq n < s$ and $\liminf_{r\to 0^+} P_{\text{out}}^{(s)}(r) > 0$ for some positive integer s and the condition of Theorem 2-2) hold if $\limsup_{r\to 0^+} (P_{\text{out}}^{-1})^{(n)}(r) = 0$ for any $1 \leq n < s$ and $\limsup_{r\to 0^+} (P_{\text{out}}^{-1})^{(s)}(r) > 0$ for some positive integer s. If $P_{\text{out}}(r)$ is a polynomial function and non-degraded, the conditions in Theorem 2-1) and 2) both holds for certain positive integer s.

Based on Theorem 2, we can obtain the following corollary.

Corollary 3: Consider an L-hop line network formed by identical outage links with the outage function P_{out} and the DF scheme is applied. If there exists $b, c, c' > 0$ such that $P_{\text{out}}(r) \geq cr$ and $P_{\text{out}}(r) \leq c'r$ for $r \leq b$, it holds that $C_L^{\text{DF}} = \Theta(1/L)$ and when $r = \Theta(\frac{1}{L})$, $R_L^{\text{DF}}(r) = \Omega(\frac{1}{L})$.

When the condition of Corollary 3 holds, the network rate $\Omega(\frac{1}{L})$ can be achieved by DF with the identical link rate $r = \Theta(\frac{1}{L})$ at each hop. It is clear from (7) that if the link rate r is a constant, the network rate using DF must decrease exponentially fast with L. In the following, we will introduce an enhanced DF scheme where the link rate can be varying hop-by-hop such that the average time of uses in each link per packet is identical. This scheme achieves higher rates than the one discussed here, but the maximum achievable rate is only $O(\frac{1}{L})$ under certain conditions on outage function, which is still worse than batched code with UniRL inner code.

1) DF+

In the DF scheme introduced in Sec. IV-A, the link rate for each link is identical. Here we introduce an enhanced DF scheme, called *DF+*, that allows the link rate to be varying hop-by-hop. At the source node, the inner code transmit a packet in $\mathcal{A}^T$ through the link 1 at link rate r_1. At the intermediate node ℓ $(1 \leq \ell < L)$, if a packet is successfully received, the node will transmit it to the outgoing link $\ell + 1$ at link rate $r_{\ell+1}$; if a packet is not received, then no packets are transmitted. We design r_ℓ such that the average time of using each link is identical for transmitting a packet in the network. Let $r_1 = r$ and

$$r_{\ell+1} = r_\ell(1 - P_{\text{out}}(r_\ell)), \quad \ell = 1, \ldots, L-1. \tag{9}$$

For $\ell = 2, \ldots, L$, the average time of using link ℓ per packet is $\frac{T}{r_\ell}\prod_{i=1}^{\ell-1}(1 - P_{\text{out}}(r_i))$ seconds. We can verify that the average time of using each link per packet is $\frac{T}{r}$.

With the DF+ scheme described above, the end-to-end batch channel is a packet erasure channel with the input alphabet $\mathcal{A}^T$ and the erasure probability $1 - \prod_{\ell=1}^{L}(1 - P_{\text{out}}(r_\ell))$. Therefore, the maximum achievable rate of DF+ is

$$R_L^{\text{DF+}}(r) \triangleq r \prod_{\ell=1}^{L}(1 - P_{\text{out}}(r_\ell)).$$

The capacity of DF+ is

$$C_L^{\text{DF+}} \triangleq \sup_{r \in [0, r_{\max}]} R_L^{\text{DF+}}(r).$$

Notice that the link rate r_ℓ in this code scheme is no more than r and then the outage probability for each link is no more than $P_{\text{out}}(r)$. Thus $R_L^{\text{DF+}}(r) \geq R_L^{\text{DF}}(r)$ and hence $C_L^{\text{DF+}} \geq C_L^{\text{DF}}$. Therefore $C_L^{\text{DF+}}$ is also $\Omega(\frac{1}{L^s})$ under condition in Theorem 2-2) for a positive integer s. In the following Theorem we will show the upper bound on $C_L^{\text{DF+}}$, which is proved in Appendix C.

Definition 1: $\{x_i\}_{i=1}^{\infty}$ is called an output-alternating input sequence of a function f defined over $\mathcal{X} \subset \mathbb{R}$ if $\{x_i\}_{i=1}^{\infty} \subset \mathcal{X}$ is strictly monotonic; and $\big(f(x_{2k-1}) - f(x_{2k})\big)\big(f(x_{2k}) - f(x_{2k+1})\big) < 0$ and $\big(f(x_{2k}) - f(x_{2k+1})\big)\big(f(x_{2k+1}) - f(x_{2k+2})\big) < 0$ for any $k \geq 1$.

Theorem 4: Consider a line network of length L formed by identical outage links with outage function $P_{\text{out}}(r)$, $r \in [0, r_{\max}]$ which satisfies the following conditions

1) $P_{\text{out}}(r)$ is non-degraded and continuous;
2) $r(1 - P_{\text{out}}(r))$ has no output-alternating input sequences;
3) P_{out} is differentiable around 0, $\lim_{r\to 0^+} P'_{\text{out}}(r) > 0$ and $\lim_{r\to 0^+} rP'_{\text{out}}(r)$ is bounded.

When DF+ is applied in the network, it holds that $C_L^{\text{DF+}} = O(\frac{1}{L})$.

B. Hop-by-Hop Retransmission

This scheme assumes the ideal hop-by-hop feedback so that the correct decoding of a packet is instantaneously known by the node sending the packet on each hop. Same as the previous scheme, a packet from $\mathcal{A}^T$ is transmitted at rate r by the source node. Due to feedback, the source node knows whether this packet is correctly decoded or not by node 1. If node 1 does not correctly decode the packet, node 0 may retransmit this packet. Let N be an integer that limits the number of transmissions for a packet. For an intermediate node, a correctly decoded packet will be transmitted at rate r for at most N times until it is correctly decoded by the next node.

With the retransmission scheme, the end-to-end transmission for a packet is a packet erasure channel with the input alphabet $\mathcal{A}^T$ and the erasure probability $1 - (1 - P_{\text{out}}(r)^N)^L$. Then the capacity of the channel per packet is $T(1 - P_{\text{out}}(r)^N)^L$. Let N_ℓ be the random variable that represents for the number of packets transmitted at node $\ell - 1$. We have $\mathbb{E}[N_1] = \frac{1-P_{\text{out}}(r)^N}{1-P_{\text{out}}(r)}$, which is largest among all the links . By (2), the maximum achievable rate of the network using the retransmission scheme is

$$R_L^{\text{RE}}(N, r) \triangleq \frac{T(1 - P_{\text{out}}(r)^N)^L}{T/r \cdot \frac{1-P_{\text{out}}(r)^N}{1-P_{\text{out}}(r)}} = r(1 - P_{\text{out}}(r))(1 - P_{\text{out}}(r)^N)^{L-1}. \tag{10}$$

The capacity of retransmission scheme is $C_L^{\text{RE}}(N) \triangleq \sup_{r\in[0,r_{\max}]} R_L^{\text{RE}}(N, r)$. We study the supremum of $C_L^{\text{RE}}(N)$ for N subject to different constraints on N.

Theorem 5: Consider an L-hop line network formed by identical outage links with the outage function P_{out} and hop-by-hop retransmission being applied.

1) When $N = \Theta(1)$, if there exists $b, c, c' > 0$ such that $P_{\text{out}}(r) \geq cr$ and $P_{\text{out}}(r) \leq c'r$ for $r \leq b$, it holds that $C_L^{\text{RE}}(N)$ is $\Theta(\frac{1}{L})$ and when $r = \Theta(\frac{1}{L})$, $R_L^{\text{RE}}(N, r) = \Theta(\frac{1}{L})$.
2) When $N = \Theta(\ln L)$, if there exists $r_0 > 0$ such that $0 < P_{\text{out}}(r_0) < 1$, it holds that $C_L^{\text{RE}}(N)$ is $\Theta(1)$. Moreover, when r is fixed with $P_{\text{out}}(r) \in (0,1)$, $R_L^{\text{RE}}(N, r) = \Theta(1)$.

Proof: When $N = \Theta(1)$, there exists a constant N' such that $N \leq N'$ for any L. Referring to the definition of $R_L^{\text{RE}}(N, r)$ in (10), we have

$$r(1 - P_{\text{out}}(r))^L \leq R_L^{\text{RE}}(N, r) \leq r(1 - P_{\text{out}}(r)^{N'})^L. \tag{11}$$

As both $P_{\text{out}}(r)$ and $P_{\text{out}}(r)^{N'}$ satisfy the condition of Corollary 3, we get

$$\sup_{r \in [0, r_{\max}]} r(1 - P_{\text{out}}(r))^L = \Theta(\frac{1}{L}) \quad \text{and} \sup_{r \in [0, r_{\max}]} r(1 - P_{\text{out}}(r)^{N'})^L = \Theta(\frac{1}{L}),$$

and when $r = \Theta(\frac{1}{L})$, $r(1 - P_{\text{out}}(r))^L = \Theta(\frac{1}{L})$ and $r(1 - P_{\text{out}}(r)^{N'})^L = \Theta(\frac{1}{L})$. The proof of 1) is completed by (11).

When $N = \Theta(\ln L)$, for any fixed r_0 such that $P_{\text{out}}(r_0) \in (0,1)$, we have $(1 - P_{\text{out}}(r_0)^N)^{L-1} = \Theta(1)$. Therefore, $R_L^{\text{RE}}(N, r_0) = \Theta(1)$. The proof is completed as $R_L^{\text{RE}}(N, r) = O(1)$. ■

When the maximum number of retransmissions is $\Theta(\ln L)$, the retransmission scheme can achieve a constant rate, which is optimal in the order of L. However, hop-by-hop feedback in practice may suffer from delay or loss, and hence both the rate and latency of the retransmission scheme can be harmed.

V. Capacity Scalability and End-to-end Latency Analysis

In this section, we compare the maximum achievable rates of various schemes and analyze the end-to-end latency for them.

A. Capacity Scalability

Based on the results in Sec. III and IV, the scalabilities of maximum achievable rate for three schemes are obtained for line network with outage links. Under certain conditions for outage function, DF scheme acheves the rate of at most $O(\frac{1}{L})$. The hop-by-hop retransmission only achieves $\Theta(\frac{1}{L})$ when the retransmission number is at most $\Theta(1)$, which is the same as DF scheme. And it achieves the rate of $\Theta(1)$ when the retransmission number is allowed to be $\Theta(\ln L)$. The batched code with UniRL inner code achieves the rate of $\Theta(\frac{1}{\ln L})$ when the batch size is $\Theta(1)$ and the number of packets transmitted for a batch is $\Theta(\ln L)$ and achieves the rate of $\Theta(1)$ when the batch size is $\Theta(\ln L)$ and the number of packets transmitted for a batch

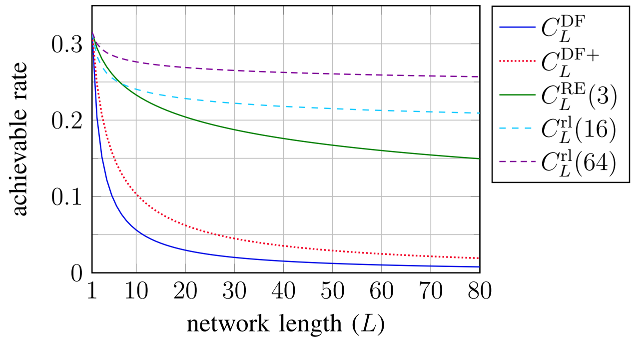

Fig. 3. Illustration of the maximum achievable rate of DF scheme, hop-by-hop retransmission and batched codes with UniRL inner codes when $P_{\mathrm{out}}(r) = \frac{e^r-1}{e-1}, r \in [0,1]$. $C_L^{\mathrm{rl}}(16)$ and $C_L^{\mathrm{rl}}(64)$ are obtained when the field size $q = 256$.

is $\Theta(\ln L)$. Therefore, the hop-by-hop retransmission and bacthed code with UniRL codes can both achieve a better rate compared with DF. However, the hop-by-hop retransmission scheme requires ideal feedback in each hop while feedback is unnecessay for the batched code with UniRL inner codes.

For batched code with UniRL inner code, we define

$$C_L^{\mathrm{rl}}(M) \triangleq \sup_{N \in \mathbb{N}^+} C_L^{\mathrm{rl}}(M,N),$$

where $C_L^{\mathrm{rl}}(M,N)$ is defined in (6). As illustrated in Fig. 3 by curves for $C_L^{\mathrm{rl}}(16)$ and $C_L^{\mathrm{rl}}(64)$, UniRL inner codes can have significant rate gain than DF even when L is small and have gain than hop-by-hop retransmission when L is slightly large.

B. *End-to-end Latency*

Now we discuss the *end-to-end latency* which is the average total time for transmitting a message from the source node to the destination node. We only consider the time for encoding and decoding operations for each packet in each link. The time used for recoding, forwarding and feedback operations are ignored. Under this model, a trivial lower bound on the end-to-end latency is $\Omega(L)$.

We give a general upper bound on the end-to-end latency of using a batched network code on the L-hop line network. Let $n_{\max}$ be a constant upper bound on $\frac{TN_\ell}{r_\ell}$ for all $\ell = 1, \ldots, L$, which bounds the time of link usage for transmitting a batch for each link. Let n_{outer} be the number of batches generated by the outer code for the successful transmission of a message over the line network and $\mathbb{E}[n_{\mathrm{outer}}]$ be its expected value. We consider the transmission of the batches

$\overbrace{\hspace{10em}}^{n_{\max} n_{\text{outer}}}\overbrace{\hspace{7em}}^{n_{\max}(L-1)}$ → time

link 1: A A A A B B B B C C C C D D D D
link 2: A A A A B B B B C C C C D D D D
link 3: A A A A B B B B C C C C D D D D
link 4: A A A A B B B B C C C C D D D D

Fig. 4. The transmission approach of batches that provides an upper bound on the end-to-end latency when $L = 4$ and $n_{\text{outer}} = 4$. The capital alphabet letters followed by "link i" ($i = 1, 2, 3, 4$) represent the packets transmitted through the link belonging to the batch indexed by the alphabet. Here $n_{\max}$ is the time of link usage for transmiting 4 packets (a batch). Extra $n_{\max}$ latency is accumulated by each hop.

TABLE I
COMPARISON TABLE FOR RATES AND LATENCY OF THREE INNER CODES

inner code scheme	case	K	M	T	r_ℓ	N_ℓ	rate	latency
DF	1a	$\Theta(1)$	1	$\Theta(1)$	$\Theta(\frac{1}{L})$	$\Theta(1)$	$\Theta(\frac{1}{L})$	$\Theta(L^2)$
retransmission	2a	$\Theta(1)$	1	$\Theta(1)$	$\Theta(1)$	$\Theta(\ln L)$	$\Theta(1)$	$\Theta(L \ln L)$
	2b	$\Theta(1)$	1	$\Theta(1)$	$\Theta(\frac{1}{L})$	$\Theta(1)$	$\Theta(\frac{1}{L})$	$\Theta(L^2)$
	2c	$\Theta(1)$	1	$\Theta(\ln L)$	$\Theta(1)$	$\Theta(\ln L)$	$\Theta(1)$	$\Theta(L(\ln L)^2)$
UniRL	3a	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(\ln L)$	$\Theta(1)$	$\Theta(L \ln L)$
	3b	$\Theta(\ln L)$	$\Theta(\ln L)$	$\Theta(\ln L)$	$\Theta(1)$	$\Theta(\ln L)$	$\Theta(1)$	$\Theta(L(\ln L)^2)$

as illustrated in Fig. 4: The packets belong to the same batch are transmitted consecutively. As batches are recoded separately, an intermediate node can start the recoding and transmission of a batch when all its packets have been received, without waiting for the packets of the other batches. Therefore, the end-to-end latency for the transmission of this message is at most

$$n_{\max}\,\mathbb{E}[n_{\text{outer}}] + n_{\max}(L-1) \text{ seconds, where } n_{\max} = \max_{\ell=1,\ldots,L} \frac{TN_\ell}{r_\ell}. \tag{12}$$

Now we analyze the end-to-end latency of the specific schemes for transmitting a message of K packets in $\mathcal{A}^T$. The comparison table is given in Table I.

1) DF

We consider case 1a in Table I. In DF scheme, $N_\ell \leq 1$ for any ℓ. We first determine the parameters to achieve the optimal rate scalability. Let $T = \Theta(1)$. Suppose P_{out} satisfy the condition in Corollary 3. Then the link rate $r_\ell = r = \Theta(\frac{1}{L})$ for any ℓ implies the rate $R_L^{\text{DF}}(r) = \Theta(\frac{1}{L})$. We analyze the end-to-end latency of the above scheme. The batch channel for the DF inner code is a packet erasure channel with the erasure probability $1 - (1 - P_{\text{out}}(r))^L$, which is upper bounded by a constant strictly smaller than 1 for any L as $P_{\text{out}}(r) = \Theta(\frac{1}{L})$. When K is a constant, the expected blocklength $\mathbb{E}[n_{\text{outer}}]$ for the outer code (an erasure code) will be upper bounded by a constant. As $r = \Theta(\frac{1}{L})$, by (12), the end-to-end latency of the above scheme is $\Theta(L^2)$.

Our latency result for DF is somehow counter-intuitive: as each packet is only forwarded once, the end-to-end latency seems to be $O(L)$. The reason we have $O(L^2)$ latency is that the link rate r must decrease with network length L to achieve the rate $\Omega(\frac{1}{L})$ of the network. Actually, if the link rate r is constant, the achievable rate decreases exponentially with L while the end-to-end latency is $O(L)$.

2) Hop-by-hop retransmission

We consider three cases 2a, 2b and 2c in Table I for the retransmission scheme studied in Sec. IV, assuming ideal feedback. In case 2a, suppose the condition in 2) of Theorem 5 holds. Let $T = \Theta(1)$, $r = \Theta(1)$ and $N_\ell = N = \Theta(\ln L)$ for any ℓ, then the rate $R_L^{\text{RE}}(N, r) = \Theta(1)$. The batch channel for the retransmission inner code is a packet erasure channel with the erasure probability $1 - (1 - P_{\text{out}}(r)^N)^L$ which is upper bounded by a constant strictly smaller than 1 for any L as $P_{\text{out}}(r)$ is strictly lower bounded by zero for any L. When K is a constant, the expected blocklength $\mathbb{E}[n_{\text{outer}}]$ for the outer code (an erasure code) will be upper bounded by a constant. As $n_{\max} = TN/r = \Theta(\ln L)$, by (12), the end-to-end latency of the above scheme is $O(Ln_{\max}) = O(L \ln L)$.

In case 2b, we can similarly show that under the condition in 1) of Theorem 5, there is a retransmission code achieving rate $\Theta(\frac{1}{L})$ while the end-to-end latency is $O(L^2)$, which has the same rate and latency scalability as DF.

In case 2c, under the condition of 2) in Theorem 5, if we let $T = \Theta(\ln L)$, $r = \Theta(1)$ and K be constant, the scheme achieves rate $\Omega(1)$ and has the end-to-end latency $O(L(\ln L)^2)$. In this case, the message size is $TK = \Theta(\ln L)$.

The practical latency of the retransmission scheme is lower than the worst case analysis as the number of packets transmitted for a batch can be less than N if the transmission succeeds before N packets are transmitted. See Fig. 5 for an illustration.

3) Batched network coding with UniRL inner codes

We first analyze batched code with UniRL inner codes for two cases 3a and 3b in Table I. In case 3a, suppose the condition in 1) of Theorem 1 holds. Let $T = \Theta(1)$, $M = \Theta(1)$ and $N_\ell = N = \Theta(\ln L)$ for any ℓ. Choose $r = \Theta(1)$ so that $P_{\text{out}}(r) \in (0, 1 - \epsilon)$ for some positive

```
——————————————————————→ time
link 1: AAABBCCCCDDDE · · ·
link 2:      AAAAB  CC  DDDE · · ·
link 3:             ABBBCCCCDDE · · ·
link 4:               AAAABBBCCDDD···
```

Fig. 5. The end-to-end latency using hop-by-hop retransmission with $N = 4$. The capital alphabet letters in the figure represent the packets transmitted through a channel belonging to the batch indexed by the alphabet. Due to the feedback information, the number of packets transmitted for a batch can be less than N.

$\epsilon < 1$. Then the rate $R_L^{\mathrm{rl}}(M, N, r)$ is $\Theta(1/\ln L)$. When $n_{\text{outer}}\,\mathbb{E}[\operatorname{rank}(\mathbf{H}_L)] > K$, there exists outer codes of blocklength n_{outer} that can decode successfully with a very high probability, e.g., the BATS code [30]. From the proof of Theorem 1, $\mathbb{E}[\operatorname{rank}(\mathbf{H}_L)] = \Theta(1)$. Let $K = \Theta(1)$, and then the outer code with $n_{\text{outer}} = \Theta(1)$ can achieve a high reliability, which implies $\mathbb{E}[n_{\text{outer}}] = \Theta(1)$. As $n_{\max} = TN/r$, by (12), the end-to-end latency of above scheme is $O(Ln_{\max}) = O(L \ln L)$. In this case, the message size is $(T - M)K = \Theta(1)$.

In case 3b, suppose the condition in 2) of Theorem 1 holds. By similar analysis, rate $\Theta(1)$ is achieved by batched network coding while the end-to-end latency is $O(L(\ln L)^2)$. In this case, the message size is $(T - M)K = \Theta((\ln L)^2)$. Therefore, UniRL inner code can achieve the same rate and latency scalability as the third case of the retransmission scheme, but has a higher order of message size.

It is possible to improve the latency performance using systematic inner code, as the received packets can be transmitted immediately without waiting for the whole batch being received as in the UniRL inner code. As an extreme case, for example, when there is no decoding error in each channel, the end-to-end delay is $O(L)$. Systematic inner code can have a better average delay than the UniRL inner code, but their worst case latency is the same, considering the case that all but the last packet in each batch are erased on each hop.

VI. Concluding Remarks

We studied the communication over a line network formed by outage links. Though the schemes we studied are all known in literature, the model of outage links in the line network, the maximum achievable rates and the corresponding end-to-end latency we obtained in this paper are new. These results close the gap between the existing theory and practice, and provide clear guidance on how to design multihop wireless communication mechanism:

- When hop-by-hop feedback is close to ideal, the hop-by-hop retransmission scheme can be used. It is necessary to allow the maximum number of retransmission to be increasing with the number of hops.
- Batched network coding with random linear inner codes provides another feasible scheme that reduces the use of hop-by-hop feedback, and hence can be applied in more general scenarios.

Both schemes have some crucial features that are different from the existing wireless networking practice, which may help us to understand the so called "multihop curse".

- First, the link level communications in the existing wireless communication systems are designed to be as reliable as possible. In our schemes, however, the hop-by-hop communication is not necessary to be highly reliable, and instead, the link level coding rate and reliability should be balanced to maximize the end-to-end rate.
- Second, the end-to-end congestion control mechanism used in the existing networking systems may face challenges due to the low link reliability and high latency. For our schemes, it is better to design hop-by-hop congestion control mechanisms.
- Last, the end-to-end retransmission mechanism can be applied with our hop-by-hop retransmission scheme to achieve reliable communication, but may generate a high end-to-end latency due to the large delay and low reliability of end-to-end feedback in a multihop network. Instead of end-to-end retransmission, end-to-end erasure coding can be employed.

APPENDIX

A. Proof of Theorem 1

In Lemma 1, 2, 3 and 4, we first investigate some results related to the expected rank function $\mathbb{E}[\mathrm{rank}(\mathbf{H}_L)]$, which is involved in the maximum achivable rate of batched code with UniRL inner codes. Based on these technical lemmas, we give the proof of Theorem 1.

1) Preliminary Results of Expected Rank Function

We present some results related to the expected rank function $\mathbb{E}[\mathrm{rank}(\mathbf{H}_L)]$. From [40, Theorem 5], we have the explicit formula

$$\mathbb{E}[\mathrm{rank}(\mathbf{H}_L)] = \lambda_1^L v_{M,1} u_{1,1} \left(1 + \sum_{i=2}^{M} \frac{\lambda_i^L v_{M,i}}{\lambda_1^L v_{M,1} u_{1,1}} \sum_{j=1}^{i} j u_{i,j}\right), \tag{13}$$

where for $0 \le i, j \le M$,

$$\lambda_j = \sum_{k=j}^{N} f\big(k; N, P_{\text{out}}(r)\big)\zeta_j^k, \quad v_{i,j} = \begin{cases} 0 & i < j, \\ \zeta_j^i & i \ge j. \end{cases}$$

Here $f(k; N, P_{\text{out}}(r))$ is the probability mass function (PMF) of a binomial distribution:

$$f\big(k; N, P_{\text{out}}(r)\big) = \binom{N}{k}\big(1 - P_{\text{out}}(r)\big)^k P_{\text{out}}(r)^{N-k},$$

and ζ_z^w is the probability of the $z \times w$ totally random matrix is full rank:

$$\zeta_z^w = \begin{cases} 1 & z = 0, \\ \prod_{i=w-z+1}^{w}(1 - q^{-i}) & 1 \le z \le w. \end{cases}$$

In particular, the coefficient $\lambda_1 = 1 - \Big(P_{\text{out}}(r) + \frac{1-P_{\text{out}}(r)}{q}\Big)^N$. Denote the matrix $\mathbf{V} = (v_{i,j})_{0 \le i,j \le M}$. Define an $(M+1) \times (M+1)$ lower triangular matrix $\mathbf{U}$ with the (i,j) entry

$$u_{i,j} = \frac{(-1)^{i-j} q^{-\frac{(i-j)(i-j-1)}{2}}}{\zeta_j^j \zeta_{i-j}^{i-j}}, \quad i \ge j.$$

Then we have the following lemma.

Lemma 1: $\mathbf{U} = \mathbf{V}^{-1}$.

Proof: We verify $\mathbf{V}\mathbf{U} = \mathbf{I}_{M+1}$, i.e., we need to show

$$\sum_{i=0}^{M} v_{m,i} u_{i,k} = \begin{cases} v_{m,m} u_{m,m} = 1 & m = k, \\ \sum_{i=k}^{m} v_{m,i} u_{i,k} = 0 & m > k. \end{cases}$$

When $m = k$, $v_{m,m} u_{m,m} = \zeta_m^m / \zeta_m^m = 1$. For $m > k$, $\sum_{i=0}^{M} v_{m,i} u_{i,k} = \sum_{i=k}^{m} v_{m,i} u_{i,k}$. When $m = k+1$, we can verify $\sum_{i=m-1}^{m} v_{m,i} u_{i,k} = \zeta_{m-1}^m \frac{1}{\zeta_{m-1}^{m-1}} + \zeta_m^m \frac{-1}{\zeta_{m-1}^{m-1}\zeta_1^1} = 0$. When $m > k+1$, for any $r \in \{k+1, k+2, \ldots, m-1\}$, let

$$S_r \triangleq \sum_{i=r}^{m} v_{m,i} u_{i,k} = \sum_{i=r}^{m} \frac{(-1)^{i-k} q^{-\frac{(i-k)(i-k-1)}{2}}}{\zeta_k^k \zeta_{i-k}^{i-k}} \zeta_i^m.$$

By induction, we can prove for $k+1 \le r \le m-1$,

$$S_r = \frac{(-1)^{r-k} q^{-\frac{(r-k)(r-k-1)}{2}} \zeta_r^m}{\zeta_{r-k-1}^{r-k-1} \zeta_k^k (1 - q^{-(m-k)})} \triangleq W(r).$$

First, we have

$$\begin{aligned} S_{m-1} &= \frac{(-1)^{m-1-k} q^{-\frac{(m-1-k)(m-2-k)}{2}}}{\zeta_k^k \zeta_{m-1-k}^{m-1-k}} \zeta_{m-1}^m + \frac{(-1)^{m-k} q^{-\frac{(m-k)(m-k-1)}{2}}}{\zeta_k^k \zeta_{m-k}^{m-k}} \zeta_m^m \\ &= \frac{(-1)^{m-k-1} q^{-\frac{(m-k-1)(m-k-2)}{2}} \zeta_{m-1}^m}{\zeta_{m-k-2}^{m-k-2} \zeta_k^k (1 - q^{-(m-k)})} = W(m-1). \end{aligned}$$

Assume that $S_r = W(r)$. Then

$$S_{r-1} = \frac{(-1)^{r-k-1} q^{-\frac{(r-k-1)(r-k-2)}{2}} \zeta_{r-1}^m}{\zeta_k^k \zeta_{r-k-1}^{r-k-1}} + S_r = \frac{(-1)^{r-k-1} q^{-\frac{(r-k-1)(r-k-2)}{2}} \zeta_{r-1}^m}{\zeta_{r-k-2}^{r-k-2} \zeta_k^k (1-q^{-(m-k)})} = W(r-1).$$

By induction on r, we have $S_{k+1} = W(k+1) = \frac{-\zeta_{k+1}^m}{\zeta_k^k(1-q^{-(m-k)})}$. Thus

$$S_k = S_{k+1} + \frac{\zeta_k^m}{\zeta_k^k} = \frac{-\zeta_{k+1}^m}{\zeta_k^k(1-q^{-(m-k)})} + \frac{\zeta_k^m}{\zeta_k^k} = 0.$$

The proof is completed. ■

Lemma 2: For any w and z, it holds that $c_0 \leq \zeta_z^w \leq 1$, where $c_0 := \lim_{w\to\infty} \zeta_w^w > 0$.

Proof: Note that

$$c_0 = \lim_{m\to\infty} \prod_{i=1}^{m} (1-q^{-i}) = \exp\left(\sum_{i=1}^{\infty} \ln(1-q^{-i})\right). \tag{14}$$

Observe that the series $\sum_{i=1}^{\infty} \ln(1-q^{-i})$ is convergent since the ratio

$$\lim_{i\to\infty} \left| \frac{\ln(1-q^{-i-1})}{\ln(1-q^{-i})} \right| = \lim_{i\to\infty} \frac{q^{-i-1}}{q^{-i}} = q^{-1} < 1.$$

As a consequence, the term c_0 in (14) is well-defined and $c_0 > 0$. Then $0 < c_0 \leq \zeta_r^m \leq 1$, $\forall m, r$.

■

Lemma 3: For $i = 2, \ldots, M$, it holds that if the field size $q \geq 2M$, $\sum_{j=1}^{i} j u_{i,j} \geq \frac{1}{2}$.

Proof: Define $J_u(i,j) := j u_{i,j}$. By calculation, for $0 \leq w \leq \left\lfloor \frac{i}{2} \right\rfloor - 1$, it holds that

$$\begin{aligned}
& J_u(i, i-2w) + J_u(i, i-2w-1) \\
&= (i-2w) \frac{(-1)^{2w} q^{-\frac{(2w)(2w-1)}{2}}}{\zeta_{i-2w}^{i-2w} \zeta_{2w}^{2w}} + (i-2w-1) \frac{(-1)^{2w+1} q^{-\frac{(2w+1)(2w)}{2}}}{\zeta_{i-2w-1}^{i-2w-1} \zeta_{2w+1}^{2w+1}} \\
&= \frac{q^{-\frac{(2w)(2w-1)}{2}}}{\zeta_{i-2w}^{i-2w} \zeta_{2w+1}^{2w+1}} \left\{ (i-2w)(1-q^{-(2w+1)}) - (i-2w-1) q^{-2w} (1-q^{-(i-2w)}) \right\} \\
&\geq q^{-\frac{(2w)(2w-1)}{2}} / 2,
\end{aligned} \tag{15}$$

where the relation in (15) holds due to Lemma 2 and $q \geq 2M$. For any $i \geq 2$, the relation (15) implies

$$\sum_{j=1}^{i} J_u(i,j) \geq J_u(i,i) + J_u(i,i-1) \geq 1/2.$$

■

Lemma 4: Suppose that $1 - P_{\text{out}}(r) - M/N \geq c$ for some constant $c > 0$. For $i = 2, \ldots, M$, it holds that $\frac{\lambda_i}{\lambda_1} \geq 1 - e^{-2Nc^2}$.

Proof: For $i = 2, \ldots, M$, we find

$$\frac{\lambda_i}{\lambda_1} = \frac{\sum_{k=i}^{N} f(k; N, P_{\text{out}}(r))\zeta_i^k}{\sum_{k=1}^{N} f(k; N, P_{\text{out}}(r))\zeta_1^k} \geq c_0 \sum_{k=i}^{N} f(k; N, P_{\text{out}}(r)),$$

where the inequality is from the relation in Lemma 2. Based on the tail bound of the cumulative density function (CDF) of the binominal distriubtion,

$$\frac{\lambda_i}{\lambda_1} \geq 1 - \exp\left\{-2N\left(1 - P_{\text{out}}(r) - \frac{i-1}{N}\right)^2\right\}.$$

Suppose $\frac{M}{N} \leq 1 - P_{\text{out}}(r) - c$, we have

$$1 - P_{\text{out}}(r) - \frac{i-1}{N} \geq 1 - P_{\text{out}}(r) - \frac{M}{N} \geq c,$$

and hence $\frac{\lambda_i}{\lambda_1} \geq 1 - \exp(-2Nc^2)$. ■

2) Proof of Theorem 1

1) By [40, Lemma 17], when $M = O(1)$ and $N = O(\ln L)$,

$$\sup_{N \in \mathbb{N}^+, r \in [0, r_{\max}]} \frac{r\,\mathbb{E}[\text{rank}(\mathbf{H}_L)]}{N} = \Theta\left(\frac{r \ln \frac{1}{P_{\text{out}}(r) + \left(1 - P_{\text{out}}(r)\right)/q}}{\ln L}\right). \tag{16}$$

Since there exists $r_0 > 0$ such that $P_{\text{out}}(r_0) < 1$ and $P_{\text{out}}(r)$ is non-decreasing, it holds that for $\epsilon_0 = \frac{1 - P_{\text{out}}(r_0)}{2}$, there exists r_1, r_2 with $0 < r_1 < r_2 \leq r_{\max}$ so that $P_{\text{out}}(r) \leq 1 - \epsilon_0$ for $r \in [r_1, r_2]$. Let $r \in [r_1, r_2]$. Then $r = \Theta(1)$ and $P_{\text{out}}(r) \leq 1 - \epsilon_0$, which implies

$$\Theta\left(r \ln \frac{1}{P_{\text{out}}(r) + \left(1 - P_{\text{out}}(r)\right)/q}\right) = \Theta(1).$$

Hence, by (16),

$$\sup_{N \in \mathbb{N}^+, r \in [0, r_{\max}]} \frac{r\,\mathbb{E}[\text{rank}(\mathbf{H}_L)]}{N} = \Theta(1/\ln L).$$

2) Let $q \geq 2M$. By Lemma 1 and 2, $v_{M,1}u_{1,1} = \zeta_1^M/\zeta_1^1 \geq c_0/(1 - q^{-1})$ and $\frac{v_{M,i}}{v_{M,1}u_{1,1}} \geq c_0(1 - q^{-1})$. Then by (13),

$$\begin{aligned}
\frac{1}{N}\,\mathbb{E}[\text{rank}(\mathbf{H}_L)] &= \frac{\lambda_1^L v_{M,1}u_{1,1}}{N}\left(1 + \sum_{i=2}^{M} \frac{\lambda_i^L v_{M,i}}{\lambda_1^L v_{M,1}u_{1,1}} \sum_{j=1}^{i} j u_{i,j}\right) \\
&= \Omega\left(\frac{\lambda_1^L}{N}\left(1 + \sum_{i=2}^{M} \frac{\lambda_i^L}{\lambda_1^L} \sum_{j=1}^{i} j u_{i,j}\right)\right).
\end{aligned}$$

Now we fix $M = \Theta(\ln L)$ and $N = \Theta(\ln L)$ so that

$$M/N \leq \epsilon_0 - c, \tag{17}$$

for some constant $c > 0$ where ϵ_0 is given in the proof of 1). Then we take $r \in [r_1, r_2]$ and $P_{\text{out}}(r) \le 1 - \epsilon_0$ for chosen M, N. Then by (17), $M/N \le 1 - P_{\text{out}}(r) - c$. From Lemma 3 and Lemma 4, it holds that $\sum_{j=1}^{i} j u_{i,j}$ and λ_i^L/λ_1^L are uniformly lower bounded for $i = 2, \ldots, M$, which implies

$$\frac{\mathbb{E}[\text{rank}(\mathbf{H}_L)]}{N} = \Omega\left(\frac{M\lambda_1^L}{N}(1 - e^{-2Nc^2})^L\right).$$

From [40, Lemma 17], we have $\frac{\lambda_1^L}{N} = \Theta(1/\ln L)$ and $(1 - e^{-2Nc^2})^L = \Theta(1)$ when $N = \Theta(\ln L)$. Thus, when $M = \Theta(\ln L)$, $N = \Theta(\ln L)$ and $r = \Theta(1)$,

$$\frac{r}{N}\,\mathbb{E}[\text{rank}(\mathbf{H}_L)] = \Omega\left(\frac{M}{\ln L}\right) = \Omega(1).$$

Therefore, when $M = \Theta(\ln L)$, $N = \Theta(\ln L)$ and $q \ge 2M$, the supremum of $R_L^{\text{rl}}(M, N, r)$ is $\Theta(1)$ as the optimal rate is always upper bounded by a constant.

B. Proof of Theorem 2

We first present Lemma 5, which provided the capacity scalability of a DF scheme with a special outage function. Based on this lemma, we give the proof of Theorem 2.

Lemma 5: Let $a > 1$, $d \ge 1$ and $P_{\text{out}}(r) = \frac{a^{r^d}-1}{a-1}$, $r \in [0, 1]$. For R_L^{DF} and C_L^{DF} defined in (7) and (8), we have $C_L^{\text{DF}} = \Theta(\frac{1}{(dL)^{1/d}})$. Moreover, $R_L^{\text{DF}} = \Theta(\frac{1}{(dL)^{1/d}})$ when $r = \Theta(\frac{1}{(dL)^{1/d}})$.

Proof: For $a > 1$, let

$$H_L(r) = r\left(1 - \frac{a^{r^d} - 1}{a - 1}\right)^L. \tag{18}$$

Solve $H_L'(r) = 0$, i.e.

$$H_L'(r) = \left(\frac{a - a^{r^d}}{a - 1}\right)^L - \frac{a^{r^d} d r^d L \ln a}{a - 1}\left(\frac{a - a^r}{a - 1}\right)^{L-1} = 0,$$

which is equivalent to $a - a^{r^d}(1 + dr^d L \ln a) = 0$. Let $g_L(r) = a - a^{r^d}(1 + dr^d L \ln a)$ and denote by r_L^* the solution of $g_L(r) = 0$, $r > 0$. $H_L(r)$ is non-decreasing in $[0, r_L^*]$ and non-increasing in $[r_L^*, 1]$. Observe that for arbitrary $\epsilon < a - 1$,

$$\lim_{L\to\infty} g_L\left(\sqrt[d]{\frac{a - 1 + \epsilon}{dL\ln a}}\right) = \lim_{L\to\infty} a - (a + \epsilon)\cdot a^{\frac{a-1+\epsilon}{dL\ln a}} = -\epsilon,$$

$$\lim_{L\to\infty} g_L\left(\sqrt[d]{\frac{a - 1 - \epsilon}{dL\ln a}}\right) = \lim_{L\to\infty} a - (a - \epsilon)\cdot a^{\frac{a-1-\epsilon}{dL\ln a}} = \epsilon.$$

So when L is large enough, $g_L(\sqrt[d]{\frac{a-1-\epsilon}{dL\ln a}}) > \epsilon/2$ and $g_L(\sqrt[d]{\frac{a-1+\epsilon}{dL\ln a}}) < -\epsilon/2$. Thus,

$$\sqrt[d]{\frac{a-1-\epsilon}{dL\ln a}} < r_L^* < \sqrt[d]{\frac{a-1+\epsilon}{dL\ln a}}.$$

Then when L is large enough,

$$\sqrt[d]{\frac{a-1-\epsilon}{dL\ln a}}\left(\frac{a-a^{\frac{a-1-\epsilon}{dL\ln a}}}{a-1}\right)^L \leq H_L(r_L^*) < \sqrt[d]{\frac{a-1+\epsilon}{dL\ln a}}.$$

Since

$$\lim_{L\to\infty}\left(\frac{a-a^{\frac{a-1-\epsilon}{dL\ln a}}}{a-1}\right)^L \geq \lim_{L\to\infty}\left(\frac{a-1-\frac{2(a-1-\epsilon)}{dL}}{a-1}\right)^L = \exp\left(-\frac{2(a-1-\epsilon)}{d(a-1)}\right),$$

we have $H_L(r_L^*) = \Theta(\frac{1}{(dL)^{1/d}})$ and the maximum value is achieved when $r_L^* = \Theta(\frac{1}{(dL)^{1/d}})$. There exists r_L' between $\sqrt[d]{\frac{a-1-\epsilon}{dL\ln a}}$ and $\sqrt[d]{\frac{a-1+\epsilon}{dL\ln a}}$ such that

$$C_L^{(0)} = \sup_{r\in[0,r_{\max}]} H_L(r) = H_L(r_L') = \Theta(\frac{1}{(dL)^{1/d}}).$$

■

Proof of Theorem 2-1) : WLOG, suppose $Q^{(s)}(0) > 0$ for a postive integer s and $Q^{(n)}(0) = 0$ for any $1 \leq n < s$. The Taylor expansion of Q at 0 is $Q(r) = \frac{Q^{(s)}(0)}{s!}r^s + o(r^s)$ and the Taylor expansion of $\frac{a^{r^s}-1}{a-1}$ at 0 is $\frac{a^{r^s}-1}{a-1} = \frac{u(s)\ln a}{a-1}r^s + o(r^s)$ for a function u with $u(s) > 0$. Then when a is large enough, we have there exists $b > 0$ such that $Q(r) > \frac{a^{r^s}-1}{a-1}$ for $r \in (0, b)$. Since $P_{\text{out}}(r) \geq Q(r)$, $P_{\text{out}}(r) > \frac{a^{r^s}-1}{a-1}$ for $r \in (0, b)$. Then for $H_L(r)$ as defined in (18) and $R_L^{\text{DF}}(r)$ as defined in (7), $R_L^{\text{DF}}(r) \leq H_L(r)$ for $r \leq b$. By Lemma 5, $\sup_{r\in[0,b]} R_L^{\text{DF}}(r) = O(\frac{1}{(sL)^{1/s}})$. Observe that

$$\sup_{r\in[b,r_{\max}]} R_L^{\text{DF}}(r) = \sup_{r\in[b,r_{\max}]} r(1-P_{\text{out}}(r))^L \leq r_{\max}(1-P_{\text{out}}(b))^L = O(\exp(-L)).$$

Thus $C_L^{\text{DF}} = \sup_{r\in[0,r_{\max}]} R_L^{\text{DF}}(r) = O(\frac{1}{(sL)^{1/s}})$. ■

Proof of Theorem 2-2): WLOG, suppose $(P^{-1})^{(s)}(0) > 0$ for a positive integer s and $(P^{-1})^{(n)}(0) = 0$ for any integer $n \in (0, s)$. Then by similar verification with proof of Theorem 2-1), we have there exists $b > 0$ such that $P^{-1}(y) > \frac{a^{y^s}-1}{a-1}$ for $y \in (0, b)$. Then for $r \in (0, P^{-1}(b))$, $P(r) < \big(\log_a(r(a-1)+1)\big)^{1/s}$. Since $P_{\text{out}}(r) \leq P(r)$, we have $P_{\text{out}}(r) < \big(\log_a(r(a-1)+1)\big)^{1/s}$ for $r \in (0, P^{-1}(b))$. When L is large enough such that $\frac{1}{L} \leq P^{-1}(b)$, for R_L^{DF} as defined in (7),

$$R_L^{\text{DF}}(1/L^s) = \frac{\big(1-P_{\text{out}}(1/L^s)\big)^L}{L^s} \geq \frac{\Big(1-\big(\log_a(\frac{a-1}{L^s}+1)\big)^{1/s}\Big)^L}{L^s}.$$

Since $\Big(1-\big(\log_a(\frac{a-1}{L^s}+1)\big)^{1/s}\Big)^L \geq \Big(1-\frac{1}{L}\big(\frac{2(a-1)}{\ln a}\big)^{1/s}\Big)^L$ when L is large, we have $R_L^{\text{DF}}(1/L) = \Omega(\frac{1}{L^s})$. Hence $C_L^{\text{DF}} = \Omega(\frac{1}{L^s})$. ■

C. *Proof of Theorem 4*

We prove Theorem 4 based on the following Lemma 6 and 7.

For convenience, we first define the following functions, which will be used in the proofs below. Define $f(r) \triangleq r(1 - P_{\text{out}}(r))$. Let

$$R_1(r) = f(r), \tag{19}$$

and for $i = 2, 3, \ldots$, $R_i(r)$ is recursively defined by

$$R_i(r) = f(R_{i-1}(r)). \tag{20}$$

Then we have $C_L^{\text{DF+}} = \max_{r \in [0, r_{\max}]} R_L(r)$.

The first three conditions on P_{out} in Theorem 4 are repeated as follows:

Condition 1: The outage function satisfies

1) P_{out} is non-degraded;
2) $P_{\text{out}}(r)$ is continuous for $r \in [0, r_{\max}]$;
3) $r(1 - P_{\text{out}}(r))$ has no alternating output sequence on $[0, r_{\max}]$.

Lemma 6: Assume Condition 1 is in force. Then $C_L^{\text{DF+}} \to 0$ as $L \to \infty$. Furthermore, there exists L_0, r^* such that $C_L^{\text{DF+}} = R_L(r^*)$ when $L \geq L_0$, i.e. r^* remains the same for all $L \geq L_0$.

Proof: According to Condition 1-2), $R_L(r)$ is a continuous function over a compact set $[0, r_{\max}]$, thus the maximizer of $R_L(r)$ always exists. For any $L \geq 1$, let $\tilde{r}_L$ satisfy $\tilde{r}_L \in \arg\max_{r \in [0, r_{\max}]} R_L(r)$. According to the definition of $C_L^{\text{DF+}}$ it holds that

$$C_{L+1}^{\text{DF+}} = R_{L+1}(\tilde{r}_{L+1}) = R_L(f(\tilde{r}_{L+1})) \leq R_L(\tilde{r}_L) = C_L^{\text{DF+}},$$

which shows $C_L^{\text{DF+}}$ is non-increasing in L. Therefore, the limit $\lim_{L \to \infty} C_L^{\text{DF+}}$ exists.

Assume that there exists fixed L_0 and r^* such that $r^* \in \arg\max_r R_L(r)$ for $L \geq L_0$. Then

$$\lim_{L \to \infty} C_L^{\text{DF+}} = \lim_{L \to \infty} R_L(r^*) \triangleq R^*,$$

which implies $f(R^*) = R^*$, i.e., R^* is a fixed point for the function $f(r)$. According to Condition 1-1), the outage function $P_{\text{out}}(r) > 0$ for $r > 0$ and therefore $f(r) < r$ for $r > 0$. As a consequence, the fixed point solution $R^* = 0$.

In the following we show that when there is no fixed L_0 and r^* such that $r^* \in \arg\max_r R_L(r)$ for $L \geq L_0$, there is an alternating output sequence for $f(r)$ on $[0, r_{\max}]$, which contradicts with Condition 1-3). The condition above can be rephrased as follows. For any L', there exists $L'' > L'$ such that for any $r \in \arg\max_r R_{L'}(r)$, it holds that $r \notin \arg\max_r R_{L''}(r)$.

Now define two infinite sequences $\{L_i\}_{i=1}^{\infty}$ and $\{r_i^*\}_{i=1}^{\infty}$. Let $L_1 = 1$ and r_1^* belongs to $\arg\max_{r\in[0,r_{\max}]} R_1(r)$. For $i \geq 2$, define L_i to be the smallest $L > L_{i-1}$ such that $r_{i-1}^* \notin \arg\max_{r\in[0,r_{\max}]} R_L(r)$, and let $r_i^* \in \arg\max_{r\in[0,r_{\max}]} R_{L_i}(r)$. According the above definition, for any $i \geq 2$,

$$R_{L_i}(r_i^*) > R_{L_i}(r_{i-1}^*), \tag{21}$$

$$r_{i-1}^* \in \arg\max_{r\in[0,r_{\max}]} R_L(r), \qquad L_{i-1} \leq L < L_i. \tag{22}$$

Let $a(i) \triangleq R_{L_i-1}(r_i^*),\ b(i) \triangleq R_{L_i-1}(r_{i-1}^*),\ c(i) \triangleq R_{L_i-2}(r_{i-1}^*)$. For $i \geq 2$, by (22), it holds that

$$b(i) \geq a(i). \tag{23}$$

If the equality in (23) holds, $R_{L_i}(r_i^*) = f\big(a(i)\big) = f\big(b(i)\big) = R_{L_i}(r_{i-1}^*)$, which contradicts with (21) and thus the inequality in (23) is strict, i.e.,

$$b(i) > a(i). \tag{24}$$

Since $f(r) < r$ for $r > 0$, we have

$$R_j(r) < R_k(r),\ \text{for any } j > k, \tag{25}$$

which together with (24) implies for $i \geq 2$,

$$f\big(a(i)\big) = R_{L_i}(r_i^*) < a(i) < b(i) = f\big(c(i)\big), \tag{26}$$

and for any $i \geq 3$,

$$a(i) < b(i) < c(i). \tag{27}$$

By (21) and (26),

$$f(b(i)) < f(a(i)) < f(c(i)), \quad \forall i \geq 3. \tag{28}$$

Due to (25) and $L_{i+1} - 2 \geq L_i - 1$, for any i,

$$c(i+1) = R_{L_{i+1}-2}(r_i^*) \leq R_{L_i-1}(r_i^*) = a(i). \tag{29}$$

Define a sequence $\{x_i\}_{i=1}^{\infty}$ by $x_1 = c(3)$ and for $k \geq 1$, $x_{2k} = b(k+2)$, and

$$x_{2k+1} = \begin{cases} a(k+2), & \text{if } f(a(k+2)) \geq f(c(k+3)), \\ c(k+3), & \text{otherwise.} \end{cases}$$

From (27) and (29), $\{x_i\}_{i=1}^{\infty}$ is strictly decreasing. From (28), it can be verified that

$$f(x_1) - f(x_2) = f\big(c(3)\big) - f\big(b(3)\big) > 0,$$

and for $k \geq 2$,

$$f(x_{2k-1}) - f(x_{2k}) = \max\Big(f\big(a(k+1)\big), f\big(c(k+2)\big)\Big) - f\big(b(k+2)\big) > 0,$$

and for $k \geq 1$,

$$f(x_{2k}) - f(x_{2k+1}) = f\big(b(k+2)\big) - \max\Big(f\big(a(k+2)\big), f\big(c(k+3)\big)\Big) < 0.$$

Thus $\{x_i\}_{i=1}^{\infty}$ is an alternating output sequence of f. The proof is completed since this implication contradicts with Condition 1-3). ■

Lemma 7: Consider a line network of length L formed by identical outage links with the outage function $P_{\text{out}}(r) = \frac{a^r-1}{a-1}$, $r \in [0,1]$, where $a > e$ is a constant. Then it holds that $C_L^{\text{DF+}} = O(1/L)$.

Proof: Let $f(r) = r\left(1 - \frac{a^r-1}{a-1}\right)$. It is easy to show that $f'(0) = 1$, $f'(1) = -a\ln a/(a-1) < 0$, and $f''(r) < 0,\ r \in [0,1]$. There exists $r^* \in (0,1)$ such that $f'(r^*) = 0$. Then $f(r)$ is strictly increasing in $[0, r^*]$ and strictly decreasing in $[r^*, 1]$. We first show that $C_L^{\text{DF+}} = R_L(r^*)$ for any L. For $r \in [0,1] \setminus \{r^*\}$, $R_1(r) = f(r) < f(r^*) < r^*$. Then as $f(r)$ is strictly increasing in $[0, r^*]$,

$$R_2(r) = f(f(r)) < f(f(r^*)) < f(r^*) < r^*.$$

Iteratively, we have $R_L(r) < R_L(r^*)$ and thus $C_L^{\text{DF+}} = R_L(r^*)$ for any L.

Now we show that there exists $\bar{L}$ such that for all $L \geq \bar{L}$, as long as $r < \frac{a-1}{L}$, it holds that $f(r) < \frac{a-1}{L+1}$. Let $L_0 = \lceil \frac{a-1}{r^*} \rceil$. Then $\frac{a-1}{L} \leq r^*$ and for $L \geq L_0$ and $r < \frac{a-1}{L}$, it holds that $f(r) < f\left(\frac{a-1}{L}\right)$. We find

$$f\left(\frac{a-1}{L}\right) - \frac{a-1}{L+1} = \frac{L(1 - a^{\frac{a-1}{L}}) + a - a^{\frac{a-1}{L}}}{L(L+1)}.$$

Since $\tau(L) := L(1 - a^{\frac{a-1}{L}}) + a - a^{\frac{a-1}{L}}$ is a continuous function of L, and $\lim_{L\to\infty} \tau(L) = (a-1)(1 - \ln a) < 0$, we can assert that there exists L_1 so that when $L \geq \bar{L} := \max(L_0, L_1)$, for $L \geq \bar{L}$,

$$f(r) < \frac{a-1}{L+1}, \text{ when } r < \frac{a-1}{L}. \tag{30}$$

In Lemma 6 we showed that $R_L(r^*) \to 0$, as $L \to \infty$, hence there exists a sufficiently large L_2 so that $R_{L_2}(r^*) < (a-1)/\bar{L}$. With $R_{L_2}(r^*)$ in place of r and $\bar{L}$ in place of L in (30),

$$R_{L_2+1}(r^*) = f(R_{L_2}(r^*)) < \frac{a-1}{\bar{L}+1}.$$

If $R_{L_2+k}(r^*) < (a-1)/(\bar{L}+k)$, by applying (30) similarly, we get

$$R_{L_2+k+1}(r^*) = f(R_{L_2+k}(r^*)) < \frac{a-1}{\bar{L}+k+1}.$$

Thus by induction, we have $R_L(r^*) < \frac{a-1}{\bar{L}+L-L_2}$, for any $L \geq L_2$. Hence $C_L^{\text{DF+}} = R_L(r^*) = O(1/L)$. ■

Proof of Theorem 4: Let $P_{\text{out}}(r)$ be an outage function so that conditions in Theorem 4 are satisfied. Note that $\lim_{a\to\infty} \left.\left(\frac{a^r-1}{a-1}\right)'\right|_{r=0} = \lim_{a\to\infty} \frac{\ln a}{a-1} = 0$, and $\lim_{r\to 0^+} P'_{\text{out}}(r) > 0$. Hence, there exists a sufficiently large a_0 and a constant $c > 0$ so that

$$P_{\text{out}}(r) \geq \tilde{P}_{\text{out}}(r) \triangleq \frac{a_0^r - 1}{a_0 - 1}, \quad r \in [0, c].$$

Let $\tilde{f}(r) = r\left(1 - \tilde{P}_{\text{out}}(r)\right)$ and $f(r) = r\left(1 - P_{\text{out}}(r)\right)$. Then

$$f(r) \leq \tilde{f}(r),\ r \in [0, c]. \tag{31}$$

$R_L(r)$ is obtained by (19) and (20) with the outage probability function $P_{\text{out}}(r)$ and $f(r)$. By similar definition, let $\tilde{R}_1(r) = \tilde{f}(r)$, and for $i = 2, 3, \ldots$, $\tilde{R}_i(r)$ is recursively defined by $\tilde{R}_i(r) = \tilde{f}(\tilde{R}_{i-1}(r))$. Take the derivative over r around 0, $f'(r) = 1 - P_{\text{out}}(r) - rP'_{\text{out}}(r)$. We find

$$0 = \lim_{r\to 0} \frac{(P_{\text{out}}(r) - P_{\text{out}}(0))r}{r} = \lim_{r\to 0} \frac{P'_{\text{out}}(r)r + P_{\text{out}}(r) - P_{\text{out}}(0)}{1} = \lim_{r\to 0} rP'_{\text{out}}(r),$$

where the second equality holds by the L'Hospital's rule. Consequently $f'(r) \to 1$ as $r \to 0$. Thus, there exists $0 < b \leq c$ such that $f(r)$ is strictly increasing over $[0, b]$.

By Lemma 6, there exists $\tilde{r} \in (0,1), r^* \in (0, r_{\max}]$ such that for sufficiently large L,

$$\max_r \tilde{R}_L(r) = \tilde{R}_L(\tilde{r}),\ \max_r R_L(r) = R_L(r^*)$$

and $\lim_{L\to\infty} \tilde{R}_L(\tilde{r}) = \lim_{L\to\infty} R_L(r^*) = 0$. Then there exists L_2, L_3 such that

$$\tilde{R}_L(\tilde{r}) < b,\ \text{for } L \geq L_2 \text{ and } 0 < R_{L_3}(r^*) < \tilde{R}_{L_2}(\tilde{r}).$$

As a result,

$$R_{L_3+1}(r^*) = f(R_{L_3}(r^*)) < f(\tilde{R}_{L_2}(\tilde{r})) \leq \tilde{f}(\tilde{R}_{L_2}(\tilde{r})) = \tilde{R}_{L_2+1}(\tilde{r}),$$

where the first inequality is due to $f(r)$ is strictly increasing over $[0, b]$ and the second inequality holds by (31). By similar argument, if $0 < R_{L_3+k}(r^*) < \tilde{R}_{L_2+k}(\tilde{r})$, we have $R_{L_3+k+1}(r^*) < \tilde{R}_{L_2+k+1}(\tilde{r})$. Using induction, for $L \geq L_3$ it holds that $C_L^{\text{DF+}} = R_L(r^*) < \tilde{R}_{L+L_2-L_3}(\tilde{r})$. Hence $C_L^{\text{DF+}} = O(1/L)$. ■

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