Lecture 2
Linear Independence, Basis, Dimension

- Linear Independence
- Basis and Dimension
- Connections with Artificial Intelligence

Contents

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- Basis and Dimension
- Connections with Artificial Intelligence

Linear Independence

$A x = 0 \Leftrightarrow c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$

$A = (v_1, \ldots, v_n)$ $\quad x = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ $\quad x = 0 \Leftrightarrow c_i = 0, \ \forall i$

- Whether the homogeneous system $A\mathbf{x} = 0$ has a unique solution or many solutions is an important question.
- The question is equivalent to whether there exists $x_1, \ldots, x_n$, not all zero, such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Linear Independence

- $Ax=0 \Rightarrow x=0$ (linear independent)
- $x \neq 0$ s.t. $Ax=0$ (linear dependent)

- The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$ are said to be **linearly independent** if

  $A = (v_1, \ldots, v_n)$ $x = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

  implies that all the scalars $c_1, \ldots, c_n$ are 0.

- The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$ are said to be **linearly dependent** if there exists scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

Exercise

Determine whether the following sets of vectors are linearly dependent or not.

1. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ dependent.

2. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} \right\}$ dependent.

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix} = 0 \iff \begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ c_2 + 7c_3 &= 0 \end{aligned}$$

$c_1 = 5 \quad c_2 = -7 \quad c_3 = 1$

3. $\{\mathbf{0}\}$ dependent, $c_1 = 0$

4. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{0}\}$ dependent

$c_1 = 0$
$c_2 = 0$
$c_3 = 1$

Linear Independence and System of Linear Equations

- To determine whether vectors $\mathbf{a}_1, \ldots \mathbf{a}_n$ are linearly dependent or not, we can check whether the system $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = A\mathbf{x} = \mathbf{0}$ has a non-trivial solution or not.
- In other words, if the columns of A are linearly independent, the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. If the columns of A are linearly dependent, the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
    - For a square matrix A, its columns are linearly dependent if and only if A is singular.
    - For an $m \times n$ matrix A with $m < n$, its columns are linearly dependent.

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$x \neq 0.$

$Ax = 0$

$A(2x) = 0$

$A(3x) = 0$

$\vdots$

$c \cdot x$

Vector Space

A set $\mathcal{V}$, on which two operations **addition** and **scalar multiplication** are defined, is a **vector space** if the following axioms are satisfied:

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

A3. There exists $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in \mathcal{V}$.

A4. For each $\mathbf{x} \in \mathcal{V}$, there exists $\mathbf{x}' \in \mathcal{V}$ such that $\mathbf{x} + \mathbf{x}' = \mathbf{0}$, where $\mathbf{x}'$ is usually denoted as $-\mathbf{x}$.

A5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for each scalar α and any $x, y \in \mathcal{V}$.

A6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x} \in \mathcal{V}$.

A7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x} \in \mathcal{V}$.

A8. $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$.

Examples of Vector Space

$p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$

$q(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$

$(p+q)(x) \triangleq (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$

$(\alpha p)(x) \triangleq (\alpha a_0) + (\alpha a_1)x + \cdots + (\alpha a_{n-1}) x^{n-1}$

- $\mathbb{R}^n$, $n \geq 1$
- $\mathbb{R}^{m \times n}$
- Let P_n denote the set of all polynomials of degree less than n.
- Let $C[a, b]$ denote the set of all real-valued functions that are defined and continuous on $[a, b]$.

$(f+g)(x) = f(x) + g(x)$

$(\alpha f)(x) = \alpha f(x)$

Opeartions on General Vector Space

Linear combination, linear span and linear independence can be defined on general vector space $\mathcal{V}$:

- **linear combination**: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \in \mathcal{V}$.
- **linear span**: $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{\sum_i c_i\mathbf{v}_i : c_i \in \mathbb{R}\} \subset \mathcal{V}$.
- **linear independence**: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ implies $c_1, \ldots, c_n$ are all zero.

Example

- How to test matrices $M_1, \ldots, M_k \in \mathbb{R}^{m\times n}$ are linearly independent?
- Are the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ linearly independent?

$$c_1 M_1 + \cdots + c_k M_k = 0 \iff \forall (i,j),\quad (M_1)_{ij}\, c_1 + \cdots + (M_k)_{ij}\, c_k = 0$$

$$c_1 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + c_2 \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 0 \iff \begin{cases} c_1 + 3c_2 = 0 \\ 2c_1 + 4c_2 = 0 \\ 3c_1 + c_2 = 0 \\ 4c_1 + 2c_2 = 0 \end{cases}$$

Example

- How to test vectors (polynomials) $p_1, p_2, \dots, p_k$ are linearly independent in P_n?
- Are the polynomials ✓

$$p_1(x) = x^2 + 3, \quad p_2(x) = 2x^2 + x, \quad p_3(x) = 8x + 7$$

in P_3 linearly independent?

$c_1 p_1 + \cdots + c_k p_k = 0 \iff c_1 p_1(x) + \cdots + c_k p_k(x) = 0, \quad \forall x \in R.$

$c_1(x^2+3) + c_2(2x^2 + x) + c_3(8x+7) = 0 \iff \begin{cases} 3c_1 + 7c_3 = 0 \\ c_2 + 8c_3 = 0 \\ c_1 + 2c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$

Linear Independence

- How to test vectors (functions) $f_1, \ldots, f_k$ are linearly independent in $C[a, b]$?
- Are the functions $x, x^2, \sin(x) \in C[-2, 2]$ linearly independent?

$c_1 x + c_2 x^2 + c_3 \sin(x) = 0 \quad \forall x.$

if $x = 1$: $c_1 + c_2 + \sin(1) c_3 = 0$

$x = -1$: $-c_1 + c_2 - \sin(1) c_3 = 0$

$x = 2$: $2c_1 + 4c_2 + \sin(2) c_3 = 0$

Linear Independence

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space $\mathcal{V}$ are linearly dependent if and only if for a certain $k \in \{1, 2, \ldots, n\}$, $\mathbf{v}_k$ is a linear combination of the other vectors.

$\Rightarrow$ (Necessary)

$c_1 v_1 + \cdots + c_n v_n = 0$, w.l.o.g., $c_1 \neq 0$

$\Rightarrow c_1 v_1 = -c_2 v_2 - \cdots - c_n v_n = -\sum_{i \neq 1} c_i v_i \Rightarrow v_1 = -\frac{1}{c_1} \sum_{i \neq 1} c_i v_i$

$\Leftarrow$ (Sufficient) $v_1 = \sum_{i \neq 1} c_i' v_i$, $v_1 - \sum_{i \neq 1} c_i' v_i = 0$

Minimum Spanning Set

- For a vector space $\mathcal{V}$, we call $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathcal{V}$ a **spanning set of** $\mathcal{V}$ if $\mathrm{Span}(\mathcal{S}) = \mathcal{V}$.
- For a vector space $\mathcal{V}$, we say $\mathcal{S} \subset \mathcal{V}$ is a **minimal spanning set of** $\mathcal{V}$ if $\mathcal{V}$ cannot be generated by any proper subset of $\mathcal{S}$.
- Suppose $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a minimal spanning set of a vector space $\mathcal{V}$. Then $\mathcal{S}$ is linearly independent.

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Proof: Assume on the contrary that S is linear dependent.

$v_k = \sum_{i \neq k} c_i' v_i$

For $\forall v \in V$, $v = \sum_{i=1}^{n} c_i v_i = c_k v_k + \sum_{i \neq k} c_i v_i = c_k \sum_{i \neq k} c_i' v_i + \sum_{i \neq k} c_i v_i = \sum_{i \neq k} (c_i + c_k c_i') v_i$.

$\Rightarrow v$ is linear combination of $S \setminus \{v_k\} \Rightarrow$ contradiction.

Contents

Basis

The vectors $v_1, v_2, \ldots, v_n$ form a **basis** for a vector space $\mathcal{V}$ if

1. $v_1, v_2, \ldots, v_n$ are linearly independent, and
2. $v_1, v_2, \ldots, v_n$ span $\mathcal{V}$.

How to determine whether a set $\mathcal{B}$ of vectors form a basis of a vector space $\mathcal{V}$?

- First, check that $\mathcal{B}$ is a subset of $\mathcal{V}$.
- Second, verify that $\mathcal{B}$ is linearly independent.
- Third, verify that for any $\mathbf{v} \in \mathcal{V}$, $\mathcal{V} \in \mathrm{Span}\{\mathcal{B}\}$.

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Example

Are the following sets a basis for $\mathbb{R}^2$ or not?

- $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
- $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$
- $\mathcal{B}_3 = \left\{ \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
- $\mathcal{B}_4 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$
- $\mathcal{B}_5 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

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Example

Given a vector space $\mathbb{R}^{2\times 2}$, the set $\mathcal{B}$ consisting of

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; B_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis of $\mathbb{R}^{2\times 2}$.

Example

Are the polynomials

$$p_1(x) = x^2 + 3, \quad p_2(x) = 2x^2 + x, \quad p_3(x) = 8x + 7$$

form a basis of P_3?

Dimension

- If a vector space $\mathcal{V}$ has a basis consisting of n vectors, we say that $\mathcal{V}$ has **dimension** n.
- The subspace $\{\mathbf{0}\}$ of $\mathcal{V}$ is said to have dimension 0.
- $\mathcal{V}$ is said to be *finite dimensional* if there is a finite set of vectors that spans $\mathcal{V}$; otherwise, we say that $\mathcal{V}$ is *infinite dimensional*.

Example

- What is the dimension of $\mathbb{R}^n$?
  - The standard basis has n vectors.
- What is the dimension of $\mathbb{R}^{m \times n}$?
  - All the $m \times n$ matrices with only one non-zero entry 1 form a basis.
- What is the dimension of P_n?
  - $\{1, x, x^2, \ldots, x^{n-1}\}$ forms a basis of P_n.
- Let P be the vector space of all polynomials.
  - If P has a finite dimension n, then any $n+1$ polynomials would be linearly dependent. Find a contradiction.
  - P is infinite dimensional.
- $C[a, b]$ is infinite dimensional.

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Contents

Regression with Linear Dependent Data

1. Linear Dependence in Features

If two features are linearly dependent, they carry redundant information, which does not improve a model.

```python
import numpy as np
import matplotlib.pyplot as plt
from sklearn.linear_model import LinearRegression

# Create data
np.random.seed(0)
X1 = np.random.rand(100, 1)  # independent feature
X2 = np.random.randn(100, 1)  # independent feature
X3 = 2 * X1 + 3 * X2          # dependent feature (perfectly correlated)
y = 3 * X1.squeeze() + 6 * X2.squeeze() + np.random.randn(100) * 0.1

# Train models
reg1 = LinearRegression().fit(np.hstack([X1, X2]), y)    # with one feature
reg2 = LinearRegression().fit(np.hstack([X1, X2, X3]), y)  # with redundant feature


print("R^2 with two features:", reg1.score(np.hstack([X1, X2]), y))
print("R^2 with redundant features (full):", reg2.score(np.hstack([X1, X2, X3]), y))
```

[17]

```
R^2 with two features: 0.999740798192029
R^2 with redundant features (full): 0.999740798192029
```

AI for Finding Basis

2. Basis & PCA for Dimensionality Reduction

PCA finds a new orthogonal basis that captures maximum variance in the data.

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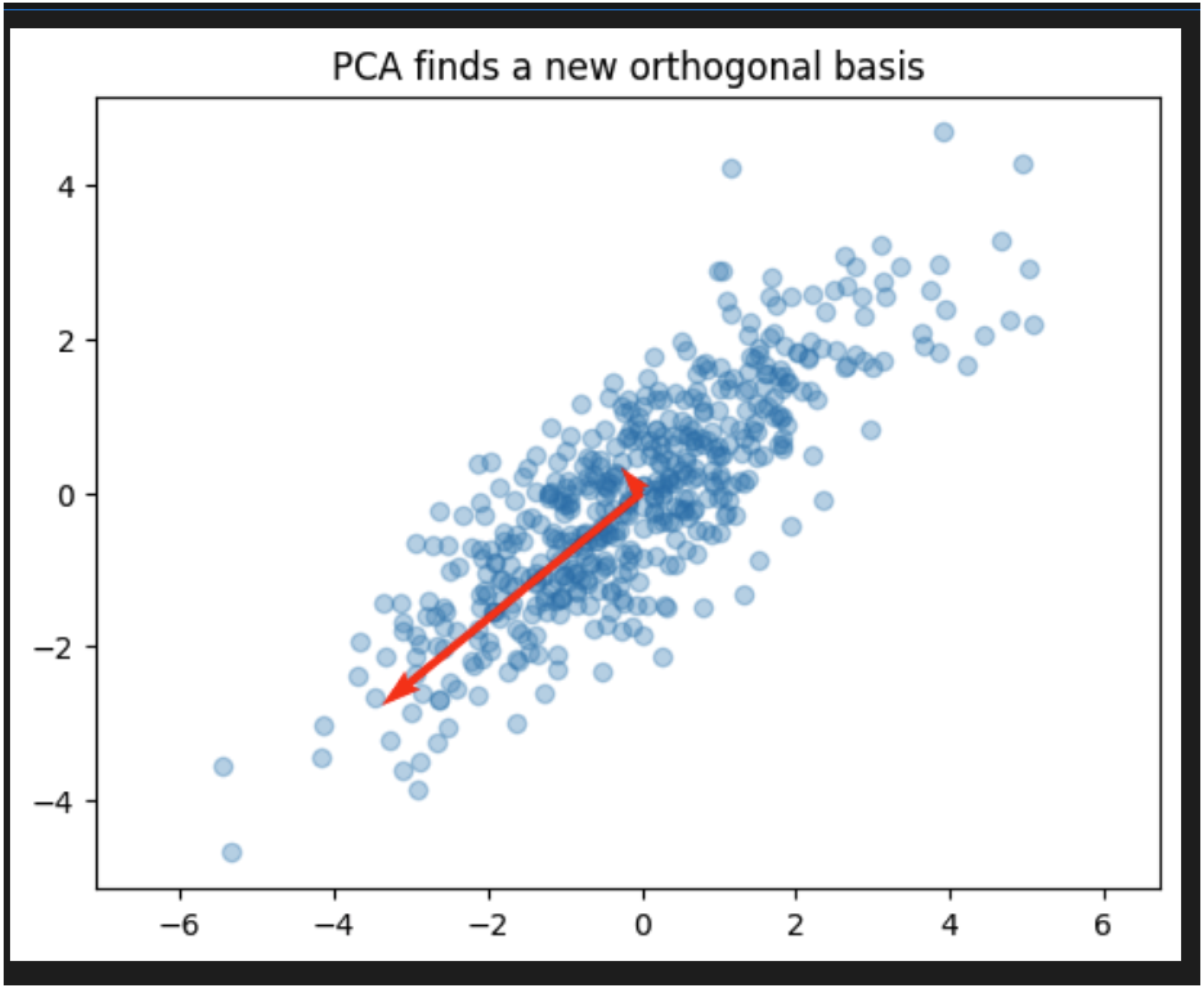
```
from sklearn.decomposition import PCA

# Create 2D correlated data
np.random.seed(1)
X = np.random.multivariate_normal([0, 0], [[3, 2], [2, 2]], size=500)

# Apply PCA
pca = PCA(n_components=2).fit(X)
X_pca = pca.transform(X)

# Plot
plt.scatter(X[:, 0], X[:, 1], alpha=0.3)
for length, vector in zip(pca.explained_variance_, pca.components_):
    plt.quiver(0, 0, vector[0]*length, vector[1]*length,
               angles='xy', scale_units='xy', scale=1, color='red')
plt.title("PCA finds a new orthogonal basis")
plt.axis("equal")
plt.show()
```

AI for Finding Basis



Infinite-Dimension AI Model

3. Infinite-Dimensional Spaces: Kernel Trick

Kernel methods (e.g., SVM with RBF kernel) operate in infinite-dimensional spaces implicitly, while computations remain finite.

```python
from sklearn.svm import SVC
from sklearn.datasets import make_moons

# Generate dataset
X, y = make_moons(n_samples=200, noise=0.1, random_state=0)

# Train SVM with RBF kernel (infinite-dimensional space)
clf = SVC(kernel='rbf', C=10).fit(X, y)

# Plot decision boundary
xx, yy = np.meshgrid(np.linspace(-2, 3, 200), np.linspace(-1.5, 2, 200))
Z = clf.predict(np.c_[xx.ravel(), yy.ravel()]).reshape(xx.shape)

plt.contourf(xx, yy, Z, alpha=0.3)
plt.scatter(X[:, 0], X[:, 1], c=y, edgecolors='k')
plt.title("SVM with RBF kernel (infinite-dimensional space)")
plt.show()
```

Recommended Reading:

- Two-sample Test with Kernel Projected Wasserstein Distance
- Statistical and Computational Guarantees of Kernel Max-Sliced Wasserstein Distances
- Variable Selection for Kernel Two-Sample Tests

Infinite-Dimension AI Model