

12.2. Monday for MAT3006

12.2.1. Remarks on MCT

■ **Example 12.2** The MCT can help us to compute the integral

$$\lim_{n \rightarrow \infty} \int_0^{n\pi} \cos\left(\frac{x}{2n}\right) x e^{-x^2} dx$$

Construct $f_n(x) = \cos\left(\frac{x}{2n}\right) x e^{-x^2} \chi_{[0, n\pi]}$.

- Since $\cos(x/2n) < \cos(x/2(n+1))$ for any $x \in [0, n\pi]$, we imply f_n is monotone increasing with n
- f_n converges pointwise to $x e^{-x^2} \chi_{[0, \infty)}$

Therefore, MCT I applies and

$$\lim_{n \rightarrow \infty} \int_0^{n\pi} \cos\left(\frac{x}{2n}\right) x e^{-x^2} dx = \int \left(\lim_{n \rightarrow \infty} f_n \right) dm$$

with

$$\lim_{n \rightarrow \infty} f_n = x e^{-x^2} \chi_{[0, \infty)}.$$

Moreover,

$$\int \left(\lim_{n \rightarrow \infty} f_n \right) dm = \lim_{m \rightarrow \infty} \int_0^m x e^{-x^2} dx \quad (12.1a)$$

$$= \int_0^\infty x e^{-x^2} dx \quad (12.1b)$$

$$= \frac{1}{2} \quad (12.1c)$$

where (12.1a) is by applying MCT I with $g_m(x) = x e^{-x^2} \chi_{[0, m]}$. ■

Then we discuss the Lebesgue integral for series:

Corollary 12.3 [Lebesgue Series Theorem] Let $\{f_n\}$ be a series of measurable functions

such that

$$\sum_{n=1}^{\infty} \int |f_n| \, dm < \infty,$$

then $\sum_{n=1}^k f_n$ converges to an integrable function $f = \sum_{n=1}^{\infty} f_n$ a.e., with

$$\int f \, dm = \sum_{n=1}^{\infty} \int f_n \, dm$$

Proof. • For each f_n , consider

$$f_n = f_n^+ - f_n^-, \text{ where } f_n^+, f_n^- \text{ are nonnegative.}$$

By proposition (11.6),

$$\int \sum_{n=1}^{\infty} f_n^+ \, dm = \sum_{n=1}^{\infty} \int f_n^+ \, dm \leq \sum_{n=1}^{\infty} \int |f_n| \, dm < \infty.$$

Therefore, $f^+ := \sum_{n=1}^{\infty} f_n^+ = \lim_{k \rightarrow \infty} \sum_{n=1}^k f_n^+$ is integrable. The same follows by replacing f^+ with f^- . By corollary (??), $f^+(x), f^-(x) < \infty, \forall x \in U$, where U^c is null.

• Therefore, construct

$$f(x) = \begin{cases} f^+(x) - f^-(x), & x \in U \\ 0, & x \in U^c \end{cases}$$

Moreover, for $x \in U$,

$$\begin{aligned} f(x) &= \left(\lim_{k \rightarrow \infty} \sum_{n=1}^k f_n^+(x) \right) - \left(\lim_{k \rightarrow \infty} \sum_{n=1}^k f_n^-(x) \right) \\ &= \lim_{k \rightarrow \infty} \left(\sum_{n=1}^k f_n^+(x) - \sum_{n=1}^k f_n^-(x) \right) \\ &= \lim_{k \rightarrow \infty} \left[\sum_{n=1}^k (f_n^+(x) - f_n^-(x)) \right] \\ &= \sum_{n=1}^{\infty} f_n(x) \end{aligned}$$

where the first equality is because that both terms are finite.

- It follows that

$$\int f \, dm = \int f^+ \, dm - \int f^- \, dm \quad (12.2a)$$

$$= \int \sum_{n=1}^{\infty} f_n^+ \, dm - \int \sum_{n=1}^{\infty} f_n^- \, dm \quad (12.2b)$$

$$= \left(\sum_{n=1}^{\infty} \int f_n^+ \, dm \right) - \left(\sum_{n=1}^{\infty} \int f_n^- \, dm \right) \quad (12.2c)$$

$$= \sum_{n=1}^{\infty} \left(\int f_n^+ \, dm - \int f_n^- \, dm \right) \quad (12.2d)$$

$$= \sum_{n=1}^{\infty} \int f_n \, dm \quad (12.2e)$$

where (12.2a),(12.2d) is because that summation/subtraction between series holds when these series are finite; (12.2c) is by proposition (11.6); (12.2e) is by definition of f_n .

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■ **Example 12.3** Compute the integral

$$\int_0^1 e^{-x} x^{\alpha-1} \, dx, \quad \alpha > 0.$$

- Construct $f_n(x) = (-1)^n \frac{x^{\alpha+n-1}}{n!} \chi_{(0,1]}, n \geq 0$, and

$$\sum_{n=0}^N f_n(x) \rightarrow e^{-x} x^{\alpha-1}, \text{ pointwisely, } x \in (0,1].$$

By applying MCT I,

$$\int |f_n| \, dm = \frac{1}{(\alpha+n)n!}$$

Therefore,

$$\sum_{n=0}^{\infty} \int |f_n| \, dm = \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)n!} < \infty$$

- Applying the Lebesgue Series Theorem,

$$\int_0^1 e^{-x} x^{\alpha-1} dx = \int_0^1 \left(\sum_{n=0}^{\infty} f_n \right) dm = \sum_{n=0}^{\infty} \int f_n dm = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\alpha+n)n!}$$

R It's essential to have $\sum \int |f| dm < \infty$ rather than $\sum \int f_n dm < \infty$ in the Lebesgue Series Theorem. For example, let

$$f_n = \frac{(-1)^{n+1}}{(n+1)} X_{[n, n+1)} \implies \sum_{n=1}^{\infty} \int f_n dm = \log(2) < \infty$$

However, $f := \sum f_n$ is not integrable.

12.2.2. Dominated Convergence Theorem

Theorem 12.2 Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ a.e., and g is integrable. Suppose that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ a.e., then

1. f is integrable,
- 2.

$$\int f dm = \lim_{n \rightarrow \infty} \int f_n dm$$

Proof. • Observe that

$$|f_n| \leq g \implies \lim_{n \rightarrow \infty} |f_n| \leq g \implies |f| \leq g$$

By comparison test, g is integrable implies $|f|$ is integrable, and further f is integrable.

- Consider the sequence of non-negative functions $\{g - f_n\}_{n \in \mathbb{N}}$ and $\{g + f_n\}_{n \in \mathbb{N}}$.

By Fatou's Lemma,

$$\begin{aligned}\liminf_{n \rightarrow \infty} \int (g - f_n) \, dm &\geq \int \liminf_{n \rightarrow \infty} (g - f_n) \, dm \\ &= \int (g - f) \, dm \\ &= \int g \, dm - \int f \, dm\end{aligned}$$

which follows that

$$\int g \, dm - \limsup_{n \rightarrow \infty} \int f_n \, dm \geq \int g \, dm - \int f \, dm$$

i.e.,

$$\int f \, dm \geq \limsup_{n \rightarrow \infty} \int f_n \, dm$$

• Similarly,

$$\liminf_{n \rightarrow \infty} \int (g + f_n) \, dm \geq \int \liminf_{n \rightarrow \infty} (g + f_n) \, dm = \int g \, dm + \int f \, dm$$

which implies

$$\liminf_{n \rightarrow \infty} \int f_n \, dm \geq \int f \, dm$$

As a result,

$$\limsup_{n \rightarrow \infty} \int f_n \, dm \leq \int f \, dm \leq \liminf_{n \rightarrow \infty} \int f_n \, dm,$$

which implies

$$\int f \, dm = \lim_n \int f_n \, dm$$

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Corollary 12.4 [Bounded Convergence Theorem] Suppose that $E \in \mathcal{M}$ be such that $m(E) < \infty$. If

- $|f_n(x)| \leq K < \infty$ for any $x \in E, n \in \mathbb{N}$

- $f_n \rightarrow f$ a.e. in E ,

then f is integrable in E with

$$\int_E f \, dm = \lim_{n \rightarrow \infty} \int f_n \, dm$$

Proof. Take $g = K\chi_E$ in DCT. ■

Proposition 12.2 Every Riemann integrable function f on $[a, b]$ is Lebesgue integrable, without the condition that f is continuous a.e.

Proof. Since f is Riemann integrable, we imply f is bounded, and there are sequences of simple functions $\{\phi_n\}$ and $\{\psi_n\}$ such that

- ϕ_n is monotone increasing a.e.; ψ_n is monotone decreasing a.e.
- $\phi_n \leq f \leq \psi_n$, and

$$\lim_{n \rightarrow \infty} \int_{[a,b]} \phi_n = \int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \int_{[a,b]} \psi_n.$$

Construct $g = \sup_n \phi_n$ and $h = \inf_n \psi_n$. Since ϕ_n is bounded on $[a, b]$, $\phi_n \rightarrow g$ on $[a, b]$, we imply g is Lebesgue integrable on $[a, b]$, with

$$\int_{[a,b]} g \, dm = \lim_{n \rightarrow \infty} \int_{[a,b]} \phi_n = \int_a^b f(x) \, dx.$$

Similarly, h is Lebesgue integrable, with

$$\int_{[a,b]} h \, dm = \lim_{n \rightarrow \infty} \int_{[a,b]} \psi_n = \int_a^b f(x) \, dx.$$

Moreover, $g \leq f \leq h$, and

$$\int_{[a,b]} (h - g) \, dm = \int_{[a,b]} h \, dm - \int_{[a,b]} g \, dm = 0,$$

which implies $h = g$ a.e., and further $f = g$ a.e., which implies

$$\int_{[a,b]} f \, dm = \int_{[a,b]} g \, dm = \int_a^b f(x) \, dx$$

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Ⓡ However, an improper Riemann integral does not necessarily has the corresponding Lebesgue integral:

$$f(x) = \sum_{n=1}^{\infty} (-1)^n n \cdot \chi_{(1/(n+1), 1/n]}, \quad x \in [0, 1]$$

In this case, f is Riemann integrable but not Lebesgue integrable.