1.2. Thursday

Reviewing for Probability Space.

- $(\Omega, \mathcal{F}, \mathbb{P})$;
- Random variable;
- Generated σ-algebra;

1.2.1. More on Probability Theory

Definition 1.9 [Distribution] A probability measure μ_X on $\mathbb{R}^n$ induced by the random variabe X isdefined as

$$\mu_X(\boldsymbol{B}) = \mathbb{P}(X^{-1}(\boldsymbol{B})),$$

where $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)$. The μ_X is called the **distribution** of X. ■

Definition 1.10 [Expectation] The expectation of X is given by

$$\mathbb{E}[X] = \int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega)$$

When $\Omega = \mathbb{R}^n$, the expectation can be written in terms of distribution function:

$$\mathbb{E}[X] = \int_{\mathbb{R}^n} y\, \mathrm{d}\mu_X(y)$$

■

Note that the expectation of the random variable X is well-defined when X is integrable:

Definition 1.11 [Integrable] The random variable X is **integrable**, if

$$\int_\Omega |X(w)|\, \mathrm{d}\mathbb{P}(w) < \infty.$$

In other words, X is said to be $\mathcal{L}^1$-integrable, denoted as $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. ■

■ **Example 1.1** If $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable, and $\int_\Omega |f(X(\omega))| \, d\mathbb{P}(\omega) < \infty$, then

$$\mathbb{E}[f(x)] = \int_\Omega f(X(\omega)) \, d\mathbb{P}(\omega) = \int_{\mathbb{R}^n} f(y) \, d\mu_X(y).$$

■

Definition 1.12 [L^p space] Suppose $X : \ \Omega \to \mathbb{R}$ is a random variable and $p \geq 1$.

- Define L^p-norm of X as

$$\|X\|_p = \left(\int_\Omega |X(\omega)|^p \, d\mathbb{P} \right)^{1/p}$$

  If $p = \infty$, define

$$\|X\|_\infty = \inf\{N \in \mathbb{R} \mid |X(w)| \leq N, \text{ a.s.}\}$$

- A random variable X is said to be in the L^p space (p-th integrable) if

$$\int_\Omega |X(\omega)|^p \, d\mathbb{P}(\omega) < \infty,$$

  denoted as $X \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$.

■

Proposition 1.2 If $p \geq q$, then $\|X\|_p \geq \|X\|_q$. Thus $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{L}^q(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. The inequality is shown by using Holder's inequality:

$$\|X\|_q^q = \int_\Omega |X|^q \, d\mathbb{P} \leq \left(\int_\Omega (|X|^q)^{p/q} \, d\mathbb{P} \right)^{q/p} = \left(\int_\Omega |X|^p \, d\mathbb{P} \right)^{\frac{1}{p} \cdot q} = \|X\|_p^q.$$

■

Then we discuss how to define independence between two random variables, by the following three steps:

Definition 1.13 [Independence]

1. Two events $A_1, A_2 \in \mathcal{F}$ are said to be **independent** if $\mathbb{P}(A_1 \bigcap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$.
2. Two σ-algebras $\mathcal{F}_1, \mathcal{F}_2$ are said to be **independent** if F_1, F_2 are independent events for $\forall F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2$
3. Two random variables X, Y are said to be **independent** if $\mathcal{H}_X, \mathcal{H}_Y$, the σ-algebra generated by X and Y, respectively, are independent.

R The independence defined above can be generalized from two events into finite number of events.

Proposition 1.3 If X and Y are two independent random variables, and $\mathbb{E}[|X|] < \infty, \mathbb{E}[|Y|] < \infty$, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] < \infty.$$

Proof. The first step is to simplify the probability distribution for the product random variable (X, Y), i.e., $\mu_{X,Y}$.

R From now on, we also write the event $\{X^{-1}(\boldsymbol{B})\}$ as $\{X \in \boldsymbol{B}\}$ for $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^n)$.

By the definition of independence, we have the following:

$$\begin{aligned} \mu_{X,Y}(A_1 \times A_2) &\triangleq \mathbb{P}(\{(X,Y) \in (A_1 \times A_2)\}) = \mathbb{P}(\{X \in A_1, Y \in A_2\}) \\ &= \mathbb{P}(\{X \in A_1\})\mathbb{P}(\{Y \in A_2\}) = \mu_X(A_1)\mu_Y(A_2). \end{aligned}$$

Now we begin to simplify the expectation of product:

$$\begin{aligned} \mathbb{E}[XY] &= \int xy \, \mathrm{d}\mu_{X,Y}(x,y) = \iint xy \, \mathrm{d}\mu_X(x)\mu_Y(y) \\ &= \int y \left[\int x \, \mathrm{d}\mu_X(x) \right] \mu_Y(y) = \int \mathbb{E}[X] y \mu_Y(y) = \mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

■

1.2.2. Stochastic Process

Consider a set T of time index, e.g., a non-negative integer set or a time interval $[0,\infty)$. We will discuss a discrete/continuous time stochastic process.

Definition 1.14 [Stochastic Process] A collection of random variables $\{X_t\}_{t\in T}$, defined on $(\Omega,\mathcal{F},\mathbb{P})$ and taking values in $\mathbb{R}^n$, is called a **stochastic process**. ■

R A stochastic process $\{X_t\}_{t\in T}$ can also be viewed as a random function, since it is a mapping $\Omega \times T \to \mathbb{R}^n$. Sometimes we omit the subscript to denote a stochastic process $\{X_t\}$.

Definition 1.15 [Sample Path] Fixing $\omega \in \Omega$, then $\{X_t(\omega)\}_{t\in T}$ (denoted as $X_\cdot(\omega)$) is called a **sample path**, or **trajectory**. ■

Definition 1.16 [Continuous] A stochastic process $\{X_t\}$ is said to be **continuous** (right-cot, left-cot, resp.) a.s., if $t \to X_t(\omega)$ is **continuous** (right-cot, left-cot, resp.) a.s., i.e.,

$$\mathbb{P}\Big(\{\omega : t \to X_t(\omega) \text{ is continuous (right-cot, left-cot, resp.)}\}\Big) = 1.$$

■

■ **Example 1.2** [Poisson Process] Consider $(\xi_j, j = 1,2,\dots)$ a sequence of i.i.d. random variables with Possion distribution with intensity $\lambda > 0$. Let $T_0 = 0$, and $T_n = \sum_{j=1}^n \xi_j$. Define $X_t = n$ if $T_n \le t < T_{n+1}$. Verify that $\{X_t\}$ is a stochastic process with right-continuity and left-limit exists. Instead of giving a mathematical proof, we provide a numerical simulation of $\{X_t\}$ plotted in Figure. 1.1. [a] ■

[a] The corresponding matlab code can be found in

https://github.com/WalterBabyRudin/Courseware/tree/master/MAT4500/week1

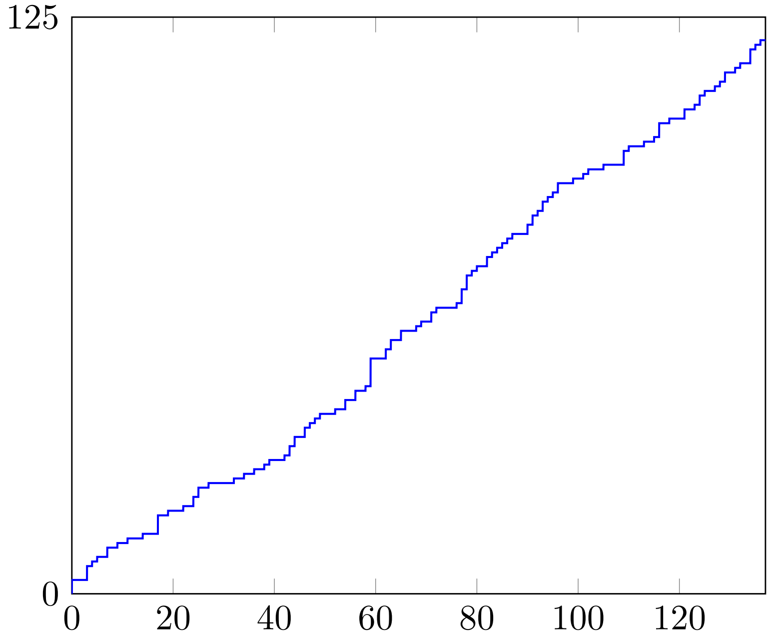

Figure 1.1: One simulation of $\{X_t\}$ with intensity $\lambda = 1.2$ and 500 samples