Chapter 1

Week1

1.1. Monday for MAT3040

1.1.1. Introduction to Advanced Linear Algebra

Advanced Linear Algebra is one of the most important course in MATH major, with pre-request MAT2040. This course will offer the really linear algebra knowledge.

What the content will be covered?.

- In MAT2040 we have studied the space $\mathbb{R}^n$; while in MAT3040 we will study the general vector space V.
- In MAT2040 we have studied the *linear transformation* between Euclidean spaces, i.e., $T:\mathbb{R}^n \to \mathbb{R}^m$; while in MAT3040 we will study the linear transformation from vector spaces to vector spaces: $T:V\to W$
- In MAT2040 we have studied the eigenvalues of $n\times n$ matrix $\boldsymbol{A}$; while in MAT3040 we will study the eigenvalues of a **linear operator** $T:V\to V$.
- In MAT2040 we have studied the dot product $\boldsymbol{x}\cdot\boldsymbol{y}=\sum_{i=1}^n x_iy_i$; while in MAT3040 we will study the **inner product** $\langle \boldsymbol{v}_1,\boldsymbol{v}_2\rangle$.

Why do we do the generalization?. We are studying many other spaces, e.g., $\mathcal{C}(\mathbb{R})$ is called the space of all functions on $\mathbb{R}$, $\mathcal{C}^\infty(\mathbb{R})$ is called the space of all infinitely differentiable functions on $\mathbb{R}$, $\mathbb{R}[x]$ is the space of polynomials of one-variable.

■ **Example 1.1** 1. Consider the Laplace equation $\Delta f = 0$ with linear operator Δ:

$$\Delta : \mathcal{C}^\infty(\mathbb{R}^3) \to \mathcal{C}^\infty(\mathbb{R}^3) \quad f \mapsto (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})f$$

The solution to the PDE $\Delta f = 0$ is the 0-eigenspace of Δ.

2. Consider the Schrödinger equation $\hat{H}f = Ef$ with the linear operator

$$\hat{H} : \mathcal{C}^\infty(\mathbb{R}^3) \to \mathbb{R}^3, \quad f \to \left[\frac{-\hbar^2}{2\mu}\nabla^2 + V(x,y,z)\right]f$$

Solving the equation $\hat{H}f = Ef$ is equivalent to finding the eigenvectors of $\hat{H}$. In fact, the eigenvalues of $\hat{H}$ are **discrete**.

■

1.1.2. Vector Spaces

Definition 1.1 [Vector Space] A **vector space** over a field $\mathbb{F}$ (in particular, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is a set of objects V equipped with vector addiction and scalar multiplication such that

1. the vector addiction $+$ is closed with the rules:
   (a) **Commutativity**: $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1$.
   (b) **Associativity**: $\boldsymbol{v}_1 + (\boldsymbol{v}_2 + \boldsymbol{v}_3) = (\boldsymbol{v}_1 + \boldsymbol{v}_2) + \boldsymbol{v}_3$.
   (c) **Addictive Identity**: $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$, $\forall \boldsymbol{v} \in V$.
2. the **scalar multiplication** is closed with the rules:
   (a) **Distributive**: $\alpha(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \alpha\boldsymbol{v}_1 + \alpha\boldsymbol{v}_2, \forall \alpha \in \mathbb{F}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$
   (b) **Distributive**: $(\alpha_1 + \alpha_2)\boldsymbol{v} = \alpha_1\boldsymbol{v} + \alpha_2\boldsymbol{v}$
   (c) **Compatibility**: $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$ for $\forall a, b \in \mathbb{F}$ and $\boldsymbol{b} \in V$.
   (d) $0\boldsymbol{v} = \mathbf{0}$, $1\boldsymbol{v} = \boldsymbol{v}$.

■

Here we study several examples of vector spaces:

■ **Example 1.2** For $V = \mathbb{F}^n$, we can define

1. Addictive Identity:
$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
2. Scalar Multiplication:
$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$
3. Vector Addiction:
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

■ **Example 1.3**

1. It is clear that the set $V = M_{n\times n}(\mathbb{F})$ (the set of all $m \times n$ matrices) is a vector space as well.
2. The set $V = \mathcal{C}(\mathbb{R})$ is a vector space:
   (a) Vector Addiction:
   $$(f+g)(x) = f(x) + g(x), \forall f, g \in V$$
   (b) Scalar Multiplication:
   $$(\alpha f)(x) = \alpha f(x), \forall \alpha \in \mathbb{R}, f \in V$$
   (c) Addictive Identity is a zero function, i.e., $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

Definition 1.2 A sub-collection $W \subseteq V$ of a vector space V is called a **vector subspace of** V if W itself forms a vector space, denoted by $W \leq V$.

■ **Example 1.4**

1. For $V = \mathbb{R}^3$, we claim that $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \leq V$
2. $W = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$ is not the vector subspace of V.

Proposition 1.1 $W \subseteq V$ is a **vector subspace** of V iff for $\forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, we have $\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2 \in W$, for $\forall \alpha, \beta \in \mathbb{F}$.

■ **Example 1.5**

1. For $V = M_{n \times n}(\mathbb{F})$, the subspace $W = \{A \in V \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\} \leq V$
2. For $V = \mathcal{C}^{\infty}(\mathbb{R})$, define $W = \{f \in V \mid \frac{\mathrm{d}^2}{\mathrm{d}x^2} f + f = 0\} \leq V$. For $f, g \in W$, we have

$$(\alpha f + \beta g)'' = \alpha f'' + \beta g'' = \alpha(-f) + \beta(-g) = -(\alpha f + \beta g),$$

which implies $(\alpha f + \beta g)'' + (\alpha f + \beta g) = 0$.

1.2. Monday for MAT3006

1.2.1. Overview on uniform convergence

Definition 1.3 [Convergence] Let $f_n(x)$ be a sequence of functions on an interval $I = [a,b]$. Then $f_n(x)$ converges **pointwise** to $f(x)$ (i.e., $f_n(x_0) \to f(x_0)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{x_0,\varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_{x_0,\varepsilon}$$

We say $f_n(x)$ converges uniformly to $f(x)$, (i.e., $f_n(x) \rightrightarrows f(x)$) for $\forall x_0 \in I$, if

$$\forall \varepsilon > 0, \exists N_{\varepsilon} \text{ such that } |f_n(x_0) - f(x_0)| < \varepsilon, \forall n \geq N_{\varepsilon}$$

■

■ **Example 1.6** It is clear that the function $f_n(x) = \frac{n}{1+nx}$ converges pointwise into $f(x) = \frac{1}{x}$ on $[0,\infty)$, and it is uniformly convergent on $[1,\infty)$. ■

Proposition 1.2 If $\{f_n\}$ is a sequence of continuous functions on I, and $f_n(x) \rightrightarrows f(x)$, then the following results hold:

1. $f(x)$ is continuous on I.
2. f is (Riemann) integrable with $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$.
3. Suppose furthermore that $f_n(x)$ is **continuously differentiable**, and $f_n'(x) \rightrightarrows g(x)$, then $f(x)$ is differentiable, with $f_n'(x) \to f'(x)$.

We can put the discussions above into the content of series, i.e., $f_n(x) = \sum_{k=1}^n S_k(x)$.

Proposition 1.3 If $S_k(x)$ is continuous for $\forall k$, and $\sum_{k=1}^n S_k \rightrightarrows \sum_{k=1}^\infty S_k$, then

1. $\sum_{k=1}^\infty S_k(x)$ is continuous,
2. The series $\sum_{k=1}^\infty S_k$ is (Riemann) integrable, with $\sum_{k=1}^\infty \int_a^b S_k(x)\,\mathrm{d}x = \int_a^b \sum_{k=1}^\infty S_k(x)\,\mathrm{d}x$
3. If $\sum_{k=1}^n S_k$ is continuously differentiable, and the derivative of which is uniform

convergent, then the series $\sum_{k=1}^{\infty} S_k$ is differentiable, with

$$\left(\sum_{k=1}^{\infty} S_k(x)\right)' = \sum_{k=1}^{\infty} S_k'(x)$$

Then we can discuss the properties for a special kind of series, say power series.

Proposition 1.4 Suppose the power series $f(x) = \sum_{k=1}^{\infty} a_k x^k$ has radius of convergence R, then

1. $\sum_{k=1}^{n} a_k x^k \rightrightarrows f(x)$ for any $[-L, L]$ with $L < R$.
2. The function $f(x)$ is continuous on $(-R, R)$, and moreover, is differentiable and (Riemann) integrable on $[-L, L]$ with $L < R$:

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{k=1}^{\infty} \frac{a_k}{k+1} x^{k+1}$$
$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

1.2.2. Introduction to MAT3006

What are we going to do.

1. (a) Generalize our study of (sequence, series, functions) on $\mathbb{R}^n$ into a metric space.

   (b) We will study spaces outside $\mathbb{R}^n$.

   Remark:

   - For (a), different metric may yield different kind of convergence of sequences. For (b), one important example we will study is $X = \mathcal{C}[a, b]$ (all continuous functions defined on $[a, b]$.) We will generalize X into $\mathcal{C}_b(E)$, which means the set of bounded continuous functions defined on $E \subseteq \mathbb{R}^n$.
   - The insights of analysis is to find a **unified** theory to study sequences/series on a metric space X, e.g., $X = \mathbb{R}^n, \mathcal{C}[a, b]$. In particular, for $\mathcal{C}[a, b]$, we will see that
     - most functions in $\mathcal{C}[a, b]$ are nowhere differentiable. (repeat part of

content in MAT2006)

- We will prove the existence and uniqueness of ODEs.
- the set poly$[a,b]$ (the set of polynomials on $[a,b]$) is dense in $\mathcal{C}[a,b]$. (analogy: $\mathbb{Q} \subseteq \mathbb{R}$ is dense)

2. Introduction to the Lebesgue Integration.

   For convergence of integration $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)$, we need the pre-conditions (a) $f_n(x)$ is continuous, and (b) $f_n(x) \rightrightarrows f(x)$. The natural question is that can we relax these conditions to

   (a) $f_n(x)$ is integrable?

   (b) $f_n(x) \to f(x)$ pointwisely?

   The answer is yes, by using the tool of Lebesgue integration. If $f_n(x) \to f(x)$ and $f_n(x)$ is Lebesgue integrable, then $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$, which is so called the **dominated convergence**.

1.2.3. Metric Spaces

We will study the **length** of an element, or the **distance** between two elements in an arbitrary set X. First let's discuss the length defined on a well-structured set, say vector space.

Definition 1.4 [Normed Space] Let X be a vector space. A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|\boldsymbol{x}\| \geq 0$ for $\forall \boldsymbol{x} \in X$, with equality iff $\boldsymbol{x} = \boldsymbol{0}$
2. $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$, for $\forall\alpha \in \mathbb{R}$ and $\boldsymbol{x} \in X$.
3. $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangular inequality)

Any vector space equipped with $\|\cdot\|$ is called a **normed space**.

Example 1.7 1. For $X = \mathbb{R}^n$, define

$$\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2} \quad \text{(Euclidean Norm)}$$

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad (p\text{-norm})$$

2. For $X = \mathcal{C}[a,b]$, define

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, \mathrm{d}x\right)^{1/p}$$

Exercise: check the norm defined above are well-defined.

Here we can define the distance in an arbitrary set:

Definition 1.5 A set X is a **metric space** with metric (X,d) if there exists a (distance) function $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x},\boldsymbol{y} \in X$, with equality iff $\boldsymbol{x} = \boldsymbol{y}$.
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$.
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$.

Example 1.8 1. If X is a normed space, then define $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|$, which is so called the metric induced from the norm $\|\cdot\|$.

2. Let X be any (non-empty) set with $\boldsymbol{x},\boldsymbol{y} \in X$, the discrete metric is given by:

$$d(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Exercise: check the metric space defined above are well-defined.

R Adopting the infinite norm discussed in Example (1.7), we can define a metric on $\mathcal{C}[a,b]$ by

$$d_\infty(f,g) = \|f-g\|_\infty := \max_{x\in[a,b]} |f(x)-g(x)|$$

which is the correct metric to study the uniform convergence for $\{f_n\} \subseteq \mathcal{C}[a,b]$.

Definition 1.6 Let (X,d) be a metric space. An **open ball** centered at $\boldsymbol{x} \in X$ of radius r is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}.$$

■ **Example 1.9** 1. For $X = \mathbb{R}^2$, we can draw the $B_1(\mathbf{0})$ with respect to the metrics d_1, d_2:

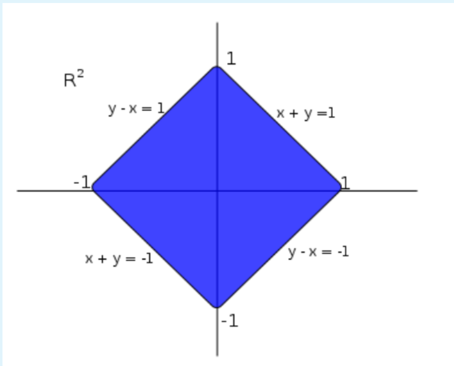


Figure 1.1: $B_1(\mathbf{0})$ w.r.t. the metric d_1

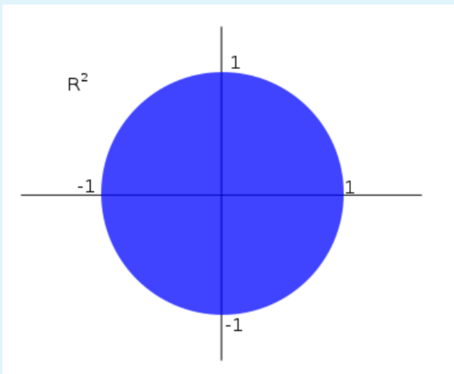


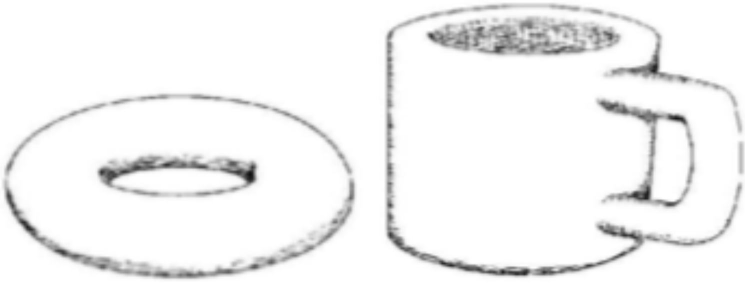
Figure 1.2: $B_1(\mathbf{0})$ w.r.t. the metric d_2

1.3. Monday for MAT4002

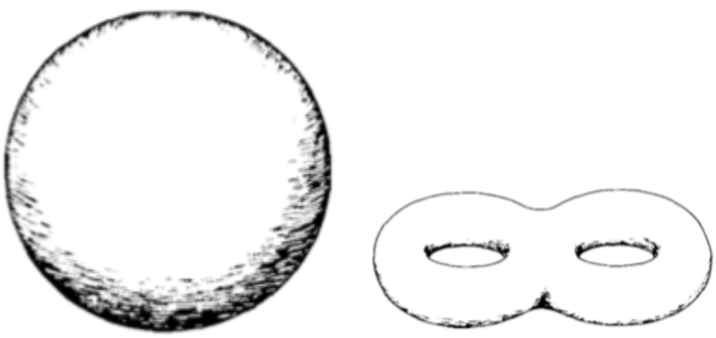
1.3.1. Introduction to Topology

We will study global properties of a geometric object, i.e., *the distrance between 2 points in an object is totally ignored*. For example, the objects shown below are essentially invariant under a certain kind of transformation:

Another example is that the coffee cup and the donut have the same topology:



However, the two objects below have the intrinsically different topologies:



In this course, we will study the phenomenon described above mathematically.

1.3.2. Metric Spaces

In order to ingnore about the distances, we need to learn about distances first.

Definition 1.7 [Metric Space] Metric space is a set X where one can measure distance between any two objects in X.

Specifically speaking, a metric space X is a non-empty set endowed with a function (distance function) $d : X \times X \to \mathbb{R}$ such that

1. $d(\boldsymbol{x},\boldsymbol{y}) \geq 0$ for $\forall \boldsymbol{x},\boldsymbol{y} \in X$ with equality iff $\boldsymbol{x} = \boldsymbol{y}$
2. $d(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{y},\boldsymbol{x})$
3. $d(\boldsymbol{x},\boldsymbol{z}) \leq d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$ (triangular inequality)

■

■ **Example 1.10**

1. Let $X = \mathbb{R}^n$, with

$$d_2(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

$$d_\infty(\boldsymbol{x},\boldsymbol{y}) = \max_{i=1,\dots,n} |x_i - y_i|$$

2. Let X be any set, and define the discrete metric

$$d(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Homework: Show that (1) and (2) defines a metric. ■

Definition 1.8 [Open Ball] An **open ball** of radius r centered at $\boldsymbol{x} \in X$ is the set

$$B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid d(\boldsymbol{x},\boldsymbol{y}) < r\}$$

■

■ **Example 1.11** 1. The set $B_1(0,0)$ defines an open ball under the metric $(X = \mathbb{R}^2, d_2)$, or the metric $(X = \mathbb{R}^2, d_\infty)$. The corresponding diagram is shown below:

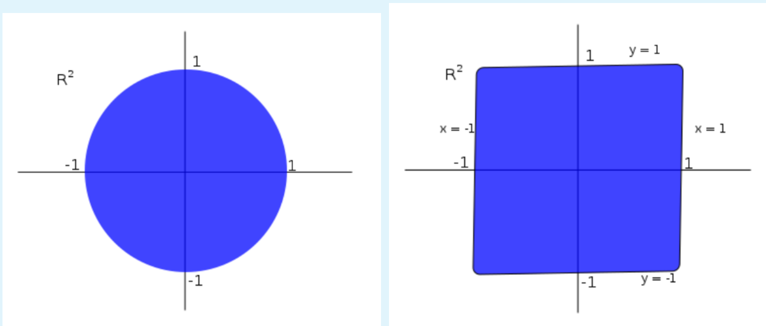


Figure 1.3: Left: under the metric $(X = \mathbb{R}^2, d_2)$; Right: under the metric $(X = \mathbb{R}^2, d_\infty)$

2. Under the metric $(X = \mathbb{R}^2, \text{discrete metric})$, the set $B_1(0,0)$ is one single point, also defines an open ball.

Definition 1.9 [Open Set] Let X be a metric space, $U \subseteq X$ is an open set in X if $\forall u \in U$, there exists $\epsilon_u > 0$ such that $B_{\epsilon_u}(u) \subseteq U$.

Definition 1.10 The **topology** induced from (X, d) is the collection of all open sets in (X, d), denoted as the symbol $\mathcal{T}$.

Proposition 1.5 All open balls $B_r(\boldsymbol{x})$ are open in (X, d).

Proof. Consider the example $X = \mathbb{R}$ with metric d_2. Therefore $B_r(x) = (x - r, x + r)$. Take $\boldsymbol{y} \in B_r(\boldsymbol{x})$ such that $d(\boldsymbol{x}, \boldsymbol{y}) = q < r$ and consider $B_{(r-q)/2}(\boldsymbol{y})$: for all $z \in B_{(r-q)/2}(\boldsymbol{y})$, we have

$$d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) < q + \frac{r-q}{2} < r,$$

which implies $\boldsymbol{z} \in B_r(x)$. ■

Proposition 1.6 Let $(X, \boldsymbol{d})$ be a metric space, and $\mathcal{T}$ is the topology induced from (X, d), then

1. let the set $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of (uncountable) open sets, i.e., $G_\alpha \in \mathcal{T}$,

then $\bigcup_{\alpha\in\mathcal{A}} G_\alpha \in \mathcal{T}$.

2. let $G_1,\ldots,G_n \in \mathcal{T}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$. The finite intersection of open sets is open.

Proof. 1. Take $x \in \bigcup_{\alpha\in\mathcal{A}} G_\alpha$, then $x \in G_\beta$ for some $\beta \in \mathcal{A}$. Since G_β is open, there exists $\epsilon_x > 0$ s.t.

$$B_{\epsilon_x}(x) \subseteq G_\beta \subseteq \bigcup_{\alpha\in\mathcal{A}} G_\alpha$$

2. Take $x \in \bigcap_{i=1}^n G_i$, i.e., $x \in G_i$ for $i = 1,\ldots,n$, i.e., there exists $\epsilon_i > 0$ such that $B_{\epsilon_i}(x) \subseteq G_i$ for $i = 1,\ldots,n$. Take $\epsilon = \min\{\epsilon_1,\ldots,\epsilon_n\}$, which implies

$$B_\epsilon(x) \subseteq B_{\epsilon_i}(x) \subseteq G_i, \forall i$$

which implies $B_\epsilon(x) \subseteq \bigcap_{i=1}^n G_i$

■

Exercise.

1. let $\mathcal{T}_2, \mathcal{T}_\infty$ be topologies induced from the metrices d_2, d_∞ in $\mathbb{R}^2$. Show that $J_2 = J_\infty$, i.e., every open set in $(\mathbb{R}^2, d_2)$ is open in $(\mathbb{R}^2, d_\infty)$, and every open set in $(\mathbb{R}^2, d_\infty)$ is open in $(\mathbb{R}_2, d_2)$.
2. Let $\mathcal{T}$ be the topology induced from the discrete metric (X, d_{discrete}). What is $\mathcal{T}$?

1.4. Wednesday for MAT3040

1.4.1. Review

1. Vector Space: e.g., $\mathbb{R}, M_{n\times n}(\mathbb{R}), \mathcal{C}(\mathbb{R}^n), \mathbb{R}[x]$.
2. Vector Subspace: $W \le V$, e.g.,
   (a) $V = \mathbb{R}^2$, the set $W := \mathbb{R}^2_+$ is not a vector subspace since W is not closed under scalar multiplication;
   (b) the set $W = \mathbb{R}^2_+ \bigcup \mathbb{R}^2_-$ is not a vector subspace since it is not closed under addition.
   (c) For $V = \mathbb{M}_{3\times 3}(\mathbb{R})$, the set of invertible 3×3 matrices is not a vector subspace, since we cannot define zero vector inside.
   (d) Exercise: How about the set of all singular matrices? Answer: it is not a vector subspace since the vector addition does not necessarily hold.

1.4.2. Spanning Set

Definition 1.11 [Span] Let V be a vector space over $\mathbb{F}$:

1. A linear combination of a subset S in V is of the form

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{s}_i, \quad \alpha_i \in \mathbb{F}, \boldsymbol{s}_i \in S$$

   Note that the summation should be finite.
2. The **span** of a subset $S \subseteq V$ is

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{n} \alpha_i \boldsymbol{s}_i \,\middle|\, \alpha_i \in \mathbb{F}, \boldsymbol{s}_i \in S \right\}$$

3. S is a spanning set of V, or say S spans V, if

$$\operatorname{span}(S) = V.$$

Example 1.12 For $V = \mathbb{R}[x]$, define the set

$$S = \{1, x^2, x^4, \dots, x^6\},$$

then $2 + x^4 + \pi x^{106} \in \text{span}(S)$, while the series $1 + x^2 + x^4 + \cdots \notin \text{span}(S)$.

It is clear that $\text{span}(S) \neq V$, but S is the spanning set of $W = \{p \in V \mid p(x) = p(-x)\}$.

Example 1.13 For $V = M_{3\times 3}(\mathbb{R})$, let $W_1 = \{\boldsymbol{A} \in V \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\}$ and $W_2 = \{\boldsymbol{B} \in V \mid \boldsymbol{B}^{\mathrm{T}} = -\boldsymbol{B}\}$ (the set of skew-symmetric matrices) be two vector subspaces. Define the set

$$\boldsymbol{S} := W_1 \bigcup W_2$$

Exercise: $\boldsymbol{S}$ spans V.

Proposition 1.7 Let S be a subset in a vector space V.

1. $S \subseteq \text{span}(S)$
2. $\text{span}(S) = \text{span}(\text{span}(S))$
3. If $\boldsymbol{w} \in \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$, then

$$\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{w}, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$$

Proof. 1. For each $\boldsymbol{s} \in S$, we have

$$\boldsymbol{s} = 1 \cdot \boldsymbol{s} \in \text{span}(S)$$

2. From (1), it's clear that $\text{span}(S) \subseteq \text{span}(\text{span}(S))$, and therefore suffices to show $\text{span}(\text{span}(S)) \subseteq \text{span}(S)$:

Pick $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i \in \text{span}(\text{span}(S))$, where $\boldsymbol{v}_i \in \text{span}(S)$. Rewrite

$$\boldsymbol{v}_i = \sum_{j=1}^{n_i} \beta_{ij} \boldsymbol{s}_j, \quad \boldsymbol{s}_j \in S,$$

which implies

$$\begin{aligned} \boldsymbol{v} &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n_i} \beta_{ij} \boldsymbol{s}_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n_i} (\alpha_i \beta_{ij}) \boldsymbol{s}_j, \end{aligned}$$

i.e., $\boldsymbol{v}$ is the finite combination of elements in S, whcih implies $\boldsymbol{v} \in \text{span}(S)$.

3. By hypothesis, $\boldsymbol{w} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n$ with $\alpha_1 \neq 0$, which implies

$$\boldsymbol{v}_1 = -\frac{\alpha_2}{\alpha_1} \boldsymbol{v}_2 + \cdots + \left(-\frac{1}{\alpha_1} \boldsymbol{w} \right)$$

which implies $\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{w}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. It suffices to show $\boldsymbol{v}_1 \notin \text{span}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. Suppose on the contrary that $\boldsymbol{v}_1 \in \text{span}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. It's clear that $\text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} = \text{span}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. (left as exercise). Therefore,

$$\emptyset = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \setminus \text{span}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\},$$

which is a contradiction.

∎

1.4.3. Linear Independence and Basis

Definition 1.12 [Linear Independence] Let S be a (not necessarily finite) subset of V. Then S is **linearly independent** (l.i.) on V if for any finite subset $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k\}$ in S,

$$\sum_{i=1}^{k} \alpha_i \boldsymbol{s}_i = 0 \Longleftrightarrow \alpha_i = 0, \forall i$$

■ **Example 1.14** For $V = \mathcal{C}(\mathbb{R})$,

1. let $S_1 = \{\sin x, \cos x\}$, which is l.i., since

$$\alpha \sin x + \beta \cos x = \mathbf{0}(\text{means zero function})$$

   Taking $x = 0$ both sides leads to $\beta = 0$; taking $x = \frac{\pi}{2}$ both sides leads to $\alpha = 0$.

2. let $S_2 = \{\sin^2 x, \cos^2 x, 1\}$, which is linearly dependent, since

$$1 \cdot \sin^2 x + 1 \cdot \cos^2 x + (-1) \cdot 1 = 0, \forall x$$

3. Exercise: For $V = \mathbb{R}[x]$, let $S = \{1, x, x^2, x^3, \dots, \}$, which is l.i.:

   Pick $x^{k_1}, \dots, x^{k_n} \in S$ with $k_1 < \cdots < k_n$. Consider that the euqation

$$\alpha_1 x^{k_1} + \cdots + \alpha_n x^{k_n} = \mathbf{0}$$

   holds for all x, and try to solve for $\alpha_1, \dots, \alpha_n$ (one way is differentation.)

■

Definition 1.13 [Basis] A subset S is a **basis** of V if

(a) S spans V;

(b) S is l.i.

■

■ **Example 1.15**

1. For $V = \mathbb{R}^n$, $S = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is a basis of V
2. For $V = \mathbb{R}[x]$, $S = \{1, x, x^2, \dots\}$ is a basis of V
3. For $V = M_{2\times 2}(\mathbb{R})$,

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis of V

Ⓡ Note that there can be many basis for a vector space V.

Proposition 1.8 Let $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, then there exists a subset of $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$, which is a basis of V.

Proof. If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ is l.i., the proof is complete.

Suppose not, then $\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_m \boldsymbol{v}_m = \boldsymbol{0}$ has a non-trivial solution. w.l.o.g., $\alpha_1 \neq 0$, which implies

$$\boldsymbol{v}_1 = -\frac{\alpha_2}{\alpha_1}\boldsymbol{v}_2 + \cdots + \left(\frac{\alpha_m}{\alpha_1}\right)\boldsymbol{v}_m \implies \boldsymbol{v}_1 \in \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$$

By the proof in (c), Proposition (1.7),

$$\text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\} = \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_m\},$$

which implies $V = \text{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$.

Continuse this argument finitely many times to guarantee that $\{\boldsymbol{v}_i, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_m\}$ is l.i., and spans V. The proof is complete. ■

Corollary 1.1 If $V = \text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ (i.e., V is finitely generated), then V has a basis. (The same holds for non-finitely generated V).

Proposition 1.9 If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is a basis of V, then every $\boldsymbol{v} \in V$ can be expressed uniquely as

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n$$

Proof. Since $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ spans V, so $\boldsymbol{v} \in V$ can be written as

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n \tag{1.1}$$

Suppose further that

$$\boldsymbol{v} = \beta_1 \boldsymbol{v}_1 + \cdots + \beta_n \boldsymbol{v}_n, \tag{1.2}$$

it suffices to show that $\alpha_i = \beta_i$ for $\forall i$:

Subtracting (1.1) into (1.2) leads to

$$(\alpha_1 - \beta_1)\boldsymbol{v}_1 + \cdots + (\alpha_n - \beta_n)\boldsymbol{v}_n = 0.$$

By the hypothesis of linear independence, we have $\alpha_i - \beta_i = 0$ for $\forall i$, i.e., $\alpha_i = \beta_i$. ■

1.5. Wednesday for MAT3006

Reviewing.

- Normed Space: a norm on a vector space
- Metric Space
- Open Ball

1.5.1. Convergence of Sequences

Since $\mathbb{R}^n$ and $C[a,b]$ are both metric spaces, we can study the convergence in $\mathbb{R}^n$ and the functions defined on $[a,b]$ at the same time.

Definition 1.14 [Convergence] Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is **convergent** to x if $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon, \forall n \geq N.$$

We can denote the convergence by

$$x_n \to x, \quad \text{or} \quad \lim_{n\to\infty} x_n = x, \quad \text{or} \quad \lim_{n\to\infty} d(x_n, x) = 0$$

Proposition 1.10 If the limit of $\{x_n\}$ exists, then it is unique.

R Note that the proposition above does not necessarily hold for topology spaces.

Proof. Suppose $x_n \to x$ and $x_n \to y$, which implies

$$0 \leq d(x,y) \leq d(x,x_n) + d(x_n,y), \forall n$$

Taking the limit $n \to \infty$ both sides, we imply $d(x,y) = 0$, i.e., $x = y$. ■

■ **Example 1.16**

1. Consider the metric space $(\mathbb{R}^k, d_\infty)$ and study the convergence

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x} \iff \lim_{n\to\infty}\left(\max_{i=1\dots,k} |x_{n_i} - x_i|\right) = 0$$
$$\iff \lim_{n\to\infty} |x_{n_i} - x_i| = 0, \forall i = 1,\dots,k$$
$$\iff \lim_{n\to\infty} x_{n_i} = x_i,$$

i.e., the convergence defined in d_∞ is the same as the convergence defined in d_2.

2. Consider the convergence in the metric space $(\mathcal{C}[a,b], d_\infty)$:

$$\lim_{n\to\infty} f_n = f \iff \lim_{n\to\infty}\left(\max_{[a,b]} |f_n(x) - f(x)|\right) = 0$$
$$\iff \forall \varepsilon > 0, \forall x \in [a,b], \exists N_\varepsilon \text{ such that } |f_n(x) - f(x)| < \varepsilon, \forall n \geq N_\varepsilon$$

which is equivalent to the uniform convergence of functions, i.e., the convergence defined in d_2.

■

Definition 1.15 [Equivalent metrics] Let d and ρ be metrics on X.

1. We say ρ is **stronger** than d (or d is **weaker** than ρ) if

$$\exists K > 0 \text{ such that } d(x,y) \leq K\rho(x,y), \forall x,y \in X$$

2. The metrics d and ρ are equivalent if there exists $K_1, K_2 > 0$ such that

$$d(x,y) \leq K_1\rho(x,y) \leq K_2 d(x,y)$$

■

R The strongerness of ρ than d is depicted in the graph below:

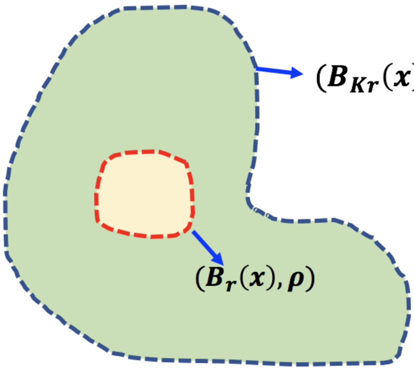


Figure 1.4: The open ball $(B_r(x), \rho)$ is contained by the open ball $(B_{Kr}(x), d)$

For each $x \in X$, consider the open ball $(B_r(x), \rho)$ and the open ball $(B_{Kr}(x), d)$:

$$B_r(x) = \{y \mid \rho(x,y) < r\}, \quad B_{Kr}(x) = \{z \mid d(x,z) < Kr\}.$$

For $y \in (B_r(x), \rho)$, we have $d(x,y) < K\rho(x,y) < Kr$, which implies $y \in (B_{Kr}(x), d)$, i.e, $(B_r(x), \rho) \subseteq (B_{Kr}(x), d)$ for any $x \in X$ and $r > 0$.

■ **Example 1.17** 1. d_1, d_2, d_∞ in $\mathbb{R}^n$ are equivalent

$$d_1(\boldsymbol{x}, \boldsymbol{y}) \le d_\infty(\boldsymbol{x}, \boldsymbol{y}) \le n d_1(\boldsymbol{x}, \boldsymbol{y})$$
$$d_2(\boldsymbol{x}, \boldsymbol{y}) \le d_\infty(\boldsymbol{x}, \boldsymbol{y}) \le \sqrt{n} d_2(\boldsymbol{x}, \boldsymbol{y})$$

We use two relation depiected in the figure below to explain these two inequalities:

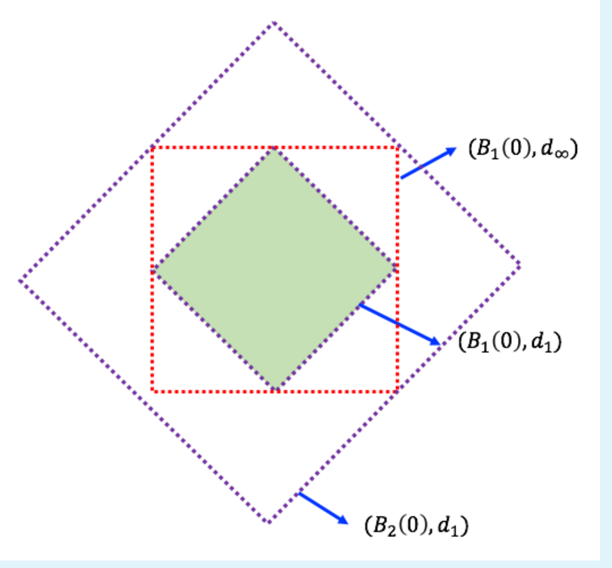


Figure 1.5: The diagram for the relation $(B_1(x), d_1) \subseteq (B_\infty(x), d_\infty) \subseteq (B_2(x), d_1)$ on $\mathbb{R}^2$

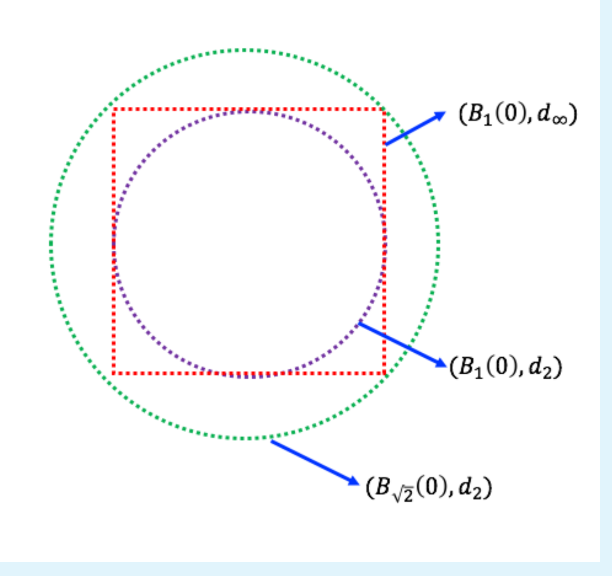


Figure 1.6: The diagram for the relation $(B_1(x), d_2) \subseteq (B_\infty(x), d_\infty) \subseteq (B_{\sqrt{2}}(x), d_2)$ on $\mathbb{R}^2$

It's easy to conclude the simple generalization for example (1.16):

Proposition 1.11 If d and ρ are equivalent, then

$$\lim_{n\to\infty} d(x_n, x) = 0 \iff \lim_{n\to\infty} \rho(x_n, x) = 0$$

Note that this does not necessarily hold for topology spaces.

2. Consider d_1, d_∞ in $\mathcal{C}[a,b]$:

$$d_1(f,g) := \int_a^b |f-g| \, \mathrm{d}x \le \int_a^b \sup_{[a,b]} |f-g| \, \mathrm{d}x = (b-a) d_\infty(f,g),$$

i.e., d_∞ is stronger than d_1. Question: Are they equivalent? **No**.

Justification. Consider $f_n(x) = n^2 x^n (1-x)$. Check that

$$\lim_{n\to\infty} d_1(f_n(x), 1) = 0, \quad \text{but } d_\infty(f_n(x), 1) \to \infty$$

The peak of f_n may go to infinite, while the integration converges to zero. Therefore d_1 and d_∞ have different limits. We will discuss this topic at Lebsegue integration again. ■

1.5.2. Continuity

Definition 1.16 [Continuity] Let $f : (X,d) \to (Y,d)$ be a function and $x_0 \in X$. Then f is continuous at x_0 if $\forall \varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon$$

The function f is continuous in X if f is continuous for all $x_0 \in X$.

Proposition 1.12 The function f is continuous at x if and only if for all $\{x_n\} \to x$ under d, $f(x_n) \to f(x)$ under ρ.

Proof. *Necessity:* Given $\varepsilon > 0$, by continuity,

$$d(x, x') < \delta \implies \rho(f(x'), f(x)) < \varepsilon. \tag{1.3}$$

Consider the sequence $\{x_n\} \to x$, then there exists N such that $d(x_n, x) < \delta$ for $\forall n \ge N$. By applying (1.3), $\rho(f(x_n), f(x)) < \varepsilon$ for $\forall n \ge N$, i.e., $f(x_n) \to f(x)$.

Sufficiency: Assume that f is not continuous at x, then there exists ε_0 such that for $\delta_n = \frac{1}{n}$, there exists x_n such that

$$d(x_n, x) < \delta_n, \text{ but } \rho(f(x_n), f(x)) > \varepsilon_0.$$

Then $\{x_n\} \to x$ by our construction, while $\{f(x_n)\}$ does not converge to $f(x)$, which is a contradiction. ■

Corollary 1.2 If the function $f : (X,d) \to (Y,\rho)$ is continuous at x, the function $g : (Y,\rho) \to (Z,m)$ is continuous at $f(x)$, then $g \circ f : (X,d) \to (Z,m)$ is continuous at x.

Proof. Note that

$$\{x_n\} \to x \overset{(a)}{\Longrightarrow} \{f(x_n)\} \to f(x) \overset{(b)}{\Longrightarrow} \{g(f(x_n))\} \to g(f(x)) \overset{(c)}{\Longrightarrow} g \circ f \text{ is continuous at } x.$$

where $(a),(b),(c)$ are all by proposition (1.12). ■

1.5.3. Open and Closed Sets

We have open/closed intervals in $\mathbb{R}$, and they are important in some theorems (e.g, continuous functions bring closed intervals to closed intervals).

Definition 1.17 [Open] Let (X,d) be a metric space. A set $U \subseteq X$ is open if for each $x \in U$, there exists $\rho_x > 0$ such that $B_{\rho_x}(x) \subseteq U$. The empty set $\emptyset$ is defined to be open.

■

■ **Example 1.18** Let $(\mathbb{R}, d_2$ or $d_\infty)$ be a metric space. The set $U = (a,b)$ is open. ■

Proposition 1.13
1. Let (X,d) be a metric space. Then all open balls $B_r(x)$ are open
2. All open sets in X can be written as a union of open balls.

Proof. 1. Let $y \in B_r(x)$, i.e., $d(x,y) := q < r$. Consider the open ball $B_{(r-q)/2}(y)$. It suffices to show $B_{(r-q)/2}(y) \subseteq B_r(x)$. For any $z \in B_{(r-q)/2}(y)$,

$$d(x,z) \leq d(x,y) + d(y,z) < q + \frac{r-q}{2} = \frac{r+q}{2} < r.$$

The proof is complete.

2. Let $U \subseteq X$ be open, i.e., for $\forall x \in U$, there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq U$. Therefore

$$\{x\} \subseteq B_{\varepsilon_x}(x) \subseteq U, \forall x \in U$$

which implies

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_{\varepsilon_x}(x) \subseteq U,$$

i.e., $U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$.

■

1.6. Wednesday for MAT4002

Reviewing.

- Metric Space (X,d)
- Open balls and open sets (note that the emoty set $\emptyset$ is open)
- Define the collection of open sets in X, say $\mathcal{T}$ is the topology.

Exercise.

1. Show that the $\mathcal{T}_2$ under $(X=\mathbb{R}^2,d_2)$ and $\mathcal{T}_\infty$ under $(X=\mathbb{R}^2,d_\infty)$ are the same.

   Ideas. Follow the procedure below:

   An open ball in d_2-metric is open in d_∞;

   Any open set in d_2-metric is open in d_∞;

   Switch d_2 and d_∞. ■

2. Describe the topology $\mathcal{T}_{\text{discrete}}$ under the metric space $(X=\mathbb{R}^2,d_{\text{discrete}})$.

   Outlines. Note that $\{x\}=B_{1/2}(x)$ is an open set.

   For any subset $W\subseteq\mathbb{R}^2$, $W=\bigcup_{w\in W}\{w\}$ is open.

   Therefore $\mathcal{T}_{\text{discrete}}$ is all subsets of $\mathbb{R}^2$. ■

1.6.1. Forget about metric

Next, we will try to define closedness, compactness, etc., without using the tool of metric:

Definition 1.18 [closed] A subset $V\subseteq X$ is **closed** if $X\setminus V$ is **open**.

■ **Example 1.19** Under the metric space $(\mathbb{R},d_1)$,

$$\mathbb{R}\setminus[b,a]=(a,\infty)\bigcup(-\infty,b)\text{ is open}\implies[b,a]\text{ is closed}$$

Proposition 1.14 Let X be a metric space.

1. $\emptyset, X$ is closed in X
2. If F_α is closed in X, so is $\bigcap_{\alpha \in A} F_\alpha$.
3. If $F_1, \dots, F_k$ is closed, so is $\bigcup_{i=1}^{k} F_i$.

Proof. 1. Note that X is open in X, which implies $\emptyset = X \setminus X$ is closed in X; Similarly, $\emptyset$ is open in X, which implies $X = X \setminus \emptyset$ is closed in X;

2. The set F_α is closed implies there exists open $U_\alpha \subseteq X$ such that $F_\alpha = X \setminus U_\alpha$. By De Morgan's Law,

$$\bigcap_{\alpha \in A} F_\alpha = \bigcap_{\alpha \in A} (X \setminus U_\alpha) = X \setminus (\bigcup_{\alpha \in A} U_\alpha).$$

By part (a) in proposition (1.6), the set $\bigcup_{\alpha \in A} U_\alpha$ is openm which implies $\bigcap_{\alpha \in A} F_\alpha$ is closed.

3. The result follows from part (b) in proposition (1.6) by taking complements.

■

We illustrate examples where open set is used to define convergence and continuity.

1. Convergence of sequences:

   Definition 1.19 [Convergence] Let (X, d) be a metric space, then $\{x_n\} \to x$ means

   $$\forall \varepsilon > 0, \exists N \text{ such that } d(x_n, x) < \varepsilon, \forall n \geq N.$$

   We will study the convergence by using open sets instead of metric.

   Proposition 1.15 Let X be a metric space, then $\{x_n\} \to x$ if and only if for $\forall$ open set $U \ni x$, there exists N such that $x_n \in U$ for $\forall n \geq N$.

   Proof. Necessity: Since $U \ni x$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Since $\{x_n\} \to x$, there exists N such that $d(x_n, x) < \varepsilon$, i.e., $x_n \in B_\varepsilon(x) \subseteq U$ for $\forall n \geq N$.

Sufficiency: Let $\varepsilon > 0$ be given. Take the open set $U = B_\varepsilon(x) \ni x$, then there exists N such that $x_n \in U = B_\varepsilon(x)$ for $\forall n \geq N$, i.e., $d(x_n, x) < \varepsilon$, $\forall n \geq N$.

■

2. Continuity:

Definition 1.20 [Continuity] Let (X, d) and (Y, ρ) be given metric spaces. Then $f : X \to Y$ is continuous at $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(x, x_0) < \delta \implies \rho(f(x), f(x_0)) < \varepsilon.$$

The function f is continuous on X if f is continous for all $x_0 \in X$.

We can get rid of metrics to study continuity:

Proposition 1.16 (a) The function f is continuous at x if and only if for all open $U \ni f(x)$, there exists $\delta > 0$ such that the set $B(x, \delta) \subseteq f^{-1}(U)$.

(b) The function f is continuous on X if and only if $f^{-1}(U)$ is open in X for each open set $U \subseteq Y$.

During the proof we will apply a small lemma:

Proposition 1.17 f is continuous at x if and only if for all $\{x_n\} \to x$, we have $\{f(x_n)\} \to f(x)$.

Proof. (a) *Necessity*:

Due to the openness of $U \ni f(x)$, there exists a ball $B(f(x), \varepsilon) \subseteq U$.

Due to the continuity of f at x, there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$, which implies

$$f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq U,$$

which implies $B(x, \delta) \subseteq f^{-1}(U)$.

Sufficiency:

Let $\{x_n\} \to x$. It suffices to show $\{f(x_n)\} \to f(x)$. For each open $U \ni f(x)$, by hypothesis, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$.

Since $\{x_n\} \to x$, there exists N such that

$$x_n \in B_\delta(x) \subseteq f^{-1}(U), \forall n \geq N \implies f(x_n) \in U, \forall n \geq N$$

Let $\varepsilon > 0$ be given, and then construct the $U = B_\varepsilon(f(x))$. The argument above shows that $f(x_n) \in B_\varepsilon(f(x))$ for $\forall n \geq N$, which implies $\rho(f(x_n), f(x)) < \varepsilon$, i.e., $\{f(x_n)\} \to f(x)$.

(b) For the forward direction, it suffices to show that each point x of $f^{-1}(U)$ is an interior point of $f^{-1}(U)$, which is shown by part (a); the converse follows trivially by applying (a).

■

R As illustracted above, convergence, continuity, (and compactness) can be defined by using open sets $\mathcal{T}$ only.

1.6.2. Topological Spaces

Definition 1.21 A **topological space** $(X, \mathcal{T})$ consists of a (non-empty) set X, and a family of subsets of X ("open sets" $\mathcal{T}$) such that

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T}$ implies $U \bigcap V \in \mathcal{T}$
3. If $U_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$.

The elements in $\mathcal{T}$ are called **open subsets** of X. The $\mathcal{T}$ is called a **topology** on X. ■

■ **Example 1.20** 1. Let (X, d) be any metric space, and

$$\mathcal{T} = \{\text{all open subsets of } X\}$$

It's clear that $\mathcal{T}$ is a topology on X.

2. Define the discrete topology

$$\mathcal{T}_{\text{dis}} = \{\text{all subsets of } X\}$$

It's clear that $\mathcal{T}_{\text{dis}}$ is a topology on X, (which also comes from the discrete metric (X, d_{discrete})).

R We say $(X, \mathcal{T})$ is induced from a metric (X, d) (or it is **metrizable**) if $\mathcal{T}$ is the faimly of open subsets in (X, d).

3. Consider the indiscrete topology $(X, \mathcal{T}_{\text{indis}})$, where X contains more than one element:

$$\mathcal{T}_{\text{indis}} = \{\emptyset, X\}.$$

Question: is $(X, \mathcal{T}_{\text{indis}})$ metrizable? No. For any metric d defined on X, let x, y be distinct points in X, and then $\varepsilon := d(x, y) > 0$, hence $B_{\frac{1}{2}\varepsilon}(x)$ is a open set belonging to the corresponding induced topology. Since $x \in B_{\frac{1}{2}\varepsilon}(x)$ and $y \notin B_{\frac{1}{2}\varepsilon}(y)$, we conclude that $B_{\frac{1}{2}\varepsilon}(x)$ is neither $\emptyset$ nor X, i.e., the topology induced by any metric d is not the indiscrete topology.

4. Consider the cofinite topology $(X, \mathcal{T}_{\text{cofin}})$:

$$\mathcal{T}_{\text{cofin}} = \{U \mid X \setminus U \text{ is a finite set}\} \bigcup \{\emptyset\}$$

Question: is $(X, \mathcal{T}_{\text{cofin}})$ metrizable?

Definition 1.22 [Equivalence] Two metric spaces are **topologically equivalent** if they give rise to the same topology.

Example 1.21 Metrics d_1, d_2, d_∞ in $\mathbb{R}^n$ are topologically equivalent.

1.6.3. Closed Subsets

Definition 1.23 [Closed] Let $(X,\mathcal{T})$ be a topology space. Then $V \subseteq X$ is **closed** if $X \setminus V \in J$

■ **Example 1.22** Under the topology space $(\mathbb{R},\mathcal{T}_{\text{usual}})$, $(b,\infty)\bigcup(-\infty,a) \in \mathcal{T}$. Therefore,

$$[a,b] = \mathbb{R} \setminus \Big((b,\infty)\bigcup(-\infty,a)\Big)$$

is closed in $\mathbb{R}$ under usual topology.

R It is important to say that V is **closed in** X**.** You need to specify the underlying the space X.