

# Chapter 1

## Week1

### 1.1. Tuesday

#### 1.1.1. Difference between ODE and SDE

We first discuss the difference between deterministic differential equations and stochastic ones by considering several real-life problems.

**Problem 1: Population Growth Model.** Consider the first-order ODE

$$\begin{cases} \frac{dN(t)}{dt} = a(t)N(t) \\ N(0) = N_0 \end{cases}$$

where  $N(t)$  denotes the **size** of the population at time  $t$ ;  $a(t)$  is the given (deterministic) function describing the **rate** of growth of population at time  $t$ ; and  $N_0$  is a given constant.

If  $a(t)$  is not completely known, e.g.,

$$a(t) = r(t) \cdot \text{noise}, \text{ or } r(t) + \text{noise},$$

with  $r(t)$  being a deterministic function of  $t$ , and the “noise” term models something random. The question arises: How to *rigorously* describe the “noise” term and solve it?

**Problem 2: Electric Circuit.** Let  $Q(t)$  denote the charge at time  $t$  in an electrical circuit, which admits the following ODE:


$$\begin{cases} LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = F(t), \\ Q(0) = Q_0, \quad Q'(0) = Q'_0 \end{cases}$$

where  $L$  denotes the inductance,  $R$  denotes the resistance,  $C$  denotes the capacity, and  $F(t)$  denotes the potential source.

Now consider the scenario where  $F(t)$  is not completely known, e.g.,

$$F(t) = G(t) + \text{noise}$$

where  $G(t)$  is deterministic. The question is how to solve the problem.

 The differential equations above involving non-deterministic coefficients are called the **stochastic differential equations** (SDEs). Clearly, the solution to an SDE should involve the randomness.

## 1.1.2. Applications of SDE

Now we discuss some applications of SDE shown in the finance area.

**Problem 3: Optimal Stopping Problem.** Suppose someone holds an asset (e.g., stock, house). He plans to sell it at some future time. Denote  $X(t)$  as the price of the asset at time  $t$ , satisfying the following dynamics:

$$\frac{dX(t)}{dt} = rX(t) + \alpha X(t) \cdot \text{noise}$$

where  $r, \alpha$  are given constants. The goal of this person is to maximize the expected selling price:

$$\sup_{\tau \geq 0} \mathbb{E}[X(\tau)]$$

where the optimal solution  $\tau^*$  is the optimal stopping time.

**Problem 4: Portfolio Selection Problem.** Suppose a person is interested in two types of assets:

- A risk-free asset which generates a deterministic return  $\rho$ , whose price  $X_1(t)$  follows a deterministic dynamics

$$\frac{dX_1(t)}{dt} = \rho X_1(t),$$

- A risky asset whose price  $X_2(t)$  satisfies the following SDE:

$$\frac{dX_2(t)}{dt} = \mu X_2(t) + \sigma X_2(t) \cdot \text{noise}$$

where  $\mu, \sigma > 0$  are given constants.

The policy of the investment is as follows. The wealth at time  $t$  is denoted as  $v(t)$ . This person decides to invest the fraction  $u(t)$  of his wealth into the risky asset, with the remaining  $1 - u(t)$  part to be invested into the safe asset. Suppose that the utility function for this person is  $U(\cdot)$ , and his goal is to maximize the expected total wealth at the terminal time  $T$ :

$$\max_{u(t), 0 \leq t \leq T} \mathbb{E}[U(v^u(T))]$$

where the decision variable is the portfolio function  $u(t)$  along whole horizon  $[0, T]$ .

**Problem 5: Option Pricing Problem.** The financial derivatives are products in the market whose value depends on the underlying asset. The European call option is a typical financial derivative. Suppose that the underlying asset is stock  $A$ , whose price at time  $t$  is  $X(t)$ . Then the call option gives the option holder the right (not the obligation) to buy one unit of stock  $A$  at a specified price (strike price)  $K$  at maturity date  $T$ . The task is to inference the fair price of the option at the current time. The formula for the price of the option is the following:

$$c_0 = \mathbb{E}[(X(T) - K)^+]$$

which is the famous Black-Sholes-Merton Formula.

### 1.1.3. Reviewing for Probability Space

Firstly, we review some basic concepts in real analysis.

**Definition 1.1** [ $\sigma$ -Algebra] A set  $\mathcal{F}$  containing subsets of  $\Omega$  is called a  $\sigma$ -algebra if:

1.  $\Omega \in \mathcal{F}$ ;
2.  $\mathcal{F}$  is closed under complement, i.e.,  $A \in \mathcal{F}$  implies  $\Omega \setminus A \in \mathcal{F}$ ;
3.  $\mathcal{F}$  is closed under countably union operation, i.e.,  $A_i \in \mathcal{F}, i \geq 1$  implies  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ .

**Definition 1.2** [Probability Measure] A function  $\mathbb{P} : \mathcal{F} \rightarrow \mathbb{R}$  is called a **probability measure** on  $(\Omega, \mathcal{F})$  if

- $\mathbb{P}(\Omega) = 1$ ;
- $\mathbb{P}(A) \geq 0, \forall A \in \mathcal{F}$ ;
- $\mathbb{P}$  is  $\sigma$ -additive, i.e., when  $A_i \in \mathcal{F}, i \geq 1$  and  $A_i \cap A_j = \emptyset, \forall i \neq j$ ,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

where  $\mathbb{P}(A)$  is called the **probability of the event**  $A$ .

**Definition 1.3** [Probability Space] A probability space is a triplet  $(\Omega, \mathcal{F}, \mathbb{P})$  defined as follows:

1.  $\Omega$  denotes the **sample space**, and a point  $\omega \in \Omega$  is called a sample point;
2.  $\mathcal{F}$  is a  $\sigma$ -algebra of  $\Omega$ , which is a collection of subsets in  $\Omega$ . The element  $A \in \mathcal{F}$  is called an “event”; and
3.  $\mathbb{P}$  is a probability measure defined in the space  $(\Omega, \mathcal{F})$ .

**Definition 1.4** [Almost Surely True] A statement  $S$  is said to be **almost surely (a.s.) true** or **true with probability 1**, if

- $\mathcal{B} := \{w : S(w) \text{ is true}\} \in \mathcal{F}$
- $\mathbb{P}(F) = 1$ .

■

**Definition 1.5** [Topological Space] A **topological space**  $(X, \mathcal{T})$  consists of a (non-empty) set  $X$ , and a family of subsets of  $X$  ("open sets"  $\mathcal{T}$ ) such that

1.  $\emptyset, X \in \mathcal{T}$
2.  $U, V \in \mathcal{T}$  implies  $U \cap V \in \mathcal{T}$
3. If  $U_\alpha \in \mathcal{T}$  for all  $\alpha \in \mathcal{A}$ , then  $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$ .

When  $A \in \mathcal{T}$ ,  $A$  is called the open subset of  $X$ . The  $\mathcal{T}$  is called a **topology** on  $X$ .

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**Definition 1.6** [Borel  $\sigma$ -Algebra] Consider a topological space  $\Omega$ , with  $\mathcal{U}$  being the topology of  $\Omega$ . The **Borel  $\sigma$ -Algebra**  $\mathcal{B}(\Omega)$  on  $\Omega$  is defined to be the *minimal*  $\sigma$ -algebra containing  $\mathcal{U}$ :

$$\mathcal{B}(\Omega) \triangleq \sigma(\mathcal{U}).$$

Any element  $B \in \mathcal{B}(\Omega)$  is called the **Borel set**.

■

**Definition 1.7** [ $\mathcal{F}$ -Measurable / Random Variable]

1. A function  $f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  is called  **$\mathcal{F}$ -measurable** if

$$f^{-1}(\mathbf{B}) = \{w \mid f(w) \in \mathbf{B}\} \in \mathcal{F},$$

for any  $\mathbf{B} \in \mathcal{B}(\mathbb{R}^n)$ .

2. A random variable  $X$  is a function  $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  and is  $\mathcal{F}$ -measurable.

■

**Definition 1.8** [Generated  $\sigma$ -Algebra] Suppose  $X$  is a random variable on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Then the  $\sigma$ -algebra generated by  $X$ , say  $\mathcal{H}_X$  is defined to be the **minimal  $\sigma$ -algebra** on  $\Omega$  to make  $X$  measurable. ■

**Proposition 1.1**  $\mathcal{H}_X = \{X^{-1}(\mathbf{B}) : \mathbf{B} \in \mathcal{B}(\mathbb{R}^n)\}$ .

*Proof.* Since  $X$  is  $\mathcal{H}_X$ -measurable, for any  $\mathbf{B} \in \mathcal{B}(\mathbb{R}^n)$ ,  $X^{-1}(\mathbf{B}) \in \mathcal{H}_X$ . Thus  $\mathcal{H}_X \supseteq \{X^{-1}(\mathbf{B}) : \mathbf{B} \in \mathcal{B}(\mathbb{R}^n)\}$ . It suffices to show that  $\{X^{-1}(\mathbf{B}) : \mathbf{B} \in \mathcal{B}(\mathbb{R}^n)\}$  is a  $\sigma$ -algebra to finish the proof, which is true since  $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{U})$ , with  $\mathcal{U}$  being the topology of  $X$ . ■