



香港中文大學(深圳)
The Chinese University of Hong Kong, Shenzhen

Advanced Linear Algebra

MAT3040 Notebook

The First Edition

A FIRST COURSE
IN
ADVANCED LINEAR ALGEBRA

A FIRST COURSE IN ADVANCED LINEAR ALGEBRA MAT3040 Notebook

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Acknowledgments

This book is taken notes from the MAT3040 in spring semester, 2019. These lecture notes were taken and compiled in \LaTeX by Jie Wang, an undergraduate student in spring 2019. The tex writter would like to thank Prof. Daniel Wong and some students for their detailed and valuable comments and suggestions, which significantly improved the quality of this notebook. Students taking this course may use the notes as part of their reading and reference materials. This version of the lecture notes were revised and extended for many times, but may still contain many mistakes and typos, including English grammatical and spelling errors, in the notes. It would be greatly appreciated if those students, who will use the notes as their reading or reference material, tell any mistakes and typos to Jie Wang for improving this notebook.

Notations and Conventions

\mathbb{F}^n	n -dimensional \mathbb{F} -valued space
$M_{m \times n}(\mathbb{F})$	set of all $m \times n$ \mathbb{F} -valued matrices
\oplus	Direct Sum
$\ker(T)$	The null space of T
$V \cong W$	vector spaces V and W are isomorphic
$(T)_{\mathcal{B}, \mathcal{A}}$	Matrix representation of T w.r.t. \mathcal{A} and \mathcal{B}
$\mathbf{v} + W$	coset of \mathbf{v} , i.e., $\{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in W\}$
\mathbf{a}_i^T	i th row of matrix \mathbf{A}
V/W	Quotient space of V by the subspace W
V^*	Dual space of V , i.e., the set of linear transformations from V to \mathbb{F}
$\text{Ann}(S)$	The annihilator of $S \subseteq V$, i.e., $\{f \in V^* \mid f(s) = 0, \forall s \in S\}$
T^*	Adjoint map $T^* : W^* \rightarrow V^*$ for the mapping $T : V \rightarrow W$
\mathbf{A}^H	Hermitian transpose of \mathbf{A} , i.e, $\mathbf{B} = \mathbf{A}^H$ means $b_{ji} = \bar{a}_{ij}$ for all i, j
$\chi_T(x)$	characteristic polynomial of T
$m_T(x)$	Minimal polynomial of the linear operator T
$m_{T, \mathbf{v}}(x)$	Minimal polynomial of a vector \mathbf{v} relative to T
T'	Hermitian Adjoint map $T' : V \rightarrow V$ for the mapping $T : V \rightarrow V$
$\langle \mathbf{v}, \mathbf{w} \rangle$	Inner product between vectors \mathbf{v} and \mathbf{w}
$V \otimes W$	Tensor product between vector spaces V and W
$V \wedge W$	Wedge product between vector spaces V and W

Chapter 1

Week1

1.1. Monday for MAT3040

1.1.1. Introduction to Advanced Linear Algebra

Advanced Linear Algebra is one of the most important course in MATH major, with pre-request MAT2040. This course will offer the really linear algebra knowledge.

What the content will be covered?.

- In MAT2040 we have studied the space \mathbb{R}^n ; while in MAT3040 we will study the general vector space V .
- In MAT2040 we have studied the *linear transformation* between Euclidean spaces, i.e., $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$; while in MAT3040 we will study the linear transformation from vector spaces to vector spaces: $T : V \rightarrow W$
- In MAT2040 we have studied the eigenvalues of $n \times n$ matrix \mathbf{A} ; while in MAT3040 we will study the eigenvalues of a **linear operator** $T : V \rightarrow V$.
- In MAT2040 we have studied the dot product $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$; while in MAT3040 we will study the **inner product** $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$.

Why do we do the generalization?. We are studying many other spaces, e.g., $C(\mathbb{R})$ is called the space of all functions on \mathbb{R} , $C^\infty(\mathbb{R})$ is called the space of all infinitely differentiable functions on \mathbb{R} , $\mathbb{R}[x]$ is the space of polynomials of one-variable.

- **Example 1.1** 1. Consider the Laplace equation $\Delta f = 0$ with linear operator Δ :

$$\Delta : C^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3) \quad f \mapsto \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

The solution to the PDE $\Delta f = 0$ is the 0-eigenspace of Δ .

2. Consider the Schrödinger equation $\hat{H}f = Ef$ with the linear operator

$$\hat{H} : C^\infty(\mathbb{R}^3) \rightarrow C^\infty(\mathbb{R}^3), \quad f \mapsto \left[\frac{-\hbar^2}{2\mu} \nabla^2 + V(x, y, z) \right] f$$

Solving the equation $\hat{H}f = Ef$ is equivalent to finding the eigenvectors of \hat{H} . In fact, the eigenvalues of \hat{H} are **discrete**.

1.1.2. Vector Spaces

Definition 1.1 [Vector Space] A **vector space** over a field \mathbb{F} (in particular, $\mathbb{F} = \mathbb{R}$ or \mathbb{C}) is a set of objects V equipped with vector addition and scalar multiplication such that

1. the vector addition $+$ is closed with the rules:

- (a) **Commutativity**: $\forall \mathbf{v}_1, \mathbf{v}_2 \in V, \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$.
- (b) **Associativity**: $\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3$.
- (c) **Additive Identity**: $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}, \forall \mathbf{v} \in V$.

2. the **scalar multiplication** is closed with the rules:

- (a) **Distributive**: $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2, \forall \alpha \in \mathbb{F} \text{ and } \mathbf{v}_1, \mathbf{v}_2 \in V$
- (b) **Distributive**: $(\alpha_1 + \alpha_2)\mathbf{v} = \alpha_1\mathbf{v} + \alpha_2\mathbf{v}$
- (c) **Compatibility**: $a(b\mathbf{v}) = (ab)\mathbf{v}$ for $\forall a, b \in \mathbb{F} \text{ and } \mathbf{v} \in V$.
- (d) $0\mathbf{v} = \mathbf{0}, 1\mathbf{v} = \mathbf{v}$.

Here we study several examples of vector spaces:

■ **Example 1.2** For $V = \mathbb{F}^n$, we can define

1. Addictive Identity:

$$\mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

2. Scalar Multiplication:

$$\alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

3. Vector Addiction:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

■ **Example 1.3** 1. It is clear that the set $V = M_{n \times n}(\mathbb{F})$ (the set of all $m \times n$ matrices) is a vector space as well.

2. The set $V = C(\mathbb{R})$ is a vector space:

(a) Vector Addiction:

$$(f + g)(x) = f(x) + g(x), \forall f, g \in V$$

(b) Scalar Multiplication:

$$(\alpha f)(x) = \alpha f(x), \forall \alpha \in \mathbb{R}, f \in V$$

(c) Addictive Identity is a zero function, i.e., $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

Definition 1.2 A sub-collection $W \subseteq V$ of a vector space V is called a **vector subspace** of V if W itself forms a vector space, denoted by $W \leq V$. ■

- **Example 1.4**
1. For $V = \mathbb{R}^3$, we claim that $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \leq V$
 2. $W = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$ is not the vector subspace of V . ■

Proposition 1.1 $W \subseteq V$ is a **vector subspace** of V iff for $\forall \mathbf{w}_1, \mathbf{w}_2 \in W$, we have $\alpha \mathbf{w}_1 + \beta \mathbf{w}_2 \in W$, for $\forall \alpha, \beta \in \mathbb{F}$.

- **Example 1.5**
1. For $V = M_{n \times n}(\mathbb{F})$, the subspace $W = \{A \in V \mid \mathbf{A}^T = \mathbf{A}\} \leq V$
 2. For $V = C^\infty(\mathbb{R})$, define $W = \{f \in V \mid \frac{d^2}{dx^2} f + f = 0\} \leq V$. For $f, g \in W$, we have

$$(\alpha f + \beta g)'' = \alpha f'' + \beta g'' = \alpha(-f) + \beta(-g) = -(\alpha f + \beta g),$$

which implies $(\alpha f + \beta g)'' + (\alpha f + \beta g) = 0$. ■

1.4. Wednesday for MAT3040

1.4.1. Review

1. Vector Space: e.g., $\mathbb{R}, M_{n \times n}(\mathbb{R}), C(\mathbb{R}^n), \mathbb{R}[x]$.
2. Vector Subspace: $W \leq V$, e.g.,
 - (a) $V = \mathbb{R}^2$, the set $W := \mathbb{R}_+^2$ is not a vector subspace since W is not closed under scalar multiplication;
 - (b) the set $W = \mathbb{R}_+^2 \cup \mathbb{R}_-^2$ is not a vector subspace since it is not closed under addition.
 - (c) For $V = M_{3 \times 3}(\mathbb{R})$, the set of invertible 3×3 matrices is not a vector subspace, since we cannot define zero vector inside.
 - (d) Exercise: How about the set of all singular matrices? Answer: it is not a vector subspace since the vector addition does not necessarily hold.

1.4.2. Spanning Set

Definition 1.11 [Span] Let V be a vector space over \mathbb{F} :

1. A linear combination of a subset S in V is of the form

$$\sum_{i=1}^n \alpha_i \mathbf{s}_i, \quad \alpha_i \in \mathbb{F}, \mathbf{s}_i \in S$$

Note that the summation should be finite.

2. The **span** of a subset $S \subseteq V$ is

$$\text{span}(S) = \left\{ \sum_{i=1}^n \alpha_i \mathbf{s}_i \mid \alpha_i \in \mathbb{F}, \mathbf{s}_i \in S \right\}$$

3. S is a spanning set of V , or say S spans V , if

$$\text{span}(S) = V.$$

■ **Example 1.12** For $V = \mathbb{R}[x]$, define the set

$$S = \{1, x^2, x^4, \dots, x^6\},$$

then $2 + x^4 + \pi x^{106} \in \text{span}(S)$, while the series $1 + x^2 + x^4 + \dots \notin \text{span}(S)$.

It is clear that $\text{span}(S) \neq V$, but S is the spanning set of $W = \{p \in V \mid p(x) = p(-x)\}$. ■

■ **Example 1.13** For $V = M_{3 \times 3}(\mathbb{R})$, let $W_1 = \{\mathbf{A} \in V \mid \mathbf{A}^T = \mathbf{A}\}$ and $W_2 = \{\mathbf{B} \in V \mid \mathbf{B}^T = -\mathbf{B}\}$ (the set of skew-symmetric matrices) be two vector subspaces. Define the set

$$\mathcal{S} := W_1 \cup W_2$$

Exercise: \mathcal{S} spans V . ■

Proposition 1.7 Let S be a subset in a vector space V .

1. $S \subseteq \text{span}(S)$
2. $\text{span}(S) = \text{span}(\text{span}(S))$
3. If $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \setminus \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$, then

$$\mathbf{v}_1 \in \text{span}\{\mathbf{w}, \mathbf{v}_2, \dots, \mathbf{v}_n\} \setminus \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$$

Proof. 1. For each $\mathbf{s} \in S$, we have

$$\mathbf{s} = 1 \cdot \mathbf{s} \in \text{span}(S)$$

2. From (1), it's clear that $\text{span}(S) \subseteq \text{span}(\text{span}(S))$, and therefore suffices to show $\text{span}(\text{span}(S)) \subseteq \text{span}(S)$:

Pick $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i \in \text{span}(\text{span}(S))$, where $\mathbf{v}_i \in \text{span}(S)$. Rewrite

$$\mathbf{v}_i = \sum_{j=1}^{n_i} \beta_{ij} \mathbf{s}_j, \quad \mathbf{s}_j \in S,$$

which implies

$$\begin{aligned}\mathbf{v} &= \sum_{i=1}^n \alpha_i \sum_{j=1}^{n_i} \beta_{ij} \mathbf{s}_j \\ &= \sum_{i=1}^n \sum_{j=1}^{n_i} (\alpha_i \beta_{ij}) \mathbf{s}_j,\end{aligned}$$

i.e., \mathbf{v} is the finite combination of elements in S , which implies $\mathbf{v} \in \text{span}(S)$.

3. By hypothesis, $\mathbf{w} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$ with $\alpha_1 \neq 0$, which implies

$$\mathbf{v}_1 = -\frac{\alpha_2}{\alpha_1} \mathbf{v}_2 + \cdots + \left(-\frac{1}{\alpha_1} \mathbf{w}\right)$$

which implies $\mathbf{v}_1 \in \text{span}\{\mathbf{w}, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. It suffices to show $\mathbf{v}_1 \notin \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$.

Suppose on the contrary that $\mathbf{v}_1 \in \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$. It's clear that $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$. (left as exercise). Therefore,

$$\emptyset = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \setminus \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\},$$

which is a contradiction. ■

1.4.3. Linear Independence and Basis

Definition 1.12 [Linear Independence] Let S be a (not necessarily finite) subset of V . Then S is **linearly independent** (l.i.) on V if for any finite subset $\{\mathbf{s}_1, \dots, \mathbf{s}_k\}$ in S ,

$$\sum_{i=1}^k \alpha_i \mathbf{s}_i = \mathbf{0} \iff \alpha_i = 0, \forall i$$

■ **Example 1.14** For $V = C(\mathbb{R})$,

1. let $S_1 = \{\sin x, \cos x\}$, which is l.i., since

$$\alpha \sin x + \beta \cos x = \mathbf{0} \text{ (means zero function)}$$

Taking $x = 0$ both sides leads to $\beta = 0$; taking $x = \frac{\pi}{2}$ both sides leads to $\alpha = 0$.

2. let $S_2 = \{\sin^2 x, \cos^2 x, 1\}$, which is linearly dependent, since

$$1 \cdot \sin^2 x + 1 \cdot \cos^2 x + (-1) \cdot 1 = 0, \forall x$$

3. Exercise: For $V = \mathbb{R}[x]$, let $S = \{1, x, x^2, x^3, \dots\}$, which is l.i.:

Pick $x^{k_1}, \dots, x^{k_n} \in S$ with $k_1 < \dots < k_n$. Consider that the equation

$$\alpha_1 x^{k_1} + \dots + \alpha_n x^{k_n} = \mathbf{0}$$

holds for all x , and try to solve for $\alpha_1, \dots, \alpha_n$ (one way is differentiation.)

Definition 1.13 [Basis] A subset S is a **basis** of V if

- (a) S spans V ;
- (b) S is l.i.

■ **Example 1.15** 1. For $V = \mathbb{R}^n$, $S = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of V

2. For $V = \mathbb{R}[x]$, $S = \{1, x, x^2, \dots\}$ is a basis of V

3. For $V = M_{2 \times 2}(\mathbb{R})$,

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis of V

R Note that there can be many basis for a vector space V .

Proposition 1.8 Let $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, then there exists a subset of $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, which is a basis of V .

Proof. If $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is l.i., the proof is complete.

Suppose not, then $\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m = \mathbf{0}$ has a non-trivial solution. w.l.o.g., $\alpha_1 \neq 0$, which implies

$$\mathbf{v}_1 = -\frac{\alpha_2}{\alpha_1} \mathbf{v}_2 + \dots + \left(\frac{\alpha_m}{\alpha_1}\right) \mathbf{v}_m \implies \mathbf{v}_1 \in \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_m\}$$

By the proof in (c), Proposition (1.7),

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} = \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_m\},$$

which implies $V = \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_m\}$.

Continue this argument finitely many times to guarantee that $\{\mathbf{v}_i, \mathbf{v}_{i+1}, \dots, \mathbf{v}_m\}$ is l.i., and spans V . The proof is complete. ■

Corollary 1.1 If $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ (i.e., V is finitely generated), then V has a basis. (The same holds for non-finitely generated V).

Proposition 1.9 If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V , then every $\mathbf{v} \in V$ can be expressed uniquely as

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$$

Proof. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V , so $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n \tag{1.1}$$

Suppose further that

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \dots + \beta_n \mathbf{v}_n, \tag{1.2}$$

it suffices to show that $\alpha_i = \beta_i$ for $\forall i$:

Subtracting (1.1) into (1.2) leads to

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n = 0.$$

By the hypothesis of linear independence, we have $\alpha_i - \beta_i = 0$ for $\forall i$, i.e., $\alpha_i = \beta_i$. ■

Chapter 2

Week2

2.1. Monday for MAT3040

Reviewing.

1. Linear Combination and Span
2. Linear Independence
3. Basis: a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is called a **basis** for V if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, and $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$.

Lemma: Given $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, we can find a basis for this set. Here V is said to be **finitely generated**.

4. Lemma: The vector $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \setminus \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$ implies that

$$\mathbf{v}_1 \in \text{span}\{\mathbf{w}, \mathbf{v}_2, \dots, \mathbf{v}_n\} \setminus \text{span}\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$$

2.1.1. Basis and Dimension

Theorem 2.1 Let V be a finitely generated vector space. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ are two basis of V . Then $m = n$. (where m is called the **dimension**)

Proof. Suppose on the contrary that $m \neq n$. Without loss of generality (w.l.o.g.), assume that $m < n$. Let $\mathbf{v}_1 = \alpha_1 \mathbf{w}_1 + \dots + \alpha_n \mathbf{w}_n$, with some $\alpha_i \neq 0$. w.l.o.g., assume $\alpha_1 \neq 0$. Therefore,

$$\mathbf{v}_1 \in \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} \setminus \text{span}\{\mathbf{w}_2, \dots, \mathbf{w}_n\} \quad (2.1)$$

which implies that $\mathbf{w}_1 \in \text{span}\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} \setminus \text{span}\{\mathbf{w}_2, \dots, \mathbf{w}_n\}$.

Then we claim that $\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is a basis of V :

1. Note that $\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ is a spanning set:

$$\begin{aligned} \mathbf{w}_1 \in \text{span}\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} &\implies \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} \\ &\implies \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} \subseteq \text{span}\{\text{span}\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} \end{aligned}$$

Since $V = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$, we have $\text{span}\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\} = V$.

2. Then we show the linear independence of $\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$. Consider the equation

$$\beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \dots + \beta_n \mathbf{w}_n = \mathbf{0}$$

- (a) When $\beta_1 \neq 0$, we imply

$$\mathbf{v}_1 = \left(-\frac{\beta_2}{\beta_1}\right) \mathbf{w}_2 + \dots + \left(-\frac{\beta_n}{\beta_1}\right) \mathbf{w}_n \in \text{span}\{\mathbf{w}_2, \dots, \mathbf{w}_n\},$$

which contradicts (2.1).

- (b) When $\beta_1 = 0$, then $\beta_2 \mathbf{w}_2 + \dots + \beta_n \mathbf{w}_n = \mathbf{0}$, which implies $\beta_2 = \dots = \beta_n = 0$, due to the independence of $\{\mathbf{w}_2, \dots, \mathbf{w}_n\}$.

Therefore, $\mathbf{v}_2 \in \text{span}\{\mathbf{v}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$, i.e.,

$$\mathbf{v}_2 = \gamma_1 \mathbf{v}_1 + \dots + \gamma_n \mathbf{w}_n,$$

where $\gamma_2, \dots, \gamma_n$ cannot be all zeros, since otherwise $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly dependent, i.e., $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ cannot form a basis. w.l.o.g., assume $\gamma_2 \neq 0$, which implies

$$\mathbf{w}_2 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_3, \dots, \mathbf{w}_n\} \setminus \text{span}\{\mathbf{v}_1, \mathbf{w}_3, \dots, \mathbf{w}_n\}.$$

Following the similar argument above, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_3, \dots, \mathbf{w}_n\}$ forms a basis of V .

Continuing the argument above, we imply $\{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{w}_{m+1}, \dots, \mathbf{w}_n\}$ is a basis of V .

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis as well, we imply

$$\mathbf{w}_{m+1} = \delta_1 \mathbf{v}_1 + \dots + \delta_m \mathbf{v}_m$$

for some $\delta_i \in \mathbb{F}$, i.e., $\{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{w}_{m+1}\}$ is linearly dependent, which is a contradiction. ■

■ **Example 2.1** A vector space may have more than one basis.

Suppose $V = \mathbb{F}^n$, it is clear that $\dim(V) = n$, and

$\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of V , where \mathbf{e}_i denotes a unit vector.

There could be other basis of V , such as

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Actually, the columns of any invertible $n \times n$ matrix forms a basis of V . ■

■ **Example 2.2** Suppose $V = M_{m \times n}(\mathbb{R})$, we claim that $\dim(V) = mn$:

$$\left\{ E_{ij} \mid \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix} \right\} \text{ is a basis of } V,$$

where E_{ij} is $m \times n$ matrix with 1 at (i, j) -th entry, and 0s at the remaining entries. ■

■ **Example 2.3** Suppose $V = \{\text{all polynomials of degree} \leq n\}$, then $\dim(V) = n + 1$. ■

■ **Example 2.4** Suppose $V = \{\mathbf{A} \in M_{n \times n}(\mathbb{R}) \mid \mathbf{A}^T = \mathbf{A}\}$, then $\dim(V) = \frac{n(n+1)}{2}$. ■

■ **Example 2.5** Let $W = \{\mathbf{B} \in M_{n \times n}(\mathbb{R}) \mid \mathbf{B}^T = -\mathbf{B}\}$, then $\dim(V) = \frac{n(n-1)}{2}$. ■

R Sometimes it should be classified the field \mathbb{F} for the scalar multiplication to define a vector space. Consider the example below:

1. Let $V = \mathbb{C}$, then $\dim(\mathbb{C}) = 1$ for the scalar multiplication defined under the field \mathbb{C} .
2. Let $V = \text{span}\{1, i\} = \mathbb{C}$, then $\dim(\mathbb{C}) = 2$ for the scalar multiplication defined under the field \mathbb{R} , since all $z \in V$ can be written as $z = a + bi$, $\forall a, b \in \mathbb{R}$.
3. Therefore, to avoid confusion, it is safe to write

$$\dim_{\mathbb{C}}(\mathbb{C}) = 1, \quad \dim_{\mathbb{R}}(\mathbb{C}) = 2.$$

2.1.2. Operations on a vector space

Note that the basis for a vector space is characterized as the **maximal linearly independent set**.

Theorem 2.2 — Basis Extension. Let V be a finite dimensional vector space, and $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a linearly independent set on V , Then we can extend it to the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V .

Proof. • Suppose $\dim(V) = n > k$, and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a basis of V . Consider the set $\{\mathbf{w}_1, \dots, \mathbf{w}_n\} \cup \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, which is linearly dependent, i.e.,

$$\alpha_1 \mathbf{w}_1 + \dots + \alpha_n \mathbf{w}_n + \beta_1 \mathbf{v}_1 + \dots + \beta_k \mathbf{v}_k = \mathbf{0},$$

with some $\alpha_i \neq 0$, since otherwise this equation will only have trivial solution. w.l.o.g., assume $\alpha_1 \neq 0$.

- Therefore, consider the set $\{\mathbf{w}_2, \dots, \mathbf{w}_n\} \cup \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. We keep removing elements from $\{\mathbf{w}_2, \dots, \mathbf{w}_n\}$ until we first get the set

$$S \cup \{\mathbf{v}_1, \dots, \mathbf{v}_k\},$$

with $S \subseteq \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ and $S \cup \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, i.e., S is a maximal subset of $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ such that $S \cup \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

- Rewrite $S = \{\mathbf{v}_{k+1}, \dots, \mathbf{v}_m\}$ and therefore $S' = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_m\}$ are linearly independent. It suffices to show S' spans V .
 - Indeed, for all $\mathbf{w}_i \in \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$, $\mathbf{w}_i \in \text{span}(S')$, since otherwise the equation

$$\alpha \mathbf{w}_i + \beta_1 \mathbf{v}_1 + \dots + \beta_m \mathbf{v}_m = \mathbf{0} \implies \alpha = 0,$$

which implies that $\beta_1 \mathbf{v}_1 + \dots + \beta_m \mathbf{v}_m = \mathbf{0}$ admits only trivial solution, i.e.,

$$\{\mathbf{w}_i\} \cup S' = \{\mathbf{w}_i\} \cup S \cup \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \text{ is linearly independent,}$$

which violates the maximality of S .

Therefore, all $\{\mathbf{w}_1, \dots, \mathbf{w}_n\} \subseteq \text{span}(S')$, which implies $\text{span}(S') = V$.

Therefore, S' is a basis of V . ■

- R** Start with a spanning set, we keep removing something to form a basis; start with independent set, we keep adding something to form a basis.

In other words, the basis is both the minimal spanning set, and the maximal linearly independent set.

Definition 2.1 [Direct Sum] Let W_1, W_2 be two vector subspaces of V , then

1. $W_1 \cap W_2 := \{\mathbf{w} \in V \mid \mathbf{w} \in W_1, \text{ and } \mathbf{w} \in W_2\}$
2. $W_1 + W_2 := \{\mathbf{w}_1 + \mathbf{w}_2 \mid \mathbf{w}_i \in W_i\}$
3. If furthermore that $W_1 \cap W_2 = \{\mathbf{0}\}$, then $W_1 + W_2$ is denoted as $W_1 \oplus W_2$, which is called **direct sum**. ■

Proposition 2.1 $W_1 \cap W_2$ and $W_1 + W_2$ are vector subspaces of V .

2.4. Wednesday for MAT3040

Reviewing.


- Basis, Dimension
- Basis Extension
- $W_1 \cap W_2 = \emptyset$ implies $W_1 \oplus W_2 = W_1 + W_2$ (Direct Sum).

2.4.1. Remark on Direct Sum

Proposition 2.13 The set $W_1 + W_2 = W_1 \oplus W_2$ iff any $\mathbf{w} \in W_1 + W_2$ can be uniquely expressed as

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_i \in W_i$ for $i = 1, 2$.

 We can also define addition among finite set of vector spaces $\{W_1, \dots, W_k\}$.

If $\mathbf{w}_1 + \dots + \mathbf{w}_k = \mathbf{0}$ implies $\mathbf{w}_i = \mathbf{0}, \forall i$, then we can write $W_1 + \dots + W_k$ as

$$W_1 \oplus \dots \oplus W_k$$

Proposition 2.14 — Complementation. Let $W \leq V$ be a vector subspace of a finite dimension vector space V . Then there exists $W' \leq V$ such that

$$W \oplus W' = V.$$

Proof. It's clear that $\dim(W) := k \leq n := \dim(V)$. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis of W .

By the basis extension proposition, we can extend it into $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$, which is a basis of V .

Therefore, we take $W' = \text{span}\{\mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$, which follows that

1. $W + W' = V$: $\forall \mathbf{v} \in V$ has the form

$$\mathbf{v} = (\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k) + (\alpha_{k+1} \mathbf{v}_{k+1} + \cdots + \alpha_n \mathbf{v}_n),$$

where $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k \in W$ and $\alpha_{k+1} \mathbf{v}_{k+1} + \cdots + \alpha_n \mathbf{v}_n \in W'$.

2. $W \cap W' = \{\mathbf{0}\}$: Suppose $\mathbf{v} \in W \cap W'$, i.e.,

$$\begin{aligned} \mathbf{v} &= (\beta_1 \mathbf{v}_1 + \cdots + \beta_k \mathbf{v}_k) + (0 \mathbf{v}_{k+1} + \cdots + 0 \mathbf{v}_n) \in W \\ &= (0 \mathbf{v}_1 + \cdots + 0 \mathbf{v}_k) + (\beta_{k+1} \mathbf{v}_{k+1} + \cdots + \beta_n \mathbf{v}_n) \in W'. \end{aligned}$$

By the uniqueness of coordinates, we imply $\beta_1 = \cdots = \beta_n = 0$, i.e., $\mathbf{v} = \mathbf{0}$.

Therefore, we conclude that $W \oplus W' = V$. ■

2.4.2. Linear Transformation

Definition 2.7 [Linear Transformation] Let V, W be vector spaces. Then $T : V \rightarrow W$ is a linear transformation if

$$T(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2),$$

for $\forall \alpha, \beta \in \mathbb{F}$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$. ■

- Proposition 2.15**
1. Suppose that $S : V \rightarrow W$ and $T : W \rightarrow U$ are linear transformations, then so is $T \circ S : V \rightarrow U$.
 2. For any linear transformation $T : V \rightarrow W$, we have

$$T(\mathbf{0}_V) = \mathbf{0}_W$$

Proof. Simply apply the definition of the linear transformation. ■

- **Example 2.12**
1. The transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined as $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ (where $\mathbf{A} \in \mathbb{R}^{m \times n}$) is a linear transformation.

2. The transformation $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ defined as

$$p(x) \mapsto T(p(x)) = p'(x), \quad p(x) \mapsto T(p(x)) = \int_0^x p(t) dt$$

is a linear transformation

3. The transformation $T : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$ defined as

$$\mathbf{A} \mapsto \text{trace}(\mathbf{A}) := \sum_{i=1}^n a_{ii}$$

is a linear transformation.

However, the transformation

$$\mathbf{A} \mapsto \det(\mathbf{A})$$

is not a linear transformation.

Definition 2.8 [Kernel/Image] Let $T : V \rightarrow W$ be a linear transformation.

1. The **kernel** of T is

$$\ker(T) = T^{-1}(\mathbf{0}) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

2. The **image** (or range) of T is

$$\text{Im}(T) = T(V) = \{T(\mathbf{v}) \in W \mid \mathbf{v} \in V\}$$

■ **Example 2.13** 1. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation with $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, then

$$\ker(T) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} = \text{Null}(\mathbf{A}) \quad \text{Null Space}$$

and

$$\text{Im}(T) = \{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^n\} = \text{Col}(\mathbf{A}) = \text{span}\{\text{columns of } \mathbf{A}\} \quad \text{Column Space}$$

2. For $T(p(x)) = p'(x)$, $\ker(T) = \{\text{constant polynomials}\}$ and $\text{Im}(T) = \mathbb{R}[x]$.

Proposition 2.16 The kernel or image for a linear transformation $T : V \rightarrow W$ also forms a vector subspace:

$$\ker(T) \leq V, \quad \text{Im}(T) \leq W$$

Proof. For $\mathbf{v}_1, \mathbf{v}_2 \in \ker(T)$, we imply

$$T(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \mathbf{0},$$

which implies $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \in \ker(T)$.

The remaining proof follows similarly. ■

Definition 2.9 [Rank/Nullity] Let V, W be finite dimensional vector spaces and $T : V \rightarrow W$ a linear transformation. Then we define

$$\text{rank}(T) = \dim(\text{Im}(T))$$

$$\text{nullity}(T) = \dim(\ker(T))$$

Ⓡ Let

$$\text{Hom}_{\mathbb{F}}(V, W) = \{\text{all linear transformations } T : V \rightarrow W\},$$

and we can define the addition and scalar multiplication to make it a vector space:

1. For $T, S \in \text{Hom}_{\mathbb{F}}(V, W)$, define

$$(T + S)(\mathbf{v}) = T(\mathbf{v}) + S(\mathbf{v}),$$

which implies $T + S \in \text{Hom}_{\mathbb{F}}(V, W)$.

2. Also, define

$$(\gamma T)(\mathbf{v}) = \gamma T(\mathbf{v}), \quad \text{for } \forall \gamma \in \mathbb{F},$$

which implies $\gamma T \in \text{Hom}_{\mathbb{F}}(V, W)$.

In particular, if $V = \mathbb{R}^n, W = \mathbb{R}^m$, then

$$\text{Hom}_{\mathbb{F}}(V, W) = M_{m \times n}(\mathbb{R}).$$

Proposition 2.17 If $\dim(V) = n, \dim(W) = m$, then $\dim(\text{Hom}_{\mathbb{F}}(V, W)) = mn$.

Proposition 2.18 There are alternative characterizations for the injectivity and surjectivity of linear transformation T :

1. The linear transformation T is injective if and only if

$$\ker(T) = \{0\} \iff \text{nullity}(T) = 0.$$

2. The linear transformation T is surjective if and only if

$$\text{im}(T) = W \iff \text{rank}(T) = \dim(W).$$

3. If T is bijective, then T^{-1} is a linear transformation.

Proof. 1. (a) For the forward direction of (1),

$$\mathbf{x} \in \ker(T) \implies T(\mathbf{x}) = \mathbf{0} = T(\mathbf{0}) \implies \mathbf{x} = \mathbf{0}$$

(b) For the reverse direction of (1),

$$T(\mathbf{x}) = T(\mathbf{y}) \implies T(\mathbf{x} - \mathbf{y}) = \mathbf{0} \implies \mathbf{x} - \mathbf{y} \in \ker(T) = \{\mathbf{0}\} \implies \mathbf{x} = \mathbf{y}$$

2. The proof follows similar idea in (1).

3. Let $T^{-1} : W \rightarrow V$. For all $\mathbf{w}_1, \mathbf{w}_2 \in W$, there exists $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$, i.e.,

$$T^{-1}(\mathbf{w}_i) = \mathbf{v}_i \quad i = 1, 2.$$

Consider the mapping

$$\begin{aligned} T(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) &= \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) \\ &= \alpha \mathbf{w}_1 + \beta \mathbf{w}_2, \end{aligned}$$

which implies $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = T^{-1}(\alpha \mathbf{w}_1 + \beta \mathbf{w}_2)$, i.e.,

$$\alpha T^{-1}(\mathbf{w}_1) + \beta T^{-1}(\mathbf{w}_2) = T^{-1}(\alpha \mathbf{w}_1 + \beta \mathbf{w}_2).$$

■

Definition 2.10 [isomorphism] We say that the vector subspaces V and W are isomorphic if there exists a bijective linear transformation $T : V \rightarrow W$. ($V \cong W$)

This mapping T is called an **isomorphism** from V to W . ■

R If $\dim(V) = \dim(W) = n < \infty$, then $V \cong W$:

Take $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}, \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ as basis of V and W , respectively. Then one can construct $T : V \rightarrow W$ satisfying $T(\mathbf{v}_i) = \mathbf{w}_i$ for $\forall i$ as follows:

$$T(\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n) = \alpha_1 \mathbf{w}_1 + \dots + \alpha_n \mathbf{w}_n \quad \forall \alpha_i \in \mathbb{F}$$

It's clear that our constructed T is a linear transformation.

R $V \cong W$ doesn't imply any linear transformations $T : V \rightarrow W$ is an isomorphism.
e.g., $T(\mathbf{v}) = \mathbf{0}$ is not an isomorphism if $W \neq \{\mathbf{0}\}$.

Theorem 2.3 — **Rank-Nullity Theorem.** Let $T : V \rightarrow W$ be a linear transformation with $\dim(V) < \infty$. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof. Since $\ker(T) \leq V$, by proposition (2.14), there exists $V_1 \leq V$ such that

$$V = \ker(T) \oplus V_1.$$

1. Consider the transformation $T|_{V_1}: V_1 \rightarrow T(V_1)$, which is an isomorphism, since:

- Surjectivity is immediate

- For $\mathbf{v} \in \ker(T|_{V_1})$,

$$T(\mathbf{v}) = \mathbf{0} \implies \mathbf{v} \in \ker(T),$$

which implies $\mathbf{v} = \mathbf{0}$ since $\mathbf{v} \in \ker(T) \cap V_1 = \{0\}$, i.e., the injectivity follows.

Therefore, $\dim(V_1) = \dim(T(V_1))$.

2. Secondly, given an isomorphism T from X to Y with $\dim(X) < \infty$, then $\dim(X) = \dim(T(X))$. The reason follows from assignment 1 questions (8-9):

$$\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \text{ is a basis of } X \implies \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\} \text{ is a basis of } Y$$

3. Note that $T(V_1) = T(V) = \text{im}(T)$, since:

- for $\forall \mathbf{v} \in V$, $\mathbf{v} = \mathbf{v}_k + \mathbf{v}_1$, where $\mathbf{v}_k \in \ker(T)$, $\mathbf{v}_1 \in V_1$, which implies

$$T(\mathbf{v}) = T(\mathbf{v}_k) + T(\mathbf{v}_1) = \mathbf{0} + T(\mathbf{v}_1),$$

i.e., $T(V) \subseteq T(V_1) \subseteq T(V)$, i.e., $T(V) = T(V_1)$.

4. We can show that $\dim(V) = \dim(\ker(T)) + \dim(V_1)$: Let $\{v_1, \dots, v_k\}$ be a basis of $\ker(T)$, and $\{v_{k+1}, \dots, v_n\}$ be a basis of V_1 , then by the proof of complementation proposition (2.14), we imply $\{v_1, \dots, v_n\}$ is a basis of V , i.e., $\dim(V) = n = k + (n - k) = \dim(\ker(T)) + \dim(V_1)$.

Therefore, we imply

$$\begin{aligned}\dim(V) &= \dim(\ker(T)) + \dim(V_1) \\ &= \text{nullity}(T) + \dim(T(V_1)) \\ &= \text{nullity}(T) + \dim(T(V)) \\ &= \text{nullity}(T) + \dim(\text{im}(T)) \\ &= \text{nullity}(T) + \text{rank}(T).\end{aligned}$$

■

Chapter 3

Week3

3.1. Monday for MAT3040

Reviewing.

1. Complementation. Suppose $\dim(V) = n < \infty$, then $W \leq V$ implies that there exists W' such that

$$W \oplus W' = V.$$

2. Given the linear transformation $T : V \rightarrow W$, define the set $\ker(T)$ and $\text{Im}(T)$.
3. Isomorphism of vector spaces: $T : V \cong W$
4. Rank-Nullity Theorem

3.1.1. Remarks on Isomorphism

Proposition 3.1 If $T : V \rightarrow W$ is an isomorphism, then

1. the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent in V if and only if $\{T\mathbf{v}_1, \dots, T\mathbf{v}_k\}$ is linearly independent.
2. The same goes if we replace the linearly independence by spans.
3. If $\dim(V) = n$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ forms a basis of V if and only if $\{T\mathbf{v}_1, \dots, T\mathbf{v}_n\}$ forms a basis of W . In particular, $\dim(V) = \dim(W)$.
4. Two vector spaces with finite dimensions are isomorphic if and only if they have the same dimension:

Proof. It suffices to show the reverse direction. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be two

basis of V, W , respectively. Define the linear transformation $T : V \rightarrow W$ by

$$T(a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n) = a_1 \mathbf{w}_1 + \cdots + a_n \mathbf{w}_n$$

Then T is surjective since $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ spans W ; T is injective since $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is linearly independent. ■

3.1.2. Change of Basis and Matrix Representation

Definition 3.1 [Coordinate Vector] Let V be a finite dimensional vector space and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ an **ordered** basis of V . Any vector $\mathbf{v} \in V$ can be uniquely written as

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n,$$

Therefore we define the map $[\cdot]_{\mathcal{B}} : V \rightarrow \mathbb{F}^n$, which maps any vector in \mathbf{v} into its **coordinate vector**:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

R Note that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\{\mathbf{v}_2, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ are distinct ordered basis.

■ **Example 3.1** Given $V = M_{2 \times 2}(\mathbb{F})$ and the ordered basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Any matrix has the coordinate vector w.r.t. \mathcal{B} , i.e.,

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 3 \end{pmatrix}$$

However, if given another ordered basis

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

the matrix may have the different coordinate vector w.r.t. \mathcal{B}_1 :

$$\left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \right]_{\mathcal{B}_1} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Theorem 3.1 The mapping $[\cdot]_{\mathcal{B}} : V \rightarrow \mathbb{F}^n$ is an isomorphism.

Proof. 1. First show the operator $[\cdot]_{\mathcal{B}}$ is well-defined, i.e., the same input gives the same output. Suppose that

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{pmatrix},$$

then we imply

$$\begin{aligned} \mathbf{v} &= \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \\ &= \alpha'_1 \mathbf{v}_1 + \cdots + \alpha'_n \mathbf{v}_n. \end{aligned}$$

By the uniqueness of coordinates, we imply $\alpha_i = \alpha'_i$ for $i = 1, \dots, n$.

2. It's clear that the operator $[\cdot]_{\mathcal{B}}$ is a linear transformation, i.e.,

$$[p\mathbf{v} + q\mathbf{w}]_{\mathcal{B}} = p[\mathbf{v}]_{\mathcal{B}} + q[\mathbf{w}]_{\mathcal{B}} \quad \forall p, q \in \mathbb{F}$$

3. The operator $[\cdot]_{\mathcal{B}}$ is surjective:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \implies \mathbf{v} = 0\mathbf{v}_1 + \dots + 0\mathbf{v}_n = \mathbf{0}.$$

4. The injective is clear, i.e., $[\mathbf{v}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$ implies $\mathbf{v} = \mathbf{w}$.

Therefore, $[\cdot]_{\mathcal{B}}$ is an isomorphism. ■

We can use the Theorem (3.1) to simplify computations in vector spaces:

■ **Example 3.2** Given a vector sapce $V = P_3[x]$ and its basis $B = \{1, x, x^2, x^3\}$.

To check if the set $\{1 + x^2, 3 - x^3, x - x^3\}$ is linearly independent, by part (1) in Proposition (3.1) and Theorem (3.1), it suffices to check whether the corresponding coordinate vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is linearly independent, i.e., do Gaussian Elimination and check the number of pivots. ■

Here gives rise to the question: if $\mathcal{B}_1, \mathcal{B}_2$ form two basis of V , then how are $[\mathbf{v}]_{\mathcal{B}_1}, [\mathbf{v}]_{\mathcal{B}_2}$ related to each other?

Here we consider an easy example first:

■ **Example 3.3** Consider $V = \mathbb{R}^n$ and its basis $\mathcal{B}_1 = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. For any $\mathbf{v} \in V$,

$$\mathbf{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \mathbf{e}_1 + \dots + \alpha_n \mathbf{e}_n \implies [\mathbf{v}]_{\mathcal{B}_1} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Also, we can construct a different basis of V :

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

which gives a different coordinate vector of \mathbf{v} :

$$[\mathbf{v}]_{\mathcal{B}_2} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 - \alpha_3 \\ \vdots \\ \alpha_{n-1} - \alpha_n \\ \alpha_n \end{pmatrix}$$

Proposition 3.2 — Change of Basis. Let $\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{A}' = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be two ordered basis of a vector space V . Define the **change of basis** matrix from \mathcal{A} to \mathcal{A}' , say $C_{\mathcal{A}', \mathcal{A}} := [\alpha_{ij}]$, where

$$\mathbf{v}_j = \sum_{i=1}^m \alpha_{ij} \mathbf{w}_i$$

Then for any vector $\mathbf{v} \in V$, the *change of basis amounts to left-multiplying the change of basis matrix*:

$$C_{\mathcal{A}', \mathcal{A}}[\mathbf{v}]_{\mathcal{A}} = [\mathbf{v}]_{\mathcal{A}'} \quad (3.1)$$

Define matrix $C_{\mathcal{A},\mathcal{A}'} := [\beta_{ij}]$, where

$$\mathbf{w}_j = \sum_{i=1}^n \beta_{ij} \mathbf{v}_i$$

Then we imply that

$$(C_{\mathcal{A},\mathcal{A}'})^{-1} = C_{\mathcal{A}',\mathcal{A}}$$

Proof. 1. First show (3.1) holds for $\mathbf{v} = \mathbf{v}_j, j = 1, \dots, n$:

$$\begin{aligned} \text{LHS of (3.1)} &= [\alpha_{ij}] \mathbf{e}_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} \\ \text{RHS of (3.1)} &= [\mathbf{v}_j]_{\mathcal{A}'} = \left[\sum_{i=1}^n \alpha_i \mathbf{w}_i \right]_{\mathcal{A}'} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} \end{aligned}$$

Therefore,

$$C_{\mathcal{A}',\mathcal{A}}[\mathbf{v}_j]_{\mathcal{A}} = [\mathbf{v}_j]_{\mathcal{A}'}, \quad \forall j = 1, \dots, n. \quad (3.2)$$

2. Then for any $\mathbf{v} \in V$, we imply $\mathbf{v} = r_1 \mathbf{v}_1 + \dots + r_n \mathbf{v}_n$, which implies that

$$C_{\mathcal{A}',\mathcal{A}}[\mathbf{v}]_{\mathcal{A}} = C_{\mathcal{A}',\mathcal{A}}[r_1 \mathbf{v}_1 + \dots + r_n \mathbf{v}_n]_{\mathcal{A}} \quad (3.3a)$$

$$= C_{\mathcal{A}',\mathcal{A}}(r_1 [\mathbf{v}_1]_{\mathcal{A}} + \dots + r_n [\mathbf{v}_n]_{\mathcal{A}}) \quad (3.3b)$$

$$= \sum_{j=1}^n r_j C_{\mathcal{A}',\mathcal{A}}[\mathbf{v}_j]_{\mathcal{A}} \quad (3.3c)$$

$$= \sum_{j=1}^n r_j [\mathbf{v}_j]_{\mathcal{A}'} \quad (3.3d)$$

$$= \left[\sum_{j=1}^n r_j \mathbf{v}_j \right]_{\mathcal{A}'} \quad (3.3e)$$

$$= [\mathbf{v}]_{\mathcal{A}'} \quad (3.3f)$$

where (3.3a) and (3.3e) is by applying the linearity of $[\cdot]_{\mathcal{A}}$ and $[\cdot]_{\mathcal{A}'}$; (3.3d) is by applying the result (3.12). Therefore (3.1) is shown for $\forall \mathbf{v} \in V$.

3. Now we show that $(C_{\mathcal{A}\mathcal{A}'}C_{\mathcal{A}'\mathcal{A}}) = \mathbf{I}_n$. Note that

$$\begin{aligned}\mathbf{v}_j &= \sum_{i=1}^n \alpha_{ij} \mathbf{w}_i \\ &= \sum_{i=1}^n \alpha_{ij} \sum_{k=1}^n \beta_{ki} \mathbf{v}_k \\ &= \sum_{k=1}^n \left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) \mathbf{v}_i\end{aligned}$$

By the uniqueness of coordinates, we imply

$$\left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) = \delta_{jk} := \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

By the matrix multiplication, the (k, j) -th entry for $C_{\mathcal{A}\mathcal{A}'}C_{\mathcal{A}'\mathcal{A}}$ is

$$[C_{\mathcal{A}\mathcal{A}'}C_{\mathcal{A}'\mathcal{A}}]_{kj} = \left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) = \delta_{jk} \implies (C_{\mathcal{A}\mathcal{A}'}C_{\mathcal{A}'\mathcal{A}}) = \mathbf{I}_n$$

Noew, suppose

$$\begin{aligned}\mathbf{v}_j &= \sum_{i=1}^n \alpha_{ij} \mathbf{w}_i \\ &= \sum_{i=1}^n \alpha_{ij} \sum_{k=1}^n \beta_{ki} \mathbf{v}_k \\ &= \sum_{k=1}^n \left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) \mathbf{v}_i\end{aligned}$$

By the uniqueness of coordinates, we imply

$$\left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

where

$$\left(\sum_{i=1}^n \beta_{ki} \alpha_{ij} \right) = (C_{AA'}C_{A'A}).$$

Therefore, $(C_{AA'}C_{A'A}) = \mathbf{I}_n$. ■

■ **Example 3.4** Back to Example (3.3), write $\mathcal{B}_1, \mathcal{B}_2$ as

$$\mathcal{B}_1 = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}, \quad \mathcal{B}_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$$

and therefore $\mathbf{w}_i = \mathbf{e}_1 + \dots + \mathbf{e}_i$. The change of basis matrix is given by

$$C_{\mathcal{B}_1, \mathcal{B}_2} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

which implies that for \mathbf{v} in the example,

$$C_{\mathcal{B}_1, \mathcal{B}_2} [\mathbf{v}]_{\mathcal{B}_2} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 - \alpha_2 \\ \vdots \\ \alpha_{n-1} - \alpha_n \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = [\mathbf{v}]_{\mathcal{B}_1}$$

■ **Definition 3.2** Let $T : V \rightarrow W$ be a linear transformation, and

$$\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}, \quad \mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$$

be basis of V and W , respectively. The **matrix representation** of T with respect to (w.r.t.) \mathcal{A} and \mathcal{B} is defined as $(T)_{\mathcal{B}\mathcal{A}} := (\alpha_{ij}) \in M_{m \times m}(\mathbb{F})$, where

$$T(\mathbf{v}_j) = \sum_{i=1}^m \alpha_{ij} \mathbf{w}_i$$

3.4. Wednesday for MAT3040

3.4.1. Remarks for the Change of Basis

Reviewing.

- $[\cdot]_{\mathcal{A}} : V \rightarrow \mathbb{F}^n$ denotes coordinate vector mapping
- Change of Basis matrix: $C_{\mathcal{A}', \mathcal{A}}$
- $T : V \rightarrow W$, $\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$.

$$\text{Hom}_{\mathbb{F}}(V, W) \rightarrow M_{m \times n}(\mathbb{F})$$

■ **Example 3.10** Let $V = \mathbb{P}_3[x]$ and $\mathcal{A} = \{1, x, x^2, x^3\}$.

Let $T : V \rightarrow V$ defined as $p(x) \mapsto p'(x)$:

$$\begin{cases} T(1) = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x) = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^2) = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x^3) = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3 \end{cases}$$

We can define the change of basis matrix for a linear transformation T as well, w.r.t. \mathcal{A} and \mathcal{A} :

$$C_{\mathcal{A}, \mathcal{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, we can define a different basis $\mathcal{A}' = \{x^3, x^2, x, 1\}$ for the output space for T , say $T : V_{\mathcal{A}} \rightarrow V_{\mathcal{A}'}$:

$$(T)_{\mathcal{A}, \mathcal{A}'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Our observation is that the corresponding coordinate vectors before and after linear transformation admits a matrix multiplication:

$$\begin{aligned}
 (2x^2 + 4x^3) &\xrightarrow{T} ((4x + 12x^2)) \\
 (2x^2 + 4x^3)_{\mathcal{A}} &= \begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \end{pmatrix} & (4x + 12x^2)_{\mathcal{A}} &= \begin{pmatrix} 0 \\ 4 \\ 12 \\ 0 \end{pmatrix} \\
 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \end{pmatrix} &= \begin{pmatrix} 0 \\ 4 \\ 12 \\ 0 \end{pmatrix} \\
 C_{\mathcal{A}\mathcal{A}} \cdot (2x^2 + 4x^3)_{\mathcal{A}} &= (4x + 12x^2)_{\mathcal{A}}
 \end{aligned}$$

Theorem 3.3 — Matrix Representation. Let $T : V \rightarrow W$ be a linear transformation of finite dimensional vector spaces. Let \mathcal{A}, \mathcal{B} the ordered basis of V, W , respectively. Then the following diagram holds:

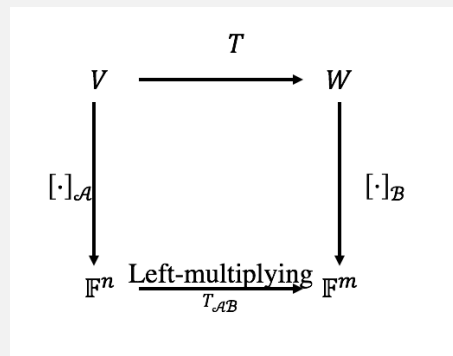


Figure 3.2: Diagram for the matrix representation, where $n := \dim(V)$ and $m := \dim(W)$

namely, for any $\mathbf{v} \in V$,

$$(T)_{\mathcal{B}, \mathcal{A}}(\mathbf{v})_{\mathcal{A}} = (T\mathbf{v})_{\mathcal{B}}$$

Therefore, we can compute $T\mathbf{v}$ by matrix multiplication.

Therefore, linear transformation corresponds to coordinate matrix multiplication.

Proof. Suppose $\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$. The proof of this theorem follows the same procedure of that in Theorem (3.1)

1. We show this result for $\mathbf{v} = \mathbf{v}_j$ first:

$$\begin{aligned} \text{LHS} &= [\alpha_{ij}] \mathbf{e}_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} \\ \text{RHS} &= (T\mathbf{v}_j)_{\mathcal{B}} = \left(\sum_{i=1}^m \alpha_{ij} \mathbf{w}_i \right)_{\mathcal{B}} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} \end{aligned}$$

2. Then we show the theorem holds for any $\mathbf{v} := \sum_{j=1}^n r_j \mathbf{v}_j$ in V :

$$(T)_{\mathcal{B}, \mathcal{A}}(\mathbf{v})_{\mathcal{A}} = (T)_{\mathcal{B}, \mathcal{A}} \left(\sum_{j=1}^n r_j \mathbf{v}_j \right)_{\mathcal{A}} \quad (3.8a)$$

$$= (T)_{\mathcal{B}, \mathcal{A}} \left(\sum_{j=1}^n r_j (\mathbf{v}_j)_{\mathcal{A}} \right) \quad (3.8b)$$

$$= \sum_{j=1}^n r_j (T)_{\mathcal{B}, \mathcal{A}}(\mathbf{v}_j)_{\mathcal{A}} \quad (3.8c)$$

$$= \sum_{j=1}^n r_j (T\mathbf{v}_j)_{\mathcal{B}} \quad (3.8d)$$

$$= \left(\sum_{j=1}^n r_j (T\mathbf{v}_j) \right)_{\mathcal{B}} \quad (3.8e)$$

$$= \left[T \left(\sum_{j=1}^n r_j \mathbf{v}_j \right) \right]_{\mathcal{B}} \quad (3.8f)$$

$$= (T\mathbf{v})_{\mathcal{B}} \quad (3.8g)$$

The justification for (3.8a) is similar to that shown in Theorem (3.1). The proof is complete. ■

- R Consider a special case for Theorem (3.3), i.e., $T = \text{id}$ and $\mathcal{A}, \mathcal{A}'$ are two ordered basis for the input and output space, respectively. Then the result in Theorem (3.3) implies

$$C_{\mathcal{A}', \mathcal{A}}(\mathbf{v})_{\mathcal{A}} = (\mathbf{v})_{\mathcal{A}'}$$

i.e., the matrix representation theorem (3.3) is a general case for the change of basis theorem (3.1)

Proposition 3.6 — Functoriality. Suppose V, W, U are finite dimensional vector spaces, and let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the ordered basis for V, W, U , respectively. Suppose that

$$T : V \rightarrow W, \quad S : W \rightarrow U$$

are given two linear transformations, then

$$(S \circ T)_{\mathcal{C}, \mathcal{A}} = (S)_{\mathcal{C}, \mathcal{B}}(T)_{\mathcal{B}, \mathcal{A}}$$

Composition of linear transformation corresponds to the multiplication of change of basis matrices.

Proof. Suppose the ordered basis $\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$, $\mathcal{C} = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$. By definition of change of basis matrices,

$$T(\mathbf{v}_j) = \sum_i (T_{\mathcal{B}, \mathcal{A}})_{ij} \mathbf{w}_i$$

$$S(\mathbf{w}_i) = \sum_k (S_{\mathcal{C}, \mathcal{B}})_{ki} \mathbf{u}_k$$

We start from the j -th column of $(S \circ T)_{C, \mathcal{A}}$ for $j = 1, \dots, n$, namely

$$(S \circ T)_{C, \mathcal{A}}(\mathbf{v}_j)_{\mathcal{A}} = (S \circ T(\mathbf{v}_j))_C \quad (3.9a)$$

$$= \left[S \circ \left(\sum_i (T_{\mathcal{B}, \mathcal{A}})_{ij} \mathbf{w}_i \right) \right]_C \quad (3.9b)$$

$$= \sum_i (T_{\mathcal{B}, \mathcal{A}})_{ij} (S(\mathbf{w}_i))_C \quad (3.9c)$$

$$= \sum_i (T_{\mathcal{B}, \mathcal{A}})_{ij} \left(\sum_k (S_{C, \mathcal{B}})_{ki} \mathbf{u}_k \right)_C \quad (3.9d)$$

$$= \sum_k \sum_i (S_{C, \mathcal{B}})_{ki} (T_{\mathcal{B}, \mathcal{A}})_{ij} (\mathbf{u}_k)_C \quad (3.9e)$$

$$= \sum_k (S_{C, \mathcal{B}} T_{\mathcal{B}, \mathcal{A}})_{kj} (\mathbf{u}_k)_C \quad (3.9f)$$

$$= \sum_k (S_{C, \mathcal{B}} T_{\mathcal{B}, \mathcal{A}})_{kj} \mathbf{e}_k \quad (3.9g)$$

$$= j\text{-th column of } [S_{C, \mathcal{B}} T_{\mathcal{B}, \mathcal{A}}] \quad (3.9h)$$

where (3.9a) is by the result in theorem (3.3); (3.9b) and (3.9d) follows from definitions of $T(\mathbf{v}_j)$ and $S(\mathbf{w}_i)$; (3.9c) and (3.9e) follows from the linearity of C ; (3.9f) follows from the matrix multiplication definition; (3.9g) is because $(\mathbf{u}_k)_C = \mathbf{e}_k$.

Therefore, $(S \circ T)_{C, \mathcal{A}}$ and $(S_{C, \mathcal{B}})(T_{\mathcal{B}, \mathcal{A}})$ share the same j -th column, and thus equal to each other. ■

Corollary 3.2 Suppose that S and T are two identity mappings $V \rightarrow V$, and consider $(S)_{\mathcal{A}', \mathcal{A}}$ and $(T)_{\mathcal{A}, \mathcal{A}'}$ in proposition (3.6), then

$$(S \circ T)_{\mathcal{A}', \mathcal{A}'} = (S)_{\mathcal{A}', \mathcal{A}} (T)_{\mathcal{A}, \mathcal{A}'}$$

Therefore,

$$\text{Identity matrix} = C_{\mathcal{A}', \mathcal{A}} C_{\mathcal{A}, \mathcal{A}'}$$

Proposition 3.7 Let $T : V \rightarrow W$ with $\dim(V) = n, \dim(W) = m$, and let

- $\mathcal{A}, \mathcal{A}'$ be ordered basis of V

- $\mathcal{B}, \mathcal{B}'$ be ordered basis of W

then the change of basis matrices admit the relation

$$(T)_{\mathcal{B}', \mathcal{A}'} = C_{\mathcal{B}', \mathcal{B}}(T)_{\mathcal{B}, \mathcal{A}}C_{\mathcal{A}, \mathcal{A}'} \quad (3.10)$$

Here note that $(T)_{\mathcal{B}', \mathcal{A}'}, (T)_{\mathcal{B}, \mathcal{A}} \in \mathbb{F}^{m \times n}$; $C_{\mathcal{B}', \mathcal{B}} \in \mathbb{F}^{m \times m}$; and $C_{\mathcal{A}, \mathcal{A}'} \in \mathbb{F}^{n \times n}$.

Proof. Let $\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}, \mathcal{A}' = \{\mathbf{v}'_1, \dots, \mathbf{v}'_n\}$. Consider simplifying the j -th column for the LHS and RHS of (3.10) and showing they are equal:

$$\begin{aligned} \text{LHS} &= (T)_{\mathcal{B}', \mathcal{A}'} \mathbf{e}_j \\ &= (T)_{\mathcal{B}', \mathcal{A}'}(\mathbf{v}'_j)_{\mathcal{A}'} \\ &= (T\mathbf{v}'_j)_{\mathcal{B}'} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= C_{\mathcal{B}', \mathcal{B}}(T)_{\mathcal{B}, \mathcal{A}}C_{\mathcal{A}, \mathcal{A}'} \mathbf{e}_j \\ &= C_{\mathcal{B}', \mathcal{B}}(T)_{\mathcal{B}, \mathcal{A}}C_{\mathcal{A}, \mathcal{A}'}(\mathbf{v}'_j)_{\mathcal{A}'} \\ &= C_{\mathcal{B}', \mathcal{B}}(T)_{\mathcal{B}, \mathcal{A}}(\mathbf{v}'_j)_{\mathcal{A}} \\ &= C_{\mathcal{B}', \mathcal{B}}(T\mathbf{v}'_j)_{\mathcal{B}} \\ &= (T\mathbf{v}'_j)_{\mathcal{B}'} \end{aligned}$$

■

R Let $T : V \rightarrow V$ be a linear operator with $\mathcal{A}, \mathcal{A}'$ being two ordered basis of V , then

$$(T)_{\mathcal{A}', \mathcal{A}'} = C_{\mathcal{A}', \mathcal{A}}(T)_{\mathcal{A}, \mathcal{A}}C_{\mathcal{A}, \mathcal{A}'} = (C_{\mathcal{A}, \mathcal{A}'})^{-1}(T)_{\mathcal{A}, \mathcal{A}}C_{\mathcal{A}, \mathcal{A}'}$$

Therefore, the change of basis matrices $(T)_{\mathcal{A}', \mathcal{A}'}$ and $(T)_{\mathcal{A}, \mathcal{A}}$ are similar to each other, which means they share the same eigenvalues, determinant, trace.

Therefore, two similar matrices corresponds to same linear transformation using different basis.

Chapter 4

Week4

4.1. Monday for MAT3040

4.1.1. Quotient Spaces

Now we aim to divide a big **vector space** into many pieces of slices.

- For example, the Cartesian plane can be expressed as union of set of vertical lines as follows:

$$\mathbb{R}^2 = \bigcup_{m \in \mathbb{R}} \left\{ \begin{pmatrix} m \\ 0 \end{pmatrix} + \text{span}\{(0,1)\} \right\}$$

- Another example is that the set of integers can be expressed as union of three sets:

$$\mathbb{Z} = Z_1 \cup Z_2 \cup Z_3,$$

where Z_i is the set of integers z such that $z \bmod 3 = i$.

Definition 4.1 [Coset] Let V be a vector space and $W \leq V$. For any element $\mathbf{v} \in V$, the **(right) coset** determined by \mathbf{v} is the set

$$\mathbf{v} + W := \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in W\}$$

For example, consider $V = \mathbb{R}^3$ and $W = \text{span}\{(1,2,0)\}$. Then the coset determined by

$\mathbf{v} = (5, 6, -3)$ can be written as

$$\mathbf{v} + W = \{(5 + t, 6 + 2t, -3) \mid t \in \mathbb{R}\}$$

It's interesting that the coset determined by $\mathbf{v}' = (4, 4, -3)$ is exactly the same as the coset shown above:

$$\mathbf{v}' + W = \{(4 + t, 4 + 2t, -3) \mid t \in \mathbb{R}\} = \mathbf{v} + W.$$

Therefore, write the exact expression of $\mathbf{v} + W$ may sometimes become tedious and hard to check the equivalence. We say \mathbf{v} is a **representative** of a coset $\mathbf{v} + W$.

Proposition 4.1 Two cosets are the same iff the subtraction for the corresponding representatives is in W , i.e.,

$$\mathbf{v}_1 + W = \mathbf{v}_2 + W \iff \mathbf{v}_1 - \mathbf{v}_2 \in W$$

Proof. Necessity. Suppose that $\mathbf{v}_1 + W = \mathbf{v}_2 + W$, then $\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_2$ for some $\mathbf{w}_1, \mathbf{w}_2 \in W$, which implies

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{w}_2 - \mathbf{w}_1 \in W$$

Sufficiency. Suppose that $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{w} \in W$. It suffices to show $\mathbf{v}_1 + W \subseteq \mathbf{v}_2 + W$. For any $\mathbf{v}_1 + \mathbf{w}' \in \mathbf{v}_1 + W$, this element can be expressed as

$$\mathbf{v}_1 + \mathbf{w}' = (\mathbf{v}_2 + \mathbf{w}) + \mathbf{w}' = \mathbf{v}_2 + \underbrace{(\mathbf{w} + \mathbf{w}')}_{\text{belong to } W} \in \mathbf{v}_2 + W.$$

Therefore, $\mathbf{v}_1 + W \subseteq \mathbf{v}_2 + W$. Similarly we can show that $\mathbf{v}_2 + W \subseteq \mathbf{v}_1 + W$. ■

Exercise: Two cosets with representatives $\mathbf{v}_1, \mathbf{v}_2$ have no intersection iff $\mathbf{v}_1 - \mathbf{v}_2 \notin W$.

Definition 4.2 [Quotient Space] The **quotient space** of V by the subspace W , is the collection of all cosets $\mathbf{v} + W$, denoted by V/W . ■

To make the quotient space a vector space structure, we define the addition and scalar

multiplication on V/W by:

$$(\mathbf{v}_1 + W) + (\mathbf{v}_2 + W) := (\mathbf{v}_1 + \mathbf{v}_2) + W$$

$$\alpha \cdot (\mathbf{v} + W) := (\alpha \cdot \mathbf{v}) + W$$

For example, consider $V = \mathbb{R}^2$ and $W = \text{span}\{(0,1)\}$. Then note that

$$\begin{aligned} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + W \right) + \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} + W \right) &= \left(\begin{pmatrix} 3 \\ 0 \end{pmatrix} + W \right) \\ \pi \cdot \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + W \right) &= \left(\begin{pmatrix} \pi \\ 0 \end{pmatrix} + W \right) \end{aligned}$$

Proposition 4.2 The addition and scalar multiplication is well-defined.

Proof. 1. Suppose that

$$\begin{cases} \mathbf{v}_1 + W = \mathbf{v}'_1 + W \\ \mathbf{v}_2 + W = \mathbf{v}'_2 + W \end{cases}, \quad (4.1)$$

and we need to show that $(\mathbf{v}_1 + \mathbf{v}_2) + W = (\mathbf{v}'_1 + \mathbf{v}'_2) + W$.

From (4.1) and proposition (4.1), we imply

$$\mathbf{v}_1 - \mathbf{v}'_1 \in W, \quad \mathbf{v}_2 - \mathbf{v}'_2 \in W$$

which implies

$$(\mathbf{v}_1 - \mathbf{v}'_1) + (\mathbf{v}_2 - \mathbf{v}'_2) = (\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}'_1 + \mathbf{v}'_2) \in W$$

By proposition (4.1) again we imply $(\mathbf{v}_1 + \mathbf{v}_2) + W = (\mathbf{v}'_1 + \mathbf{v}'_2) + W$

2. For scalar multiplication, similarly, we can show that $\mathbf{v}_1 + W = \mathbf{v}'_1 + W$ implies

$\alpha \mathbf{v}_1 + W = \alpha \mathbf{v}'_1 + W$ for all $\alpha \in \mathbb{F}$.

■

Proposition 4.3 The canonical projection mapping

$$\pi_W : V \rightarrow V/W,$$

$$\mathbf{v} \mapsto \mathbf{v} + W,$$

is a **surjective linear transformation** with $\ker(\pi_W) = W$.

Proof. 1. First we show that $\ker(\pi_W) = W$:

$$\pi_W(\mathbf{v}) = 0 \implies \mathbf{v} + W = \mathbf{0}_{V/W} \implies \mathbf{v} + W = \mathbf{0} + W \implies \mathbf{v} = (\mathbf{v} - \mathbf{0}) \in W$$

Here note that the zero element in the quotient space V/W is the coset with representative $\mathbf{0}$.

2. For any $\mathbf{v}_0 + W \in V/W$, we can construct $\mathbf{v}_0 \in V$ such that $\pi_W(\mathbf{v}_0) = \mathbf{v}_0 + W$. Therefore the mapping π_W is surjective.
3. To show the mapping π_W is a linear transformation, note that

$$\begin{aligned} \pi_W(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) &= (\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) + W \\ &= (\alpha \mathbf{v}_1 + W) + (\beta \mathbf{v}_2 + W) \\ &= \alpha(\mathbf{v}_1 + W) + \beta(\mathbf{v}_2 + W) \\ &= \alpha \pi_W(\mathbf{v}_1) + \beta \pi_W(\mathbf{v}_2) \end{aligned}$$

■

4.1.2. First Isomorphism Theorem

The key of linear algebra is to solve the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{m \times n}$. The general step for solving this linear system is as follows:

1. Find the solution set for $\mathbf{A}\mathbf{x} = \mathbf{0}$, i.e., the set $\ker(\mathbf{A})$
2. Find a particular solution \mathbf{x}_0 such that $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$.

Then the general solution set to this linear system is $\mathbf{x}_0 + \ker(\mathbf{A})$, which is a coset in

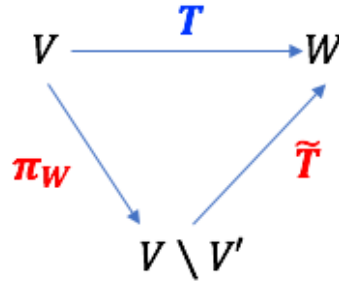
the space $\mathbb{R}^n / \ker(\mathbf{A})$. Therefore, to solve the linear system $\mathbf{Ax} = \mathbf{b}$ suffices to study the quotient space $\mathbb{R}^n / \ker(\mathbf{A})$:

Proposition 4.4 — Universal Property I. Suppose that $T : V \rightarrow W$ is a linear transformation, and that $V' \leq \ker(T)$. Then the mapping

$$\tilde{T} : V/V' \rightarrow W$$

$$\mathbf{v} + V' \mapsto T(\mathbf{v})$$

is a well-defined linear transformation. As a result, the diagram below commutes:



In other words, we have $T = \tilde{T} \circ \pi_W$.

Proof. First we show the well-definedness. Suppose that $\mathbf{v}_1 + V' = \mathbf{v}_2 + V'$ and suffices to show $\tilde{T}(\mathbf{v}_1 + V') = \tilde{T}(\mathbf{v}_2 + V')$, i.e., $T(\mathbf{v}_1) = T(\mathbf{v}_2)$. By proposition (4.1), we imply

$$\mathbf{v}_1 - \mathbf{v}_2 \in V' \leq \ker(T) \implies T(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0} \implies T(\mathbf{v}_1) - T(\mathbf{v}_2) = \mathbf{0}.$$

Then we show \tilde{T} is a linear transformation:

$$\begin{aligned}
 \tilde{T}(\alpha(\mathbf{v}_1 + V') + \beta(\mathbf{v}_2 + V')) &= \tilde{T}((\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) + V') \\
 &= T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) \\
 &= \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) \\
 &= \alpha \tilde{T}(\mathbf{v}_1 + V') + \beta \tilde{T}(\mathbf{v}_2 + V')
 \end{aligned}$$

■

Actually, if we let $V' = \ker(T)$, the mapping $\tilde{T} : V/V' \rightarrow T(V)$ forms an isomorphism. In particular, if further T is surjective, then $T(V) = W$, i.e., the mapping $\tilde{T} : V/V' \rightarrow W$ forms an isomorphism.

Theorem 4.1 — First Isomorphism Theorem. Let $T : V \rightarrow W$ be a surjective linear transformation. Then the mapping

$$\begin{aligned}\tilde{T} : V/\ker(T) &\rightarrow W \\ \mathbf{v} + \ker(T) &\mapsto T(\mathbf{v})\end{aligned}$$

is an isomorphism.

Proof. Injectivity. Suppose that $\tilde{T}(\mathbf{v}_1 + \ker(T)) = \tilde{T}(\mathbf{v}_2 + \ker(T))$, then we imply

$$T(\mathbf{v}_1) = T(\mathbf{v}_2) \implies T(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}_W \implies \mathbf{v}_1 - \mathbf{v}_2 \in \ker(T),$$

i.e., $\mathbf{v}_1 + \ker(T) = \mathbf{v}_2 + \ker(T)$.

Surjectivity. For $\mathbf{w} \in W$, due to the surjectivity of T , we can find a \mathbf{v}_0 such that $T(\mathbf{v}_0) = \mathbf{w}$. Therefore, we can construct a set $\mathbf{v}_0 + \ker(T)$ such that

$$\tilde{T}(\mathbf{v}_0 + \ker(T)) = \mathbf{w}.$$

■

4.4. Wednesday for MAT3040

Reviewing.

- Quotient Space:

$$V/W = \{v + W \mid v \in V\}$$

The elements in V/W are cosets. Note that V/W does not mean a subset of V .

- Define the canonical projection mapping

$$\pi_W : V \rightarrow V/W,$$

$$\text{with } v \mapsto v + W,$$

then we imply π_W is a surjective linear transformation with $\ker(\pi_W) = W$.

If $\dim(V) < \infty$, then by Rank-Nullity Theorem (2.3), we imply that

$$\dim(V) = \dim(W) + \dim(V/W),$$

i.e., $\dim(V/W) = \dim(V) - \dim(W)$.

- **(Universal Property I)** Every linear transformation $T : V \rightarrow W$ with $V' \leq \ker(T)$ can be descended to the composition of the canonical projection mapping $\pi_{V'}$ and the mapping

$$\tilde{T} : V/V' \rightarrow W$$

$$\text{with } v + V' \mapsto T(v).$$

In other words, the diagram (2.1) commutes:

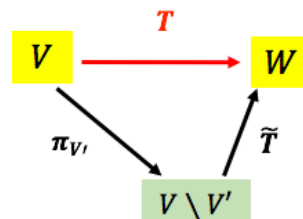


Diagram (2.1)

In other words, the mapping starting from either the black or red line gives the same result, i.e., $T(\mathbf{v}) = \tilde{T} \circ \pi_{V'}(\mathbf{v}) = \tilde{T}(\mathbf{v} + V')$ for any $\mathbf{v} \in V$.

- **(First Isomorphism Theorem)** Under the setting of Universal Property I (UPI), if T is a surjective linear transformation with $V' = \ker(T)$, then the \tilde{T} is an isomorphism.

■ **Example 4.2** Suppose that $U, W \leq V$ with $U \cap W = \{\mathbf{0}\}$, then define the mapping

$$\begin{aligned} \phi : U \oplus W &\rightarrow U \\ \text{with } \phi(\mathbf{u} + \mathbf{w}) &= \mathbf{u} \end{aligned}$$

Ⓡ Exercise: if $U, W \leq V$ but $U \cap W \neq \{\mathbf{0}\}$, then the mapping

$$\begin{aligned} \phi : U + W &\rightarrow U \\ \text{with } \mathbf{u} + \mathbf{w} &\mapsto \mathbf{u} \end{aligned} \quad \text{is not well-defined:}$$

Suppose that $\mathbf{0} \neq \mathbf{v} \in U \cap W$ and for any $\mathbf{u} \in U, \mathbf{w} \in W$, we construct

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} \in U, \quad \mathbf{w}' = \mathbf{w} + \mathbf{v} \in W \implies \phi(\mathbf{u}' + \mathbf{w}') = \mathbf{u} - \mathbf{v}$$

Therefore we get $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$ but $\phi(\mathbf{u} + \mathbf{w}) \neq \phi(\mathbf{u}' + \mathbf{w}')$.

Back to the situation $U \cap W = \{\mathbf{0}\}$, then it's clear that $\phi : U \oplus W \rightarrow U$ is surjective linear transformation with $\ker(\phi) = W$. Therefore, construct the new mapping

$$\begin{aligned} \tilde{\phi} : U \oplus W / W &\rightarrow U \\ \text{with } \mathbf{u} + \mathbf{w} + W &\mapsto \phi(\mathbf{u} + \mathbf{w}) \end{aligned}$$

We imply $\tilde{\phi}$ is an isomorphism by First Isomorphism Theorem. ■

Now we study the generalized quotients, which is defined to satisfy the generalized version of universal property I.

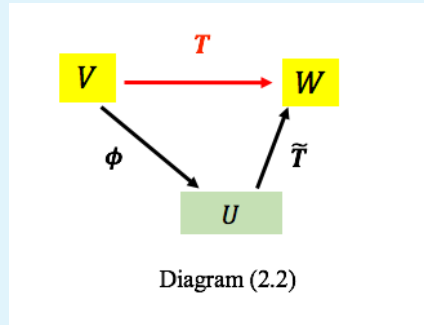
Definition 4.7 [Universal Property for Quotients] Let V be a vector space and $V' \leq V$. Consider the collection of linear transformations

$$\text{Obj} = \left\{ T : V \rightarrow W \left| \begin{array}{l} T \text{ is a linear transformation} \\ V' \leq \ker(T) \end{array} \right. \right\}$$

(For example, $\pi_{V'} : V \rightarrow V/V'$ is an element from the set Obj .)

An element $(\phi : V \rightarrow U) \in \text{Obj}$ is said to satisfy the **universal property** if it satisfies the following:

Given any element $(T : V \rightarrow W) \in \text{Obj}$, we can extend the transformation ϕ with a **uniquely existing** $\tilde{T} : U \rightarrow W$ so that the diagram (2.2) commutes:



Or equivalently, for given $(T : V \rightarrow W) \in \text{Obj}$, there exists the **unique** mapping $\tilde{T} : U \rightarrow W$ such that $T = \tilde{T} \circ \phi$.

Theorem 4.3 — Universal Property II.

1. The mapping $(\pi_{V'} : V \rightarrow V/V') \in \text{Obj}$ is a universal object, i.e., it satisfies the universal property.
2. If $(\phi : V \rightarrow U)$ is a universal object, then $U \cong V/V'$, i.e., there is intrinsically “one” element in the set of universal objects.

Proof. 1. Consider any linear transformation $T : V \rightarrow W$ such that $V' \leq \ker(T)$, then define (construct) the same $\tilde{T} : V/V' \rightarrow W$ as that in UPI. Therefore, for given T , applying the result of UPI, we imply $T = \tilde{T} \circ \pi_{V'}$, i.e., $\pi_{V'}$ satisfies the diagram (2.2).

To show the uniqueness of \tilde{T} , suppose there exists $\tilde{S} : V/V' \rightarrow W$ such that the diagram (2.3) commutes.

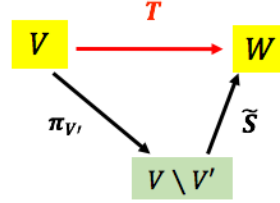


Diagram (2.3)

It suffices to show the mapping $\tilde{S} = \tilde{T}$: for any $\mathbf{v} + V' \in V/V'$, we have

$$\tilde{S}(\mathbf{v} + V') := \tilde{S} \circ \pi_{V'}(\mathbf{v}) = T(\mathbf{v}),$$

where the first equality is due to the surjectivity of $\pi_{V'}$. By the result of UPI, $T(\mathbf{v}) = \tilde{T}(\mathbf{v} + V')$. Therefore $\tilde{T}(\mathbf{v} + V') = \tilde{S}(\mathbf{v} + V')$ for all $\mathbf{v} + V' \in V/V'$. The proof is complete.

2. Suppose that $(\phi : V \rightarrow U)$ satisfies the universal property. In particular, the following two diagrams hold:

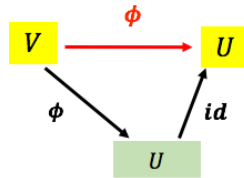


Diagram (2.4)

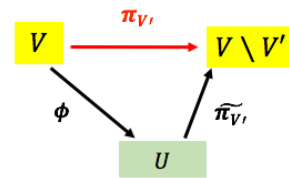


Diagram (2.5)

Since $(\pi_{V'})$ satisfies the universal property, in particular, the following two diagrams hold:

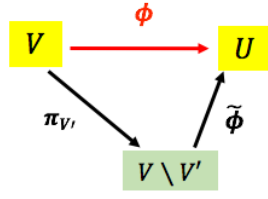


Diagram (2.6)

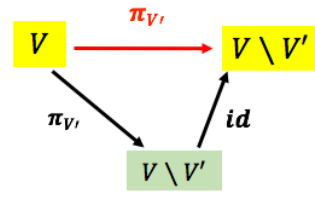


Diagram (2.7)

Then we claim that: Combining Diagram (2.5) and (2.6), we imply the diagram (2.8):

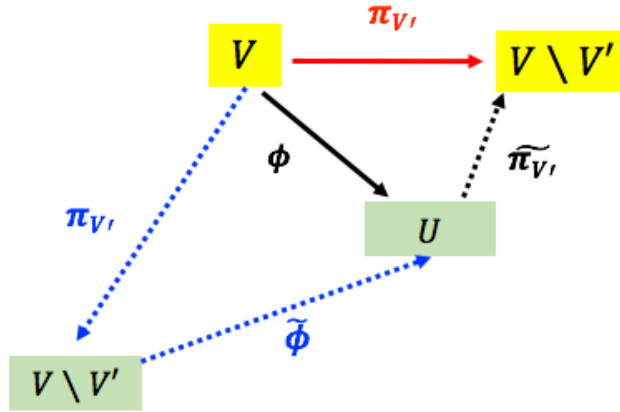


Diagram (2.8)

Graph Description: Note that this diagram commutes, i.e., the mapping starting from either the red line or the dash line gives the same result, i.e., $\pi_{V'} = \tilde{\pi}_{V'} \circ \tilde{\phi} \circ \pi_{V'}$. Comparing Diagram (2.7) and Diagram (2.8), we have $\tilde{\pi}_{V'} \circ \tilde{\phi} = id$, by the **uniqueness** of the universal object.

Therefore, $\tilde{\pi}_{V'} \circ \tilde{\phi} = id$ implies $\tilde{\pi}_{V'}$ is surjective and $\tilde{\phi}$ is injective.

Also, combining Diagram (2.6) and (2.5), we imply diagram (2.9):

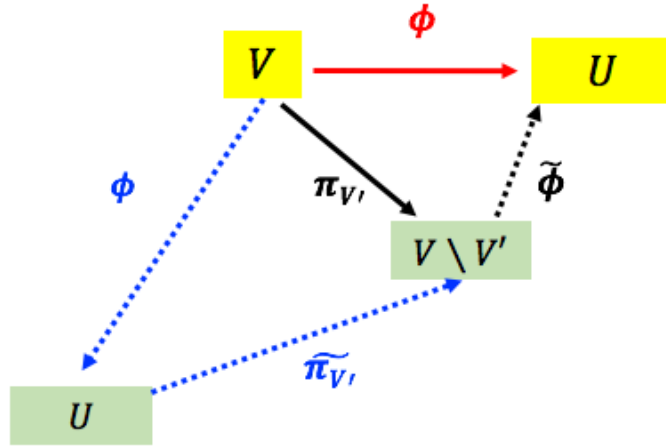


Diagram (2.9)

Graph Description: Note that this diagram commutes, i.e., the mapping starting from either the red line or the dash line gives the same result, i.e., $\phi = \tilde{\phi} \circ \pi_{V'} \circ \phi$. Comparing Diagram (2.9) and Diagram (2.4), we have $\tilde{\phi} \circ \pi_{V'} = id$, by the **uniqueness** of the universal object

Therefore, $\tilde{\phi} \circ \pi_{V'} = id$ implies $\tilde{\phi}$ is surjective and $\pi_{V'}$ is injective.

Therefore, both $\tilde{\phi} : U \rightarrow V/V'$ and $\pi_{V'} : V/V' \rightarrow U$ are bijective, i.e., $U \cong V/V'$. The proof is complete.

■

4.4.1. Dual Space

Definition 4.8 Let V be a vector space over a field \mathbb{F} . The **dual vector space** V^* is defined as

$$\begin{aligned} V^* &= \text{Hom}_{\mathbb{F}}(V, \mathbb{F}) \\ &= \{f : V \rightarrow \mathbb{F} \mid f \text{ is a linear transformation}\} \end{aligned}$$

■

- **Example 4.3** 1. Consider $V = \mathbb{R}^n$ and define $\phi_i : V \rightarrow \mathbb{R}$ as the i -th component of input:

$$\phi_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i,$$

Then we imply $\phi_i \in V^*$. On the contrary, $\phi_i^2 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i^2$ is not in V^*

2. Consider $V = \mathbb{F}[x]$ and define $\phi : V \rightarrow \mathbb{F}$ as:

$$\phi(p(x)) = p(1),$$

It's clear that $\phi \in V^*$:

$$\begin{aligned} \phi(ap(x) + bq(x)) &= ap(1) + bq(1) \\ &= a\phi(p(x)) + b\phi(q(x)) \end{aligned}$$

3. Also, $\psi : V \rightarrow \mathbb{F}$ by $\psi(p(x)) = \int_0^1 p(x) dx$ is in V^* .
 4. Also, for $V = M_{n \times n}(\mathbb{F})$, the mapping $\text{tr} : V \rightarrow \mathbb{F}$ by $\text{tr}(M) = \sum_{i=1}^n M_{ii}$ is in V^* . However, the $\det : V \rightarrow \mathbb{F}$ is not in V^*

Definition 4.9 Let V be a vector space, with basis $B = \{v_i \mid i \in I\}$ (I can be finite or countable, or uncountable). Define

$$B^* = \{f_i : V \rightarrow \mathbb{F} \mid i \in I\},$$

where f_i 's are defined on the basis B :

$$f_i(v_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Then we extend f_i 's linearly, i.e., for $\sum_{j=1}^N \alpha_j v_j \in V$,

$$f_i\left(\sum_{j=1}^N \alpha_j v_j\right) = \sum_{j=1}^N \alpha_j f_i(v_j).$$

It's clear that $f_i \in V^*$ is well-defined. ■

Our question is that whether the B^* can be the basis of V^* ?

Chapter 5

Week5

5.1. Monday for MAT3040

Reviewing.

- Dual space: the set of linear transformations from V to \mathbb{F} , denoted as $\text{Hom}(V, \mathbb{F})$.
- Suppose $B = \{\mathbf{v}_i \mid i \in I\}$ is the basis of V , define $B^* = \{f_i \mid i \in I\}$ by

$$f_i(\mathbf{v}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Actually, the above recipe uniquely defines a linear transformation $f_i : V \rightarrow \mathbb{F}$:

For any $\mathbf{v} \in V$, it can be written as $\mathbf{v} = \sum_{i \in I} \alpha_i \mathbf{v}_i$, and therefore

$$f_i(\mathbf{v}) = f_i\left(\sum_{i \in I} \alpha_i \mathbf{v}_i\right) = \sum_{i \in I} \alpha_i f_i(\mathbf{v}_i).$$

■ **Example 5.1** Consider $V = \mathbb{R}^n, B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then we imply $B^* = \{\phi_i\}_{i=1}^n$, where ϕ_i is the mapping $V \rightarrow \mathbb{R}$ defined by

$$\phi_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \phi(x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n) = \sum_{j=1}^n x_j \phi_i(\mathbf{e}_j) = x_i$$

5.1.1. Remarks on Dual Space

- Proposition 5.1**
1. B^* is always linearly independent, i.e., any finite subset of B^* is linearly independent.
 2. If V has finite dimension, then B^* is a basis of V^* .

Proof. 1. Suppose that

$$\alpha_1 f_{i_1} + \alpha_2 f_{i_2} + \cdots + \alpha_k f_{i_k} = \mathbf{0}_{V^*}.$$

In particular, let the input of these linear transformations be \mathbf{v}_{i_1} , we imply

$$\begin{aligned} \alpha_1 f_{i_1}(\mathbf{v}_{i_1}) + \alpha_2 f_{i_2}(\mathbf{v}_{i_1}) + \cdots + \alpha_k f_{i_k}(\mathbf{v}_{i_1}) &= \mathbf{0}(\mathbf{v}_{i_1}) \equiv \mathbf{0} \\ &= \alpha_1 \cdot 1 + \cdots + 0 \\ &= \alpha_1 \end{aligned}$$

Applying the same trick, one can show that $\alpha_2 = \cdots = \alpha_k = 0$. Therefore, $\{f_{i_1}, \dots, f_{i_k}\}$ is linearly independent.

2. Suppose that $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $B^* = \{f_1, \dots, f_n\}$. For any $f \in V^*$, construct the linear transformation

$$g := \sum_{i=1}^n f(\mathbf{v}_i) \cdot f_i \in \text{span}\{B^*\}.$$

It follows that for $j = 1, 2, \dots, n$,

$$g(\mathbf{v}_j) = \sum_{i=1}^n f(\mathbf{v}_i) \cdot f_i(\mathbf{v}_j) = f(\mathbf{v}_j).$$

It's clear that $g(\mathbf{v}) = f(\mathbf{v})$ for all $\mathbf{v} \in V$, i.e., $f \equiv g \in \text{span}(B^*)$. Therefore B^* spans V^* , i.e., forms a basis of V^* .

■

Corollary 5.1 If $\dim(V) = n$, then $\dim(V^*) = n$.

Proof. It's easy to show the mapping defined as

$$V \rightarrow V^*$$

with $v_i \mapsto f_i$

is an isomorphism from $V \rightarrow V^*$. Note that this constructed isomorphism depends on the choice of basis B in V . (We say this is not a **natural isomorphism**.) ■

- Ⓡ The part 2 for proposition (5.1) does not hold for V with infinite dimension. The reason is that the spanning set is defined with **finite** linear combinations. Check the example below for a counter-example.

■ **Example 5.2** Suppose that $V = \mathbb{F}[x]$, and $B^* = \{1, x, x^2, \dots\}$ forms a basis of V . We imply that $B^* = \{\phi_0, \phi_1, \phi_2, \dots\}$, where ϕ_i is the mapping defined as

$$\phi_i(x^j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

Consider a special element $\phi \in V^*$ with $f(p(x)) = p(1)$:

$$\phi(1) = 1, \quad \phi(x) = 1, \quad \phi(x^2) = 1, \quad \dots \quad \phi(x^n) = 1, \quad \forall n \in \mathbb{N}.$$

If following the proof in proposition (5.1), we expect that

$$g := \sum_{n=0}^{\infty} \phi(x^n) \phi_n = \sum_{n=0}^{\infty} \phi_n \in \text{span}\{B^*\},$$

which is a contradiction, since $\text{span}\{B^*\}$ consists of finite sum of ϕ_i 's only. ■

- Ⓡ Therefore, if V is not finite-dimensional, we can say the cardinality of V is strictly less than the cardinality of V^* .

Any subspace of a given vector space has some gap. Now we want to describe this gap formally from the perspective of the dual space.

5.1.2. Annihilators

Definition 5.1 Let V be a vector space, $S \subseteq V$ be a subset. The **annihilator** of S is defined as

$$\text{Ann}(S) = \{f \in V^* \mid f(s) = 0, \forall s \in S\}$$

■ **Example 5.3** Consider $V = \mathbb{R}^4$, $B = \{e_1, \dots, e_4\}$. Let $B^* = \{f_1, \dots, f_4\}$, $S = \{e_3, e_4\}$.

- Then $f_1 \in \text{Ann}(S)$, since

$$f_1(e_3) = 0, \quad f_1(e_4) = 0$$

Indeed, any $a \cdot f_1 + b \cdot f_2 \in V^*$ is in $\text{Ann}(S)$. ■

Proposition 5.2

1. The set $\text{Ann}(S)$ is a vector subspace of V^*
2. The mapping $\text{Ann}(\cdot)$ is **inclusion-reversing**, i.e., if $W_1 \subseteq W_2 \subseteq V$, then

$$\text{Ann}(W_1) \supseteq \text{Ann}(W_2)$$

3. The mapping $\text{Ann}(\cdot)$ is **idempotent**, i.e., $\text{Ann}(S) = \text{Ann}(\text{span}(S))$.
4. If V has finite dimension, and $W \leq V$, then $\text{Ann}(W)$ fills in the gap, i.e.,

$$\dim(W) + \dim(\text{Ann}(W)) = \dim(V)$$

Proof. 1. Suppose that $f, g \in \text{Ann}(S)$, i.e., $f(s) = g(s) = 0, \forall s \in S$. It's clear that $(af + bg) \in \text{Ann}(S)$.

2. Suppose that $f \in \text{Ann}(W_2)$, we imply $f(w) = 0$ for any $w \in W_2$. Therefore, $f(w_1) = 0$ for any $w_1 \in W_1 \subseteq W_2$, i.e., $f \in \text{Ann}(W_1)$.

3. Note that $S \subseteq \text{span}(S)$. Therefore we imply $\text{Ann}(S) \supseteq \text{Ann}(\text{span}(S))$ from part (b). It suffices to show $\text{Ann}(S) \subseteq \text{Ann}(\text{span}(S))$:

For any $f \in \text{Ann}(S)$ and any $\sum_{i=1}^n k_i \mathbf{s}_i \in \text{span}(S)$, we imply

$$\begin{aligned} f\left(\sum_{i=1}^n k_i \mathbf{s}_i\right) &= \sum_{i=1}^n k_i f(\mathbf{s}_i) \\ &= \sum_{i=1}^n k_i \cdot 0 \\ &= 0, \end{aligned}$$

i.e., $f \in \text{Ann}(\text{span}(S))$.

4. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis of W . By basis extension, we construct a basis of V :

$$B = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}.$$

Let $B^* = \{f_1, \dots, f_k, f_{k+1}, \dots, f_n\}$ be a basis of V^* . We claim that $\{f_{k+1}, \dots, f_n\}$ is a basis of $\text{Ann}(W)$:

- Firstly, f_j 's are the elements in $\text{Ann}(W)$ for $j = k+1, \dots, n$, since for any $\mathbf{w} = \sum_{i=1}^k \alpha_i(\mathbf{v}_i) \in W$, we have

$$\begin{aligned} f_j(\mathbf{w}) &= \sum_{i=1}^k \alpha_i f_j(\mathbf{v}_i) \\ &= \sum_{i=1}^k \alpha_i \cdot 0 \\ &= 0, \quad j = k+1, k+2, \dots, n \end{aligned}$$

- Secondly, the set $\{f_{k+1}, \dots, f_n\}$ is linearly independent, since the set $B^* = \{f_1, \dots, f_n\}$ is linearly independent.

- Thirdly, $\{f_{k+1}, \dots, f_n\}$ spans $\text{Ann}(W)$: for any $g \in \text{Ann}(W) \subseteq V^*$, it can be

expressed as $g = \sum_{i=1}^n \beta_i f_i$. It follows that

$$\begin{aligned} g(\mathbf{v}_1) &= \sum_{i=1}^n \beta_i f_i(\mathbf{v}_1) = 0 \implies \beta_1 = 0 \\ &\vdots \\ g(\mathbf{v}_k) &= \sum_{i=1}^n \beta_i f_i(\mathbf{v}_k) = 0 \implies \beta_k = 0 \end{aligned}$$

Substituting $\beta_1 = \dots = \beta_k = 0$ into $g = \sum_{i=1}^n \beta_i f_i$, we imply

$$g = \beta_{k+1} f_{k+1} + \dots + \beta_n f_n \in \text{span}\{f_{k+1}, \dots, f_n\}.$$

Therefore, $\{f_{k+1}, \dots, f_n\}$ forms a basis for $\text{Ann}(W)$, i.e., $\dim(\text{Ann}(W)) = n - k$. ■

R Let $W \leq V$, where V has finite dimension, recall that we have obtained two relations below:

$$\dim(\text{Ann}(W)) = \dim(V) - \dim(W)$$

$$\dim((V/W)^*) = \dim(V/W) = \dim(V) - \dim(W)$$

Therefore, $\dim((V/W)^*) = \dim(\text{Ann}(W))$, i.e.,

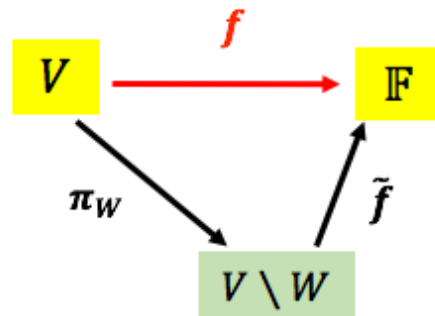
$$(V/W)^* \cong \text{Ann}(W).$$

The question is that can we construct an isomorphism explicitly? We claim that the mapping defined below is an isomorphism:

$$\begin{aligned} &\text{Ann}(W) \rightarrow (V/W)^* \\ \text{with } &f \mapsto \tilde{f}, \end{aligned}$$

where $\tilde{f}: V/W \rightarrow \mathbb{F}$ is constructed from the **universal property I**, i.e., given

the mapping $f \in \text{Ann}(W)$, since $W \leq \ker(f)$, there exists $\tilde{f} : V/W \rightarrow \mathbb{F}$ such that the diagram below commutes:



i.e., $\tilde{f}(\mathbf{v} + W) = f(\mathbf{v})$.

5.4. Wednesday for MAT3040

There will be a quiz on next Monday.

Scope : From Week 1 up to (including) the definition of B^* .

Reviewing.

1. If V is finite dimensional, and B a basis of V , then B^* is a basis of the dual space V^* .
2. Define the Annihilator $\text{Ann}(S) \leq V^*$:

$$\text{Ann}(S) = \{f \in V^* \mid f(s) = 0, \forall s \in S\}$$

3. If V is finite dimensional, and $W \leq V$, then $\text{Ann}(W)$ fills the gap, i.e.,

$$\dim(\text{Ann}(W)) = \dim(V) - \dim(W)$$

4. Define a map

$$\Phi : \text{Ann}(W) \rightarrow (V/W)^*$$

$$f \mapsto \tilde{f}$$

where \tilde{f} is defined such that the diagram (5.1) below commutes

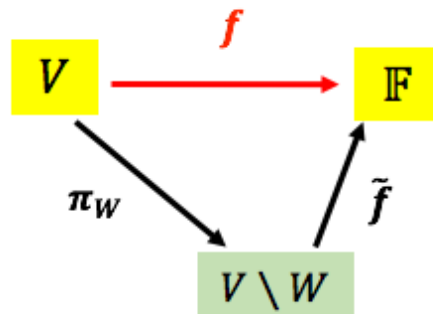


Figure 5.1: Construction of \tilde{f}

Or equivalently, $\tilde{f} : V/W \rightarrow \mathbb{F}$ is such that $\tilde{f}(\mathbf{v} + W) = f(\mathbf{v})$.

5.4.1. Adjoint Map

The natural question is that whether Φ is the isomorphism between $\text{Ann}(W)$ and $(V/W)^*$:

Proposition 5.4 Φ is a linear transformation, i.e.,

$$\Phi(af + bg) = a \cdot \Phi(f) + b \cdot \Phi(g).$$

Proof. It suffices to show that

$$\overline{af + bg} = a\overline{f} + b\overline{g}$$

■

Therefore, we need to answer whether Φ a bijective map. We will show this conjecture at the end of this lecture. The definition of Φ is **natural**, i.e., we do not need to specify any basis to define this Φ . However, as studied in Monday, the constructed isomorphism $V \rightarrow V^*$ with $v_i \mapsto f_i$ is not natural.

Definition 5.3 [Adjoint Map] Let $T : V \rightarrow W$ be a linear transformation. Define the **adjoint** of T by

$$T^* : W^* \rightarrow V^*$$

such that for any $f \in W^*$,

$$[T^*(f)](\mathbf{v}) := f(T(\mathbf{v})), \quad \forall \mathbf{v} \in V.$$

■

R

1. In other words, $T^*(f) = f \circ T$, i.e., a linear transformation from V to \mathbb{F} , i.e., belongs to V^* .
2. Moreover, the mapping T^* itself is a linear transformation: For $f, g \in W^*$,

and $\forall \mathbf{v} \in V$,

$$\begin{aligned}
[T^*(af + bg)](\mathbf{v}) &= (af + bg)[T(\mathbf{v})] \\
&= af(T(\mathbf{v})) + bg(T(\mathbf{v})) && \text{definition of } W^* \text{ as a vector space} \\
&= a[T^*(f)](\mathbf{v}) + b[T^*(g)](\mathbf{v}) \\
&= [aT^*(f) + bT^*(g)](\mathbf{v}) && \text{definition of } V^* \text{ as a vector space}
\end{aligned}$$

Proposition 5.5 Let $T : V \rightarrow W$ be a linear transformation.

1. If T is **injective**, then T^* is **surjective**.
2. If T is **surjective**, then T^* is **injective**.

This statement is quite intuitive, since T^* reverses the dual of output into the dual of input:

$$T : V \rightarrow W$$

$$T^* : W^* \rightarrow V^*$$

Proof. We only give a proof of (2), i.e., suffices to show $\ker(T) = \{\mathbf{0}\}$.

Consider any $g \in W^*$ such that $T^*(g) = \mathbf{0}_{V^*}$. It follows that

$$[T^*(g)](\mathbf{v}) = \mathbf{0}_{V^*}(\mathbf{v}), \quad \forall \mathbf{v} \in V. \iff g(T(\mathbf{v})) = \mathbf{0}, \quad \forall \mathbf{v} \in V. \quad (5.4)$$

To show $g = \mathbf{0}_{W^*}$, it suffices to show $g(\mathbf{w}) = \mathbf{0}$ for $\forall \mathbf{w} \in W$. For all $\mathbf{w} \in W$, by the surjectivity of T , there exists $\mathbf{v}' \in V$ such that

$$\mathbf{w} = T(\mathbf{v}').$$

By substituting \mathbf{w} with $T(\mathbf{v}')$ and (5.4), we imply

$$g(\mathbf{w}) = g(T(\mathbf{v}')) = \mathbf{0}.$$

The proof is complete. ■

Proposition 5.6 Let $T : V \rightarrow W$ be a linear transformation, and $\mathcal{A} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}, \mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be the bases of V and W , respectively. Let $\mathcal{A}^* = \{f_1, \dots, f_n\}, \mathcal{B}^* = \{g_1, \dots, g_m\}$

be bases of dual spaces V^* and W^* , respectively. Then $T^* : W^* \rightarrow V^*$ admits a matrix representation

$$(T^*)_{\mathcal{A}^* \mathcal{B}^*} = \text{transpose}((T)_{\mathcal{B} \mathcal{A}})$$

where $(T^*)_{\mathcal{A}^* \mathcal{B}^*} \in \mathbb{F}^{n \times m}$ and $(T)_{\mathcal{B} \mathcal{A}} \in \mathbb{F}^{m \times n}$

Proof. Let $(T)_{\mathcal{B} \mathcal{A}} = (\alpha_{ij})$ and $(T^*)_{\mathcal{A}^* \mathcal{B}^*} = (\beta_{ij})$. By definition of matrix representation,

$$T(\mathbf{v}_j) = \sum_{i=1}^m \alpha_{ij} \mathbf{w}_i, \quad T^*(g_i) = \sum_{k=1}^n \beta_{ki} f_k \in V^*$$

As a result,

$$\begin{aligned} [T^*(g_i)](\mathbf{v}_j) &= g_i(T(\mathbf{v}_j)) \\ &= g_i\left(\sum_{\ell=1}^m \alpha_{\ell j} \mathbf{w}_\ell\right) \\ &= \sum_{\ell=1}^m \alpha_{\ell j} g_i(\mathbf{w}_\ell) \\ &= \alpha_{ij} \end{aligned}$$

and

$$\begin{aligned} [T^*(g_i)](\mathbf{v}_j) &= \left(\sum_{k=1}^n \beta_{ki} f_k\right)(\mathbf{v}_j) \\ &= \sum_{k=1}^n \beta_{ki} f_k(\mathbf{v}_j) \\ &= \beta_{ji} \end{aligned}$$

Therefore, $\beta_{ji} = \alpha_{ij}$. The proof is complete. ■

5.4.2. Relationship between Annihilator and dual of quotient spaces

■ **Example 5.5** Consider the canonical projection mapping $\pi_W : V \rightarrow V/W$ with its **adjoint** mapping:

$$(\pi_W)^* : (V/W)^* \rightarrow V^*$$

The understanding of $(\pi_W)^*$ is as follows:

1. Take $h \in (V/W)^*$ and study $(\pi_W)^*(h) \in V^*$
2. Take $\mathbf{v} \in V$ and understand

$$[(\pi_W)^*(h)](\mathbf{v}) = h(\pi_W(\mathbf{v})) = h(\mathbf{v} + W)$$

(a) In particular, for all $\mathbf{w} \in W \leq V$, we have

$$[(\pi_W)^*(h)](\mathbf{w}) = h(\mathbf{w} + W) = h(\mathbf{0}_{V/W}) = \mathbf{0}_{\mathbb{F}}$$

Therefore,

$$(\pi_W)^*(h) \in \text{Ann}(W).$$

i.e., $(\pi_W)^*$ is a mapping from $(V/W)^*$ to $\text{Ann}(W)$.

(b) By proposition (5.5), π_W is surjective implies $(\pi_W)^*$ is injective.

Combining (a) and (b), it's clear that (i.e., left as homework problem)

$$\Phi \circ \pi_W^* = \text{id}_{(V/W)^*} \text{ and } \pi_W^* \circ \Phi = \text{id}_{\text{Ann}(W)}$$

This relationship implies Φ is an isomorphism. ■

Chapter 6

Week6

6.1. Monday for MAT3040

6.1.1. Polynomials

We recall some useful properties of polynomial before studying eigenvalues/eigenvectors.

Definition 6.1 [Polynomial]

1. A polynomial over \mathbb{F} has the form

$$p(z) = a_m z^m + \cdots + a_1 z + a_0, \quad (a_m \neq 0).$$

Here $a_m z^m$ is called the **leading term** of $p(z)$; m is called the degree; a_m is called the **leading coefficient**; a_m, \dots, a_0 are called the coefficients of this polynomial.

2. A polynomial over \mathbb{F} is monic if its leading coefficient is $1_{\mathbb{F}}$.
3. A polynomial $p(z) \in \mathbb{F}[z]$ is **irreducible** if for any $a(z), b(z) \in \mathbb{F}[z]$,

$$p(z) = a(z)b(z) \implies \text{either } a(z) \text{ or } b(z) \text{ is a constant polynomial.}$$

Otherwise $p(z)$ is **reducible**.

■ **Example 6.1** For example, the polynomial $p(x) = x^2 + 1$ is irreducible over \mathbb{R} ; but $p(x) = (x - i)(x + i)$ is **reducible** over \mathbb{C} . ■

Theorem 6.1 — Division Theorem. For all $p, q \in \mathbb{F}[z]$ such that $p \neq 0$, there exists unique $s, r \in \mathbb{F}[z]$ satisfying $\deg(r) < \deg(p)$, such that

$$p(z) = s(z) \cdot q(z) + r(z).$$

Here $r(z)$ is called the **remainder**.

■ **Example 6.2** Given $p(x) = x^4 + 1$ and $q(x) = x^2 + 1$, the junior school knowledge tells us that uniquely

$$x^4 + 1 = (x^2 - 1)(x^2 + 1) + 2.$$

Theorem 6.2 — Root Theorem. For $p(x) \in \mathbb{F}[x]$, and $\lambda \in \mathbb{F}$, $x - \lambda$ divides p if and only if $p(\lambda) = 0$.

Proof. 1. If $(x - \lambda)$ divides p , then $p = (x - \lambda)q$ for some $q \in \mathbb{F}[x]$. Thus clearly $p(\lambda) = 0$.
 2. For the other direction, suppose that $p(\lambda) = 0$. By division theorem, there exists $s, r \in \mathbb{F}[x]$ such that

$$p = (x - \lambda)s + r \quad \text{with } \deg(r) < \deg(x - \lambda) = 1. \quad (6.1)$$

Therefore, the polynomial r must be constant.

Substituting λ into x both sides in (6.1), we have

$$0 = p(\lambda) = 0 \cdot s + r \implies r = 0.$$

Therefore, $p = (x - \lambda) \cdot s$, i.e., $(x - \lambda)$ divides p . ■

6.4. Wednesday for MAT3040

Reviewing: Root Theorem: $p(\lambda) = 0$ iff $(x - \lambda)$ divides $p(x)$.

Corollary 6.2 A polynomial with degree n has at most n roots counting multiplicity.

For example, the polynomial $(x - 3)^2$ has one root $x = 3$ with multiplicity 2. When counting multiplicity, we say the polynomial $(x - 3)^2$ has two roots.

Definition 6.5 [Algebraically Closed] A field \mathbb{F} is called **algebraically closed** if every non-constant polynomial $p(x) \in \mathbb{F}[x]$ has a root $\lambda \in \mathbb{F}$. ■

Theorem 6.5 — Fundamental Theorem of Algebra. The set of complex numbers \mathbb{C} is algebraically closed.

Proof. One way is by complex analysis; Another way is by the topology on $\mathbb{C} \setminus \{0\}$. ■

R By induction, we can show that every polynomial with degree n on algebraically closed field \mathbb{F} has **exactly** n roots, counting multiplicity. Therefore, for any $p(x)$ on algebraically closed field \mathbb{F} ,

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_n) \quad (6.3)$$

for $c, \lambda_1, \dots, \lambda_n \in \mathbb{F}$.

The polynomials on general field \mathbb{F} may not necessarily be factorized as in (6.3), but still admit unique factorization property:

Theorem 6.6 — Unique Factorization. Every $f(x) = a_n x^n + \cdots + a_0$ in $\mathbb{F}[x]$ can be factorized as

$$f(x) = a_n [p_1(x)]^{e_1} \cdots [p_k(x)]^{e_k}$$

where p_i 's are **monic, irreducible, distinct**. Furthermore, this expression is unique up to the permutation of factors.

Definition 6.6 [Factor] If $p(x) = q(x)s(x)$ with $p, q, s \in \mathbb{F}[x]$, then we say

- $p(x)$ is **divisible** by $s(x)$;
- $s(x)$ is a **factor** of $p(x)$;
- $s(x) | p(x)$
- $s(x)$ **divides** $p(x)$
- $p(x)$ is **multiple** of $s(x)$

Definition 6.7 [Common Factor]

1. The polynomial $g(x)$ is said to be a **common factor** of $f_1, \dots, f_k \in \mathbb{F}[x]$ if

$$g | f_i, i = 1, \dots, k$$

2. The polynomial $g(x)$ is said to be a **greatest common divisor** of f_1, \dots, f_k if
 - g is **monic**.
 - g is common factor of f_1, \dots, f_k
 - g is of largest possible (maximal) degree.



- $\gcd(f_1, \dots, f_k) = \gcd(\gcd(f_1, f_2), f_3, \dots, f_k) = \gcd(\gcd(f_1, f_2, f_3), \dots, f_k)$
- $\gcd(f_1, \dots, f_k)$ is unique.
- If $\gcd(f_1, \dots, f_k) = 1$, we say f_1, \dots, f_k is **relatively prime**
- Polynomials f_1, \dots, f_k are relatively prime does not necessarily mean $\gcd(f_i, f_j) = 1$ for any $i \neq j$.

Counter-example: Let a_1, \dots, a_n distinct irreducible polynomials, and

$$f_i(x) = a_1(x) \cdots \hat{a}_i(x) \cdots a_n(x) := a_1 \cdots a_{i-1} a_{i+1} \cdots a_n,$$

then $\gcd(f_1, \dots, f_n) = 1$, but $\gcd(f_i, f_j) = a_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_n$, which does not necessarily equal to 1.

■ **Example 6.6** The $\gcd(f_1, f_2)$ is easy to compute for factorized polynomials. For example, let $f_1(x) = (x^2 + x + 1)^3(x - 3)^2x^4$ and $f_2(x) = (x^2 + 1)(x - 3)^4x^2$ in $\mathbb{R}[x]$, then

$$\gcd(f_1, f_2) = (x - 3)^2x^2$$

The question is how to find $\gcd(f_1, f_2)$ for given un-factorized polynomials?

Theorem 6.7 — Bezout. Let $g = \gcd(f_1, f_2)$, then there exists $r_1, r_2 \in \mathbb{F}[x]$ such that

$$g(x) = r_1(x)f_1(x) + r_2(x)f_2(x)$$

More generally, $g = \gcd(f_1, \dots, f_k)$ implies there exists r_1, \dots, r_k such that

$$g = r_1f_1 + \cdots + r_kf_k$$

The derivation of r_i 's is by applying **Euclidean algorithm**. For example, given $x^3 + 6x + 7$ and $x^2 + 3x + 2$, we imply

$$x^3 + 6x + 7 - (x - 3)(x^2 + 3x + 2) = 13x + 13$$

and

$$x^2 + 3x + 2 - \frac{x+2}{13}(13x + 13) = 0$$

Therefore, $\gcd(x^3 + 6x + 7, x^2 + 3x + 2) = \gcd(x^2 + 3x + 2, 13x + 13) = x + 2$.

6.4.1. Eigenvalues & Eigenvectors

Definition 6.8 [Eigenvalues] Let $T : V \rightarrow V$ be a linear operator.

1. We say $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ is an eigenvector of T with eigenvalue λ if $T(\mathbf{v}) = \lambda\mathbf{v}$;
2. Or equivalently, $\mathbf{v} \in \ker(T - \lambda I)$, the λ -eigenspace of T . Here the mapping $I : V \rightarrow V$ denotes identity map, i.e., $I(\mathbf{v}) = \mathbf{v}, \forall \mathbf{v} \in V$.

Definition 6.9 A vector $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ is a **generalized eigenvector** of T with **generalized eigenvalue** λ if $\mathbf{v} \in \ker((T - \lambda I)^k)$ for some $k \in \mathbb{N}^+$.

Note that an eigenvector is a generalized eigenvector of T ; while the converse does not necessarily hold.

■ **Example 6.7** Consider the linear transformation $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

$$\begin{aligned} A : \quad & \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ \text{with} \quad & \mathbf{x} \rightarrow \mathbf{A}\mathbf{x} \\ \text{where} \quad & \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

1. Note that $[1, 0]^T$ is an eigenvector with eigenvalue 1, since

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. However, $[0, 1]^T$ is not an eigenvector, since

$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that

$$(A - I)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (A - I)^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \ker(A - I)^2,$$

i.e., a generalized eigenvector with generalized eigenvalue 1. ■

■ **Example 6.8** Consider $V = C^\infty(\mathbb{R})$, which is a set of all infinitely differentiable functions.

Define the linear operator $T : V \rightarrow V$ as $T(f) = f''$. Then the (-1) -eigenspace of T has $f \in V$ satisfying

$$f'' = -f$$

From ODE course, we imply $\{\sin x, \cos x\}$ forms a basis of (-1) -eigenspace. ■

Assumption. From now on, we assume V has finite dimension by default.

Definition 6.10 [Determinant] Let $T : V \rightarrow V$ be a linear operator. The **determinant** of T is given by

$$\det(T) = \det((T)_{\mathcal{A}, \mathcal{A}})$$

where \mathcal{A} is some basis of V . ■

Ⓡ Assume we have complete knowledge about $\det(M)$ for matrices for now. The determinant is well-defined, i.e., independent of the choice of basis \mathcal{A} . For another basis \mathcal{B} , we imply

$$\det(T_{\mathcal{B}, \mathcal{B}}) = \det(C_{\mathcal{B}, \mathcal{A}} T_{\mathcal{A}, \mathcal{A}} C_{\mathcal{A}, \mathcal{B}}) = \det(C_{\mathcal{B}, \mathcal{A}}) \det(T_{\mathcal{A}, \mathcal{A}}) \det(C_{\mathcal{A}, \mathcal{B}}) = \det(T_{\mathcal{A}, \mathcal{A}})$$

Definition 6.11 [characteristic polynomial] The **characteristic polynomial** $\chi_T(x)$ of $T : V \rightarrow V$ is defined as

$$\chi_T(x) = \det((T)_{\mathcal{A},\mathcal{A}} - xI)$$

for any basis \mathcal{A} ■

In the next few lectures, we will study

- Cayley-Hamilton Theorem
- Jordan Canonical Form

These theorems can be stated using matrices, and they both hold up to change of basis. We have a unified statement of these theorems using vector space rather than \mathbb{R}^n .

Chapter 7

Week7

7.1. Monday for MAT3040

Reviewing. Define the characteristic polynomial for an linear operator T :

$$\chi_T(x) = \det((T)_{\mathcal{A},\mathcal{A}} - xI)$$

We will use the notation “ I/I ” in two different occasions:

1. I denotes the identity transformation from V to V with $I(\mathbf{v}) = \mathbf{v}, \forall \mathbf{v} \in V$
2. I denotes the identity matrix $(I)_{\mathcal{A},\mathcal{A}}$, defined based on any basis \mathcal{A} .

7.1.1. Minimal Polynomial

Definition 7.1 [Linear Operator Induced From Polynomial] Let $f(x) := a_mx^m + \cdots + a_0$ be a polynomial in $\mathbb{F}[x]$, and $T : V \rightarrow V$ be a linear operator. Then the mapping

$$f(T) = a_mT^m + \cdots + a_1T + a_0I : V \rightarrow V,$$

is called a linear operator induced from the polynomial $f(x)$. ■



1. The composition of linear operators is not abelian, e.g., in general $S \circ T = T \circ S$ does not hold. The reason follows similarly from the fact that square-matrix multiplication is not abelian in general.

2. However, we always have $f(T)T = Tf(T)$, where $f(T)$ is a linear operator induced from the polynomial $f(x)$:

Proof. We can show that $T^nT = TT^n, \forall n$ by induction. Suppose that $f(x) = \sum_i a_i x^i$, which follows that

$$f(T)T = \sum_i a_i T^i T = \sum_i a_i T T^i = T \sum_i a_i T^i = T f(T).$$

■

3. We can generalize the statement in (2) into the fact that the composition of linear operators induced from polynomials is abelian, i.e.,

$$f(T)g(T) = g(T)f(T)$$

for any polynomials $f(x), g(x)$.

Definition 7.2 [Minimal Polynomial] Let $T : V \rightarrow V$ be a linear operator. The **minimal polynomial** $m_T(x)$ is a **nonzero monic polynomial** of least (minimal) degree such that

$$m_T(T) = \mathbf{0}_{V \rightarrow V}.$$

where $\mathbf{0}_{V \rightarrow V}$ denotes the zero vector in $\text{Hom}(V, V)$. ■

■ **Example 7.1** 1. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then \mathbf{A} defines a linear operator:

$$A : \mathbb{F}^2 \rightarrow \mathbb{F}^2$$

$$\text{with } \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$$

Here $\chi_A(x) = (x - 1)^2$ and $\mathbf{A} - \mathbf{I} = \mathbf{0}$, which gives $m_A(x) = x - 1$.

2. Let $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which implies

$$\chi_{\mathbf{B}}(x) = (x - 1)^2,$$

The question is that can we get the minimal polynomial with degree 1?

The answer is no, since $\mathbf{B} - k\mathbf{I} = \begin{pmatrix} 1-k & 1 \\ 0 & 1-k \end{pmatrix} \neq \mathbf{0}$.

In fact, $m_{\mathbf{B}}(x) = (x - 1)^2$, since

$$(\mathbf{B} - \mathbf{I})^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Two questions naturally arises:

1. Does $m_T(x)$ exist? If exists, is it unique?
2. What's the relationship between $m_T(x)$ and $\chi_T(x)$?

Regarding to the first question, the minimal polynomial $m_T(x)$ may not exist, if V has infinite dimension:

■ **Example 7.2** Consider $V = \mathbb{R}[x]$ and the mapping

$$T : V \rightarrow V$$

$$p(x) \mapsto \int_0^x p(t) dt$$

In particular, $T(x^n) = \frac{1}{n+1}x^{n+1}$. Suppose $m_T(x)$ is with degree n , i.e.,

$$m_T(x) = x^n + \cdots + a_1x + a_0,$$

then

$$m_T(T) = T^n + \cdots + a_0I \text{ is a zero linear transformation}$$

It follows that

$$[m_T(T)](x) = \frac{1}{n!}x^n + a_{n-1}\frac{1}{(n-1)!}x^{n-1} + \cdots + a_1x + a_0 = 0_{\mathbb{F}},$$

which is a contradiction since the coefficients of x^k is nonzero on LHS for $k = 1, \dots, n$, but zero on the RHS. ■

Proposition 7.1 The minimal polynomial $m_T(x)$ always exists for $\dim(V) = n < \infty$.

Proof. It's clear that $\{I, T, \dots, T^n, T^{n+1}, \dots, T^{n^2}\} \subseteq \text{Hom}(V, V)$. Since $\dim(\text{Hom}(V, V)) = n^2$, we imply $\{I, T, \dots, T^n, T^{n+1}, \dots, T^{n^2}\}$ is linearly dependent, i.e., there exists a_i 's that are not all zero such that

$$a_0I + a_1T + \cdots + a_{n^2}T^{n^2} = 0$$

i.e., there is a polynomial $g(x)$ of degree less than n^2 such that $g(T) = 0$.

The proof is complete. ■

Proposition 7.2 The minimal polynomial $m_T(x)$, if exists, then it exists uniquely.

Proof. Suppose f_1, f_2 are two distinct minimal polynomials with $\deg(f_1) = \deg(f_2)$. It follows that

- $\deg(f_1 - f_2) < \deg(f_1)$.
- $f_1 - f_2 \neq 0$
- $(f_1 - f_2)(T) = f_1(T) - f_2(T) = 0_{V \rightarrow V}$

By scaling $f_1 - f_2$, there is a monic polynomial g with lower degree satisfying $g(T) = 0$, which contradicts the definition for minimal polynomial. ■

Proposition 7.3 Suppose $f(x) \in \mathbb{F}[x]$ satisfying $f(T) = \mathbf{0}$, then

$$m_T(x) \mid f(x).$$

Proof. It's clear that $\deg(f) \geq \deg(m_T)$. The division algorithm gives

$$f(x) = q(x)m_T(x) + r(x).$$

Therefore, for any $\mathbf{v} \in V$

$$[r(T)](\mathbf{v}) = [f(T)](\mathbf{v}) - [q(T)m_T(T)](\mathbf{v}) = \mathbf{0}_V - q(T)\mathbf{0}_V = \mathbf{0}_V - \mathbf{0}_V = \mathbf{0}_V$$

Therefore, $r(T) = \mathbf{0}_{V \rightarrow V}$. By definition of minimal polynomial, we imply $r(x) \equiv 0$. ■

Proposition 7.4 If $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{n \times n}$ are similar to each other, then $m_A(x) = m_B(x)$.

Proof. Suppose that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, and that

$$m_A(x) = x^k + \cdots + a_1x + a_0, \quad m_B(x) = x^\ell + \cdots + b_0.$$

It follows that

$$\begin{aligned} m_A(\mathbf{B}) &= \mathbf{B}^k + \cdots + a_0\mathbf{I} \\ &= \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P} + \cdots + a_0\mathbf{P}^{-1}\mathbf{P} \\ &= \mathbf{P}^{-1}(\mathbf{A}^k + \cdots + a_0\mathbf{I})\mathbf{P} \\ &= \mathbf{P}^{-1}(m_A(\mathbf{A}))\mathbf{P} \end{aligned}$$

Therefore, $m_A(\mathbf{B}) = \mathbf{0}$ since $m_A(\mathbf{A}) = \mathbf{0}$. By proposition (7.3), we imply $m_B(x) \mid m_A(x)$.

Similarly, $m_A(x) \mid m_B(x)$. Since $m_A(x)$ and $m_B(x)$ are monic, we imply $m_A(x) = m_B(x)$. ■

R Proposition (7.4) claims that the minimal polynomial is **similarity-invariant**; actually, the characteristic polynomial is **similarity-invariant** as well.

Assumption. We will assume V has finite dimension from now on. Now we study the vanishing of a single vector $\mathbf{v} \in V$.

Notation. The $m_T(x)$ is a nonzero monic polynomial of least degree such that

$$m_T(T) = \mathbf{0}_{V \rightarrow V}.$$

7.1.2. Minimal Polynomial of a vector

Definition 7.3 [Minimal Polynomial of a vector] Similar to the minimal polynomial, we define the **minimal polynomial of a vector \mathbf{v} relative to T** , say $m_{T,\mathbf{v}}(x)$, as the monic polynomial of least degree such that

$$m_{T,\mathbf{v}}(T)(\mathbf{v}) = 0$$

The existence of minimal polynomial of a vector is due to the existence of minimal polynomial; the uniqueness follows similarly as in proposition (7.2).

Proposition 7.5 Let $T : V \rightarrow V$ be a linear operator and $\mathbf{v} \in V$. The degree of the minimal polynomial of a vector is upper bounded by:

$$\deg(m_{T,\mathbf{v}}(x)) \leq \dim(V).$$

Proof. It's clear that $\{\mathbf{v}, T\mathbf{v}, \dots, T^n\mathbf{v}\} \subseteq V$ and the proof follows similarly as in proposition (7.1). ■

Similar to the division property in proposition (7.3), we have the division property for minimal polynomial of a vector:

Proposition 7.6 Suppose $f(x) \in \mathbb{F}[x]$ satisfying $f(T)(\mathbf{v}) = \mathbf{0}_V$, then

$$m_{T,\mathbf{v}}(x) \mid f(x).$$

In particular, $m_{T,\mathbf{v}} \mid m_T(x)$.

Proof. The proof follows similarly as in proposition (7.3). ■

Proposition 7.7 Suppose that $m_{T,\mathbf{v}}(x) = f_1(x)f_2(x)$, where f_1, f_2 are both monic. Let $\mathbf{w} = f_1(T)\mathbf{v}$, then

$$m_{T,\mathbf{w}}(x) = f_2(x)$$

Proof. 1.

$$f_2(T)\mathbf{w} = f_2(T)f_1(T)\mathbf{v} = m_{T,\mathbf{v}}(T)\mathbf{v} = \mathbf{0}$$

By the proposition (7.3), we imply $m_{T,\mathbf{w}}|f_2$.

2. On the other hand,

$$\mathbf{0} = m_{T,\mathbf{w}}(T)(\mathbf{w}) = m_{T,\mathbf{w}}(T)f_1(T)\mathbf{v} = f_1(T)m_{T,\mathbf{w}}(T)\mathbf{v},$$

which implies that $m_{T,\mathbf{v}}(x) | f_1(x)m_{T,\mathbf{w}}(x)$, i.e.,

$$f_1 \cdot f_2 | f_1 \cdot m_{T,\mathbf{w}} \implies f_2 | m_{T,\mathbf{w}}.$$

The proof is complete.

■

7.4. Wednesday for MAT3040

Reviewing.

- Given the polynomial $f(x) \in \mathbb{F}[x]$, we extend it into the linear operator $f(T) : V \rightarrow V$.
- The minimal polynomial $m_T(x)$ is defined to be the polynomial with least degree such that

$$m_T(T) = \mathbf{0}_{V \rightarrow V},$$

$$\text{i.e., } [m_T(T)]\mathbf{v} = \mathbf{0}_V, \forall \mathbf{v} \in V.$$

- The minimal polynomial of a vector \mathbf{v} relative to T is defined to be the polynomial $m_{T,\mathbf{v}}(x)$ with the least degree such that

$$m_{T,\mathbf{v}}(T)(\mathbf{v}) = \mathbf{0}$$

- If $f(T) = \mathbf{0}_{V \rightarrow V}$, then we imply $m_T(x) \mid f(x)$. If $[g(T)](\mathbf{w}) = \mathbf{0}_V$, following the similar argument, we imply $m_{T,\mathbf{w}}(x) \mid g(x)$.
- In particular, $m_T(T)\mathbf{w} = \mathbf{0}$, which implies $m_{T,\mathbf{w}}(x) \mid m_T(x)$.

7.4.1. Cayley-Hamilton Theorem

Let's raise an motivative example first:

■ **Example 7.8** Consider the matrix and its induced mapping $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. It has the characteristic polynomial

$$\chi_A = (x - 1)(x - 2).$$

- Note that $m_A(x)$ cannot be with degree one, since otherwise $m_A(x) = x - k$ with

some k , and

$$m_A(\mathbf{A}) = \mathbf{A} - k\mathbf{I} = \begin{pmatrix} 1-k & 0 \\ 0 & 2-k \end{pmatrix} \neq \mathbf{0}, \quad \forall k,$$

which is a contradiction.

- However, one can verify that the $m_A(x)$ is with degree 2:

$$m_A(x) = (x-1)(x-2).$$

- The minimal polynomial with eigenvectors can be with degree 1:

$$\mathbf{w} = [0, 1]^T \implies (A - 2I)\mathbf{w} = \mathbf{0} \implies m_{A,\mathbf{w}}(x) = x - 2$$

R More generally, given an eigen-pair (λ, \mathbf{v}) , the minimal polynomial of an \mathbf{v} has the explicit form

$$m_{T,\mathbf{v}}(x) = (x - \lambda) \implies (x - \lambda) \mid m_T(x)$$

Now we want to relate the characteristic polynomial $m_T(x)$ with $\chi_T(x)$. Suppose that

$$\chi_T(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k} \in \mathbb{F}[x]. \quad (7.1)$$

Then we imply

- λ_i is an eigenvalue of T ;
- $(x - \lambda_i) \mid m_T(x)$;

which implies that $(x - \lambda_1) \cdots (x - \lambda_k) \mid m_T(x)$.

Furthermore, (a). does $m_T(x)$ possess other factors, e.g., does there exist $\mu \neq \lambda_i, i = 1, \dots, k$ such that $(x - \mu) \mid m_T(x)$? (b). does $(x - \lambda_i)^{f_i} \mid m_T(x)$ when $f_i > e_i$?

The answer is no for both question (a) and (b).

Theorem 7.1 — Cayley-Hamilton. $m_T(x) \mid \chi_T(x)$. In particular, $\chi_T(T) = \mathbf{0}$.

The nice equality in (7.1) does not necessarily hold. Sometimes $\chi_T(x)$ cannot be factorized into linear factors in $\mathbb{F}[x]$, e.g., $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in \mathbb{R} .

However, for every $f(x) \in \mathbb{F}[x]$, we can extend \mathbb{F} into the algebraically closed set $\overline{\mathbb{F}} \supseteq \mathbb{F}$ such that

$$f(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_k)^{e_k}$$

where $\lambda_i \in \overline{\mathbb{F}}$.

For example, for $f(x) = x^2 + 1 \in \mathbb{R}[x]$, we can extend \mathbb{R} into \mathbb{C} to obtain

$$f(x) = (x + i)(x - i).$$

Therefore, the general proof outline for the Cayley-Hamilton Theorem is as follows:

- Consider the case where $m_T(x), \chi_T(x)$ are both in $\overline{\mathbb{F}}[x]$
- Show that $m_T(x) \mid \chi_T(x)$ under $\overline{\mathbb{F}}[x]$.

Before the proof, let's study the invariant subspaces, which leads to the decomposition of characteristic polynomial:

Assumption. From now on, we assume that V is finite dimensional by default.

Definition 7.12 [Invariant Subspace] An **invariant subspace** of a linear operator $T : V \rightarrow V$ is a subspace $W \leq V$ that is preserved by T , i.e., $T(W) \subseteq W$. We also call W as T -invariant. ■

- Ⓡ If $W \leq V$ is T -invariant, then the restriction of the linear operator $T : V \rightarrow V$ induces the linear operator

$$T|_W : W \rightarrow W.$$

- **Example 7.9**
1. V itself is T -invariant.
 2. For the eigenvalue λ , the associated λ -eigenspace $U = \ker(T - \lambda I)$ is T -invariant.
 3. More generally, $U = \ker(g(T))$ is T -invariant for any polynomial g :
If $\mathbf{v} \in \ker(g(T))$, i.e., $g(T)\mathbf{v} = \mathbf{0}$, it suffices to show $T(\mathbf{v}) \in \ker(g(T))$:

$$\begin{aligned}
 g(T)[T(\mathbf{v})] &= (a_m T^m + \cdots + a_0 I)[T(\mathbf{v})] \\
 &= (a_m T \circ T^{m-1} + \cdots + a_1 T \circ T + a_0 T \circ I)(\mathbf{v}) \\
 &= T[g(T)\mathbf{v}] = T(\mathbf{0}) = \mathbf{0}
 \end{aligned}$$

4. For $\mathbf{v} \in \ker(T - \lambda I)$, $U = \text{span}\{\mathbf{v}\}$ is T -invariant.

Proposition 7.10 Suppose that $T : V \rightarrow V$ is a linear transformation and $W \leq V$ is T -invariant, then we construct the subspace mapping and the recipe mapping

$$\begin{aligned}
 T|_W : W &\rightarrow W \\
 \text{with } \mathbf{w} &\mapsto T(\mathbf{w})
 \end{aligned} \tag{7.2a}$$

$$\begin{aligned}
 \tilde{T} : V/W &\rightarrow V/W \\
 \text{with } \mathbf{v} + W &\mapsto T(\mathbf{v}) + W
 \end{aligned} \tag{7.2b}$$

(Here the well-definedness of the recipe mapping \tilde{T} is shown in Hw2, Exercise 4),

which leads to the decomposition of the characteristic polynomial:

$$\chi_T(x) = \chi_{T|_W}(x) \chi_{\tilde{T}}(x).$$

Proof. Suppose $C = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis of W , and extend it into the basis of V , denoted as

$$\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$$

Therefore, $\overline{\mathcal{B}} = \{\mathbf{v}_{k+1} + W, \dots, \mathbf{v}_n + W\}$ is a basis of V/W . By Homework 2, Question 5,

the representation $(T)_{\mathcal{B},\mathcal{B}}$ can be written as the block matrix

$$(T)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} (T|_W)_{C,C} & \times \\ \mathbf{0} & (\tilde{T})_{\overline{\mathcal{B}},\overline{\mathcal{B}}} \end{pmatrix}_{(k+(n-k)) \times (k+(n-k))}$$

Therefore, the characteristic polynomial of T can be calculated as:

$$\begin{aligned} \chi_T(x) &= \det((T)_{\mathcal{B},\mathcal{B}} - xI) \\ &= \det((T|_W)_{C,C} - xI) \cdot \det((\tilde{T})_{\overline{\mathcal{B}},\overline{\mathcal{B}}} - xI) \end{aligned}$$

■

Proposition 7.11 Suppose that

$$\chi_T(x) = (x - \lambda_1) \cdots (x - \lambda_n)$$

where λ_i 's are not necessarily distinct. Then there exists a basis of V , say \mathcal{A} , such that

$$(T)_{\mathcal{A},\mathcal{A}} = \begin{pmatrix} \lambda_1 & \times & \times & \times \\ 0 & \lambda_2 & \cdots & \times \\ 0 & \cdots & \ddots & \times \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Proof. The proof is by induction on n , i.e., suppose the results hold for all vector spaces with dimension no more than $n - 1$, and we aim to show this result holds for dimension n .

1. **Step 1:** Argue that there exists the associated eigenvector \mathbf{v} of λ_1 under the linear operator T .

Consider any basis \mathcal{M} , by MAT2040, there exists associated eigenvector of λ_1 , say $\mathbf{y} \in \mathbb{C}^n$ such that

$$(T)_{\mathcal{M},\mathcal{M}} \cdot \mathbf{y} = \lambda_1 \mathbf{y}$$

Since the operator $(\cdot)_{\mathcal{M}} : V \rightarrow \mathbb{C}^n$ is an isomorphism, there exists $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such

that $(\mathbf{v})_{\mathcal{M}} = \mathbf{y}$. It follows that

$$(T)_{\mathcal{M},\mathcal{M}}(\mathbf{v})_{\mathcal{M}} = \lambda_1(\mathbf{v})_{\mathcal{M}} \implies (T\mathbf{v})_{\mathcal{M}} = (\lambda_1\mathbf{v})_{\mathcal{M}} \implies T\mathbf{v} = \lambda_1\mathbf{v}$$

2. **Step 2:** Dimensionality reduction of $\mathcal{X}_T(x)$: Construct $W = \text{span}\{\mathbf{v}\}$, which is T -invariant. By the proof of proposition (7.11), we have $\tilde{T} : V/W \rightarrow V/W$ admits the characteristic polynomial

$$\mathcal{X}_{\tilde{T}}(x) = (x - \lambda_2) \cdots (x - \lambda_n)$$

3. **Step 3:** Applying the induction, there exists basis \overline{C} of V/W , i.e.,

$$\overline{C} = \{\mathbf{w}_2 + W, \dots, \mathbf{w}_n + W\}$$

such that

$$(\tilde{T})_{\overline{C},\overline{C}} = \begin{pmatrix} \lambda_2 & \times & \times & \times \\ 0 & \lambda_3 & \cdots & \times \\ 0 & \cdots & \ddots & \times \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

4. **Step 4:** Therefore, we construct the set $\mathcal{A} := \{\mathbf{v}, \mathbf{w}_2, \dots, \mathbf{w}_n\}$. We claim that

- \mathcal{A} is a basis of V (left as exercise in Hw2, Question 2)


•

$$(T)_{\mathcal{A},\mathcal{A}} = \begin{pmatrix} \lambda_1 & \times \\ \mathbf{0} & (\tilde{T})_{\overline{C},\overline{C}} \end{pmatrix} = \begin{pmatrix} \lambda_1 & \times & \times & \times \\ 0 & \lambda_2 & \cdots & \times \\ 0 & \cdots & \ddots & \times \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

(This statement is also left as exercise in Hw2, Question 5.)

■

Proposition 7.12 Suppose that $\mathcal{X}_T(x) = (x - \lambda_1) \cdots (x - \lambda_n)$, then $\mathcal{X}_T(T) = \mathbf{0}$.

 One special case is that $\mathbf{A} = \text{diag}(\lambda_1, \dots, \lambda_n)$. The results for proposition (7.12)

gives

$(A - \lambda_1 I) \cdots (A - \lambda_n I)$ is a zero matrix

Chapter 8

Week8


8.1. Monday for MAT3040

Reviewing.

- If $\chi_T(x) = (x - \lambda_1) \cdots (x - \lambda_n)$, then

$$(T)_{\mathcal{A},\mathcal{A}} = \begin{pmatrix} \lambda_1 & \times & \times & \times \\ 0 & \lambda_2 & \cdots & \times \\ 0 & \cdots & \ddots & \times \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

for some basis \mathcal{A} . In other words, T is **triangularizable** with the diagonal entries $\lambda_1, \dots, \lambda_n$.

-  I hope you appreciate this result. Consider the example below: In linear algebra we have studied that the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable, and the characteristic polynomial is given by

$$\chi_{\mathbf{A}}(x) = (x - 1)^2.$$

However, the theorem above claims that \mathbf{A} is *triangularizable*, with diagonal entries 1 and 1. The diagonalization of \mathbf{A} only uses the eigenvector of \mathbf{A} , but the 1-eigenspace has only 1 dimension. Fortunately, the triangularization gives a rescue such that we can make use of the generalized eigenvector

$(0,1)^T$ (but not an eigenvector) of \mathbf{A} by considering the mapping below:

$$U = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$\bar{A}: V/U \rightarrow V/U$$

Here $(0,1)^T + U$ is an eigenvector of \bar{A} , with eigenvalue 1.

Theorem 8.1 The linear operator T is triangularizable with diagonal entries $(\lambda_1, \dots, \lambda_n)$ if and only if

$$\chi_T = (x - \lambda_1) \cdots (x - \lambda_n)$$

Proof. It suffices to show only the sufficiency. Suppose that there exists basis \mathcal{A} such that

$$(T)_{\mathcal{A},\mathcal{A}} = \begin{pmatrix} \lambda_1 & \times & \times & \times \\ 0 & \lambda_2 & \cdots & \times \\ 0 & \cdots & \ddots & \times \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Then we compute the characteristic polynomial directly:

$$\begin{aligned} \chi_T(x) &= \det[(xI - T)_{\mathcal{A},\mathcal{A}}] \\ &= \det \begin{pmatrix} x - \lambda_1 & \times & \times & \times \\ 0 & x - \lambda_2 & \cdots & \times \\ 0 & \cdots & \ddots & \times \\ 0 & 0 & \cdots & x - \lambda_n \end{pmatrix} \\ &= (x - \lambda_1) \cdots (x - \lambda_n) \end{aligned}$$

■

8.1.1. Cayley-Hamilton Theorem

Proposition 8.1 — A Useful Lemma. Suppose that $\chi_T(x) = (x - \lambda_1) \cdots (x - \lambda_n)$, then $\chi_T(T) = 0$.

Proof. Since $\chi_T(x) = (x - \lambda_1) \cdots (x - \lambda_n)$, we imply T is triangularizable under some basis \mathcal{A} . Note that

- $T \mapsto (T)_{\mathcal{A},\mathcal{A}}$ is an isomorphism between $\text{Hom}(V, V)$ and $M_{n \times n}(\mathbb{F})$,
- $\underbrace{(T \circ T \circ \cdots \circ T)}_{m \text{ times}}_{\mathcal{A},\mathcal{A}} = [(T)_{\mathcal{A},\mathcal{A}}]^m$, for any m ,

It suffices to show $\chi_T((T)_{\mathcal{A},\mathcal{A}})$ is the zero matrix (why?):

$$\chi_T((T)_{\mathcal{A},\mathcal{A}}) = ((T)_{\mathcal{A},\mathcal{A}} - \lambda_1 \mathbf{I}) \cdots ((T)_{\mathcal{A},\mathcal{A}} - \lambda_n \mathbf{I}).$$

Observe the matrix multiplication

$$((T)_{\mathcal{A},\mathcal{A}} - \lambda_i \mathbf{I}) \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 - \lambda_i & \times & \times & \times \\ 0 & \lambda_2 - \lambda_i & \cdots & \times \\ 0 & \cdots & \ddots & \times \\ 0 & 0 & \cdots & \lambda_n - \lambda_i \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_{i-1}\}$$

Therefore, for any $\mathbf{v} \in V$,

$$((T)_{\mathcal{A},\mathcal{A}} - \lambda_n \mathbf{I}) \mathbf{v} \in \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}.$$

Applying the same trick, we conclude that

$$((T)_{\mathcal{A},\mathcal{A}} - \lambda_1 \mathbf{I}) \cdots ((T)_{\mathcal{A},\mathcal{A}} - \lambda_n \mathbf{I}) \mathbf{v} = \mathbf{0}, \quad \forall \mathbf{v} \in V,$$

i.e., $\chi_T((T)_{\mathcal{A},\mathcal{A}}) = ((T)_{\mathcal{A},\mathcal{A}} - \lambda_1 \mathbf{I}) \cdots ((T)_{\mathcal{A},\mathcal{A}} - \lambda_n \mathbf{I})$ is a zero matrix. ■

Now we are ready to give a proof for the Cayley-Hamilton Theorem:

Proof. Suppose that $\mathcal{X}_T(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{F}[x]$. By considering algebraically closed field $\overline{\mathbb{F}} \supseteq \mathbb{F}$, we imply

$$\mathcal{X}_T(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad (8.1a)$$

$$= (x - \lambda_1) \cdots (x - \lambda_n), \quad \lambda_i \in \overline{\mathbb{F}} \quad (8.1b)$$

By applying proposition (8.1), we imply $\mathcal{X}_T(T) = 0$, where the coefficients in the formula $\mathcal{X}_T(T) = 0$ w.r.t. T are in $\overline{\mathbb{F}}$.

Then we argue that these coefficients are essentially in \mathbb{F} . Expand the whole map of $\mathcal{X}_T(T)$:

$$\mathcal{X}_T(T) = (T - \lambda_1 I) \cdots (T - \lambda_n I) \quad (8.2a)$$

$$= T^n - (\lambda_1 + \cdots + \lambda_n)T^{n-1} + \cdots + (-1)^n \lambda_1 \cdots \lambda_n I \quad (8.2b)$$

$$= T^n + a_{n-1}T^{n-1} + \cdots + a_0 I \quad (8.2c)$$

where the derivation of (8.2c) is because that the polynomial coefficients for (8.1a) and (8.1b) are all identical.

Therefore, we conclude that $\mathcal{X}_T(T) = 0$, under the field \mathbb{F} . ■

Corollary 8.1 $m_T(x) \mid \mathcal{X}_T(x)$. More precisely, if

$$\mathcal{X}_T(x) = [p_1(x)]^{e_1} \cdots [p_k(x)]^{e_k}, \quad e_i > 0, \forall i$$

where p_i 's are distinct, monic, and irreducible polynomials. Then

$$m_T(x) = [p_1(x)]^{f_1} \cdots [p_k(x)]^{f_k}, \quad \text{for some } 0 < f_i \leq e_i, \forall i$$

Proof. The statement $m_T(x) \mid \mathcal{X}_T(x)$ is from Cayley-Hamilton Theorem. Therefore, $0 \leq f_i \leq e_i, \forall i$. Suppose on the contrary that $f_i = 0$ for some i . w.l.o.g., $i = 1$.

It's clear that $\gcd(p_1, p_j) = 1$ for $\forall j \neq 1$, which implies

$$a(x)p_1(x) + b(x)p_j(x) = 1, \quad \text{for some } a(x), b(x) \in \mathbb{F}[x].$$

Considering the field extension $\overline{\mathbb{F}} \supseteq \mathbb{F}$, we have $p_1(x) = (x - \mu_1) \cdots (x - \mu_\ell)$. For any root μ_m of p_1 , $m = 1, \dots, \ell$, we have

$$a(\mu_m)p_1(\mu_m) + b(\mu_m)p_j(\mu_m) = 1 \implies b(\mu_m)p_j(\mu_m) = 1 \implies p_j(\mu_m) \neq 0,$$

i.e., μ_m is not a root of p_j , $\forall j \neq 1$.

Therefore, μ_m is a root of $\chi_T(x)$, but not a root of $m_T(x)$. Then μ_m is an eigenvalue of T , e.g., $T\mathbf{v} = \mu_m\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Recall that $m_{T,\mathbf{v}} = x - \mu_m$, we imply $m_{T,\mathbf{v}} = x - \mu_m \mid m_T(x)$, which is a contradiction. ■

■ **Example 8.1** We can use Corollary (8.1), a stronger version of Cayley-Hamilton Theorem to determine the minimal polynomials:

1. For matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, we imply $\chi_{\mathbf{A}}(x) = (x^2 + x + 1)^1$. Since $x^2 + x + 1$ is irreducible in \mathbb{R} , we have $m_{\mathbf{A}}(x) = x^2 + x + 1$.

2. For matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

we imply $\chi_{\mathbf{A}}(x) = (x - 1)^2(x - 2)^2$.

By Corollary (8.1), we imply both $(x - 1)$ and $(x - 2)$ should be roots of $m_T(x)$, i.e., $m_{\mathbf{A}}(x)$ may have the four options:

$$(x - 1)^2(x - 2)^2, \text{ or}$$

$$(x - 1)(x - 2)^2, \text{ or}$$

$$(x - 1)^2(x - 2), \text{ or}$$

$$(x - 1)(x - 2).$$

By trial and error, one sees that $m_{\mathbf{A}}(x) = (x - 1)^2(x - 2)$.

8.1.2. Primary Decomposition Theorem

We know that not every linear operator is diagonalizable, but diagonalization has some nice properties:

Definition 8.1 [diagonalizable] The linear operator $T : V \rightarrow V$ is diagonalizable over \mathbb{F} if and only if there exists a basis \mathcal{A} of V such that

$$(T)_{\mathcal{A},\mathcal{A}} = \text{diag}(\lambda_1, \dots, \lambda_n),$$

where λ_i 's are not necessarily distinct.

Proposition 8.2 If the linear operator $T : V \rightarrow V$ is diagonalizable, then

$$m_T(x) = (x - \mu_1) \cdots (x - \mu_k),$$

where μ_i 's are **distinct**.

Proof. Suppose T is diagonalizable, then there exists a basis \mathcal{A} of V such that

$$(T)_{\mathcal{A},\mathcal{A}} = \text{diag}(\mu_1, \dots, \mu_1, \mu_2, \dots, \mu_2, \dots, \mu_k, \dots, \mu_k)$$

It's clear that $((T)_{\mathcal{A},\mathcal{A}} - \mu_1 \mathbf{I}) \cdots ((T)_{\mathcal{A},\mathcal{A}} - \mu_k \mathbf{I}) = \mathbf{0}$, i.e., $m_T(x) \mid (x - \mu_1) \cdots (x - \mu_k)$.

Then we show the minimality of $(x - \mu_1) \cdots (x - \mu_k)$. In particular, if $(x - \mu_i)$ is omitted for any $1 \leq i \leq k$, then it's easy to show

$$(T_{\mathcal{A},\mathcal{A}} - \mu_1 \mathbf{I}) \cdots (T_{\mathcal{A},\mathcal{A}} - \mu_{i-1} \mathbf{I})(T_{\mathcal{A},\mathcal{A}} - \mu_{i+1} \mathbf{I}) \cdots (T_{\mathcal{A},\mathcal{A}} - \mu_k \mathbf{I}) \neq \mathbf{0},$$

since all μ_i 's are distinct. Therefore, $m_T(x)$ will not divide $(x - \mu_1) \cdots (x - \mu_{i-1})(x - \mu_{i+1}) \cdots (x - \mu_k)$ for any i , i.e.,

$$m_T(x) = (x - \mu_1) \cdots (x - \mu_k)$$

- Ⓡ The converse of proposition (8.2) is also true, which is a special case for the Primary Decomposition Theorem.

Theorem 8.2 — Primary Decomposition Theorem. Let $T : V \rightarrow V$ be a linear operator with

$$m_T(x) = [p_1(x)]^{e_1} \cdots [p_k(x)]^{e_k},$$

where p_i 's are distinct, monic, and irreducible polynomials. Let $V_i = \ker([p_i(x)]^{e_i}) \leq V, i = 1, \dots, k$, then

1. Each V_i is T -invariant ($T(V_i) \leq V_i$)
2. $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$
3. Consider $T|_{V_i} : V_i \rightarrow V_i$, then

$$m_{T|_{V_i}}(x) = [p_i(x)]^{e_i}$$

Chapter 9

Week9

9.1. Monday for MAT3040

Reviewing.

- $\chi_T(x) = (x - \lambda_1) \cdots (x - \lambda_n)$ over \mathbb{F} if and only if T is triangularizable over \mathbb{F} .
- $m_T(x) = (x - \mu_1) \cdots (x - \mu_k)$, where μ_i 's are distinct over \mathbb{F} if and only if T is diagonalizable over \mathbb{F} .

The converse for this statement is the proposition (8.2). Let's focus on the proof for the forward direction.

9.1.1. Remarks on Primary Decomposition Theorem

Theorem 9.1 — Primary Decomposition Theorem. Let $T : V \rightarrow V$ be a linear operator with $\dim(V) < \infty$, and

$$m_T(x) = [p_1(x)]^{e_1} \cdots [p_k(x)]^{e_k}$$

where p_i 's are distinct, monic, irreducible polynomials. Let $V_i = \ker(p_i(T)^{e_i})$, then

1. each V_i is T -invariant (i.e., $T(V_i) \leq V_i$)
2. $V = V_1 \oplus \cdots \oplus V_k$
3. $T|_{V_i}$ has the minimal polynomial $p_i(x)^{e_i}$.

Proof. 1. (1) follows from part (2) for example (??).

2. Let $q_i(x) = [p_1(x)]^{e_1} \cdots \widehat{[p_i(x)]^{e_i}} \cdots [p_k(x)]^{e_k} := m_T(x)/[p_i(x)]^{e_i}$, then it is clear that

- (a) $\gcd(q_1, \dots, q_k) = 1$
- (b) $\gcd(q_i, p_i^{e_i}) = 1$
- (c) $q_i \cdot p_i^{e_i} = m_T$
- (d) If $i \neq j$, then $m_T(x) \mid q_i(x)q_j(x)$

- By (a) and Bezout's Theorem (6.7), there exists polynomials a_1, \dots, a_k such that

$$a_1(x)q_1(x) + \cdots + a_k(x)q_k(x) = 1,$$

which implies

$$\underbrace{a_1(T)q_1(T)\mathbf{v} + \cdots + a_k(T)q_k(T)\mathbf{v}}_{\mathbf{v}_1} = \mathbf{v}$$

Therefore, $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_k$ for our constructed $\mathbf{v}_1, \dots, \mathbf{v}_k$.

- Note that

$$p_i(T)^{e_i} \mathbf{v}_i = p_i(T)^{e_i} a_i(T) q_i(T) \mathbf{v} = a_i(T) [q_i(T) p_i(T)^{e_i}] \mathbf{v} = a_i(T) m_T(T) \mathbf{v} = \mathbf{0},$$

whcih implies $\mathbf{v}_i \in \ker([p_i(T)]^{e_i}) := V_i$, and therefore

$$V = V_1 + \cdots + V_k \tag{9.1}$$

- To show that the summation in (9.3) is essentially the direct sum, consider

$$\mathbf{0} = \mathbf{v}'_1 + \cdots + \mathbf{v}'_k, \quad \forall \mathbf{v}'_i \in V_i. \tag{9.2}$$

By (a) and Bezout's Theorem (6.7), there exists $b_i(x), c_i(x)$ such that

$$b_i(x)q_i(x) + c_i(x)p_i(x)^{e_i} = 1 \implies b_i(T)q_i(T) + c_i(T)p_i(T)^{e_i} = I,$$

i.e.,

$$b_i(T)q_i(T)\mathbf{v}'_i + c_i(T)p_i(T)^{e_i} \mathbf{v}'_i = b_i(T)q_i(T)\mathbf{v}'_i = \mathbf{v}'_i.$$

Applying the mapping $b_i(T)q_i(T)$ into equality (9.4) both sides, $i = 1, \dots, k$, we obtain

$$\mathbf{0} = b_i(T)q_i(T)\mathbf{0} = b_i(T)q_i(T)\mathbf{v}'_1 + \dots + b_i(T)q_i(T)\mathbf{v}'_k$$

Note that all terms on RHS vanish except for $b_i(T)q_i(T)\mathbf{v}'_i = \mathbf{v}'_i$, since $q_i(x) = [p_1(x)]^{e_1} \dots [\widehat{p_i(x)}]^{e_i} \dots [p_k(x)]^{e_k}$ and $\mathbf{v}'_j \in \ker([p_j(x)]^{e_j})$. Therefore, $\mathbf{v}'_i = 0$ for $i = 1, \dots, k$, i.e., $V = V_1 \oplus \dots \oplus V_k$.

3. For any $\mathbf{v}_i \in V_i$, we have $p_i(T)^{e_i} \mathbf{v}_i = \mathbf{0}$, which implies $m_{T|V_i}(x) \mid p_i(x)^{e_i}$. Together with Corollary (8.1), $m_{T|V_i}(x) = p_i(x)^{f_i}$ for some $1 \leq f_i \leq e_i$.

Suppose on the contrary that there exists $f_i < e_i$ for some i , consider any $\mathbf{v} := \mathbf{v}_1 + \dots + \mathbf{v}_k \in V$, and

$$p_1(T)^{f_1} \dots p_k(T)^{f_k} \mathbf{v} = p_1(T)^{f_1} \dots p_k(T)^{f_k} (\mathbf{v}_1 + \dots + \mathbf{v}_k)$$

The term on the RHS vanishes since $p_j(T)^{f_j} \mathbf{v}_j = \mathbf{0}$, which implies

$$m_T \mid p_1^{f_1} \dots p_k^{f_k},$$

but there exists i such that $e_i > f_i$, which is a contradiction. ■

Corollary 9.1 If $m_i(x) = (x - \mu_1) \dots (x - \mu_k)$ over \mathbb{F} , where μ_i 's are distinct, then T is diagonalizable over \mathbb{F} . (the converse actually also holds, see proposition (8.2))

Proof. By primary decomposition theorem,

$$V = \underbrace{\ker(T - \mu_1 I)}_{V_1} \oplus \dots \oplus \underbrace{\ker(T - \mu_k I)}_{V_k}$$

Take B_i as a basis of V_i , an μ_i -eigenspace of T . Then $B := \cup_{i=1}^k B_i$ is a basis consisting of eigenvectors of T .

It's clear that $(T|_{V_i})_{\mathcal{B},\mathcal{B}} = \text{diag}(\mu_i, \dots, \mu_i)$, and T is diagonalizable with

$$(T)_{\mathcal{B},\mathcal{B}} = \text{diag}((T|_{V_1})_{\mathcal{B},\mathcal{B}}, \dots, (T|_{V_k})_{\mathcal{B},\mathcal{B}}).$$

■

Corollary 9.2 [Spectral Decomposition] Suppose $T : V \rightarrow V$ is diagonalizable, then there exists a linear operator $p_i : V \rightarrow V$ for $1 \leq i \leq k$ such that

- $p_i^2 = p_i$ (idempotent)
- $p_i p_j = 0, \forall i \neq j$
- $\sum_{i=1}^k p_i = I$
- $p_i T = T p_i, \forall i$

and scalars μ_1, \dots, μ_k such that

$$T = \mu_1 p_1 + \dots + \mu_k p_k$$

Proof. Diagonalization of T is equivalent to say that $m_T(x) = (x - \mu_1) \cdots (x - \mu_k)$, where μ_i 's are distinct. Construct

- $V_i := \ker(T - \mu_i I)$
- $p_i : V \rightarrow V$ given by $p_i = a_i(T)q_i(T)$ as in the proof of primary decomposition theorem

Then:

- $p_i T = T p_i$ is obvious
- $\sum_{i=1}^k p_i = \sum_{i=1}^k a_i(T)q_i(T) = I$
- $p_i p_j = a_i(T)a_j(T)q_i(T)q_j(T) := a_i(T)a_j(T)s(T)m_T(T) = \mathbf{0}$
- $p_i^2 = p_i(p_1 + \dots + p_k) = p_i \cdot I = p_i$

For the last part, note that

- $p_i V \leq V_i, \forall i$: for $\forall \mathbf{v} \in V$,

$$(T - \mu_i I)p_i \mathbf{v} = (T - \mu_i I)a_i(T)q_i(T)\mathbf{v} = a_i(T)m_T(x)\mathbf{v} = \mathbf{0}$$

Therefore, $p_i V \leq \ker(T - \mu_i I) = V_i$

- Now, for all $\mathbf{w} \in V$,

$$\begin{aligned} T\mathbf{w} &= T(p_1 + \cdots + p_k)\mathbf{w} \\ &= Tp_1\mathbf{w} + \cdots + Tp_k\mathbf{w} \\ &= (\mu_1 p_1)\mathbf{w} + \cdots + (\mu_k p_k)\mathbf{w} \end{aligned}$$

and therefore $T = \mu_1 p_1 + \cdots + \mu_k p_k$

■

Organization of future two weeks. We are interested in under which condition does the T is diagonalizable. One special case is $T = A$, where \mathbf{A} is a symmetric matrix. We will study normal operators, which includes the case for symmetric matrices.

Question: what happens if $m_T(x)$ contains repeated linear factors? We will spend the next whole class to show the Jordan Normal Form:

Theorem 9.2 — Jordan Normal Form. Let \mathbb{F} be algebraically closed field such that every linear operator $T : V \rightarrow V$ has the form

$$m_T(x) = \prod_{i=1}^k (x - \lambda_i)^{e_i}$$

where λ_i 's are distinct.

Then there exists basis \mathcal{A} of V such that

$$(T)_{\mathcal{A}, \mathcal{A}} = \text{diag}(\mathbf{J}_1, \dots, \mathbf{J}_k)$$

where

$$\mathbf{J}_i = \begin{pmatrix} \mu & 1 & 0 & 0 \\ 0 & \mu & 1 & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu \end{pmatrix}$$

for some $\mu \in \{\lambda_1, \dots, \lambda_k\}$

9.4. Wednesday for MAT3040

9.4.1. Jordan Normal Form

Theorem 9.3 — Jordan Normal Form. Suppose that $T : V \rightarrow V$ has minimal polynomial

$$m_T(x) = \prod_{i=1}^k (x - \lambda_i)^{e_i},$$

then there exists a basis \mathcal{A} such that

$$(T)_{\mathcal{A}, \mathcal{A}} = \text{diag}(J_1, \dots, J_\ell),$$

where each block J_i is a square matrix of the form

$$J_i = \begin{bmatrix} \mu_i & 1 & & \\ & \mu_i & \ddots & \\ & & \ddots & 1 \\ & & & \mu_i \end{bmatrix}.$$

R By primary decomposition theorem,

$$V = V_1 \oplus \dots \oplus V_k, \quad \text{where } V_i = \ker((T - \lambda_i I)^{e_i}), \quad i = 1, \dots, k,$$

and each V_i is T -invariant.

We pick basis \mathcal{B}_i for each subspace V_i , then $\mathcal{B} := \cup_{i=1}^k \mathcal{B}_i$ is a basis of V , and

$$(T)_{\mathcal{B}, \mathcal{B}} = \begin{pmatrix} (T|_{V_1})_{\mathcal{B}_1, \mathcal{B}_1} & 0 & \dots & 0 \\ 0 & (T|_{V_2})_{\mathcal{B}_2, \mathcal{B}_2} & \vdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \vdots & (T|_{V_k})_{\mathcal{B}_k, \mathcal{B}_k} \end{pmatrix}$$

with $m_{T|_{V_i}}(x) = (x - \lambda_i)^{e_i}$.

Therefore, it suffices to show the Jordan normal form holds for the linear operator

T with minimal polynomial $m_T(x) = (x - \lambda)^e$.

Firstly, we consider the case where the minimal polynomial has the form x^m :

Proposition 9.6 Suppose $T : V \rightarrow V$ is such that $m_T(x) = x^m$, then the theorem (9.3) holds, i.e., there exists a basis \mathcal{A} such that

$$(T)_{\mathcal{A}, \mathcal{A}} = \text{diag}(\mathbf{J}_1, \dots, \mathbf{J}_\ell),$$

where each block J_i is a square matrix of the form

$$J_i = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Proof. • Suppose that $m_T(x) = x^m$, then it is clear that

$$\{0\} := \ker(T^0) \leq \ker(T) \leq \ker(T^2) \leq \dots \leq \ker(T^m) := V$$

Furthermore, we have $\ker(T^{i-1}) \subsetneq \ker(T^i)$ for $i = 1, \dots, m$: Note that $\ker(T^{m-1}) \subsetneq \ker(T^m) := V$ due to the minimality of $m_T(x)$; and $\ker(T^{m-2}) \subsetneq \ker(T^{m-1})$ since otherwise for any $\mathbf{x} \in \ker(T^m)$,

$$T^{m-1}(T\mathbf{x}) = \mathbf{0} \implies T\mathbf{x} \in \ker(T^{m-1}) = \ker(T^{m-2}) \implies T^{m-2}(T\mathbf{x}) = T^{m-1}(\mathbf{x}) = \mathbf{0},$$

i.e., $\mathbf{x} \in \ker(T^{m-1})$, which contradicts to the fact that $\ker(T^{m-1}) \subsetneq \ker(T^m)$. Proceeding this trick sequentially for $i = m, m-1, \dots, 1$, we proved the desired result.

- Then construct the quotient space $W_i = \ker(T^i)/\ker(T^{i-1})$ and define \mathcal{B}'_i to be a basis of W_i :

$$\mathcal{B}'_i = \{a_1^i + \ker(T^{i-1}), \dots, a_{\ell_i}^i + \ker(T^{i-1})\}$$

Construct $\mathcal{B}_i = \{a_1^i, \dots, a_{\ell_i}^i\}$, then we claim that $B := \cup_{i=1}^m \mathcal{B}_i$ forms a basis of V :

- First proof the case $m = 2$ first: let $U \leq V$ ($\dim(V) < \infty$), and $\mathcal{B}_1 = \{a_1^1, \dots, a_{k_1}^1\}$ be a basis of U , and

$$\mathcal{B}_2' = \{a_1^2 + U, \dots, a_{k_2}^2 + U\}$$

be a basis of V/U . Then to show the statement suffices to show that

$$\bigcup_{i=1}^2 \{a_1^i, \dots, a_{k_i}^i\} \text{ forms a basis of } V.$$

It's clear that $\cup_{i=1}^2 \{a_1^i, \dots, a_{k_i}^i\}$ spans V . Furthermore, $\dim(V) = \dim(U) + \dim(V/U) = k_1 + k_2$, i.e., $\cup_{i=1}^2 \{a_1^i, \dots, a_{k_i}^i\}$ contains correct amount of vectors. The proof is complete.

- This result can be extended from 2 to general m , thus the claim is shown.
- For $i < m$, consider the set $S_i = \{T(\mathbf{w}_j) + \ker(T^{i-1}) \mid \mathbf{w}_j \in B_{i+1}\}$. Note that
 - Since $T^{i+1}(\mathbf{w}_j) = \mathbf{0}$, $T^i(T(\mathbf{w}_j)) = \mathbf{0}$, we imply $T(\mathbf{w}_j) \in \ker(T^i)$, i.e., $S_i \subseteq W_i$.
 - The set S_i is linearly independent: consider the equation

$$\sum_j k_j(T(\mathbf{w}_j) + \ker(T^{i-1})) = \mathbf{0}_{W_i} \iff T\left(\sum_j k_j \mathbf{w}_j\right) + \ker(T^{i-1}) = \mathbf{0}_{W_i}$$

i.e.,

$$T\left(\sum_j k_j \mathbf{w}_j\right) \in \ker(T^{i-1}) \iff T^{i-1}(T(\sum_j k_j \mathbf{w}_j)) = \mathbf{0}_V,$$

i.e., $\sum_j k_j \mathbf{w}_j \in \ker(T^i)$, i.e.,

$$\sum_j k_j \mathbf{w}_j + \ker(T^i) = \mathbf{0}_{W_{i+1}} \iff \sum_j k_j(\mathbf{w}_j + \ker(T^i)) = \mathbf{0}_{W_{i+1}}.$$

Since $\{\mathbf{w}_j + \ker(T^i), \forall j\}$ forms a basis of W_{i+1} , we imply $k_j = 0, \forall j$.

From \mathcal{B}_{i+1} we construct S_i , which is linearly independent in W_i . Therefore, we imply $|T(\mathcal{B}_{i+1})| \leq |\mathcal{B}_i|$ for $\forall i < m$ (why?).

- Now we start to construct a basis \mathcal{A} of V :
 - Start with $\mathcal{B}_m' := \{u_1^m + \ker(T^{m-1}), \dots, u_{\ell_m}^m + \ker(T^{m-1})\}$, and $\mathcal{B}_m = \{u_1^m, \dots, u_{\ell_m}^m\}$.

– By the previous result,

$$\{T(u_1^m) + \ker(T^{m-2}), \dots, T(u_{\ell_m}^m) + \ker(T^{m-2})\}$$

is linear independent in W_{m-1} . By basis extension, we get a basis \mathcal{B}'_{m-1} of W_{m-1} , and let

$$\mathcal{B}_{m-1} = \{T(u_1^m), \dots, T(u_{\ell_m}^m)\} \cup \xi_{m-1}$$

where $\xi_{m-1} := \{u_1^{m-1}, \dots, u_{\ell_{m-1}}^{m-1}\}$

– Continue the process above to obtain $\mathcal{B}_{m-2}, \dots, \mathcal{B}_1$, and $\cup_{i=1}^m \mathcal{B}_i$ forms a basis of V :

\mathcal{B}_1	\mathcal{B}_2	\dots	\mathcal{B}_{m-1}	\mathcal{B}_m
$\{T^{m-1}(u_1^m), \dots, T^{m-1}(u_{\ell_m}^m)\}$	$\{T^{m-2}(u_1^m), \dots, T^{m-2}(u_{\ell_m}^m)\}$	\dots	$\{T(u_1^m), \dots, T(u_{\ell_m}^m)\}$	$\{u_1^m, \dots, u_{\ell_m}^m\}$
$\{T^{m-2}(u_1^{m-1}), \dots, T^{m-2}(u_{\ell_{m-1}}^{m-1})\}$	$\{T^{m-3}(u_1^{m-1}), \dots, T^{m-3}(u_{\ell_{m-1}}^{m-1})\}$	\dots	$\{u_1^{m-1}, \dots, u_{\ell_{m-1}}^{m-1}\}$	
\vdots	\vdots			
$\{T(u_1^2), \dots, T(u_{\ell_2}^2)\}$	$\{u_1^2, \dots, u_{\ell_2}^2\}$			
$\{u_1^1, \dots, u_{\ell_1}^1\}$				

– Now construct the ordered basis \mathcal{A} :

$$\mathcal{A} = \left(\begin{array}{cccccc} T^{m-1}(u_1^m) & \dots & T^2(u_1^m) & T(u_1^m) & u_1^m & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ T^{m-1}(u_{\ell_m}^m) & \dots & T^2(u_{\ell_m}^m) & T(u_{\ell_m}^m) & u_{\ell_m}^m & \\ & T^{m-2}(u_1^{m-1}) & \dots & T(u_1^{m-1}) & u_1^{m-1} & \\ & \vdots & \ddots & \vdots & \vdots & \\ & T^{m-2}(u_{\ell_{m-1}}^{m-1}) & \dots & T(u_{\ell_{m-1}}^{m-1}) & u_{\ell_{m-1}}^{m-1} & \\ & & \vdots & \ddots & \vdots & \\ & & & & u_1^1 & \\ & & & & \vdots & \\ & & & & u_{\ell_1}^1 & \end{array} \right)$$

- Then the diagonal entries of $(T)_{\mathcal{A},\mathcal{A}}$ should be all zero, since

$$T(T^{i-1}(u_j^i)) = T^i(u_j^i) = 0, \forall i = 1, \dots, m, j = 1, \dots, \ell_i,$$

and every entry on the superdiagonal is 1:

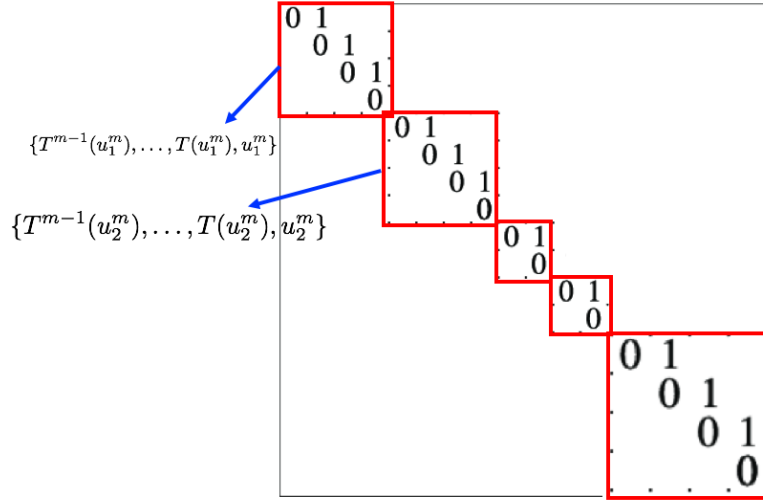


Figure 9.2: Illustration for $(T)_{\mathcal{A},\mathcal{A}}$

■

Then we consider the case where $m_T(x) = (x - \lambda)^e$:

Corollary 9.3 Suppose $T : V \rightarrow V$ is such that $m_T(x) = (x - \lambda)^e$, then the theorem (9.3) holds, i.e., there exists a basis \mathcal{A} such that

$$(T)_{\mathcal{A},\mathcal{A}} = \text{diag}(J_1, \dots, J_\ell),$$

where each block J_i is a square matrix of the form

$$J_i = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

Proof. Suppose that $m_T(x) = (x - \lambda)^e$. Consider the operator $U := T - \lambda I$, then $m_U(x) = x^e$.

By applying proposition (9.6),

$$(U)_{\mathcal{A},\mathcal{A}} = \text{diag}(\mathbf{J}_1, \dots, \mathbf{J}_\ell),$$

where

$$\mathbf{J}_i = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Or equivalently,

$$(T)_{\mathcal{A},\mathcal{A}} - \lambda(I)_{\mathcal{A},\mathcal{A}} = \text{diag}(\mathbf{J}_1, \dots, \mathbf{J}_\ell)$$


i.e.,

$$(T)_{\mathcal{A},\mathcal{A}} = \text{diag}(\mathbf{K}_1, \dots, \mathbf{K}_\ell),$$

where

$$\mathbf{K}_i = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

■

 The Jordan Normal Form Theorem (9.3) follows from our arguments using the primary decomposition.

Corollary 9.4 Any matrix $A \in M_{n \times n}(\mathbb{C})$ is similar to a matrix of the Jordan normal form

$$\text{diag}(\mathbf{J}_1, \dots, \mathbf{J}_\ell).$$

9.4.2. Inner Product Spaces

Definition 9.8 [Bilinear] Let V be a vector space over \mathbb{R} . A bilinear form on V is a mapping

$$F : V \times V \rightarrow \mathbb{R}$$

satisfying

1. $F(\mathbf{u} + \mathbf{v}, \mathbf{w}) = F(\mathbf{u}, \mathbf{w}) + F(\mathbf{v}, \mathbf{w})$
2. $F(\mathbf{u}, \mathbf{v} + \mathbf{w}) = F(\mathbf{u}, \mathbf{v}) + F(\mathbf{u}, \mathbf{w})$
3. $F(\lambda \mathbf{u}, \mathbf{v}) = \lambda F(\mathbf{u}, \mathbf{v}) = F(\mathbf{u}, \lambda \mathbf{v})$

We say

- F is symmetric if $F(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}, \mathbf{u})$
- F is non-degenerate if $F(\mathbf{u}, \mathbf{w}) = 0$ for $\forall \mathbf{u} \in V$ implies $\mathbf{w} = \mathbf{0}$
- F is positive definite if $F(\mathbf{v}, \mathbf{v}) > 0$ for $\forall \mathbf{v} \neq \mathbf{0}$

R If F is positive-definite, then F is non-degenerate: Suppose that $F(\mathbf{v}, \mathbf{v}) > 0, \forall \mathbf{v} \neq \mathbf{0}$. If we have $F(\mathbf{u}, \mathbf{w}) = 0$ for any $\mathbf{u} \in V$, then in particular, when $\mathbf{u} = \mathbf{w}$, we imply $F(\mathbf{w}, \mathbf{w}) = 0$. By positive-definiteness, $\mathbf{w} = \mathbf{0}$, i.e., F is non-degenerate.

Chapter 10

Week10

10.1. Monday for MAT3040

10.1.1. Inner Product Space

- Symmetric: $F(\mathbf{u}, \mathbf{w}) = F(\mathbf{w}, \mathbf{u}), \forall \mathbf{u}, \mathbf{w}$
- Non-degenerate: $F(\mathbf{u}, \mathbf{w}) = 0, \forall \mathbf{w}$ implies $\mathbf{u} = \mathbf{0}$
- Positive definite: $F(\mathbf{v}, \mathbf{v}) > 0, \forall \mathbf{v} \neq \mathbf{0}$

Classification. When we say V be a vector space over \mathbb{F} , we treat $\alpha \in \mathbb{F}$ as a scalar.


Definition 10.1 [Sesqui-linear Form] Let V be a vector space over \mathbb{C} . A **sesquilinear form** on V is a function $F : V \times V \rightarrow \mathbb{C}$ such that

1. $F(\mathbf{u} + \mathbf{v}, \mathbf{w}) = F(\mathbf{u}, \mathbf{w}) + F(\mathbf{v}, \mathbf{w})$
2. $F(\mathbf{u}, \mathbf{v} + \mathbf{w}) = F(\mathbf{u}, \mathbf{v}) + F(\mathbf{u}, \mathbf{w})$
3. $F(\bar{\lambda}\mathbf{v}, \mathbf{w}) = F(\mathbf{v}, \lambda\mathbf{w}) = \lambda F(\mathbf{v}, \mathbf{w}), \forall \lambda \in \mathbb{C}$

In this case, we say F is **conjugate symmetric** if

$$F(\mathbf{v}, \mathbf{w}) = \overline{F(\mathbf{w}, \mathbf{v})}, \quad \forall \mathbf{v}, \mathbf{w} \in V.$$

The definition for non-degenerateness, and positive definiteness is the same as that in bilinear form. ■

 In the sesquilinear form, why there is a $\bar{\lambda}$ shown in condition (3)?

Partial Answer: We want our F to be positive definite in many cases:

- Suppose that $F(\mathbf{v}, \mathbf{v}) > 0$ and we do not have $\bar{\lambda}$ in sesquilinear form F , it follows that

$$F(i\mathbf{v}, i\mathbf{v}) = i^2 F(\mathbf{v}, \mathbf{v}) = -F(\mathbf{v}, \mathbf{v}) < 0$$

As a result, there will be no positive bilinear form for vector space over \mathbb{C} .

Therefore, $\bar{\lambda}$ is essential to guarantee that we have a positive definite form on vector space over \mathbb{C} , i.e.,

$$F(i\mathbf{v}, i\mathbf{v}) = \bar{i}i F(\mathbf{v}, \mathbf{v}) = F(\mathbf{v}, \mathbf{v})$$

■ **Example 10.1** Consider $V = \mathbb{C}^n$, and a basic sesquilinear form is the Hermitian inner product:

$$F(\mathbf{v}, \mathbf{u}) = \mathbf{v}^H \mathbf{u} = \begin{pmatrix} \bar{v}_1 & \cdots & \bar{v}_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^n \bar{v}_i w_i$$

In this case, we do not have symmetric property $F(\mathbf{v}, \mathbf{w}) = F(\mathbf{w}, \mathbf{v})$ any more, instead, we have the conjugate symmetric property $F(\mathbf{v}, \mathbf{w}) = \overline{F(\mathbf{w}, \mathbf{v})}$. ■

Definition 10.2 [Inner Product] A real (complex) vector space V with a bilinear (sesquilinear) form with symmetric (conjugate symmetric) and positive definite property is called an **inner product** on V . Any vector space equipped with inner product is called an **inner product space**. ■

Notation. We write $\langle \cdot, \cdot \rangle$ instead of $F(\cdot, \cdot)$ to denote inner product.

■ **Definition 10.3** [Norm] The **norm** of a vector \mathbf{v} is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. ■

Ⓡ As a result, $\|\alpha \mathbf{v}\| = \sqrt{\langle \alpha \mathbf{v}, \alpha \mathbf{v} \rangle} = \sqrt{\bar{\alpha} \alpha \langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{|\alpha|^2 \langle \mathbf{v}, \mathbf{v} \rangle} = |\alpha| \|\mathbf{v}\|$.

The norm is well-defined since $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ (positive definiteness of inner product).

Definition 10.4 [Orthogonal] We say a family of vectors $S = \{\mathbf{v}_i \mid i \in I\}$ is **orthogonal** if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \forall i \neq j$$

If furthermore $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1, \forall i$, then we say S is an **orthonormal** set. ■



1. The Cauchy-Scharwz inequality holds for inner product space:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|, \forall \mathbf{u}, \mathbf{v} \in V.$$

Proof. The proof for $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{R}$ is the same as in MAT2040 course. Check Theorem (6.1) in the note

<https://walterbabyrudin.github.io/information/Notes/MAT2040.pdf>

However, for $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{C} \setminus \mathbb{R}$, we need the re-scaling technique:

Let $\mathbf{w} = \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \mathbf{u}$, then $\langle \mathbf{w}, \mathbf{v} \rangle \in \mathbb{R}$:

$$\langle \mathbf{w}, \mathbf{v} \rangle = \left\langle \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \mathbf{u}, \mathbf{v} \right\rangle = \overline{\left(\frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \right)} \langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \langle \mathbf{u}, \mathbf{v} \rangle = 1.$$

Applying the Cauchy-Scharwz inequality for $\langle \mathbf{w}, \mathbf{v} \rangle \in \mathbb{R}$ gives

$$\begin{aligned} \left| \left\langle \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \mathbf{u}, \mathbf{v} \right\rangle \right| &= |\langle \mathbf{w}, \mathbf{v} \rangle| \\ &\leq \|\mathbf{w}\| \|\mathbf{v}\| = \left\| \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \mathbf{u} \right\| \|\mathbf{v}\| \end{aligned}$$

Or equivalently,

$$\left| \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| |\langle \mathbf{u}, \mathbf{v} \rangle| \leq \left| \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| \|\mathbf{u}\| \|\mathbf{v}\|$$

Since $\left| \frac{1}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| = \left| \frac{1}{\overline{\langle \mathbf{u}, \mathbf{v} \rangle}} \right|$, we imply

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

■

2. The triangle inequality also holds for inner product process:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

3. The Gram-Schmidt process holds for finite set of vectors: let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be (finite) linearly independent. Then we can construct an orthonormal set from S :

$$\mathbf{w}_1 = \mathbf{v}_1, \quad \mathbf{w}_{i+1} = \mathbf{v}_{i+1} - \frac{\langle \mathbf{v}_{i+1}, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 - \frac{\langle \mathbf{v}_{i+1}, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2} \mathbf{w}_2 - \dots - \frac{\langle \mathbf{v}_{i+1}, \mathbf{w}_i \rangle}{\|\mathbf{w}_i\|^2} \mathbf{w}_i, \quad i = 1, \dots, n-1$$

Then after normalization, we obtain the constructed orthonormal set.

Consequently, every finite dimensional inner product space has an orthonormal basis.

10.1.2. Dual spaces

Theorem 10.1 — Riesz Representation. Consider the mapping

$$\phi : \quad V \rightarrow V^*$$

$$\text{with} \quad \mathbf{v} \mapsto \phi_{\mathbf{v}}$$

$$\text{where} \quad \phi_{\mathbf{v}}(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle, \quad \forall \mathbf{w} \in V$$

Then the mapping ϕ is well-defined and it is an \mathbb{R} -linear transformation.

Moreover, if V is finite dimensional, then ϕ is an isomorphism.

The \mathbb{R} -linear transformation $V \rightarrow V^*$ means that, when V, V^* are vector space over \mathbb{R} , the \mathbb{R} -linear transformation deduces into exactly the linear transformation.

- Ⓡ The \mathbb{R} -linear transformation $V \rightarrow V^*$ is **not** necessarily linear if V, V^* are vector spaces over \mathbb{C} .

However, we can transform a vector space over \mathbb{C} into a vector space over \mathbb{R} :

- For example, suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V over \mathbb{C} , i.e.,

$$\mathbf{v} = \sum_{j=1}^n \alpha_j \mathbf{v}_j$$

where $\alpha_j = p_j + iq_j, \forall p_j, q_j \in \mathbb{R}$, then

$$\mathbf{v} = \sum_j p_j \mathbf{v}_j + \sum_j q_j (i\mathbf{v}_j), \quad p_j, q_j \in \mathbb{R}$$

Therefore, $\{\mathbf{v}_1, \dots, \mathbf{v}_n, i\mathbf{v}_1, \dots, i\mathbf{v}_n\}$ forms a basis of V over \mathbb{R} .

Note that $i\mathbf{v}_1$ cannot be considered as a linear combination of \mathbf{v}_1 over \mathbb{R} , but a linear combination of \mathbf{v}_1 over \mathbb{C} .

In particular, if $\phi : V \rightarrow V^*$ is a \mathbb{R} -linear transformation, then

$$\phi(i\mathbf{v}) \neq i\phi(\mathbf{v}), \text{ but } \phi(2\mathbf{v}) = 2\phi(\mathbf{v}).$$

Proof. 1. Well-definedness: We need to show $\phi_{\mathbf{v}} \in V^*$, i.e., for scalars a, b ,

$$\phi_{\mathbf{v}}(a\mathbf{w}_1 + b\mathbf{w}_2) = \langle \mathbf{v}, a\mathbf{w}_1 + b\mathbf{w}_2 \rangle = a\langle \mathbf{v}, \mathbf{w}_1 \rangle + b\langle \mathbf{v}, \mathbf{w}_2 \rangle = a\phi_{\mathbf{v}}(\mathbf{w}_1) + b\phi_{\mathbf{v}}(\mathbf{w}_2)$$

Therefore, $\phi_{\mathbf{v}} \in V^*$.

2. \mathbb{R} -linearity of ϕ : it suffices to show

$$\phi(c\mathbf{v}_1 + d\mathbf{v}_2) = c\phi(\mathbf{v}_1) + d\phi(\mathbf{v}_2), \quad \forall c, d \in \mathbb{R}, \mathbf{v}_1, \mathbf{v}_2 \in V.$$

For all $\mathbf{w} \in V$, we have

$$\phi_{c\mathbf{v}_1 + d\mathbf{v}_2}(\mathbf{w}) = \langle c\mathbf{v}_1 + d\mathbf{v}_2, \mathbf{w} \rangle = c\langle \mathbf{v}_1, \mathbf{w} \rangle + d\langle \mathbf{v}_2, \mathbf{w} \rangle = c\phi_{\mathbf{v}_1}(\mathbf{w}) + d\phi_{\mathbf{v}_2}(\mathbf{w})$$

where the second equality holds because $c, d \in \mathbb{R}$.

Therefore,

$$\phi(c\mathbf{v}_1 + d\mathbf{v}_2) = c\phi(\mathbf{v}_1) + d\phi(\mathbf{v}_2).$$

■

10.4. Wednesday for MAT3040

Reviewing. Consider the mapping

$$\begin{aligned}\phi : & \quad V \rightarrow V^* \\ \text{with} & \quad \phi(\mathbf{v}) = \phi_{\mathbf{v}} \\ \text{where} & \quad \phi_{\mathbf{v}}(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

The Riesz Representation Theorem claims that

1. ϕ is a \mathbb{R} -linear transformation.
2. ϕ is injective.
3. If $\dim(V) < \infty$, then ϕ is an isomorphism.

Proof for Claim (2). Consider the equality $\phi(\mathbf{v}) = \phi_{\mathbf{v}} = 0_{V^*}$, which implies

$$\phi_{\mathbf{v}}(\mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in V$$

By the non-degeneracy property, $\mathbf{v} = 0_{\mathbf{v}}$, i.e., ϕ is injective. ■

Proof for Claim (3). Since $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{R}}(V^*)$, and ϕ is injective as a \mathbb{R} -linear transformation, we imply ϕ is an isomorphism from V to V^* , where V, V^* are treated as vector spaces over \mathbb{R} . ■

10.4.1. Orthogonal Complement

Definition 10.5 [Orthogonal Complement] Let $U \leq V$ be a subspace of an inner product space. Then the **orthogonal complement** of U is

$$U^{\perp} = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0, \forall \mathbf{u} \in U\}$$

The analysis for orthogonal complement for vector spaces over \mathbb{C} is quite similar as what we have studied in MAT2040.

Proposition 10.7 1. U^\perp is a subspace of V

2. $U \cap U^\perp = \{0\}$
3. $U_1 \subseteq U_2$ implies $U_2^\perp \subseteq U_1^\perp$.

Proof. 1. Suppose that $\mathbf{v}_1, \mathbf{v}_2 \in U^\perp$, where $a, b \in K$ ($K = \mathbb{C}$ or \mathbb{R}), then for all $\mathbf{u} \in U$,

$$\begin{aligned}\langle a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{u} \rangle &= \bar{a}\langle \mathbf{v}_1, \mathbf{u} \rangle + \bar{b}\langle \mathbf{v}_2, \mathbf{u} \rangle \\ &= \bar{a} \cdot 0 + \bar{b} \cdot 0 = 0\end{aligned}$$

Therefore, $a\mathbf{v}_1 + b\mathbf{v}_2 \in U^\perp$.

2. Suppose that $\mathbf{u} \in U \cap U^\perp$, then we imply $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. By the positive-definiteness of inner product, $\mathbf{u} = \mathbf{0}$.
3. The statement (3) is easy.

■

Proposition 10.8 1. If $\dim(V) < \infty$ and $U \leq V$, then $V = U \oplus U^\perp$

2. If $U, W \leq V$, then

$$\begin{aligned}(U + W)^\perp &= U^\perp \cap W^\perp \\ (U \cap W)^\perp &\supseteq U^\perp + W^\perp \\ (U^\perp)^\perp &\supseteq U\end{aligned}$$

Moreover, if $\dim(V) < \infty$, then these are equalities.

Proof. 1. Suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ forms a basis for U , and by basis extension, we obtain $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ is a basis for V .

By Gram-Schmidt Process, any finite basis induces an orthonormal basis.

Therefore, suppose that $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ forms an orthonormal basis for U , and $\{\mathbf{e}_{k+1}, \dots, \mathbf{e}_n\}$ forms an orthonormal basis for U^\perp .

It's easy to show $V = U + U^\perp$ using orthonormal basis.

2. (a) The reverse part $(U + W)^\perp \supseteq U^\perp \cap W^\perp$ is trivial; for the forward part, suppose

$\mathbf{v} \in (U + W)^\perp$, then

$$\langle \mathbf{v}, \mathbf{u} + \mathbf{w} \rangle = 0, \forall \mathbf{u} \in U, \mathbf{w} \in W$$

Taking $\mathbf{u} \equiv \mathbf{0}$ in the equality above gives $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, i.e., $\mathbf{v} \in U^\perp$. Similarly, $\mathbf{v} \in W^\perp$.

- (b) Follow the similar argument as in (2a). If $\dim(V) < \infty$, then write down the orthonormal basis for $U^\perp + W^\perp$ and $(U \cap W)^\perp$.
- (c) Follow the similar argument as in (2a). If $\dim(V) < \infty$, then

$$V = U^\perp \oplus (U^\perp)^\perp = U \oplus U^\perp.$$

Therefore, $(U^\perp)^\perp = U$.

■

Proposition 10.9 The mapping $\phi : V \rightarrow V^*$ maps $U^\perp \leq V$ injectively to $\text{Ann}(U) \leq V^*$. If $\dim(V) < \infty$, then $U^\perp \cong \text{Ann}(U)$ as \mathbb{R} -vector spaces

Proof. The injectivity of ϕ has been shown at the beginning of this lecture. For any $\mathbf{v} \in U^\perp$, we imply $\phi_{\mathbf{v}}(\mathbf{u}) = 0, \forall \mathbf{u} \in U$, i.e., $\phi_{\mathbf{v}} \in \text{Ann}(U)$.

Therefore, $\phi(U^\perp) \leq \text{Ann}(U)$.

Provided that $\dim(V) < \infty$, by (1) in proposition (10.8),

$$\dim(U) + \dim(U^\perp) = \dim(V)$$

Since $\dim(U) + \dim(\text{Ann}(U)) = \dim(V)$, we imply $\dim(U^\perp) = \dim(\text{Ann}(U))$.

Moreover,

$$\phi : U^\perp \rightarrow \text{Ann}(U)$$

is an isomorphism between \mathbb{R} -vector spaces U^\perp and $\text{Ann}(U)$.

■

10.4.2. Adjoint Map

Motivation. Then we study the induced mapping based on a given linear operator T , denoted as T' . This induced mapping essentially plays the similar role as taking the Hermitian for a complex matrix.

Notation. Previously we have studied the **adjoint** of $T : V \rightarrow W$, denoted as $T^* : W^* \rightarrow V^*$. However, from now on, we use the same terminology but with different meaning. If $T : V \rightarrow V$ is a linear operator, then the **adjoint** of T is the linear operator $T' : V \rightarrow V$ defined as follows.

Definition 10.6 [Adjoint] Let $T : V \rightarrow V$ be a linear operator between inner product spaces. The **adjoint** of T is defined as $T' : V \rightarrow V$ satisfying

$$\langle T'(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle, \quad \forall \mathbf{w} \in V \quad (10.1)$$

Proposition 10.10 If $\dim(V) < \infty$, then T' exists, and it is unique. Moreover, T' is a linear map.

Proof. Fix any $\mathbf{v} \in V$. Consider the mapping

$$\alpha_{\mathbf{v}} : \mathbf{w} \xrightarrow{T} T(\mathbf{w}) \xrightarrow{\phi_{\mathbf{v}}} \langle \mathbf{v}, T(\mathbf{w}) \rangle$$

This is a linear transformation from V to \mathbb{F} , i.e., $\alpha_{\mathbf{v}} \in V^*$

By Riesz representation theorem, ϕ is an isomorphism from V to V^* . Therefore, for any $\alpha_{\mathbf{v}} \in V^*$, there exists a vector $T'(\mathbf{v}) \in V$ such that

$$\phi(T'(\mathbf{v})) = \alpha_{\mathbf{v}} \in V^*$$

Or equivalently, $\phi_{T'(\mathbf{v})}(\mathbf{w}) = \alpha_{\mathbf{v}}(\mathbf{w}), \forall \mathbf{w} \in V$, i.e., $\langle T'(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle$.

Therefore, from \mathbf{v} we have constructed $T'(\mathbf{v})$ satisfying (10.1). Now define $T' : V \rightarrow V$ by $\mathbf{v} \mapsto T'(\mathbf{v})$.

- Since the choice of $T'(\mathbf{v})$ is unique by the injectivity of ϕ , T' is well-defined.
- Now we show T' is a linear transformation: Let $\mathbf{v}_1, \mathbf{v}_2 \in V, a, b \in K$. For all $\mathbf{w} \in V$, we have

$$\begin{aligned}
 \langle T'(a\mathbf{v}_1 + b\mathbf{v}_2), \mathbf{w} \rangle &= \langle a\mathbf{v}_1 + b\mathbf{v}_2, T(\mathbf{w}) \rangle \\
 &= \bar{a} \langle \mathbf{v}_1, T(\mathbf{w}) \rangle + \bar{b} \langle \mathbf{v}_2, T(\mathbf{w}) \rangle \\
 &= \bar{a} \langle T'(\mathbf{v}_1), \mathbf{w} \rangle + \bar{b} \langle T'(\mathbf{v}_2), \mathbf{w} \rangle \\
 &= \langle aT'(\mathbf{v}_1) + bT'(\mathbf{v}_2), \mathbf{w} \rangle
 \end{aligned}$$

Therefore,

$$\langle T'(a\mathbf{v}_1 + b\mathbf{v}_2) - [aT'(\mathbf{v}_1) + bT'(\mathbf{v}_2)], \mathbf{w} \rangle = 0, \forall \mathbf{w} \in V$$

By the non-degeneracy of inner product,

$$T'(a\mathbf{v}_1 + b\mathbf{v}_2) - [aT'(\mathbf{v}_1) + bT'(\mathbf{v}_2)] = \mathbf{0},$$

$$\text{i.e., } T'(a\mathbf{v}_1 + b\mathbf{v}_2) = aT'(\mathbf{v}_1) + bT'(\mathbf{v}_2)$$

■

■ **Example 10.2** Let $V = \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ as the usual inner product. Consider the matrix-multiplication mapping

$$T: V \rightarrow V$$

$$T(\mathbf{v}) = A\mathbf{v}$$

Then $\langle T'(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle$ implies

$$\begin{aligned}
 (T'(\mathbf{v}))^T \mathbf{w} &= \langle \mathbf{v}, A\mathbf{w} \rangle \\
 &= \mathbf{v}^T A\mathbf{w} \\
 &= (A^T \mathbf{v})^T \mathbf{w}
 \end{aligned}$$

Therefore, $T'(\mathbf{v}) = A^T \mathbf{v}$.

■

Proposition 10.11 Let $T : V \rightarrow V$ be a linear transformation, V a inner product space.

Suppose that $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is an orthonormal basis of V , then

$$(T')_{\mathcal{B}, \mathcal{B}} = \overline{((T)_{\mathcal{B}, \mathcal{B}})^T}$$

Proof. Suppose that $(T)_{\mathcal{B}, \mathcal{B}} = (a_{ij})$, where $T(\mathbf{e}_j) = \sum_{k=1}^n a_{kj} \mathbf{e}_k$, then

$$\begin{aligned} \langle \mathbf{e}_i, T(\mathbf{e}_j) \rangle &= \langle \mathbf{e}_i, \sum_{k=1}^n a_{kj} \mathbf{e}_k \rangle \\ &= \sum_{k=1}^n a_{kj} \langle \mathbf{e}_i, \mathbf{e}_k \rangle \\ &= a_{ij} \end{aligned}$$

Also, suppose $(T')_{\mathcal{B}, \mathcal{B}} = (b_{ij})$, we imply $T'(\mathbf{e}_j) = \sum_{k=1}^n b_{kj} \mathbf{e}_k$, which follows that

$$\langle \mathbf{e}_i, T'(\mathbf{e}_j) \rangle = b_{ij} \implies \overline{\langle T'(\mathbf{e}_j), \mathbf{e}_i \rangle} = b_{ij} \implies \overline{\langle \mathbf{e}_j, T(\mathbf{e}_i) \rangle} = b_{ij},$$

i.e., $\overline{a_{ji}} = b_{ij}$. ■

R Proposition (10.11) does not hold if \mathcal{B} is not an orthonormal basis.

Chapter 11

Week11

11.1. Monday for MAT3040

Reviewing. Adjoint Operator: $\langle T'(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle$.

11.1.1. Self-Adjoint Operator

Definition 11.1 [Self-Adjoint] Let V be an inner product space and $T : V \rightarrow V$ be a linear operator. Then T is **self-adjoint** if $T' = T$. ■

■ **Example 11.1** Let $V = \mathbb{C}^n$, and $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a orthonormal basis. Let $T : V \rightarrow V$ be given by

$$T(\mathbf{v}) = \mathbf{A}\mathbf{v}, \quad \text{where } \mathbf{A} \in M_{n \times n}(\mathbb{C}).$$

Or equivalently, there exists basis \mathcal{B} such that $(T)_{\mathcal{B}, \mathcal{B}} = \mathbf{A}$.

In such case, T is self-adjoint if and only if $(T')_{\mathcal{B}, \mathcal{B}} = (T)_{\mathcal{B}, \mathcal{B}}$, i.e., $\overline{(T)_{\mathcal{B}, \mathcal{B}}^T} = (T)_{\mathcal{B}, \mathcal{B}}$, i.e., $\mathbf{A}^H = \mathbf{A}$.

Therefore, $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$ is self-adjoint if and only if $\mathbf{A}^H = \mathbf{A}$.

Moreover, if \mathbb{C} is replaced by \mathbb{R} , then T is self-adjoint if and only if \mathbf{A} is symmetric. ■

R The notion of self-adjoint for linear operator is essentially the generalized notion of Hermitian for matrix that we have studied in MAT2040.

We also have some nice properties for self-adjoint, and the proof for which are essentially the same for the proof in the case of Hermitian matrices.

Proposition 11.1 If λ is an eigenvalue of a self-adjoint operator T , then $\lambda \in \mathbb{R}$.

Proof. Suppose there is an eigen-pair (λ, \mathbf{w}) for $\mathbf{w} \neq \mathbf{0}$, then

$$\begin{aligned}\lambda \langle \mathbf{w}, \mathbf{w} \rangle &= \langle \mathbf{w}, \lambda \mathbf{w} \rangle \\ &= \langle \mathbf{w}, T(\mathbf{w}) \rangle = \langle T'(\mathbf{w}), \mathbf{w} \rangle \\ &= \langle T(\mathbf{w}), \mathbf{w} \rangle = \langle \lambda \mathbf{w}, \mathbf{w} \rangle \\ &= \bar{\lambda} \langle \mathbf{w}, \mathbf{w} \rangle\end{aligned}$$

Since $\langle \mathbf{w}, \mathbf{w} \rangle \neq 0$ by non-degeneracy property, we have $\lambda = \bar{\lambda}$, i.e., $\lambda \in \mathbb{R}$. ■

Proposition 11.2 If $U \leq V$ is T -invariant over the self-adjoint operator T , then so is U^\perp .

Proof. It suffices to show $T(\mathbf{v}) \in U^\perp, \forall \mathbf{v} \in U^\perp$, i.e., for any $\mathbf{u} \in U$, check that

$$\langle \mathbf{u}, T(\mathbf{v}) \rangle = \langle T'(\mathbf{u}), \mathbf{v} \rangle = \langle T(\mathbf{u}), \mathbf{v} \rangle = 0,$$

where the last equality is because that $T(\mathbf{u}) \in U$ and $\mathbf{v} \in U^\perp$. Therefore, $T(\mathbf{v}) \in U^\perp$. ■

Theorem 11.1 If $T : V \rightarrow V$ is self-adjoint, and $\dim(V) < \infty$, then there exists an orthonormal basis of eigenvectors of T , i.e., an orthonormal basis of V such that any element from this basis is an eigenvector of T .

Proof. We use the induction on $\dim(V)$:

- The result is trivial for $\dim(V) = 1$.
- Suppose that this theorem holds for all vector spaces V with $\dim(V) \leq k$, then we want to show the theorem holds when $\dim(V) = k + 1$:

Suppose that $T : V \rightarrow V$ is self-adjoint with $\dim(V) = k + 1$, then consider

$$\chi_T(x) = x^{k+1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{K}, \text{ where } \mathbb{K} \text{ denotes } \mathbb{R} \text{ or } \mathbb{C}.$$

- If $\mathbb{K} = \mathbb{C}$, then $\mathcal{X}_T(x)$ can be decomposed as

$$\mathcal{X}_T(x) = (x - \lambda_1) \cdots (x - \lambda_{k+1})$$

In particular, we obtain the eigen-pair (λ_1, \mathbf{v})

- If $\mathbb{K} = \mathbb{R}$, i.e., we treat real number as scalars, then

$$\mathcal{X}_T(x) = (x - \lambda_1) \cdots (x - \lambda_{k+1}), \text{ where } \lambda_i \in \mathbb{C}.$$

By proposition (11.1), we imply all λ_i 's are in \mathbb{R} . Moreover, we also obtain the eigen-pair (λ_1, \mathbf{v})

Consider $U = \text{span}\{\mathbf{v}\}$, then

- U is T -invariant
- $V = U \oplus U^\perp$, since V is finite dimensional
- U^\perp is T -invariant.

Consider $T|_{U^\perp}$, which is a self-adjoint operator on U^\perp , with $\dim(U^\perp) = k$.

By induction, there exists an orthonormal basis $\{\mathbf{e}_2, \dots, \mathbf{e}_{k+1}\}$ of eigenvectors of $T|_{U^\perp}$.

Consider the basis $\mathcal{B} = \{\mathbf{v}' = \mathbf{v}/\|\mathbf{v}\|, \mathbf{e}_2, \dots, \mathbf{e}_{k+1}\}$. As a result,

1. \mathcal{B} forms a basis of V
2. All $\mathbf{v}', \mathbf{e}_i$ are of norm 1 eigenvectors of T .
3. \mathcal{B} is an orthonormal set, e.g., $\langle \mathbf{v}', \mathbf{e}_i \rangle = 0$, where $\mathbf{v}' \in U$ and $\mathbf{e}_i \in U^\perp$.

Therefore, \mathcal{B} is a basis of orthonormal eigenvectors of V .

■

Corollary 11.1 If $\dim(V) < \infty$, and $T : V \rightarrow V$ is self-adjoint, then there exists orthonormal basis \mathcal{B} such that

$$(T)_{\mathcal{B}, \mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_n)$$

In particular, for all real symmetric matrix $\mathbf{A} \in \mathbb{S}^n$, there exists orthogonal matrix P ($P^T P = \mathbf{I}_n$) such that

$$P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$$

Proof. 1. By applying theorem (11.1), there exists orthonormal basis of V , say $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$. Directly writing the basis representation gives

$$(T)_{\mathcal{B}, \mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_n).$$

2. For the second part, consider $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$. Since $\mathbf{A}^T = \mathbf{A}$, we imply T is self-adjoint. There exists orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that

$$(T)_{\mathcal{B}, \mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_n).$$

In particular, if $\mathcal{A} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, then $(T)_{\mathcal{A}, \mathcal{A}} = \mathbf{A}$. We construct $P := C_{\mathcal{A}, \mathcal{B}}$, which is the change of basis matrix from \mathcal{B} to \mathcal{A} , then

$$P = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$$

and

$$P^{-1}(T)_{\mathcal{A}, \mathcal{A}}P = (T)_{\mathcal{B}, \mathcal{B}}$$

Or equivalently, $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n)$, with

$$P^T P = \begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix} = \mathbf{I}$$

■

11.1.2. Orthonormal/Unitary Operators

Definition 11.2 A linear operator $T : V \rightarrow V$ over \mathbb{K} with $\langle T(\mathbf{w}), T(\mathbf{v}) \rangle = \langle \mathbf{w}, \mathbf{v} \rangle, \forall \mathbf{v}, \mathbf{w} \in V$, is called

1. **Orthogonal** if $\mathbb{K} = \mathbb{R}$
2. **Unitary** if $\mathbb{K} = \mathbb{C}$

Proposition 11.3 T is orthogonal / unitary if and only if $T' \circ T = I$

Proof. The reverse direction is by directly checking that

$$\langle T(\mathbf{w}), T(\mathbf{v}) \rangle = \langle T' \circ T(\mathbf{w}), \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$$

The forward direction is by checking $T' \circ T(\mathbf{w}) = \mathbf{w}, \forall \mathbf{w} \in V$:

$$\langle T' \circ T(\mathbf{w}), \mathbf{v} \rangle = \langle T(\mathbf{w}), T(\mathbf{v}) \rangle = \langle \mathbf{w}, \mathbf{v} \rangle \implies \langle T' \circ T(\mathbf{w}) - \mathbf{w}, \mathbf{v} \rangle = 0, \forall \mathbf{v} \in V$$

By non-degeneracy, $T' \circ T(\mathbf{w}) - \mathbf{w} = 0$, i.e., $T' \circ T(\mathbf{w}) = \mathbf{w}, \forall \mathbf{w} \in V$. ■

■ **Example 11.2** Let $T : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be given by $T(\mathbf{v}) = A\mathbf{v}$. Then T is orthogonal implies $(T')_{\mathcal{B}, \mathcal{B}}(T)_{\mathcal{B}, \mathcal{B}} = I$.

(Orthogonal) When $\mathbb{K} = \mathbb{R}$, then $A^T A = I$

(Unitary) When $\mathbb{K} = \mathbb{C}$, then $A^H A = I$. ■

Definition 11.3 [Orthogonal/Unitary Group]

Orthogonal Group : $O(n, \mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) \mid A^T A = I\}$

Unitary Group : $U(n, \mathbb{C}) = \{A \in M_{n \times n}(\mathbb{C}) \mid A^H A = I\}$ ■

11.4. Wednesday for MAT3040

Reviewing. Unitary Operators

$$\langle T\mathbf{v}, T\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle, \quad \forall \mathbf{v}, \mathbf{w} \in V.$$

11.4.1. Unitary Operator

■ **Example 11.8** Let $V = \mathbb{R}^n$ with usual inner product. For the linear operator $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$, T is orthogonal if and only if $\mathbf{A}^T \mathbf{A} = \mathbf{I}$.

Let $V = \mathbb{C}^n$ with usual inner product. For the linear operator $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$, T is unitary if and only if $\mathbf{A}^H \mathbf{A} = \mathbf{I}$. ■

Proposition 11.8 Let $T : V \rightarrow V$ be a linear operator on a vector space over \mathbb{K} satisfying $T'T = I$. Then for all eigenvalues λ of T , we have $|\lambda| = 1$.

Proof. Suppose we have the eigen-pair (λ, \mathbf{v}) , then

$$\begin{aligned} \langle T\mathbf{v}, T\mathbf{v} \rangle &= \langle \mathbf{v}, \mathbf{v} \rangle \\ \iff \langle \lambda\mathbf{v}, \lambda\mathbf{v} \rangle &= \langle \mathbf{v}, \mathbf{v} \rangle \\ \iff \bar{\lambda}\lambda \langle \mathbf{v}, \mathbf{v} \rangle &= \langle \mathbf{v}, \mathbf{v} \rangle \end{aligned}$$

Since $\langle \mathbf{v}, \mathbf{v} \rangle \neq 0$ ($\mathbf{v} \neq \mathbf{0}$), we imply $|\lambda|^2 = 1$, i.e., $|\lambda| = 1$. ■

Proposition 11.9 Let $T : V \rightarrow V$ be an operator on a finite dimension V over \mathbb{K} satisfying $T'T = I$. If $U \leq V$ is T -invariant, then U is also T^{-1} -invariant.

Proof. Since $T'T = I$, i.e., T is invertible, we imply 0 is not a root of $\chi_T(x)$, i.e., 0 is not a root of $m_T(x)$. Since $m_T(0) \neq 0$, $m_T(x)$ has the form

$$m_T(x) = x^m + \cdots + a_1x + a_0, \quad a_0 \neq 0,$$

which follows that

$$m_T(T) = T^m + \cdots + a_0 I = 0 \implies T(T^{m-1} + \cdots + a_1 I) = -a_0 I$$

Or equivalently,

$$T \left(-\frac{1}{a_0} (T^{m-1} + \cdots + a_1 I) \right) = I$$

Therefore,

$$T^{-1} = -\frac{1}{a_0} T^{m-1} - \cdots - \frac{a_2}{a_0} T - \frac{a_1}{a_0} I,$$

i.e., the inverse T^{-1} can be expressed as a polynomial involving T only.

Since U is T -invariant, we imply U is T^m -invariant for $m \in \mathbb{N}$, and therefore U is T^{-1} -invariant since T^{-1} is a polynomial of T . ■

Proposition 11.10 Let $T : V \rightarrow V$ satisfies $T'T = I$ ($\dim(V) < \infty$), then $U \leq V$ is T -invariant implies U^\perp is T -invariant.

Proof. Let $v \in U^\perp$, it suffices to show $T(v) \in U^\perp$.

For all $u \in U$, we have

$$\langle u, T(v) \rangle = \langle T'(u), v \rangle = \langle T^{-1}(u), v \rangle$$

Since U is T^{-1} -invariant, we imply $T^{-1}(u) \in U$, and therefore

$$\langle u, T(v) \rangle = \langle T^{-1}(u), v \rangle = 0 \implies T(v) \in U^\perp.$$

■

Theorem 11.2 Let $T : V \rightarrow V$ be a unitary operator on finite dimension V (over \mathbb{C}), then there exists an orthonormal basis \mathcal{A} such that

$$(T)_{\mathcal{A}, \mathcal{A}} = \text{diag}(\lambda_1, \dots, \lambda_n), \quad |\lambda_i| = 1, \quad \forall i.$$

Proof Outline. Note that $\chi_T(x)$ always admits a root in \mathbb{C} , so we can always find an

eigenvector $\mathbf{v} \in V$ of T .

Then the theorem follows by the same argument before on self-adjoint operators.

- Consider $U = \text{span}\{\mathbf{v}\}$
- $V = U \oplus U^\perp$ and U^\perp is T -invariant
- Use induction on the unitary operator $T|_{U^\perp}: U^\perp \rightarrow U^\perp$

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- The argument fails for orthogonal operators

$$\begin{aligned} T &: \mathbb{R} \rightarrow \mathbb{R}^2, \\ \text{with } T(\mathbf{v}) &= \mathbf{A}\mathbf{v} \\ \text{where } \mathbf{A} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \end{aligned}$$

The matrix \mathbf{A} is not diagonalizable over \mathbb{R} . It has no real eigenvalues.

However, if we treat \mathbf{A} as $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ with $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$, then $\mathbf{A}^H \mathbf{A} = \mathbf{I}$, and therefore T is unitary. Then \mathbf{A} is diagonalizable over \mathbb{C} with eigenvalues $e^{i\theta}, e^{-i\theta}$

- As a corollary of the theorem, for all $\mathbf{A} \in M_{n \times n}(\mathbb{C})$ satisfying $\mathbf{A}^H \mathbf{A} = \mathbf{I}$, there exists $P \in M_{n \times n}(\mathbb{C})$ such that

$$P^{-1} \mathbf{A} P = \text{diag}(\lambda_1, \dots, \lambda_n), \quad |\lambda_i| = 1,$$

where $P = (\mathbf{u}_1, \dots, \mathbf{u}_n)$, with $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ forming orthonormal basis of \mathbb{C}^n .

In fact,

$$P^H P = \begin{pmatrix} \mathbf{u}_1^H \\ \vdots \\ \mathbf{u}_n^H \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle & \cdots & \langle \mathbf{u}_1, \mathbf{u}_n \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_n, \mathbf{u}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{u}_n \rangle \end{pmatrix}$$

Conclusion: all matrices $\mathbf{A} \in M_{n \times n}(\mathbb{C})$ with $\mathbf{A}^H \mathbf{A} = \mathbf{I}$ can be written as

$$\mathbf{A} = \mathbf{P}^{-1} \text{diag}(\lambda_1, \dots, \lambda_n) \mathbf{P},$$

with some \mathbf{P} satisfying $\mathbf{P}^H \mathbf{P} = \mathbf{I}$.

Notation. Let $U(n) = \{\mathbf{A} \in M_{n \times n}(\mathbb{C}) \mid \mathbf{A}^H \mathbf{A} = \mathbf{I}\}$ be the unitary group, then all $\mathbf{A} \in U(n)$ can be diagonalized by

$$\mathbf{A} = \mathbf{P}^{-1} \text{diag}(\lambda_1, \dots, \lambda_n) \mathbf{P}, \quad \mathbf{P} \in U(n).$$

11.4.2. Normal Operators

Definition 11.10 [Normal] Let $T : V \rightarrow V$ be a linear operator over a \mathbb{C} inner product vector space V . We say T is **normal**, if

$$T'T = TT'$$

■ **Example 11.9** • All self-adjoint operators are normal:

$$T = T' \implies TT' = T'T = T^2$$

• All (finite-dimensional) unitary operators are normal:

$$T'T = TT' = I$$

Proposition 11.11 Let T be a normal operator on V . Then

1. $\|T(\mathbf{v})\| = \|T'(\mathbf{v})\|, \forall \mathbf{v} \in V$.

In particular, $T(\mathbf{v}) = 0$ if and only if $T'(\mathbf{v}) = 0$

2. $(T - \lambda I)$ is also a normal operator, for any $\lambda \in \mathbb{C}$
3. $T(\mathbf{v}) = \lambda \mathbf{v}$ if and only if $T'(\mathbf{v}) = \bar{\lambda} \mathbf{v}$.

Proof. 1.

$$\begin{aligned}
 \langle T\mathbf{v}, T\mathbf{v} \rangle &= \langle T'T\mathbf{v}, \mathbf{v} \rangle \\
 &= \langle TT'\mathbf{v}, \mathbf{v} \rangle \\
 &= \overline{\langle \mathbf{v}, TT'\mathbf{v} \rangle} \\
 &= \overline{\langle T'\mathbf{v}, T'\mathbf{v} \rangle} \\
 &= \langle T'\mathbf{v}, T'\mathbf{v} \rangle
 \end{aligned}$$

Therefore, $\|T(\mathbf{v})\|^2 = \|T'(\mathbf{v})\|^2$, i.e., $\|T(\mathbf{v})\| = \|T'(\mathbf{v})\|$.

2. By hw4, $(T - \lambda I)' = T' - \bar{\lambda}I$. It suffices to check

$$(T - \lambda I)'(T - \lambda I) = (T - \lambda I)(T - \lambda I)',$$

Expanding both sides out gives the desired result, i.e.,

$$(T - \lambda I)'(T - \lambda I) = (T' - \bar{\lambda}I)(T - \lambda I) = T'T - \bar{\lambda}T - \lambda T' + \lambda \bar{\lambda}I$$

and

$$(T - \lambda I)(T - \lambda I)' = (T - \lambda I)(T' - \bar{\lambda}I) = TT' - \bar{\lambda}T - \lambda T' + \lambda \bar{\lambda}I$$

3. The proof for (3) will be discussed in the next lecture.

■

Chapter 12

Week12

12.1. Monday for MAT3040

12.1.1. Remarks on Normal Operator

Proposition 12.1 If T is normal, then

1. $\|T(\mathbf{v})\| = \|T'(\mathbf{v})\|$ for any $\mathbf{v} \in V$
2. $(T - \lambda I)$ is normal for any $\lambda \in \mathbb{C}$
3. $T(\mathbf{v}) = \lambda \mathbf{v}$ if and only if $T'(\mathbf{v}) = \bar{\lambda} \mathbf{v}$
4. If $T(\mathbf{v}) = \lambda \mathbf{v}$ and $T(\mathbf{w}) = \mu \mathbf{w}$ with $\lambda \neq \mu$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

Proof. (3) • For the forward direction, if $(T - \lambda I)\mathbf{v} = 0$, then by part (2), $(T - \lambda I)$ is normal, which follows that

$$\|(T - \lambda I)'(\mathbf{v})\| = 0 \implies (T - \lambda I)'(\mathbf{v}) = 0 \implies T'\mathbf{v} = \bar{\lambda} \mathbf{v}.$$

- For the reverse direction, suppose that $(T' - \bar{\lambda} I)\mathbf{v} = 0$. Since T is normal, we imply T' is normal. Then by part (2), $(T' - \bar{\lambda} I)$ is normal. By applying the same trick,

$$(T' - \bar{\lambda} I)' \mathbf{v} = 0 \implies ((T')' - \bar{\bar{\lambda}} I)\mathbf{v} = 0.$$

By hw4, $(T')' = T$. Therefore, $(T - \lambda I)\mathbf{v} = 0$.

(4) Observe that

$$\lambda \langle \mathbf{v}, \mathbf{w} \rangle = \langle \bar{\lambda} \mathbf{v}, \mathbf{w} \rangle \xrightarrow{\text{by (3)}} \lambda \langle \mathbf{v}, \mathbf{w} \rangle = \langle T'(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle = \langle \mathbf{v}, \mu \mathbf{w} \rangle = \mu \langle \mathbf{v}, \mathbf{w} \rangle$$

Since $\lambda \neq \mu$, we imply $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. The proof is complete. ■

Theorem 12.1 Let T be an operator on a finite dimensional ($\dim(V) = n$) \mathbb{C} -inner product vector space V satisfying $T'T = TT'$. Then there is an orthonormal basis of eigenvectors of V , i.e., an orthonormal basis of V such that any element from this basis is an eigenvector of T .

Proof. Since $\chi_T(x)$ must have a root in \mathbb{C} , there must exist an eigen-pair (\mathbf{v}, λ) of T .

- Construct $U = \text{span}\{\mathbf{v}\}$, and it follows that

$$T\mathbf{v} = \lambda\mathbf{v} \implies U \text{ is } T\text{-invariant.}$$

$$T'\mathbf{v} = \bar{\lambda}\mathbf{v} \implies U \text{ is } T'\text{-invariant.}$$

- Moreover, we claim that U^\perp is T and T' invariant: let $\mathbf{w} \in U^\perp$, and for all $\mathbf{u} \in U$, we have

$$\langle \mathbf{u}, T(\mathbf{w}) \rangle = \langle T'(\mathbf{u}), \mathbf{w} \rangle = \langle \bar{\lambda}\mathbf{u}, \mathbf{w} \rangle = \lambda \langle \mathbf{u}, \mathbf{w} \rangle = 0,$$

i.e., U^\perp is T invariant.

$$\langle \mathbf{u}, T'(\mathbf{w}) \rangle = \langle T(\mathbf{u}), \mathbf{w} \rangle = \langle \lambda\mathbf{u}, \mathbf{w} \rangle = \bar{\lambda} \langle \mathbf{u}, \mathbf{w} \rangle = 0,$$

which implies U^\perp is T' invariant.

- Therefore, we construct the operator $T|_{U^\perp}: U^\perp \rightarrow U^\perp$, and

$$TT' = T'T \implies (T|_{U^\perp})(T'|_{U^\perp}) = (T'|_{U^\perp})(T|_{U^\perp}),$$

i.e., $(T|_{U^\perp})$ is normal on U^\perp . Moreover, $\dim(U^\perp) = n - 1$.

- Applying the same trick as in Theorem (11.1), we imply there exists an orthonor-

mal basis $\{\mathbf{e}_2, \dots, \mathbf{e}_n\}$ of eigenvectors of $(T|_{U^\perp})$. Then we can argue that

$$\mathcal{B} = \{\mathbf{v}' = \mathbf{v}/\|\mathbf{v}\|, \mathbf{e}_2, \dots, \mathbf{e}_{k+1}\}$$

is a basis of orthonormal eigenvectors of V . ■

Corollary 12.1 [Spectral Theorem for Normal Operator] Let $T : V \rightarrow V$ be a normal operator on a \mathbb{C} -inner product space with $\dim(V) < \infty$. Then there exists self-adjoint operators P_1, \dots, P_k such that

$$P_i^2 = P_i, \quad P_i P_j = 0, i \neq j, \quad \sum_{i=1}^k P_i = I,$$

and $T = \sum_{i=1}^k \lambda_i P_i$, where λ_i 's are the eigenvalues of T .

R These P_i 's are the **orthogonal projections** from V to the λ_i -eigenspace $\ker(T - \lambda_i I)$ of T , i.e., we have

$$v = P_i(v) + (v - P_i(v)),$$

where $P_i(v) \in \ker(T - \lambda_i I)$, and $v - P_i(v) \in (\ker(T - \lambda_i I))^\perp$.

You should know how to compute P_i 's when $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$ in the course MAT2040.

Proof. Since T has a basis of eigenvectors, by definition, T is diagonalizable. By proposition (8.2),

$$m_T(x) = (x - \lambda_1) \cdots (x - \lambda_k),$$

where λ_i 's are distinct. By spectral decomposition corollary (9.2), it suffices to show P_i 's are self-disjoint.

- Recall that $P_i = a_i(T)q_i(T) := b_m T^m + \cdots + b_1 T + b_0 I$, i.e., a polynomial of T , and therefore

$$P_i' = \bar{b}_m (T')^m + \cdots + \bar{b}_1 (T') + \bar{b}_0 I.$$

We claim that P_i is normal: Since $T'T = TT'$, we imply

$$(T')^p T^q = T^q (T')^p, \forall p, q \in \mathbb{N}$$

which follows that

$$\begin{aligned} P_i P_i' &= (b_m T^m + \cdots + b_0 I)(\bar{b}_m (T')^m + \cdots + \bar{b}_1 (T') + \bar{b}_0 I) \\ &= \sum_{1 \leq x, y \leq m} b_x \bar{b}_y (T)^x (T')^y \\ &= \sum_{1 \leq x, y \leq m} \bar{b}_y b_x (T')^y (T)^x \\ &= (\bar{b}_m (T')^m + \cdots + \bar{b}_1 (T') + \bar{b}_0 I)(b_m T^m + \cdots + b_0 I) \\ &= P_i' P_i \end{aligned}$$

- In general, S is self-adjoint, which implies S is normal, but not vice versa. However, the converse holds if further all eigenvalues of S are real numbers:

By Theorem (12.1), we imply S is orthonormally diagonalizable, and its diagonal representation is of the form

$$(S)_{\mathcal{B}, \mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_k).$$

Note that \mathcal{B} is also a basis for S' and elements of \mathcal{B} are eigenvalues of S' , by part (3) in proposition (12.1). Therefore,

$$(S')_{\mathcal{B}, \mathcal{B}} = \text{diag}(\lambda_1, \dots, \lambda_k).$$

Therefore, $S = S'$.

In particular, for $S = P_i$, we can easily show all eigenvalues of P_i are 0 or 1, which are real. Therefore, P_i 's are self-adjoint. ■

Corollary 12.2 Let $T : V \rightarrow V$ be a linear operator on \mathbb{C} -inner product space with $\dim(V) < \infty$. Then T is normal if and only if $T' = f(T)$ for some polynomial $f(x) \in \mathbb{C}[x]$.

Proof. • For the reverse direction, if $T' = f(T)$, then $T'T = f(T)T = Tf(T) = TT'$.
 • For the forward direction, suppose that T is normal, then by corollary (12.1),

$$T = \sum_{i=1}^k \lambda_i P_i, \quad P_i = f_i(T), \quad \text{where } P_i \text{'s are self-adjoint,}$$

which follows that

$$T' = \left(\sum_{i=1}^k \lambda_i P_i \right)' = \sum_{i=1}^k \bar{\lambda}_i P_i' = \sum_{i=1}^k \bar{\lambda}_i P_i = \sum_{i=1}^k \bar{\lambda}_i f_i(T)$$

■

R The normal operator is a generalization of Hermitian matrices, and it inherits many nice properties of Hermitian.

12.1.2. Tensor Product

Motivation. Let U, V, W be vector spaces. We want to study bilinear maps $f : U \times W \rightarrow U$, i.e.,

$$f(av_1 + bv_2, w) = af(v_1, w) + bf(v_2, w)$$

$$f(v, cw_1 + dw_2) = cf(v, w_1) + df(v, w_2)$$

Unfortunately, bilinear form usually is not a linear transformation!

- **Example 12.1** • Let $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be with $(u, v) \mapsto \langle u, v \rangle$.
- Let $f : M_{n \times n}(\mathbb{F}) \times M_{n \times n}(\mathbb{F}) \rightarrow M_{n \times n}(\mathbb{F})$ be with $f(A, B) = AB$.
 - Let $f : \mathbb{F}[x] \times \mathbb{F}[x] \rightarrow \mathbb{F}$ be with $f(p(x), q(x)) = p(1)q(2)$

- Let $f : \mathbb{F}[x] \times \mathbb{F}[x] \rightarrow \mathbb{F}[x]$ be with $f(p(x), q(x)) = p(x)q(x)$.

■

12.4. Wednesday for MAT3040

12.4.1. Introduction to Tensor Product

Reviewing. Bilinear map: $f : V \times W \rightarrow U$, e.g.,

$$f : \mathbb{R}^3 \times \mathbb{R}^3$$

with $f(u, v) = u \times v$

Note that f is usually not a linear transformation, e.g.,

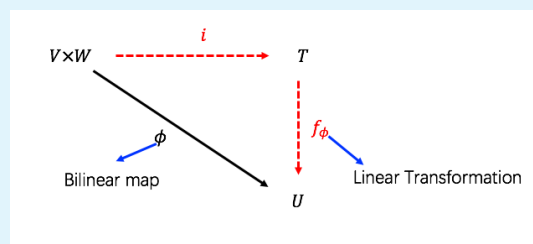
$$f(3\mathbf{v}, 3\mathbf{w}) = f(3\mathbf{v}, 3\mathbf{w}) = (3\mathbf{v}) \times (3\mathbf{w}) = 9\mathbf{v} \times \mathbf{w} \neq 3f(\mathbf{v}, \mathbf{w}).$$

The vector space structure of $V \times W$ is not suited to study bilinear map, and the proper way is to study its induced linear transformation.

Definition 12.4 [Universal Property of Tensor Product] Let V, W be vector spaces. Consider the set

$$\text{Obj} := \{\phi : V \times W \rightarrow U \mid \phi \text{ is a bilinear map}\}$$

We say T , or $(i : V \times W \rightarrow T) \in \text{Obj}$ satisfies the **universal property** if for any $(\phi : V \times W \rightarrow U) \in \text{Obj}$, there exists a unique linear transformation $f_\phi : T \rightarrow U$ such that the diagram below commutes:



$$\text{i.e., } \phi = f_\phi \circ i.$$

Therefore, rather than studying bilinear map ϕ , it is better to study the linear transformation f_ϕ instead.

Question: does T exist?

Definition 12.5 [Spanning Set] Let V, W be vector spaces. Let $S = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in V, \mathbf{w} \in W\}$, then we define

$$\mathfrak{X} = \text{span}(S).$$

(R)

1. The spanning set \mathfrak{X} is not additive, e.g., $\mathfrak{x}_1 = 3(0, \mathbf{w}) \in \mathfrak{X}$ and $\mathfrak{x}_2 = 1(0, \mathbf{w}) + 1(0, 2\mathbf{w}) \in \mathfrak{X}$, but $\mathfrak{x}_1 \neq \mathfrak{x}_2$.
2. Note that we assume no relations on the elements $(\mathbf{v}, \mathbf{w}) \in S$. More precisely, the set $S = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in V, \mathbf{w} \in W\}$ is linearly independent in \mathfrak{X} . For example, $(0, \mathbf{w}) \perp (0, 2\mathbf{w})$.
3. The only legitimate relationship is

$$2(\mathbf{v}_1, \mathbf{w}_1) + 3(\mathbf{v}_1, \mathbf{w}_1) = 5(\mathbf{v}, \mathbf{w}),$$

which is not equal to $(5\mathbf{v}, 5\mathbf{w})$

4. S is a basis of \mathfrak{X} , and therefore \mathfrak{X} is of uncountable dimension.

Definition 12.6 [Special subspace of \mathfrak{X}] Let $\mathfrak{y} \leq \mathfrak{X}$ be a vector subspace spanned by vectors of the form

$$\{1(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}) - 1(\mathbf{v}_1, \mathbf{w}) - 1(\mathbf{v}_2, \mathbf{w})\}, \quad \text{and} \quad \{1(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) - 1(\mathbf{v}, \mathbf{w}_1) - 1(\mathbf{v}, \mathbf{w}_2)\}$$

and

$$\{1(k\mathbf{v}, \mathbf{w}) - k(\mathbf{v}, \mathbf{w}) \mid k \in \mathbb{F}\}$$

and

$$\{1(\mathbf{v}, k\mathbf{w}) - k(\mathbf{v}, \mathbf{w}) \mid k \in \mathbb{F}\}$$

Definition 12.7 [Tensor Product] We define the **tensor product** $V \otimes W$ by

$$V \otimes W = \mathcal{X}/y.$$

Therefore, $\mathbf{v} \otimes \mathbf{w} = (\mathbf{v}, \mathbf{w}) + y \in \mathcal{X}/y$ ■

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1. As a result, the tensor product is finitely additive:

$$\begin{aligned} (\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} &= (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) + y \\ &= (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) - [(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) - (\mathbf{v}_1, \mathbf{w}) - (\mathbf{v}_2, \mathbf{w})] + y \\ &= 0(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) + (\mathbf{v}_1, \mathbf{w}) + (\mathbf{v}_2, \mathbf{w}) + y \\ &= [(\mathbf{v}_1, \mathbf{w}) + y] + [(\mathbf{v}_2, \mathbf{w}) + y] \\ &= \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w} \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) &= (\mathbf{v} \otimes \mathbf{w}_1) + (\mathbf{v} \otimes \mathbf{w}_2) \\ (k\mathbf{v}) \otimes \mathbf{w} &= k(\mathbf{v} \otimes \mathbf{w}) \\ \mathbf{v} \otimes (k\mathbf{w}) &= k(\mathbf{v} \otimes \mathbf{w}) \end{aligned}$$

2. The product space $V \times W$ is different from the tensor product space $V \otimes W$:

(a) $(\mathbf{v}, \mathbf{0}) \neq \mathbf{0}_{V \times W}$ in $V \times W$; but $\mathbf{v} \otimes \mathbf{0} \in 0_{V \otimes W}$:

$$\begin{aligned} V \otimes \mathbf{0} &= V \otimes (\mathbf{0}\mathbf{w}) \\ &= 0(V \otimes \mathbf{w}) \\ &= 0_{V \otimes W} \end{aligned}$$

Moreover, f is bilinear implies $f(\mathbf{v}, \mathbf{0}) = \mathbf{0}$.

(b) $(\mathbf{v}_1, \mathbf{w}_1) + (\mathbf{v}_2, \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 + \mathbf{w}_2)$; but $\mathbf{v}_1 \otimes \mathbf{w}_1 + \mathbf{v}_2 \otimes \mathbf{w}_2$ cannot be simplified further, unless $\mathbf{v}_1 = \mathbf{v}_2$:

$$\mathbf{v} \otimes \mathbf{w}_1 + \mathbf{v} \otimes \mathbf{w}_2 = \mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2)$$

Theorem 12.3 The bilinear map

$$i: V \times W \rightarrow V \otimes W \quad (i \in \text{Obj})$$

$$\text{with } (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \otimes \mathbf{w}$$

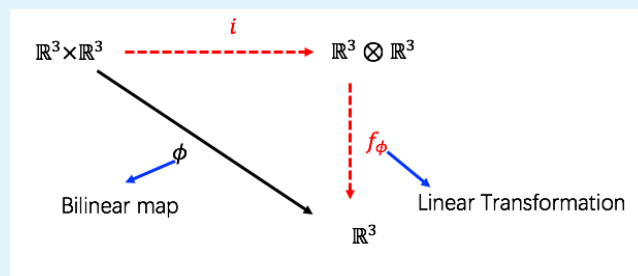
satisfies the universal property of tensor products.

■ **Example 12.5** Consider a common bilinear map

$$\phi: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\text{with } (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \times \mathbf{w}$$

By the universal property, there exists the linear transformation $f_\phi: \mathbb{R}^3 \otimes \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that the diagram below commutes:



Chapter 13

Week13

13.1. Monday for MAT3040

Reviewing.

1. Define $S = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in V, \mathbf{w} \in W\}$ and $\mathfrak{X} = \text{span}(S)$. In \mathfrak{X} , there are no relations between distinct elements of S , e.g.,

$$2(\mathbf{v}, 0) + 3(0, \mathbf{w}) \neq 1(2\mathbf{v}, 3\mathbf{w})$$

General element in \mathfrak{X} :

$$a_1(\mathbf{v}_1, \mathbf{w}_1) + \cdots + a_n(\mathbf{v}_n, \mathbf{w}_n),$$

where $(\mathbf{v}_i, \mathbf{w}_i)$ are distinct.

2. Define the space $V \otimes W = \mathfrak{X}/y$, with

$$\mathbf{v} \otimes \mathbf{w} = 1(\mathbf{v}, \mathbf{w}) + y \in V \otimes W.$$

General element in $\mathfrak{X}/y := V \otimes W$:

$$\begin{aligned} a_1(\mathbf{v}_1, \mathbf{w}_1) + \cdots + a_n(\mathbf{v}_n, \mathbf{w}_n) + y &= a_1((\mathbf{v}_1, \mathbf{w}_1) + y) + \cdots + a_n((\mathbf{v}_n, \mathbf{w}_n) + y) \\ &= a_1(\mathbf{v}_1 \otimes \mathbf{w}_1) + \cdots + a_n(\mathbf{v}_n \otimes \mathbf{w}_n) \\ &= (a_1 \mathbf{v}_1) \otimes \mathbf{w}_1 + \cdots + (a_n \mathbf{v}_n) \otimes \mathbf{w}_n \end{aligned}$$

Therefore, a general element in $V \otimes W$ is of the form

$$\mathbf{v}'_1 \otimes \mathbf{w}_1 + \cdots + \mathbf{v}'_n \otimes \mathbf{w}_n, \quad \mathbf{v}'_i \in V, \mathbf{w}_i \in W. \quad (13.1)$$

Note that $V \otimes W$ is different from $V \times W$, where all elements in $V \times W$ can be expressed as (\mathbf{v}, \mathbf{w}) .

3. The tensor product mapping

$$\begin{aligned} i: \quad V \times W &\rightarrow V \otimes W \\ \text{with } (\mathbf{v}, \mathbf{w}) &\mapsto \mathbf{v} \otimes \mathbf{w} \end{aligned}$$

satisfies the universal property.

Here we present an example for computing tensor product by making use of the rules below:

$$(\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} = \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w}$$

$$\mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) = (\mathbf{v} \otimes \mathbf{w}_1) + (\mathbf{v} \otimes \mathbf{w}_2)$$

$$(k\mathbf{v}) \otimes \mathbf{w} = k(\mathbf{v} \otimes \mathbf{w})$$

$$\mathbf{v} \otimes (k\mathbf{w}) = k(\mathbf{v} \otimes \mathbf{w})$$

■ **Example 13.1** Let $V = W = \mathbb{R}^2$, with

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here we have

$$\begin{aligned}
 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -4 \\ 2 \end{pmatrix} &= (3\mathbf{e}_1 + 2\mathbf{e}_2) \otimes (-4\mathbf{e}_1 + 2\mathbf{e}_2) \\
 &= (3\mathbf{e}_1) \otimes (-4\mathbf{e}_1 + 2\mathbf{e}_2) + (\mathbf{e}_2) \otimes (-4\mathbf{e}_1 + 2\mathbf{e}_2) \\
 &= (3\mathbf{e}_1) \otimes (-4\mathbf{e}_1) + (3\mathbf{e}_1) \otimes (2\mathbf{e}_2) + (\mathbf{e}_2) \otimes (-4\mathbf{e}_1) + \mathbf{e}_2 \otimes (2\mathbf{e}_2) \\
 &= -12(\mathbf{e}_1 \otimes \mathbf{e}_1) + 6(\mathbf{e}_1 \otimes \mathbf{e}_2) - 4(\mathbf{e}_2 \otimes \mathbf{e}_1) + 2(\mathbf{e}_2 \otimes \mathbf{e}_2)
 \end{aligned}$$

Exercise: Check that $\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1$ cannot be re-written as

$$(a\mathbf{e}_1 + b\mathbf{e}_2) \otimes (c\mathbf{e}_1 + d\mathbf{e}_2), \quad a, b, c, d \in \mathbb{R}.$$

13.1.1. Basis of $V \otimes W$

Motivation. Given that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V , and $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis of W , we aim to find a basis of $V \otimes W$ using \mathbf{v}_i 's and \mathbf{w}_j 's.

Proposition 13.1 The set $\{\mathbf{v}_i \otimes \mathbf{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ spans the tensor product space $V \otimes W$.

Proof. Consider any $\mathbf{v} \in V$ and $\mathbf{w} \in W$, and we want to express $\mathbf{v} \otimes \mathbf{w}$ in terms of $\mathbf{v}_i, \mathbf{w}_j$. Suppose that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$ and $\mathbf{w} = \beta_1 \mathbf{w}_1 + \dots + \beta_m \mathbf{w}_m$.

Substituting $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$ into the expression $\mathbf{v} \otimes \mathbf{w}$, we imply

$$\begin{aligned}
 \mathbf{v} \otimes \mathbf{w} &= (\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n) \otimes \mathbf{w} \\
 &= (\alpha_1 \mathbf{v}_1) \otimes \mathbf{w} + \dots + (\alpha_n \mathbf{v}_n) \otimes \mathbf{w} \\
 &= \alpha_1(\mathbf{v}_1 \otimes \mathbf{w}) + \dots + \alpha_n(\mathbf{v}_n \otimes \mathbf{w})
 \end{aligned}$$

For each $\mathbf{v}_i \otimes \mathbf{w}$, $i = 1, \dots, n$, similarly,

$$\mathbf{v}_i \otimes \mathbf{w} = \beta_1(\mathbf{v}_i \otimes \mathbf{w}_1) + \dots + \beta_m(\mathbf{v}_i \otimes \mathbf{w}_m).$$

Therefore,

$$\mathbf{v} \otimes \mathbf{w} = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j (\mathbf{v}_i \otimes \mathbf{w}_j) \quad (13.2)$$

By (13.4), any vector in $V \otimes W$ is of the form

$$\mathbf{v}^{(1)} \otimes \mathbf{w}^{(1)} + \cdots + \mathbf{v}^{(\ell)} \otimes \mathbf{w}^{(\ell)}$$

By (13.5), each $\mathbf{v}^{(k)} \otimes \mathbf{w}^{(k)}, k = 1, \dots, \ell$, can be expressed as

$$\mathbf{v}^{(k)} \otimes \mathbf{w}^{(k)} = \sum_{i=1}^n \sum_{j=1}^m \alpha_i^{(k)} \beta_j^{(k)} (\mathbf{v}_i \otimes \mathbf{w}_j)$$

Therefore,

$$\mathbf{v}^{(1)} \otimes \mathbf{w}^{(1)} + \cdots + \mathbf{v}^{(\ell)} \otimes \mathbf{w}^{(\ell)} = \sum_{k=1}^{\ell} \sum_{i=1}^n \sum_{j=1}^m \alpha_i^{(k)} \beta_j^{(k)} (\mathbf{v}_i \otimes \mathbf{w}_j)$$

In other words, $\{\mathbf{v}_i \otimes \mathbf{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ spans $V \otimes W$. ■

Theorem 13.1 A basis of $V \otimes W$ is $\{\mathbf{v}_i \otimes \mathbf{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$

Proof. By proposition (13.1), it suffices to show that the set $\{\mathbf{v}_i \otimes \mathbf{w}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is linear independent. Suppose that

$$\sum_{i=1}^n \sum_{j=1}^m \alpha_{ij} (\mathbf{v}_i \otimes \mathbf{w}_j) = \mathbf{0} \quad (13.3)$$

Suppose that $\{\phi_1, \dots, \phi_n\}$ is a dual basis of V^* , and $\{\psi_1, \dots, \psi_m\}$ is a dual basis of W^* .

Construct the mapping

$$\pi_{p,q} : V \times W \rightarrow \mathbb{F}$$

$$\text{with } \pi_{p,q} = \phi_p(\mathbf{v})\psi_q(\mathbf{w})$$

- The mapping $\pi_{p,q}$ is actually bilinear: for instance,

$$\begin{aligned}
\pi_{p,q}(a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w}) &= \phi_p(a\mathbf{v}_1 + b\mathbf{v}_2)\psi_q(\mathbf{w}) \\
&= (a\phi_p(\mathbf{v}_1) + b\phi_p(\mathbf{v}_2))\psi_q(\mathbf{w}) \\
&= a\phi_p(\mathbf{v}_1)\psi_q(\mathbf{w}) + b\phi_p(\mathbf{v}_2)\psi_q(\mathbf{w}) \\
&= a\pi_{p,q}(\mathbf{v}_1, \mathbf{w}) + b\pi_{p,q}(\mathbf{v}_2, \mathbf{w}).
\end{aligned}$$

Following the similar ideas, we can check that $\pi_{p,q}(\mathbf{v}, a\mathbf{w}_1 + b\mathbf{w}_2) = a\pi_{p,q}(\mathbf{v}, \mathbf{w}_1) + b\pi_{p,q}(\mathbf{v}, \mathbf{w}_2)$.

- Therefore, $\pi_{p,q} \in \text{Obj}$. By the universal property of the tensor product, $\pi_{p,q}$ induces the unique linear transformation

$$\begin{aligned}
\Pi_{p,q} : V \otimes W &\rightarrow \mathbb{F} \\
\text{with } \Pi_{p,q}(\mathbf{v} \otimes \mathbf{w}) &= \pi_{p,q}(\mathbf{v}, \mathbf{w})
\end{aligned}$$

In other words, $\Pi_{p,q}(\mathbf{v} \otimes \mathbf{w}) = \phi_p(\mathbf{v})\psi_q(\mathbf{w})$.

- Applying the mapping $\Pi_{p,q}$ on both sides of (13.3), we imply

$$\Pi_{p,q} \left(\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} (\mathbf{v}_i \otimes \mathbf{w}_j) \right) = \Pi_{p,q}(\mathbf{0})$$

Or equivalently,

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} \Pi_{p,q}(\mathbf{v}_i \otimes \mathbf{w}_j) = 0,$$

i.e.,

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} \phi_p(\mathbf{v}_i) \psi_q(\mathbf{w}_j) = \alpha_{p,q} = 0$$

Following this procedure, we can argue that $\alpha_{ij} = 0, \forall i, \forall j$.

■

Corollary 13.1 If $\dim(V), \dim(W) < \infty$, then $\dim(V \otimes W) = \dim(V) \dim(W)$

Proof. Check dimension of the basis of $V \otimes W$.

■

R The universal property can be very helpful. In particular, given a bilinear mapping, say $\phi : V \times W \rightarrow U$, we imply $\phi \in \text{Obj}$. By theorem (12.3), since i satisfies the universal property of tensor product, we can induce an unique linear transformation $\psi : V \otimes W \rightarrow U$.

Let's try another example for making use of the universal property:

Theorem 13.2 For finite dimension U and V ,

$$V \otimes U \cong U \otimes V$$

Proof. Construct the mapping

$$\begin{aligned} \phi : \quad V \times U &\rightarrow U \otimes V \\ \text{with } \phi(\mathbf{v}, \mathbf{u}) &= \mathbf{u} \otimes \mathbf{v} \end{aligned}$$

Indeed, ϕ is bilinear: for instance,

$$\begin{aligned} \phi(a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{u}) &= \mathbf{u} \otimes (a\mathbf{v}_1 + b\mathbf{v}_2) \\ &= a(\mathbf{u} \otimes \mathbf{v}_1) + b(\mathbf{u} \otimes \mathbf{v}_2) \\ &= a\phi(\mathbf{v}_1, \mathbf{u}) + b\phi(\mathbf{v}_2, \mathbf{u}) \end{aligned}$$

Therefore, $\phi \in \text{Obj}$. By the universal property of tensor product, we induce an unique linear transformation

$$\begin{aligned} \Phi : \quad V \otimes U &\rightarrow U \otimes V \\ \text{with } \Phi(\mathbf{v} \otimes \mathbf{u}) &= \mathbf{u} \otimes \mathbf{v} \end{aligned}$$

Similarly, we may induce the linear transformation

$$\begin{aligned} \Psi : \quad U \otimes V &\rightarrow V \otimes U \\ \text{with } \Psi(\mathbf{u} \otimes \mathbf{v}) &= \mathbf{v} \otimes \mathbf{u} \end{aligned}$$

Given any $\sum_i \mathbf{u}_i \otimes \mathbf{v}_i \in U \otimes V$, observe that

$$\begin{aligned}
 (\Phi \circ \Psi) \left(\sum_i \mathbf{u}_i \otimes \mathbf{v}_i \right) &= \Phi \left(\sum_i \Psi(\mathbf{u}_i \otimes \mathbf{v}_i) \right) \\
 &= \Phi \left(\sum_i \mathbf{v}_i \otimes \mathbf{u}_i \right) \\
 &= \sum_i \Phi(\mathbf{v}_i \otimes \mathbf{u}_i) \\
 &= \sum_i \mathbf{u}_i \otimes \mathbf{v}_i
 \end{aligned}$$

Therefore, $\Phi \circ \Psi = \text{id}_{U \otimes V}$. Similarly, $\Psi \circ \Phi = \text{id}_{V \otimes U}$. Therefore,

$$U \otimes V \cong V \otimes U.$$

■

13.1.2. Tensor Product of Linear Transformation

Motivation. Given two linear transformations $T : V \rightarrow V'$ and $S : W \rightarrow W'$, we want to construct the tensor product

$$T \otimes S : V \otimes W \rightarrow V' \otimes W'$$

Question: is $T \otimes S$ a linear transformation?

Answer: Yes. Universal property plays a role!

13.4. Wednesday for MAT3040

13.4.1. Tensor Product for Linear Transformations

Proposition 13.5 Suppose that $T : V \rightarrow V'$ and $S : W \rightarrow W'$ are linear transformations, then there exists a unique linear transformation

$$T \otimes S : V \otimes W \rightarrow V' \otimes W'$$
$$\text{satisfying } (T \otimes S)(v \otimes w) = T(v) \otimes S(w)$$

Proof. We construct the mapping

$$T \times S : V \times W \rightarrow V' \otimes W'$$
$$\text{with } (T \times S)(v, w) = T(v) \otimes S(w)$$

This mapping is indeed bilinear: for instance, we can show that

$$(T \times S)(av_1 + bv_2, w) = a(T \times S)(v_1, w) + b(T \times S)(v_2, w)$$

Therefore, $T \times S \in \text{Obj.}$ Since the tensor product satisfies the universal property, we imply there exists a unique linear transformation

$$T \otimes S : V \otimes W \rightarrow V' \otimes W'$$
$$\text{satisfying } (T \otimes S)(v \otimes w) = T(v) \otimes S(w)$$

■

Notation Warning. Does the notion $T \otimes S$ really form a tensor product, i.e., do we obtain the additive rules for tensor product such as

$$(aT_1 + bT_2) \otimes S = a(T_1 \otimes S) + b(T_2 \otimes S)?$$

■ **Example 13.2** Let $V = V' = \mathbb{F}^2$ and $W = W' = \mathbb{F}^3$. Define the matrix-multiply mappings:

$$\left\{ \begin{array}{l} T: V \rightarrow V \\ \text{with } \mathbf{v} \mapsto \mathbf{A}\mathbf{v} \\ \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \right\} \quad \left\{ \begin{array}{l} S: W \rightarrow W \\ \text{with } \mathbf{w} \mapsto \mathbf{B}\mathbf{w} \\ \mathbf{B} = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} \end{array} \right\}$$

How does $T \otimes S : V \otimes W \rightarrow V \otimes W$ look like?

- Suppose $\{e_1, e_2\}, \{f_1, f_2, f_3\}$ are usual basis of V, W , respectively. Then the basis of $V \otimes W$ is given by:

$$C = \{e_1 \otimes f_1, e_1 \otimes f_2, e_1 \otimes f_3, e_2 \otimes f_1, e_2 \otimes f_2, e_2 \otimes f_3\}.$$

- As a result, we can compute $(T \otimes S)(e_i \otimes f_j)$ for $i = 1, 2$ and $j = 1, 2, 3$. For instance,

$$\begin{aligned} (T \otimes S)(e_1 \otimes e_1) &= T(e_1) \otimes S(e_1) \\ &= (ae_1 + ce_2) \otimes (pe_1 + se_2 + ve_3) \\ &= (ap)e_1 \otimes e_1 + (as)e_1 \otimes e_2 + (av)e_1 \otimes e_3 + (cp)e_2 \otimes e_1 + (cs)e_2 \otimes e_2 + (cv)e_2 \otimes e_3 \end{aligned}$$

- Therefore, we obtain a matrix representation for the linear transformation $(T \otimes S)$:

We want a matrix representation for $(T \otimes S)$:

$$(T \otimes S)_{C,C} = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix},$$

which is a large matrix formed by taking all possible products between the elements of \mathbf{A} and those of \mathbf{B} . This operation is called the **Kronecker Tensor Product**, see the command *kron* in MATLAB for detail.

■

Proposition 13.6 More generally, given the linear operator $T : V \rightarrow V$ and $S : W \rightarrow W$, let $\mathcal{A} = \{v_1, \dots, v_n\}, \mathcal{B} = \{w_1, \dots, w_m\}$ be a basis of V, W respectively, with

$$(T)_{\mathcal{A}, \mathcal{A}} = (a_{ij}) \quad (S)_{\mathcal{B}, \mathcal{B}} = (b_{ij}) := B$$

As a result, $(T \otimes S)_{C, C} = A \otimes B$, where $C = \{v_1 \otimes w_1, \dots, v_n \otimes w_m\}$, and $A \otimes B$ denotes the Kronecker tensor product, defined as the matrix

$$\begin{pmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \cdots & a_{n,n}B \end{pmatrix}.$$

Proof. Following the similar procedure as in Example (13.2) and applying the relation

$$\begin{aligned} (T \otimes S)(v_i \otimes w_j) &= T(v_i) \otimes S(w_j) \\ &= \left(\sum_{k=1}^n a_{ki} v_k \right) \otimes \left(\sum_{\ell=1}^m b_{\ell j} w_\ell \right) \\ &= \sum_{k=1}^n \sum_{\ell=1}^m (a_{ki} b_{\ell j}) v_k \otimes w_\ell \end{aligned}$$

■

Proposition 13.7 The operation $T \otimes S$ satisfies all the properties of tensor product. For example,

$$(aT_1 + bT_2) \otimes S = a(T_1 \otimes S) + b(T_2 \otimes S)$$

$$T \otimes (cS_1 + dS_2) = c(T \otimes S_1) + d(T \otimes S_2)$$

Therefore, the usage of the notion “ \otimes ” is justified for the definition of $T \otimes S$.

Proof using matrix multiplication. For instance, consider the operation $(T + T') \otimes S$, with $(T)_{\mathcal{A}, \mathcal{A}} = (a_{ij}), (T')_{\mathcal{A}, \mathcal{A}} = (c_{ij}), (S)_{\mathcal{B}, \mathcal{B}} = B$.

We compute its matrix representation directly:

$$\begin{aligned}
((T + T') \otimes S)_{C,C} &= (T + T')_{\mathcal{A},\mathcal{A}} \otimes (S)_{\mathcal{B},\mathcal{B}} \\
&= [(T)_{\mathcal{A},\mathcal{A}} + (T')_{\mathcal{A},\mathcal{A}}] \otimes (S)_{\mathcal{B},\mathcal{B}} \\
&= (T)_{\mathcal{A},\mathcal{A}} \otimes (S)_{\mathcal{B},\mathcal{B}} + (T')_{\mathcal{A},\mathcal{A}} \otimes (S)_{\mathcal{B},\mathcal{B}}
\end{aligned}$$

where the last equality is by the additive rule for kronecker product for matrices.

Therefore,

$$((T + T') \otimes S)_{C,C} = (T \otimes S)_{C,C} + (T' \otimes S)_{C,C} \implies (T + T') \otimes S = T \otimes S + T' \otimes S$$

■

Proof using basis of $T \otimes S$. Another way of the proof is by computing

$$((T + T') \otimes S)(v_i \otimes w_j),$$

where $\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ forms a basis of $(T + T') \otimes S$:

$$\begin{aligned}
((T + T') \otimes S)(v_i \otimes w_j) &= (T + T')(v_i) \otimes S(w_j) \\
&= (T(v_i) + T'(v_i)) \otimes S(w_j) \\
&= T(v_i) \otimes S(w_j) + T'(v_i) \otimes S(w_j) \\
&= (T \otimes S)(v_i \otimes w_j) + (T' \otimes S)(v_i \otimes w_j)
\end{aligned}$$

Since $((T + T') \otimes S)(v_i \otimes w_j)$ coincides with $(T \otimes S + T' \otimes S)(v_i \otimes w_j)$ for all basis vectors $v_i \otimes w_j \in C$, we imply

$$(T + T') \otimes S = T \otimes S + T' \otimes S$$

■

Proposition 13.8 Let A, C be linear operators from V to V , and B, D be linear operators from W to W , then

$$(A \otimes B) \circ (C \otimes D) = (AC) \otimes (BD)$$

Proposition 13.9 Define linear operators $A : V \rightarrow V$ and $B : W \rightarrow W$ with $\dim(V), \dim(W) < \infty$. Then

$$\det(A \otimes B) = (\det(A))^{\dim(W)} (\det(B))^{\dim(V)}$$

Corollary 13.3 There exists a linear transformation

$$\Phi : \text{Hom}(V, V) \otimes \text{Hom}(W, W) \rightarrow \text{Hom}(V \otimes W, V \otimes W)$$

$$\text{with } A \otimes B \mapsto A \otimes B$$

where the input of Φ is the tensor product of linear transformations, and the output is the linear transformation.

Proof. Construct the mapping

$$\Phi : \text{Hom}(V, V) \times \text{Hom}(W, W) \rightarrow \text{Hom}(V \otimes W, V \otimes W)$$

$$\text{with } \Phi(A, B) = A \otimes B$$

The Φ is indeed bilinear: for instance,

$$\begin{aligned} \Phi(pA + qC, B) &= (pA + qC) \otimes B \\ &= p(A \otimes B) + q(C \otimes B) \\ &= p\Phi(A, B) + q\Phi(C, B) \end{aligned}$$

This corollary follows from the universal property of tensor product. ■

R If assuming that $\dim(V), \dim(W) < \infty$, we imply

$$\begin{aligned} \dim(\text{Input space of } \Phi) &= \dim(\text{Hom}(V, V)) \dim(\text{Hom}(W, W)) \\ &= [\dim(V) \dim(V)] \cdot [\dim(W) \dim(W)] = [\dim(V) \dim(W)]^2 \\ &= [\dim(V \otimes W)]^2 \\ &= \dim(\text{Hom}(V \otimes W, V \otimes W)) \\ &= \dim(\text{Output space of } \Phi) \end{aligned}$$

Therefore, is Φ an isomorphism? If so, then every linear operator $\alpha : V \otimes W \rightarrow V \otimes W$ can be expressed as

$$\alpha = A_1 \otimes B_1 + \cdots + A_k \otimes B_k$$

where $A_i : V \rightarrow V$ and $B_j : W \rightarrow W$.

Chapter 14

Week14

14.1. Monday for MAT3040

14.1.1. Multilinear Tensor Product

Definition 14.1 [Tensor Product among More spaces] Let V_1, \dots, V_p be vector spaces over \mathbb{F} . Let $S = \{(v_1, \dots, v_p) \mid v_i \in V_i\}$ (We assume no relations among distinct elements in S), and define $\mathfrak{X} = \text{span}(S)$.

1. Then define the tensor product space $V_1 \otimes \dots \otimes V_p = \mathfrak{X}/y$, where y is the vector subspace of \mathfrak{X} spanned by vectors of the form

$$(v_1, \dots, v_i + v'_i, \dots, v_p) - (v_1, \dots, v_i, \dots, v_p) - (v_1, \dots, v'_i, \dots, v_p),$$

and

$$(v_1, \dots, \alpha v_i, \dots, v_p) - \alpha(v_1, \dots, v_i, \dots, v_p)$$

where $i = 1, 2, \dots, p$.

2. The tensor product for vectors is defined as

$$v_1 \otimes \dots \otimes v_p := \{(v_1, \dots, v_p) + y\} \in V_1 \otimes \dots \otimes V_p$$

1. We have

$$v_1 \otimes \cdots \otimes (\alpha v_i + \beta v'_i) \otimes \cdots \otimes v_p = \alpha(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_p) + \beta(v_1 \otimes \cdots \otimes v'_i \otimes \cdots \otimes v_p)$$

2. A general vector in $V_1 \otimes \cdots \otimes V_p$ is

$$\sum_{i=1}^n (W_1^{(i)} \otimes \cdots \otimes W_p^{(i)}), \quad \text{where } W_j^{(i)} \in V_j, j = 1, \dots, p$$

3. Let $\mathcal{B}_i = \{v_i^{(1)}, \dots, v_i^{(\dim(V_i))}\}$ be a basis of $V_i, i = 1, \dots, p$, then

$$\mathcal{B} = \{V_1^{(\alpha_1)} \otimes \cdots \otimes V_p^{(\alpha_p)} \mid 1 \leq \alpha_i \leq \dim(V_i)\}$$

is a basis of $V_1 \otimes \cdots \otimes V_p$. As a result,

$$\dim(V_1 \otimes \cdots \otimes V_p) = (\dim(V_1)) \times \cdots \times (\dim(V_p))$$

Theorem 14.1 — Universal Property of multi-linear tensor. Let $\text{Obj} = \{\phi : V_1 \times \cdots \times V_p \rightarrow W \mid \phi \text{ is a } p\text{-linear map}\}$, i.e.,

$$\begin{aligned} \phi(v_1, \dots, \alpha v_i + \beta v'_i, \dots, v_p) &= \alpha \phi(v_1, \dots, v_i, \dots, v_p) + \beta \phi(v_1, \dots, v'_i, \dots, v_p), \\ \forall v_i, v'_i \in V_i, i &= 1, \dots, p, \forall \alpha, \beta \in \mathbb{F}. \end{aligned}$$

For instance, the multiplication of p matrices is a p -linear map.

Then the mapping in the Obj ,

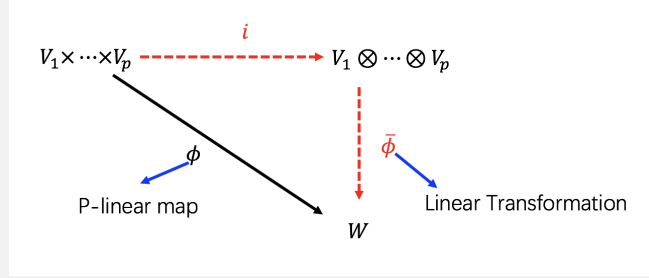
$$\begin{aligned} i : \quad V_1 \times V_p &\rightarrow V_1 \otimes \cdots \otimes V_p \\ \text{with } (v_1, \dots, v_p) &\mapsto v_1 \otimes \cdots \otimes v_p \end{aligned}$$

satisfies the universal property. In other words, for any $\phi : V_1 \times \cdots \times V_p \in \text{Obj}$, there

exists the unique linear transformation

$$\bar{\phi} : V_1 \otimes \cdots \otimes V_p \rightarrow W$$

such that the diagram below commutes:



In other words, $\phi = \bar{\phi} \circ i$.

Corollary 14.1 Let $T_i : V_i \rightarrow V'_i$ be a linear transformation, $1 \leq i \leq p$. There is a unique linear transformation

$$(T_1 \otimes \cdots \otimes T_p) : V_1 \otimes \cdots \otimes V_p \rightarrow V'_1 \otimes \cdots \otimes V'_p$$

$$\text{satisfying} \quad (T_1 \otimes \cdots \otimes T_p)(v_1 \otimes \cdots \otimes v_p) = T_1(v_1) \otimes \cdots \otimes T_p(v_p)$$

Proof. Construct the mapping

$$\phi : V_1 \times \cdots \times V_p \rightarrow V'_1 \otimes \cdots \otimes V'_p$$

$$\text{with } (v_1, \dots, v_p) \mapsto T_1(v_1) \otimes \cdots \otimes T_p(v_p)$$

which is indeed p -linear.

By the universal property, we induce the unique linear transformation

$$\bar{\phi} : V_1 \otimes \cdots \otimes V_p \rightarrow V'_1 \otimes \cdots \otimes V'_p$$

■

Notation. To make life easier, from now on, we only consider $V_1 = \cdots = V_p = V$. Then for any linear transformation $T : V \rightarrow W$, we have

$$T^{\otimes p} : V \otimes \cdots \otimes V \rightarrow W \otimes \cdots \otimes W$$

We use the short-hand notation $V^{\otimes p}$ to denote $\underbrace{V \otimes \cdots \otimes V}_{p \text{ terms in total}}$

Final Exam Ends Here.

14.1.2. Exterior Power

Definition 14.2 A p -linear map $\phi : V \times \cdots \times V \rightarrow W$ is called **alternating** if

$$\phi(v_1, \dots, v_i, \dots, v_j, \dots, v_p) = \mathbf{0}_W, \quad \text{provided that there exists some } v_i = v_j \text{ for } i \neq j.$$

Also, we say ϕ is p -alternating ■

■ **Example 14.1** 1. The cross product mapping

$$\phi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\text{with } (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \times \mathbf{w}$$

is alternating:

- ϕ is bilinear
- $\phi(\mathbf{v}, \mathbf{v}) = \mathbf{v} \times \mathbf{v} = \mathbf{0}$.

2. The determinant mapping

$$\phi : \underbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}_{n \text{ terms in total}} \rightarrow \mathbb{F}$$

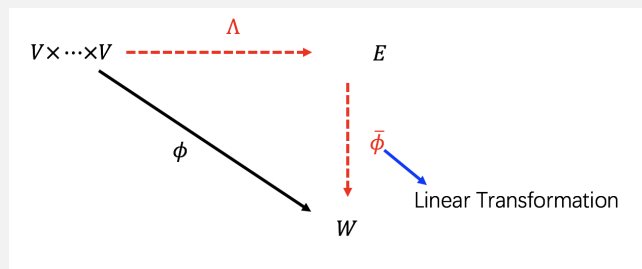
$$\text{with } (\mathbf{v}_1, \dots, \mathbf{v}_n) \mapsto \det([\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n])$$

is alternating:

- ϕ is n -linear by MAT2040 knowledge
- ϕ is alternating by MAT2040 knowledge

Theorem 14.2 — Universal Property for exterior power. Let $\text{Obj} := \{\phi : \underbrace{V \times \cdots \times V}_{p \text{ terms}} \rightarrow W \mid \phi \text{ is } p\text{-alternating map}\}$. Then there exists $\{\Lambda : V \times \cdots \times V \rightarrow E\} \in \text{Obj}$ satisfying the following:

- For all $\phi : V \times \cdots \times V \rightarrow W \in \text{Obj}$, there exists unique linear transformation $\bar{\phi} : E \rightarrow W$ satisfying



In other words, $\phi = \bar{\phi} \circ \Lambda$.

Chapter 15

Week15

15.1. Monday for MAT3040

15.1.1. More on Exterior Power

Reviewing. Let $\text{Obj} := \{\phi : V \times \cdots \times V \rightarrow W \mid \phi \text{ is alternating}\}$, then there exists

$$\{\Lambda : V \times \cdots \times V \rightarrow E\} \in \text{Obj}$$

such that

$$\phi = \bar{\phi} \circ \Lambda, \quad \text{where } \bar{\phi} : E \rightarrow W \text{ is the unique linear transformation}$$

Here we give one way for constructing E :

$$E = V^{\otimes p} / U,$$

where U is spanned by vectors of the form

$$v_1 \otimes \cdots \otimes v_p \in V^{\otimes p}, \quad v_i = v_j \text{ where for some } i \neq j.$$

For instance, $v \otimes v \otimes \cdots \otimes v_p \in U$.

Definition 15.1 [Wedge Product] Define the wedge product space

$$\wedge^p V := V^{\otimes p} / U = E,$$

with the wedge product among vectors

$$v_1 \wedge \cdots \wedge v_p = v_1 \otimes \cdots \otimes v_p + U \in \wedge^p V$$

As a result, the mapping

$$\begin{aligned} \wedge : V \times \cdots \times V &\rightarrow E := \wedge^p V \\ (v_1, \dots, v_p) &\mapsto v_1 \wedge \cdots \wedge v_p \end{aligned}$$

will satisfy the universal property of exterior power.

Proposition 15.1 1. We have the p -linearity for $\wedge^p V$, i.e.,

$$v_1 \wedge \cdots \wedge (av_i + bv'_i) \wedge \cdots \wedge v_p = a(v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_p) + b(v_1 \wedge \cdots \wedge v'_i \wedge \cdots \wedge v_p)$$

for $i = 1, \dots, p$.

2. The wedge product is alternating:

$$\begin{aligned} v_1 \wedge \cdots \wedge v \wedge \cdots \wedge v \wedge \cdots \wedge v_p &:= v_1 \otimes \cdots \otimes v \otimes \cdots \otimes v \otimes \cdots \otimes v_p + U \\ &= 0 + U \\ &= 0_{\wedge^p V} \end{aligned}$$

3. The wedge product reverses sign reversal property:

$$v_1 \wedge \cdots \wedge v \wedge \cdots \wedge w \wedge \cdots \wedge v_p = -v_1 \wedge \cdots \wedge w \wedge \cdots \wedge v \wedge \cdots \wedge v_p$$

Reason: $(v + w) \wedge (v + w) = 0$, which implies $v \wedge w + w \wedge v = 0$.

Proposition 15.2

1. If $\dim(V) = n$, and $0 \leq p \leq n$, then

$$\dim(\wedge^p V) = \binom{n}{p}$$

2. For all linear operators $T : V \rightarrow V$, there is a unique linear operator from $\wedge^p V$ to $\wedge^p V$:

$$T^{\wedge p} : \wedge^p V \rightarrow \wedge^p V$$

$$\text{with } v_1 \wedge \cdots \wedge v_p \mapsto T(v_1) \wedge \cdots \wedge T(v_p)$$

Proof. 1. Let $\{v_1, \dots, v_n\}$ be basis of V , then $\{v_{i_1} \otimes \cdots \otimes v_{i_p} \mid 1 \leq i_k \leq n\}$ forms basis of $V^{\otimes p}$. Note that $\{v_{i_1} \wedge \cdots \wedge v_{i_p} \mid 1 \leq i_k \leq n\}$ spans $\wedge^p V$, since $\pi_V : V \rightarrow V/U$ is surjective. We claim that

$$\mathcal{B} = \{v_{i_1} \wedge \cdots \wedge v_{i_p} \mid 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}$$

is a basis of $\wedge^p V$

- \mathcal{B} spans $\wedge^p V$: we can use (3) in proposition (15.1) to “rearrange” the indices j_1, \dots, j_p into ascending order, and $\text{span}(\mathcal{B}) = \text{span}\{v_{i_1} \wedge \cdots \wedge v_{i_p} \mid 1 \leq i_k \leq n\}$.
- We omit the proof that \mathcal{B} is linear independent due to time limit.

The number of vectors in \mathcal{B} is equal to $\binom{n}{p}$.

■

15.1.2. Determinant

Previous Approach for defining determinant. We define the determinant for $\mathbf{A} = M_{n \times n}(\mathbb{F})$ directly. From such complicated definition, we come up with $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$, which implies that the similar matrices share with the same determinant, then we define the determinant for any linear operator $T : V \rightarrow V$ as

$$\det(T) = \det((T)_{\mathcal{B}, \mathcal{B}}), \quad \text{for some basis } \mathcal{B} \text{ of } V$$

New Approach. We will define $\det(T)$ for linear operators without fixing a basis, and then we will imply $\det(T \circ S) = \det(T)\det(S)$ easily. Then $\det(\mathbf{A})$ for $\mathbf{A} \in M_{n \times n}(\mathbb{F})$ belongs to our special case.

Definition 15.2 [Determinant for Linear Operators]

1. Suppose that $\dim(V) = n$, then

$$\dim(\wedge^n V) = \binom{n}{n} = 1$$

More precisely, for any basis $\{v_1, \dots, v_n\}$ of V , we have $\wedge^n(V) = \text{span}\{v_1 \wedge \dots \wedge v_n\}$.

2. Note that $T^{\wedge n} : \wedge^n V \rightarrow \wedge^n V$ is a linear operator on $\wedge^n V \cong \mathbb{F}$. Therefore, for all $\tau \in \wedge^n V$, there exists $\alpha_T \in \mathbb{F}$ such that

$$T^{\wedge n}(\tau) = \alpha_T \tau$$

3. Now we define

$$\det(T) = \alpha_T$$

This definition of determinant does not depend on any choice of basis of V .

- **Example 15.1** 1. Suppose that $T = I : V \rightarrow V$ be identity. Take a basis $\{v_1, \dots, v_n\}$ of V , then

$$T^{\wedge n}(v_1 \wedge \dots \wedge v_n) = T(v_1) \wedge \dots \wedge T(v_n)$$

Or equivalently,

$$\det(T) \cdot (v_1 \wedge \dots \wedge v_n) = v_1 \wedge \dots \wedge v_n$$

Therefore, $\det(T) = 1$.

2. Suppose that $T : V \rightarrow V$ is diagonalizable with $\{w_1, \dots, w_n\}$ forming eigen-basis of T .

As a result,

$$T^{\wedge n}(w_1 \wedge \cdots \wedge w_n) = T(w_1) \wedge T(w_2) \cdots \wedge T(w_n),$$

which implies

$$\det(T)(w_1 \wedge \cdots \wedge w_n) = (\lambda_1 w_1) \wedge \cdots \wedge (\lambda_n w_n),$$

which implies

$$\det(T)w_1 \wedge \cdots \wedge w_n = (\lambda_1 \cdots \lambda_n)w_1 \wedge \cdots \wedge w_n,$$

i.e., $\det(T) = \lambda_1 \cdots \lambda_n$.

Proposition 15.3 Let $T, S : V \rightarrow V$ be linear transformations, then

$$(T \circ S)^{\wedge p} : \wedge^p V \rightarrow \wedge^p V$$

$$\text{with } T^{\wedge p}, S^{\wedge p} : \wedge^p V \rightarrow \wedge^p V$$

satisfies

$$(T \circ S)^{\wedge p} = (T^{\wedge p}) \circ (S^{\wedge p})$$

Proof. Pick any basis $\{v_{i_1} \wedge \cdots \wedge v_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}$ of $\wedge^p V$. Then

$$(T \circ S)^{\wedge p}(v_{i_1} \wedge \cdots \wedge v_{i_p}) = (T \circ S)(v_{i_1}) \wedge \cdots \wedge (T \circ S)(v_{i_p})$$

On the other hand,

$$\begin{aligned} (T^{\wedge p}) \circ (S^{\wedge p})(v_{i_1} \wedge \cdots \wedge v_{i_p}) &= (T^{\wedge p})(S(v_{i_1}) \wedge \cdots \wedge S(v_{i_p})) \\ &= (T \circ S)(v_{i_1}) \wedge \cdots \wedge (T \circ S)(v_{i_p}) \end{aligned}$$

■

Corollary 15.1

$$\det(T \circ S) = \det(T)\det(S)$$

Proof. Pick any basis $\{v_1 \wedge \cdots \wedge v_n\}$ of $\wedge^n V$, then

$$\begin{aligned}
 \det(T \circ S)v_1 \wedge \cdots \wedge v_n &= (T \circ S)^{\wedge n} v_1 \wedge \cdots \wedge v_n \\
 &= (T^{\wedge n}) \circ ((S^{\wedge n})v_1 \wedge \cdots \wedge v_n) \\
 &= (T^{\wedge n})(\det(S)v_1 \wedge \cdots \wedge v_n) \\
 &= \det(S)T^{\wedge n}(v_1 \wedge \cdots \wedge v_n) \\
 &= \det(S)\det(T)v_1 \wedge \cdots \wedge v_n
 \end{aligned}$$

Therefore, $\det(T \circ S) = \det(T)\det(S)$. ■

Theorem 15.1 Let $V = \mathbb{F}^n$, and

$$\begin{aligned}
 T : V &\rightarrow V \\
 \text{with } T(\mathbf{v}) &= \mathbf{A}\mathbf{v}, \quad \mathbf{A} \in M_{n \times n}(\mathbb{F})
 \end{aligned}$$

Then $\det(T) = \det(\mathbf{A})$

Proof. Take $\{e_1, \dots, e_n\}$ as the usual basis of $V \equiv \mathbb{F}^n$, then

$$\begin{aligned}
 \det(T)e_1 \wedge \cdots \wedge e_n &= T(e_1) \wedge \cdots \wedge T(e_n) \\
 &= a_1 \wedge \cdots \wedge a_n
 \end{aligned}$$

where a_i denotes the i -th column of \mathbf{A} .

As we have studied before [c.f. p141 in MAT2040 Notebook], the previous definition of determinant is based on three basic properties. It suffices to show these three basic properties:

1. The determinant of the n by n identity matrix is 1: See part (1) in Example (15.1)
2. The determinant changes sign when two columns (w.l.o.g., “rows” are related with “columns”) are exchanged: due to the sign reversal property for wedge product

3. The determinant is a linear function of each column separately, i.e.,

$$a_1 \wedge \cdots \wedge (ta_i) \wedge \cdots \wedge a_n = t(a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_n)$$

Once we verify these three properties, we conclude that the explicit formula for $\det(\mathbf{A})$ is a special case for our new definition. ■