On the Capacity Scalability of Line Networks with Buffer Size Constraints

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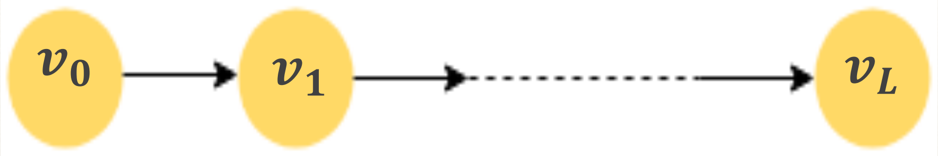
Slides available on the link https://walterbabyrudin.github.io

Why Multi-hop Line Networks?

Problem Setup

Consider a line network with L hops:



Zero-error Capacity is zero

- Adjacent Nodes are connected with the same channel
- Intermediate nodes have practical buffer constraints:
  - ➢ Buffer: contain relevant information for recoding
  - ➢ The space for storing the processing space is constant
- How does the communication capacity scale with hop number L?

Related Work

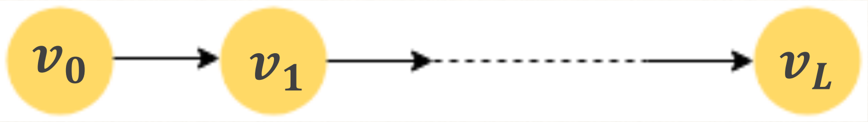
Consider a line network with L hops with Buffer Size B

How does the communication capacity scale with hop number L?

- B is unbounded : min-cut capacity [Cover, 2006]
- $B = O(\log L)$: constant capacity [Niesen, 2007]
- $\boxed{B = O(1)}$: $\Omega(e^{-cL})$ [Niesen, 2007]

Main Result

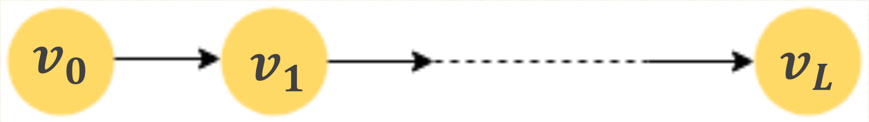
Consider a line network with L hops with Buffer Size B



How does the communication capacity scale with hop number L?

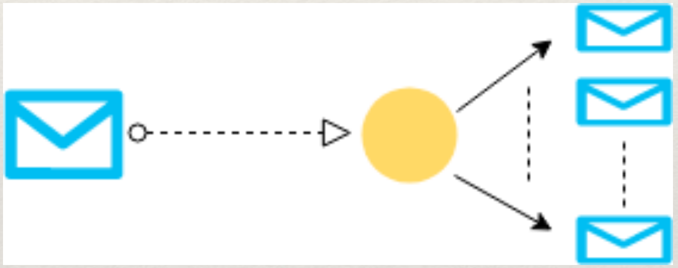
	Buffer Size	Capacity
[Cover, 2006]	unbounded	min-cut Capacity
[Niesen, 2007]	$B = O(1)$	$\Omega(e^{-cL})$
Our Work	$\boldsymbol{B = O(\log\log L)}$	$\boldsymbol{\Omega(1/\log L)}$
[Niesen, 2007]	$B = O(\log L)$	$\Theta(1)$

Batched Code for Line Networks of BEC



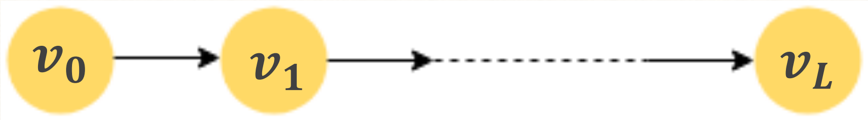
BEC with erasure probability ϵ

- The batched code contains an outer code and an inner code:
  - ◻ The outer code at v_0 encodes the message into several batches.



  - ➢ The inner code is performed on different batches separately.
  - ◻ The destination node v_L uses all received symbols to decode the message at source node.

Performance of proposed Code for Line Networks of BEC



BEC with erasure probability ϵ

- The destination node v_L can recover the message with probability

$$(1-\epsilon^N)^L$$

- The achievable rate of our coding scheme is

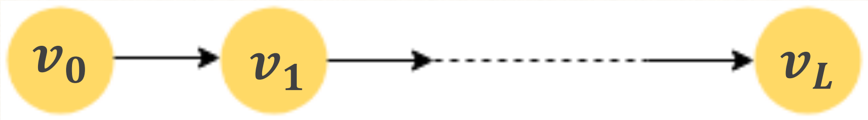
$$\frac{(1-\epsilon^N)^L}{N}$$

- The best performance of our coding scheme should be

$$\max_N \frac{(1-\epsilon^N)^L}{N} = \Theta(1/\log L)$$

which is achieved when $N = \Theta(\log L)$.

Batched Code for Line Networks of BSC

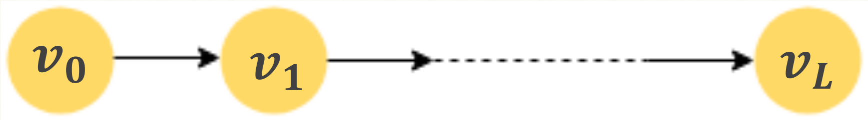


BSC with crossover probability ϵ

- Inner code: repetition codes for BSC;
  Outer code: capacity achieving code for BSCs

  - The outer code at v_0 encodes the message into several batches.
  - Inner code: Source node, intermediate nodes perform on different batch separately.
  - The destination node v_L uses all received symbols to decode the message at source node.

Performance of proposed Code for Line Networks of BSC



BSC with crossover probability ϵ

- The transmission between adjacent nodes can be viewed as a new BSC:

$$\tilde{Q} = \begin{pmatrix} 1 - P_e & P_e \\ P_e & 1 - P_e \end{pmatrix}, \quad \text{where } P_e = \sum_{i \leq N/2} \binom{N}{i} (1-\epsilon)^i \epsilon^{N-i}$$

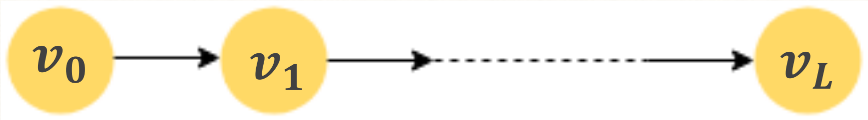
- The transmission from v_0 to v_L is the L-concatenation of $\tilde{Q}$:

$$\tilde{Q}^L = \begin{pmatrix} 1 - \tilde{\epsilon} & \tilde{\epsilon} \\ \tilde{\epsilon} & 1 - \tilde{\epsilon} \end{pmatrix}, \quad \text{where } \tilde{\epsilon} = \frac{1 - (1 - 2P_e)^L}{2}$$

- The achievable rate of our coding scheme is

$$\frac{\mathcal{C}(\tilde{Q}^L)}{N} = \frac{1 - h_2(\tilde{\epsilon})}{N}$$

Best Performance of proposed Code for Line Networks of BSC (I)



BSC with crossover probability ϵ

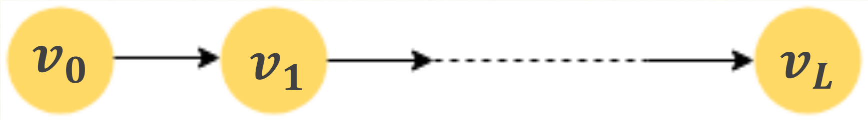
- The best performance of our coding scheme should be

$$
\begin{aligned}
\max \quad & \frac{\mathcal{C}(\tilde{Q}^L)}{N} \triangleq \frac{1-h_2(\tilde{\epsilon})}{N} \\
\text{s.t.} \quad & \tilde{\epsilon} = \frac{1-(1-2P_e)^L}{2} \\
& P_e = \sum_{i \le N/2} \binom{N}{i} (1-\epsilon)^i \epsilon^{N-i}
\end{aligned}
$$

- Global quadratic lower bound by Taylor Expansion:

$$
h_2(x) = 1 - \frac{1}{2\ln 2} \sum_{n=1}^{\infty} \frac{(1-2x)^{2n}}{n(2n-1)} \le 1 - \frac{1}{2\ln 2}(1-2x)^2
$$

Best Performance of proposed Code for Line Networks of BSC (II)



BSC with crossover probability ϵ

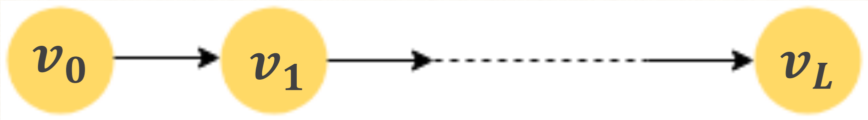
- The lower bound for the optimization problem should be

$$
\begin{aligned}
\max \quad & \frac{1}{2\ln 2}\frac{(1-2\tilde{\epsilon})^2}{N} \\
\text{s.t.} \quad & \tilde{\epsilon} = \frac{1-(1-2P_e)^L}{2} \\
& P_e = \sum_{i \le N/2} \binom{N}{i} (1-\epsilon)^i \epsilon^{N-i}
\end{aligned}
$$

- Or equivalently,

$$
\begin{aligned}
\max \quad & \frac{1}{2\ln 2}\frac{(1-2P_e)^{2L}}{N} \\
\text{s.t.} \quad & P_e = \sum_{i \le N/2} \binom{N}{i} (1-\epsilon)^i \epsilon^{N-i}
\end{aligned}
$$

Best Performance of proposed Code for Line Networks of BSC (III)



BSC with crossover probability ϵ

- Apply the tail bound for cdf of binomial distribution:

$$P_e \leq \exp(-ND), \text{ where } D = D(0.5\|1-\epsilon)$$

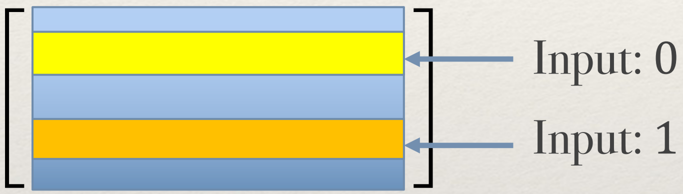
- This leads to a easy-solved optimization problem:

$$\max_N \quad \frac{1}{2\ln 2}\frac{(1-2e^{-ND})^{2L}}{N} = \Theta(1/\log L)$$

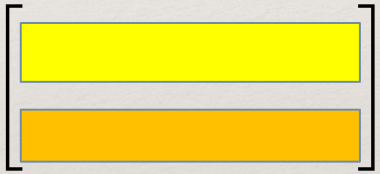
Generalizing into Line Networks of DMC

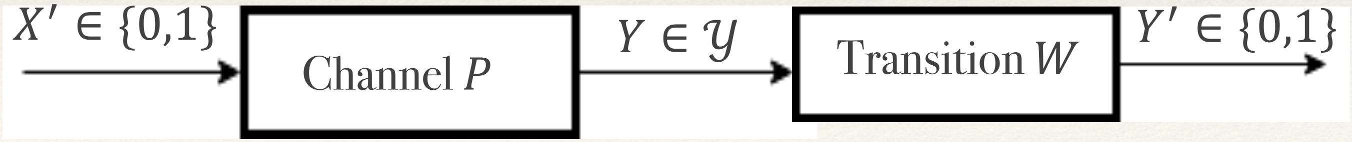
- Any line network of a general DMC Q can be converted into that of a non-trivial BSC:

$Q_{|\mathcal{X}|\times|\mathcal{Y}|} \triangleq$



$P_{2\times|\mathcal{Y}|} \triangleq$





Generalizing into Line Networks of DMC

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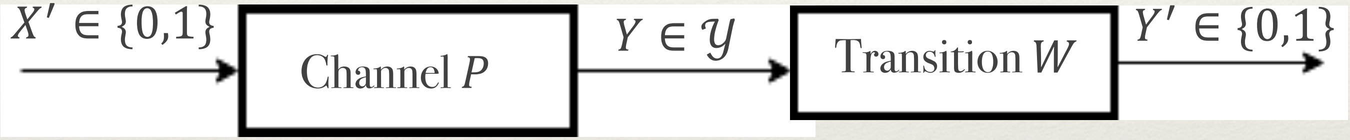

$$P_{2\times|\mathcal{Y}|} \triangleq$$

$$W = \begin{pmatrix} w_1 & 1-w_1 \\ \vdots & \vdots \\ w_{|\mathcal{Y}|} & 1-w_{|\mathcal{Y}|} \end{pmatrix} \quad \text{where} \quad w_k = \begin{cases} p_{1k}/(p_{1k}+p_{2k}), & \text{if } p_{1k}+p_{2k}>0 \\ 0, & \text{otherwise} \end{cases}$$

The Channel PW is the desired **BSC** with **positive** capacity

Remarks

- The proposed coding scheme achieve $\frac{1}{\log L}$ scalability of line network capacity with buffer size $O(\log \log L)$
- Practical codes may achieve a higher rate under the same buffer size constraint:
    - Random Linear Codes for Packet Erasure Channels
    - Convolutional codes for Binary Symmetric Channels

Future Study

- Our Scalability Results can be extended into more general channels:
  - ➢Continuous Channels
  - ➢The Channel between adjacent nodes can be different
  - ➢The channel can have memory

	Buffer Size	Capacity
[Cover, 2006]	Unlimited	Min-cut Capacity
[Niesen, 2007]	$B = O(1)$	$\Omega(e^{-cL})$
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Future Study

- We only give lower bound for capacity scalability results

- How about the converse:
    - if the buffer is finite?
    - if the buffer scales with L?

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Thanks!

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