14.2. Monday for MAT3006

Proposition 14.1 For all $E \in \mathcal{M} \otimes \mathcal{M}$,we have

$$\int m_Y(E_x)\,\mathrm{d}x = \int m_X(E_y)\,\mathrm{d}y = \pi(E), \tag{14.1}$$

where $\pi(\cdot)$ is a measure on $\mathcal{M} \otimes \mathcal{M}$.

Here note that

$$\begin{aligned} m_X(E_y) &:= \int (\mathcal{X}_E)_y(x)\,\mathrm{d}x \\ m_Y(E_x) &:= \int (\mathcal{X}_E)_x(y)\,\mathrm{d}y \end{aligned}$$

Proof. Construct

$$\mathcal{A} = \left\{ E \in \mathcal{M} \otimes \mathcal{M} \;\middle|\; \begin{array}{l} x \mapsto m_Y(E_x) \text{ measurable} \\ y \mapsto m_X(E_y) \text{ measurable} \\ \text{(14.1) holds for } E \end{array} \right\}$$

Following the proof given in the last lecture, it suffices to show $\mathcal{A}$ is a monotone class:

- Construct

$$\mathcal{A}_k = \mathcal{A} \cap \{E \in \mathcal{M} \otimes \mathcal{M} \mid E \subseteq [-k,k] \times [-k,k]\}.$$

  We first show that $\mathcal{A}_k$ is a monotone class for all $k \in \mathbb{N}$:

  1. Suppose that $E_n \subseteq E_{n+1}, \forall n$ and $E_n \in \mathcal{A}_k$, and we aim to show $E := \cup_{n=1}^{\infty} E_n \in \mathcal{A}_k$. Consider the function $f_n(x) = m_Y((E_n)_x)$, which is measurable for all n, and $f_n(x) \le f_{n+1}(x)$ for all n, since $E_n \subseteq E_{n+1}$.
     The MCT I implies that $f(x) = m_Y(E_x)$ is measurable with

$$\int m_Y(E_x)\,\mathrm{d}x = \lim_{n\to\infty} \int m_Y((E_n)_x)\,\mathrm{d}x \overset{(a)}{=} \lim_{n\to\infty} \pi(E_n) \overset{(b)}{=} \pi(E)$$

     where (a) is because that $E_n \in \mathcal{A}$; and (b) is due to the exercise in Hw3. Similarly, $y \mapsto m_X(E_y)$ is measurable, with $\int m_X(E_y)\,\mathrm{d}y = \pi(E)$. Therefore,

$E \in \mathcal{A}$, i.e., $E \in \mathcal{A}_k$ as well.

2. Suppose that $F_i \in \mathcal{A}_k, F_i \supseteq F_{i+1}$, and we aim to show $F := \cap_{i=1}^{\infty} F_i \in \mathcal{A}_k$. Construct the measurable function $g_n(x) = m_Y((F_n)_x)$, and $g_n(x) \geq g_{n+1}(x)$; $|g_n(x)| \leq g_1(x)$, with $g_1(x)$ integrable. (You may see the bounded rectangle in $\mathcal{A}_k$ matters here)

   The DCT implies that $g(x) = m_Y(F_x)$ is measurable, with

$$\int m_Y(F_x)\,\mathrm{d}x = \lim_{n\to\infty} \int g_n\,\mathrm{d}x = \lim_{n\to\infty} \pi(F_n) = \pi(F).$$

   Similarly, $y \mapsto m_X(F_y)$ is measurable, with $\int m_X(F_y)\,\mathrm{d}y = \pi(F)$. Therefore, $F \in \mathcal{A}_k$.

- Then we show $\mathcal{A}$ is a monotone class, i.e., closed under countable decreasing intersections.

  Suppose that $F_i \in \mathcal{A}, F_i \supseteq F_{i+1}$, we aim to show that $F := \cap F_i \in \mathcal{A}$.

  Construct

$$F_i^{(k)} = F_i \cap ([-k,k] \times [-k,k]),$$

  which implies $F_i^{(k)} \supseteq F_{i+1}^{(k)}, F_i^{(k)} \in \mathcal{A}_k$. We denote $F^{(k)} = \cap_{i=1}^{\infty} F_i^{(k)}$. The previous result implies that $F^{(k)} \in \mathcal{A}_k$, i.e.,

$$\int m_Y((F^{(k)})_x)\,\mathrm{d}x = \pi(F^{(k)})$$

  Now note that $F^{(1)} \subseteq F^{(2)} \subseteq \cdots$, and $F = \cup_{k\in\mathbb{N}} F^{(k)}$. Therefore, applying MCT gives

$$\int m_Y(F_x)\,\mathrm{d}x = \lim_{k\to\infty} \int m_Y((F^{(k)})_x)\,\mathrm{d}x = \lim_{k\to\infty} \pi(F^{(k)}) = \pi(F).$$

  Therefore, F satisfies (14.1), i.e., $F \in \mathcal{A}$

■

Theorem 14.3 — Tonelli's Theorem. Let $F : \mathbb{R}^2 \to [0, \infty]$ be measurable under the space $(\mathbb{R}^2, \mathcal{M} \otimes \mathcal{M}, \pi)$. Then

$$\begin{cases} x \mapsto \int F(x,y)\,\mathrm{d}y \\ y \mapsto \int F(x,y)\,\mathrm{d}x \end{cases} \text{is measurable,}$$

and

$$\int F\,\mathrm{d}\pi = \int \left(\int F(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int \left(\int F(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

Proof. Let

$$\phi_n(x,y) = \sum_{k=0}^{4^n} (k \cdot 2^{-n}) \mathcal{X}_{F^{-1}([k\cdot 2^{-n},(k+1)\cdot 2^{-n}])} + 2^n \mathcal{X}_{F^{-1}(2^n,\infty]}$$

We just re-write the terms above as $\sum_k \alpha_k \mathcal{X}_{E_k}$. Our constructed $\phi_n(x,y)$ is a monotone increasing simple function such that $\phi_n \to F$ pointwise. It follows that

$$\begin{aligned}
\int F\,\mathrm{d}\pi &= \lim_{n\to\infty} \int \phi_n\,\mathrm{d}\pi && (14.2a)\\
&= \lim_{n\to\infty} \int \left(\sum_k \alpha_k \mathcal{X}_{E_k} \right) \mathrm{d}\pi && (14.2b)\\
&= \lim_{n\to\infty} \sum_k \alpha_k \int \mathcal{X}_{E_k}\,\mathrm{d}\pi = \lim_{n\to\infty} \sum_k \alpha_k \pi(E_k) && (14.2c)\\
&= \lim_{n\to\infty} \sum_k \alpha_k \int \left(\int \mathcal{X}_{E_k}(x,y)\,\mathrm{d}x \right) \mathrm{d}y && (14.2d)\\
&= \lim_{n\to\infty} \int \int \left(\sum_k \alpha_k \mathcal{X}_{E_k}(x,y) \right) \mathrm{d}x\,\mathrm{d}y && (14.2e)\\
&= \lim_{n\to\infty} \int \left(\int \phi_n(x,y)\,\mathrm{d}x \right) \mathrm{d}y && (14.2f)\\
&= \int \lim_{n\to\infty} \left(\int \phi_n(x,y)\,\mathrm{d}x \right) \mathrm{d}y && (14.2g)\\
&= \int \int \lim_{n\to\infty} \phi_n(x,y)\,\mathrm{d}x\,\mathrm{d}y && (14.2h)\\
&= \int \int F(x,y)\,\mathrm{d}x\,\mathrm{d}y && (14.2i)
\end{aligned}$$

where (14.2a) is by the MCT I on ϕ_n; (14.2c) is by the linearity of integral; (14.2d) is by proposition (14.1) (14.2e) is by the linearity of integral; (14.2g) is by the MCT I on

$f_n(y) = \int \phi_n(x,y)\,\mathrm{d}x$; (14.2h) is by the MCT I on $g_n(x) = \phi_n(x,y)$; (14.2i) is because that $\phi_n(x,y) \to F(x,y)$. ■

Theorem 14.4 — Fubini's Theorem. Suppose that $F : \mathbb{R}^2 \to [-\infty, \infty]$ is integrable, then

$$\int F\,\mathrm{d}\pi = \int \left(\int F(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int \left(\int F(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

Proof. Suppose $F = F^+ - F^-$, where $F^\pm$ are both integrable. Applying Tonell's theorem on both F^- and F^+ and the linearity of integrals gives the desired result. ■