2.3. Monday for MAT4002

Reviewing.

1. Topological Space $(X, \mathcal{J})$: a special class of topological space is that induced from metric space (X, d):
$$(X, \mathcal{T}), \quad \text{with } \mathcal{T} = \{\text{all open sets in } (X, d)\}$$
2. Closed Sets $(X \setminus U)$ with U open.

Proposition 2.8 Let $(X, \mathcal{T})$ be a topological space,

1. $\emptyset, X$ are closed in X
2. V_1, V_2 closed in X implies that $V_1 \bigcup V_2$ closed in X
3. $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ closed in X implies that $\bigcap_{\alpha \in \mathcal{A}} V_\alpha$ closed in X

Proof. Applying the De Morgan's Law
$$(X \setminus \bigcup_{i \in I} U_i) = \bigcap_{i \in I} (X \setminus U_i)$$
■

2.3.1. Convergence in topological space

Definition 2.4 [Convergence] A sequence $\{x_n\}$ of a topological space $(X, \mathcal{T})$ converges to $x \in X$ if $\forall U \ni x$ is open, there $\exists N$ such that $x_n \in U, \forall n \geq N$.

Example 2.9 1. The topology for the space $(X = \mathbb{R}^n, d_2) \to (X, \mathcal{T})$ (i.e., a topological space induced from meric space $(X = \mathbb{R}^n, d_2)$) is called a **usual topology** on $\mathbb{R}^n$.

When I say $\mathbb{R}^n$ (or subset of $\mathbb{R}^n$) is a topological space, it is equipeed with usual topology.

Convergence of sequence in $(\mathbb{R}^n, \mathcal{T})$ is the usual convergence in analysis.

For $\mathbb{R}^n$ or metric space, the limit of sequence (if exists) is unique.

2. Consider the topological space $(X, \mathcal{T}_{\text{indiscrete}})$. Take any sequence $\{x_n\}$ in X, it is convergent to any $x \in X$. Indeed, for $\forall U \ni x$ open, $U = X$. Therefore,

$$x_n \in U(= X), \forall n \geq 1.$$

3. Consider the topological space $(X, \mathcal{T}_{\text{cofinite}})$, where X is infinite. Consider $\{x_n\}$ is a sequence satisfying $m \neq n$ implies $x_m \neq x_n$. Then $\{x_n\}$ is convergent to any $x \in X$. (Question: how to define openness for $\mathcal{T}_{\text{cofinite}}$ and $\mathcal{T}_{\text{indiscrete}}$)?
4. Consider the topological space $(X, \mathcal{T}_{\text{discrete}})$, the sequence $\{x_n\} \to x$ is equivalent to say $x_n = x$ for all sufficiently large n.

R The limit of sequences may not be unique. The reason is that "$\mathcal{T}$ is not big enough". We will give a criterion to make sure the limit is unique in the future. (Hausdorff)

Proposition 2.9 If $F \subseteq (X, \mathcal{T})$ is closed, then for any convergent sequence $\{x_n\}$ in F, the limit(s) are also in F.

Proof. Let $\{x_n\}$ be a sequence in F with limit $x \in X$. Suppose on the contrary that $x \notin F$ (i.e., $x \in X \setminus F$ that is open). There exists N such that

$$x_n \in X \setminus F, \forall n \geq N,$$

i.e., $x_n \notin F$, which is a contradiction. ■

R The converse may not be true. If the $(X, \mathcal{T})$ is metrizable, the converse holds. Counter-example: Consider the co-countable topological space $(X, \mathcal{T}_{\text{co-co}})$, where

$$\mathcal{T}_{\text{co-co}} = \{U \mid X \setminus U \text{ is a countable set}\} \bigcup \{\emptyset\},$$

and X is uncontable. Let $F \subsetneqq$ be an un-countable set such that is closed under limits, e.g., $[0,1]$. It's clear that $X \setminus F \notin \mathcal{T}_{\text{co-co}}$, i.e., F is not closed.

2.3.2. Interior, Closure, Boundary

Definition 2.5 Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.

1. The **interior** of A is

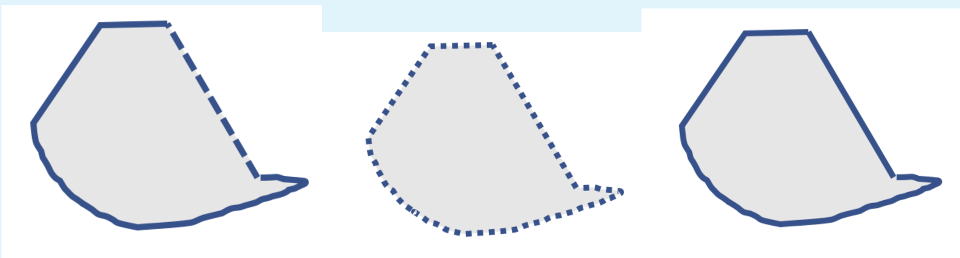
$$A^\circ = \bigcup_{U \subseteq A, U \text{ is open}} U$$

2. The **closure** of A is

$$\overline{A} = \bigcap_{A \subseteq V, V \text{ is closed}} V$$

If $\overline{A} = X$, we say that A is dense in X.

The graph illustration of the definition above is as follows:



(a) Illustration of A (b) Illustration of A° (c) Illustration of $\overline{A}$

Figure 2.1: Graph Illustrations

■ **Example 2.10**

1. For $[a,b) \subseteq \mathbb{R}$, we have:

$$[a,b)^\circ = (a,b), \quad \overline{[a,b)} = [a,b]$$

2. For $X = \mathbb{R}$, $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$.
3. Consider the discrete topology $(X, \mathcal{T}_{\text{discrete}})$, we have

$$S^\circ = S, \quad \overline{S} = S$$

The insights behind the definition (2.5) is as follows

Proposition 2.10 1. A° is the largest open subset of X contained in A;

$\overline{A}$ is the smallest closed subset of X containing A.

2. If $A \subseteq B$, then $A^\circ \subseteq B$ and $\overline{A} \subseteq \overline{B}$
3. A is open in X is equivalent to say $A^\circ = A$; A is closed in X is equivalent to say $\overline{A} = A$.

■ **Example 2.11** Let (X,d) be a metric space. What's the closure of an open ball $B_r(x)$?

The direct intuition is to define the closed ball

$$\bar{B}_r(x) = \{y \in X \mid d(x,y) \le r\}.$$

Question: is $\bar{B}_r(x) = \overline{B_r(x)}$?

1. Since $\bar{B}_r(x)$ is a closed subset of X, and $B_r(x) \subseteq \bar{B}_r(x)$, we imply that

$$\overline{B_r(x)} \subseteq \bar{B}_r(x)$$

2. However, we may find an example such that $\overline{B_r(x)}$ is a proper subset of $\bar{B}_r(x)$: Consider the discrete metric space (X, d_{discrete}) and for $\forall x \in X$,

$$B_1(x) = \{x\} \implies \overline{B_1(x)} = \{x\}, \quad \bar{B}_1(x) = X$$

The equality $\bar{B}_r(x) = \overline{B_r(x)}$ holds when (X,d) is a normed space.

Here is another characterization of $\overline{A}$:

Proposition 2.11

$$\overline{A} = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \emptyset\}$$

Proof. Define

$$S = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \emptyset\}$$

It suffices to show that $\overline{A} = S$.

1. First show that S is closed:

$$X \setminus S = \{x \in X \mid \exists U_x \ni x \text{ open s.t. } U_x \bigcap A = \emptyset\}$$

Take $x \in X \setminus S$, we imply there exists open $U_x \ni x$ such that $U_x \bigcap A = \emptyset$. We claim $U_x \subseteq X \setminus S$:

- For $\forall y \in U_x$, note that $U_x \ni y$ that is open, such that $U_x \bigcap A = \emptyset$. Therefore, $y \in X \setminus S$.

Therefore, we have $x \in U_x \subseteq X \setminus S$ for any $\forall x \in X \setminus S$.

Note that

$$X \setminus S = \bigcup_{x \in X \setminus S} \{x\} \subseteq \bigcup_{x \in X \setminus S} U_x \subseteq X \setminus S,$$

which implies $X \setminus S = \bigcup_{x \in X \setminus S} U_x$ is open, i.e., S is closed in X.

2. By definition, it is clear that $A \subseteq S$:

$$\forall a \in A, \forall \text{open } U \ni a, U \bigcap A \supseteq \{a\} \neq \emptyset \implies a \in S.$$

Therefore, $\overline{A} \subseteq \overline{S} = S$.

3. Suppose on the contrary that there exists $y \in S \setminus \overline{A}$.

Since $y \notin \overline{A}$, by definition, there exists $F \supseteq A$ closed such that $y \notin F$.

Therefore, $y \in X \setminus F$ that is open, and

$$(X \setminus F) \bigcap A \subseteq (X \setminus A) \bigcap A = \emptyset \implies y \notin S,$$

which is a contradiction. Therefore, $S = \overline{A}$.

■

Definition 2.6 [accumulation point] Let $A \subseteq X$ be a subset in a topological space. We call $x \in X$ are an **accumulation point (limit point)** of A if

$$\forall U \subseteq X \text{ open s.t. } U \ni x, (U \setminus \{x\}) \bigcap A \neq \emptyset.$$

The set of accumulation points of A is denoted as A' ■

Proposition 2.12 $\overline{A} = A \bigcup A'$.