10.3. Monday for MAT4002

Proposition 10.6 Let K and L be two simplifical complexes, and $f:|K|\to|L|$ be a continuous mapping. If there exists a simplicial mapping $g:K\to L$ such that $f(\mathrm{st}_K(\boldsymbol{v}))\subseteq \mathrm{st}_L(g(\boldsymbol{v})),\forall \boldsymbol{v}\in V(K)$, then

$$|g|\cong f$$

Recall the definition

$$\mathrm{st}_K(\boldsymbol{v})=\bigcup\{\mathrm{inside}(\sigma):\sigma\text{ is a simplex of }|K|\text{ and }x\in\sigma\}$$

Proof.

- We first show a statement: Suppose that $\sigma=\{v_0,\dots,v_n\}\in\Sigma(K)$, and $x\in\mathrm{inside}(\sigma)\subseteq|K|$. If $f(x)\in|L|$ lies in the inside of the (unique) simplex $\tau\in\Sigma_L$, then $g(v_0),\dots,g(v_n)$ are vertices of τ.

  By definition of inside(σ), $x=\sum_{i=0}^n\alpha_iv_i$ with $\alpha_i>0$ and $\sum_{i=1}^n\alpha_i=1$. Therefore, $x\in\mathrm{st}_K(v_i)$ for $i=1,\dots,n$, where

$$\mathrm{st}_K(v_i):=\left\{av_i+\sum_{j=1}^m b_jw_j\mid a>0,b_j>0,a+\sum_{j=1}^m b_j=1,\{v_i,w_1,\dots,w_m\}\in\Sigma_K\right\}.$$

  Therefore, $f(x)\in\mathrm{int}(\mathrm{st}_K(v_i))\subseteq\mathrm{st}_L(g(v_i))$, which follows that

$$f(x)=ag(v_i)+\sum_{j=1}^m b_ju_j,\text{ where }a>0,b_j>0,a+\sum_{j=1}^m b_j=1,\ \{g(v_i),u_1,\dots,u_m\}\in\Sigma_L$$

  Therefore, $g(v_i)$ is a vertex of the simplex τ, $i=1,\dots,n$. Moreover, $\{g(v_0),\dots,g(v_n)\}$ spans a simplex, which is a face of τ, and therefore $\{g(v_0),\dots,g(v_n)\}\in\Sigma_L$.

- Therefore, the mapping $g:K\to L$ maps simplicies to simplicies, which is a simplicial mapping. We can construct a homotopy between f and $|g|$ as follows: Consider any $x\in|K|$, and let $\tau\in\Sigma_L$ be such that $f(x)\in\mathrm{inside}(\tau)$. We write $x=\sum_{i=0}^n\lambda_iv_i$ for some $\{v_0,\dots,v_n\}\in\Sigma_K$ and $\lambda_i\ge0,\sum_{i=1}^n\lambda_i=1$. Applying our claim,

$$|g|(x)=\sum_{i=0}^n\lambda_ig(v_i),$$

where $g(v_0), \ldots, g(v_n)$ are all vertices of τ.

We can directly construct a homotopy between f and $|g|$. Before that, we need some reformulations. Since $f(x) \in \text{inside}(\tau)$, we let $f(x) = \sum_{i=0}^{m} \mu_i \tau_i$. Since $|g|(x) = \sum_{i=0}^{n} \lambda_i g(v_i) \in \text{inside}(\tau)$, we rewrite $|g|(x) = \sum_{i=0}^{m} \lambda_i' \tau_i$. We define the map

$$\begin{aligned} H: \quad & |K| \times I \longrightarrow |L| \\ \text{with} \quad & (x,t) \mapsto \textstyle\sum_{i=0}^{m} t\lambda_i' + (1-t)\mu_i \end{aligned}$$

which follows that $f \simeq |g|$.

■

Theorem 10.2 — **Simplicial Approximation Theorem.** Let K, L be simplicial complexes with V_K finite, and $f : |K| \to |L|$ be continuous. Then there exists a subdivison $|K'|$ of $|K|$ together with a simplicial map g such that $|g| \simeq f$.

Here the way for constructing subdivison $|K'|$ is as follows. There exists a constant $\delta > 0$. As long as the coarseness of K' is less than δ, our constructed subdivision satisfies the condition.

Proof. The sets $\{\text{st}_L(w) \mid w \in V(L)\}$ forms an open cover of $|L|$, which implies $\{f^{-1}(\text{st}_L(w))\}$ forms an open cover of $|K|$. By compactness, there exists a finite subcover of $|K|$, denoted as

$$|K| \subseteq \bigcup_{i=1}^{n} f^{-1}(\text{st}_L(w_i))$$

There exists a small number $\delta > 0$ such that for any $x, y \in |K|$ with $d(x,y) < \delta$, $x, y \in f^{-1}(\text{st}_L(w_i))$ for some i. Then we construct a simplicial subdivision $|K'|$ of $|K|$ with coarseness less than δ, i.e., $\forall x, y \in \text{st}_{K'}(v)$, $d(x,y) < \delta$.

Therefore, $\text{st}_{K'}(v) \subseteq f^{-1}(\text{st}_L(w_i))$ for any $v \in V(K;)$ and some $w_i \in V(L)$, i.e., $f(\text{st}_{K'}(v)) \subseteq \text{st}_L(w_i)$.

Setting $g(v) = w_i$ and applying proposition (10.6) gives the desired result. ■

10.3.1. Group Presentations

Group is a highlight of our course, which interwises topology and algebra. I assume that most students have learnt abstract algebra course MAT3004, and encourage those without this knowledge to read the notes for group posted on blackboard.