Sinkhorn Distributionally Robust Optimization

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Contributions

- Distributionally robust optimization with entropic regularized Wasserstein distance (Sinkhorn distance).
- Ambiguity set contains only absolutely continuous distributions.
- Computationally efficient first-order optimization algorithm.

Decision-Making Under Uncertainty

- Objective: Find decision θ to minimize the risk

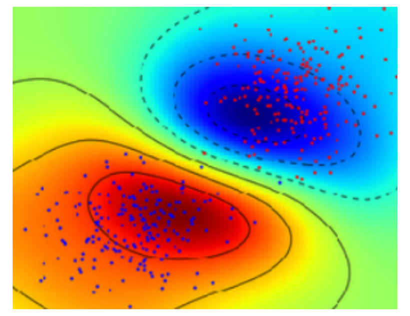
$$\mathcal{R}(\theta;\mathbb{P}) = \mathbb{E}_{\mathbb{P}}[f_\theta(z)].$$

- Available Information:

Structual : $\mathbb{P}$ is supported on $\Omega \subseteq \mathbb{R}^d$
Statistical : $\hat{x}_1,\ldots,\hat{x}_n \sim \mathbb{P}$

Supply Chain Mgmt.

Portfolio Mgmt.



Machine Learning

- Sample Average Approximation (SAA):

$$\inf_{\theta\in\Theta}\ \{\mathcal{R}(\theta;\widehat{\mathbb{P}}_n) \triangleq \mathbb{E}_{\widehat{\mathbb{P}}_n}[f_\theta(z)]\},\quad \text{where } \widehat{\mathbb{P}}_n = \tfrac{1}{n}\textstyle\sum_{i=1}^n \delta_{\hat{x}_i}.$$

- Wasserstein Distributionally Robust Optimization (DRO):

$$\inf_{\theta\in\Theta}\left\{\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[f_\theta(z)]\right\},\quad \text{where } \mathcal{P} = \{\mathbb{P}:\ W(\widehat{\mathbb{P}},\mathbb{P}) \le \rho\}.$$

- Facts about Wasserstein DRO:
  - For WDRO with n-point nominal distribution, the worst-case distribution is supported on $n+1$ points.
  - Finite-dimensional convex reformulation is available if the objective is a pointwise maximum of finitely many concave functions.
  - Some cases the same performance as SAA.

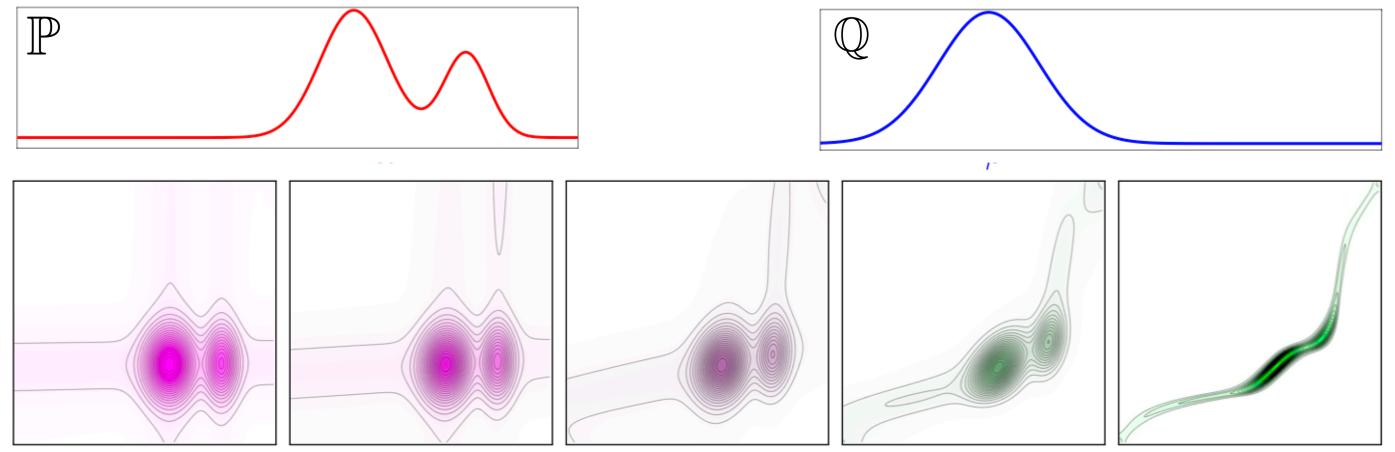
Sinkhorn Distance and Robust Formulation

- Sinkhorn Distance [Cuturi 2013]:

$$\mathcal{W}_\varepsilon(\mathbb{P},\mathbb{Q}) = \inf_{\gamma\in\Gamma(\mathbb{P},\mathbb{Q})} \left\{\mathbb{E}_{(X,Y)\sim\gamma}[c(X,Y)] + \varepsilon H(\gamma \mid \mathbb{P}\otimes\nu)\right\}.$$

- Relative Entropy between γ and $\mathbb{P}\otimes\nu$:

$$H(\gamma \mid \mathbb{P}\otimes\nu) = \int \log\left(\frac{\mathrm{d}\gamma(x,y)}{\mathrm{d}\mathbb{P}(x)\,\mathrm{d}\nu(y)}\right)\mathrm{d}\gamma(x,y).$$



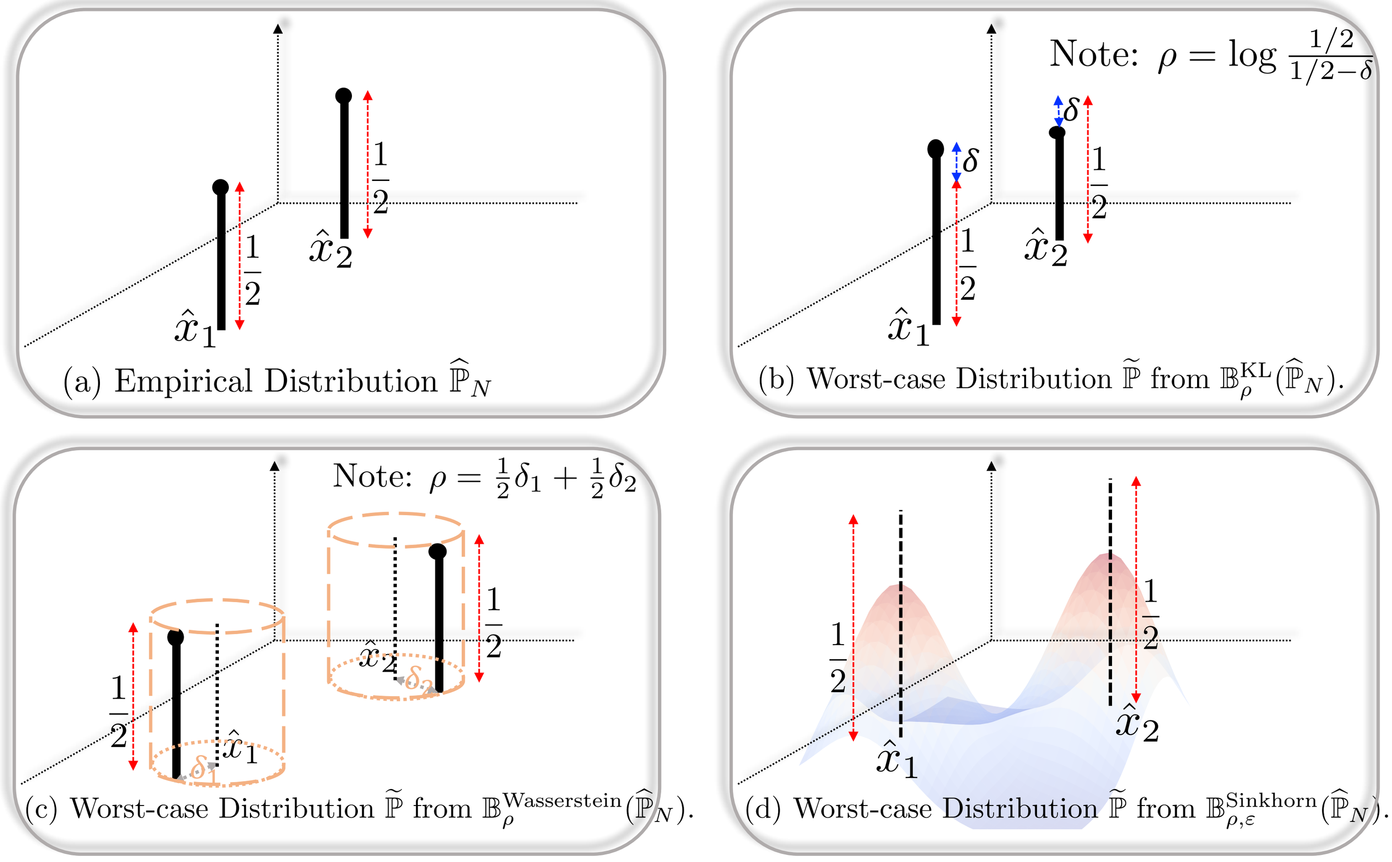
$\varepsilon = 10$ $\varepsilon = 1$ $\varepsilon = 0.5$ $\varepsilon = 10^{-1}$ $\varepsilon = 10^{-2}$

- Sinkhorn DRO:

$$V^* = \inf_{\theta\in\Theta}\ \sup_{\mathbb{P}\in\mathbb{B}_{\rho,\varepsilon}(\widehat{\mathbb{P}})} \mathbb{E}_{z\sim\mathbb{P}}[f_\theta(z)],$$

where $\mathbb{B}_{\rho,\varepsilon}(\widehat{\mathbb{P}}) = \{\mathbb{P}:\ W_\varepsilon(\widehat{\mathbb{P}},\mathbb{P}) \le \rho\}$.

- Visualization of Worst-Case Distributions:



(a) Empirical Distribution $\widehat{\mathbb{P}}_N$ (b) Worst-case Distribution $\widetilde{\mathbb{P}}$ from $\mathbb{B}_\rho^{\mathrm{KL}}(\widehat{\mathbb{P}}_N)$. (c) Worst-case Distribution $\widetilde{\mathbb{P}}$ from $\mathbb{B}_\rho^{\mathrm{Wasserstein}}(\widehat{\mathbb{P}}_N)$. (d) Worst-case Distribution $\widetilde{\mathbb{P}}$ from $\mathbb{B}_{\rho,\varepsilon}^{\mathrm{Sinkhorn}}(\widehat{\mathbb{P}}_N)$.

Theorem: Strong Dual Reformulation

Assume that

- $\nu\{z:\ 0\le c(x,z)<\infty\} = 1$ for $\widehat{\mathbb{P}}$-almost every x;
- $\int e^{-c(x,z)/\varepsilon}\,\mathrm{d}\nu(z) < \infty$ for $\widehat{\mathbb{P}}$-almost every x;
- $\mathcal{Z}$ is a measurable space, and the function $f:\ \mathcal{Z}\to\mathbb{R}\cup\{\infty\}$ is measurable.

Then $V_{\mathsf{P}} = V_{\mathsf{D}}$:

$$V_{\mathsf{P}} = \sup_{\mathbb{P}}\ \left\{\mathbb{E}_{z\sim\mathbb{P}}[f(z)]:\quad W_\varepsilon(\widehat{\mathbb{P}},\mathbb{P})\le\rho\right\},$$

$$V_{\mathsf{D}} = \inf_{\lambda>0}\ \lambda\overline{\rho} + \lambda\varepsilon\int_\Omega \log\left(\mathbb{E}_{\mathbb{Q}_x}\left[e^{f(z)/(\lambda\varepsilon)}\right]\right)\mathrm{d}\widehat{\mathbb{P}}(x),$$

where

$$\overline{\rho} = \rho + \varepsilon\int_\Omega \log\left(\int_\Omega e^{-c(x,z)/\varepsilon}\,\mathrm{d}\nu(z)\right)\mathrm{d}\widehat{\mathbb{P}}(x),$$

$$\mathrm{d}\mathbb{Q}_x(z) = \frac{e^{-c(x,z)/\varepsilon}}{\int_\Omega e^{-c(x,u)/\varepsilon}\,\mathrm{d}\nu(u)}\,\mathrm{d}\nu(z).$$

Proof Sketch of Strong Duality

1. First show the weak duality result $V_{\mathsf{P}} \le V_{\mathsf{D}}$.
2. Construct primal feasible solution $\widetilde{\mathbb{P}}$ with $V_{\mathsf{P}} \ge \mathbb{E}_{z\sim\widetilde{\mathbb{P}}}[f(z)] = V_{\mathsf{D}}$.

Geometry of Worst-Case Distribution:

- For each $x\in\mathrm{supp}(\widehat{\mathbb{P}})$, optimal transport maps it to a (conditional) distribution γ_x:

$$\frac{\mathrm{d}\gamma_x(z)}{\mathrm{d}\nu(z)} = \alpha_x\cdot\exp\left(\left(f(z)-\lambda^* c(x,z)\right)/(\lambda^*\varepsilon)\right).$$

- Worst-case distribution $\widetilde{\mathbb{P}} = \int \gamma_x\,\mathrm{d}\widehat{\mathbb{P}}(x)$.

Algorithm for Sinkhorn Robust Learning

$$V^* = \min_{\lambda\ge0}\left\{\lambda\overline{\rho} + \underbrace{\min_{\theta\in\Theta}\ \mathbb{E}_{x\sim\widehat{\mathbb{P}}}\left[\lambda\varepsilon\log\left(\mathbb{E}_{z\sim\mathbb{Q}_x}\left[e^{f_\theta(z)/(\lambda\varepsilon)}\right]\right)\right]}_{V(\lambda)}\right\}$$

Bisection Search on λ: Estimating $V(\lambda)$ up to accuracy $O(\delta)$ for $O(\mathrm{Poly}(\log\frac{1}{\delta}))$ times to find δ-optimal solution of V^*.

Stochastic Approximation for Solving $V(\lambda)$

- Goal: to solve the optimization

$$\min_{\theta\in\Theta}\ \left\{F(\theta) := \mathbb{E}_{x\sim\widehat{\mathbb{P}}}\left[\lambda\varepsilon\log\left(\mathbb{E}_{z\sim\mathbb{Q}_x}\left[e^{f_\theta(z)/(\lambda\varepsilon)}\right]\right)\right]\right\}.$$

- Biased Stochastic Mirror Descent (BSMD): for $t=1,\ldots,T$,

$$\begin{cases} v(\theta_t) \leftarrow \text{(biased) gradient/subgradient estimate of } F(\theta_t) \\ \theta_{t+1} \leftarrow \mathbf{Prox}_{\theta_t}(\gamma v(\theta_t)) \end{cases}$$

Remark: Gradient estimators should optimally balance the bias-variance trade-off.

- Complexity of finding δ-optimal solution or δ-critical point:

Estimators	Convex Nonsmooth	Convex Smooth	Nonconvex Smooth
Vanilla SGD	$O(\delta^{-3})$	$O(\delta^{-3})$	$O(\delta^{-6})$
V-MLMC	N/A	$\tilde{O}(\delta^{-2})$	$\tilde{O}(\delta^{-4})$
RT-MLMC	N/A	$\tilde{O}(\delta^{-2})$	$\tilde{O}(\delta^{-4})$

Example: Mean-Risk Portfolio Optimization