7.3. Monday for MAT4002

7.3.1. Quotient Map

Definition 7.6 [Quotient Map] A mapping $q : X \to Y$ between topological spaces is a **quotient map** if

1. q is surjective
2. For any $U \subseteq Y$, U is open iff $q^{-1}(U)$ is open.

R

1. The canonical projection mapping $p : X \to X/\sim$ is a quotient mapping
2. We say f is an open mapping if U is open in X implies $f(U)$ is open in Y. Note that a continuous open mapping satisfies condition (2) in definition (7.6).

In proposition (6.5) we show the homeomorphism between $X/\sim$ and Y given the compactness of X and Hausdorffness of Y. Now we show the homeomorphism by replacing these conditions with the quotient mapping q:

Proposition 7.9 Suppose $q : X \to Y$ is a quotient map, and that $\sim$ is an equivalence relation on X given by the partition $\{q^{-1}(y) \mid y \in Y\}$. Then $X/\sim$ and Y are **homeomorphic**.

Proof. Construct the mapping

$$\begin{aligned} h: \quad & X/\sim \to Y \\ \text{with} \quad & h([x]) = q(x) \end{aligned}$$

Note that:

1. The mapping h is well-defined and injective.
2. Surjective is easy to shown.
3. The quotient mapping $q := h \circ p$, by definition, is continuous. By applying proposition (6.4), h is continuous.

It suffices to show h^{-1} is continuous:

- For any open $\tilde{U} \subseteq X/\sim$, it suffices to show $h(\tilde{U})$ is open in Y.

  Note that
  $$q^{-1}(h(\tilde{U})) = p^{-1}h^{-1}(h(\tilde{U})) = p^{-1}(\tilde{U}),$$
  which is open by the definition of quotient topology (check proposition (6.1)).

  Therefore, $h(\tilde{U})$ is open by (2) in definition (7.6).

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■ **Example 7.4** The $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the unit circle S^1:

Define the mapping
$$\begin{aligned} q: \quad \mathbb{R} &\to S^1 \\ x &\mapsto e^{2\pi i x} \end{aligned}$$

It's clear that

1. q is a continuous open mapping (why?)
2. q is surjective

Therefore, $\mathbb{R}/\sim \cong S^1$, provided that $x \sim y$ iff $q(x) = q(y)$, i.e., $x - y \in \mathbb{Z}$. Therefore,
$$\mathbb{R}/\mathbb{Z} \cong S^1$$

■

7.3.2. Simplicial Complex

Combinatorics is the slums of topology. — J. H. C. Whitehead

The idea is to build some new spaces from some "fundamental" objects. The combinatorialists often study topology by the combinatorics of these fundamental objects. First we define what are the "fundamental" objects:

Definition 7.7 [n-simplex] The standard n-simplex is the set

$$\Delta^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \middle| x_i \geq 0, \forall i \text{ and } \sum_{i=1}^{n+1} x_i = 1 \right\}$$

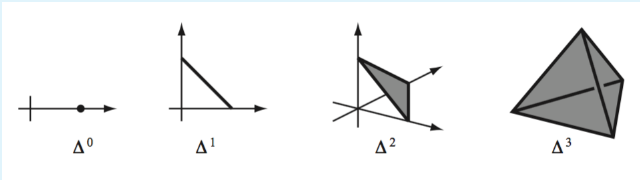


Figure 7.1: Simplices on $\mathbb{R}^2$ are the triangles, so you may consider simplexes as the "triangles" in general spaces

1. The non-negative integer n is the **dimension** of this simplex
2. Its **vertices**, denoted as $V(\Delta^n)$, are those points $(x_1, \ldots, x_{n+1})$ in Δ^n such that $x_i = 1$ for some i.
3. For each given non-empty $\mathcal{A} \subseteq \{1, \ldots, n+1\}$, its **face** is defined as

$$\{(x_1, \ldots, x_{n+1}) \in \Delta^n \mid x_i = 0, \ \forall i \notin \mathcal{A}\}$$

   In particular, Δ^n is a face of itself
4. The **inside** of Δ^n is

$$\text{inside}(\Delta^n) := \{(x_1, \ldots, x_{n+1}) \in \Delta^n \mid x_i > 0, \forall i\}$$

   In particular, the inside of Δ^0 is Δ^0.

Definition 7.8 [Face Inclusion] A face inclusion of Δ^m into Δ^n $(m < n)$ is a function $\Delta^m \to \Delta^n$ which comes from the restriction of an **injective linear** map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that maps vertices in Δ^m into vertices in Δ^n.

For example, the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined below is a face inclusion:

$$f(1,0) = (0,1,0), \quad f(0,1) = (0,0,1).$$

R Any injection mapping from $\{1, \ldots, m+1\} \to \{1, \ldots, n+1\}$ gives a face inclusion $\Delta^m \to \Delta^n$, and vice versa.

Motivation. Now we build new spaces by making use of simplices. This new space is called the **abstract complex**. If a simplex is a part of the complex, so are all its faces.

Definition 7.9 [Abstract Simplicial Complex] An (abstract) **simplicial complex** is a pair $K = (V, \Sigma)$, where V is a set of vertices and Σ is a collection of non-empty finite subsets of V (simplices) such that

1. For any $\boldsymbol{v} \in V$, the 1-element set $\{\boldsymbol{v}\}$ is in Σ
2. If σ is an element of Σ, then so is any non-empty subset of σ.

For example, if $V = \{1,2,3,4\}$, then

$$\Sigma = \{\{1\},\{2\},\{3\},\{4\},\{1,3,4\},\{2,4\},\{1,3\},\{3,4\},\{1,4\}\}$$

We can associate to an abstract simplicial complex K a topological space $|K|$, which is called its **geometric realization**:

Definition 7.10 [Topological Realization] The **topological realization** of $K = (V, \Sigma)$ is a topological space $|K|$ (or denoted as $|(V, \Sigma)|$), where

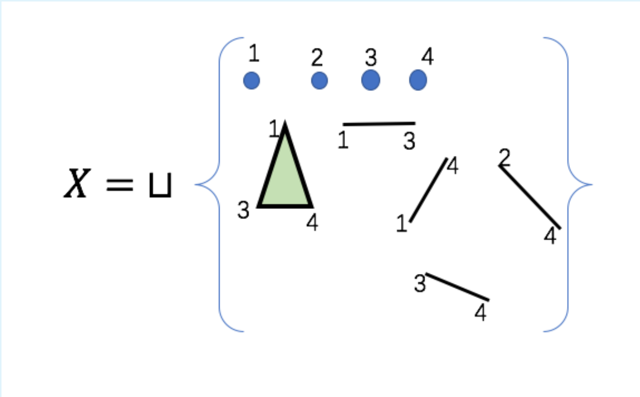
1. For each $\sigma \in \Sigma$ with $|\sigma| = n+1$, take a copy of n-simplex and denote it as Δ_σ
2. Whenever $\sigma \subset \tau \in \Sigma$, identify Δ_σ with a face of Δ_τ through face inclusion.

R Or equivalently, $|K|$ is a quotient space of the disjoint union

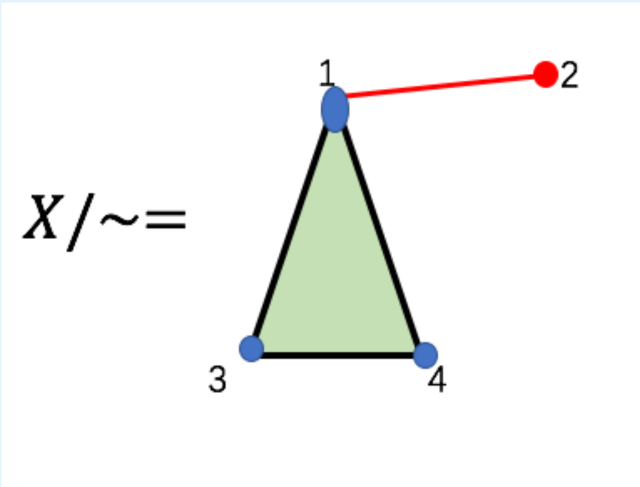
$$\coprod_{\sigma \in \Sigma} \sigma$$

by the equivalence relation which identifies a point $y \in \sigma$ with its image under the face inclusion $\sigma \to \tau$, for any $\sigma \subset \tau$.

■ **Example 7.5** Take



As a result,



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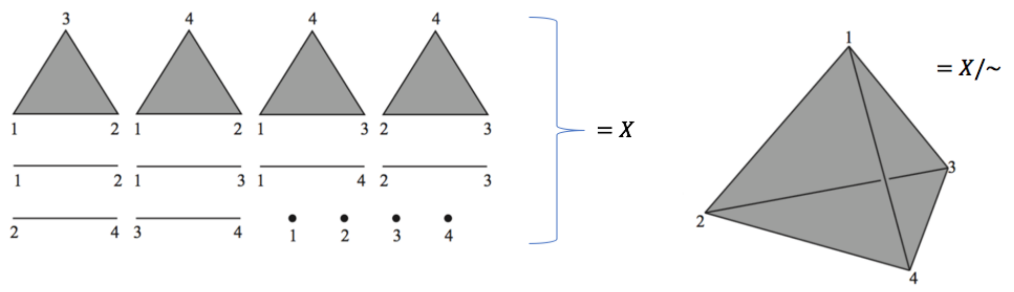
▪ **Example 7.6** Take $V = \{1,2,3,4\}$ and

$$\Sigma = \{\text{all subsets of } V \text{ except } V\}$$

As shown in the figure below, $|(V,\Sigma)| = \Delta^3$:



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Definition 7.11 [Triangulation] A **triangulation** of a topological space X is a simplicial complex $K = (V,\Sigma)$ together with a choice of homeomorphism $|K| \to X$. ▪

▪ **Example 7.7** The triangulation of $S^1 \times S^1$ can be realized by using nine vertices given below:

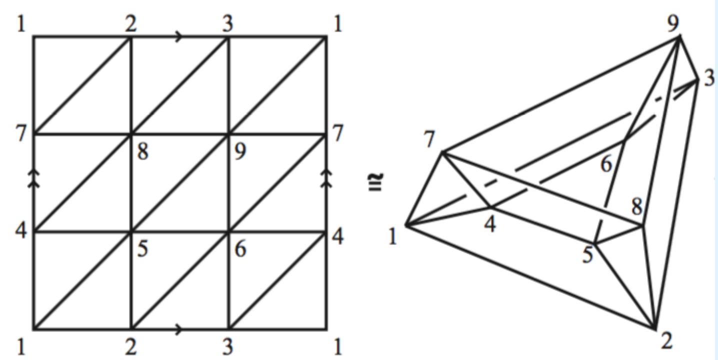

Figure 7.2: The quotient space $|K| := X/\sim$

(Try to identify X) ▪