# Lecture 4
# Basics of Probability

- Probability
- Conditional Probability
- Discrete and Continuous Distributions
- Conditional Expectation and Variance
- Order Statistics

# Contents

# Elements of Probability

- Outcome $(\omega)$: the outcome of an experiment or trial
- Sample space/Probability space $(\Omega)$: the set of all possible outcomes of an experiment

$\Omega \to \circlearrowleft$

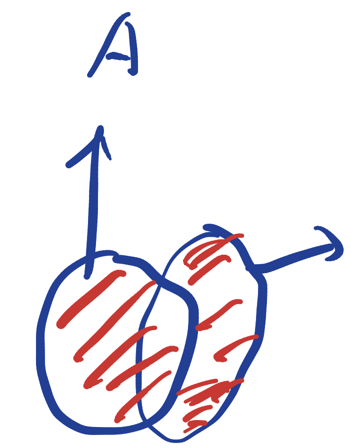
- $P(\omega)$: the probability of an outcome
- $P(\omega) \geq 0, \forall \omega \in \Omega, \sum_{\omega \in \Omega} P(\omega) = 1.$
- Random variable: a function that maps each outcome $(\omega)$ in the sample space $(\Omega)$ into the set of real numbers.

$$X: \Omega \to \{0, 1\}$$

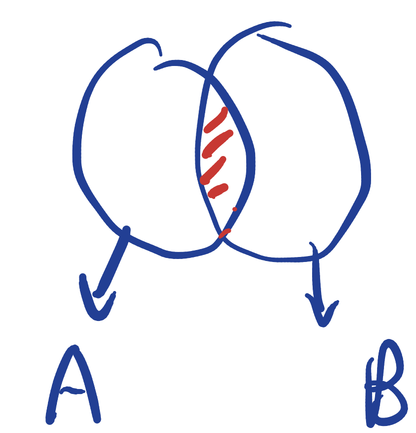
$$X(\omega) = \begin{cases} 0, & \text{if } \omega \text{ is tail} \\ 1, & \text{if } \omega \text{ is head} \end{cases}$$

# Event

- Event: A set of outcomes, a subset of the sample space.
- $P(A)$ :Probability of an event $A$, i.e., $P(A) = \sum_{\omega \in A} P(\omega)$.
- $A \cup B$: set of outcomes in event $A$ or in event $B$ (or both)

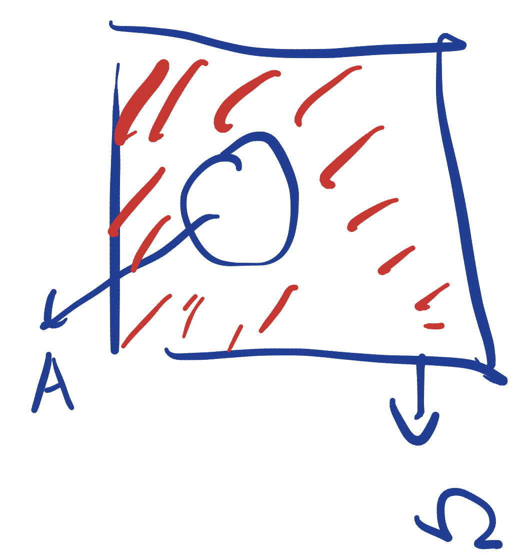



- $A \cap B$: set of outcomes in both $A$ and $B$, also written as $AB$




- $A^c$: the complement of $A$, which is the event "not $A$"
- Mutually exclusive: $A \cap B = \emptyset$.
- For any mutually exclusive events $E_1, \ldots, E_N$,

$$P\Big(\bigcup_{i=1}^{N} E_i\Big) = \sum_{i=1}^{N} P(E_i).$$



# Example

Consider rolling a six-sided dice.

- $\Omega = \{1,2,3,4,5,6\}$, and the probability of each outcome is $1/6$.
- Assume that $A = \{1,3,5\}$. Calculate $A^c, P(A)$.

  $A^c = \{2, 4, 6\}$

  $P(A) = \sum_{\omega \in A} P(\omega) = \frac{3}{6}$

- Let $B$ denote the event that the outcome is larger than $3$. Calculate $A \cup B, A \cap B$.
- Define a special random variable $I_A$ such that

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in \{1,3,5\}, \\ 0, & \text{if } \omega \notin \{1,3,5\}. \end{cases}$$

  Calculate $\mathbb{E}[I_A]$.

# Example

Consider rolling a six-sided dice.

- $\Omega = \{1, 2, 3, 4, 5, 6\}$, and the probability of each outcome is $1/6$.
- Assume that $A = \{1, 3, 5\}$. Calculate $A^c, P(A)$.
- Let $B$ denote the event that the outcome is larger than $3$. Calculate $A \cup B, A \cap B$.

  $B = \{4, 5, 6\}$ $A \cup B = \{1, 3, 4, 5, 6\}$

  $A \cap B = \{5\}$.

- Define a special random variable $I_A$ such that

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in \{1, 3, 5\}, \\ 0, & \text{if } \omega \notin \{1, 3, 5\}. \end{cases}$$

Calculate $\mathbb{E}[I_A]$.

# Example

Consider rolling a six-sided dice.

- $\Omega = \{1, 2, 3, 4, 5, 6\}$, and the probability of each outcome is $1/6$.
- Assume that $A = \{1, 3, 5\}$. Calculate $A^c, P(A)$.
- Let $B$ denote the event that the outcome is larger than $3$. Calculate $A \cup B, A \cap B$.
- Define a special random variable $I_A$ such that

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in \{1, 3, 5\}, \\ 0, & \text{if } \omega \notin \{1, 3, 5\}. \end{cases}$$

Calculate $\mathbb{E}[I_A]$.

$$E[I_A] = \sum_{w \in \Omega} I_A(w) P(w) = \sum_{w \in \{1,3,5\}} \cancel{I_A(w)} P(w)$$

$$= \frac{3}{6} = P(A).$$

# Example

$x + 0.5y = \frac{1}{2}(2x+y) = 0.5$

Two gamblers are playing a coin toss game. Suppose Gambler $A$ has $2$ fair coins and $B$ has $1$ fair coin. What is the probability that $A$ will have more heads than $B$ if both flip all their coins?

$E_1$ : A's first fair coin have more heads than B ↙ $P(E_1) = x$

$E_2$ : - - - - - - have equal head as B ↙ $P(E_2) = y$

$E_3$ : - - - - - - less heads than B

$2x + y = 1 \leftarrow \sum_{w \in \Omega} P(w) = 1$

# Contents

# Definition

- Conditional probability: If $P(B) > 0$, then $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ is the fraction of $B$ outcomes that are also $A$ outcomes.
- Multiplication rule:

$P(A|B) = \frac{P(AB)}{P(B)} \Leftrightarrow P(AB) = P(B)\,P(A|B)$

$$P(E_1 E_2 \cdots E_n)$$

$= P(E_1)\,P(E_2 \cdots E_n | E_1) = P(E_1)\,P(E_2|E_1)\,P(E_3 \cdots E_n | E_1E_2)$

$$= P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 E_2) \cdots P(E_n \mid E_1 \cdots E_{n-1}).$$

- Law of total probability: for any mutually exclusive events $\{F_i\}, i = 1, \ldots, n$ whose union is the entire sample space,

$F_i \cap F_j = \phi$

$\bigcup_{i=1}^{n} F_i = \Omega$

$$P(E) = P(EF_1) + \cdots + P(EF_n)$$

$$= P(E \mid F_1)P(F_1) + \cdots + P(E \mid F_n)P(F_n)$$

$E = E \cap \left( \bigcup_{i=1}^{n} F_i \right)$

$= \bigcup_{i=1}^{n} (E \cap F_i)$

# Definition

- Independent events: $P(EF) = P(E)P(F)$. Also,

$$P(EF^c) = P(E)P(F^c).$$

- Bayes' formula:

$$P(F_j \mid E) = \frac{P(E \mid F_j)P(F_j)}{\sum_{i=1}^{n} P(E \mid F_i)P(F_i)},$$

provided that $F_i, i = 1, \ldots, n$ are mutually exclusive events whose union is the entire sample space.

$\Omega = \{(b,b), (b,g), (g,b), (g,g)\}$

# Example

$A = \{(b,b)\}$ $B = \{(b,b), (b,g), (g,b)\}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$

1. A company is holding a dinner for working mothers **with at least one son**. Ms. Jackson, a mother with two children, is invited. What is the probability that both children are boys?

B

A

2. Your new colleague, Ms. Parker is known to have two children. Suppose you see her walking with **one of her children and that child is a boy**. What is the probability that both children are boys?
3. Part 1 asks given there is at least one boy in two children, what is the conditional probability that both children are boys. Part 2 asks that given one child is a boy, what is the conditional probability that the other child is also a boy.

# Example

1. A company is holding a dinner for working mothers **with at least one son**. Ms. Jackson, a mother with two children, is invited. What is the probability that both children are boys?

   $A = \{(b,b), (b,g)\}$

2. Your new colleague, Ms. Parker is known to have two children. Suppose you see her walking with **one of her children and that child is a boy**. What is the probability that both children are boys?

   A

   B

3. Part 1 asks given there is at least one boy in two children, what is the conditional probability that both children are boys. Part 2 asks that given one child is a boy, what is the conditional probability that the other child is also a boy.

# Example

1. A company is holding a dinner for working mothers **with at least one son**. Ms. Jackson, a mother with two children, is invited. What is the probability that both children are boys?
2. Your new colleague, Ms. Parker is known to have two children. Suppose you see her walking with **one of her children and that child is a boy**. What is the probability that both children are boys?
3. Part 1 asks given there is at least one boy in two children, what is the conditional probability that both children are boys. Part 2 asks that given one child is a boy, what is the conditional probability that the other child is also a boy.

# Example

In a primitive society, let's assume every couple prefers to have a baby boy. Assume there is a $50\%$ chance that each child they have is a boy,and the genders of their children are mutually independent. If each couple insists on having more children until they get a boy and once they have a boy, they will stop having more children. What will eventually happen to the fraction of boys in this society?

# Example

$P(A)$: prior probability $\frac{1}{1000}$

$P(B|A) = 1$  $P(A^c) = \frac{999}{1000}$  $P(B|A^c) = (\frac{1}{2})^{10} = \frac{1}{1024}$

You are given 1000 coins. Among them, $1$ coin has heads on both sides. The other $999$ coins are fair coins. You randomly choose a coin and toss it $10$ times. Each time, the coin turns up heads. What is the probability that the coin you choose is the unfair one?

A: chosen coin is unfair  B: coin turn heads for 10 times

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B|A)}{\underbrace{P(A)P(B|A)}_{P(BA)} + \underbrace{P(A^c)P(B|A^c)}_{P(BA^c)}} = \frac{\frac{1}{1000} \times 1}{\frac{1}{1000} \times 1 + \frac{999}{1000} \times \frac{1}{1024}} \approx \frac{1}{2}$$

# Contents

# Common functions of random variables

proof: $E[(X-E[X])^2] = E[X^2 + (E[X])^2 - 2X E[X]] = E[X^2] + E[(E[X])^2] - E[2X E[X]]$

$= E[X^2] + (E[X])^2 - 2E[X]E[X]$

$= E[X^2] - (E[X])^2$

| Random Variable $X$ | Discrete | Continuous |
|---|---|---|
| Cumulative distribution function (cdf) | $F(a) = P\{X \le a\}$ | $F(a) = \int_{-\infty}^{a} f(x)\mathrm{d}x$ |
| Probability mass function (pmf) or Probability density function (pdf) | $p(x) = P\{X = x\}$ | $f(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x)$ |
| Expected value $\mathbb{E}[X]$ | $\sum_{x:\, p(x)>0} xp(x)$ | $\int_{-\infty}^{\infty} xf(x)\mathrm{d}x$ |
| Expected value of $g(x)$, $\mathbb{E}[g(x)]$ | $\sum_{x:\, p(x)>0} g(x)p(x)$ | $\int_{-\infty}^{\infty} g(x)f(x)\mathrm{d}x$ |
| Variance of $X$, $\mathrm{Var}(X)$ | $\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ | |
| Standard deviation of $X$, $\mathrm{std}(X)$ | $\sqrt{\mathrm{Var}(X)}$ | |

# Common discrete random variables

| Name | Probability mass function (pmf) | $E[X]$ | var($X$) |
|---|---|---|---|
| Uniform | $P(x)=\dfrac{1}{b-a+1}, \quad x=a,a+1,\cdots,b$ | $\dfrac{b+a}{2}$ | $\dfrac{(b-a+1)^2-1}{12}$ |
| Binomial | $P(x)=\dbinom{n}{x}p^x(1-p)^{n-x}, \quad x=0,1,\cdots,n$ | $np$ | $np(1-p)$ |
| Poisson | $P(x)=\dfrac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x=0,1,\cdots$[20] | $\lambda t$ | $\lambda t$ |
| Geometric | $P(x)=(1-p)^{x-1}p, \ x=1,2,\cdots$ | $\dfrac{1}{p}$ | $\dfrac{1-p}{p^2}$ |
| Negative Binomial | $P(x)=\dbinom{x-1}{r-1}p^r(1-p)^{x-r}, \ x=r,r+1,\cdots$ | $\dfrac{r}{p}$ | $\dfrac{r(1-p)}{p^2}$ |

rate $\lambda t$, pmf: $\dfrac{e^{-\lambda t}(\lambda t)^x}{x!}$, $x=0,1,\cdots$.

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

# Common continuous random variables

| Name | Probability density function (pdf) | $E[X]$ | var($X$) |
|---|---|---|---|
| Uniform | $\frac{1}{b-a}$, $a \le x \le b$ | $\frac{b+a}{2}$ | $\frac{(b-a)^2}{12}$ |
| Normal | $\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $x \in (-\infty, \infty)$ | $\mu$ | $\sigma^2$ |
| Exponential | $\lambda e^{-\lambda x}$, $x \ge 0$ | $1/\lambda$ | $1/\lambda^2$ |
| Gamma | $\frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$, $x \ge 0$, $\Gamma(a) = \int_0^\infty e^{-y} y^{a-1}$ | $\alpha/\lambda$ | $\alpha/\lambda^2$ |
| Beta | $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$ |

$f(x) = \lambda e^{-\lambda x}$ is well-defined pdf:

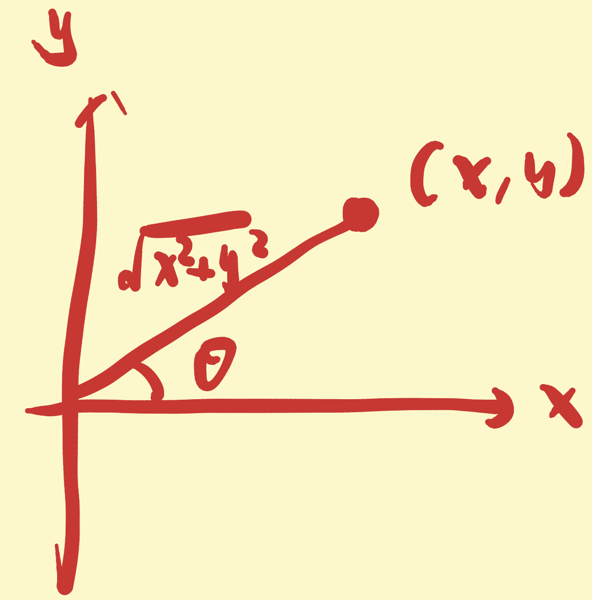
$$\int_0^{+\infty} f(x)\, dx = \int_0^{+\infty} \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_0^{+\infty} = 1$$

proof: $f(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$. $x\in(-\infty,+\infty)$ is well-defined pdf.

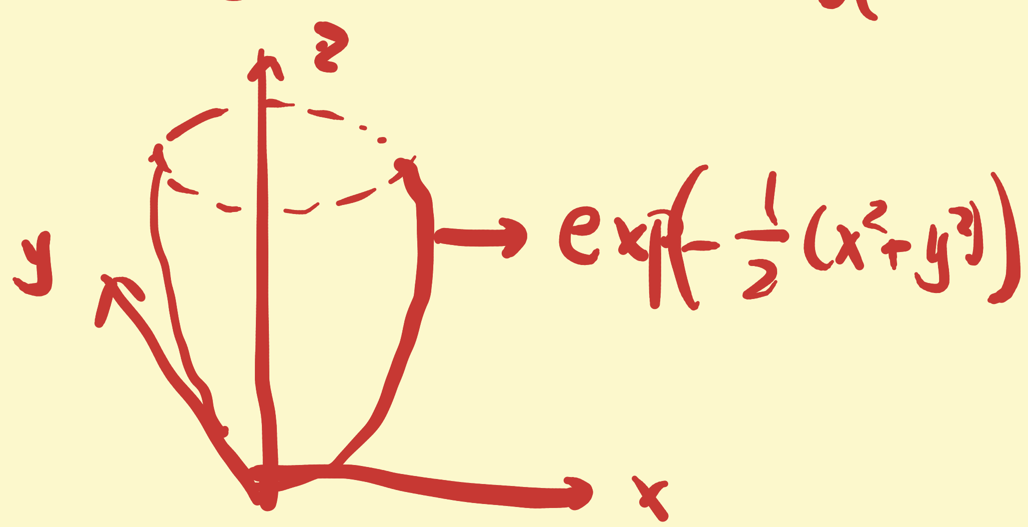
$$\Leftarrow \int_{-\infty}^{+\infty} f(x)\,dx=1 \Leftarrow \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

$$= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cancel{\sigma}}\exp\left(-\frac{1}{2}y^2\right)\cdot\cancel{\sigma}\,dy \qquad y=\frac{x-\mu}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}y^2\right)dy}_{I}$$




$$I^2=\left(\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}y^2\right)dy\right)\left(\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}x^2\right)dx\right)=\int_{R^2}\exp\left(-\frac{1}{2}(x^2+y^2)\right)d(x,y)$$




$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} = \int_0^{2\pi}\int_0^{+\infty}\exp\left(-\frac{1}{2}r^2\right)r\,dr\,d\theta$$

$dxdy=r\,dr\,d\theta$

$$=\int_0^{2\pi}d\theta\int_0^{+\infty}\exp\left(-\frac{1}{2}r^2\right)r\,dr$$

$$=(2\pi)\cdot 1=2\pi$$

# Example

$$E[X \mid X>0] = \int_0^{+\infty} x \cdot f(x \mid X>0)\, dx = \int_0^{+\infty} \frac{2x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} x \exp\left(-\frac{x^2}{2}\right) dx$$

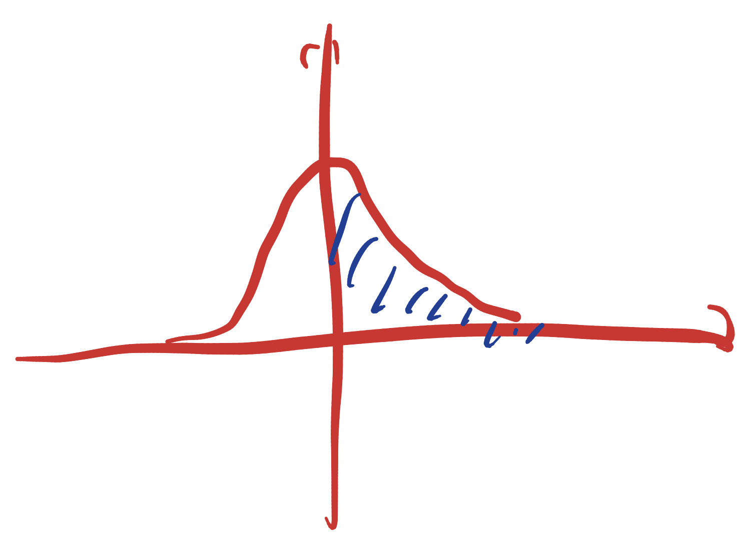
$$f(x \mid X>0) = \frac{f(x, X>0)}{P(X>0)} = \frac{f(x)}{1/2} = 2f(x) = \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in (0, +\infty)$$

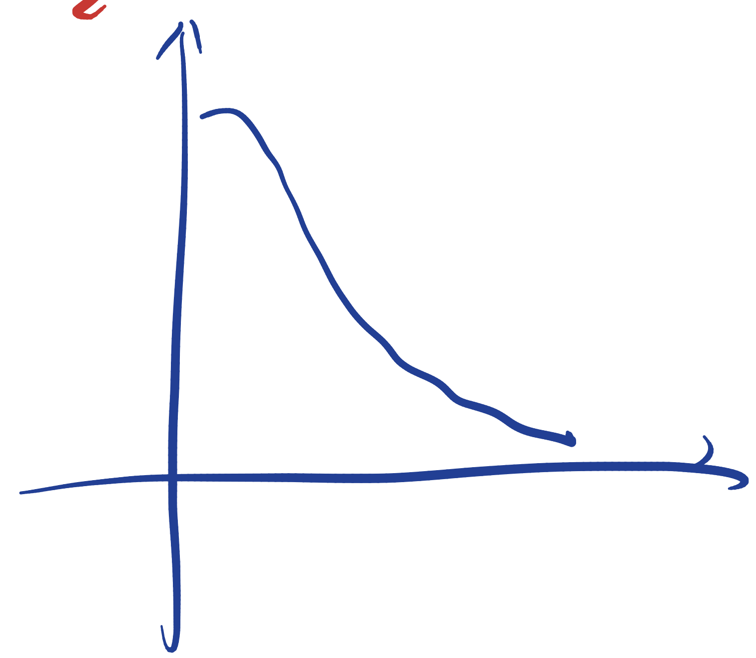
$$= \frac{2}{\sqrt{2\pi}} \left[-\exp\left(-\frac{x^2}{2}\right)\right]_0^{+\infty}$$

$$= \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$$

If $X$ is a standard normal random variable, what is $\mathbb{E}[X \mid X > 0]$?

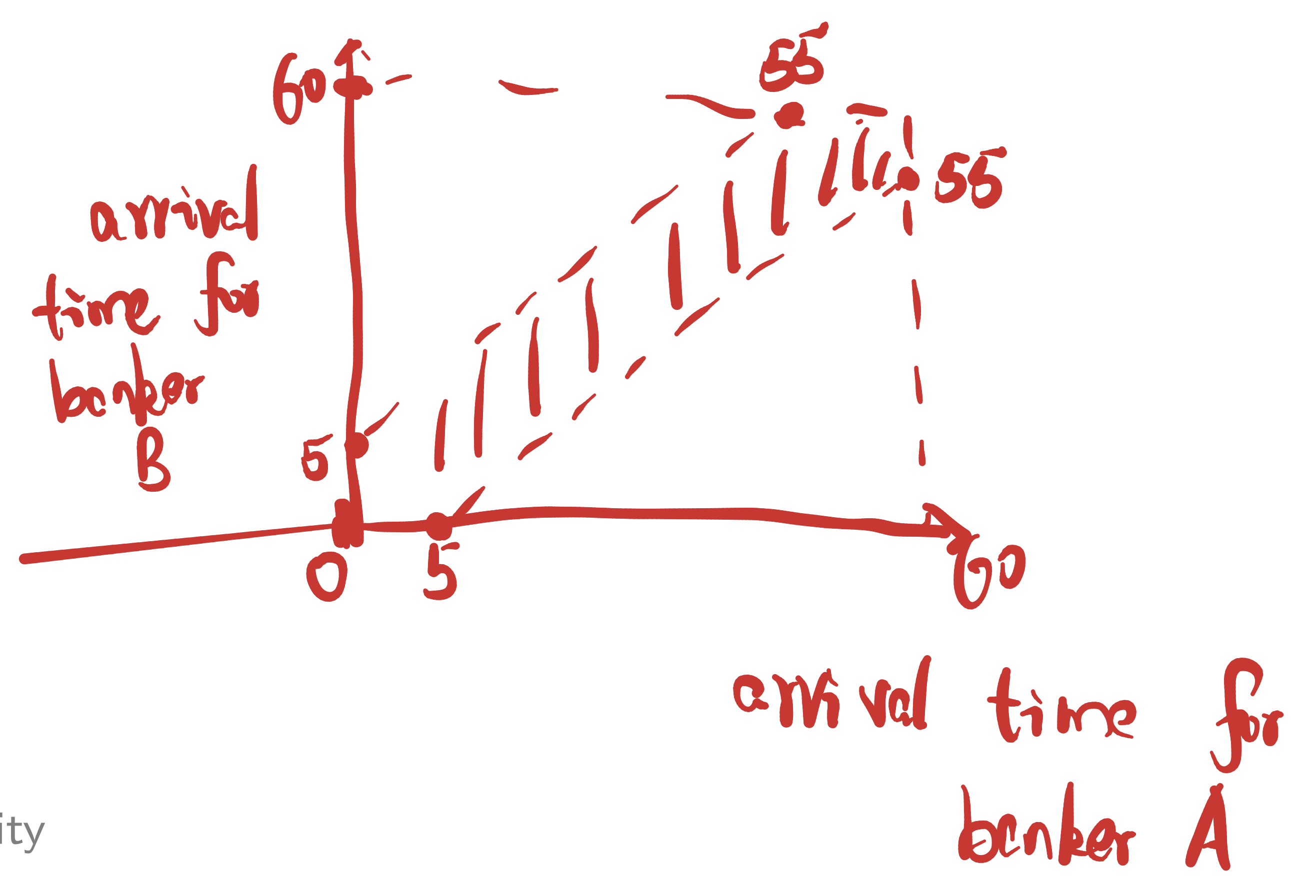
$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$





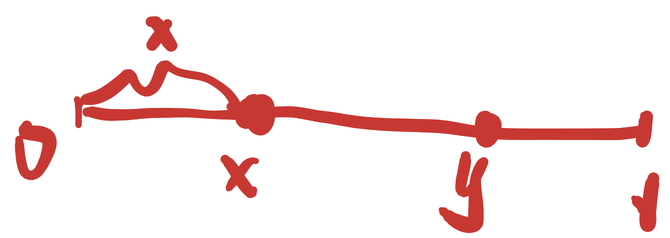
# Example

Two bankers each arrive at the station at some random time between 5:00am and 6:00am (arrival time for either banker is uniformly distributed). They stay exactly five minutes and then leave. What is the probability they will meet on a given day?




$$\frac{60^2 - 2\times\frac{55^2}{2}}{60^2} = \frac{60^2 - 55^2}{60^2} = \frac{12^2 - 11^2}{12^2} = \frac{23}{144}$$

# Example




$x, \quad y-x \quad 1-y$

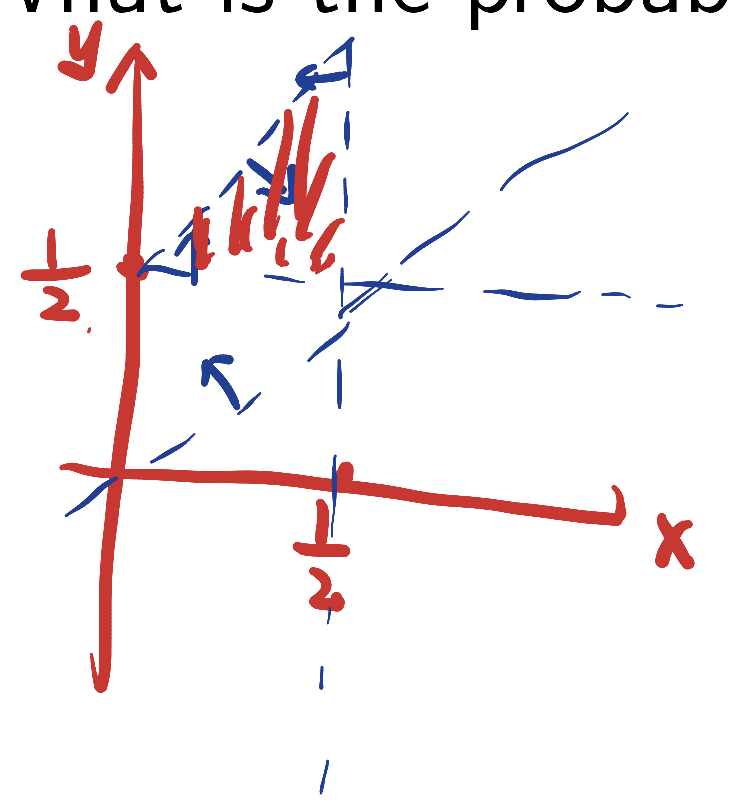
Case 1: $x<y$

① $x+(y-x)>1-y \Rightarrow y>1/2$

② $x+(1-y)>y-x \Rightarrow 2y<x+1 \Rightarrow y<\frac{1}{2}+x$

③ $(y-x)+(1-y)>x \Rightarrow x<\frac{1}{2}$

A stick is cut twice randomly (each cut point follows a uniform distribution on the stick). What is the probability that the 3 segments can form a triangle?




$\downarrow$

$\frac{1}{4}$

Def: $\{N(t): t \geq 0\}$ is a Poisson Process

# Example

- $N(t) - N(s) \sim \text{Poisson}(\lambda \cdot (t-s))$
- $N(0) = 0$
- $N(t)$ has independent increments

$$f(x) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}$$

$$\Pr\{X > t+s \mid X > s\} = \Pr\{X > t\}$$

You are waiting for a bus at a bus station. The bus arrive at the station according to a Poisson process with an average arrival time of $10$ minutes ($\lambda = 0.1/\text{min}$). If the buses have been running for a long time and you arrive at the bus station at a random time, what is your expected waiting time? On average, how many minutes ago did the last bus leave?

event: first arrival of bus will take $s$ minutes $A$

$$P\{A\} = \Pr\{N(s) - N(0) = 0\} = \Pr\{\text{Poisson}(\lambda s) = 0\} = \frac{e^{-\lambda s} (\lambda s)^0}{0!} = e^{-\lambda s}$$

Expected waiting time = 10 minutes $\sim \exp(\lambda)$

$N(t)$ Poisson Process with $\lambda = 2$ /minute

- $N(t) - N(s) \sim \text{Poisson}(\lambda(t-s))$ $\qquad \Pr\{N(t)-N(s) = x\} = \dfrac{e^{-\lambda(t-s)}(\lambda(t-s))^x}{x!}$
- $N(0) = 0$
- $N(t)$ has ind. increments. e.g.

  $N(t) - N(s) \perp N(t') - N(s')$ as long as $[s, t]$ and $[s', t']$ does not overlap.

1. Probability there're exactly 4 arrivals in first 3 minutes.

$$\Pr\{N(3) - N(0) = 4\} = \Pr\{\text{Poisson}(6) = 4\} = \frac{e^{-6} \cdot 6^4}{4!}$$

2. Probability there're no arrivals in first 3 minutes?

$$e^{-6}$$

# Example

$M'(t) = t e^{t^2/2} \Rightarrow E[X] = M'(0) = 0$

$M''(t) = e^{t^2/2} + t^2 e^{t^2/2} \Rightarrow E[X^2] = M''(0) = 1$

$M^{(3)}(t) = t e^{t^2/2} + 2t e^{t^2/2} + t^3 e^{t^2/2} = 3t e^{t^2/2} + t^3 e^{t^2/2} \quad E[X^3] = 0$

$M^{(4)}(t) = 3 e^{t^2/2} + 3t^2 e^{t^2/2} + 3t^2 e^{t^2/2} + 3t^4 e^{t^2/2} \Rightarrow E[X^4] = M^{(4)}(0) = 3$

Assume $X$ follows standard normal distribution, then what is $\mathbb{E}[X^n]$ for $n = 1, 2, 3, 4$?

$E[X^n] = \int x^n \cdot \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)\, dx$

completing the squares

$M(t) = E[e^{tX}] = \int e^{tx} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(tx - \frac{1}{2}x^2)\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2)\, dx$

$M^{(k)}(0) = E[X^k]$

$= \exp(\frac{1}{2}t^2) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x-t)^2)\, dx = \exp(\frac{t^2}{2})$

# Contents

# Definition

- For discrete distribution,

$$\mathbb{E}[g(X) \mid Y = y] = \sum_x g(x) p_{X|Y}(x \mid y) = \sum_x g(x) p(X = x \mid Y = y).$$

- For continuous distribution,

$$\mathbb{E}[g(X) \mid Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y) \mathrm{d}x.$$

- Law of total expectation:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \begin{cases} \sum_y \mathbb{E}[X \mid Y = y] p(Y = y), & \text{for discrete } Y, \\ \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y] f_Y(y) \mathrm{d}y, & \text{for continuous } Y. \end{cases}$$

# Example

$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

$\downarrow$

$2\,Corr(X,Y)\,Std(X)\,Std(Y)$

optimal variance

$= \sigma_A^2 + \left(\rho\frac{\sigma_A}{\sigma_B}\right)^2\sigma_B^2 - 2\left(\rho\frac{\sigma_A}{\sigma_B}\right)$ $\sigma_A\sigma_B$

$= \sigma_A^2 - \rho^2\sigma_A^2$

variance reduction / Control covariate

You just brought one share of stock $A$ and wish to hedge it by shorting stock $B$. How many shares of $B$ should you short in order to minimize the variance of the hedged position? Assume that the variance of stock $A$'s return is $\sigma_A^2$, and that of $B$'s return is $\sigma_B^2$. Their correlation coefficient is $\rho$.

$r_A \quad r_B$

$\min_h \; Var(r_A - h r_B)$

$\begin{cases} Var(r_A) = \sigma_A^2 \\ Var(r_B) = \sigma_B^2 \\ Corr(r_A, r_B) = \rho \end{cases}$

$= Var(r_A) + Var(h r_B) - 2\,Corr(r_A, h r_B)\,Std(r_A)\,Std(h r_B)$

$= \sigma_A^2 + h^2\sigma_B^2 - 2h\rho\sigma_A\sigma_B \Rightarrow 2h\sigma_B^2 - 2\rho\sigma_A\sigma_B = 0 \Rightarrow h = \rho\frac{\sigma_A}{\sigma_B}$

# Example

Suppose that you roll a dice. For each roll, you are paid the face value. if the roll gives $4, 5$, or $6$, you can roll the dice again. Once you get $1, 2$, or $3$, the game stops. What is the expected payoff of this game?

$X$: payoff

$Y$: outcome for first roll

$E[X \mid Y \in \{1,2,3\}] = 2$

$E[X \mid Y \in \{4,5,6\}] = 5 + E[X]$

$$E[X] = \Pr\{Y \in \{1,2,3\}\} E[X \mid Y \in \{1,2,3\}] + \Pr\{Y \in \{4,5,6\}\} E[X \mid Y \in \{4,5,6\}]$$

$$= \frac{1}{2} \times 2 + \frac{1}{2} \times (5 + E[X])$$

$$= \frac{7}{2} + \frac{1}{2} E[X]$$

# Contents

# Definition

- Let $X$ be a random variable with cumulative distribution function $F_X(x)$.

  • $X_1, \cdots, X_n$ are i.i.d. copies of $X$

- Define $Y_n = \min(X_1, \ldots, X_n)$ and $Z_n = \max(X_1, \ldots, X_n)$.
- $f_{Y_n}(x) = nf_X(x)(1 - F_X(x))^{n-1}$

  $F_{Y_n}(x) = 1 - (1 - F_X(x))^n$

- $f_{Z_n}(x) = nf_X(x)(F_X(x))^{n-1}$

  $f_{Y_n}(x) = -n(1-F_X(x))^{n-1} \cdot (-f_X(x))$

- Proof: $\Pr\{Y_n \geq x\} = 1 - \Pr\{Y_n \leq x\} = 1 - F_{Y_n}(x)$

  $= \Pr\{\min(X_1, \cdots, X_n) \geq x\} = \Pr\{X_1 \geq x, X_2 \geq x, \cdots, X_n \geq x\}$

  $= \Pr\{X_1 \geq x\} \Pr\{X_2 \geq x\} \cdots \Pr\{X_n \geq x\} = (1 - F_X(x))^n$

# Example

$$\int_0^1 f_{Z_n}(x)\,dx = \int_0^1 n x^{n-1}\,dx = x^n \Big|_0^1 = 1$$

Let $X_1, \ldots, X_n$ be i.i.d. random variables with uniform distribution between $0$ and $1$.

- What are the cumulative distribution funciton, the probability density function, and expeted value of $Z_n = \max(X_1, \ldots, X_n)$?
- What are the cumulative distribution function, the probability density function, and expected value of $Y_n = \min(X_1, \ldots, X_n)$?

- $f_X(x) = 1, \quad x \in [0,1]$

  $F_X(x) = x, \quad x \in [0,1]$

- $f_{Z_n}(x) = n f_X(x) \left(F_X(x)\right)^{n-1} = n x^{n-1}, \quad x \in [0,1]$
- $E[Z_n] = \int_0^1 x f_{Z_n}(x)\,dx = \int_0^1 n x^n\,dx = \frac{n}{n+1} x^{n+1} \Big|_0^1 = \frac{n}{n+1}$