4.4. Wednesday for MAT3040

Reviewing.

- Quotient Space:

$$V \setminus W = \{\boldsymbol{v} + W \mid \boldsymbol{v} \in V\}$$

  The elements in $V \setminus W$ are cosets. Note that $V \setminus W$ does not mean a subset of V.

- Define the canonical projection mapping

$$\begin{aligned} \pi_W : V &\to V \setminus W, \\ \text{with} \quad \boldsymbol{v} &\mapsto \boldsymbol{v} + W, \end{aligned}$$

  then we imply π_W is a surjective linear transformation with $\ker(\pi_W) = W$.

  If $\dim(V) < \infty$, then by Rank-Nullity Theorem (2.3), we imply that

$$\dim(V) = \dim(W) + \dim(V \setminus W),$$

  i.e., $\dim(V \setminus W) = \dim(V) - \dim(W)$.

- **(Universal Property I)** Every linear transformation $T : V \to W$ with $V' \leq \ker(T)$ can be descended to the composition of the canonical projection mapping $\pi_{V'}$ and the mapping

$$\begin{aligned} \tilde{T} &: V \setminus V' \\ \text{with} \quad \boldsymbol{v} + V' &\mapsto T(\boldsymbol{v}). \end{aligned}$$

  In other words, the diagram (2.1) commutes:

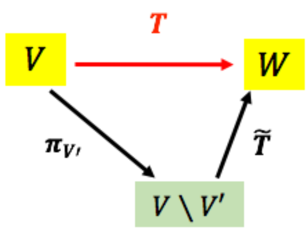

Diagram (2.1)

In other words, the mapping starting from either the black or red line gives the same result, i.e., $T(\boldsymbol{v}) = \tilde{T} \circ \pi_{V'}(\boldsymbol{v}) = \tilde{T}(\boldsymbol{v} + V')$ for any $\boldsymbol{v} \in V$.

- **(First Isomorphism Theorem)** Under the setting of Universal Property I (UPI), if T is a surjective linear transformation with $V' = \ker(T)$, then the $\tilde{T}$ is an isomorphism.

■ **Example 4.2** Suppose that $U, W \leq V$ with $U \cap W = \{\mathbf{0}\}$, then define the mapping

$$\begin{aligned} &\phi : U \oplus W \to U \\ \text{with} \quad &\phi(\boldsymbol{u} + \boldsymbol{w}) = \boldsymbol{u} \end{aligned}$$

Ⓡ Exercise: if $U, W \leq V$ but $U \cap W \neq \{\mathbf{0}\}$, then the mapping

$$\begin{aligned} &\phi : U + W \to U \\ \text{with} \quad &\boldsymbol{u} + \boldsymbol{w} \mapsto \boldsymbol{u} \end{aligned} \qquad \text{is not well-defined:}$$

Suppose that $\mathbf{0} \neq \boldsymbol{v} \in U \cap W$ and for any $\boldsymbol{u} \in U, \boldsymbol{w} \in W$, we construct

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v} \in U, \quad \boldsymbol{w}' = \boldsymbol{w} + \boldsymbol{v} \in V \quad \Longrightarrow \phi(\boldsymbol{u}' + \boldsymbol{w}') = \boldsymbol{u} - \boldsymbol{v}$$

Therefore we get $\boldsymbol{u} + \boldsymbol{w} = \boldsymbol{u}' + \boldsymbol{w}'$ but $\phi(\boldsymbol{u} + \boldsymbol{w}) \neq \phi(\boldsymbol{u}' + \boldsymbol{w}')$.

Back to the situation $U \cap W = \{\mathbf{0}\}$, then it's clear that $\phi : U \oplus W \to U$ is surjective linear transformation with $\ker(\phi) = W$. Therefore, construct the new mapping

$$\begin{aligned} &\tilde{\phi} : U \oplus W \setminus W \to U \\ \text{with} \quad &\boldsymbol{u} + \boldsymbol{w} + W \mapsto \phi(\boldsymbol{u} + \boldsymbol{w}) \end{aligned}$$

We imply $\tilde{\phi}$ is an isomorphism by First Isomorphism Theorem. ■

Now we study the generalized quotients, which is defined to satisfy the generalized version of universal property I.

Definition 4.7 [Universal Property for Quotients] Let V be a vector space and $V' \leq V$. Consider the collection of linear transformations

$$\text{Obj} = \left\{ T : V \to W \,\middle|\, \begin{array}{l} T \text{ is a linear transformation} \\ V' \leq \ker(T) \end{array} \right\}$$

(For example, $\pi_{V'} : V \to V \setminus V'$ is an element from the set Obj.)

An element $(\phi : V \to U) \in \text{Obj}$ is said to satisfy the **universal property** if it satisfies the following:

> Given any element $(T : V \to W) \in \text{Obj}$, we can extend the transformation ϕ with a uniquely existing $\tilde{T} : U \to W$ so that the diagram (2.2) commutes:

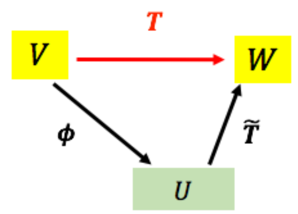

Diagram (2.2)

> Or equivalently, for given $(T : V \to W) \in \text{Obj}$, there exists the unique mapping $\tilde{T} : U \to W$ such that $T = \tilde{T} \circ \phi$.

Theorem 4.3 — Universal Property II.

1. The mapping $(\pi_{V'} : V \to V \setminus V') \in \text{Obj}$ is a universal object, i.e., it satisfies the universal property.
2. If $(\phi : V \to U)$ is a universal object, then $U \cong V \setminus V'$, i.e., there is intrinsically "one" element in the set of universal objects.

Proof. 1. Consider any linear transformation $T : V \to W$ such that $V' \leq \ker(T)$, then define (construct) the same $\tilde{T} : V \setminus V' \to W$ as that in UPI. Therefore, for given T, applying the result of UPI, we imply $T = \tilde{T} \circ \pi_{V'}$, i.e., $\pi_{V'}$ satisfies the diagram (2.2).

To show the uniqueness of $\tilde{T}$, suppose there exists $\tilde{S}: V \setminus V' \to W$ such that the diagram (2.3) commutes.

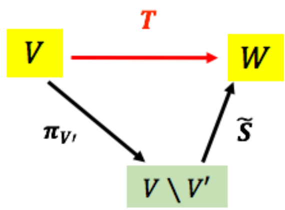

Diagram (2.3)

It suffices to show the mapping $\tilde{S} = \tilde{T}$: for any $\boldsymbol{v} + V' \in V \setminus V'$, we have

$$\tilde{S}(\boldsymbol{v} + V') := \tilde{S} \circ \pi_{V'}(\boldsymbol{v}) = T(\boldsymbol{v}),$$

where the first equality is due to the surjectivity of $\pi_{V'}$. By the result of UPI, $T(\boldsymbol{v}) = \tilde{T}(\boldsymbol{v} + V')$. Therefore $\tilde{T}(\boldsymbol{v} + V') = \tilde{S}(\boldsymbol{v} + V')$ for all $\boldsymbol{v} + V' \in V \setminus V'$. The proof is complete.

2. Suppose that $(\phi : V \to U)$ satisfies the universal property. In particular, the following two diagrams hold:

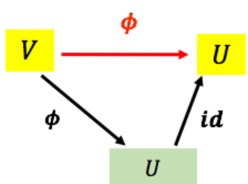

Diagram (2.4)

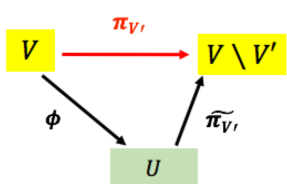

Diagram (2.5)

Since $(\pi_{V'})$ satisfies the universal property, in particular, the following two diagrams hold:

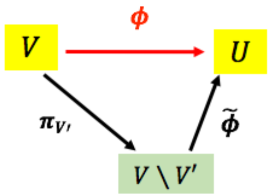

Diagram (2.6)

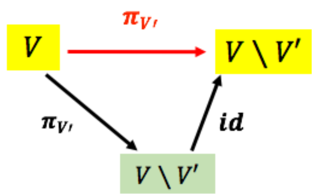

Diagram (2.7)

Then we claim that: Combining Diagram (2.5) and (2.6), we imply the diagram (2.8):

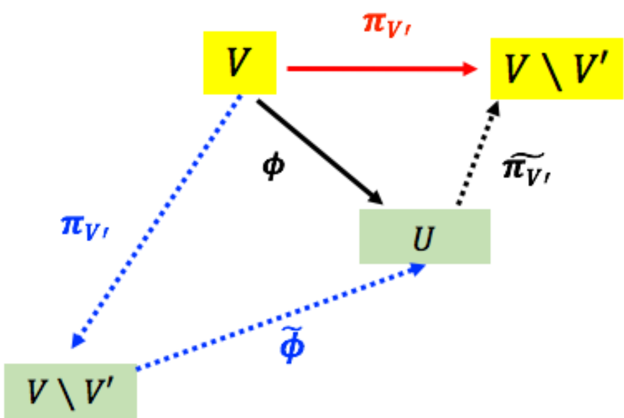

Diagram (2.8)

Graph Description: Note that this diagram commutes, i.e., the mapping starting from either the red line or the dash line gives the same result, i.e., $\pi_{V'} = \tilde{\pi}_{V'} \circ \tilde{\phi} \circ \pi_{V'}$, i.e., $\tilde{\pi}_{V'} \circ \tilde{\phi} = id$

Therefore, $\tilde{\pi}_{V'} \circ \tilde{\phi} = id$ implies $\tilde{\pi}_{V'}$ is surjective and $\tilde{\phi}$ is injective.

Also, combining Diagram (2.6) an (2.5), we imply diagram (2.9):

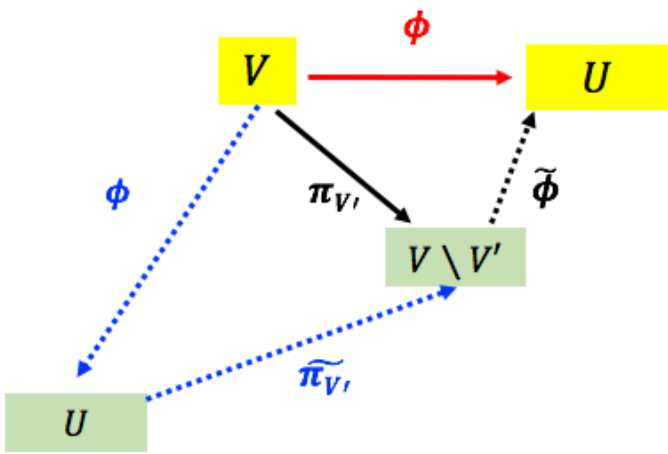

Diagram (2.9)

Graph Description: Note that this diagram commutes, i.e., the mapping starting from either the red line or the dash line gives the same result, i.e., $\phi = \tilde{\phi} \circ \tilde{\pi}_{V'} \circ \phi$, i.e., $\tilde{\phi} \circ \tilde{\pi}_{V'} = id$

Therefore, $\tilde{\phi} \circ \tilde{\pi}_{V'} = id$ implies $\tilde{\phi}$ is surjective and $\tilde{\pi}_{V'}$ is injective.

Therefore, both $\tilde{\phi} : U \to V \setminus V'$ and $\tilde{\pi}_{V'} : V \setminus V' \to U$ are bijective, i.e., $U \cong V \setminus V'$.

The proof is complete.

■

4.4.1. Dual Space

Definition 4.8 Let V be a vector space over a field $\mathbb{F}$. The **dual vector space** V^* is defined as

$$\begin{aligned} V^* &= \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) \\ &= \{f : V \to \mathbb{F} \mid f \text{ is a linear transformation}\} \end{aligned}$$

■ **Example 4.3** 1. Consider $V = \mathbb{R}^n$ and define $\phi_i : V \to \mathbb{R}$ as the i-th component of input:

$$\phi_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i,$$

Then we imply $\phi_i \in V^*$. On the contrary, $\phi_i^2 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i^2$ is not in V^*

2. Consider $V = \mathbb{F}[x]$ and define $\phi : V \to \mathbb{F}$ as:

$$\phi(p(x)) = p(1),$$

It's clear that $\phi \in V^*$:

$$\begin{aligned} \phi(ap(x) + bq(x)) &= ap(1) + bq(1) \\ &= a\phi(p(x)) + b\phi(q(x)) \end{aligned}$$

3. Also, $\psi : V \to \mathbb{F}$ by $\psi(p(x)) = \int_0^1 p(x)\,\mathrm{d}x$ is in V^*.
4. Also, for $V = M_{n\times n}(\mathbb{F})$, the mapping $\mathrm{tr} : V \to \mathbb{F}$ by $\mathrm{tr}(M) = \sum_{i=1}^n M_{ii}$ is in V^*. However, the $\det : V \to \mathbb{F}$ is not in V^*

■

Definition 4.9 Let V be a vector space, with basis $B = \{v_i \mid i \in I\}$ (I can be finite or countable, or uncountable). Define

$$B^* = \{f_i : V \to \mathbb{F} \mid i \in I\},$$

where f_i's are defined on the basis B:

$$f_i(v_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Then we extend f_i's linearly, i.e., for $\sum_{j=1}^{N} \alpha_j v_j \in V$,

$$f_i(\sum_{j=1}^{N} \alpha_j v_j) = \sum_{i=1}^{N} \alpha_j f_i(v_j).$$

It's clear that $f_i \in V^*$ is well-defined. ■

Our question is that whether the B^* can be the basis of V^*?