9.3. Monday for MAT4002

Reviewing.

1. Homotopy: we denote the homotopic function pair as $f \cong g$.
2. If $Y \subseteq \mathbb{R}^n$ is convex, then the set of continuous functions $f : X \to Y$ form a single equivalence class, i.e., $\{\text{continuous functions } f : X \to Y\}/\sim$ has only one element

9.3.1. Remarks on Homotopy

Proposition 9.4 Consider four continous mappings

$$W \xrightarrow{f} X, \quad X \xrightarrow{g} Y, \quad X \xrightarrow{h} Y, \quad Y \xrightarrow{k} Z.$$

If $g \cong h$, then

$$g \circ f \cong h \circ f, \quad k \circ g \cong k \circ h$$

Proof. Suppose there exists the homotopy $H : g \cong h$, then $k \circ H : X \times I \to Z$ gives the momotopy between $k \circ g$ and $k \circ h$.

Simiarly, $H \circ (f \times \text{id}_I) : W \times I \to Y$ gives the homotopy $g \circ f \simeq h \circ f$. ■

Definition 9.4 [Homotopy Equivalence] Two topological spaces X and Y are **homotopy equivalent** if there are continuous maps $f : X \to Y$, and $g : Y \to X$ such that

$$g \circ f \simeq \text{id}_{X \to X}$$
$$f \circ g \simeq \text{id}_{Y \to Y},$$

which is denoted as $X \simeq Y$.

R

1. If $X \cong Y$ are homeomorphic, then they are homotopic equivalent.
2. The homotopy equivalence $X \simeq Y$ gives a bijection between $\{\phi : \text{continuous } W \to X\}/\sim$ and $\{\phi : \text{continuous } W \to Y\}/\sim$, for any given topological space W.

Proof. Since $X \simeq Y$, we can find $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We construct a mapping

$$\begin{aligned} \phi : \quad & \{\phi : \text{continuous } W \to X\}/\sim \to \{\phi : \text{continuous } W \to Y\}/\sim \\ \text{with} \quad & [\phi] \mapsto [f \circ \phi] \end{aligned}$$

ϕ is well-defined since $\phi_1 \sim \phi_2$ implies $f \circ \phi_1 \sim f \circ \phi_2$

Also, we can construct a mapping

$$\begin{aligned} \beta : \quad & \{\phi : \text{continuous } W \to Y\}/\sim \to \{\phi : \text{continuous } W \to X\}/\sim \\ \text{with} \quad & [\psi] \mapsto [g \circ \phi] \end{aligned}$$

Similarly, β is well-defined.

Also, we can check that $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$. For example,

$$\alpha \circ \beta[\psi] = [f \circ g \circ \psi] = [\psi],$$

where the last equality is because that $f \circ g \simeq \mathrm{id}_Y$. ■

3. The homotopy equivalence $X \simeq Y$ forms an equivalence relation between topological spaces

Compared with homeomorphism, some properties are lost when consider the homotopy equivalence.

Definition 9.5 [Contractible] The topological space X is **contractible** if it is homotopy equivalent to any point $\{c\}$.

R In other words, there exists continuous mappings f, g such that

$$\{c\} \xrightarrow{f} X \xrightarrow{g} \{c\}, \; g \circ f \simeq \mathrm{id}_{\{c\}}$$
$$X \xrightarrow{g} \{c\} \xrightarrow{f} X, \; f \circ g \simeq \mathrm{id}_X$$

Note that $g \circ f \simeq \mathrm{id}_{\{c\}}$ follows naturally; and since $X \cong X$, we can find f, g

such that $f \circ g = c_y$ for some $y \in X$, where $c_y : X \to X$ is a constant function $c_y(x) = y, \forall x \in X$. Therefore, to check X is contractible, it suffices to check $c_y \simeq \mathrm{id}_X, \forall y \in X$.

Therefore, X is contractible if its identity map id_X is homotopic to any constant map $c_y, \forall y \in X$.

Proposition 9.5 The definition for contractible can be simplified further:

1. X is contractible if it is homotopy equivalent to some point $\{c\}$
2. X is contractible if the identity map id_X is homotopic to some constant map $c_y(x) = y$.

Proof. The only thing is to show that $c_y \simeq c_{y'}, \forall y, y' \in X$. By hw 3, X is path-connected, and therefore there exists continous $p(t)$ such that

$$p(0) = y, \quad p(1) = y'$$

Therefore, we construct the homotopy between c_y and $c_{y'}$ as follows:

$$H(x,t) = p(t).$$

■

Example 9.1 1. $X = \mathbb{R}^2$ is contractible:

It suffices to show that the mapping $f(\boldsymbol{x}) = \boldsymbol{x}, \forall \boldsymbol{x} \in \mathbb{R}^2$ is homotopic to the constant function $g(x) = (0,0), \forall x \in \mathbb{R}^2$, i.e., $g = c_{(0,0)}$.

Consider the continuous mapping $H(\boldsymbol{x},t) = tf(\boldsymbol{x})$, with

$$H(\boldsymbol{x},0) = c_{(0,0)}, \quad H(\boldsymbol{x},1) = \mathrm{id}_X$$

Therefore, $c_{(0,0)} \simeq \mathrm{id}_X$. Since $c_{(0,0)} \simeq c_{\boldsymbol{y}}, \forall \boldsymbol{y} \in \mathbb{R}^2$, we imply $c_{\boldsymbol{y}} \simeq \mathrm{id}_X$ for any $\boldsymbol{y} \in \mathbb{R}^2$. Therefore, X is contractible.

More generally, any convex $X \subseteq \mathbb{R}^n$ is contractible.

R S^1 is not contractible, and we will see it in 3 weeks' time. In particular, we are not able to construct the continuous mapping

$$H : S^1 \times [0,1] \to S^1$$

such that

$$H(e^{2\pi ix},0) = e^{2\pi ix}, \quad H(e^{2\pi ix},1) = e^{2\pi i(0)} = 1$$

How about the mapping $H(e^{2\pi ix},t) = e^{2\pi ixt}$? Unfortunately, it is not well-defined, since

$$H(e^{2\pi i(1)},t) = e^{2\pi it} = H(e^{2\pi i(0)},t) = 1$$

and the equality is not true for $t \neq 0,1$.

Definition 9.6 [Homotopy Retract] Let $A \subseteq X$ and $i : A \hookrightarrow X$ be an inclusion. We say A is a **homotopy retract** of X if there exists continuous mapping $r : X \to A$ such that

$$r \circ i : A \hookrightarrow X \xrightarrow{r} A = \mathrm{id}_A$$
$$i \circ r : X \xrightarrow{r} A \hookrightarrow X \simeq \mathrm{id}_X$$

In particualr, $A \simeq X$.

■ **Example 9.2** The 1-sphere S^1 is a homotopy retract of Mobius band M.

Let $M = [0,1]^2/\sim$ and $S^1 = [0,1]/\sim$. Define the inclusion i and r as:

$$\begin{aligned} i : \quad & S^1 \hookrightarrow M \\ \text{with} \quad & [x] \mapsto [(x, \tfrac{1}{2})] \end{aligned}$$

$$
\begin{aligned}
r: \quad & M \to S^1 \\
\text{with} \quad & [(x,y)] \mapsto [x]
\end{aligned}
$$

As a result,

$$
r \circ i = \text{id}_{S^1}, \qquad i \circ r([(x,y)]) = [(x,1/2)]
$$

It suffices to show $i \circ r \simeq \text{id}_M$, where $\text{id}_M([(x,y)]) = [(x,y)]$.

Construct the continous mapping $H : M \times I \to M$ with

$$
H([(x,y)],t) := [(x,(1-t)y + t/2)]
$$

To show the well-definedness of H, we need to check

$$
H([(0,y)],t) = H([(1,1-y)],t), \quad \forall y \in [0,1]
$$

It's clear that H gives a homotopy between $i \circ r$ and id_M, i.e., $i \circ r \simeq \text{id}_M$ ▪

▪ **Example 9.3** The $n-1$-sphere S^{n-1} is a homotopy retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$:

We have the inclusion $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ and

$$
\begin{aligned}
r: \quad & \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1} \\
\text{with} \quad & x \mapsto \frac{x}{\|x\|}
\end{aligned}
$$

Therefore, $r \circ i = \text{id}_{S^{n-1}}$ and $i \circ r(x) = \frac{x}{\|x\|}$.

It suffices to show that $i \circ r \simeq \text{id}_{\mathbb{R}^n \setminus \{0\}}$. Consider the homotopy $H(x,t) = t\boldsymbol{x} + (1-t)\boldsymbol{x}/\|\boldsymbol{x}\|$ such that

$$
H(\boldsymbol{x},0) = i \circ r(\boldsymbol{x}), \quad H(\boldsymbol{x},1) = \boldsymbol{x} = \text{id}(\boldsymbol{x})
$$

To show the well-definedness of H, we need to check $H(x,t) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $t \in [0,1]$. ▪

Definition 9.7 [Homotopic Relative] Let $A \subseteq X$ be topological spaces. We say $f, g : X \to Y$

are homotopic relative to A if there eixsts $H : X \times I \longrightarrow Y$ such that

$$\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \end{cases} \quad \text{and } H(a,t) = f(a) = g(a), \forall a \in A$$