Chapter 2

Week2

2.1. Monday for MAT3040

Reviewing.

1. Linear Combination and Span
2. Linear Independence
3. Basis: a set of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is called a **basis** for V if $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is linearly independent, and $V = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$.

   Lemma: Given $V = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$, we can find a basis for this set. Here V is said to be **finitely generated**.
4. Lemma: The vector $\boldsymbol{w} \in \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \setminus \operatorname{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ implies that

$$\boldsymbol{v}_1 \in \operatorname{span}\{\boldsymbol{w}, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \setminus \operatorname{span}\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$$

2.1.1. Basis and Dimension

Theorem 2.1 Let V be a finitely generated vector space. Suppose $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ are two basis of V. Then $m = n$. (where m is called the **dimension**)

Proof. Suppose on the contrary that $m \neq n$. Without loss of generality (w.l.o.g.), assume that $m < n$. Let $\boldsymbol{v}_1 = \alpha_1 \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n$, with some $\alpha_i \neq 0$. w.l.o.g., assume $\alpha_1 \neq 0$. Therefore,

$$\boldsymbol{v}_1 \in \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \setminus \operatorname{span}\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \tag{2.1}$$

which implies that $\boldsymbol{w}_1 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \setminus \operatorname{span}\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$.

Then we claim that $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ is a basis of V:

1. Note that $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ is a spannning set:

$$\begin{aligned}\boldsymbol{w}_1 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} &\implies \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \\ &\implies \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \subseteq \operatorname{span}\{\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}\end{aligned}$$

   Since $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$, we have $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} = V$.

2. Then we show the linear independence of $\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$. Consider the equation

$$\beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 + \cdots + \beta_n \boldsymbol{w}_n = \mathbf{0}$$

   (a) When $\beta_1 \neq 0$, we imply

$$\boldsymbol{v}_1 = \left(-\frac{\beta_2}{\beta_1}\right) \boldsymbol{w}_2 + \cdots + \left(-\frac{\beta_n}{\beta_1}\right) \boldsymbol{w}_n \in \operatorname{span}\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\},$$

   which contradicts (2.1).

   (b) When $\beta_1 = 0$, then $\beta_2 \boldsymbol{w}_2 + \cdots + \beta_n \boldsymbol{w}_n = \mathbf{0}$, which implies $\beta_2 = \cdots = \beta_n = 0$, due to the independence of $\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$.

Therefore, $\boldsymbol{v}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$, i.e.,

$$\boldsymbol{v}_2 = \gamma_1 \boldsymbol{v}_1 + \cdots + \gamma_n \boldsymbol{v}_n,$$

where $\gamma_2, \dots, \gamma_n$ cannot be all zeros, since otherwise $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ are linearly dependent, i.e., $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ cannot form a basis. w.l.o.g., assume $\gamma_2 \neq 0$, which implies

$$\boldsymbol{w}_2 \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_3, \dots, \boldsymbol{w}_n\} \setminus \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{w}_3, \dots, \boldsymbol{w}_n\}.$$

Following the simlar argument above, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_3, \dots, \boldsymbol{w}_n\}$ forms a basis of V.

Continuing the argument above, we imply $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m, \boldsymbol{w}_{m+1}, \dots, \boldsymbol{w}_n\}$ is a basis of V.

Since $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ is a basis as well, we imply

$$\boldsymbol{w}_{m+1} = \delta_1 \boldsymbol{v}_1 + \cdots + \delta_m \boldsymbol{v}_m$$

for some $\delta_i \in \mathbb{F}$, i.e., $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m, \boldsymbol{w}_{m+1}\}$ is linearly dependent, which is a contradction.

■

■ **Example 2.1** A vector space may have more than one basis.

Suppose $V = \mathbb{F}^n$, it is clear that $\dim(V) = n$, and

$$\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\} \text{ is a basis of } V, \text{ where } \boldsymbol{e}_i \text{ denotes a unit vector.}$$

There could be other basis of V, such as

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \right\}$$

Actually, the columns of any invertible $n \times n$ matrix forms a basis of V. ■

■ **Example 2.2** Suppose $V = M_{m \times n}(\mathbb{R})$, we claim that $\dim(V) = mn$:

$$\left\{ E_{ij} \middle| \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix} \right\} \text{ is a basis of } V,$$

where E_{ij} is $m \times n$ matrix with 1 at (i,j)-th entry, and 0s at the remaining entries. ■

■ **Example 2.3** Suppose $V = \{\text{all polynomials of degree} \le \text{n}\}$, then $\dim(V) = n + 1$. ■

■ **Example 2.4** Supppose $V = \{\boldsymbol{A} \in M_{n \times n}(\mathbb{R}) \mid \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\}$, then $\dim(V) = \frac{n(n+1)}{2}$. ■

■ **Example 2.5** Let $W = \{\boldsymbol{B} \in M_{n\times n}(\mathbb{R}) \mid \boldsymbol{B}^{\mathrm{T}} = -\boldsymbol{B}\}$, then $\dim(V) = \frac{n(n-1)}{2}$. ■

R Sometimes it should be classified the field $\mathbb{F}$ for the scalar multiplication to define a vector space. Conside the example below:

1. Let $V = \mathbb{C}$, then $\dim(\mathbb{C}) = 1$ for the scalar multiplication defined under the field $\mathbb{C}$.
2. Let $V = \operatorname{span}\{1, i\} = \mathbb{C}$, then $\dim(\mathbb{C}) = 2$ for the scalar multiplication defined under the field $\mathbb{R}$, since all $z \in V$ can be written as $z = a + bi$, $\forall a, b \in \mathbb{R}$.
3. Therefore, to aviod confusion, it is safe to write

$$\dim_{\mathbb{C}}(\mathbb{C}) = 1, \quad \dim_{\mathbb{R}}(\mathbb{C}) = 2.$$

2.1.2. Operations on a vector space

Note that the basis for a vector space is characterized as the **maximal linearly independent set**.

Theorem 2.2 — Basis Extension. Let V be a finite dimensional vector space, and $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a linearly independent set on V, Then we can extend it to the basis $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$ of V.

Proof.

- Suppose $\dim(V) = n > k$, and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ is a basis of V. Consider the set $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\} \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$, which is linearly dependent, i.e.,

$$\alpha_1 \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n + \beta_1 \boldsymbol{v}_1 + \cdots + \beta_k \boldsymbol{v}_k = \boldsymbol{0},$$

with some $\alpha_i \neq 0$, since otherwise this equation will only have trivial solution. w.l.o.g., assume $\alpha_1 \neq 0$.

- Therefore, consider the set $\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\} \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$. We keep removing elements from $\{\boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ until we first get the set

$$S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\},$$

with $S \subseteq \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_n\}$ and $S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is linearly independent, i.e., S is a maximal subset of $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ such that $S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is linearly independent.

- Rewrite $S = \{\boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_m\}$ and therefore $S' = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_m\}$ are linearly independent. It suffices to show S' spans V.
  - Indeed, for all $\boldsymbol{w}_i \in \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$, $\boldsymbol{w}_i \in \text{span}(S')$, since otherwise the equation

    $$\alpha \boldsymbol{w}_i + \beta_1 \boldsymbol{v}_1 + \cdots + \beta_m \boldsymbol{v}_m = \boldsymbol{0} \implies \alpha = 0,$$

    which implies that $\beta_1 \boldsymbol{v}_1 + \cdots + \beta_m \boldsymbol{v}_m = \boldsymbol{0}$ admits only trivial solution, i.e.,

    $$\{\boldsymbol{w}_i\} \bigcup S' = \{\boldsymbol{w}_i\} \bigcup S \bigcup \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} \text{ is linearly independent,}$$

    which violetes the maximality of S.

  Therefore, all $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\} \subseteq \text{span}(S')$, which implies $\text{span}(S') = V$.

  Therefore, S' is a basis of V.

■

R Start with a spanning set, we keep removing something to form a basis; start with independent set, we keep adding something to form a basis.

In other words, the basis is both the minimal spanning set, and the maximal linearly independent set.

Definition 2.1 [Direct Sum] Let W_1, W_2 be two vector subspaces of V, then

1. $W_1 \bigcap W_2 := \{\boldsymbol{w} \in V \mid \boldsymbol{w} \in W_1, \text{ and } \boldsymbol{w} \in W_2\}$
2. $W_1 + W_2 := \{\boldsymbol{w}_1 + \boldsymbol{w}_2 \mid \boldsymbol{w}_i \in W_i\}$

3. If furthermore that $W_1 \bigcap W_2 = \{\mathbf{0}\}$, then $W_1 + W_2$ is denoted as $W_1 \oplus W_2$, which is called **direct sum**.

Proposition 2.1 $W_1 \bigcap W_2$ and $W_1 + W_2$ are vector subspaces of V.

2.2. Monday for MAT3006

Reviewing.

1. Equivalent Metric:
$$d_1(\boldsymbol{x},\boldsymbol{y}) \le Kd_2(\boldsymbol{x},\boldsymbol{y}) \le K'd_1(\boldsymbol{x},\boldsymbol{y})$$
In $\mathcal{C}[0,1]$, the metric d_1 and d_∞ are not equivalent:
For $f_n(x) = x^n n^2(1-x)$, $d_1(f_n,0) \to 1$ and $d_\infty(f_n,0) \to \infty$. Suppose on contrary that
$$d_1(f_n,0) \le Kd_\infty(\boldsymbol{x},\boldsymbol{y}) \le K'd_1(\boldsymbol{x},\boldsymbol{y}).$$
Taking limit both sides, we imply the immediate term goes to infinite, which is a contradiction.
2. Continuous functions: the function f is continuous is equivalent to say for $\forall x_n \to x$, we have $f(x_n) \to f(x)$.
3. Open sets: Let (X,d) be a metric space. A set $U \subseteq X$ is open if for each $x \in U$, there exists $\rho_x > 0$ such that $B_{\rho_x}(x) \subseteq U$.

R Unless stated otherwise, we assume that
$$\mathcal{C}[a,b] \longleftrightarrow (\mathcal{C}[a,b], d_\infty)$$
$$\mathbb{R}^n \longleftrightarrow (\mathbb{R}^n, d_2)$$

2.2.1. Remark on Open and Closed Set

■ **Example 2.6** Let $X = \mathcal{C}[a,b]$, show that the set
$$U := \{f \in X \mid f(x) > 0, \forall x \in [a,b]\} \quad \text{is open.}$$

Take a point $f \in U$, then

$$\inf_{[a,b]} f(x) = m > 0.$$

Consider the ball $B_{m/2}(f)$, and for $\forall g \in B_{m/2}(f)$,

$$\begin{aligned}
|g(x)| &\geq |f(x)| - |f(x) - g(x)| \\
&\geq \inf_{[a,b]} |f(x)| - \sup_{[a,b]} |f(x) - g(x)| \\
&\geq m - \frac{m}{2} \\
&= \frac{m}{2} > 0, \ \forall x \in [a,b]
\end{aligned}$$

Therefore, we imply $g \in U$, i.e., $B_{m/2}(f) \subseteq U$, i.e., U is open in X. ■

Proposition 2.2 Let (X,d) be a metric space. Then

1. $\emptyset, X$ are open in X
2. If $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ are open in X, then $\bigcup_{\alpha \in \mathcal{A}}$ is also open in X
3. If $U_1, \ldots, U_n$ are open in X, then $\bigcap_{i=1}^n U_i$ are open in X

R Note that $\bigcap_{i=1}^\infty U_i$ is not necessarily open if all U_i's are all open:

$$\bigcap_{i=1}^\infty \left(-\frac{1}{i}, 1 + \frac{1}{i}\right) = [0,1]$$

Definition 2.2 [Closed] The closed set in metric space (X,d) are the complement of open sets in X, i.e., any closed set in X is of the form $V = X \setminus U$, where U is oepn. ■

For example, in $\mathbb{R}$,

$$[a,b] = \mathbb{R} \setminus \{(-\infty, a) \bigcup (b, \infty)\}$$

Proposition 2.3
1. $\emptyset, X$ are closed in X
2. If $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ are closed subsets in X, then $\bigcap_{\alpha \in \mathcal{A}} V_\alpha$ is also closed in X
3. If $V_1, \ldots, V_n$ are closed in X, then $\bigcup_{i=1}^n V_i$ is also closed in X.

R Whenever you say U is open or V is closed, you need to specify the underlying space, e.g.,

$$\textbf{Wrong}: U \text{ is open}$$

$$\textbf{Right}: U \text{ is open in } X$$

Proposition 2.4 The following two statements are equivalent:

1. The set V is closed in metric space (X,d).
2. If the sequence $\{v_n\}$ in V converges to x, then $x \in V$

Proof. Necessity.

Suppose on the contrary that $\{v_n\} \to x \notin V$. Since $X \setminus V \ni x$ is open, there exists an open ball $B_\varepsilon(x) \subseteq X \setminus V$.

Due to the convergence of sequence, there exists N such that $d(v_n,x) < \varepsilon$ for $\forall n \geq N$, i.e., $v_n \in B_\varepsilon(x)$, i.e., $v_n \notin V$, which contradicts to $\{v_n\} \subseteq V$.

Sufficiency.

Suppose on the contrary that V is not closed in X, i.e., $X \setminus V$ is not open, i.e., there exists $x \notin V$ such that for all open $U \ni x$, $U \bigcap V \neq \emptyset$. In particular, take

$$U_n = B_{1/n}(x), \implies \exists v_n \in B_{1/n}(x) \bigcap V,$$

i.e., $\{v_n\} \to x$ but $x \notin V$, which is a contradiction. ■

Proposition 2.5 Given two metric space (X,d) and (Y,ρ), the following statements are equivalent:

1. A function $f : (X,d) \to (Y,\rho)$ is continuous on X
2. For $\forall U \subseteq Y$ open in Y, $f^{-1}(U)$ is open in X.
3. For $\forall V \subseteq Y$ closed in Y, $f^{-1}(V)$ is closed in X.

■ **Example 2.7** The mapping $\Psi : \mathcal{C}[a,b] \to \mathbb{R}$ is defined as:

$$f \mapsto f(c)$$

where Ψ is called a **functional**.

Show that Ψ is continuous by using d_∞ metric on $\mathcal{C}[a,b]$:

1. Any open set in $\mathbb{R}$ can be written as countably union of open disjoint intervals, and therefore suffices to consider the pre-image $\Psi^{-1}(a,b) = \{f \mid f(c) \in (a,b)\}$. Following the similar idea in Example (2.6), it is clear that $\Psi^{-1}(a,b)$ is open in $(\mathcal{C}[a,b], d_\infty)$. Therefore, Ψ is continuous.
2. Another way is to apply definition.

We now study open sets in a subspace $(Y,d_Y) \subseteq (X,d_X)$, i.e.,

$$d_Y(y_1,y_2) := d_X(y_1,y_2).$$

Therefore, the open ball is defined as

$$\begin{aligned}
B_\varepsilon^Y(y) &= \{y' \in Y \mid d_Y(y,y') < \varepsilon\} \\
&= \{y' \in Y \mid d_X(y,y') < \varepsilon\} \\
&= \{y' \in X \mid d_X(y,y') < \varepsilon, y' \in Y\} \\
&= B_\varepsilon^X(y) \bigcap Y
\end{aligned}$$

Proposition 2.6 All open sets in the subspace $(Y,d_Y) \subseteq (X,d_X)$ are of the form $U \bigcap Y$, where U is open in X.

Corollary 2.1 For the subspace $(Y,d_Y) \subseteq (X,d_X)$, the mapping $i : (Y,d_Y) \to (X,d_X)$ with $i(y) = y, \forall y \in Y$ is continuous.

Proof. $i^{-1}(U) = U \bigcap Y$ for any subset $U \subseteq X$. The results follows from proposition (2.5). ∎

R It's important to specify the underlying space to describe an open set.

For example, the interval $[0, \frac{1}{2})$ is not open in $\mathbb{R}$, while $[0, \frac{1}{2})$ is open in $[0,1]$, since

$$[0,\frac{1}{2}) = (-\frac{1}{2},\frac{1}{2})\bigcap[0,1].$$

2.2.2. Boundary, Closure, and Interior

Definition 2.3 Let (X,d) be a metric space, then

1. A point x is a **boundary point** of $S \subseteq X$ (denoted as $x \in \partial S$) if for any open $U \ni x$, then both $U\bigcap S, U \setminus S$ are non-empty.
   (one can replace U by $B_{1/n}(x)$, with $n = 1,2,\dots$)
2. The **closure** of S is defined as $\overline{S} = S\bigcup\partial S$.
3. A point x is an **interior point** of S (denoted as $x \in S^\circ$) if there $\exists U \ni x$ open such that $U \subseteq S$. We use S° to denote the set of interior points.

Proposition 2.7 1. The closure of S can be equivalently defined as

$$\overline{S} = \bigcap\{C \in X \mid C \text{ is closed and } C \supseteq S\}$$

Therefore, $\overline{S}$ is the smallest closed set containing S.

2. The interior set of S can be equivalently defined as

$$S^\circ = \bigcup\{U \subseteq X \mid U \text{ is open and } U \subseteq S\}$$

Therefore, S° is the largest open set contained in S.

Example 2.8 For $S = [0,\frac{1}{2}] \subseteq X$, we have

1. $\partial S = \{0,\frac{1}{2}\}$
2. $\overline{S} = [0,\frac{1}{2}]$
3. $S^\circ = (0,\frac{1}{2})$

Proof. 1. (a) Firstly, we show that $\overline{S}$ is closed, i.e., $X \setminus \overline{S}$ is open.

- Take $x \notin \overline{S}$. Since $x \notin \partial S$, there $\exists B_r(x) \ni x$ such that

$$B_r(x) \bigcap S, \quad \text{or} \quad B_r(x) \setminus S \text{ is } \emptyset.$$

- Since $x \notin S$, the set $B_r(x) \setminus S$ is not empty. Therefore, $B_r(x) \bigcap S = \emptyset$.
- It's clear that $B_{r/2}(x) \bigcap S = \emptyset$. We claim that $B_{r/2}(x) \bigcap \overline{S}$ is empty. Suppose on the contrary that

$$y \in B_{r/2}(x) \bigcap \partial S,$$

which implies that $B_{r/2}(y) \bigcap S \neq \emptyset$. Therefore,

$$B_{r/2}(y) \subseteq B_r(x) \implies B_r(x) \bigcap S \supseteq B_{r/2}(y) \bigcap S \neq \emptyset,$$

which is a contradiction.

Therefore, $x \in X \setminus \overline{S}$ implies $B_{r/2}(x) \bigcap \overline{S} = \emptyset$, i.e., $X \setminus \overline{S}$ is open, i.e., $\overline{S}$ is closed.

(b) Secondly, we show that $\overline{S} \subseteq C$, for any closed $C \supseteq S$, i.e., suffices to show $\partial S \subseteq C$.

Take $x \in \partial S$, and construct a sequence

$$x_n \in B_{1/n}(x) \bigcap S.$$

Here $\{x_n\}$ is a sequence in $S \subseteq C$ converging to x, which implies $x \in C$, due to the closeness of C in X.

Combining (a) and (b), the result follows naturally. (Question: do we need to show the well-defineness?)

2. Exercise. Show that

$$S^\circ = S \setminus \partial S = X \setminus (\overline{X \setminus S}).$$

Then it's clear that S° is open, and contained in S.

■

The next lecture we will talk about compactness and sequential compactness.

2.3. Monday for MAT4002

Reviewing.

1. Topological Space $(X, \mathcal{J})$: a special class of topological space is that induced from metric space (X,d):

$$(X,\mathcal{T}), \quad \text{with } \mathcal{T} = \{\text{all open sets in } (X,d)\}$$

2. Closed Sets $(X \setminus U)$ with U open.

Proposition 2.8 Let $(X,\mathcal{T})$ be a topological space,

1. $\emptyset, X$ are closed in X
2. V_1, V_2 closed in X implies that $V_1 \bigcup V_2$ closed in X
3. $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ closed in X implies that $\bigcap_{\alpha\in\mathcal{A}} V_\alpha$ closed in X

Proof. Applying the De Morgan's Law

$$(X \setminus \bigcup_{i\in I} U_i) = \bigcap_{i\in I}(X \setminus U_i)$$

∎

2.3.1. Convergence in topological space

Definition 2.4 [Convergence] A sequence $\{x_n\}$ of a topological space $(X,\mathcal{T})$ converges to $x \in X$ if $\forall U \ni x$ is open, there $\exists N$ such that $x_n \in U, \forall n \geq N$.

Example 2.9 1. The topology for the space $(X = \mathbb{R}^n, d_2) \to (X,\mathcal{T})$ (i.e., a topological space induced from meric space $(X = \mathbb{R}^n, d_2)$) is called a **usual topology** on $\mathbb{R}^n$.

When I say $\mathbb{R}^n$ (or subset of $\mathbb{R}^n$) is a topological space, it is equipeed with usual topology.

Convergence of sequence in $(\mathbb{R}^n,\mathcal{T})$ is the usual convergence in analysis.

For $\mathbb{R}^n$ or metric space, the limit of sequence (if exists) is unique.

2. Consider the topological space $(X, \mathcal{T}_{\text{indiscrete}})$. Take any sequence $\{x_n\}$ in X, it is convergent to any $x \in X$. Indeed, for $\forall U \ni x$ open, $U = X$. Therefore,

$$x_n \in U (= X), \forall n \geq 1.$$

3. Consider the topological space $(X, \mathcal{T}_{\text{cofinite}})$, where X is infinite. Consider $\{x_n\}$ is a sequence satisfying $m \neq n$ implies $x_m \neq x_n$. Then $\{x_n\}$ is convergent to any $x \in X$. (Question: how to define openness for $\mathcal{T}_{\text{cofinite}}$ and $\mathcal{T}_{\text{indiscrete}}$)?
4. Consider the topological space $(X, \mathcal{T}_{\text{discrete}})$, the sequence $\{x_n\} \to x$ is equivalent to say $x_n = x$ for all sufficiently large n.

R The limit of sequences may not be unique. The reason is that "$\mathcal{T}$ is not big enough". We will give a criterion to make sure the limit is unique in the future. (Hausdorff)

Proposition 2.9 If $F \subseteq (X, \mathcal{T})$ is closed, then for any convergent sequence $\{x_n\}$ in F, the limit(s) are also in F.

Proof. Let $\{x_n\}$ be a sequence in F with limit $x \in X$. Suppose on the contrary that $x \notin F$ (i.e., $x \in X \setminus F$ that is open). There exists N such that

$$x_n \in X \setminus F, \forall n \geq N,$$

i.e., $x_n \notin F$, which is a contradiction. ■

R The converse may not be true. If the $(X, \mathcal{T})$ is metrizable, the converse holds. Counter-example: Consider the co-countable topological space $(X, \mathcal{T}_{\text{co-co}})$, where

$$\mathcal{T}_{\text{co-co}} = \{U \mid X \setminus U \text{ is a countable set}\} \bigcup \{\emptyset\},$$

and X is uncontable. Let $F \subsetneq$ be an un-countable set such that is closed under limits, e.g., $[0,1]$. It's clear that $X \setminus F \notin \mathcal{T}_{\text{co-co}}$, i.e., F is not closed.

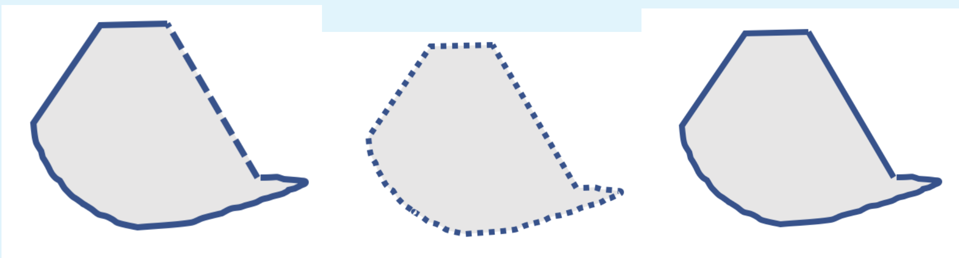
2.3.2. Interior, Closure, Boundary

Definition 2.5 Let $(X,\mathcal{T})$ be a topological space, and $A \subseteq X$ a subset.

1. The **interior** of A is
$$A^\circ = \bigcup_{U \subseteq A, U \text{ is open}} U$$
2. The **closure** of A is
$$\overline{A} = \bigcap_{A \subseteq V, V \text{ is closed}} V$$

If $\overline{A} = X$, we say that A is dense in X.

The graph illustration of the definition above is as follows:



(a) Illustration of A (b) Illustration of A° (c) Illustration of $\overline{A}$

Figure 2.1: Graph Illustrations

■ **Example 2.10** 1. For $[a,b) \subseteq \mathbb{R}$, we have:
$$[a,b)^\circ = (a,b), \quad \overline{[a,b)} = [a,b]$$
2. For $X = \mathbb{R}$, $\mathbb{Q}^\circ = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$.
3. Consider the discrete topology $(X, \mathcal{T}_{\text{discrete}})$, we have
$$S^\circ = S, \quad \overline{S} = S$$

The insights behind the definition (2.5) is as follows

Proposition 2.10

1. A° is the largest open subset of X contained in A; $\overline{A}$ is the smallest closed subset of X containing A.
2. If $A \subseteq B$, then $A^\circ \subseteq B$ and $\overline{A} \subseteq \overline{B}$
3. A is open in X is equivalent to say $A^\circ = A$; A is closed in X is equivalent to say $\overline{A} = A$.

■ **Example 2.11** Let (X,d) be a metric space. What's the closure of an open ball $B_r(x)$? The direct intuition is to define the closed ball

$$\bar{B}_r(x) = \{y \in X \mid d(x,y) \le r\}.$$

Question: is $\bar{B}_r(x) = \overline{B_r(x)}$?

1. Since $\bar{B}_r(x)$ is a closed subset of X, and $B_r(x) \subseteq \bar{B}_r(x)$, we imply that

$$\overline{B_r(x)} \subseteq \bar{B}_r(x)$$

2. However, we may find an example such that $\overline{B_r(x)}$ is a proper subset of $\bar{B}_r(x)$: Consider the discrete metric space (X, d_{discrete}) and for $\forall x \in X$,

$$B_1(x) = \{x\} \implies \overline{B_1(x)} = \{x\}, \quad \bar{B}_1(x) = X$$

The equality $\bar{B}_r(x) = \overline{B_r(x)}$ holds when (X,d) is a normed space.

Here is another characterization of $\overline{A}$:

Proposition 2.11

$$\overline{A} = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \varnothing\}$$

Proof. Define

$$S = \{x \in X \mid \forall \text{open } U \ni x, U \bigcap A \neq \varnothing\}$$

It suffices to show that $\overline{A} = S$.

1. First show that S is closed:

$$X \setminus S = \{x \in X \mid \exists U_x \ni x \text{ open s.t. } U_x \bigcap A = \emptyset\}$$

Take $x \in X \setminus S$, we imply there exists open $U_x \ni x$ such that $U_x \bigcap A = \emptyset$. We claim $U_x \subseteq X \setminus S$:

   - For $\forall y \in U_x$, note that $U_x \ni y$ that is open, such that $U_x \bigcap A = \emptyset$. Therefore, $y \in X \setminus S$.

   Therefore, we have $x \in U_x \subseteq X \setminus S$ for any $\forall x \in X \setminus S$.

   Note that
$$X \setminus S = \bigcup_{x \in X \setminus S} \{x\} \subseteq \bigcup_{x \in X \setminus S} U_x \subseteq X \setminus S,$$
   which implies $X \setminus S = \bigcup_{x \in X \setminus S} U_x$ is open, i.e., S is closed in X.

2. By definition, it is clear that $A \subseteq S$:

$$\forall a \in A, \forall \text{open } U \ni a, U \bigcap A \supseteq \{a\} \neq \emptyset \implies a \in S.$$

   Therefore, $\overline{A} \subseteq \overline{S} = S$.

3. Suppose on the contrary that there exists $y \in S \setminus \overline{A}$.

   Since $y \notin \overline{A}$, by definition, there exists $F \supseteq A$ closed such that $y \notin F$.

   Therefore, $y \in X \setminus F$ that is open, and

$$(X \setminus F) \bigcap A \subseteq (X \setminus A) \bigcap A = \emptyset \implies y \notin S,$$

   which is a contradiction. Therefore, $S = \overline{A}$.

∎

Definition 2.6 [accumulation point] Let $A \subseteq X$ be a subset in a topological space. We call $x \in X$ are an **accumulation point (limit point)** of A if

$$\forall U \subseteq X \text{ open s.t. } U \ni x, (U \setminus \{x\}) \bigcap A \neq \emptyset.$$

The set of accumulation points of A is denoted as A' ■

Proposition 2.12 $\overline{A} = A \bigcup A'$.

2.4. Wednesday for MAT3040

Reviewing.

- Basis, Dimension
- Basis Extension
- $W_1 \bigcap W_2 = \emptyset$ implies $W_1 \oplus W_2 = W_1 + W_2$ (Direct Sum).

2.4.1. Remark on Direct Sum

Proposition 2.13 The set $W_1 + W_2 = W_1 \oplus W_2$ iff any $\boldsymbol{w} \in W_1 + W_2$ can be uniquely expressed as

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2,$$

where $\boldsymbol{w}_i \in W_i$ for $i = 1,2$.

(R) We can also define addiction among finite set of vector spaces $\{W_1, \dots, W_k\}$. If $\boldsymbol{w}_1 + \cdots + \boldsymbol{w}_k = \mathbf{0}$ implies $\boldsymbol{w}_i = 0, \forall i$, then we can write $W_1 + \cdots + W_k$ as

$$W_1 \oplus \cdots \oplus W_k$$

Proposition 2.14 — **Complementation.** Let $W \leq V$ be a vector subspace of a fintie dimension vector space V. Then there exists $W' \leq V$ such that

$$W \oplus W' = V.$$

Proof. It's clear that $\dim(W) := k \leq n := \dim(V)$. Suppose $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is a basis of W.

By the basis extension proposition, we can extend it into $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$, which is a basis of V.

Therefore, we take $W' = \operatorname{span}\{\boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_n\}$, which follows that

1. $W + W' = V$: $\forall \boldsymbol{v} \in V$ has the form

$$\boldsymbol{v} = (\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k) + (\alpha_{k+1} \boldsymbol{v}_{k+1} + \cdots + \alpha_n \boldsymbol{v}_n),$$

   where $\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k \in W$ and $\alpha_{k+1} \boldsymbol{v}_{k+1} + \cdots + \alpha_n \boldsymbol{v}_n \in W'$.
2. $W \bigcap W' = \{\mathbf{0}\}$: Suppose $\boldsymbol{v} \in W \bigcap W'$, i.e.,

$$\begin{aligned} \boldsymbol{v} &= (\beta_1 \boldsymbol{v}_1 + \cdots + \beta_k \boldsymbol{v}_k) + (0\boldsymbol{v}_{k+1} + \cdots + 0\boldsymbol{v}_n) \in W \\ &= (0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_k) + (\beta_{k+1} \boldsymbol{v}_{k+1} + \cdots + \beta_n \boldsymbol{v}_n) \in W'. \end{aligned}$$

   By the uniqueness of coordinates, we imply $\beta_1 = \cdots = \beta_n = 0$, i.e., $\boldsymbol{v} = \mathbf{0}$.

Therefore, we conclude that $W \oplus W' = V$. ■

2.4.2. Linear Transformation

Definition 2.7 [Linear Transformation] Let V, W be vector spaces. Then $T : V \to W$ is a **linear transformation** if

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2),$$

for $\forall \alpha, \beta \in \mathbb{F}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$.

■ **Example 2.12** 1. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$ (where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$) is a linear transformation.

2. The transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined as

$$p(x) \mapsto T(p(x)) = p'(x), \quad p(x) \mapsto T(p(x)) = \int_0^x p(t)\,\mathrm{d}t$$

   is a linear transformation

3. The transformation $T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined as

$$\boldsymbol{A} \mapsto \mathrm{trace}(\boldsymbol{A}) := \sum_{i=1}^{n} a_{ii}$$

is a linear transformation.

However, the transformation

$$\boldsymbol{A} \mapsto \det(\boldsymbol{A})$$

is not a linear transformation.

Definition 2.8 [Kernel/Image] Let $T : V \to W$ be a linear transfomation.

1. The **kernel** of T is

$$\ker(T) = T^{-1}(\mathbf{0}) = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \mathbf{0}\}$$

2. The **image** (or range) of T is

$$\mathsf{Im}(T) = T(\boldsymbol{v}) = \{T(\boldsymbol{v}) \in W \mid \boldsymbol{v} \in V\}$$

Example 2.13

1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$, then

$$\ker(T) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\} = \mathsf{Null}(\boldsymbol{A}) \qquad \text{Null Space}$$

and

$$\mathsf{Im}(T) = \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n\} = \mathsf{Col}(\boldsymbol{A}) = \mathrm{span}\{\text{columns of } \boldsymbol{A}\} \qquad \text{Column Space}$$

2. For $T(p(x)) = p'(x)$, $\ker(T) = \{\text{constant polynomials}\}$ and $\mathsf{Im}(T) = \mathbb{R}[x]$.

Proposition 2.15 The kernel or image for a linear transformation $T : V \to W$ also forms a vector subspace:

$$\ker(T) \le V, \quad \text{Im}(T) \le W$$

Proof. For $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \ker(T)$, we imply

$$T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) = \mathbf{0},$$

which implies $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 \in \ker(T)$.

The remaining proof follows similarly. ■

Definition 2.9 [Rank/Nullity] Let V, W be finite dimensional vector spaces and $T : V \to W$ a linear transformation. Then we define

$$\text{rank}(T) = \dim(\text{im}(T))$$
$$\text{nullity}(T) = \dim(\ker(T))$$

R Let

$$\text{Hom}_{\mathbb{F}}(V, W) = \{\text{all linear transformations } T : V \to W\},$$

and we can define the addiction and scalar multiplication to make it a vector space:

1. For $T, S \in \text{Hom}_{\mathbb{F}}(V, W)$, define

$$(T + S)(\boldsymbol{v}) = T(\boldsymbol{v}) + S(\boldsymbol{v}),$$

which implies $T + S \in \text{Hom}_{\mathbb{F}}(V, W)$.

2. Also, define

$$(\gamma T)(\boldsymbol{v}) = \gamma T(\boldsymbol{v}), \qquad \text{for } \forall \gamma \in \mathbb{F},$$

which implies $\gamma T \in \text{Hom}_{\mathbb{F}}(V, W)$.

In particular, if $V=\mathbb{R}^n, W=\mathbb{R}^m$, then

$$\text{Hom}_{\mathbb{F}}(V,W)=M_{m\times n}(\mathbb{R}).$$

Proposition 2.16 If $\dim(V)=n,\dim(W)=m$, then $\dim(\text{Hom}_{\mathbb{F}}(V,W))=mn$.

Proposition 2.17 There are anternative characterizations for the injectivity and surjectivity of lienar transformation T:

1. The linear transformation T is injective if and only if

$$\ker(T)=0, \Longleftrightarrow \text{nullity}(T)=0.$$

2. The linear transformation T is surjective if and only if

$$\text{im}(T)=W, \Longleftrightarrow \text{rank}(T)=\dim(W).$$

3. If T is bijective, then T^{-1} is a linear transformation.

Proof. 1. (a) For the forward direction of (1),

$$\boldsymbol{x}\in\ker(T) \implies T(\boldsymbol{x})=0=T(\mathbf{0}) \implies \boldsymbol{x}=\mathbf{0}$$

(b) For the reverse direction of (1),

$$T(\boldsymbol{x})=T(\boldsymbol{y}) \implies T(\boldsymbol{x}-\boldsymbol{y})=\mathbf{0} \implies \boldsymbol{x}-\boldsymbol{y}\in\ker(T)=\mathbf{0} \implies \boldsymbol{x}=\boldsymbol{y}$$

2. The proof follows similar idea in (1).
3. Let $T^{-1}:W\to V$. For all $\boldsymbol{w}_1,\boldsymbol{w}_2\in W$, there exists $\boldsymbol{v}_1,\boldsymbol{v}_2\in V$ such that $T(\boldsymbol{v}_i)=\boldsymbol{w}_i$, i.e., $T^{-1}(\boldsymbol{w}_i)=\boldsymbol{v}_i$ $i=1,2$.

Consider the mapping

$$
\begin{aligned}
T(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2) &= \alpha T(\boldsymbol{v}_1) + \beta T(\boldsymbol{v}_2) \\
&= \alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2,
\end{aligned}
$$

which implies $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 = T^{-1}(\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2)$, i.e.,

$$
\alpha T^{-1}(\boldsymbol{w}_1) + \beta T^{-1}(\boldsymbol{w}_2) = T^{-1}(\alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2).
$$

■

Definition 2.10 [isomorphism]

We say the vector subspaces V and W are isomorphic if there exists a bijective linear transfomation $T : V \to W$. ($V \cong W$)

This mapping T is called an **isomorphism** from V to W.

R If $\dim(V) = \dim(W) = n < \infty$, then $V \cong W$:

Take $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}, \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$ as basis of V and W, respectively. Then one can construct $T : V \to W$ satisfying $T(\boldsymbol{v}_i) = \boldsymbol{w}_i$ for $\forall i$ as follows:

$$
T(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n) = \alpha_n \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n \ \forall \alpha_i \in \mathbb{F}
$$

It's clear that our constructed T is a linear transformation.

R $V \cong W$ doesn't imply any linear transformations $T : V \to W$ is an isomorphism. e.g., $T(\boldsymbol{v}) = \boldsymbol{0}$ is not an isomorphic if $W \neq \{\boldsymbol{0}\}$.

Theorem 2.3 — **Rank-Nullity Theorem.** Let $T : V \to W$ be a linear transformation with $\dim(V) < \infty$. Then

$$
\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).
$$

Proof. Since $\ker(T) \leq V$, by proposition (2.14), there exists $V_1 \leq V$ such that

$$V = \ker(T) \oplus V_1.$$

1. Consider the transformation $T|_{V_1}: V_1 \to T(V_1)$, which is an isomorphism, since:
   - Surjectivity is immediate
   - For $\boldsymbol{v} \in \ker(T|_{V_1})$,
   $$T(\boldsymbol{v}) = \mathbf{0} \implies \boldsymbol{v} \in \ker(T),$$
   which implies $\boldsymbol{v} = \mathbf{0}$ since $\boldsymbol{v} \in \ker(T) \cap V_1 = 0$, i.e., the injectivity follows.

   Therefore, $\dim(V_1) = \dim(T(V_1))$.
2. Secondly, given an isomorphism T from X to Y with $\dim(X) < \infty$, then $\dim(X) = \dim(T(X))$. The reason follows from assignment 1 questions (8-9):
$$\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} \text{ is a basis of } X \implies \{T(\boldsymbol{v}_1), \dots, T(\boldsymbol{v}_k)\} \text{ is a basis of } Y$$
3. Note that $T(V_1) = T(V) = \text{im}(T)$, since:
   - for $\forall \boldsymbol{v} \in V$, $\boldsymbol{v} = \boldsymbol{v}_k + \boldsymbol{v}_1$, where $\boldsymbol{v}_k \in \ker(T), \boldsymbol{v}_1 \in V_1$, which implies
   $$T(\boldsymbol{v}) = T(\boldsymbol{v}_k) + T(\boldsymbol{v}_1) = \mathbf{0} + T(\boldsymbol{v}_1),$$
   i.e., $T(V) \subseteq T(V_1) \subseteq T(V)$, i.e., $T(V) = T(V_1)$.
4. By the proof of complementation,
$$\begin{aligned}
\dim(V) &= \dim(\ker(T)) + \dim(V_1) \\
&= \text{nullity}(T) + \dim(T(V_1)) \\
&= \text{nullity}(T) + \dim(T(V)) \\
&= \text{nullity}(T) + \dim(\text{im}(T)) \\
&= \text{nullity}(T) + \text{rank}(T).
\end{aligned}$$

2.5. Wednesday for MAT3006

2.5.1. Compactness

This lecture will talk about the generalization of closeness and boundedness property in $\mathbb{R}^n$. First let's review some simple definitions:

Definition 2.11 [Compact] Let (X, d) be a metric space, and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of open sets.

1. $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is called an **open cover** of $E \subseteq X$ if $E \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$
2. A **finite subcover** of $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a finite sub-collection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\} \subseteq \{U_\alpha\}$ covering E.
3. The set $E \subseteq X$ is **compact** if every open cover of E has a finite subcover.

A well-known result is talked in MAT2006:

Theorem 2.4 — Heine-Borel Theorem. The set $E \subseteq \mathbb{R}^n$ is **compact** if and only if E is closed and bounded.

However, there's a notion of sequentially compact, and we haven't identify its gap and relation with compactness.

Definition 2.12 [Sequentially Compact] Let (X, d) be a metric space. Then $E \subseteq X$ is **sequentially compact** if every sequence in E has a convergent subsequence with limit in E.

A well-known result is talked in MAT2006:

Theorem 2.5 — Bolzano-Weierstrass Theorem. The set $E \subseteq \mathbb{R}^n$ is closed and bounded if and only if E is sequentially compact.

Actually, the definitions of comapctness and the sequential compactness are equivalent under a metric space.

Theorem 2.6 Let (X,d) be a metric space, then $E \subseteq X$ is compact if and only if E is sequentially compact.

Proof. Necessity

Suppose $\{x_n\}$ is a sequence in E, it suffices to show it has a convergent subsequence. Consider the tail of $\{x_n\}$, say

$$F_n = \{x_k \mid k \geq n\} \implies F_1 \supseteq F_2 \supseteq \cdots.$$

- Note that $\cap_{i=1}^{\infty} F_i \neq \emptyset$. Assume not, then we imply $\cup_{i=1}^{\infty}(E \setminus F_i) = E$, i.e., $\{E \setminus F_i\}_{i=1}^{\infty}$ a open cover of E. By the compactness of E, we imply there exists a finite subcover of E:
$$E = \bigcup_{j=1}^{r}(E \setminus F_{i_j}) \implies \bigcap_{j=1}^{r} F_{i_j} = \emptyset \implies F_{i_j} = \emptyset, \forall j$$
which is a contradiction, and there must exist an element $x \in \cap_{n=1}^{\infty} F_i$.
- For any $n \geq 1$, the open ball $B_{1/n}(x)$ must intersect with the n-th tail of the sequence $\{x_n\}$:
$$B_{1/n}(x) \cap \{x_k \mid k \geq n\} \neq \emptyset$$
Pick the r-th intersection, say x_{n_r}, which implies that the subsequence $x_{n_r} \to x$ as $r \to \infty$. The proof for necessity is complete.

Sufficiency

Firstly, let's assume the claim below hold (which will be shown later):

Proposition 2.18 If $E \subseteq X$ is sequentially compact, then for any$\varepsilon > 0$, there exists finitely many open balls, say $\{B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)\}$, covering E.

Suppose on the contrary that there exists an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of E, that has no finite subcover.

- By proposition (2.18), for $n \geq 1$, there are finitely many balls of radius $1/n$ covering E. Due to our assumption, there exists a open ball $B_{1/n}(y_n)$ such that $B_{1/n}(y) \cap E$ cannot be covered by finitely many members in $\{U_\alpha\}_{\alpha \in \mathcal{A}}$.

- Pick $x_n \in B_{1/n}(y_n)$ to form a sequence. Due to the sequential compactness of E, there exists a subsequence $\{x_{n_j}\} \to x$ for some $x \in E$.
- Since $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ covers E, there exists a U_β containing x. Since U_β is open and the radius of $B_{1/n_j}(y_{n_j})$ tends to 0, we imply that, for sufficiently large n_j, the set $B_{1/n_j}(y_{n_j}) \cap E$ is contained in U_β.

  In oteher words, U_β forms a **single** subcover of $B_{1/n}(y) \cap E$, which contradicts to our choice of $B_{1/n_j}(y_{n_j}) \cap E$. The proof for sufficiency is complete.

■

Proof for proposition (2.18). Pick $B_\varepsilon(x_1)$ for some $x_1 \in E$. Suppose $E \setminus B_\varepsilon(x_1) \neq \varnothing$. We can find $x_2 \notin B_\varepsilon(x_1)$ such that $d(x_2, x_1) \geq \varepsilon$.

Suppose $E \setminus (B_\varepsilon(x_1) \bigcup B_\varepsilon(x_2))$ is non-empty, then we can find $x_3 \notin B_\varepsilon(x_1) \bigcup B_\varepsilon(x_2)$ so that $d(x_j, x_3) \geq \varepsilon$, $j = 1, 2$.

Keeping this procedure, we obtain a sequence $\{x_n\}$ in E such that

$$E \setminus \bigcup_{j=1}^{n} B_\varepsilon(x_j) \neq \varnothing, \qquad \text{and} \qquad d(x_j, x_n) \geq \varepsilon, j = 1, 2, \dots, n-1.$$

By the sequential compactness of E, there exists $\{x_{n_j}\}$ and $x \in E$ so that $x_{n_j} \to x$ as $j \to \infty$. But then $d(x_{n_j}, x_{n_k}) < d(x_{n_j}, x) + d(x_{n_k}, x) \to 0$, which contradicts that $d(x_j, x_n) \geq \varepsilon$ for $\forall j < n$.

Therefore, one must have $E \setminus \bigcup_{j=1}^{N} B_\varepsilon(x_j) = \varnothing$ for some finite N.

The proof is complete. ■

R

1. Given the condition metric space,

$$\text{Sequential Compactness} \Longleftrightarrow \text{Compactness}$$

2. Given the condition metric space, we will show that

$$\text{Compactness} \implies \text{Closed and Bounded}$$

However, the converse may not necessarily hold. Given the condition the metric space is $\mathbb{R}^n$, then

$$\text{Compactness} \Longleftrightarrow \text{Closed and Bounded}$$

Proposition 2.19 Let (X,d) be a metric space. Then $E \subseteq X$ is compact implies that E is closed and bounded.

Proof.

1. Let $\{x_n\}$ be a convergent sequence in E. By sequential compactness, $\{x_{n_j}\} \to x$ for some $x \in E$. By the uniqueness of limits, under metric space, $\{x_n\} \to x$ for $x \in E$. The closeness is shown
2. Take $x \in E$ and consider the open cover $\bigcup_{n=1}^{\infty} B_n(x)$ of E. By compactness,

$$E \subseteq \bigcup_{i=1}^{k} B_{n_i}(x) = B_{n_k}(x),$$

which implies that for any $y,z \in E$,

$$d(y,z) \leq d(y,x) + d(x,z) \leq n_k + n_k = 2n_k.$$

The boundness is shown.

■

Here we raise several examples to show that the coverse does not necessarily hold under a metric space.

■ **Example 2.14** Given the metric space $\mathcal{C}[0,1]$ and a set $E = \{f \in \mathcal{C}[0,1] \mid 0 \leq f(x) \leq 1\}$. Notice that E is closed and bounded:

- $E = \cap_{x \in [0,1]} \Psi_x^{-1}([0,1])$, where $\Psi_x(f) = f(x)$, which implies that E is closed.
- Note that $E \subseteq B_2(\mathbf{0}) = \{f \mid |f| < 2\}$, i.e., E is bounded.

However, E may not be compact. Consider a sequence $\{f_n\}$ with

$$f_n(x) = \begin{cases} nx, & 0 \le x \le \frac{1}{n} \\ 1, & \frac{1}{n} \le x \le 1 \end{cases}$$

Suppose on the contrary that E is sequentially compact, therefore there exists a subsequence $\{f_{n_k}\} \to f$ under d_∞ metric, which implies, $\{f_{n_k}\}$ uniformly converges to f.

By the definition of $f_n(x)$, we imply

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \in (0,1] \end{cases}$$

However, since d_∞ indicates uniform convergence, the limit for $\{f_{n_k}\}$, say f, must be continuous, which is a contradiction.

■

Theorem 2.7 Let the set E be compact in (X,d) and the function $f : (X,d) \to (Y,\rho)$ is continuous. Then $f(E)$ is compact in Y.

Note that the technique to show compactness by using the sequential compactness is very useful. However, this technique only applies to the metric space, but fail in general topological spaces.

Proof. Let $\{y_n\} = \{f(x_n)\}$ be any sequence in $f(E)$.

- By the compactness of X, $\{x_n\}$ has a convergent subsequence $\{x_{n_r}\} \to x$ as $r \to \infty$.
- Therefore, $\{y_{n_r}\} := \{f(x_{n_r})\} \to f(x)$ by the continuity of f.
- Therefore, $f(E)$ is sequentially compact, i.e., compact.

■

R The Theorem (2.7) is a generalization of the statement that *a continuous function on $\mathbb{R}^n$ admits its minimum and maximum*. Note that such an extreme value property no longer holds for arbitrary closed, bounded sets in a general metric space, but it continues to hold when the sets are strengthened to compact ones.

Another characterization of compactness in $\mathcal{C}[a,b]$ is shown in the Ascoli-Arzela Theorem (see Theorem (14.1) in MAT2006 Notebook).

2.5.2. Completeness

Definition 2.13 [Complete] Let (X,d) be metric space.

1. A sequence $\{x_n\}$ in (X,d) is a **Cauchy sequence** if for every $\varepsilon>0$, there exists some N such that $d(x_n,x_m)<\varepsilon$ for all $n,m\geq N$.
2. A subset $E\subseteq X$ is said to be **complete** if every Cauchy sequence in E is convergent.

■ **Example 2.15** The set $X=\mathcal{C}[a,b]$ is complete:

- Suppose $\{f_n\}$ is Cauchy in $\mathcal{C}[a,b]$, i.e., $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$ for $\forall x\in[a,b]$.
- By the compactness of $\mathbb{R}$, the sequence $f_n(x)\to f(x)$ for some $f(x)\in\mathbb{R},\forall x\in[a,b]$. It suffices to show $f_n\to f$ uniformly:
  - For fixed $\varepsilon>0$, there exists $N>0$ such that

$$d_\infty(f_n,f_{n+k})<\frac{\varepsilon}{2},\qquad \forall n\geq N,k\in\mathbb{N}$$

which implies that for $\forall x\in[a,b]$, $\forall n\geq N,k\in\mathbb{N}$,

$$|f_n(x)-f_{n+k}(x)|<\frac{\varepsilon}{2}\implies\lim_{k\to\infty}|f_n(x)-f_{n+k}(x)|\leq\frac{\varepsilon}{2}$$

Therefore, we imply

$$|f_n(x) - f(x)| = \lim_{k \to \infty} |f_n(x) - f_{n+k}(x)| \le \frac{\varepsilon}{2} < \varepsilon, \qquad \forall n \ge N, x \in [a,b]$$

The proof is complete.

∎

2.6. Wednesday for MAT4002

Reviewing.

1. Interior, Closure:
$$\overline{A} = \{x \mid \forall U \ni x \text{ open}, U \bigcap A \neq \emptyset\}$$
2. Accumulation points

2.6.1. Remark on Closure

Definition 2.14 [Sequential Closure] Let A_S be the set of limits of any convergent sequence in A, then A_S is called the **sequential closure** of A.

Definition 2.15 [Accumulation/Cluster Points] The set of accumulation (limit) points is defined as
$$A' = \{x \mid \forall U \ni x \text{ open }, (U \setminus \{x\}) \bigcap A \neq \emptyset\}$$

R

1. (a) There exists some point in A but not in A':
$$A = \{1,2,3,\dots,n,\dots\}$$
Then any point in A is not in A'
   (b) There also exists some point in A' but not in A:
$$A = \{\frac{1}{n} \mid n \geq 1\}$$
Then the point 0 is in A' but not in A.
2. The closure $\overline{A} = A \bigcup A'$.
3. The size of the sequentical closure A_S is between A and $\overline{A}$, i.e., $A \subseteq A_S \subseteq \overline{A}$:

It's clear that $A \subseteq A_S$, since the sequence $\{a_n := a\}$ is convergent to a for $\forall a \in A$.

For all $a \in A_S$, we have $\{a_n\} \to a$. Then for any open $U \ni a$, there exists N such that $\{a_N, a_{N+1}, \dots\} \subseteq U \bigcap A \neq \emptyset$. Therefore, $a \in \overline{A}$, i.e., $A_S \subseteq \overline{A}$.

Question: Is $A_S = \overline{A}$?

Proposition 2.20 Let (X,d) be a metric space, then $A_S = \bar{A}$.

Proof. Let $a \in \overline{A}$, then there exists $a_n \in B_{1/n}(a) \bigcap A$, which implies $\{a_n\} \to a$, i.e., $a \in A_S$. ■

R If $(X,\mathcal{T})$ is metrizable, then $A_S = \overline{A}$. The same goes for first countable topological spaces. However, A_S is a proper subset of A in general:

Let $A \subseteq X$ be the set of continuous functions, where $X = \mathbb{R}^{\mathbb{R}}$ denotes the set of all real-valued functions on $\mathbb{R}$, with the topology of pointwise convergence.

Then $A_S = B_1$, the set of all functions of first Baire-Category on $\mathbb{R}$; and $[A_S]_S = B_2$, the set of all functions of second Baire-Category on $\mathbb{R}$. Since $B_1 \neq B_2$, we have $[A_S]_S = A_S$. Note that $\overline{\overline{A}} = \overline{A}$. We conclude that A_S cannot equal to $\overline{A}$, since the sequential closure operator cannot be idemotenet.

Definition 2.16 [Boundary] The **boundary of A** is defined as

$$\partial \boldsymbol{A} = \overline{A} \setminus A^{\circ}$$

■

Proposition 2.21 Let $(X,\mathcal{T})$ be a topological space with $A,B \subseteq X$.

$$\overline{X \setminus A} = X \setminus A^{\circ}, \quad (X \setminus B)^{\circ} = X \setminus \overline{B} \quad \partial A = \overline{A} \bigcap (\overline{X \setminus A})$$

Proof.

$$X \setminus A^\circ = X \setminus \left(\bigcup_{U \text{ is open}, U \subseteq A} U \right) \tag{2.2a}$$
$$= \bigcap_{U \text{ is open}, U \subseteq A} (X \setminus U) \tag{2.2b}$$
$$= \bigcap_{V \text{ is closed}, F \supseteq X \setminus A} F \tag{2.2c}$$
$$= \overline{X \setminus A} \tag{2.2d}$$

Denoting $X \setminus A$ by B, we obtain:

$$(X \setminus B)^\circ = A^\circ \tag{2.3a}$$
$$= X \setminus (X \setminus A^\circ) \tag{2.3b}$$
$$= X \setminus \overline{X \setminus A} \tag{2.3c}$$
$$= X \setminus \overline{B} \tag{2.3d}$$

By definition of ∂A,

$$\partial A = \overline{A} \setminus A^\circ \tag{2.4a}$$
$$= \overline{A} \bigcap (X \setminus A^\circ) \tag{2.4b}$$
$$= \overline{A} \bigcap (\overline{X \setminus A}) \tag{2.4c}$$

■

2.6.2. Functions on Topological Space

Definition 2.17 [Continuous] Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map. Then the function f is continuous, if

$$U \in \mathcal{T}_Y \implies f^{-1}(U) \in \mathcal{T}_X$$

■ **Example 2.16**

1. The identity map $\text{id} : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined as $x \mapsto x$ is continuous
2. The identity map $\text{id} : (X, \mathcal{T}_{\text{discrete}}) \to (X, \mathcal{T}_{\text{indiscrete}})$ defined as $x \mapsto x$ is continuous. Since $\text{id}^{-1}(\emptyset) = \emptyset$ and $\text{id}^{-1}(X) = X$
3. The identity map $\text{id} : (X, \mathcal{T}_{\text{indiscrete}}) \to (X, \mathcal{T}_{\text{discrete}})$ defined as $x \mapsto x$ is not continuous.

Proposition 2.22 If $f : X \to Y$, and $g : Y \to Z$ be continuous, then $g \circ f$ is continuous

Proof. For given $U \in \mathcal{T}_Z$, we imply

$$g^{-1}(U) \in \mathcal{T}_Y \implies f^{-1}(g^{-1}(U)) \in \mathcal{T}_X,$$

i.e., $(g \circ f)^{-1}(U) \in \mathcal{T}_X$ ∎

Proposition 2.23 Suppose $f : X \to Y$ is continuous between two topological spaces. Then $\{x_n\} \to X$ implies $\{f(x_n)\} \to f(x)$.

Proof. Take open $U \ni f(x)$, which implies $f^{-1}(U) \ni x$. Since $f^{-1}(U)$ is open, we imply there exists N such that

$$\{x_n \mid n \geq N\} \subseteq f^{-1}(U),$$

i.e., $\{f(x_n) \mid n \geq N\} \subseteq U$ ∎

We use the notion of Homeomorphism to describe the equivalence between two topological spaces.

Definition 2.18 [Homeomorphism] A **homeomorphism** between spaces topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a bijection

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y),$$

such that both f and f^{-1} are continuous.

2.6.3. Subspace Topology

Definition 2.19 Let $A \subseteq X$ be a non-empty set. The **subspace topology** of A is defined as:

1. $\mathcal{T}_A := \{U \bigcap A \mid U \in \mathcal{T}_A\}$
2. The **coarsest** topology on A such that the **inclusion map**

$$i : (A, \mathcal{T}_A) \to (X, \mathcal{T}_X), \quad i(x) = x$$

   is continuous.

   (We say the topology $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, or $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ e.g., $\mathcal{T}_{\text{discrete}}$ is the finest topology, and $\mathcal{T}_{\text{indiscrete}}$ is coarsest topology.)
3. The (**unique**) topology such that for any $(Y, \mathcal{T}_Y)$,

$$f : (Y, \mathcal{T}_Y) \to (A, \mathcal{T}_A)$$

   is continuous iff $i \circ f : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ (where i is the inclusion map) is continuous.

Proposition 2.24 The definition (1) and (2) in (2.19) are equivalent.

Outline. The proof is by applying

$$i^{-1}(S) = S \bigcap A, \quad \forall S$$

■

■ **Example 2.17** Let all English and numerical letters be subset of $\mathbb{R}^2$:

P,6

The homeomorphism can be constrcuted between these two English letters.

Proposition 2.25 The definition (2) and (3) in (2.19) are equivalent.

Proof. Necessity.

- For $\forall U \in \mathcal{T}_X$, consider that

$$(i \circ f)^{-1}(U) = f^{-1}(i^{-1}(U)) = f^{-1}(U \bigcap A)$$

since $U \bigcap A \in \mathcal{T}_A$ and f is continuous, we imply $(i \circ f)^{-1}(U) \in \mathcal{T}_Y$

- For $\forall U' \in \mathcal{T}_A$, we have $U' = U \bigcap A$ for some $U \in \mathcal{T}_X$. Therefore,

$$f^{-1}(U') = f^{-1}(U \bigcap A) = f^{-1}(i^{-1}(U)) = (i \circ f)^{-1}(U) \in \mathcal{T}_Y.$$

The sufficiency is left as exercise. ∎

Proposition 2.26 1. The definition (1) in (2.19) does define a topology of A

2. Closed sets of A under subspace topology are of the form $V \bigcap A$, where V is closed in X

Proposition 2.27 Suppose $(A, \mathcal{T}_A) \subseteq (X, \mathcal{T}_X)$ is a subspace topology, and $B \subseteq A \subseteq X$. Then

1. $\bar{B}^A = \bar{B}^X \bigcap A$.
2. $B^{\circ A} \supseteq B^{\circ X}$

Proof. By proposition (2.26), $\bar{B}^X \bigcap A$ is closed in A, and $\bar{B}^X \bigcap A \supset B$, which implies

$$\bar{B}^A \subseteq \bar{B}^X \bigcap A$$

Note that $\bar{B}^A \supset B$ is closed in A, which implies $\bar{B}^A = V \bigcap A \subseteq V$, where V is closed in X. Therefore,

$$\bar{B}^X \subseteq V \implies \bar{B}^X \bigcap A \subseteq V \bigcap A = \bar{B}^A$$

Therefore, $\bar{B}^A = \bar{B}^X \subseteq V$

∎

Can we have $B^{\circ X} = B^{\circ A}$?

2.6.4. Basis (Base) of a topology

Roughly speaking, a basis of a topology is a family of "generators" of the topology.

Definition 2.20 Let $(X, \mathcal{T})$ be a topological space. A family of subsets $\mathcal{B}$ in X is a **basis** for $\mathcal{T}$ if

1. $\mathcal{B} \subseteq \mathcal{T}$, i.e., everything in $\mathcal{B}$ is open
2. Every $U \in \mathcal{T}$ can be written as union of elements in $\mathcal{B}$.

■ **Example 2.18**

1. $\mathcal{B} = \mathcal{T}$ is a basis.
2. For $X = \mathbb{R}^n$,

$$\mathcal{B} = \{B_r(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{Q}^n, r \in \mathbb{Q} \bigcap (0, \infty)\}$$

   Exercise: every $(a, b) = \bigcup_{i \in I} (p_i, q_i)$ for $p_i, q_i \in \mathbb{Q}$.

   Therefore, $\mathcal{B}$ is countable.

Proposition 2.28 If $(X, \mathcal{T})$ has a countable basis, e.g., $\mathbb{R}^n$, then $(X, \mathcal{T})$ has a second-countable space.