On the Capacity Scalability of Line Networks with Buffer Size Constraints

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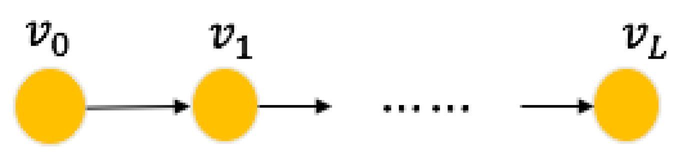
Problem Definition and Contribution

Problem: Study the communication capacity of a line network with L hops

Assumption:

- Adjacent nodes are connected by the same channel Q with zero-error capacity 0
- The intermediate network nodes have a buffer size $\mathcal{O}(\log\log L)$, which is minimal for the intermediate nodes to synchronize different batches



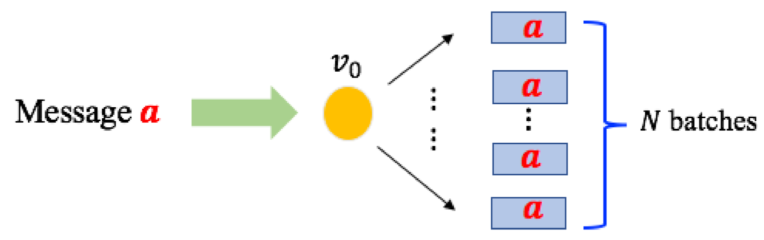
Summarization of Results: (In the table c is a constant value)

	buffer size	achievable rate
[1]	unbounded	min-cut capacity
[2]	$B = O(1)$	$\Omega(e^{-cL})$
this paper	$B = O(\ln\ln L)$	$\Omega(1/\ln L)$
[2]	$B = O(\ln L)$	$\Theta(1)$

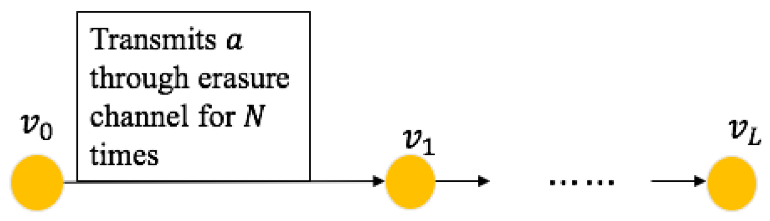
Line Networks of Packet Erasure Channels

The coding scheme for this case is a $(1, N)$ batched code, where the inner code consists of repetition codes for the erasure channel with erasure probability ϵ, and the outer code is an erasure correction code:

1. The source node duplicates the message a into N batches:



2a. The source node v_0 transmits the symbol a N times through the channel (v_0, v_1)



2b. Intermediate nodes tries to recover the symbol a from the N outputs from (v_{i-1}, v_i). If successful, the node v_i transmits the recovered symbol N times through the channel (v_i, v_{i+1}).

The destination node v_L can recover a with probability $(1-\epsilon^N)^L$. Normalized by the channel uses N, the achievable rate is given by $(1-\epsilon^N)^L/N$. Therefore, the best performance of our coding scheme is given by:

$$\max_N \frac{(1-\epsilon^N)^L}{N} = \Theta(1/\log L),$$

which is achieved when $N = \Theta(\log L)$.

Line Networks of Binary Symmetric Channels

The coding scheme for this case is the $(1, N)$ batched code. Here the outer code is a capacity achieving code for BSCs instead. The transmission between two adjacent nodes per N channel uses can be regarded as a BSC with the transition matrix

$$\tilde{Q} = \begin{pmatrix} 1-P_e & P_e \\ P_e & 1-P_e \end{pmatrix}, \text{ where } P_e = \sum_{i \le N/2} \binom{N}{i} (1-\epsilon)^i \epsilon^{N-i}$$

The transmission of one bit using above inner code from v_0 to v_L is the L-concatenation of $\tilde{Q}$, with the transition matrix

$$\tilde{Q}^L = \begin{pmatrix} 1-\tilde{\epsilon} & \tilde{\epsilon} \\ \tilde{\epsilon} & 1-\tilde{\epsilon} \end{pmatrix}, \text{ where } \tilde{\epsilon} = \frac{1-(1-2P_e)^L}{2}$$

Thus the achievable rate for this line network is $[1-h_2(\tilde{\epsilon})]/N$, and the best performance of our coding scheme is given by:

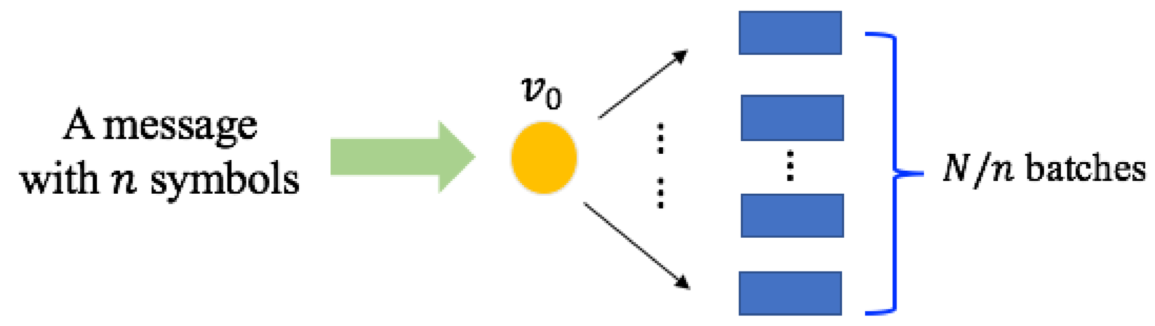
$$\max_N \frac{1-h_2(\tilde{\epsilon})}{N} = \Omega(1/\log L),$$

where the order $1/\log L$ is achieved when $N = \Theta(\log L)$.

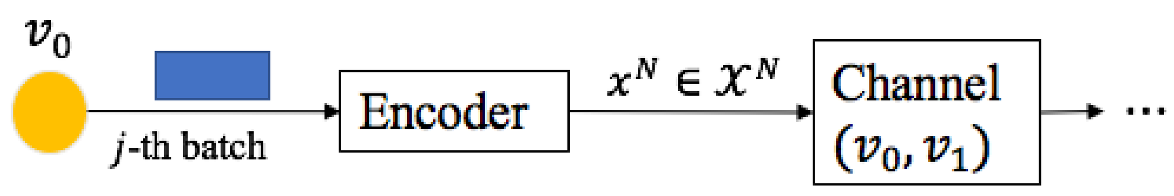
Batched Code Description

A (m, N) batched code has an outer code and an inner code:

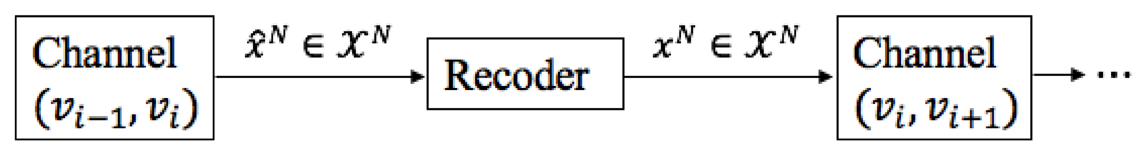
- At the source node v_0, the outer code encodes a message (with n input symbols) into N/n batches, each of which has m symbols.



- The inner code performs on different batches separately, i.e., for fixed $j = 1, \dots, N/n$,
  - The source v_0 generates and transmits N symbols using the j-th batch



  - Intermediate nodes v_i generates and transmits N symbols using received symbols belonging to the j-th batch:



- The destination node v_L uses all the N received symbols to decode the message of the source node.

Line Networks of General DMCs

We can covert any line network of a general DMC Q with positive capacity to one of a non-trivial BSC:

- Since the DMC Q has the positive capacity, we imply $|\mathcal{X}|, |\mathcal{Y}| \ge 2$, and there exists two inputs, say 0 and 1, such that the corresponding rows of Q are not identical.
- Let $P = (p_{ij})_{2\times|\mathcal{Y}|}$ be the transition matrix formed by the two rows of Q corresponding to 0 and 1
- Construct an inner code scheme that uses only 0 and 1 as input for channel Q, and the output of the channel can be converted to 0 or 1 using the transition matrix

$$W = \begin{pmatrix} w_1 & 1-w_1 \\ \vdots & \vdots \\ w_{|\mathcal{Y}|} & 1-w_{|\mathcal{Y}|} \end{pmatrix}$$

where $w_k = p_{1k}/(p_{1k}+p_{2k})$ for k satisfying $p_{1k}+p_{2k} > 0$ and otherwise $w_k = 1$.

- The above operation gives a binary-input-binary-output transition matrix

$$PW = \begin{pmatrix} \sum_k p_{1k}w_k & 1-\sum_k p_{1k}w_k \\ \sum_k p_{2k}w_k & 1-\sum_k p_{2k}w_k \end{pmatrix}$$

We can verify that PW is our desired BSC with positive capacity.

References

[1] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. John Wiley & Sons, Inc, 2006.

[2] U. Niesen, C. Fragouli and D. Tuninetti, “On Capacity of Line Networks,” in *IEEE Transactions on Information Theory*, vol. 53, no. 11, pp. 4039-4058, Nov. 2007.