Lecture 3
Eigenvalue, Matrix Decomposition

- Motivation
- Eigenvalues and Eigenvectors
- Properties about Eigenvalues
- Eigenvalue Decomposition
- Singular Value Decomposition

Contents

Example

- In a certain town, 30% married men get divorced each year and 20% single men get married each year.
- There are 8000 married men and 2000 single men at the beginning. Assume the total population always remains constant.
- Let $\mathbf{w}_0 = \begin{pmatrix} 8000 \\ 2000 \end{pmatrix}$ be the initial status, and $\mathbf{w}_i$ denote the status after i years.
- What is the marital status when time goes to infinity? How about we change $\mathbf{w}_0$?

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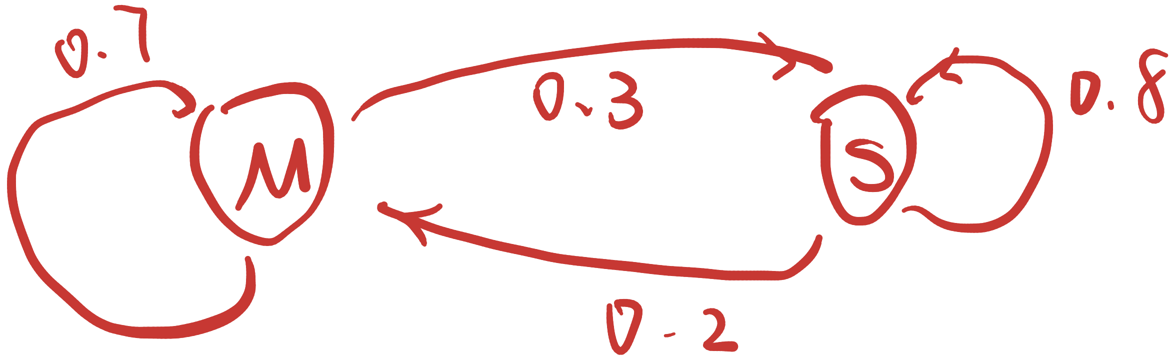

Example

$w^* = \begin{pmatrix} 4000 \\ 6000 \end{pmatrix}$ $Aw^* = w^*$

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- There are 8000 married men and 2000 single men at the beginning. Assume the total population always remains constant.
- Let $\mathbf{w}_0 = \begin{pmatrix} 8000 \\ 2000 \end{pmatrix}$ be the initial status, and $\mathbf{w}_i$ denote the status after i years.

$$W_i = \begin{pmatrix} \text{Married after i-th year} \\ \text{Single after i-th year} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} W_{i-1}$$

$0.7 W_{i-1,1} + 0.2 W_{i-1,2}$

$0.3 W_{i-1,1} + 0.8 W_{i-1,2}$

- What is the marital status when time goes to infinity? How about we change $\mathbf{w}_0$?

$$W_t = AW_{t-1} = A^2 W_{t-2} = \cdots = A^t W_0$$

Example

- Let
$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}.$$
and $\mathbf{w}_i = A\mathbf{w}_{i-1} = A^i\mathbf{w}_0$.
- Using computer, we find $\mathbf{w}_n \to \begin{pmatrix} 4000 \\ 6000 \end{pmatrix}$ as $n \to \infty$.
- We represent a general initial marital status as
$$\mathbf{w}_0 = x_1\mathbf{u}_1 + x_2\mathbf{u}_2, \text{ where } \mathbf{u}_1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
We check that $A\mathbf{u}_1 = \mathbf{u}_1$ and $A\mathbf{u}_2 = 0.5\mathbf{u}_2$
- Then the marital status after n years is
$$A^n\mathbf{w}_0 = x_1A^n\mathbf{u}_1 + x_2A^n\mathbf{u}_2 = x_1\mathbf{u}_1 + x_2 0.5^n\mathbf{u}_2 \to x_1\mathbf{u}_1.$$

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$0.4x_1 + x_2 = 8000$
$0.6x_1 - x_2 = 2000$

Example

$A u_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.28 + 0.12 \\ 0.12 + 0.48 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$

- Let

$\Rightarrow x_1 = 10000$

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}.$$

and $\mathbf{w}_i = A\mathbf{w}_{i-1} = A^i \mathbf{w}_0$.

$A u_2 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.7 - 0.2 \\ 0.3 - 0.8 \end{pmatrix} = \begin{pmatrix} +0.5 \\ -0.5 \end{pmatrix}$

- Using computer, we find $\mathbf{w}_n \to \begin{pmatrix} 4000 \\ 6000 \end{pmatrix}$ as $n \to \infty$.
- We represent a general initial marital status as

$$\mathbf{w}_0 = x_1\mathbf{u}_1 + x_2\mathbf{u}_2, \text{ where } \mathbf{u}_1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$A^n u_1 = A^{n-1}(Au_1) = A^{n-1} u_1$

We check that $A\mathbf{u}_1 = \mathbf{u}_1$ and $A\mathbf{u}_2 = 0.5\mathbf{u}_2$

$A^n u_2 = A^{n-1}(Au_2) = 0.5 A^{n-1} u_2$

- Then the marital status after n years is

$A^n(x_1 u_1 + x_2 u_2)$

$$A^n\mathbf{w}_0 = x_1 A^n \mathbf{u}_1 + x_2 A^n \mathbf{u}_2 = x_1\mathbf{u}_1 + x_2 0.5^n \mathbf{u}_2 \to x_1\mathbf{u}_1.$$

Contents

Definition

- Let A be an $n \times n$ matrix.
- A scalar λ is said to be an **eigenvalue** of A if there exists a nonzero vector $\mathbf{x}$ such that

eigenvalue

$$A\mathbf{x} = \lambda\mathbf{x}.$$

eigenvector associated with λ

- The vector $\mathbf{x}$ is said to be an **eigenvector belonging to** λ.

(λ, x) is an eigen-pair of A

Implications

$$Ax = \lambda x$$

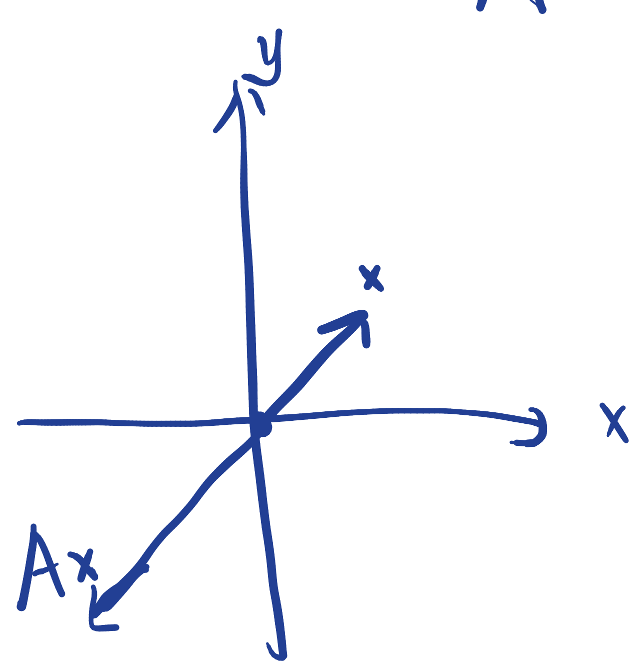
$$cAx = \lambda x \qquad c \neq 0 \iff A(cx) = \lambda(cx)$$

- An eigenvector $\mathbf{x}$ and $A\mathbf{x}$ have the same direction.
- If $\mathbf{x}$ is an eigenvector belonging to λ, so is $c\mathbf{x}$ for any $c \neq 0$.
- If $\mathbf{x}$ is an eigenvector of A belonging to λ, then $\mathbf{x}$ is an eigenvector of A^s belonging to λ^s.

$$Ax = \lambda x \Rightarrow A^s x = A^{s-1}(Ax) = \lambda A^{s-1} x = \cdots = \lambda^s x$$

$s \in \mathbb{N}_+$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & -3 \\ 1 & -4 \end{pmatrix} \qquad Ax = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

Example

- Let

$$A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Then

$$A\mathbf{u} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = (-3)\mathbf{u}.$$

- Thus, -3 is the eigenvalue of A and the corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Equivalent Characterizations

- Note that $A\mathbf{x} = \lambda\mathbf{x}$ is equivalent to

$$\boxed{A}\,\boxed{x} = \lambda\,\boxed{x} \qquad\qquad (A - \lambda I)\mathbf{x} = \mathbf{0}. \qquad\qquad I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

  Note that it is not $(A - \lambda)\mathbf{x} = \mathbf{0}$.

- The following statements are equivalent:
  - λ is an eigenvalue of A.
  - $(A - \lambda I)\mathbf{x} = 0$ has a nontrivial solution.
  - $\mathcal{N}(A - \lambda I) \neq \{\mathbf{0}\}$.
  - $A - \lambda I$ is singular.
  - $\det(A - \lambda I) = 0$.

- $\mathcal{N}(A - \lambda I)$ is called the **eigenspace** of eigenvalue λ.

- All nonzero vectors in $\mathcal{N}(A - \lambda I)$ are eigenvectors corresponding to λ.

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Characteristic polynomial

• Step 2: Solve chara. equation.

Find λ s.t. $p(\lambda)=0$

$\lambda_1=4 \qquad \lambda_2=-3$

- $p(\lambda) = \det(A - \lambda I)$ is an nth degree polynomial in λ.
- $p(\lambda)$ is called the **characteristic polynomial** of A.
- $p(\lambda) = 0$ is called the **characteristic equation** of A.
- A scalar λ is an eigenvalue of A if and only if $p(\lambda) = 0$.

$A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$

• Step 1: write down characteristic polynomial.

$p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) - 6$

$= \lambda^2 - \lambda - 12 = (\lambda-4)(\lambda+3)$

Example

- The characteristic polynomial of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ is

$$p(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3) = 0.$$

- Hence, the the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -3$.

- The eigenvectors belonging to λ_1 are nonzero solutions of

$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow x_1 = 2x_2$

$$(A - 4I)\mathbf{x} = \mathbf{0}.$$

$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}$

$c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where $c \neq 0$

The eigenvectors belonging to λ_2 are nonzero solutions of

$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$(A + 3I)\mathbf{x} = \mathbf{0}.$$

$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$

$c \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, where $c \neq 0$

Example

$\lambda_1 = 0$ $\lambda_2 = 1$

$(A - \lambda_1 I)x = 0 = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\left.\begin{array}{l} 2x_1 - 3x_2 + x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \\ x_1 - 3x_2 + 2x_3 = 0 \end{array}\right\} \Rightarrow x_1 = x_2 = x_3$

$\Rightarrow c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ c \neq 0$

Find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

① $A - \lambda I = \begin{pmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{pmatrix}$

$\det(A - \lambda I) = (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2-\lambda \\ 1 & -3 \end{vmatrix}$

$= -\lambda(\lambda - 1)^2$

$(A - \lambda_2 I)x = 0$

$\Rightarrow \begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix} x = 0$

$\Rightarrow x_1 - 3x_2 + x_3 = 0.$

$\left\{ \begin{pmatrix} 3x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} : x_2, x_3 \in R \right\}$

Example

$i = \sqrt{-1}$

Find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 0$

$(1-\lambda)^2 = -4$

$\lambda - 1 = \pm\sqrt{-4} = 2i$

$\lambda_1 = 1+2i \qquad \lambda_2 = 1-2i$

Complex Eigenvalues of Real Matrices

- As $p(\lambda)$ has degree n, $p(\lambda)$ can be factored into the product of n linear terms:

$$p(\lambda) = c(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda), \tag{1}$$

  where λ_i is the root of $p(\lambda)$.

- For real valued matrices,
  - Complex eigenvalues occur in **conjugate pairs**, i.e., if λ is an eigenvalue, so is $\bar{\lambda}$.
  - If $\mathbf{z}$ is an eigenvector belonging to a complex eigenvalue λ, then $\bar{\mathbf{z}}$ is an eigenvector belonging to $\bar{\lambda}$.

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- For real valued matrices, $Ax = \lambda x \Rightarrow \overline{Ax} = \overline{\lambda x}$
  $\bar{A}\bar{x} = \bar{\lambda}\bar{x}$
  - Complex eigenvalues occur in **conjugate pairs**, i.e., if λ is an eigenvalue, so is $\bar{\lambda}$. $\lambda = a + bi$ $\bar{\lambda} = a - bi$ $\Rightarrow A\bar{x} = \bar{\lambda}\bar{x}$
  - If $\mathbf{z}$ is an eigenvector belonging to a complex eigenvalue λ, then $\bar{\mathbf{z}}$ is an eigenvector belonging to $\bar{\lambda}$.

(λ, x) is eigen-pair

$\Rightarrow (\bar{\lambda}, \bar{x})$ is another eigen-pair

Contents

Multiplicity of Eigenvalues

- We know that $\lambda_1, \ldots, \lambda_n$ may not be all distinct.
- Let $\lambda_1, \ldots, \lambda_p$ be the p distinct eigenvalues.
- The eigenvalue λ_k has *multiplicity* m_k. We know that $\sum_k m_k = n$.
- The characteristic polynomial can be written as

$$p(\lambda) = c(\lambda_1 - \lambda)^{m_1} \cdots (\lambda_p - \lambda)^{m_p}. \tag{2}$$

- Example: For $p(\lambda) = (1-\lambda)^2(4-\lambda)^3$, the multiplicity of 1 is 2 and the multiplicity of 4 is 3.

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$p(\lambda) = -\lambda(\lambda-1)^2$

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Product and Sum of the Eigenvalues

- Consider an $n \times n$ square matrix $A = (a_{ij})$.
- Let $\lambda_1, \ldots, \lambda_n$ be the n eigenvalues of A.
- $\prod_{i=1}^{n} \lambda_i = \det(A)$.
- $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$.
- Proof:

Product and Sum of the Eigenvalues

① degree n term: $c(-\lambda)^n = (-\lambda)^n \Rightarrow c=1$

③ $p(\lambda) = \det(A-\lambda I)$

$p(0) = \det(A) = c\,\lambda_1 \cdots \lambda_n = \prod_{i}^{n} \lambda_i$

② degree $(n-1)$-term: $c \sum \lambda_i (-\lambda)^{n-1} = \sum a_{ii} (-\lambda)^{n-1} \Rightarrow c \sum \lambda_i = \sum a_{ii} \Rightarrow \sum \lambda_i = \mathrm{Tr}(A)$

- Consider an $n \times n$ square matrix $A = (a_{ij})$.
- Let $\lambda_1, \dots, \lambda_n$ be the n eigenvalues of A.
- $\prod_{i=1}^{n} \lambda_i = \det(A)$.
- $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$.
- Proof:

$p(\lambda) = c(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$

$= \det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$

$(a_{11}-\lambda) \cdots (a_{nn}-\lambda)$ accounts for degree n and degree $(n-1)$ terms

Transpose and Inverse

- As $|A - \lambda I| = |A^\top - \lambda I|$, A and $A^\top$ have same characteristic polynomial, and hence the same eigenvalues.
- If A is singular, 0 is an eigenvalue of A.
- If A is invertible, λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.
- Proof:

Transpose and Inverse

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- If A is singular, 0 is an eigenvalue of A. $Ax = 0 = 0 \cdot x$
- If A is invertible, λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.
- Proof: $Ax = \lambda x$

$$x = A^{-1}(Ax) = A^{-1}(\lambda x) = \lambda A^{-1} x$$

$$\Rightarrow A^{-1} x = \lambda^{-1} x$$

Stochastic Matrix

$A = \begin{pmatrix} 0.3 & 0.2 \\ 0.7 & 0.8 \end{pmatrix}$

- An $n \times n$ matrix A is a stochastic matrix if
  1. all the entries are non-negative ($a_{ij} \geq 0$);
  2. the summation of each column is 1 ($\mathbf{1}^\top A = \mathbf{1}^\top$).

$A = [a_1, \dots, a_n]$

- For any vector $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x}$ and $\mathbf{x}$ have the same sum.

$1^T A = [1^T a_1, \dots, 1^T a_n]$

$= [1, \dots, 1]$

$= 1^T$

- 1 is an eigenvalue of A (and $A^\top$).
- All the eigenvalues λ of A have $|\lambda| \leq 1$.
- Proof:

$1^T(Ax) = (1^T A)x = 1^T x$

$A^T 1 = 1 \Rightarrow A^T 1 = (1) \cdot 1$

$A^n x = \lambda^n x$

LHS absolute values are bounded

Contents

Spectral Theorem

- If A is a real symmetric matrix, the spectral theorem shows that there exists an orthogonal matrix that diagonalize A.
- Every real symmetric matrix A can be factored into $Q\Lambda Q^\top$ where Q is an orthogonal matrix and Λ is a real diagonal matrix.
- - The diagonal entries λ_i of Λ are eigenvalues of A.
  - The columns $\mathbf{q}_i$ of Q are eigenvectors belonging to λ_i, respectively.

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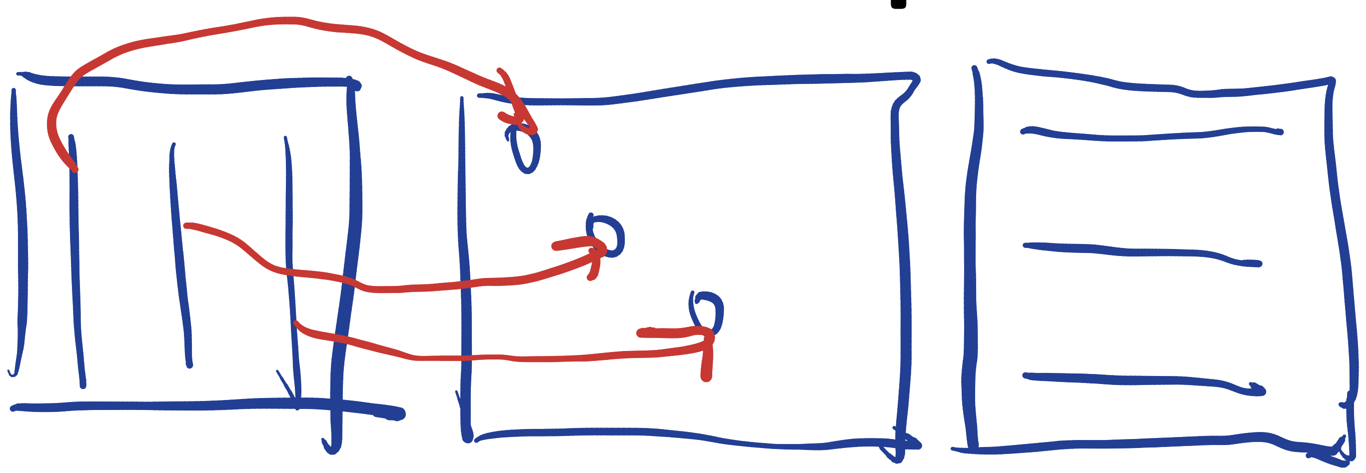
Spectral Theorem

1. Assume A is diagonalizable
$A = Q \Lambda Q^{-1}$

2. Assume A is real, symmetric
$A = Q \Lambda Q^{T}$

$A =$ 

$A = Q \Lambda Q^T$

- If A is a real symmetric matrix, the spectral theorem shows that there exists an orthogonal matrix that diagonalize A.

Lemma: Q is orthogonal, and square matrix, $Q^T = Q^{-1}$ ($Q^TQ = I$)

- Every real symmetric matrix A can be factored into $Q\Lambda Q^\top$ where Q is an orthogonal matrix and Λ is a real diagonal matrix.

$\det(Q^TQ) = \det^2(Q)$
$= \det(I) = 1$

- - The diagonal entries λ_i of Λ are eigenvalues of A.
  - The columns $\mathbf{q}_i$ of Q are eigenvectors belonging to λ_i, respectively.

Orthogonal matrix: $Q \in R^{n \times n}$ $\quad Q^TQ = I_n \quad = (q_i^T q_j)_{i,j}$

$Q = [q_1, \cdots, q_n]$

$$Q^TQ = \begin{pmatrix} q_1^T \\ \vdots \\ q_n^T \end{pmatrix} (q_1, \cdots, q_n) = \begin{pmatrix} q_1^Tq_1 & q_1^Tq_2 & \cdots & q_1^Tq_n \\ & \ddots & & \\ & & \ddots & \\ q_n^Tq_1 & \cdots & \cdots & q_n^Tq_n \end{pmatrix}$$

Spectral Theorem

We can also write

$$A = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^\top \\ \vdots \\ \mathbf{q}_n^\top \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \mathbf{q}_1^\top \\ \vdots \\ \lambda_n \mathbf{q}_n^\top \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i \mathbf{q}_i \mathbf{q}_i^\top$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$\mathrm{Tr}(q_i q_i^T) = \mathrm{Tr}(q_i^T q_i) = \|q_i\|_2^2 = 1$

Properties of $A^\top A$

Let A be any $m \times n$ matrix A of real numbers.

- $A^\top A$ is symmetric.
- $A^\top A$ is diagonalizable by an orthogonal matrix, and the eigenvalues of $A^\top A$ are real.
- $\mathcal{N}(A^\top A) = \mathcal{N}(A)$.
- $\operatorname{rank}(A) = \operatorname{rank}(A^\top A)$.
- The eigenvalues of $A^\top A$ are nonnegative.

Proof: Let's assume $x \in \mathcal{N}(A)$

$\Rightarrow Ax = 0 \Rightarrow A^T A x = 0 \Rightarrow x \in \mathcal{N}(A^T A)$

$\mathcal{N}(A) \subseteq \mathcal{N}(A^T A)$

Let's assume $x \in \mathcal{N}(A^T A)$

$\Rightarrow A^T A x = 0 \Rightarrow x^T A^T A x = 0$

$\Leftrightarrow \|Ax\|_2^2 = 0$

$\mathcal{N}(A^T A) \subseteq \mathcal{N}(A)$ $\Rightarrow Ax = 0$

$\Rightarrow x \in \mathcal{N}(A)$

$A^T A x = \lambda x$

$\Rightarrow x^T A^T A x = \lambda x^T x$

$\Rightarrow \|Ax\|_2^2 = \lambda \|x\|_2^2$.

Contents

Singular Value Decomposition

The **singular-value decomposition (SVD)** of an $m \times n$ matrix A of real numbers is a factorization of the form $U\Sigma V^\top$, where

- U is an $m \times m$ orthogonal matrix;
- V is an $n \times n$ orthogonal matrix;
- Σ is an $m \times n$ matrix whose off-diagonal entries are all 0's, and whose diagonal entries σ_i, $i = 1, \dots, n$, called the **singular values** satisfy

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$$

Here singular values $\sigma_k = 0$ if $k > \min\{m, n\}$.

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ 0 & & \sigma_n & 0 & \cdots & 0 \end{bmatrix}$$

SVD exists for any real matrix.

Linear Transformation View

- Suppose A has the SVD: $A = U\Sigma V^\top$. $A \in R^{m \times n}: R^n \to R^m$
- The columns of U form an orthonormal basis of $\mathbb{R}^m$. e.g. $A \in R^{2\times 3}$
- The columns of V form an orthonormal basis of $\mathbb{R}^n$. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
- The linear transformation $L(\mathbf{x}) = A\mathbf{x}$ has the matrix representation Σ with respect to the above bases of $\mathbb{R}^m$ and $\mathbb{R}^n$.
- In other words,

$$L(\mathbf{x}) = U\Sigma V^\top \mathbf{x}$$

$$[L(\mathbf{x})]_U = \Sigma[\mathbf{x}]_V$$

$$L(\mathbf{x}) = U$$

$$[\mathbf{x}]_V = V^\top \mathbf{x}$$

$$\boxed{x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad Ax = \begin{pmatrix} a \\ b \end{pmatrix}}$$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

$\Sigma A x = \begin{pmatrix} 4a \\ 2b \end{pmatrix}, \quad x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\Sigma A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$x = \begin{pmatrix} a \\ b \end{pmatrix}$

$Ax = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

Linear Transformation View

$\mathrm{rk}(S) \le k$

$\Leftrightarrow$ $B = U^\top S V$

$\mathrm{rk}(B) \le k$

- Suppose A has the SVD: $A = U\Sigma V^\top$.
- The columns of U form an orthonormal basis of $\mathbb{R}^m$.
- The columns of V form an orthonormal basis of $\mathbb{R}^n$.
- The linear transformation $L(\mathbf{x}) = A\mathbf{x}$ has the matrix representation Σ with respect to the above bases of $\mathbb{R}^m$ and $\mathbb{R}^n$.
- In other words,

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$$

$$L(\mathbf{x}) = U\Sigma V^\top \mathbf{x}$$

$$[L(\mathbf{x})]_U = \Sigma[\mathbf{x}]_V$$

$$L(\mathbf{x}) = U\,[L(\mathbf{x})]_U$$

$$[\mathbf{x}]_V = V^\top \mathbf{x}$$

Visualization of SVD

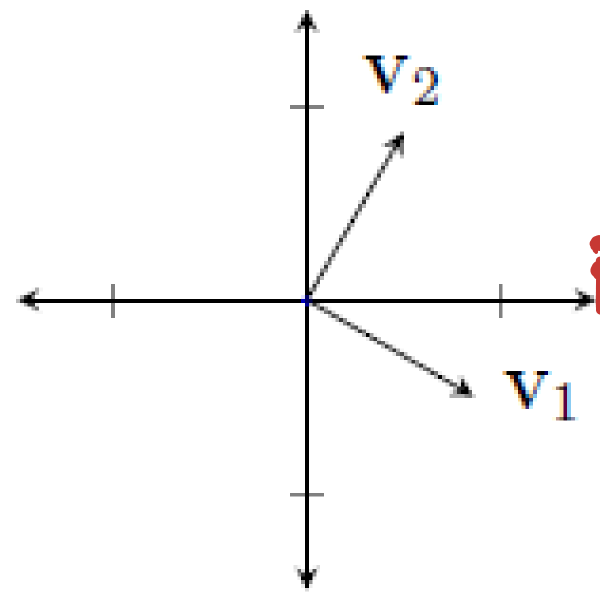
$A: \mathbb{R}^2 \to \mathbb{R}^2$

$A = U\Sigma V^T$

$(V^T x)$?

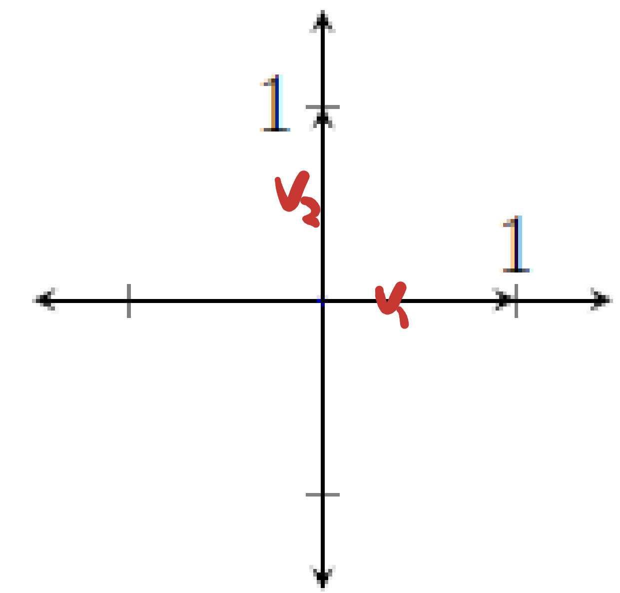
$V = (v_1, v_2)$

$V^T x$: rotate right singular vectors to standard basis



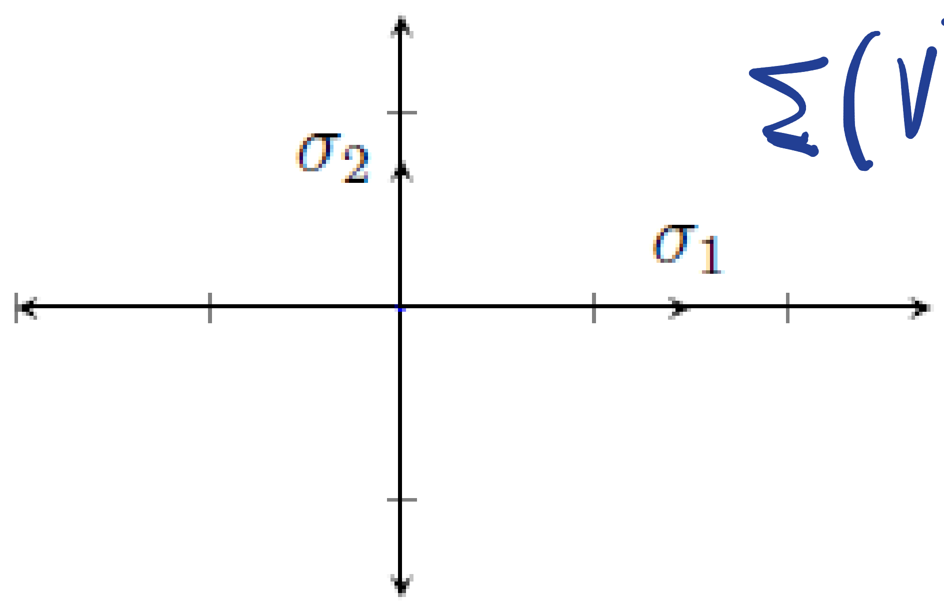
(a) standard axis

$V^T x$



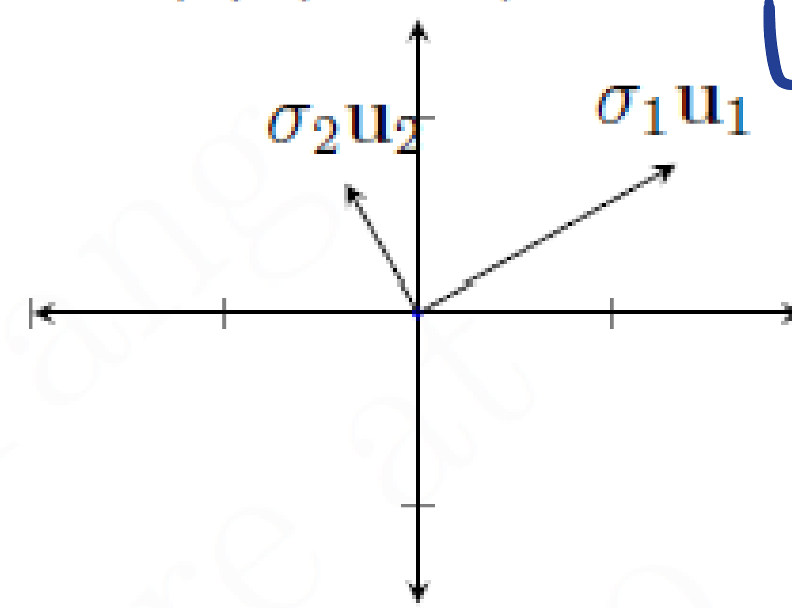
(b) $(\mathbf{v}_1, \mathbf{v}_2)$ axis

$\Sigma(V^T x)$



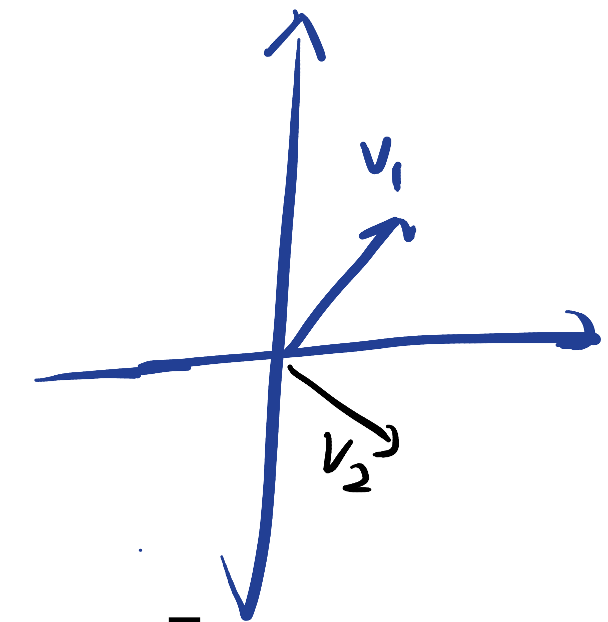
(c) scaled by σ_1 and σ_2, $(\mathbf{u}_1, \mathbf{u}_2)$ axis

$U(\Sigma V^T x)$: rotate standard basis to left singular vectors



(d) standard axis

Example



1.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Example

1.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

SVD and Rank

- If $A = U\Sigma V^\top$, then the rank of A is equal to the number of **nonzero singular values**.

$A \in R^{m \times n}$

$U = [(u_1, \ldots,) \; u_m]$ $V = [(v_1, \ldots), v_n]$

- Proof Technique:
  - Let r be the number of nonzero singular values of A.
  - Let U_r and V_r be the first r columns of U and V, respectively.
  - Let $\Sigma_r = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$.

We have $U_r^\top U_r = V_r^\top V_r = I_r$ and

$$A = U_r \Sigma_r V_r^\top. \tag{3}$$

$\Rightarrow rk(A) = \#$ of nonzero singular values.

$A = U_r \Sigma_r V_r^\top \Rightarrow rk(A) \le rk(U_r)$

$A V_r = U_r \Sigma_r \Rightarrow A V_r \Sigma_r^{-1} = U_r \Rightarrow rk(U_r) \le rk(A)$.

Outer Product Expansion

$$
\begin{aligned}
A &= U_r \Sigma_r V_r^T \\
&= \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^\top \\ \vdots \\ \mathbf{v}_r^\top \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 \mathbf{v}_1^\top \\ \vdots \\ \sigma_r \mathbf{v}_r^\top \end{bmatrix} \\
&= \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top
\end{aligned}
$$

Eigenvalues *vs* Singular Values

- For an $n \times n$ square matrix A with SVD and rank r

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i, i = 1, \ldots, r.$$

- The rank of a matrix is always the same as the number of non-zero singular values.

Eigenvalues *vs* Singular Values

- If A is diagonalizable, i.e., A has n linearly independent eigenvectors $\mathbf{x}_i$

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i, i = 1, \ldots, n$$

  then the rank of the matrix is equal to the number of non-zero eigenvalues.

- But this may not be the case when the matrix is not diagonalizable.

  Consider $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Property of SVD

Let an $m \times n$ matrix A having SVD $U\Sigma V^\top$. Then

- $\sigma_i = \sqrt{\lambda_i}$, $i = 1, \ldots, n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are eigenvalues of $A^\top A$.

  $A^TA = V\Sigma^TU^TU\Sigma V^T = V\,\Sigma^T\Sigma\,V^T$

- V diagonalizes $A^\top A$, and hence $\mathbf{v}_j$'s are eigenvectors of $A^\top A$.
- the columns of U satisfy:

$$\mathbf{u}_j = \frac{1}{\sigma_j} A\mathbf{v}_j, \quad j = 1, \ldots, r = \operatorname{rank}(A)$$

$$A^\top \mathbf{u}_j = \mathbf{0}, \quad j = r+1, \ldots, m.$$

Exercise.

$A = U\Sigma V^T$

$AV = U\Sigma V^TV = U\Sigma$

$V = (v_1, \ldots, v_n)$

$Av_j = \sigma_j u_j \Rightarrow u_j = (Av_j)/\sigma_j$

Matrix Norm

$$\|A\|_F^2 = \sum_{ij} |a_{ij}|^2 = \mathrm{Tr}(A^T A)$$
$$= \mathrm{Tr}(AA^T)$$

- Matrix norm $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$, which is also called *Frobenius* norm.
- If $A = U\Sigma V^\top$, then $\|A\|_F^2 = \sigma_1^2 + \cdots + \sigma_n^2$.

$$\|A\|_F^2 = \|U\Sigma V^T\|_F^2 = \|\Sigma V^T\|_F^2 = \|\Sigma\|_F^2$$
$$= \sigma_1^2 + \cdots + \sigma_n^2$$

Lemma

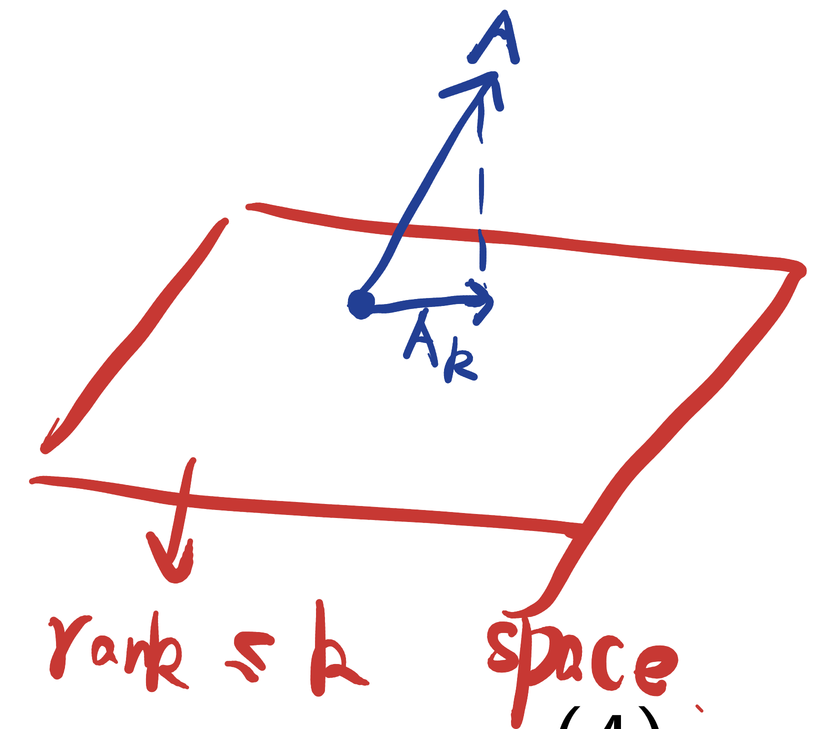
If Q is an orthogonal matrix, then $\|QA\|_F = \|A\|_F$.

proof of lemma:
$$\|QA\|_F^2 = \mathrm{Tr}(QAA^TQ^T)$$
$$= \mathrm{Tr}(AA^TQ^TQ) = \mathrm{Tr}(AA^T) = \|A\|_F^2$$

$\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$

Low rank approximation



For a fixed $m \times n$ matrix A and an integer k, solve

★

$$\min_{\mathrm{rank}(S) \leq k} \|A - S\|_F. \tag{4}$$

Theorem

$U^T S V = \Sigma_k \Rightarrow S^* = U \Sigma_k V^T$

Let $A = U\Sigma V^\top$ be an $m \times n$ matrix and let $A_k = U\Sigma_k V^\top$ where Σ_k is same as Σ except that the (j, j) entry is 0 for $j > k$. Then

$$\min_{\mathrm{rank}(S) \leq k} \|A - S\|_F = \|A - A_k\|_F = (\sigma_{k+1}^2 + \cdots + \sigma_n^2)^{1/2}.$$

In other words, A_k is the best rank k approximation of A in Frobenius norm.

Proof: For S with $\mathrm{rk}(S) \leq k$, $\|A - S\|_F = \|U\Sigma V^T - S\|_F = \|U^T(U\Sigma V^T - S)V\|_F = \|\Sigma - U^T S V\|_F$

$\Leftrightarrow$ It suffices to find $B \triangleq U^T S V$ s.t. $\mathrm{rk}(B) \leq k$, $\|\Sigma - B\|_F$ is minimized $\Rightarrow B = \Sigma_k$

In the proof, we used the fact that " there is no better k-rank approximation for Σ than $\Sigma_k = \mathrm{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)$

Justification:

Consider $\min_{B:\, \mathrm{rank}(B) \le k} \|B - \Sigma\|_F^2 = \sum_{i=1}^{r} (B_{ii} - \sigma_i)^2 + \sum_{i \ne j} B_{ij}^2$

- If optimal B has any nonzero off-diagonal entry, set it to zero $\Rightarrow$ obtain better optimal solution $\Rightarrow$ Contradiction

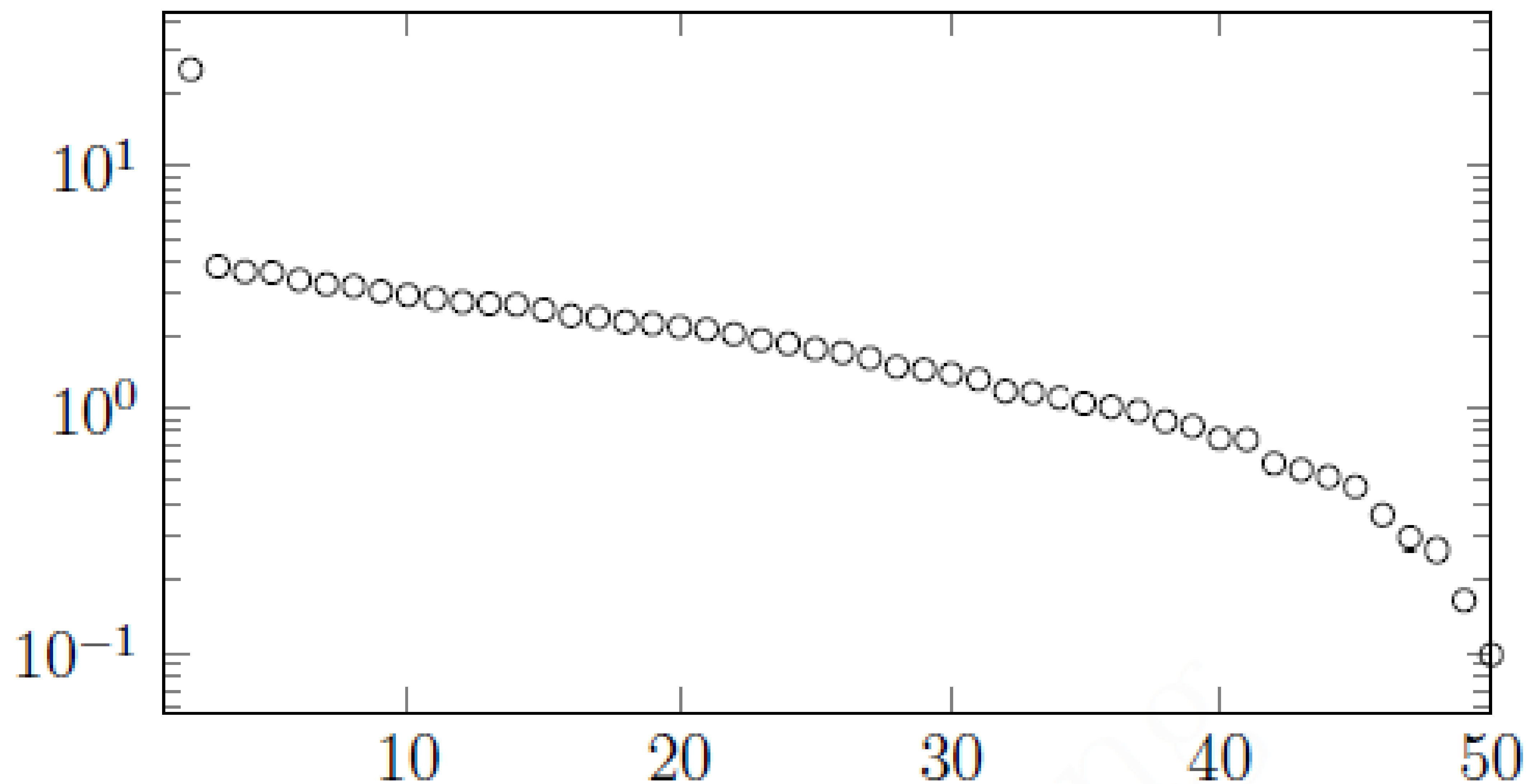
- Hence, B must be diagonal matrix, i.e., $B = \mathrm{diag}(B_{11}, \dots, B_{rr})$

$$\|B - \Sigma\|_F^2 = \sum_{i=1}^{r} (B_{ii} - \sigma_i)^2$$

$$\Rightarrow B^* = \mathrm{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)$$

Example

- Generate a random 50×50 matrix A using Julia
- Check the rank of matrix A, which should be $\cancel{64}$ most of the case.
  50
- Plot the singular values:



Applications

$Tr(AB) = Tr(BA)$

$Tr(QA\underbrace{A^TQ^T}_{B})$

$= Tr(AA^T(Q^TQ))$

$= Tr(AA^T)$

- Eigenvalues with PCA;
- Eigenvalues for extracting information from graph;
- SVD with recommender systems.