13.3. Monday for MAT4002

13.3.1. Isomorphsim between Edge Loop Group and the Fundamental Group

Recall that

$$\pi_1(X,b) := \{[\ell] \mid \ell : [0,1] \to X \text{ denotes the loops based at } b\}$$

and

$$E(K,b) = \{[\alpha] \mid \alpha \text{ is an edge loop in } K \text{ based at } b\}$$

Now we show that the mapping defined below is injective:

$$\begin{aligned} \theta : \quad & E(K,b) \to \pi_1(|K|,b) \\ \text{with} \quad & [\alpha] \mapsto [|g_\alpha|] \end{aligned}$$

- Let $\alpha = (v_0,\dots,v_n)$ be an edge loop based at b such that $\theta([\alpha]) = e$, i.e., $|g_\alpha| \simeq c_b$. It suffices to show that $[\alpha]$ is the identity element of $E(K,b)$.
- Choose a homotopy $H : |g_\alpha| \simeq c_b$ such that $H : I \times I \to |K|$. The graphic illustration for H is shown in Fig. (13.1).

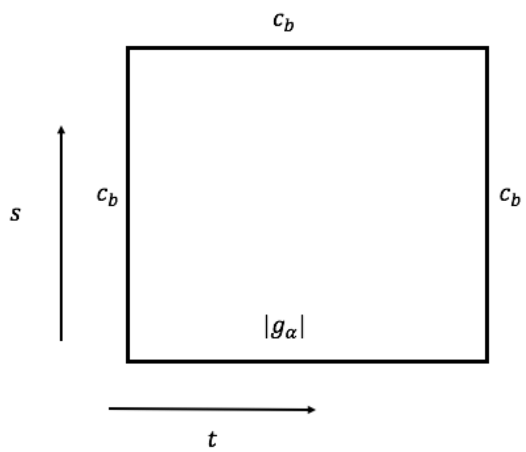

Figure 13.1: Graphic illustration for $H : I \times I \to |K|$

Now apply the simplicial approximation theorem, there exists a subdivision of $I \times I$, denoted as $(I \times I)_{(r)}$ (for sufficiently large r), shown in the Fig. (13.2)

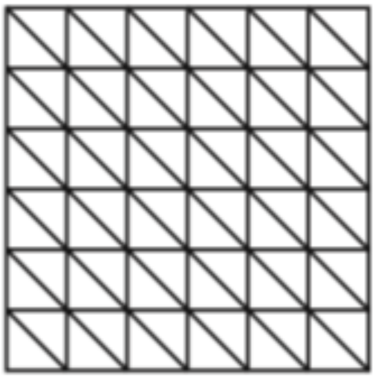

Figure 13.2: Graphic illustration for $(I \times I)_{(r)}$. In particular, divide $I \times I$ into r^2 congruent squares, and then further divide each of these squares along the diagonal to form $(I \times I)_{(r)}$.

such that $|(I \times I)_{(r)}| = I \times I$, and there exists the simplicial map

$$\begin{aligned} G: \quad & (I \times I)_{(r)} \to K \\ \text{such that} \quad & |G| \simeq H. \end{aligned}$$

Without loss of generality, assume r is a sufficiently large multiple of n.

The graphic illustration of $|G|$ is shown in Fig. (13.3):

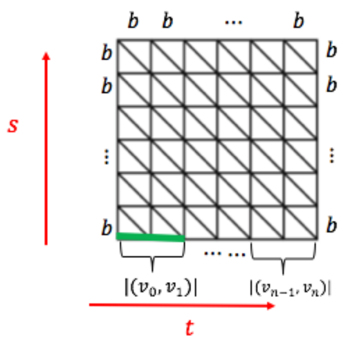

Figure 13.3: Graphic illustration for the mapping $|G|$.

In particular, $|G|$ maps $\{0,1\} \times I$ into $\{b\}$; $I \times \{1\}$ into $\{b\}$; $(i/n, 0)$ into $\{v_i\}, i =$

$0, \ldots, n$, and $[i/n, (i+1)/n]$ into $|(v_i, v_{i+1})|, i = 0, \ldots, n-1$.

- Consider the simplicial subcomplex of $(I \times I)_{(r)}$ shown in Fig. (13.4)

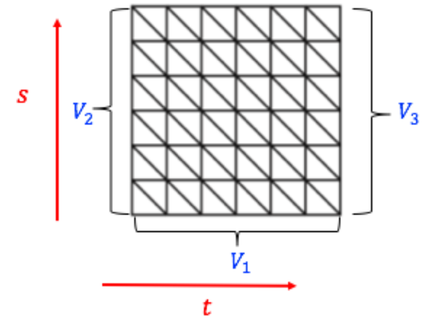

Figure 13.4: Graphic illustration for the simplicial subcomplex V_1, V_2, V_3.

For instance, V_1 has $(r+1)$ 0-simplicies and r 1-simplies. It follows that

$$H(|V_1|) = H(|V_2|) = H(|V_3|) = \{b\}.$$

By proposition (10.6), we can pick G be such that

$$G(V_1) = G(V_2) = G(V_3) = \{b\}.$$

Consider W_1 as the simplicial subcomplex of $(I \times I)_{(r)}$ given by the green line shown in Fig. (13.3), which follows that

$$H(|W_1|) = \{v_0, v_1\} \implies G(W_1) = \{v_0, v_1\}$$

Similarly,

$$H(|W_i|) = \{v_{i-1}, v_i\} \implies G(W_i) = \{v_{i-1}, v_i\}, \forall 1 \leq i \leq n.$$

As a result, $|G|(|V_1|) = \beta := (bv_0 \cdots v_0 v_1 \cdots v_1 \cdots v_n \cdots v_n b)$, and clearly,

$$\begin{aligned} \beta &\sim (bv_0 v_1 v_2 \cdots v_{n-1} v_n b) \\ &\sim (bv_1 v_2 \cdots v_{n-1} b) = \alpha \end{aligned}$$

- Now it suffices to show $\beta \simeq e$. This is true by the sequence of elementary contractions and expansions as shown in the Fig. (13.5).

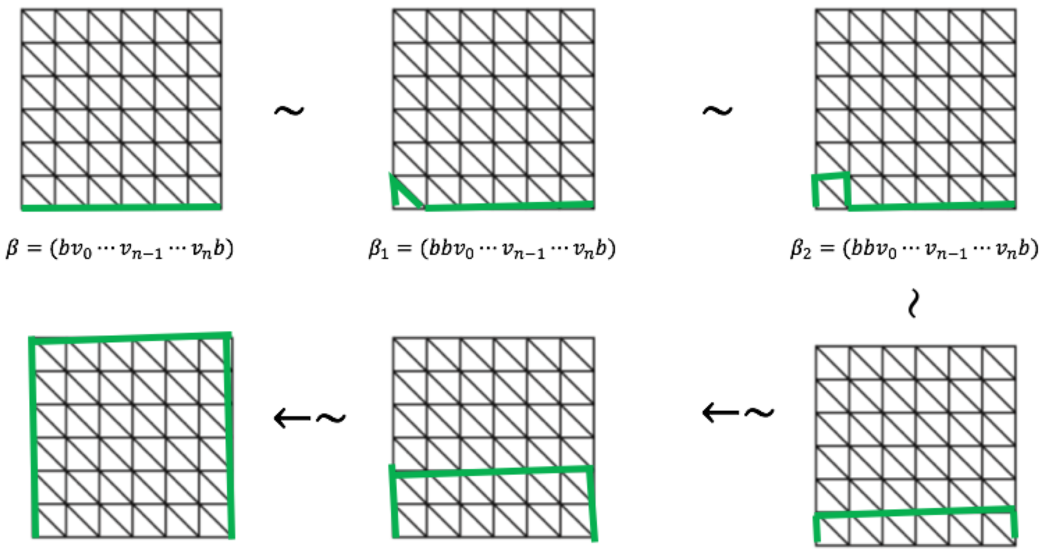

Figure 13.5: A sequence of elementary contractions and expansions to show that $\beta \sim (b \cdots b) = (b)$.

Ⓡ The definition of $E(K,b)$ only involves n-simplicials for $n \le 2$.

Proposition 13.4 For any simplicial complex K, consider the simplicial subcomplex $\text{Skel}^n(K) = (V_k, \Sigma_K^n)$, where Σ_K^n consists of $\sigma \in \Sigma_K$ with $|\sigma| \le n+1$ (this is the n-skeleton of K). Then

$$\pi_1(|K|,b) \cong \pi_1(|\text{Skel}^2(K)|,b)$$

Proof. Since $E(K,b)$ only involves n-simplicials for $n \le 2$, we imply $E(K,b) \cong E(\text{Skel}^2(K),b)$.

Moreover, $\pi_1(|K|,b) \cong E(K,b)$ and $\pi_1(|\text{Skel}^2(K)|,b) \cong E(\text{Skel}^2(K),b)$.

The proof is complete. ■

Corollary 13.2 For $n \ge 2$, $\pi_1(S^n)$ is a trivial fundamental group.

Proof. Consider the simplicial complex K with

$$V = \{1,2,\dots,n+2\}, \quad \Sigma = \{\text{all proper subsets of } V\}$$

It's clear that $|K| \cong S^n$, and $\text{Skel}^2(K)$ has

- $V : \{1, \ldots, n+2\}$
- Σ^2 : all subsets of V with less or equal to 3 elements.

For any edge loop a in $\pi_1(|\text{skel}^2(K)|)$, we have

$$\begin{aligned}
a &= (bv_0v_1v_2\cdots v_n) \\
&\sim (bv_1v_2\cdots v_{n-2}v_{n-1}b) \\
&\sim \cdots \\
&\sim (b)
\end{aligned}$$

Therefore, all edge loops α in $\pi_1(|\text{skel}^2(K)|)$ satisfies $[\alpha] = [(b)] = e.$, i.e.,

$$\pi_1(|\text{skel}^2(K)|) \cong \{e\},$$

which implies $\pi_1(|K|) \cong \pi_1(|\text{skel}^2(K)|) \cong \{e\}$. Since $|K| \cong S^n$, we imply

$$\pi_1(S^n) \cong \pi_1(|K|) \cong \{e\}.$$

■

R The Corollary (13.2) does not hold for S^1 since the constructed Σ^2 for S^1 does not contain $\{1, 2, 3\}$.

Theorem 13.4 $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. Construct the triangle K shown in Fig. (13.6), and it's clear that $|K| \cong S^1$.

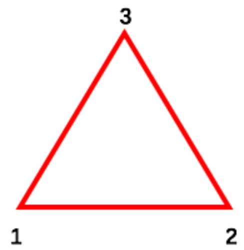

Figure 13.6: Triangle K such that $|K| \cong S^1$

It suffices to show $E(K,1) \cong \mathbb{Z}$. Define the orientation of $|K|$ as shown in Fig. (13.7).

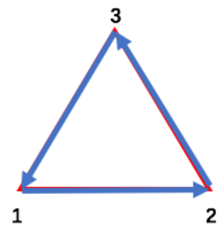

Figure 13.7: Orientation of $|K|$

Any edge loop α based at 1 is equivalent to the canonical form

$$\alpha \sim (1bc1bc\cdots 1bc1), \quad \text{where } bc = 32 \text{ or } 23.$$

We construct the isomorphism between $E(K,b)$ and $\mathbb{Z}$ directly:

$$\begin{aligned} \phi: \quad & E(K,b) \to \mathbb{Z} \\ \text{with} \quad & [\alpha] \mapsto \text{winding number of } \alpha \end{aligned}$$

where the winding number of α is the number of times it traverses $(2,3)$ in the forwards direction minus the number of times it traverses $(3,2)$ in the backwards direction.

The difficult part is to show the well-definedness of ϕ, which can be done by using canonical form of α. ■