Entropic Regularization for Adversarial Robust Learning

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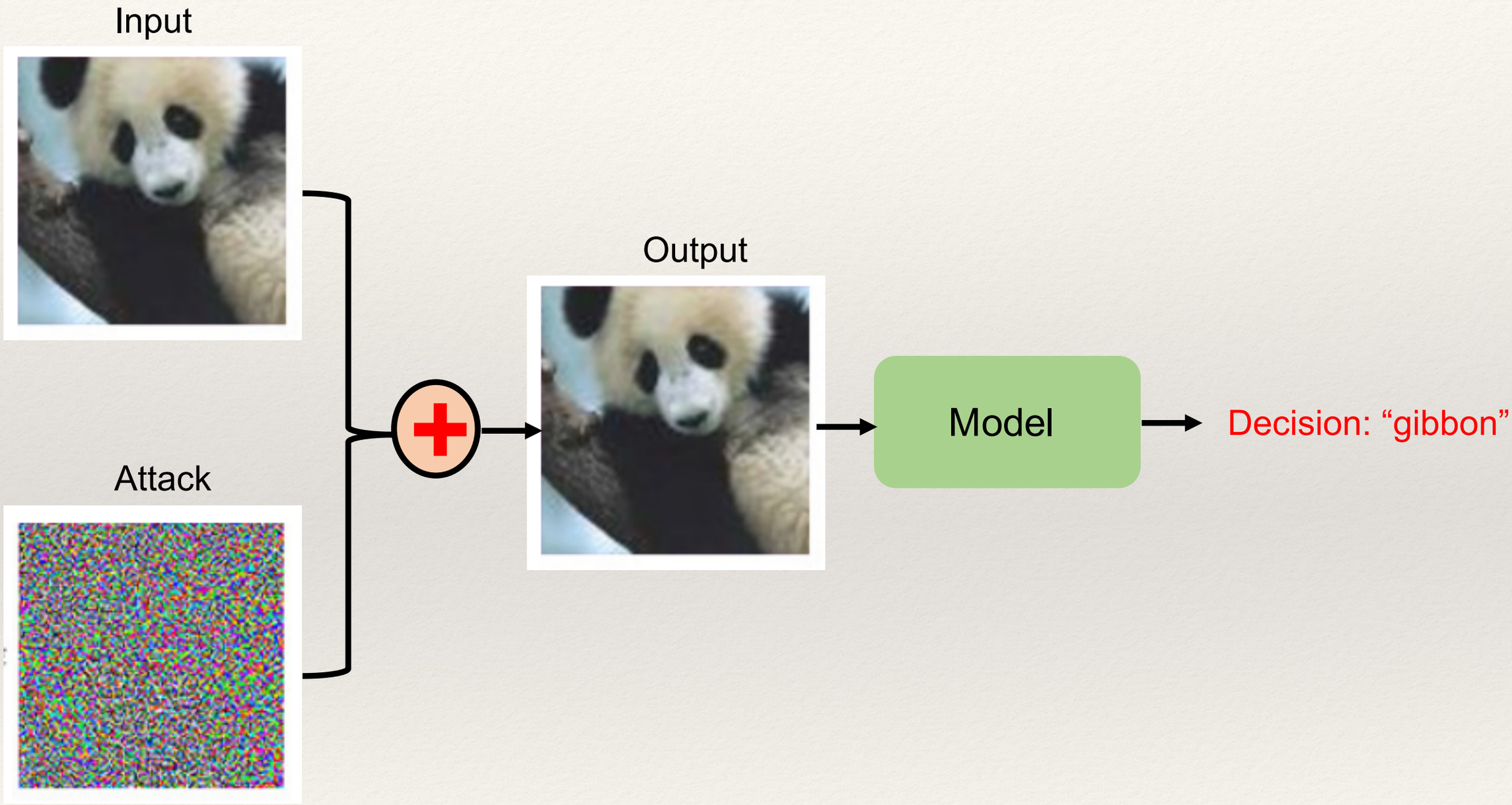
On the Robustness of ML Models



[Goodfellow et al. 2015]

Adversarial Risk Minimization

- Formulation:

$$\min_{\theta\in\Theta} \left\{ F(\theta;\rho,\widehat{\mathbb{P}}) = \mathbb{E}_{x\sim\widehat{\mathbb{P}}}\left[\sup_{z\in\mathbb{B}_\rho(x)} f_\theta(z) \right] \right\}$$

- **Intractability**:

When $f_\theta(z)$ is convex in θ: Convex-Nonconcave Game

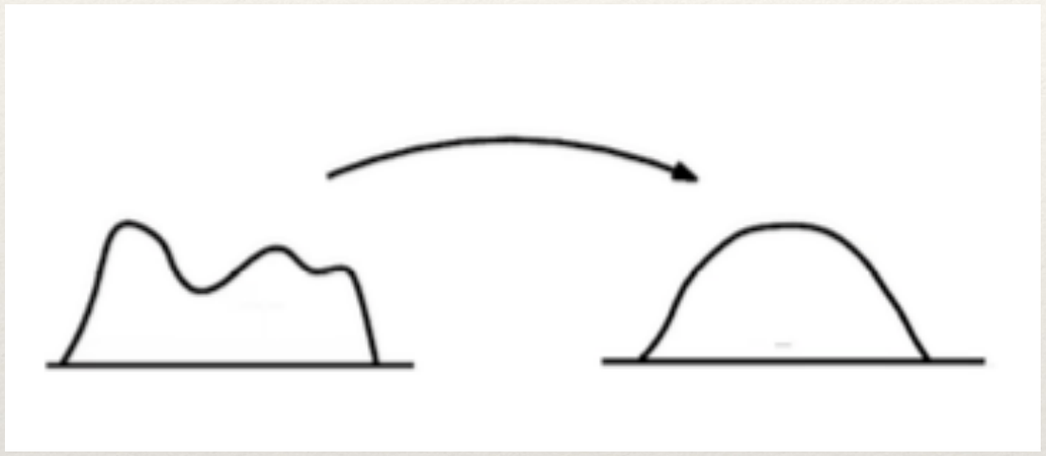
When $f_\theta(z)$ is nonconvex in θ: Nonconvex-Nonconcave Game

- Connections with Optimal Transport:

$$\min_{\theta\in\Theta} \left\{ \sup_{\mathbb{P}} \ \mathbb{E}_{z\sim\mathbb{P}}[f_\theta(z)] : \ \mathcal{W}_\infty(\mathbb{P},\widehat{\mathbb{P}}) \le \rho \right\}$$

p-Type Wasserstein Distance

$$\mathcal{W}_p(\mathbb{P},\mathbb{Q}) := \left(\min_{\gamma\in\mathcal{P}(\Omega^2)} \left\{ \mathbb{E}_{(\omega,\omega')\sim\gamma}[\|\omega-\omega'\|^p] : \right.\right.$$



$$\left.\left. \gamma \text{ has marginal distributions } \mathbb{P} \text{ and } \mathbb{Q} \right\} \right)^{1/p}.$$

$$\mathcal{W}_\infty(\mathbb{P},\mathbb{Q}) = \inf_{\gamma} \left\{ \text{ess.sup } \mathsf{d}(\zeta_1,\zeta_2) : \begin{array}{l} \gamma \text{ is a joint distribution of } \zeta_1 \text{ and } \zeta_2 \\ \text{with marginals } \mathbb{P} \text{ and } \mathbb{Q}, \text{ respectively} \end{array} \right\}$$

- ✓ Geometric properties;
- ✓ Flexibility: non-overlapping support, discrete and continuous.

Our Proposed Formulation

- Original Formulation:

$$\min_{\theta\in\Theta} \sup_{\mathbb{P},\gamma} \left\{ \mathbb{E}_{\mathbb{P}}[f_\theta(z)] : \begin{array}{l} \mathrm{Proj}_{1\#\gamma} = \widehat{\mathbb{P}}, \mathrm{Proj}_{2\#\gamma} = \mathbb{P} \\ \mathrm{ess.sup}_\gamma \mathsf{d}(\zeta_1, \zeta_2) \leq \rho \end{array} \right\}$$

- Formulation with Entropic Regularization:

$$\min_{\theta\in\Theta} \sup_{\mathbb{P},\gamma} \left\{ \mathbb{E}_{\mathbb{P}}[f_\theta(z)] - \eta H(\gamma \mid \pi) : \begin{array}{l} \mathrm{Proj}_{1\#\gamma} = \widehat{\mathbb{P}}, \mathrm{Proj}_{2\#\gamma} = \mathbb{P} \\ \mathrm{ess.sup}_\gamma \mathsf{d}(\zeta_1, \zeta_2) \leq \rho \end{array} \right\}$$

Entropy: $$H(\gamma \mid \pi) = \mathbb{E}_{(x,z)\sim\gamma} \left[\log \left(\frac{\mathrm{d}\gamma(x,z)}{\mathrm{d}\pi(x,z)} \right) \right]$$

$$= \mathbb{E}_{x\sim\widehat{\mathbb{P}}} \mathbb{E}_{z\sim\gamma_x} \left[\log \left(\frac{\mathrm{d}\widehat{\mathbb{P}}(x)\mathrm{d}\gamma_x(z)}{\mathrm{d}\widehat{\mathbb{P}}(x)\mathrm{d}\nu_x(z)} \right) \right]$$

Our Contribution (I)

- Strong Dual Formulation:

$$V_{\text{Primal}} = \sup_{\mathbb{P},\gamma} \left\{ \mathbb{E}_{z\sim\mathbb{P}}\left[f_\theta(z)\right] - \eta H(\gamma \mid \pi) : \begin{array}{l} \text{Proj}_{1\#\gamma} = \widehat{\mathbb{P}}, \text{Proj}_{2\#\gamma} = \mathbb{P} \\ \text{ess.sup}_\gamma \mathsf{d}(\zeta_1, \zeta_2) \le \rho \end{array} \right\}.$$

$$V_{\text{Dual}} = \mathbb{E}_{x\sim\widehat{\mathbb{P}}}\left[\eta \log \mathbb{E}_{z\sim\mathbb{Q}_x}\left[\exp\left(\frac{f_\theta(z)}{\eta}\right)\right]\right] + \text{Constant}.$$

- Kernel Probability Distribution:

$$\frac{\mathrm{d}\mathbb{Q}_x}{\mathrm{d}\nu_x}(z) = \text{vol}(\nu_x)^{-1}, \quad z \in \mathbb{B}_\rho(x).$$

Our Contribution (II)

$$V_{\text{Primal}} = \sup_{\mathbb{P},\gamma} \left\{ \mathbb{E}_{z\sim\mathbb{P}}\left[f_\theta(z)\right] - \eta H(\gamma \mid \pi) : \begin{array}{l} \text{Proj}_{1\#\gamma} = \widehat{\mathbb{P}}, \text{Proj}_{2\#\gamma} = \mathbb{P} \\ \text{ess.sup}_\gamma \mathsf{d}(\zeta_1, \zeta_2) \le \rho \end{array} \right\}.$$

- Recover Worst-case Distribution:

The worst-case distribution maps each $x \in \text{supp}(\widehat{\mathbb{P}})$ to a conditional distribution γ_x whose density value (with respect to measure ν_x) is proportional to

$$\alpha_x \cdot \exp\left(\frac{f_\theta(z)}{\eta}\right),$$

$$\mathrm{d}\mathbb{P}_*(z) = \mathbb{E}_{x\sim\widehat{\mathbb{P}}}\left[\alpha_x \cdot \exp\left(\frac{f_\theta(z)}{\eta}\right) \mathrm{d}\nu_x(z)\right]$$

Our Contribution (III)

- Tractable Algorithm for Dual Formulation:

$$\min_{\theta\in\Theta} \left\{ F_\eta(\theta) \triangleq \mathbb{E}_{x\sim\widehat{\mathbb{P}}} \left[\eta \log \mathbb{E}_{z\sim\mathbb{Q}_x} \left[\exp\left(\frac{f_\theta(z)}{\eta} \right) \right] \right] \right\}$$

Algorithm 1 BSMD for solving (8)

Require: maximum iterations T, constant step size γ, initial guess θ_0.

1: **for** $t = 0, 1, \ldots, T-1$ **do**
2: Formulate (biased) gradient estimate of $F_\eta(\theta_t)$, denoted as $v(\theta_t)$.
3: Update $\theta_{t+1} = \mathrm{Prox}_{\theta_t}\left(\gamma v(\theta_t)\right)$.
4: **end for**

Output $\tilde{\theta}$ **randomly selected from** $\{\theta_0, \theta_1, \ldots, \theta_T\}$.

Our Contribution (III)

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Estimators	Convex Nonsmooth	Convex Smooth	Nonconvex Smooth
Vanilla SGD	$O(\delta^{-3})$	$O(\delta^{-3})$	$O(\delta^{-6})$
V-MLMC	N/A	$\tilde{O}(\delta^{-2})$	$\tilde{O}(\delta^{-4})$
RT-MLMC	N/A	$\tilde{O}(\delta^{-2})$	$\tilde{O}(\delta^{-4})$

Vanilla SGD Estimator

$$\min_{\theta\in\Theta} \left\{ F_\eta(\theta) \triangleq \mathbb{E}_{x\sim\widehat{\mathbb{P}}} \left[\eta \log \mathbb{E}_{z\sim\mathbb{Q}_x} \left[\exp\left(\frac{f_\theta(z)}{\eta} \right) \right] \right] \right\}$$

- Construct approximation function:

$$F_\eta^\ell(\theta) = \mathbb{E}_{x^\ell}\ \mathbb{E}_{\{z_j^\ell\}_{j\in[2^\ell]}|x^\ell} \left[\eta \log \left(\frac{1}{2^\ell} \sum_{j\in[2^\ell]} \exp\left(\frac{f_\theta(z_j^\ell)}{\eta} \right) \right) \right].$$

- Oracle: Sample random parameters $\zeta^L = \{x^L\} \cup \{z_j^\ell\}_j$ and compute

$$g(\theta, \zeta^\ell) = \nabla_\theta \left\{ \eta \log \left(\frac{1}{2^L} \sum_{j\in[2^L]} \exp\left(\frac{f_\theta(z_j^\ell)}{\eta} \right) \right) \right\}$$

- V-SGD: call oracle for several times and return avg

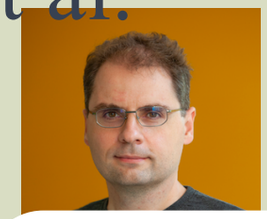
Connections with Existing Literature

Approximation using p-Wasserstein DRO

Sinha, et al. 2020

Origin of Adversarial Training

Aleksander, et al. 2017

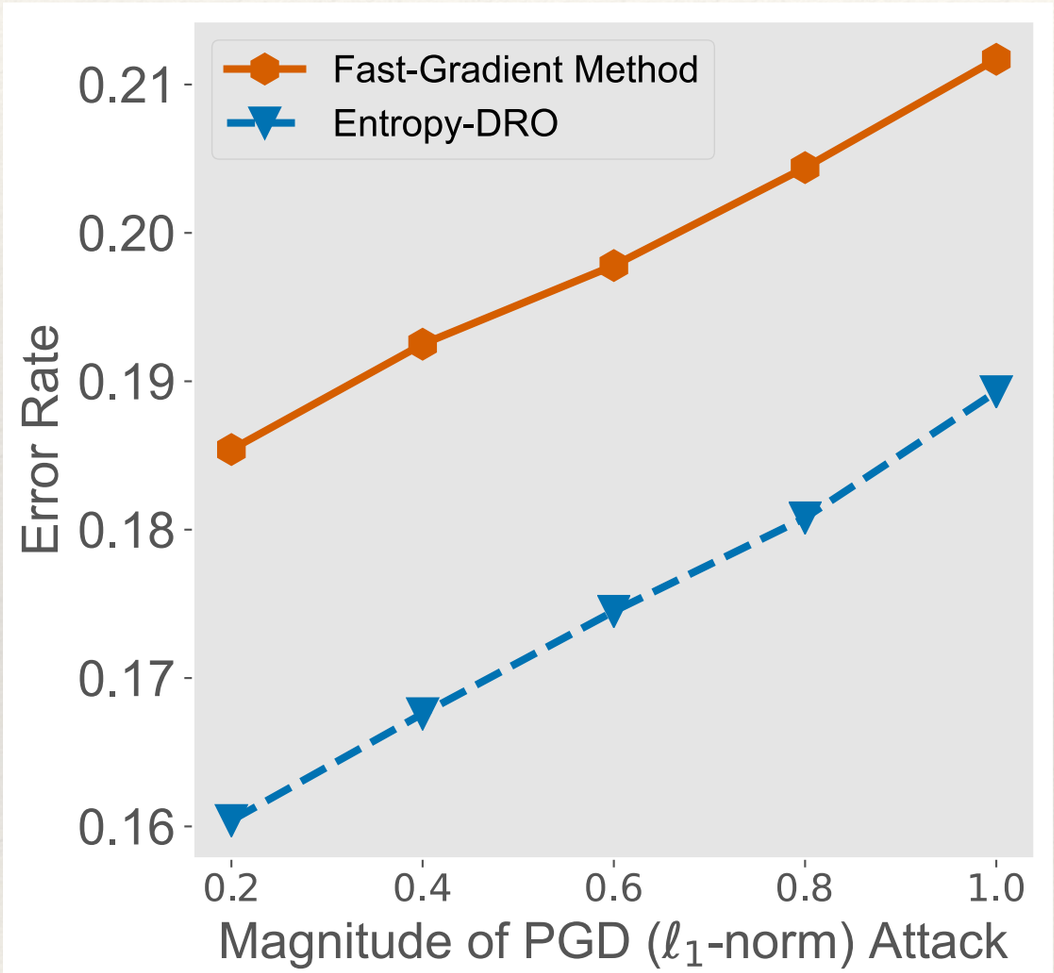


Adding Entropic Regularization Directly

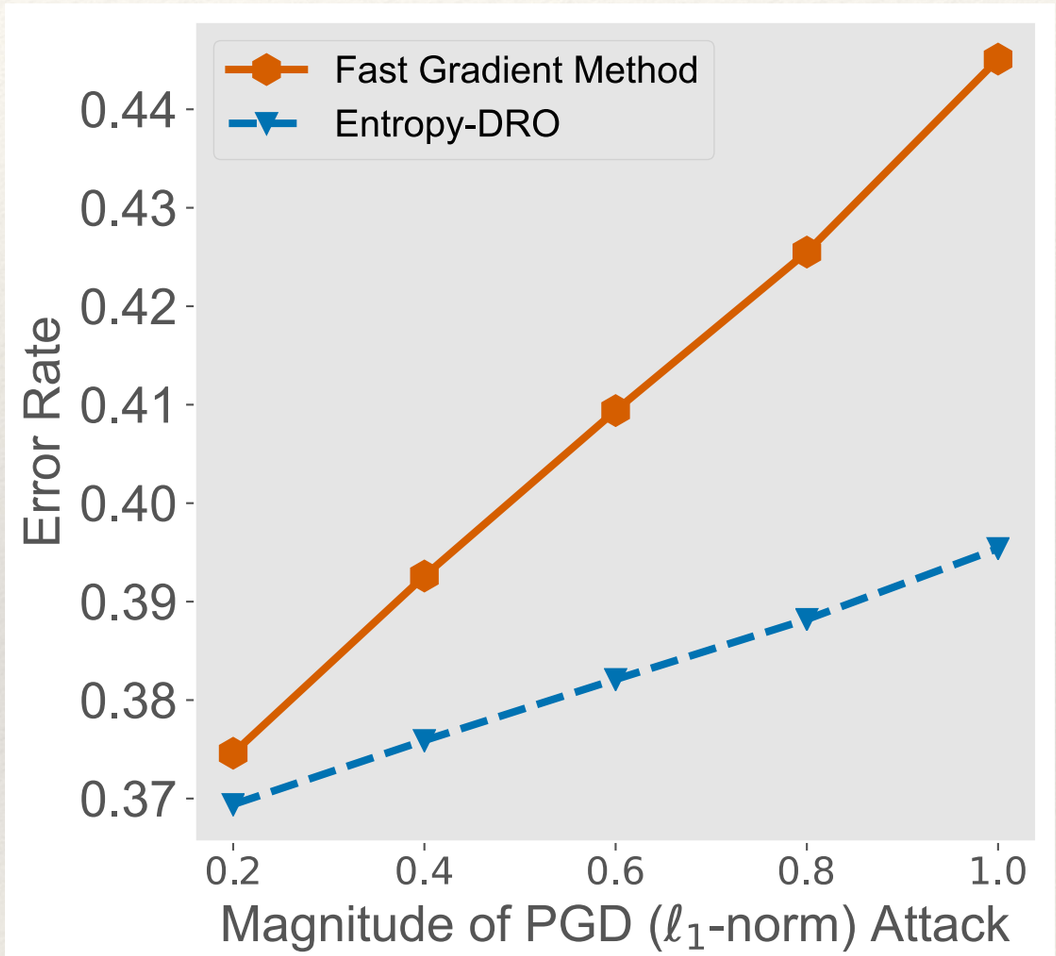
Entropic Regularization for p-Wasserstein DRO

Wang, et al. 2021

Numerical Study



Cifar-10 Dataset



Cifar-100 Dataset