11.4. Wednesday for MAT3040

Reviewing. Unitary Operators

$$\langle T\boldsymbol{v}, T\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle,\ \forall \boldsymbol{v}, \boldsymbol{w} \in V.$$

11.4.1. Unitary Operator

■ **Example 11.8** Let $V = \mathbb{R}^n$ with usual inner product. For the linear operator $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$, T is orthogonal if and only if $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}$.

Let $V = \mathbb{C}^n$ with usual inner product. For the linear operator $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$, T is unitary if and only if $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}$. ■

Proposition 11.7 Let $T : V \to V$ be a linear operator on a vector space over $\mathbb{K}$ satisfying $T'T = I$. Then for all eigenvalues λ of T, we have $|\lambda| = 1$.

Proof. Suppose we have the eigen-pair $(\lambda, \boldsymbol{v})$, then

$$\begin{aligned} \langle T\boldsymbol{v}, T\boldsymbol{v}\rangle &= \langle \boldsymbol{v}, \boldsymbol{v}\rangle \\ \iff \langle \lambda\boldsymbol{v}, \lambda\boldsymbol{v}\rangle &= \langle \boldsymbol{v}, \boldsymbol{v}\rangle \\ \iff \bar{\lambda}\lambda\langle \boldsymbol{v}, \boldsymbol{v}\rangle &= \langle \boldsymbol{v}, \boldsymbol{v}\rangle \end{aligned}$$

Since $\langle \boldsymbol{v}, \boldsymbol{v}\rangle \neq 0$ ($\boldsymbol{v} \neq \boldsymbol{0}$), we imply $|\lambda|^2 = 1$, i.e., $|\lambda| = 1$. ■

Proposition 11.8 Let $T : V \to V$ be an operator on a finite dimension V over $\mathbb{K}$ satisfying $T'T = I$. If $U \le V$ is T-invariant, then U is also T^{-1}-invariant.

Proof. Since $T'T = I$, i.e., T is invertible, we imply 0 is not a root of $\mathcal{X}_T(x)$, i.e., 0 is not a root of $m_T(x)$. Since $m_T(0) \neq 0$, $m_T(x)$ has the form

$$m_T(x) = x^m + \cdots + a_1 x + a_0,\ a_0 \neq 0,$$

which follows that

$$m_T(T) = T^m + \cdots + a_0 I = 0 \implies T(T^{m-1} + \cdots + a_1 I) = -a_0 I$$

Or equivalently,

$$T\left(-\frac{1}{a_0}(T^{m-1} + \cdots + a_1 I)\right) = I$$

Therefore,

$$T^{-1} = -\frac{1}{a_0}T^{m-1} - \cdots - \frac{a_2}{a_0}T - \frac{a_1}{a_0}I,$$

i.e., the inverse T^{-1} can be expressed as a polynomial involving T only.

Sicne U is T-invariant, we imply U is T^m-invariant for $m \in \mathbb{N}$, and therefore U is T^{-1}-invariant since T^{-1} is a polynomial of T. ■

Proposition 11.9 Let $T : V \to V$ satisfies $T'T = I$ ($\dim(V) < \infty$), then $U \le V$ is T-invariant implies $U^\perp$ is T-invariant.

Proof. Let $v \in U^\perp$, it suffices to show $T(v) \in U^\perp$.

For all $u \in U$, we have

$$\langle u, T(v)\rangle = \langle T'(u), v\rangle = \langle T^{-1}(u), v\rangle$$

Since U is T^{-1}-invaraint, we imply $T^{-1}(u) \in U$, and therefore

$$\langle u, T(v)\rangle = \langle T^{-1}(u), v\rangle = 0 \implies T(v) \in U^\perp.$$

■

Theorem 11.2 Let $T : V \to V$ be a unitary operator on finite dimension V (over $\mathbb{C}$), then there exists an orthonormal basis $\mathcal{A}$ such that

$$(T)_{\mathcal{A},\mathcal{A}} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),\ |\lambda_i| = 1,\ \forall i.$$

Proof Outline. Note that $\mathcal{X}_T(x)$ always admits a root in $\mathbb{C}$, so we can always find an eigenvector $\boldsymbol{v} \in V$ of T.

Then the theorem follows by the same argument before on seld-adjoint operators.

- Consider $U = \text{span}\{\boldsymbol{v}\}$
- $V = U \oplus U^{\perp}$ and $U^{\perp}$ is T-invariant
- Use induction on the unitary operator $T\mid_{U^{\perp}}: U^{\perp} \to U^{\perp}$

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- The argument fails for orthogonal operators

$$\begin{aligned} T \quad & : \mathbb{R} \to \mathbb{R}^2, \\ \text{with} \quad & T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v} \\ \text{where} \quad & \boldsymbol{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

  The matrix $\boldsymbol{A}$ is not diagonalizable over $\mathbb{R}$. It has no real eigenvalues. However, if we treat $\boldsymbol{A}$ as $T : \mathbb{C}^2 \to \mathbb{C}^2$ with $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$, then $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} = \boldsymbol{I}$, and therefore T is unitary. Then $\boldsymbol{A}$ is diagonalizable over $\mathbb{C}$ with eigenvalues $e^{i\theta}, e^{-i\theta}$

- As a corollary of the theorem, for all $\boldsymbol{A} \in M_{n\times n}(\mathbb{C})$ satisfying $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} = \boldsymbol{I}$, there exists $P \in M_{n\times n}(\mathbb{C})$ such that

$$P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n), \quad |\lambda_i| = 1,$$

  where $P = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_n)$, with $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ forming orthonormal basis of $\mathbb{C}^n$. In fact,

$$P^{\mathrm{H}}P = \begin{pmatrix} \boldsymbol{u}_1^{\mathrm{H}} \\ \vdots \\ \boldsymbol{u}_n^{\mathrm{H}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_n \end{pmatrix} = \begin{pmatrix} \langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle & \cdots & \langle \boldsymbol{u}_1, \boldsymbol{u}_n \rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{u}_n, \boldsymbol{u}_1 \rangle & \cdots & \langle \boldsymbol{u}_n, \boldsymbol{u}_n \rangle \end{pmatrix}$$

Conclusion: all matrices $\boldsymbol{A} \in M_{n\times n}(\mathbb{C})$ with $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} = \boldsymbol{I}$ can be written as

$$\boldsymbol{A} = \boldsymbol{P}^{-1}\operatorname{diag}(\lambda_1, \dots, \lambda_n)\boldsymbol{P},$$

with some $\boldsymbol{P}$ satisfying $\boldsymbol{P}^{\mathrm{H}}\boldsymbol{P} = \boldsymbol{I}$.

Notation. Let $U(n) = \{\boldsymbol{A} \in M_{n\times n}(\mathbb{C}) \mid \boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} = \boldsymbol{I})\}$ be the unitary group, then all $\boldsymbol{A} \in U(n)$ can be diagonalized by

$$A = P^{-1}\operatorname{diag}(\lambda_1, \dots, \lambda_n)P, \quad P \in U(n).$$

11.4.2. Normal Operators

Definition 11.10 [Normal] Let $T : V \to V$ be a linear operator over a $\mathbb{C}$ inner product vector space V. We say T is **normal**, if

$$T'T = TT'$$

■

▪ **Example 11.9**

- All self-adjoint operators are normal:

$$T = T' \implies TT' = T'T = T^2$$

- All unitary operators are normal:

$$T'T = TT' = I$$

■

Proposition 11.10 Let T be a normal operator on V. Then

1. $\|T(\boldsymbol{v})\| = \|T'(\boldsymbol{v})\|, \forall \boldsymbol{v} \in V$.

   In particular, $T(\boldsymbol{v}) = 0$ if and only if $T'(\boldsymbol{v}) = 0$

2. $(T - \lambda I)$ is also a normal operator, for any $\lambda \in \mathbb{C}$
3. $T(\boldsymbol{v}) = \lambda \boldsymbol{v}$ if and only if $T'(\boldsymbol{v}) = \bar{\lambda}\boldsymbol{v}$.

Proof. 1.

$$\begin{aligned}\langle T\boldsymbol{v}, T\boldsymbol{v}\rangle &= \langle T'T\boldsymbol{v}, \boldsymbol{v}\rangle \\ &= \langle TT'\boldsymbol{v}, \boldsymbol{v}\rangle \\ &= \overline{\langle \boldsymbol{v}, TT'\boldsymbol{v}\rangle} \\ &= \overline{\langle T'\boldsymbol{v}, T'\boldsymbol{v}\rangle} \\ &= \langle T'\boldsymbol{v}, T'\boldsymbol{v}\rangle\end{aligned}$$

Therefore, $\|T(\boldsymbol{v})\|^2 = \|T'(\boldsymbol{v})\|^2$, i.e., $\|T(\boldsymbol{v})\| = \|T'(\boldsymbol{v})\|$.

2. By hw4, $(T - \lambda I)' = T' - \overline{\lambda}I$. It suffices to check

$$(T - \lambda I)'(T - \lambda I) = (T - \lambda I)(T - \lambda I)',$$

Expanding both sides out gives the desired result, i.e.,

$$(T - \lambda I)'(T - \lambda I) = (T' - \bar{\lambda}I)(T - \lambda I) = T'T - \bar{\lambda}T - \lambda T' + \lambda\bar{\lambda}I$$

and

$$(T - \lambda I)(T - \lambda I)' = (T - \lambda I)(T' - \bar{\lambda}I) = TT' - \bar{\lambda}T - \lambda T' + \lambda\bar{\lambda}I$$

3. The proof for (3) will be discussed in the next lecture.

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