Lecture 4
Basics of Probability

- Probability

- Conditional Probability

- Discrete and Continuous Distributions

- Conditional Expectation and Variance

- Order Statistics

Contents

Elements of Probability

- Outcome (ω): the outcome of an experiment or trial

- Sample space/Probability space (Ω): the set of all possible outcomes of an experiment $\quad \Omega \to \cup$

- $P(\omega)$: the probability of an outcome

- $P(\omega) \geq 0, \forall \omega \in \Omega, \sum_{\omega \in \Omega} P(\omega) = 1.$

- Random variable: a function that maps each outcome (ω) in the sample space (Ω) into the set of real numbers.
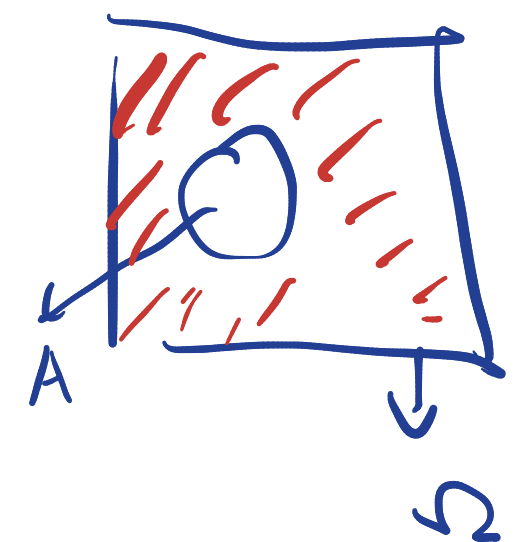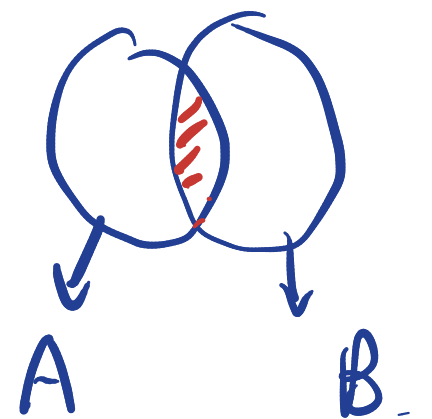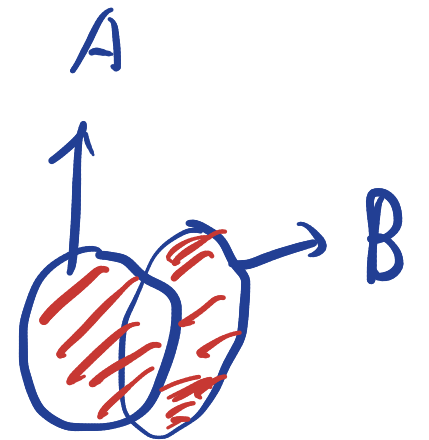
$$X: \Omega \to \{0, 1\}$$

$$X(w) = \begin{cases} 0, & \text{if } w \text{ is tail} \\ 1, & \text{if } w \text{ is head} \end{cases}$$

Event

- Event: A set of outcomes, a subset of the sample space.

- $P(A)$:Probability of an event A, i.e., $P(A) = \sum_{\omega \in A} P(\omega)$.

- $A \cup B$: set of outcomes in event A or in event B (or both)

- $A \cap B$: set of outcomes in both A and B, also written as AB

- A^c: the complement of A, which is the event "not A"

- Mutually exclusive: $A \cap B = \emptyset$.

- For any mutually exclusive events $E_1, \ldots, E_N$,

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i).$$

Example

Consider rolling a six-sided dice.

- $\Omega = \{1, 2, 3, 4, 5, 6\}$, and the probability of each outcome is $1/6$.

- Assume that $A = \{1, 3, 5\}$. Calculate $A^c, P(A)$.

$$A^c = \{2, 4, 6\}$$

$$P(A) = \sum_{w \in A} P(w)$$
$$= \frac{3}{6}$$

- Let B denote the event that the outcome is larger than 3. Calculate $A \cup B, A \cap B$.

- Define a special random variable I_A such that

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in \{1, 3, 5\}, \\ 0, & \text{if } \omega \notin \{1, 3, 5\}. \end{cases}$$

Calculate $\mathbb{E}[I_A]$.

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  $B = \{4, 5, 6\}$ $A \cup B = \{1, 3, 4, 5, 6\}$

  $A \cap B = \{5\}$

- Define a special random variable I_A such that

$$
I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in \{1, 3, 5\}, \\ 0, & \text{if } \omega \notin \{1, 3, 5\}. \end{cases}
$$

Calculate $\mathbb{E}[I_A]$.

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Calculate $\mathbb{E}[I_A]$.

$$E[I_A] = \sum_{\omega \in \Omega} I_A(\omega) P(\omega) = \sum_{\omega \in \{1,3,5\}} I_A(\omega) P(\omega)$$

$$= \frac{3}{6} := P(A)$$

Example

$$x + 0.5y = \frac{1}{2}(2x+y) = 0.5$$

Two gamblers are playing a coin toss game. Suppose Gambler A has 2 fair coins and B has 1 fair coin. What is the probability that A will have more heads than B if both flip all their coins?

E_1: A's first fair coins have more heads than B $P(E_1) = x$

E_2: — — — — — — have equal head as B $P(E_2) = y$

E_3: — — — — — — less heads than B

$$2x + y = 1 \quad \leftarrow \sum_{w \in \Omega} P(w) = 1$$

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- **Conditional Probability**

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Definition

- Conditional probability: If $P(B) > 0$, then $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ is the fraction of B outcomes that are also A outcomes.

- Multiplication rule:

$$P(A \mid B) = \frac{P(AB)}{P(B)} \iff P(AB) = P(B) P(A \mid B)$$

$$P(E_1 E_2 \cdots E_n) = P(E_1) P(E_2 \cdots E_n \mid E_1) = P(E_1) P(E_2 \mid E_1) P(E_3 \cdots E_n \mid E_1 E_2)$$

$$= P(E_1) P(E_2 \mid E_1) P(E_3 \mid E_1 E_2) \cdots P(E_n \mid E_1 \cdots E_{n-1}).$$

- Law of total probability: for any mutually exclusive events $\{F_i\}, i = 1, \ldots, n$ whose union is the entire sample space,

$$F_i \cap F_j = \emptyset$$

$$\bigcup_{i=1}^{n} F_i = \Omega$$

$$P(E) = P(EF_1) + \cdots + P(EF_n)$$

$$E = E \cap \left(\bigcup_{i=1}^{n} F_i \right) = \bigcup_{i=1}^{n} (E \cap F_i) \qquad = P(E \mid F_1) P(F_1) + \cdots + P(E \mid F_n) P(F_n)$$

Definition

- Independent events: $P(EF) = P(E)P(F)$. Also,

$$P(EF^c) = P(E)P(F^c).$$

- Bayes' formula:

$$P(F_j \mid E) = \frac{P(E \mid F_j)P(F_j)}{\sum_{i=1}^{n} P(E \mid F_i)P(F_i)},$$

provided that $F_i, i = 1, \ldots, n$ are mutually exclusive events whose union is the entire sample space.

$\Omega = \{(b, b), (b, g), (g, b), (g, g)\}$

Example

$A = \{(b, b)\}$ $B = \{(b, b), (b, g), (g, b)\}$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$

1. A company is holding a dinner for working mothers **with at least one son**. Ms. Jackson, a mother with two children, is invited. What is the probability that both children are boys?

B A

2. Your new colleague, Ms. Parker is known to have two children. Suppose you see her walking with **one of her children and that child is a boy**. What is the probability that both children are boys?

3. Part 1 asks given there is at least one boy in two children, what is the conditional probability that both children are boys. Part 2 asks that given one child is a boy, what is the conditional probability that the other child is also a boy.

Example

1. A company is holding a dinner for working mothers **with at least one son**. Ms. Jackson, a mother with two children, is invited. What is the probability that both children are boys?

   $A = \{ (b,b), \ (b,g) \}$

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   A ←

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Example

In a primitive society, let's assume every couple prefers to have a baby boy. Assume there is a 50% chance that each child they have is a boy,and the genders of their children are mutually independent. If each couple insists on having more children until they get a boy and once they have a boy, they will stop having more children. What will eventually happen to the fraction of boys in this society?

Example

$P(A) = $ prior Probability $\frac{1}{1000}$

$P(B|A) = 1$ $P(A^c) = \frac{999}{1000}$ $P(B|A^c) = \left(\frac{1}{2}\right)^{10}$

You are given 1000 coins. Among them, 1 coin has heads on both sides. $= \frac{1}{1024}$

The other 999 coins are fair coins. You randomly choose a coin and toss

it 10 times. Each time, the coin turns up heads. What is the probability

that the coin you choose is the unfair one?

$A = $ chosen coin is unfair $\qquad B: $ coin turn heads for 10 times

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)\,P(B|A)}{P(A)\,P(B|A) + P(A^c)\,P(B|A^c)} = \frac{\frac{1}{1000} \times 1}{\frac{1}{1000} \times 1 + \frac{999}{1000} \times \frac{1}{1024}} \approx \frac{1}{2}$$

$\qquad\qquad\qquad P(BA) \qquad\qquad P(BA^c)$

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Common functions of random variables

$$\text{proof: } \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 + (\mathbb{E}[X])^2 - 2X\mathbb{E}[X]] = \mathbb{E}[X^2] + \mathbb{E}[(\mathbb{E}[X])^2]$$

$$- \mathbb{E}[2X\mathbb{E}[X]]$$

$$= \mathbb{E}[X^2] + (\mathbb{E}[X])^2$$

$$- 2\mathbb{E}[X]\mathbb{E}[X]$$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Random Variable X	Discrete	Continuous
Cumulative distribution function (cdf)	$F(a) = P\{X \leq a\}$	$F(a) = \int_{-\infty}^{a} f(x)\mathrm{d}x$
Probability mass function (pmf) or Probability density function (pdf)	$p(x) = P\{X = x\}$	$f(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}F(x)$
Expected value $\mathbb{E}[X]$	$\displaystyle\sum_{x:\ p(x)>0} xp(x)$	$\displaystyle\int_{-\infty}^{\infty} xf(x)\mathrm{d}x$
Expected value of $g(x)$, $\mathbb{E}[g(x)]$	$\displaystyle\sum_{x:\ p(x)>0} g(x)p(x)$	$\displaystyle\int_{-\infty}^{\infty} g(x)f(x)\mathrm{d}x$
Variance of X, $\mathrm{Var}(X)$		$\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
Standard deviation of X, $\mathrm{std}(X)$		$\sqrt{\mathrm{Var}(X)}$

Common discrete random variables

Name	Probability mass function (pmf)	$E[X]$	var(X)
Uniform	$P(x) = \dfrac{1}{b-a+1}, \quad x = a, a+1, \cdots, b$	$\dfrac{b+a}{2}$	$\dfrac{(b-a+1)^2 - 1}{12}$
Binomial	$P(x) = \dbinom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \cdots, n$	np	$np(1-p)$
Poisson	$P(x) = \dfrac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, \cdots \, ^{20}$	λt	λt
Geometric	$P(x) = (1-p)^{x-1} p, \; x = 1, 2, \cdots$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$
Negative Binomial	$P(x) = \dbinom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \cdots$	$\dfrac{r}{p}$	$\dfrac{r(1-p)}{p^2}$

Common continuous random variables

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Name	Probability density function (pdf)	$E[X]$	$\text{var}(X)$
Uniform	$\dfrac{1}{b-a}$, $a \leq x \leq b$	$\dfrac{b+a}{2}$	$\dfrac{(b-a)^2}{12}$
Normal	$\dfrac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $x \in (-\infty, \infty)$	μ	σ^2
Exponential	$\lambda e^{-\lambda x}$, $x \geq 0$	$1/\lambda$	$1/\lambda^2$
Gamma	$\dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$, $x \geq 0$, $\Gamma(a) = \int_0^\infty e^{-y} y^{a-1}$	α/λ	α/λ^2
Beta	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$	$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

Example

$$B = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$
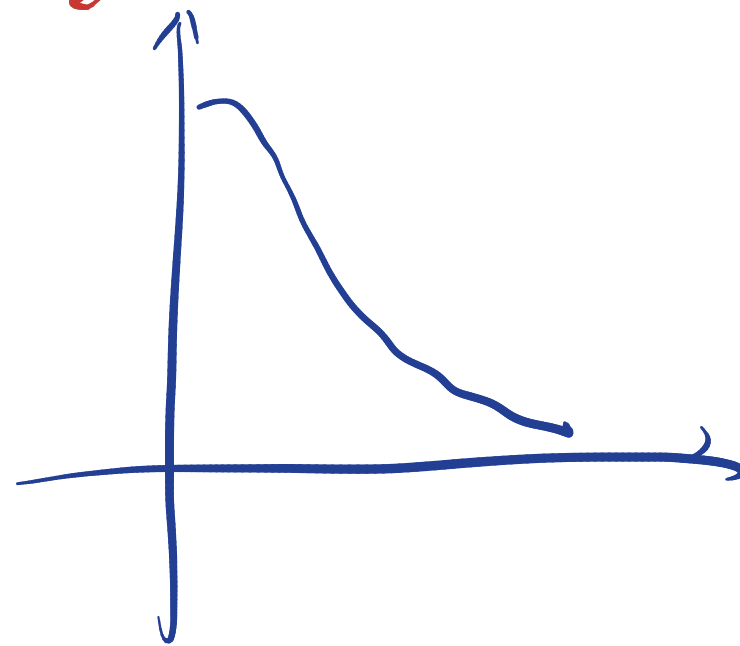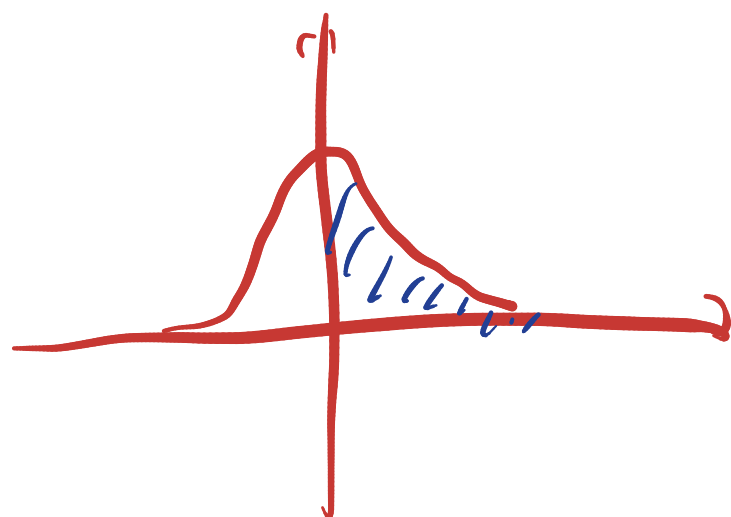
$$\Sigma = \begin{pmatrix} 5 & & \\ & 4 & \\ & & 3 \end{pmatrix}$$

$$\| \Sigma - B \|_F^2$$

$$= \sum_{i=1}^{3} (\Sigma_{ii} - B_{ii})^2 + \sum_{i \neq j} B_{ij}^2$$

If X is a standard normal random variable, what is $\mathbb{E}[X \mid X > 0]$?

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$B' = \begin{pmatrix} 1 & & \\ & 3 & \\ & & 5 \end{pmatrix}$$

$$\| \Sigma - B' \|_F^2$$

$$= \sum_i (\Sigma_{ii} - B_{ii})^2 < \| \Sigma - B \|_F^2$$

Example

Two bankers each arrive at the station at some random time between 5:00am and 6:00am (arrival time for either banker is uniformly distributed). They stay exactly five minutes and then leave. What is the probability they will meet on a given day?

Example

A stick is cut twice randomly (each cut point follows a uniform distribution on the stick). What is the probability that the 3 segments can form a triangle?

Example

You are waiting for a bus at a bus station. The bus arrive at the station according to a Poisson process with an average arrival time of 10 minutes ($\lambda = 0.1/\text{min}$). If the buses have been running for a long time and you arrive at the bus station at a random time, what is your expected waiting time? On average, how many minutes ago did the last bus leave?

Example

Assume X follows standard normal distribution, then what is $\mathbb{E}[X^n]$ for

$n = 1, 2, 3, 4$?

Contents

Definition

- For discrete distribution,

$$\mathbb{E}[g(X) \mid Y = y] = \sum_x g(x) p_{X|Y}(x \mid y) = \sum_x g(x) p(X = x \mid Y = y).$$

- For continuous distribution,

$$\mathbb{E}[g(X) \mid Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y) \mathrm{d}x.$$

- Law of total expectation:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \begin{cases} \sum_y \mathbb{E}[X \mid Y = y] p(Y = y), & \text{for discrete } Y, \\ \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y] f_Y(y) \mathrm{d}y, & \text{for continuous } Y. \end{cases}$$

Example

You just brought one share of stock A and wish to hedge it by shorting stock B. How many shares of B should you short in order to minimize the variance of the hedged position? Assume that the variance of stock A's return is σ_A^2, and that of B's return is σ_B^2. Their correlation coefficient is ρ.

Example

Suppose that you roll a dice. For each roll, you are paid the face value. if the roll gives $4, 5$, or 6, you can roll the dice again. Once you get $1, 2$, or 3, the game stops. What is the expected payoff of this game?

Contents

Definition

- Let X be a random variable with cumulative distribution function $F_X(x)$.

- Define $Y_n = \min(X_1, \ldots, X_n)$ and $Z_n = \max(X_1, \ldots, X_n)$.

- $f_{Y_n}(x) = n f_X(x)(1 - F_X(x))^{n-1}$

- $f_{Z_n}(x) = n f_X(x)(F_X(x))^{n-1}$

- Proof:

Example

Let $X_1, \ldots, X_n$ be i.i.d. random variables with uniform distribution between 0 and 1.

- What are the cumulative distribution funciton, the probability density function, and expeted value of $Z_n = \max(X_1, \ldots, X_n)$?

- What are the cumulative distribution function, the probability density function, and expected value of $Y_n = \min(X_1, \ldots, X_n)$?