3.3. Monday for MAT4002

3.3.1. Remarks on Basis and Homeomorphism

Reviewing.

1. $A \subseteq A_S \subseteq \overline{A}$, where A_S is sequential closure and $\overline{A}$ denotes closure.
2. Subspace topology.
3. Homeomorphism. Consider the mapping $f : X \to Y$ with the topogical space X, Y shown below, with the standard topology, the question is whether f is continuous?

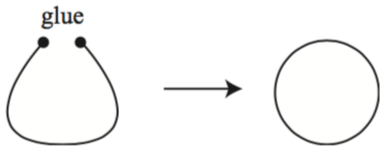

Figure 3.1: Diagram for mapping f

The answer is no, since the left in (3.1) can be isomorphically mapped into $(0,1)$; the right can be isomorphically mapped into $[0,1]$, and the mapping $(0,1) \to [0,1]$ cannot be isomorphism:

Proof. Assume otherwise the mapping $g : (0,1) \to [0,1]$ is isomorphism, and therefore $f^{-1}(U)$ is open for any open set U in the space $[0,1]$.
Construct $U = (1-\delta, 1]$ for $\delta \le 1$, and therfore $f^{-1}((1-\delta,1])$ is open, and therfore for the point $x = f^{-1}(1)$, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq f^{-1}((1-\delta,1]) \implies [x-\varepsilon, x) \subseteq f^{-1}((1-\delta,1)), \text{ and } (x, x+\varepsilon] \subseteq f^{-1}((1-\delta,1)).$$

which implies that there exists a, b such that $[x-\varepsilon, x) = f^{-1}((a,1))$ and $(x, x+\varepsilon] = f^{-1}((b,1))$, i.e., $f^{-1}((a,b) \cap (b,1))$ admits into two values in $[x-\varepsilon, x)$ and $(x, x+\varepsilon]$, which is a contradiction. ∎

4. Basis of a topology $\mathcal{B} \subseteq (X, \mathcal{T})$ is a collection of open sets in the space such that the whole space can be recovered, or equivalently

(a) $\mathcal{B} \subseteq \mathcal{T}$

(b) Every set in $\mathcal{T}$ can expressed as a union of sets in $\mathcal{B}$

Example: Let $\mathbb{R}^n$ be equipped with usual topology, then

$$B = \{B_q(x) \mid x \in \mathbb{Q}^n, q \in \mathbb{Q}^+\} \text{ is a basis of } \mathbb{R}^n.$$

It suffices to show $U \subseteq \mathbb{R}^n$ can be written as

$$U = U_{x \in \mathbb{Q}} B_{q_x}(x)$$

Proposition 3.4 Let X, Y be topological spaces, and $\mathcal{B}$ a basis for topology on Y. Then

$$f : X \to Y \text{ is continuous} \iff f^{-1}(B) \text{ is open in X}, \forall B \in \mathcal{B}$$

Therefore checking $f^{-1}(U)$ is open for all $U \in \mathcal{T}_Y$ suffices to checking $f^{-1}(N)$ is open for all $B \in \mathcal{B}$.

Proof. The forward direction follows from the fact $B \subseteq \mathcal{T}_Y$.

To show the reverse direction, let $U \in \mathcal{T}_Y$, then $U = \bigcup_{i \in I} B_i$, where $B_i \in \mathcal{B}$, which implies

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$$

which is open in X by our hypothesis. ∎

Corollary 3.1 Let $f : X \to Y$ be a bijection. Suppose there is a basis $\mathcal{B}_X$ of $\mathcal{T}_X$ such that $\{f(B) \mid B \in \mathcal{B}_X\}$ forms a basis of $\mathcal{T}_Y$. Then $X \cong Y$.

Proof. Suppose $W \in \mathcal{T}_Y$, then by our hypothesis,

$$W = \bigcup_{i \in I} f(B_i),\ B_i \in \mathcal{B}_X \implies f^{-1}(W) = \bigcup_{i \in I} B_i \in \mathcal{T}_X,$$

which implies f is continuous.

Suppose $U \in \mathcal{T}_X$, then

$$U = \bigcup_{i \in I} B_i \implies f(U) = \bigcup_{i \in I} f(B_i) \in \mathcal{T}_Y \implies [f^{-1}]^{-1}(U) \in \mathcal{T}_Y,$$

i.e., f is continuous. ■

Question: *how to recognise whether a family of subsets is a basis for some given topology?*

Proposition 3.5 Let X be a set, $\mathcal{B}$ is a collection of subsets satisfying

1. X is a union of sets in $\mathcal{B}$, i.e., every $x \in X$ lies in some $B_x \in \mathcal{B}$
2. The intersection $B_1 \cap B_2$ for $\forall B_1, B_2 \in \mathcal{B}$ is a union of sets in $\mathcal{B}$, i.e., for each $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the collection of subsets $\mathcal{T}_{\mathcal{B}}$, formed by taking any union of sets in $\mathcal{B}$, is a topology, and $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{B}}$.

Proof. 1. $\emptyset \in \mathcal{T}_{\mathcal{B}}$ (taking nothing from $\mathcal{B}$); for $x \in X, B_x \in \mathcal{B}$, by hypothesis (1),

$$X = \bigcup_{x \in X} B_x \in \mathcal{T}_{\mathcal{B}}$$

2. Suppose $T_1, T_2 \in \mathcal{T}_{\mathcal{B}}$. Let $x \in T_1 \cap T_2$, where T_i is a union of subsets in $\mathcal{B}$. Therefore,

$$\begin{cases} x \in B_1 \subseteq T_1, & B_1 \in \mathcal{B} \\ x \in B_2 \subseteq T_2, & B_2 \in \mathcal{B} \end{cases}$$

which implies $x \in B_1 \cap B_2$, i.e., $x \in B_x \subseteq B_1 \cap B_2$ for some $B_x \in \mathcal{B}$. Therefore,

$$\bigcup_{x \in B_1 \cap B_2} \{x\} \subseteq \bigcup_{x \in B_1 \cap B_2} B_x \subseteq B_1 \cap B_2,$$

i.e., $B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_x$, i.e., $B_1 \cap B_2 \in \mathcal{T}_{\mathcal{B}}$.

3. The property that $\mathcal{T}_{\mathcal{B}}$ is closed under union operations can be checked directly.

The proof is complete. ■

3.3.2. Product Space

Now we discuss how to construct new topological spaces out of given ones is by taking Cartesian products:

Definition 3.4 Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. Consider the family of subsets in $X \times Y$:

$$\mathcal{B}_{X\times Y} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_y\}$$

This $\mathcal{B}_{X\times Y}$ forms a basis of a topology on $X \times Y$. The induced topology from $\mathcal{B}_{X\times Y}$ is called **product topology**.

For example, for $X = \mathbb{R}, Y = \mathbb{R}$, the elements in $\mathcal{B}_{X\times Y}$ are rectangles.

Proof for well-definedness in definition (3.4). We apply proposition (3.5) to check whether $B_{X\times Y}$ forms a basis:

1. For any $(x, y) \in X \times Y$, we imply $x \in X, y \in Y$. Note that $X \in \mathcal{T}_X, Y \in \mathcal{T}_Y$, we imply $(x, y) \in X \times Y \in \mathcal{B}_{X\times Y}$.
2. Suppose $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}_{X\times Y}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

where $U_1 \cap U_2 \in \mathcal{T}_X, V_1 \cap V_2 \in \mathcal{T}_Y$. Therefore, $(U_1 \times V_1) \cap (U_2 \times V_2) \in \mathcal{B}_{X\times Y}$.

■

R However, the product topology may not necessarily become the largest topology in the space $X \times Y$. Consider $X = \mathbb{R}, Y = \mathbb{R}$, the open set in the space $X \times Y$ may not necessarily be rectangles. However, all elements in $\mathcal{B}_{X\times Y}$ are rectangles.

Example 3.8 The space $\mathbb{R} \times \mathbb{R}$ is isomorphic to $\mathbb{R}^2$, where the product topology is defined on $\mathbb{R} \times \mathbb{R}$ and the standard topology is defined on $\mathbb{R}^2$:

Construct the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ with $(a, b) \to (a, b)$.

Obviously, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ is a bijection.

Take the basis of the topology on $\mathbb{R}$ as open intervals,

$$B_X = \{(a,b) \mid a < b \text{ in } \mathbb{R}\}$$

Therefore, one can verify that the set $\mathcal{B} := \{(a,b) \times (c,d) \mid a < b, c < d\}$ forms a basis for the product topology, and

$$\{f(B) \mid B \in \mathcal{B}\} = \{(a,b) \times (c,d) \mid a < b, c < d\}$$

forms a basis of the usual topology in $\mathbb{R}^2$.

By Corollary (3.1), we imply $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$. ■

We also raise an example on the homeomorphism related to product spaces:

■ **Example 3.9** Let $S^1 = \{(\cos x, \sin x \mid x \in [0, 2\pi])\}$ be a unit circle on $\mathbb{R}^2$.

Consider $f : S^1 \times (0, \infty) \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ defined as

$$f(\cos x, \sin x, r) \mapsto (r\cos x, r\sin x)$$

It's clear that f is a bijection, and f is continuous. Moreover, the inverse $g := f^{-1}$ is defined as

$$g(a,b) = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, \sqrt{a^2+b^2})$$

which is continuous as well. Therefore, the $f : \mathcal{S}^1 \times (0, \infty) \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ is a homeomorphism.

■