Alternating Maximization: An Unified View of EM and Blahut-Arimoto Algorithms

Jie Wang
Fourth Year Undergraduate Student
Pure Mathematics Major
Network Coding Lab

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Motivation

- Alternating Optimization is applicable to a special class of problems: Those here varibales can be split into two sets, i.e., $v = (x, y)$ such that

$$\max_x f(x,y) \qquad \text{is easy for fixed } y$$
$$\max_y f(x,y) \qquad \text{is easy for fixed } x$$

- Alternating Optimization Framework:

$$x_+ = \arg\max_x f(x,y)$$
$$y_+ = \arg\max_y f(x_+,y)$$

- Comment:
  - "big" global steps, and much faster than "local" gradient descent
  - No step size to be tuned / chosen

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Problem Setting

- Consider optimization for groups of variables:

$$\max_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}} f(x,y)$$

where $\mathcal{X},\mathcal{Y}$ are *convex*, and the objective function f is such that:

  - $f:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ is bounded above;
  - f is continuous and has continuous partial derivatives on $\mathcal{X}\times\mathcal{Y}$;
  - For any fixed $y\in\mathcal{Y}$, there exists an unique $c_1(y)\in\mathcal{X}$ such that

$$f(c_1(y),y)=\max_{x\in\mathcal{X}} f(x,y)$$

For any fixed $x\in\mathcal{X}$, there exists an unique $c_2(x)\in\mathcal{Y}$ such that

$$f(x,c_2(x))=\max_{y\in\mathcal{Y}} f(x,y)$$

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Alternating Optimization (AO) and its Variants

Algorithm 1 Alternating Optimization

Require: Objective function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$;

Ensure: A near-optimal point $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$;

$(x^1, y^1) \leftarrow$ INITALIZE$()$;

for $t = 1, 2, \ldots, T$ **do**

$\quad x^{t+1} \leftarrow \arg\max_{x \in \mathcal{X}} f(x, y^t)$;

$\quad y^{t+1} \leftarrow \arg\max_{y \in \mathcal{Y}} f(x^{t+1}, y)$;

end for

return (x^T, y^T)

1. Question 1: What point does AO Algorithm converge to?
2. Question 2: How fast for the convergence of AO Algorithm?
3. Question 3: Is the limit point guaranteed to be (*global*) optimal?

Alternating Optimization (AO) and its Variants

Algorithm 2 Alternating Optimization

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Ensure: A near-optimal point $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$;

1: $(x^1, y^1) \leftarrow$ INITALIZE$()$;

2: **for** $t = 1, 2, \ldots, T$ **do**

3: $\quad x^{t+1} \leftarrow \arg\max_{x \in \mathcal{X}} f(x, y^t)$;

4: $\quad y^{t+1} \leftarrow \arg\max_{y \in \mathcal{Y}} f(x^{t+1}, y)$;

5: **end for**

6: **return** (x^T, y^T)

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Remarks for Alternating Optimization

- Alternating Optimization Algorithm converges to *Bistable Point*.

Definition (Bistable Point)

A point $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ is said to be a *bistable point* if

$$y^* = \arg\max_{y \in \mathcal{Y}} f(x^*, y), \qquad \text{and} \qquad x^* = \arg\max_{x \in \mathcal{X}} f(x, y^*)$$

- All bistable points are *globally optimal* for convex problems.
  For marginally-convex problems, bi-stable points are equivalent to FOSP.
- Alternating Optimization has sublinear convergence rate.

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Concluding Remarks

1. Successful for a collection of machine learning problems:
   - Phase retrieval
   - Matrix sensing
   - Robust PCA
   - Matrix Completion
   - Mixed linear regression
2. Special initialization is designed, such as *spectral initialization*
3. Empirically: memory efficient and parallelly faster than convex methods such as trace-norm minimization
4. **Open Problem:** a more general theory of AltMin and its convergence ...

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Problem Setting

- An statistical model $\mathcal{F}$ is generating $X \in \mathcal{X}, Z \in \mathcal{Z}$:

$$\mathcal{F} = \{f_\theta = p(\cdot, \cdot \mid \theta) : \theta \in \Theta\}$$

- f_{θ^*} generates sample pair $(x_i, z_i)_{i=1}^n$, but only $\{x_i\}_{i=1}^n$ is observed.
- The goal is to recover θ^*, by using only samples $\{x_i\}_{i=1}^n$.
  The most popular method is by likelihood maximization:

$$\mathcal{L}(\theta; x_1, \dots, x_n) \triangleq p(x_1, \dots, x_n \mid \theta) = \prod_{i=1}^n \sum_{z_i \in \mathcal{Z}} p(x_i, z_i \mid \theta)$$

$$\hat{\theta}_{\text{MLE}} = \arg\max_{\theta \in \Theta} \log \mathcal{L}(\theta; x_1, \dots, x_n) \triangleq \sum_{i=1}^n \log \mathcal{L}(\theta; x_i)$$

- Intractability: The objective contains $|\mathcal{Z}|^n$ conditional probability terms!

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Description of EM Algorithm

- The likelihood maximization is to solve

$$\hat{\theta}_{\text{MLE}} = \arg\max_{\theta\in\Theta} \log \mathcal{L}(\theta; x_1, \ldots, x_n) \triangleq \sum_{i=1}^{n} \underbrace{\log \sum_{z_i\in\mathcal{Z}} p(x_i, z_i \mid \theta)}_{\log \mathcal{L}(\theta, x_i)}$$

- The EM Algorithm is separated into two steps:
  1. E-step: Construct the Q-function $Q(\theta \mid \theta^t)$;
     (A lower bound of the objective function for current iteration)
  2. M-step: Maximize the Q-function such that

$$\theta^{t+1} \leftarrow \arg\max Q(\theta \mid \theta^t).$$

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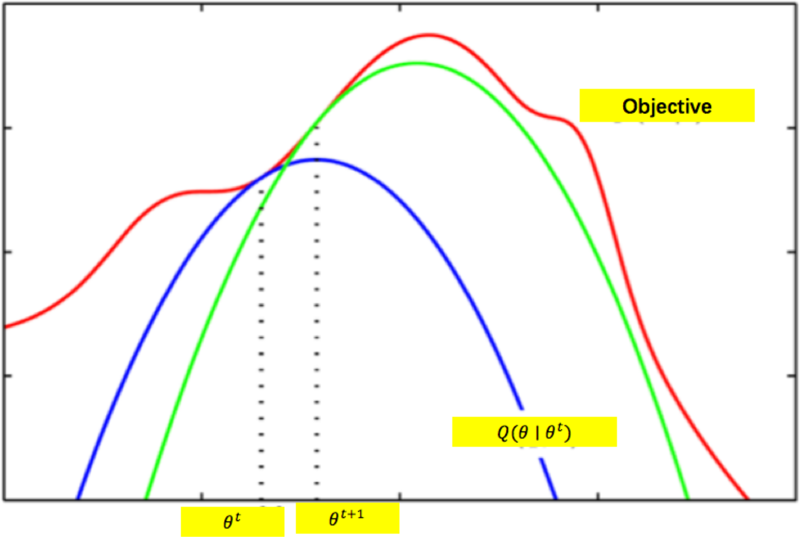
EM is a successive approximation method



EM Algorithm in Detail

Key inequality:

$$\log \mathcal{L}(\theta; x_i) \triangleq \log \sum_{z_i \in \mathcal{Z}} p(x_i, z_i \mid \theta) \geq Q_{x_i}(\theta \mid \theta^t) \triangleq \mathbb{E}_{z \sim p(\cdot \mid x_i, \theta^t)}[\log p(x_i, z \mid \theta)]$$

$$\text{Objective} \triangleq \sum_{i=1}^{n} \log \mathcal{L}(\theta; x_i) \geq Q(\theta \mid \theta^t) \triangleq \sum_{i=1}^{n} Q_{x_i}(\theta \mid \theta^t)$$

EM Algorithm Description

- E-step: Compute the distribution $p(Z \mid x_i, \theta^t)$.
  (Then the Q-function can be constructed)
- M-Step: Choose θ^{t+1} maximizing the objective function above.

Remark: The construction of Q-function only requires $p(\cdot \mid x, \theta^t)$ and $p(\cdot, \cdot \mid \theta)$.

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EM as maximization-maximization

Consider the following function:

$$F(\theta, q) \equiv \mathbb{E}_q[\log p(z, x_{1:n} \mid \theta)] + H(q), \quad q \text{ is some distribution of } z.$$

Or equivalently,

$$F(\theta, q) = -D(q \| q_\theta) + \mathcal{L}(\theta; x_{1:n}), \qquad q_\theta(z) = p(z \mid x_{1:n}, \theta)$$

Theorem (Alternative Maximimzation of F reduces to EM)

1. *For fixed θ, the unique maximizer of $F(\theta, q)$ is given by*

$$q = q_\theta \triangleq p(z \mid x_{1:n}, \theta).$$

2. *For fixed $q = q_\theta$, the M-step is equivalent to maximizing $F(\theta, q_\theta)$.*

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Landscape of Alternating Maximization

Alternating Maximimzation & EM Algorithm

Suppose that the function $F(\theta, q)$ has a local maxima (θ^*, q^*):

$$\begin{aligned}(\theta^*, q^*) &= \arg\max F(\theta, q)\\ &:= \arg\max \mathbb{E}_q[\log p(z, x_{1:n} \mid \theta)] + H(q)\\ &:= \arg\max -D(q \| q_\theta) + \mathcal{L}(\theta; x_{1:n}).\end{aligned}$$

Then the point θ^* is a local maxima of objective function $\mathcal{L}(\theta; x_{1:n})$:

$$\theta^* = \arg\max \mathcal{L}(\theta; x_1, \ldots, x_n) \triangleq p(x_1, \ldots, x_n \mid \theta).$$

Proof.

$$\text{local maxima } (\theta^*, q^*) \implies q^* = q_\theta \implies \theta^* = \arg\max \mathcal{L}(\theta; x_{1:n})$$

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□

EM Algorithm as Alternating Maximization

Algorithm 3 Expectation Maximization as Alternating Maximization

1: **Input:** Solvers of $\max\ F(\theta, q)$ for fixed q and $\max\ F(\theta, q)$ for fixed θ;
2: **Output:** A good parameter $\hat{\theta} \in \Theta$;
3: $\theta^1 \leftarrow$ INITALIZE();
4: **for** $t = 1, 2, \ldots$ **do**
5: $\quad q^t \leftarrow \arg\max F(\theta^{t-1}, q)$;
6: $\quad \theta^t \leftarrow \arg\max F(\theta, q^t)$;
7: **end for**

Comment:

1. The Alternating Maximization blindly searches for FOSP.
2. Pay attention to Initialization, unless for unique bi-stable point;

EM Algorithm as Alternating Maximization

Algorithm 4 Expectation Maximization as Alternating Maximization

Input: Solvers of $\max\ F(\theta, q)$ for fixed q and $\max\ F(\theta, q)$ for fixed θ;
Output: A good parameter $\hat{\theta} \in \Theta$;
$\theta^1 \leftarrow$ INITALIZE();
for $t = 1, 2, \ldots$ **do**
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    $\theta^t \leftarrow \arg\max F(\theta, q^t)$;
end for

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Incremental Variants of EM Algorithm (IVEM)

Key Observation:

$$\begin{aligned} F(\theta, q) &= \mathbb{E}_q[\log p(z, x_{1:n} \mid \theta)] + H(q) \\ &= \sum_{i=1}^{n} \mathbb{E}_q[\log p(z_i, x_i \mid \theta)] + H(q_i) \\ &\triangleq \sum_{i=1}^{n} F_i(\theta, q_i), \end{aligned}$$

Incremental Variants of EM Algorithm (IVEM)

$$F(\theta, q) = \sum_{i=1}^{n} F_i(\theta, q_i)$$

Perform the maximization of $q := (q_i)_{i=1}^{n}$ *lazily*:

Algorithm 5 An Incremental Variant of EM Algorithm

```
Input: Solvers of max F_i(θ, q) for fixed θ;
Output: A good parameter θ̂ ∈ Θ;
θ^1 ← INITALIZE();
for t = 1, 2, ... do
    Pick Index i_t from {1, ..., n};
    q^t_{i_t} ← arg max F_{i_t}(θ^{t−1}, q_{i_t});
    q^t_j ← q^{t−1}_j for any j ≠ i_t;
    θ^t ← arg max F(θ, q^t) = Σ_{i=1}^{n} F_i(θ, q^t_i)        (Not Memory Efficient);
end for
```

- **Input:** Solvers of $\max \; F_i(\theta, q)$ for fixed θ;
- **Output:** A good parameter $\hat{\theta} \in \Theta$;
- $\theta^1 \leftarrow$ INITALIZE$()$;
- **for** $t = 1, 2, \ldots$ **do**
  - Pick Index i_t from $\{1, \ldots, n\}$;
  - $q_{i_t}^t \leftarrow \arg\max F_{i_t}(\theta^{t-1}, q_{i_t})$;
  - $q_j^t \leftarrow q_j^{t-1}$ for any $j \neq i_t$;
  - $\theta^t \leftarrow \arg\max F(\theta, q^t) = \sum_{i=1}^{n} F_i(\theta, q_i^t)$ (Not Memory Efficient);
- **end for**

Finite-Sum Problem Setting

Consider the optimization for a finite-sum objective:

$$\max_x \quad \sum_{i=1}^{n} f_i(x)$$

$$\begin{aligned} \text{Full GD}: \quad & x^{t+1} = x^t + \eta_t \frac{1}{n} \sum_i \nabla f_i(x^t) \quad && \mathcal{O}(n) \text{ computations per iteration} \\ \text{SGD}: \quad & x^{t+1} = x^t + \eta_t \nabla f_{i_t}(x^t) && \mathcal{O}(1) \text{ computation per iteration} \\ & && \text{(but many more iterations)} \end{aligned}$$

Stochastic Average Gradient (SAG)

- At each iteration t, update the gradient estimate lazily:
  - Pick Index i_t from $\{1, \ldots, n\}$,

$$g_{i_t}^t = \nabla f_{i_t}(x^{t-1})$$
$$g_j^t = g_j^{t-1}, \quad \text{for any } j \neq i_t$$

  - The gradient estimate for objective $\sum_{i=1}^{n} f_i(x)$ is given by

$$\sum_{i=1}^{n} g_i^t$$

- At each iteration t perform the update

$$x^t = x^{t-1} + \eta_t \sum_{i=1}^{n} g_i^t$$

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Memoery Efficient Implementation of SAG

The gradient update of SAG only takes *contant* time, independ of n:

$$\sum_{i=1}^{n} g_i^t = g_{i_t}^t - g_{i_t}^{t-1} + \sum_{i=1}^{n} g_i^{t-1}$$

$$a^t = g_{i_t}^t - g_{i_t}^{t-1} + a^{t-1}$$

At each iteration t perform the update

$$x^t = x^{t-1} + \eta_t a^t$$

Memoery Efficient Implementation of SAG

The gradient update of SAG only takes *contant* time, independ of n:

$$\sum_{i=1}^{n} g_i^t = g_{i_t}^t - g_{i_t}^{t-1} + \sum_{i=1}^{n} g_i^{t-1}$$

$$a^t = g_{i_t}^t - g_{i_t}^{t-1} + a^{t-1}$$

At each iteration t perform the update

$$x^t = x^{t-1} + \eta_t a^t$$

Memoery Efficient Implementation of IVEM

$$\max \sum_{i=1}^{n} F_i(\theta, q_i^t) = \max \sum_{i=1}^{n} F_i(\theta, q_i^{t-1}) - F_{i_t}(\theta, q_{i_t}^{t-1}) + F_{i_t}(\theta, q_{i_t}^t)$$

Algorithm 6 Another Incremental Variant of EM Algorithm

1: **Input:** Solvers of $\max \ F_i(\theta, q)$ for fixed θ;
2: **Output:** A good parameter $\hat{\theta} \in \Theta$;
3: $\theta^1 \leftarrow$ INITALIZE();
4: **for** $t = 1, 2, \ldots$ **do**
5: Pick Index i_t from $\{1, \ldots, n\}$;
6: $q_{i_t}^t \leftarrow \arg\max F_{i_t}(\theta^{t-1}, q_{i_t})$;
7: $\theta^t \leftarrow \arg\max F(\theta, q^{t-1}) - F_{i_t}(\theta, q_{i_t}^{t-1}) + F_{i_t}(\theta, q_{i_t}^t)$;
8: **end for**

Exponentially decaying variants of IVEM

$$\max \sum_{i=1}^{n} F_i(\theta, q_i^t) \approx \max \ \gamma \sum_{i=1}^{n} F_i(\theta, q_i^{t-1}) + F_{i_t}(\theta, q_{i_t}^t)$$

Algorithm 7 Exponentially decaying version of IVEM

1: **Input:** Solvers of $\max \ F_i(\theta, q)$ for fixed θ;
2: **Output:** A good parameter $\hat{\theta} \in \Theta$;
3: $\theta^1 \leftarrow$ INITALIZE();
4: **for** $t = 1, 2, \ldots$ **do**
5: Pick Index i_t from $\{1, \ldots, n\}$;
6: $q_{i_t}^t \leftarrow \arg\max F_{i_t}(\theta^{t-1}, q_{i_t})$;
7: $\theta^t \leftarrow \arg\max \gamma F(\theta, q^{t-1}) + F_{i_t}(\theta, q_{i_t}^t)$;
8: **end for**

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Preliminary Definitions

- The KL-Divergence between two distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is given by:

$$D(\boldsymbol{p}\|\boldsymbol{q}) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$$

  where $\boldsymbol{p}$ and $\boldsymbol{q}$ are assumed to have the same support set $\{1, \ldots, n\}$
- The mutual information between a pair of random variables (X, Y) is:

$$I(X;Y) = D(p_{X,Y} \| p_X \otimes p_Y)$$

- Suppose that the joint distribution of (X, Y) is $p_X \cdot p_{Y|X}$. For fixed conditional distribution $p_{Y|X}$, the channel capacity of (X, Y) is defined as

$$C = \max_{p_X} I(X;Y)$$

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Variantional Charactization of Mutual Information

- Suppose that $X \sim \boldsymbol{p}$ and $Y \mid X \sim \boldsymbol{Q}$, then

$$I(X;Y) \triangleq I(\boldsymbol{p}, \boldsymbol{Q}) = \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{Y}|} p_i Q_{i,j} \log \frac{Q_{i,j}}{q_j}$$

where $Y \sim \boldsymbol{q} \triangleq \boldsymbol{p} \cdot \boldsymbol{Q}$.

Theorem (Global Tight Lower Bound of Mutual Information)

$$I(\boldsymbol{p}, \boldsymbol{Q}) \geq \tilde{I}(\boldsymbol{p}, \boldsymbol{Q}; \phi) \triangleq \sum_{i=1}^{|\mathcal{X}|} \sum_{j=1}^{|\mathcal{Y}|} p_i Q_{i,j} \log \frac{\phi(i \mid j)}{p_i}$$

$$I(\boldsymbol{p}, \boldsymbol{Q}) = \max_{\phi \in \Phi} \tilde{I}(\boldsymbol{p}, \boldsymbol{Q}; \phi),$$

where the optimal transition matrix ϕ is given by:

$$\phi^*(i \mid j) = p_i \frac{Q_{i,j}}{q_j} \triangleq p_i \frac{Q_{i,j}}{\sum_{i=1}^{|\mathcal{X}|} p_i Q_{i,j}}$$

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Variants of Channel Capacity Problem

- Channel Capacity Problem: for fixed transition matrix Q, we aim to solve

$$C(Q) := \max_{p_X} I_Q(X;Y)$$

- Rate Distortion Functions: for fixed distortion D,

$$R(D) := \min_{Q(y|x)} I(X;Y), \quad \text{subject to } \mathbb{E}d(X;Y) \leq D.$$

- Information Bottleneck Problem.

Comment: The method to be discussed can be applied to all those problems.

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Channel Capacity by Alternating Optimization (1)

The Channel Capacity is by maximizing over two groups of variables:

$$\mathcal{C} = \max_{\boldsymbol{p}} \max_{\phi \in \Phi} \tilde{I}(\boldsymbol{p}, \boldsymbol{Q}; \phi)$$

Key observation

1. For fixed $\boldsymbol{p}$ and $\boldsymbol{Q}$, we have

$$\arg\max_{\phi \in \Phi} \tilde{I}(\boldsymbol{p}, \boldsymbol{Q}; \phi) = \phi^*, \quad \phi^*(i \mid j) = p_i \frac{Q_{i,j}}{\sum_{i=1}^{|\mathcal{X}|} p_i Q_{i,j}}$$

2. For fixed ϕ and $\boldsymbol{Q}$, we have

$$\arg\max_{\boldsymbol{p}} \tilde{I}(\boldsymbol{p}, \boldsymbol{Q}; \phi) = \boldsymbol{p}^*, \quad p^*(i) = \frac{r_i}{\sum_{i=1}^{|\mathcal{X}|} r_i},$$

$$r_i \triangleq \exp\left(\sum_{j=1}^{|\mathcal{Y}|} Q_{i,j} \log \phi(i \mid j)\right)$$

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Channel Capacity by Alternating Optimization (2)

The Channel Capacity can be computed using alternating maximization:

$$\phi^{t+1} = \arg\max_{\phi \in \Phi} \tilde{I}(\boldsymbol{p}^t; \boldsymbol{Q}; \phi)$$
$$\boldsymbol{p}^{t+1} = \arg\max_{\boldsymbol{p}} \tilde{I}(\boldsymbol{p}; \boldsymbol{Q}; \phi^{t+1})$$

Key observation

1. $$\phi^{t+1}(i \mid j) = p_i^t \frac{Q_{i,j}}{\sum_{i=1}^{|\mathcal{X}|} p_i^t Q_{i,j}} \triangleq \frac{p_i^t Q_{i,j}}{q_j^{t+1}}$$
2. Substituting $\phi := \phi^{t+1}$ and for fixed $\boldsymbol{Q}$,

$$p^{t+1}(i) = \frac{r_i^{t+1}}{\sum_{i=1}^{|\mathcal{X}|} r_i^{t+1}}, \qquad \text{with } r_i^{t+1} \triangleq p_i^t \exp\left(D(\boldsymbol{Q}_i \| \boldsymbol{q}^{t+1}) \right)$$

Blahut-Arimoto Algorithm

Algorithm 8 Blahut-Arimoto Algorithm for Computing Channel Capacity

1: $\boldsymbol{p}^0 \leftarrow$ INITALIZE$()$;
2: **for** $t = 1, 2, \ldots$ **do**
3: $\quad \boldsymbol{q}^{t+1} \leftarrow \boldsymbol{p}^t \boldsymbol{Q}$;
4:

$$p_i^{t+1} \leftarrow \frac{p_i^t \exp(D(\boldsymbol{Q}_i \| \boldsymbol{q}^{t+1}))}{\sum_{i=1}^{|\mathcal{X}|} p_i^t \exp(D(\boldsymbol{Q}_i \| \boldsymbol{q}^{t+1}))}$$

5: **end for**

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Re-consideration of Blahut-Arimoto Algorithm

- The Blahut-Arimoto Algorithm is essentially the alternating maximization:

$$\phi^{t+1} = \arg\max_{\phi\in\Phi} \tilde{I}(\boldsymbol{p}^t; \boldsymbol{Q}; \phi)$$
$$\boldsymbol{p}^{t+1} = \arg\max_{\boldsymbol{p}} \tilde{I}(\boldsymbol{p}; \boldsymbol{Q}; \phi^{t+1})$$

Theorem (Reformulation for the update of $\boldsymbol{p}^{t+1}$)

$$\boldsymbol{p}^{t+1} = \arg\max_{\boldsymbol{p}} \left(I(\boldsymbol{p}^t, \boldsymbol{Q}) + \sum_{i=1}^{N} (p_i - p_i^t) \cdot D(\boldsymbol{Q}_i \| \boldsymbol{q}^t) - D(\boldsymbol{p} \| \boldsymbol{p}^t) \right)$$

Each update of $\boldsymbol{p}^{t+1}$ suffices to maximize the first-order Taylor-expansion of $I(\boldsymbol{p}, \boldsymbol{Q})$, with penalty term $D(\boldsymbol{p} \| \boldsymbol{p}^t)$.

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Proof.

By substituting the term ϕ^{t+1} into the update of $\boldsymbol{p}^{t+1}$:

$$\begin{aligned}
\boldsymbol{p}^{t+1} :&= \arg\max_{\boldsymbol{p}} \tilde{I}(\boldsymbol{p}; \boldsymbol{Q}; \phi^{t+1}) = \arg\max_{\boldsymbol{p}} \sum_i \sum_j p_i Q_{i,j} \log \frac{\phi^{t+1}(i \mid j)}{p_i} \\
&= \arg\max_{\boldsymbol{p}} \sum_i \sum_j p_i Q_{i,j} \log \frac{p_i^t Q_{i,j}}{p_i q_j^{t+1}} \\
&= \arg\max_{\boldsymbol{p}} \left(\sum_{i=1}^{N} p_i \cdot D(\boldsymbol{Q}_i \| \boldsymbol{q}^t) - D(\boldsymbol{p} \| \boldsymbol{p}^t) \right)
\end{aligned}$$

$\square$

Proximal Point (Mirror Descent) Algorithm

Consider the maximization of a concave function $f(x)$:

$$\max_{x \in \mathcal{X}} \ f(x)$$

- Let $f_k(x)$ be a certain concave approximation of $f(x)$.
- Let $\{t_k > 0 \mid k = 0, 1, \dots\}$ be a sequence of parameters.
- The Bregman distance is a generalized choice of Euclidean distance:

$$B(y, x) = \Phi(y) - \Phi(x) - \nabla^{\mathrm{T}}\Phi(x)(y - x),$$

where Φ is a smooth and strongly convex function

Algorithm 9 Proximal Point (Mirror Descent) Algorithm

$x^0 \leftarrow$ INITALIZE();
for $t = 0, 1, 2, \dots$ **do**

$$x^{t+1} \leftarrow \arg\max_{x \in \mathcal{X}} f_k(x) - \frac{1}{\gamma_t} B(x, x^t).$$

end for

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end for

Blahut-Arimoto Algorithm as a Proximal Point Algorithm

- The Blahut-Arimoto Algorithm is essentially the proximal point algorithm:

$$\boldsymbol{p}^{t+1} = \arg\max_{\boldsymbol{p}} \left(I_t(\boldsymbol{p}, \boldsymbol{Q}) - \frac{1}{\gamma_t} D(\boldsymbol{p} \| \boldsymbol{p}^t) \right)$$

  where
  - $\gamma_t \equiv 1$;
  - I_t is the first order Taylor expansion of $I(\boldsymbol{p}, \boldsymbol{Q})$ around $\boldsymbol{p}^t$.
- A suitable choice of step size γ_t could accelerate this algorithm.

Theorem

$$\gamma_t \le \frac{1}{\lambda_{KL}^2(\boldsymbol{Q})} \triangleq 1 / \left[\sup_{\boldsymbol{p} \neq \boldsymbol{p}'} \frac{D(\boldsymbol{p}\boldsymbol{Q} \| \boldsymbol{p}'\boldsymbol{Q})}{D(\boldsymbol{p} \| \boldsymbol{p}')} \right]$$

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Concluding Remarks

- EM and Blahut-Arimoto algorithms all belong to *alternating optimization* methods.
- Blahut-Arimoto algorithm can be understood as a generalized EM Algorithm:

$$\phi^{t+1} = \arg\max_{\phi\in\Phi} \tilde{I}(\boldsymbol{p}^t; \boldsymbol{Q}; \phi) \qquad \text{E-step}$$

$$\boldsymbol{p}^{t+1} = \arg\max_{\boldsymbol{p}} \tilde{I}(\boldsymbol{p}; \boldsymbol{Q}; \phi^{t+1}) \qquad \text{M-Step}$$

$$q^{t+1} = \arg\max -D(q \| q_{\theta^t})$$

$$\theta^{t+1} = \arg\max \mathbb{E}_{q^{t+1}}[\log p(z, x_{1:n} \mid \theta)]$$

The first step of them are searching for the poserior probability of latent variable; the second step of them are all maximzing some likelihood function based on the estimation in the first step.

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