On the Partial Convexification for Low-Rank Spectral Optimization: Rank Bounds and Algorithms

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Low-Rank Spectral Optimization Problem (LSOP)

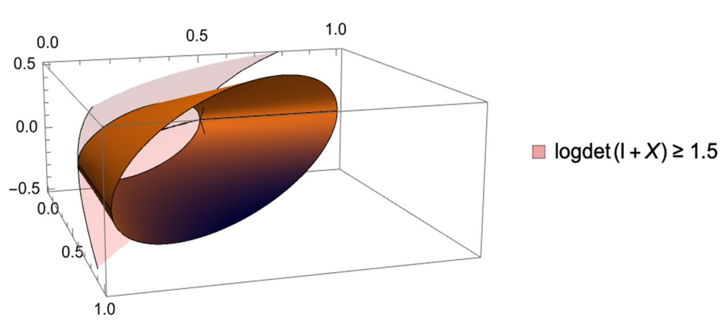
$$\text{(LSOP)} \quad \mathbf{V}_{\text{opt}} := \min_{X\in\mathcal{D}} \left\{ \langle A_0, X\rangle : b_i^l \le \langle A_i, X\rangle \le b_i^u, \forall i\in[m] \right\}$$

- Domain set: $\mathcal{D} := \{X \in \mathcal{S}_+^n : \text{rank}(X) \le k, F(X) := f(\lambda(X)) \le 0\}$
- $F(X)$ is closed convex spectral that depends on eigenvalues $\lambda(X)$
- Can cover multiple spectral functions

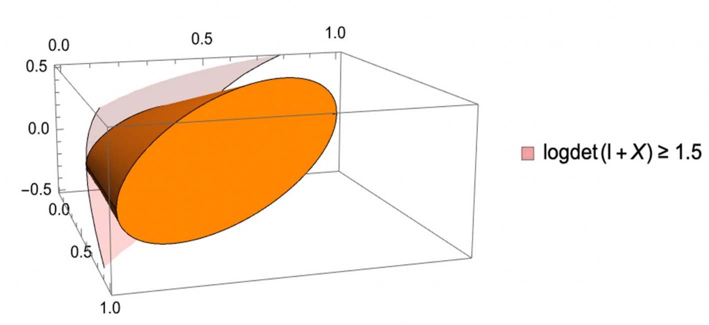
LSOP-R: Partial Convexification of Doamin Set $\mathcal{D}$

$$\text{(LSOP-R)} \quad \mathbf{V}_{\text{rel}} := \min_{X\in\text{conv}(\mathcal{D})} \left\{ \langle A_0, X\rangle : b_i^l \le \langle A_i, X\rangle \le b_i^u, \forall i\in[m] \right\}$$

- Replace $\mathcal{D}$ by $\text{conv}(\mathcal{D})$
- **Example 1.** $\mathcal{D} = \left\{X \in \mathcal{S}_+^2 : \text{rank}(X) \le 1, \text{tr}(X) \le 1, \log\det(I+X) \ge \frac{3}{2}\right\}$



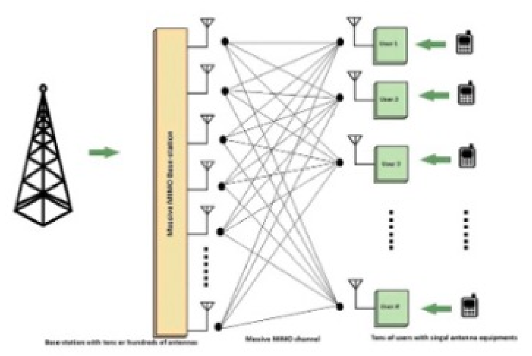
LSOP: $\mathcal{D}$



LSOP-R: $\text{conv}(\mathcal{D})$

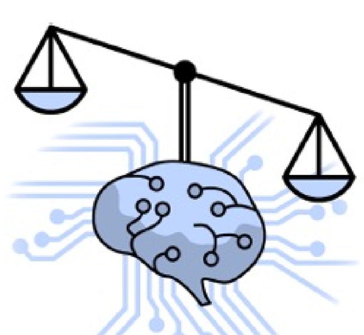
Research Gap: No theoretical rank bound for general LSOP-R

Applications



I: Radio network

II: Power grid

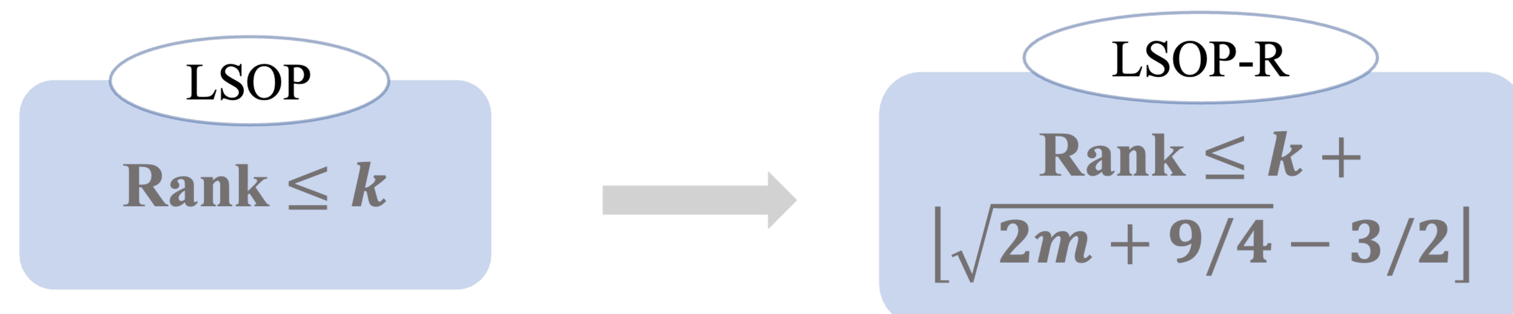


III: Fair PCA

- II: optimal power flow formulated by QCQP
- Besides, sparse ridge regression, matrix completion, etc.

Theoretical Rank Bounds for LSOP-R

Theorem 1. Suppose $\mathbf{V}_{\text{rel}} > -\infty$, there is an optimal extreme point to LSOP-R with rank at most $k + \left\lfloor \sqrt{2m + \frac{9}{4}} - \frac{3}{2} \right\rfloor$.

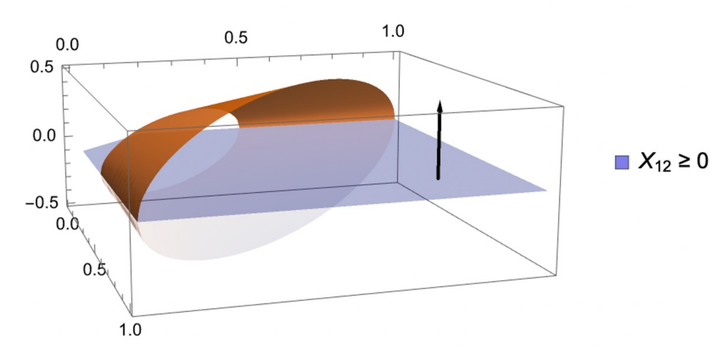


- Proof Idea: describe set $\text{conv}(\mathcal{D})$ and bound the rank of its faces
- Be independent of the domain set $\mathcal{D}$
- Recover the existing ones for QCQP and Fair PCA

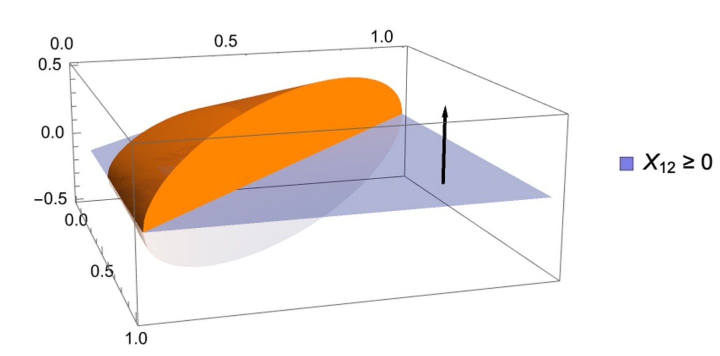
When LSOP-R Matches LSOP ?

Theorem 2. Suppose $\mathbf{V}_{\text{rel}} > -\infty$, LSOP-R yields the same optimal solution and value as LSOP when $m \le 1$.

- Proof Idea: let $\lfloor \sqrt{2m+9/4} - 3/2 \rfloor = 0$
- For **Example 1**, consider $m = 1$ linear inequality: $X_{12} \ge 0$



LSOP feasible set



LSOP-R feasible set

- Same extreme points

Extensions

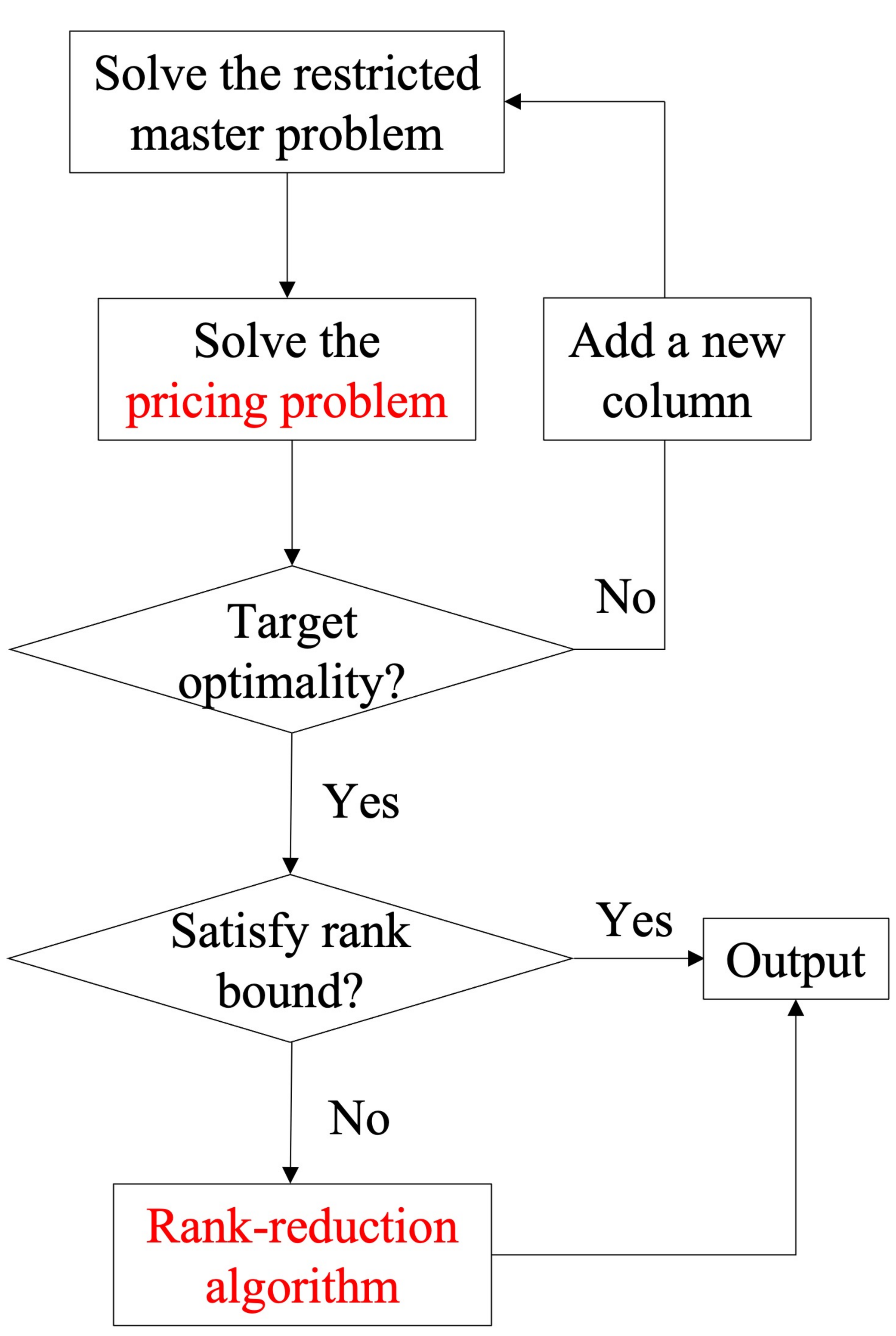
Matrix space	Symmetric	Non-symmetric	Diagonal
Rank bound	$k + \lfloor \sqrt{4m+9} - 3 \rfloor$	$k + \lfloor \sqrt{2m + \frac{9}{4}} - \frac{3}{2} \rfloor$	$k + m$

Algorithms: Column Generation (CG) + Rank-Reduction and Numerical Study

$$\text{(Pricing problem)} \quad \max_{X\in\mathcal{D}} \langle C_t, X\rangle$$

- Equivalent to vector-based convex program

Theorem 3. The output solution is optimal to LSOP-R and satisfies the rank bound.



Case I. Complex domain set $\mathcal{D} := \{X \in \mathcal{S}_+^n : \text{rank}(X) \le k, \text{tr}(X) \le U, \log\det(I_n + X) \ge L\}$

- Naive CG: directly use $\text{conv}(\mathcal{D})$
- "–": cannot be solved within one hour

n	m	k	MOSEK		Naive CG		Proposed Algorithms: Our CG		Proposed Algorithms: Rank-reduction		Rank bound
			rank	time(s)	rank	time(s)	rank	time(s)	reduced rank	time(s)	(Theorem 1)
50	5	5	48	17	3	223	3	1	2	1	7
50	10	5	29	19	5	1261	5	1	3	1	8
50	10	10	32	183	–	–	5	1	3	1	13
100	10	10	–	–	–	–	2	2	1	1	13
100	15	10	–	–	–	–	5	2	3	1	14
100	15	15	–	–	–	–	5	3	3	1	19
500	25	25	–	–	–	–	10	24	8	2	30
500	50	25	–	–	–	–	21	99	20	2	33
500	50	50	–	–	–	–	22	104	20	2	58

Case II: QCQP. Simple domain set $\mathcal{D} := \{X \in \mathcal{S}_+^n : \text{rank}(X) \le 1, L \le \text{tr}(X) \le U\}$

n	m	k	MOSEK		Naive CG		Proposed Algorithms: Our CG		Proposed Algorithms: Rank-reduction		Rank bound
			rank	time(s)	rank	time(s)	rank	time(s)	reduced rank	time(s)	(Theorem 1)
1000	50	1	28	160	–	–	3	42	2	3	9
1000	60	1	32	195	–	–	5	80	3	10	10
1500	60	1	27	642	–	–	3	113	2	11	10
1500	75	1	186	724	–	–	6	344	4	35	11
2000	75	1	40	1850	–	–	5	594	3	67	11
2000	90	1	12	2236	–	–	4	483	2	27	13
2500	90	1	–	–	–	–	5	1323	3	122	13
2500	100	1	–	–	–	–	4	1326	2	114	13