AIE6001 Assignment 4

Due date: 11:59 PM, Sunday, December 7, 2025.

Question 1 (Coin Flips)**.** *A fair coin is flipped until the first head occurs. Let X denote the number of flips required.*

1) *Write down the probabiltiy mass function of the random variable X.*
2) *Find the entropy $H(X)$ in bits. Hint: you will find the following expressions useful:*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad \sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}.$$

(20 points)

Question 2 (Entropy of functions of a random variable)**.** *Entropy of functions of a random variable. Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:*

$$H(X, g(X)) \overset{(a)}{=} H(X) + H(g(X) \mid X) \tag{2.1}$$

$$\overset{(b)}{=} H(X), \tag{2.2}$$

$$H(X, g(X)) \overset{(c)}{=} H(g(X)) + H(X \mid g(X)) \tag{2.3}$$

$$\overset{(d)}{\geq} H(g(X)). \tag{2.4}$$

Thus, $H(g(X)) \leq H(X)$. *(20 points)*

Question 3 (Measure of correlation)**.** *Let X_1 and X_2 be identically distributed but not necessarily independent. Let*

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}.$$

1) *Show that $\rho = \frac{I(X_1;X_2)}{H(X_1)}$.*
2) *Show that $0 \leq \rho \leq 1$.*
3) *When is $\rho = 0$?*
4) *When is $\rho = 1$?*

(20 points)

Question 4 (Entropy and Mutual Information)**.** *Let the joint pmf of (X, Y), denoted as $p(x, y)$, be given by*

	Y	
X	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find:

(a) $H(X), H(Y)$.

(b) $H(X \mid Y), H(Y \mid X)$.

(c) $H(X, Y)$.

(d) $H(Y) - H(Y \mid X)$.

(e) $I(X; Y)$.

(f) Draw a Venn diagram for the quantities in parts (a) through (e).

(20 points)

Question 5 (Entropy)**.** *Let* X *and* Y *be two independent integer-valued random variables. Let* X *be uniformly distributed over* $\{1, 2, \ldots, 8\}$*, and let* $\Pr\{Y = k\} = 2^{-k}$*,* $k = 1, 2, 3, \ldots$.

(a) Find $H(X)$.

(b) Find $H(Y)$.

(20 points)