The Underlying Decision under Epidemic Models

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1 Introduction

The COVID-19 pandemic has caused millions of deaths worldwide, and nearly 800 thousand deaths in the United States. To analyze the pandemic, many researchers has formulate epidemic models to simulate the effect of the pandemic using differential equations. There are different variations of such model. They share the same characteristic where they all split the total population into multiple groups that represents things like infected population and recovered population. Each of those models has its own unique assumption about the population changes between each group. The epidemic model can potentially support decision making of the governors by considering some parameters like vaccination parameter and infectious parameter as decision variables in our model.

The project will presented in the following order :

1. I will first explore some epidemic models, explain their assumptions, analyze the meaning of each parameter and how they might fall short of formulating the real world-case. Finally, I continue with the SIRV model with immunity decay that I think makes sense.
2. Using the model, I will formulate different mathematical programming models to perform decision making on vaccination parameter. One interesting thing is that since I will use a modeling language call GAMS, the numerical methods will be formulated as a combination of constraints in my model. I will also talk about how to use constraints to construct explicit methods
3. Lastly, I will make an interactive application using R-Shiny framework that allows users to do interactive optimization by changing parameters in the model. The interactive application will be constructed with the simplest SIR model. Because running such application will require GAMS license, the demonstration application will be presented with a video,

2 Exploration of Epidemic Models

In this section, I will explore three epidemic models and analyze their performance under different parameter settings. One assumption I am going to make throughout this project is that there is no population change in all the models.

2.1 SIR Model

Under the assumption of SIR model [2],the population is split into three groups.

1. Susceptible individuals $S(t)$: this group represents the population that is not infected at time t, but can be infected in the future.
2. Infected individuals $I(t)$: this group represents the population that is infected at time t.
3. Removed individuals $R(t)$: These individuals are assumed to have recovered from the infection and be immune.

The SIR model is formulated as follow :

$$\frac{dS(t)}{dt} = -\beta S(t)I(t)$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t)$$

$$\frac{R(t)}{dt} = -\gamma I(t)$$

In the model, β represents the infection rate, and γ represents the recovery rate. According to the structure of the SIR model in figure 1, we can see that the main assumption of it is that the immunity of of the recovered population is permanent. Nonetheless, the concept of permanent immunity is not realistic in the real world setting. Also, immunity can come from both recovering from an infection and getting vaccinated.

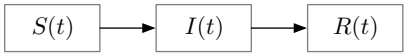


Figure 1: SIR model

2.2 SISV Model

Another more realistic model is the SISV model [1]. In the SISV model, the total population is split into three categories, $S(t)$, $I(t)$, and $V(t)$. The former two categories are the same in the SIR model, and the $V(t)$ represents the vaccinated individuals. The SISV model is formulated as follow :

$$\frac{dS(t)}{dt} = -\beta S(t)I(t) + \alpha I(t) - U(t)$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \alpha I(t)$$

$$\frac{V(t)}{dt} = U(t)$$

In the model, β represents the infection rate, and α represents the recovery rate of infected population. $U(t)$ is the vaccination control at time t. According to figure 2, we can see that the recovered population is no longer permanent immune. The model also assumes that the vaccinated people are permanently immune under this setting. The SISV model therefore has the following drawback :

1. They assume that the immunity for recovered people is weak. This can be observed from the fact that the recovered people will immediately be susceptible after the recovery.
2. They assume that the vaccinated people will be free from infection permanently.

As mentioned in the above two points, the problem mainly lies in the formulation of immunity. We know that immunity can decay, and how can we simulate the immunity decay ? Therefore, we need the next model.

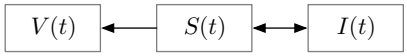


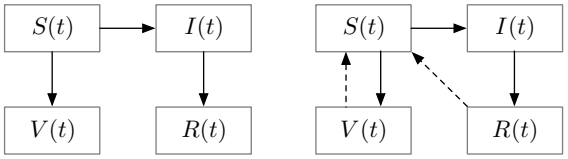
Figure 2: SISV model

2.3 SIRV model

The SIRV model [3] assumes there are removed individuals $R(t)$ and vaccinated individuals $V(t)$. Two different variations of SIRV models are formulated as follow :

$$\begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t)I(t) - uS(t) \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - \alpha I(t) \\ \frac{dR(t)}{dt} &= \alpha I(t) \\ \frac{dV(t)}{dt} &= uS(t) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \frac{dS(t)}{dt} &= -\beta S(t)I(t) - uS(t) + \lambda R(t) + \lambda V(t) \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - \alpha I(t) \\ \frac{dR(t)}{dt} &= \alpha I(t) - \lambda R(t) \\ \frac{dV(t)}{dt} &= uS(t) - \lambda V(t) \end{aligned}$$

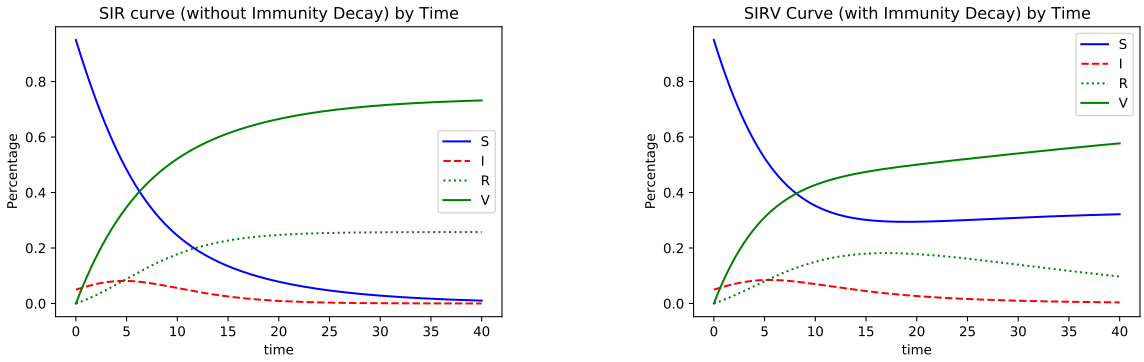
The differential equations on the left is the SIRV model without immunity decay, while the formulation on the right is the SIRV model with immunity decay, which is inspired by the model presented in homework 8. β is the infection rate, α represents the recovery rate, u represents the vaccination rate, and λ is the rate of immunity decay. The differences of the two model can be visualized in the two graphs in figure 3.



(a) SIRV model (b) SIRV model with immunity decay

Figure 3: SIRV models

We let parameters $(\beta, \gamma, u, \lambda) = (0.5, 0.25, 0.1, 0.05)$. Also, we have $(S_0, I_0, R_0, V_0) = (0.95, 0.05, 0, 0)$, $T = 40$, and $N = 8000$. Using second-order Adam Bashforth method, we can see that because of the immunity decay, there are less people staying vaccinated and more people are staying in susceptible group in figure 4.



(a) SIRV Curve

(b) SIRV Curve with immunity decay

Figure 4: SIRV models

Parameter	Definition
β	Infection Rate : the decisions related to lockdown or quarantine that decrease the contact
α	Recovery Rate : speed of recovery is harder for human to control
u	Vaccination Rate : government can spur the vaccination by providing incentive
λ	Immunity Decay Rate : speed of immunity decay, which is harder for us to control

Table 1: Definition of each parameter

3 Decision Making with Optimization Model

SIRV with immunity decay model can simulate the epidemic model nicely. Therefore, we will stick discuss SIRV model in the following sections. Here, I want to explore the possibility for the epidemic model to support decision in disease control. The main idea comes from the SISV model introduced in section 2.2. The interesting feature in the SISV model is that it formulates the vaccination $U(t)$ as a control variable, while in the SIRV model it uses the vaccination parameter, u , to represent the vaccination speed. The idea is to treat u as multiple decision variable $u_t, t \in 0...N-1$ at every time stamp in multi-step (or one-step) methods. After we let u be represented by variables across time stamps, the differential equations can be rewritten as :

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\beta S(t)I(t) - \boldsymbol{u(t)}S(t) + \lambda R(t) + \lambda V(t) \\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - \alpha I(t) \\
\frac{dR(t)}{dt} &= \alpha I(t) - \lambda R(t) \\
\frac{dV(t)}{dt} &= \boldsymbol{u(t)}S(t) - \lambda V(t)
\end{aligned}$$

The main thing we can observe here is that u is changed to $u(t)$. By doing this transformation, we allow the decision makers to treat $u(t)$ as decision variables at each time stamp. According to table 1, we can see that β and u are all good candidates for decision variables. We will only explore the decision of u in the following sections, and the simple interactive application at the end will be designed to make decision on β.

3.1 Formulating Numerical Methods in Modeling Language - Using Second-Order Adam Bashforth as An Example

In the following section, I am going to use a modeling language called GAMS to formulate mathematical models to solve for optimal decision. First, I would like to introduce how **explicit** numerical methods such as Euler Method and Second-Order Adam Bashforth Method can be incorporated in the modeling language. For simplicity, we let $f_S(t) = \frac{dS(t)}{dt}, f_I(t) = \frac{dI(t)}{dt}, f_R(t) = \frac{dR(t)}{dt}$, and $f_V(t) = \frac{dV(t)}{dt}$. Also, the computation in $f_S(t), f_I(t), f_R(t)$ and $f_V(t)$ uses predicted value of $S(t), I(t), R(t)$ and $V(t)$.

A model formulated with modeling language has two parts - objective function and constraints. Since the the solution of explicit method can be expressed in explicit form unlike implicit methods that requires iterative algorithm such as Newton method, it is easier to formulate it in the modeling language. Here I formulate second-order Adam BashForth Method as a mathematical programming model . We uses SIRV model with immunity decay . S_t, I_t, R_t, V_t are the population of each group at time t. The objective function and the Adam Bashforth constraints are written as :

$$\min_{u_t, t=0\ldots N-1} \quad \text{Objective Function} \tag{1}$$

$$\text{s.t.} \quad S_{t+1} = S_t + h * (\frac{3}{2} f_S(t) - \frac{1}{2} f_S(t-1)) \qquad \forall t = 1\ldots N-1 \tag{2}$$

$$I_{t+1} = I_t + h * (\frac{3}{2} f_I(t) - \frac{1}{2} f_I(t-1)) \qquad \forall t = 1\ldots N-1 \tag{3}$$

$$R_{t+1} = R_t + h * (\frac{3}{2} f_R(t) - \frac{1}{2} f_R(t-1)) \qquad \forall t = 1\ldots N-1 \tag{4}$$

$$V_{t+1} = V_t + h * (\frac{3}{2} f_V(t) - \frac{1}{2} f_V(t-1)) \qquad \forall t = 1\ldots N-1 \tag{5}$$

$$S_1 = S_0 + h * f_S(0) \tag{6}$$

$$I_1 = I_0 + h * f_I(0) \tag{7}$$

$$R_1 = R_0 + h * f_R(0) \tag{8}$$

$$V_1 = V_0 + h * f_V(0) \tag{9}$$

$$S_0 = \bar{S_0},\ I_0 = \bar{I_0},\ R_0 = \bar{R_0},\ V_0 = \bar{V_0} \tag{10}$$

,Where N is the number of mesh points and h is the distance cut. (1) is the objective function we want to minimize. More details about the objective will be discussed in the later sections. Constraints (2) to (5) is the main implementation of AB2. Constraints (6) to (9) are the one step inference using Euler method from the original point at time t_0 because AB2 requires two starting points. Constraint (10) is the given initial condition of the model. We can see that the logic of AB2 can be represented by the constraints regardless of the definition of the objective function. The model has demonstrated that AB2 can be represented by a set of constraints. If we think more deeply, we can also see that other explicit methods such as improved Euler (with predictor-corrector inference) and RK4 methods can also be expressed by a set of constraints in a similar manner.

3.2 Maximizing Continuity on Vaccination Variable

In this section, I am going to make the vaccination parameters u_t at each time stamp as decision variables. The model is introduced at the start of section 3.

The definition of maximizing continuity on vaccination is that we want the population that gets vaccinated at time t is as close as those at time $t+1$. Maximizing continuity of vaccination can give the health system

Scenarios	
Scenario 1 constraints (lower vaccination rate at the end time)	Scenario 2 constraints (higher vaccination rate at the end time)
$(\beta, \alpha, \lambda) = (0.5, 0.25, 0.05)$	$(\beta, \alpha, \lambda) = (0.5, 0.25, 0.05)$
$(S_0, I_0, R_0, V_0) = (0.95, 0.05, 0, 0)$	$(S_0, I_0, R_0, V_0) = (0.95, 0.05, 0, 0)$
$(S_T, V_T) = (0.4, 0.4)$ (ending value)	$(S_T, V_T) = (0.3, 0.5)$ (ending value)
$V_{T/2} \geq 0.3$ (intermediate goal)	$V_{T/2} \geq 0.4$ (intermediate goal)
$u_0 = 0.05$	$u_0 = 0.05$
$u_t \leq 0.2, \forall t$	$u_t \leq 0.2, \forall t$
$T = 40$	$T = 40$
$N = 8000$	$N = 8000$

Table 2: Two scenarios

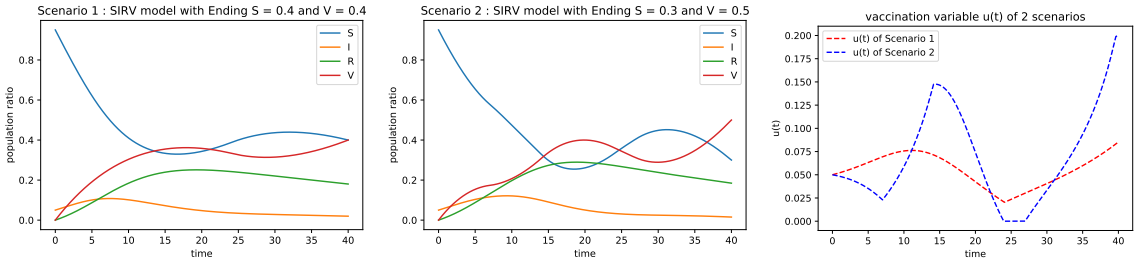


Figure 5: results of SIRV population of vaccination continuity objective. The left two figures are SIRV population ratio. The right one is the value of vaccination variable for both scenarios.

time to provide enough resources to administer the vaccination. I consider this objective because it the vaccine supply is normally a gradual change instead of a sudden change. The following is the continuity model :

$$\begin{aligned} & \min_{u_t, t=0\ldots N-1} \quad x \\ \text{s.t.} \quad & x \geq u_t S_t - u_{t-1} S_{t-1} \quad \forall t = 1\ldots N-1 \\ & x \geq -u_t S_t + u_{t-1} S_{t-1} \quad \forall t = 1\ldots N-1 \\ & \text{Adam Bashforth Constraints in section 3.1.} \end{aligned}$$

$u_t S_t$ is the population that got vaccination at time t. I let x be bigger than $u_t S_t - u_(t-1) S_(t-1)$ and $-u_t S_t + u_(t-1) S_(t-1)$ to capture the difference between time t and $t-1$. By minimizing x, difference of the vaccinated population between two consecutive timestamp can be minimized.

In the real world scenario, decision makers will try to control the vaccination parameters in order to ensure that some population goals are reached at some given time. In the experiment, I created two scenarios (in table 2). In each scenario, it has initial values, boundary(ending) values, intermediate goal, and vaccination parameter bounds.

I ran the model with the two scenarios. In the experiment result, we can see from figure 5 that the change of vaccination variables are continuous. Scenario 2 has stricter ending value requirement and stricter intermediate requirement, that leads to higher fluctuation in vaccination variables. One thing worth notice is that at around time 25, the vaccination variable of scenario 2 goes to zero, which is highly unlikely in the real world. Sometimes the solution of optimization model may be unrealistic. More constraints such as lower

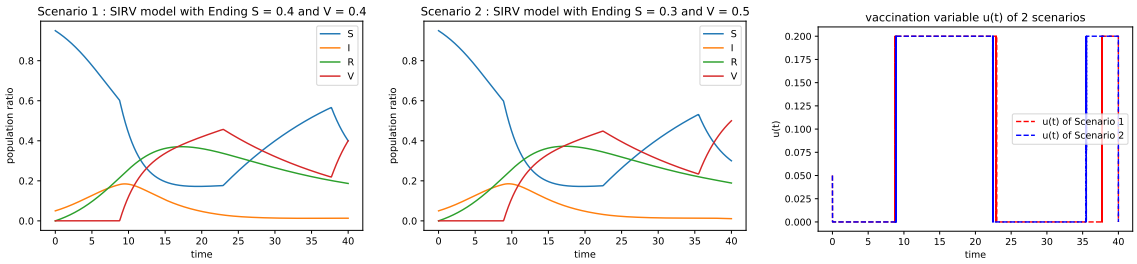


Figure 6: results of SIRV population of vaccination cost objective. The left two figures are SIRV population ratio. The right one is the value of vaccination variable for both scenarios. Note that the vaccination variable change drastically over iterations. If we compare the vaccination variable here with that in figure 5, we can see the difference between maximizing contiguity and minimizing cost. The solution in the contiguity model is much smoother.

bounds of vaccination variables might help us get more natural result.

3.3 Minimizing Cost on Vaccination Variable

Here I am going to study the total cost of vaccination rate in u_t over all time steps. Like the last section, the model is the same one in section 3.

As decision makers try to control vaccination rate, the corresponding cost by administering more vaccinations should also be considered. Therefore, another objective we can consider for this variable is the cost. When vaccination speed is higher, it may mean that the government should hire or deploy more health workers to finish the work, which can incur more cost, Therefore, in order to minimize the cost, I formulated the objective as follow :

$$\min_{u_t, t=0\ldots N-1} \quad \sum_{t=00}^{N-1} u_t S_t$$

s.t. Adam Bashforth Constraints in section 3.1.

Using the sum of $u_t S_t$ as objective means when there are more people getting vaccinated, the cost will become higher. Similar to the result in section 3.2, I solve the optimization problem of minimizing cost under the two scenarios in table 2.

According to the result figure 6, we can see that the population in scenario 1 and 2 are really similar except in the end. We can also see that the SIRV populations in both can have some sharp change. That is due to the sharp change in vaccination variable (the right figure). One thing worth notice is that if we only consider to minimize the cost, the model will minimize the resource it spends to " just meet the goal." Therefore, there is a lot of time when the vaccination rate is zero. Maybe incorporate the cost objective with continuity objective might help.

3.4 Maximizing Continuity and Minimizing Cost on Vaccination Variable

Sometimes we might want to pursue both low cost and continuity in our decision making process. One way to incorporate two objectives is to use the weighted sum of the two objectives as the new objective function.

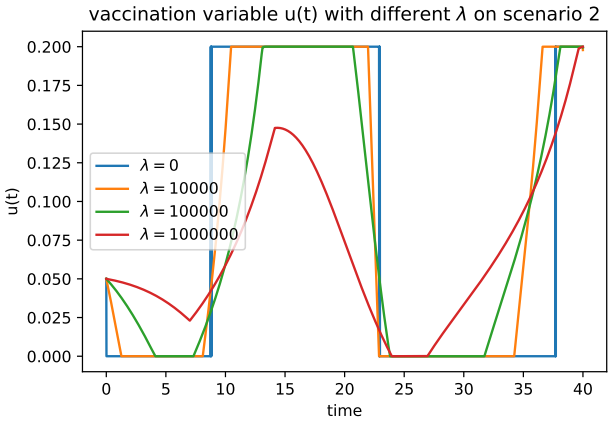


Figure 7: Vaccination variable values under different λ weights.

In our problem, the objective function will look like this :

$$\begin{aligned} \min_{u_t, t=0\ldots N-1} \quad & \sum_{0}^{N-1} u_t S_t + \lambda x \\ \text{s.t.} \quad x \geq u_t S_t - u_{t-1} S_{t-1} \quad & \forall t = 1 \ldots N-1 \\ x \geq -u_t S_t + u_{t-1} S_{t-1} \quad & \forall t = 1 \ldots N-1 \end{aligned}$$

Adam Bashforth Constraints in section 3.1.

, where λ is the weight we assign for the continuity objective function. If λ is bigger, more importance will be put on finding the optimal solution of the continuity problem.

Here we use the basic setting of scenario 2 in table 2 as an example, and we will try to see how the change of weight λ will affect the result for vaccination variable. In figure 7, we can see that by increasing the value of λ, the change of vaccination variable gradually becomes smoother. Therefore, there can be a trade-off between cutting down the cost and maintaining the "smoothness" of vaccination strategy in the model.

4 Lessons Learned and Things That Can Be Done Better

1. Explicit method can be implemented in modeling languages : when first introduced with Euler Method in class, I imagined that it would be easy to write it in modeling language. After more explicit methods are introduced, I really wanted to explore the possibility to implement them in modeling language, and this project is a perfect opportunity to do that. The result proves that my thought is right.

2. Setting boundary values and constraints for SIRV model is not as easy as for SIR model : For the experiment, I coded up Euler Method in GAMS and tried to solve SIR boundary problem with recovery rate as decision variables. Since there are only three components, it is easy to set the boundary value in SIR model, while setting more than 2 boundary values in SIRV model often leads to infeasible result. Therefore, I only specified two ending values which are S_T and V_T to avoid that situation, which I think is far from perfect.

3. More details need to be considered when we are constructing the model: in the experiment results, there can be a time when the vaccination rate is zero, which is not natural in the real world. Also, since I am not sure what a typical decision maker would like to consider when they are determining the vaccination rate, I made up a lot of details when I was building the 2 scenarios in table 2.

Though the model details and constraints are not perfect in this project, I think it still provides some flavor of incorporating optimization language with numerical methods to support decision making process.

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