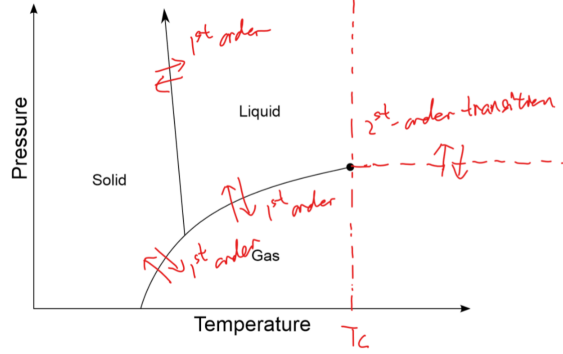


2d CFT

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Week 1

Exercies 0.0.1. The first order transitions and second order transitions show in the diagram



Exercies 0.0.2. By the homogeneous relation

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

we have

$$f(t, h) = t^{\frac{d}{y_t}} g(\alpha)$$

where $g(\alpha) = f(1, \alpha)$ and $\alpha = t^{-\frac{y_h}{y_t}} h$. It is easy to see that α is invariance under scaling transformation $x \rightarrow x/b$. Hence we have

$$C(t, 0) = -T \frac{\partial^2 f}{\partial T^2} \Big|_{h=0} = -\frac{1}{T_c} t^{\frac{d}{y_t}-2} g''(0)$$

$$M(t, 0) = -\frac{\partial f}{\partial B} \Big|_{h=0} = t^{\frac{d-y_h}{y_t}} g'(0)$$

$$\chi(t, 0) = \frac{\partial^2 f}{\partial B^2} \Big|_{h=0} = t^{(d-2y_h)/y_t} g''(0)$$

As function with single variable h , $\lim_{t \rightarrow 0} M(t, h) \sim h^{\frac{1}{\delta}}$, which implies that $g'(\alpha) \sim \alpha^{\frac{1}{\delta}}$ since α is linear function of h . Hence we have

$$\lim_{t \rightarrow 0} M = \lim_{t \rightarrow 0} t^{(d-y_h-\frac{y_h}{\delta})} h^{1/\delta}$$

since it is non-zero, we have $d - y_h - y_h \frac{1}{\delta} = 0$. Hence we have

$$\delta = \frac{y_h}{d - y_h}$$

Exercies 0.0.3. We have following relation

$$G_\sigma(\mathbf{r}; t, h) = t^{-2x_\sigma} G_\sigma\left(\frac{\mathbf{r}}{b}; b^{y_t} t, b^{y_h} h\right) \quad (1)$$

Let $h = 0, K = b^{y_t} t$,

$$G_\sigma(\mathbf{r}; t, 0) = t^{2x_\sigma/y_t} G_\sigma\left(\frac{\mathbf{r}}{K t^{-1/y_t}}; K, 0\right)$$

Since $G_\sigma(\mathbf{r}) \sim r^{-\tau} e^{-\frac{r}{\xi}}$, we have $\xi \sim t^{-1/y_t}$. It implies $\nu = 1/y_t$. With relation 1, we have

$$\chi(t, h) = \frac{1}{T} \int d^d \mathbf{r} G_\sigma(\mathbf{r}; t, h) = t^{d-2x_\sigma} \chi(b^{y_t} t, b^{y_h} h)$$

So $\gamma = (d - 2x_\sigma)/y_t$. But we have $\eta = 2x_\sigma + 2 - d$ for finite limit of $G(r)$ when $t \rightarrow 0$ and $h = 0$. Therefore, we get

$$\gamma = \nu(2 - \eta)$$

With scaling relations

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \alpha + \beta(1 + \delta) &= 2 \end{aligned}$$

and $\alpha = 2 - d\nu$, we have

$$\begin{aligned} \beta &= \frac{d\nu - 2\nu + \nu\eta}{2} \\ \delta &= \frac{d - \eta + 2}{d + \eta - 2} \end{aligned}$$

Exercies 0.0.4. By listed commutation relations, we have, for $r, s > 0$,

$$\begin{aligned} [D, J_{rs}] &= [D, L_{rs}] = \frac{i}{2} [D, [K_r, P_s]] \\ &= -\frac{i}{2} ([P_s, [D, K_r]] + [K_r, [P_s, D]]) \\ &= \frac{1}{2} [P_s, K_r] - \frac{1}{2} [K_r, P_s] \\ &= 0 \end{aligned}$$

For $r = -1, s = 0$, $[D, J_{rs}] = [D, D] = 0$. For $r = -1, s \neq 0$, $[D, J_{-1,s}] = [D, \frac{1}{2}(P_s - K_s)] = \frac{i}{2}(P_s + K_s)$. For $r = 0$, $[D, J_{0s}] = \frac{i}{2}(P_s - K_s)$. Hence (2,25) is satisfied when $(m, n) = (-1, 0)$.

If $(m, n) = (-1, n)$, then we have

$$[J_{mn}, J_{rs}] = \frac{1}{2} [P_n, J_{rs}] - \frac{1}{2} [K_n, J_{rs}]$$

With listed commutation relations, we can easily check it coincides with (2,25) respectively. Similarly check in the case of $(m, n) = (0, n)$.

2d CFT

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1 $SL_2(\mathbb{C})$

Exercies 1.0.1. We have $\det X = t^2 - (x^2 + y^2 + z^2)$. Since points in $\mathbb{R}^{1,3}$ can be written with Pauli matrix as base. Elements in $SO(1,3)$ can be viewed as action on $M_2(\mathbb{C})$ with form $P \mapsto PXP^*$, which preserve \det of X . We have exact sequence of groups as follows:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{C}) \xrightarrow{sp} SO(1,3) \longrightarrow 1$$

where sp is map $P \mapsto (X \mapsto PXP^*)$. Since for $P \in SL_2(\mathbb{C})$, $\det(PXP^*) = \det(X) = t^2 - (x^2 + y^2 + z^2)$, sp is well-defined. Hence $SO(1,3) \cong SL_2(\mathbb{C})/\mathbb{Z}_2$.

Exercies 1.0.2. •

$$z \mapsto \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}$$

- We have

$$\begin{aligned} (w_1 - w_3) &= \frac{(az_1 + b)(cz_3 + d) - (az_2 + b)(cz_4 + d)}{(cz_1 + d)(cz_3 + d)} \\ &= \frac{z_1 - z_3}{(cz_1 + d)(cz_3 + d)} \end{aligned}$$

Hence we have $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$.

2 Three-point function

Exercies 2.0.1. Let $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle = f(z_{12}, z_{23}, z_{13})$. Under scalar transformation $z_i \mapsto \lambda z_i$, we have

$$f(z_{12}, z_{23}, z_{13}) = \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13})$$

Therefore, f is with form

$$f(z_{12}, z_{23}, z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$$

where $a + b + c = h_1 + h_2 + h_3$. Then under comformal transformation $z_i \mapsto \frac{1}{z_i}$, we have

$$z_1^{-2h_1} z_2^{-2h_2} z_3^{-2h_3} \frac{(z_1 z_2)^a (z_2 z_3)^b (z_1 z_3)^c}{z_{12}^a z_{23}^b z_{13}^c} = \frac{1}{z_{12}^a z_{23}^b z_{13}^c}$$

Hence $a = h_1 + h_2 - h_3, b = h_2 + h_3 - h_1, c = h_1 + h_3 - h_2$.

3 Energy-momentum tensor

Exercies 3.0.1. •

$$T^{\mu\nu} = -\eta^{\mu\nu} \partial_k \varphi \partial^k \varphi + 2\partial^\mu \varphi \partial^\nu \varphi$$

- We have

$$\delta \sqrt{g} = -\frac{1}{2} \sqrt{g} g_{\mu\nu} \delta g^{\mu\nu}$$

Therefore,

$$\begin{aligned} \tilde{T}^{\mu\nu} &= \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{2} (-\delta_{\mu\nu} + 2) \partial_\mu \varphi \partial_\nu \varphi \end{aligned}$$

4 Derivations

4.1 Energy-momentum tensor in complex coordinate

Since

$$\begin{aligned}\partial_0\Phi &= \partial_z\Phi + \partial_{\bar{z}}\Phi \\ \partial_1\Phi &= i\partial_z\Phi - i\partial_{\bar{z}}\Phi\end{aligned}$$

we have

$$\begin{aligned}\partial_z\Phi &= \frac{1}{2}\partial_0\Phi - \frac{i}{2}\partial_1\Phi \\ \partial_{\bar{z}}\Phi &= \frac{1}{2}\partial_0\Phi + \frac{i}{2}\partial_1\Phi\end{aligned}$$

Also, since there are metric tensors in complex coordinates

$$g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

, we have $\partial^z\Phi = 2\partial_{\bar{z}}\Phi$ and $\partial^{\bar{z}}\Phi = 2\partial_z\Phi$. Therefore, from definition of energy-momentum tensor

$$T_{\alpha\beta} = -g_{\alpha\beta}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial^\alpha\Phi)}\partial_\beta\Phi$$

we get expression of them in complex coordinates

$$\begin{aligned}T_{zz} &= \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right) \\ T_{z\bar{z}} &= -\frac{1}{2}\mathcal{L} + \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right) \\ T_{\bar{z}z} &= -\frac{1}{2}\mathcal{L} + \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right)\end{aligned}$$

Hence

$$\begin{aligned}T_{zz} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11})\end{aligned}$$

4.2 Schwarzian derivative

$$\tilde{T}(z + \epsilon(z)) = (1 + \partial_z\epsilon(z))^{-2} \left[T(z) - \frac{c}{12} \left(\frac{\partial_z^3\epsilon(z)}{1 + \partial_z\epsilon(z)} - \frac{2}{3} \frac{\partial_z^2\epsilon(z)}{1 + \partial_z\epsilon(z)} \right) \right]$$

Since

$$\frac{1}{1 + \partial_z\epsilon(z)} = 1 - \partial_z\epsilon(z) + (\partial_z\epsilon)^2 + \dots$$

we have

$$\begin{aligned}\tilde{T}(z + \epsilon(z)) &\approx T(z)(1 - 2\partial_z \epsilon(z)) - \frac{c}{12}(\partial_z^3 \epsilon(z) - \frac{2}{3}\partial_z^2 \epsilon(z)) \\ &\approx T(z) - 2\partial_z \epsilon(z)T(z) - \frac{c}{12}\partial_z^3 \epsilon(z)\end{aligned}$$

Hence

$$\tilde{T}(z + \epsilon(z)) - [\epsilon(z)\partial_z T(z) + T(z)\epsilon'(z)] \approx -\frac{c}{12}\partial_z^3 \epsilon(z) - 2\partial_z \epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$$

It implies that $\delta_\epsilon(T(z)) = -\frac{c}{12}\partial_z^3 \epsilon(z) - 2\partial_z \epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$.

4.3 Virasoro algebra

The Laurent expansion of z^{n+1} around ω is

$$z^{n+1} = (z - \omega)^{n+1} + \binom{n+1}{1}\omega(z - \omega)^n + \dots + \binom{n+1}{i}\omega^i(z - \omega)^{n+1-i} + \dots + \omega^{n+1}$$

Hence we have following residues:

$$\begin{aligned}\text{Res}_\omega \frac{z^{n+1}}{(z - \omega)^4} &= 2\pi i \frac{(n+1)n(n-1)}{6} \omega^{n-2} \\ \text{Res}_\omega \frac{z^{n+1}}{(z - \omega)^2} &= 2\pi i (n+1) \omega^n \\ \text{Res}_\omega \frac{z^{n+1}}{z - \omega} &= 2\pi i \omega^{n+1}\end{aligned}$$

Hence we have

$$\begin{aligned}[L_n, L_m] &= \frac{1}{(2\pi i)^2} \oint_0 d\omega \omega^{m+1} \oint_\omega dz z^{n+1} \left(\frac{c}{2(z - \omega)^4} + \frac{2T(\omega)}{(z - \omega)^2} + \frac{\partial_\omega T(\omega)}{(z - \omega)} + \text{regular part} \right) \\ &= \frac{1}{2\pi i} \oint_0 d\omega \omega^{m+1} \left(\frac{c}{12}(n+1)n(n-1)\omega^{n-2} + 2(n+1)\omega^n T(\omega) + \omega^{n+1}\partial_\omega T(\omega) \right) \\ &= \frac{1}{2\pi i} \left\{ \oint_0 d\omega \left(\frac{c(n+1)n(n-1)}{12} \omega^{m+n-1} \right) - (m-n) \oint_0 d\omega \omega^{m+n+1} T(\omega) \right\} \\ &= \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} - (m-n) L_{n+m}\end{aligned}$$

4.4 Commutation relations in free boson

4.5 Hamiltonian in free boson

4.6 Action of free fermion

$$S = \frac{1}{4\pi} \int d^2x \psi^\dagger \gamma^0 (\gamma^0 \partial_0 \psi + \gamma^1 \partial_1 \psi)$$

But

$$\gamma^0 (\gamma^0 + \gamma^1) \psi = \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} \psi$$

Write ψ as $\begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix}$, then we have

$$\gamma^0 (\gamma^0 + \gamma^1) \psi = \begin{pmatrix} 2\partial_{\bar{z}} \varphi \\ 2\partial_z \bar{\varphi} \end{pmatrix}$$

Hence

$$S = \frac{1}{2\pi} \int d^2x (\bar{\varphi} \partial_z \bar{\varphi} + \varphi \partial_{\bar{z}} \varphi)$$

4.7 TT OPE in free fermion

By derivative, we get

$$\begin{aligned}\langle \psi(z), \partial_\omega \psi(\omega) \rangle &\sim \frac{1}{(z-\omega)^2} \\ \langle \partial_z \psi, \partial_\omega \psi \rangle &\sim \frac{-2}{(z-\omega)^3}\end{aligned}$$

Hence

$$\begin{aligned}T(z)\partial_\omega \psi(\omega) &= \frac{1}{2} : \psi(z)\partial_z \psi(z) : \partial_\omega \psi(\omega) \\ &\sim -\frac{\psi(\omega)}{(z-\omega)^3} - \frac{1}{2} \frac{\partial_\omega \psi(\omega)}{(z-\omega)^2}\end{aligned}$$

and

$$\begin{aligned}T(z)T(\omega) &= \frac{1}{4} : \psi(z)\partial_z \psi(z) :: \psi(\omega)\partial_\omega \psi(\omega) \\ &\sim \frac{1}{4} \left\{ -\frac{: \partial_z \psi(z)\partial_\omega \psi(\omega) :}{z-\omega} + \frac{2 : \psi(z)\psi(\omega) :}{(z-\omega)^3} - \frac{: \psi(z)\partial_\omega \psi(\omega) + \partial_z \psi(z)\psi(\omega) :}{(z-\omega)^2} + \frac{1}{(z-\omega)^4} \right\} \\ &\sim \frac{1}{4} \left\{ \frac{2\partial_\omega T(\omega)}{z-\omega} + \frac{4T(\omega)}{(z-\omega)^2} + \frac{1}{(z-\omega)^4} \right\}\end{aligned}$$

5 Vertex operator and OPE

6 bc ghost system

$$\begin{aligned}T(z)b(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :)b(\omega) \\ &\quad 2\frac{b(z)}{(z-\omega)^2} - \frac{\partial_z b(z)}{z-\omega}\end{aligned}$$

Take Taylor expansion of $b(z)$ and $\partial_z b(z)$ around ω , we have

$$T(z)b(\omega) \sim 2\frac{b(\omega)}{(z-\omega)^2} + \frac{\partial_\omega b(\omega)}{z-\omega}$$

Hence the conformal dimension of b is 2.

Similarly, we have

$$\begin{aligned}T(z)c(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :)c(\omega) \\ &\sim -\frac{c(z)}{(z-\omega)^2} + 2\frac{\partial_z c(z)}{z-\omega} \\ &\sim -\frac{c(\omega)}{(z-\omega)^2} + \frac{\partial_\omega c(\omega)}{z-\omega}\end{aligned}$$

Hence conformal dimension of c is -1 .

$$\begin{aligned}T(z)T(\omega) &= 4 : b(z)\partial c(\omega) :: b(\omega)\partial c(\omega) : \\ &\quad - 2 : c(z)\partial b(z) :: b(\omega)\partial c(\omega) : - 2 : b(z)\partial c(z) :: c(\omega)\partial b(\omega) : \\ &\quad + c(z)\partial b(z) :: c(\omega)\partial b(\omega) : \end{aligned}$$

We will calculate it term by term

$$\begin{aligned}
& 4 : b(z) \partial c(z) :: b(\omega) \partial c(\omega) : \\
& \sim 4 \left(: \overbrace{b(z) \partial c(z) b(\omega) \partial c(\omega)} : + b(z) \overbrace{\partial c(z) b(\omega)} \partial c(\omega) + : \overbrace{b(z) \partial c(z) b(\omega)} \partial c(\omega) : \right) \\
& \sim \frac{4(- : \partial_z c(z) b(\omega) : + : b(z) \partial_\omega c(\omega) :)}{(z - \omega)^2} - \frac{4}{(z - \omega)^4} \\
& \sim -\frac{4}{(z - \omega)^4} + \frac{8b(\omega) \partial_\omega c(\omega)}{(z - \omega)^2} + \frac{-4 : \partial_\omega^2 c(\omega) b(\omega) : + 4 \partial_\omega b(z) \partial_\omega c(\omega)}{z - \omega}
\end{aligned}$$

and

$$\begin{aligned}
& 2 : c(z) \partial b(z) :: b(\omega) \partial c(\omega) : \\
& \sim 2 \left(- : \overbrace{c(z) \partial b(z) b(\omega)} \partial c(\omega) - : c(z) \overbrace{\partial b(z) b(\omega) \partial c(\omega)} - : \overbrace{\partial b(z) c(z) b(\omega)} \partial c(\omega) \right) \\
& \sim \frac{4}{(z - \omega)^4} + \frac{4 : c(z) b(\omega) :}{(z - \omega)^3} - \frac{2 : \partial b(z) \partial c(\omega)}{(z - \omega)} \\
& \sim \frac{4}{(z - \omega)^4} + \frac{4 : c(\omega) b(\omega) :}{(z - \omega)^3} + \frac{4 : \partial_\omega c(\omega) b(\omega) :}{(z - \omega)^2} + \frac{2 : \partial_\omega^2 c(\omega) b(\omega) : - 2 : \partial_\omega b(\omega) \partial_\omega c(\omega) :}{z - \omega}
\end{aligned}$$

and symmetrically

$$\begin{aligned}
& 2 : b(z) \partial c(z) :: c(\omega) \partial b(\omega) : \\
& \sim \frac{4}{(z - \omega)^4} + \frac{4 : b(\omega) c(\omega) :}{(z - \omega)^3} + \frac{4 : \partial_\omega b(\omega) c(\omega) :}{(z - \omega)^2} + \frac{2 : \partial_\omega^2 b(\omega) c(\omega) : - 2 : \partial_\omega c(\omega) \partial_\omega b(\omega) :}{z - \omega}
\end{aligned}$$

and

$$\begin{aligned}
& c(z) \partial b(z) :: c(\omega) \partial b(\omega) : \\
& \sim \frac{2 : c(\omega) \partial_\omega b(\omega) :}{(z - \omega)^2} + \frac{-\partial_\omega^2 b(\omega) c(\omega) + \partial_\omega c(\omega) \partial_\omega b(\omega)}{z - \omega} - \frac{1}{(z - \omega)^4}
\end{aligned}$$

Hence we have

$$T(z)T(\omega) \sim -\frac{13}{(z - \omega)^4} + \frac{2T(\omega)}{(z - \omega)^2} + \frac{\partial T(\omega)}{z - \omega}$$