2d CFT

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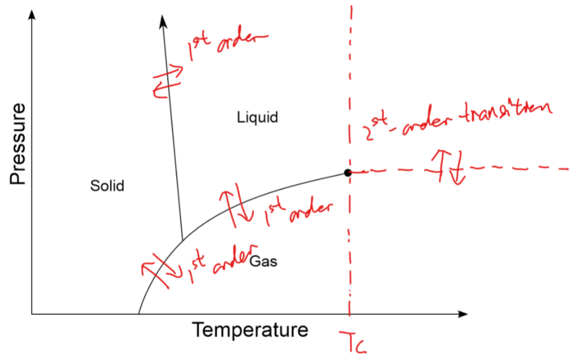
Week 1

Exercies 0.0.1. The first order transitions and second order transitions show in the diagram



Exercies 0.0.2. By the homogeneous relation

$$f(t,h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

we have

$$f(t,h) = t^{\frac{d}{y_t}} g(\alpha)$$

where $g(\alpha) = f(1,\alpha)$ and $\alpha = t^{-\frac{y_h}{y_t}} h$. It is easy to see that α is invariance under scaling transformation $x \to x/b$. Hence we have

$$C(t,0) = -T\frac{\partial^2 f}{\partial T^2}|_{h=0} = -\frac{1}{T_c} t^{\frac{d}{y_t}-2} g''(0)$$

$$M(t,0) = -\frac{\partial f}{\partial B}|_{h=0} = t^{\frac{d-y_h}{y_t}} g'(0)$$

$$\chi(t,0) = \frac{\partial^2 f}{\partial B^2}|_{h=0} = t^{(d-2y_h)/y_t} g''(0)$$

As function with single variable h, $\lim_{t\to 0} M(t,h) \sim h^{\frac{1}{\delta}}$, which implies that $g'(\alpha) \sim \alpha^{\frac{1}{\delta}}$ since α is linear function of h. Hence we have

$$\lim_{t\to 0} M = \lim_{t\to 0} t^{(d-y_n-\frac{y_n}{\delta})} h^{1/\delta}$$

since it is non-zero, we have $d - y_n - y_n\frac{1}{\delta} = 0$. Hence we have

$$\delta = \frac{y_h}{d-y_h}$$

Exercies 0.0.3. We have following relation

$$G_\sigma(\mathbf{r}; t, h) = t^{-2x_\sigma} G_\sigma(\frac{\mathbf{r}}{b}; b^{y_t} t, b^{y_h} h) \tag{1}$$

Let $h = 0, K = b^{y_t} t$,

$$G_\sigma(\mathbf{r}; t, 0) = t^{2x_\sigma/y_t} G_\sigma(\frac{\mathbf{r}}{Kt^{-1/y_t}}; K, 0)$$

Since $G_\sigma(\mathbf{r}) \sim r^{-\tau} e^{-\frac{r}{\xi}}$, we have $\xi \sim t^{-1/y_t}$. It implies $\nu = 1/y_t$. With relation 1, we have

$$\chi(t,h) = \frac{1}{T}\int d^d\mathbf{r} G_\sigma(\mathbf{r};t,h) = t^{d-2x_\sigma}\chi(b^{y_t}t, b^{y_h}h)$$

So $\gamma = (d-2x_\sigma)/y_t$. But we have $\eta = 2x_\sigma + 2 - d$ for finite limit of $G(r)$ when $t \to 0$ and $h = 0$. Therefore, we get

$$\gamma = \nu(2-\eta)$$

With scaling relations

$$\alpha + 2\beta + \gamma = 2$$
$$\alpha + \beta(1+\delta) = 2$$

and $\alpha = 2 - d\nu$, we have

$$\beta = \frac{d\nu - 2\nu + \nu\eta}{2}$$
$$\delta = \frac{d-\eta+2}{d+\eta-2}$$

Exercies 0.0.4. By listed commutation relations, we have, for $r, s > 0$,

$$\begin{aligned}
[D, J_{rs}] = [D, L_{rs}] &= \frac{i}{2}[D,[K_r,P_s]] \\
&= -\frac{i}{2}\big([P_s,[D,K_r]] + [K_r,[P_s,D]]\big) \\
&= \frac{1}{2}[P_s,K_r] - \frac{1}{2}[K_r,P_s] \\
&= 0
\end{aligned}$$

For $r=-1, s=0$, $[D,J_{rs}] = [D,D] = 0$. For $r=-1, s\neq 0$, $[D,J_{-1,s}] = [D,\frac{1}{2}(P_s-K_s)] = \frac{i}{2}(P_s+K_s)$. For $r=0$, $[D,J_{0s}] = \frac{i}{2}(P_s-K_s)$. Hence (2,25) is satisfied when $(m,n)=(-1,0)$.

If $(m,n) = (-1,n)$, then we have

$$[J_{mn}, J_{rs}] = \frac{1}{2}[P_n, J_{rs}] - \frac{1}{2}[K_n, J_{rs}]$$

With listed commutation relations, we can easily check it coincides with (2,25) respectively. Similarly check in the case of $(m,n) = (0,n)$.

2d CFT (Week 3)

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1 $SL_2(\mathbb{C})$

Exercies 1.0.1. We have $\det X = t^2 - (x^2+y^2+z^2)$. Since points in $\mathbb{R}^{1,3}$ can be written with Pauli matrix as base. Elements in $SO(1,3)$ can be viewed as action on $M_2(\mathbb{C})$ with form $P \mapsto PXP^*$, which preserve det of X. We have exact sequence of groups as follows:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{C}) \xrightarrow{sp} SO(1,3) \longrightarrow 1$$

where sp is map $P \mapsto (X \mapsto PXP^*)$. Since for $P \in SL_2(\mathbb{C})$, $\det(PXP*) = \det(X) = t^2 - (x^2+y^2+z^2)$, sp is well-defined. Hence $SO(1,3) \cong SL_2(\mathbb{C})/\mathbb{Z}_2$.

Exercies 1.0.2.

- $$z \mapsto \frac{(z_2-z_3)(z-z_1)}{(z_2-z_1)(z-z_3)}$$

- We have
$$\begin{aligned}(w_1-w_3) =& \frac{(az_1+b)(cz_3+d)-(az_2+b)(cz_4+d)}{(cz_1+d)(cz_3+d)}\\ =& \frac{z_1-z_3}{(cz_1+d)(cz_3+d)}\end{aligned}$$
Hence we have $[z_1,z_2,z_3,z_4] = [w_1,w_2,w_3,w_4]$.

2 Three-point function

Exercies 2.0.1. Let $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = f(z_{12},z_{23},z_{13})$. Under scalar transformation $z_i \mapsto \lambda z_i$, we have

$$f(z_{12},z_{23},z_{13}) = \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13})$$

Therefore, f is with form

$$f(z_{12},z_{23},z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$$

where $a+b+c = h_1+h_2+h_3$. Then under comformal transformation $z_i \mapsto \frac{1}{z_i}$, we have

$${z_1}^{-2h_1}{z_2}^{-2h_2}{z_3}^{-2h_3}\frac{(z_1z_2)^a(z_2z_3)^b(z_1z_3)^c}{z_{12}^a z_{23}^b z_{13}^c} = \frac{1}{z_{12}^a z_{23}^b z_{13}^c}$$

Hence $a = h_1+h_2-h_3, b = h_2+h_3-h_1, c = h_1+h_3-h_2$.

3 Energy-momentum tensor

Exercies 3.0.1.

- $$T^{\mu\nu} = -\eta^{\mu\nu}\partial_k\varphi\partial^k\varphi + 2\partial^\mu\varphi\partial^\nu\varphi$$

- We have
$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}$$
Therefore,
$$\begin{aligned}\tilde{T}^{\mu\nu} &= \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\\ &= \frac{1}{2}(-\delta_{\mu\nu}+2)\partial_\mu\varphi\partial_\nu\varphi\end{aligned}$$

2d CFT

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4 Derivations

4.1 Energy-momentum tensor in complex coordinate

Since

$$\begin{aligned}\partial_0\Phi &= \partial_z\Phi + \partial_{\bar{z}}\Phi \\ \partial_1\Phi &= i\partial_z\Phi - i\partial_{\bar{z}}\Phi\end{aligned}$$

we have

$$\begin{aligned}\partial_z\Phi &= \frac{1}{2}\partial_0\Phi - \frac{i}{2}\partial_1\Phi \\ \partial_{\bar{z}}\Phi &= \frac{1}{2}\partial_0\Phi + \frac{i}{2}\partial_1\Phi\end{aligned}$$

Also, since there are metric tensors in complex coordinates

$$g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad g_{\alpha\beta}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

, we have $\partial^z\Phi = 2\partial_{\bar{z}}$ and $\partial^{\bar{z}}\Phi = 2\partial_{\bar{z}}\Phi$. Therefore, from definition of energy-momentum tensor

$$T_{\alpha\beta} = -g_{\alpha\beta}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial^\alpha\Phi)}\partial_\beta\Phi$$

we get expression of them in complex coordinates

$$\begin{aligned}T_{zz} &= \frac{1}{4}\Big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\Big) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}\Big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\Big) \\ T_{z\bar{z}} &= -\frac{1}{2}\mathcal{L} + \frac{1}{4}\Big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\Big) \\ T_{z\bar{z}} &= -\frac{1}{2}\mathcal{L} + \frac{1}{4}\Big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\Big)\end{aligned}$$

Hence

$$\begin{aligned}T_{zz} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11})\end{aligned}$$

4.2 Schwarzian derivative

$$\tilde{T}(z+\epsilon(z)) = (1+\partial_z\epsilon(z))^{-2}\Big[T(z) - \frac{c}{12}\Big(\frac{\partial_z^3\epsilon(z)}{1+\partial_z\epsilon(z)} - \frac{2}{3}\frac{\partial_z^2\epsilon(z)}{1+\partial_z\epsilon(z)}\Big)\Big]$$

Since

$$\frac{1}{1+\partial_z\epsilon(z)} = 1 - \partial_z\epsilon(z) + (\partial_z\epsilon)^2 + \cdots$$

we have

$$\begin{aligned}\tilde{T}(z+\epsilon(z)) &\approx T(z)(1-2\partial_z\epsilon(z)) - \frac{c}{12}(\partial_z^3\epsilon(z) - \frac{2}{3}\partial_z^2\epsilon(z)) \\ &\approx T(z) - 2\partial_z\epsilon(z)T(z) - \frac{c}{12}\partial_z^3\epsilon(z)\end{aligned}$$

Hence

$$\tilde{T}(z+\epsilon(z)) - \big[\epsilon(z)\partial_z T(z) + T(Z)\big] \approx -\frac{c}{12}\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$$

It implies that $\delta_\epsilon(T(z)) = -\frac{c}{12}\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$.

4.3 Virasoro algebra

The Larrent expansion of z^{n+1} around ω is

$$z^{n+1} = (z-\omega)^{n+1} + \binom{n+1}{1}\omega(z-\omega)^n + \cdots + \binom{n+1}{i}\omega^i(z-\omega)^{n+1-i} + \cdots + \omega^{n+1}$$

Hence we have following residues:

$$\begin{aligned}\mathrm{Res}_\omega \frac{z^{n+1}}{(z-\omega)^4} &= 2\pi i\frac{(n+1)n(n-1)}{6}\omega^{n-2} \\ \mathrm{Res}_\omega \frac{z^{n+1}}{(z-\omega)^2} &= 2\pi i(n+1)\omega^n \\ \mathrm{Res}_\omega \frac{z^{n+1}}{z-\omega} &= 2\pi i\omega^{n+1}\end{aligned}$$

Hence we have

$$\begin{aligned}&[L_n, L_m] = \frac{1}{(2\pi i)^2}\oint_0 d\omega\ \omega^{m+1}\oint_\omega dz\ z^{n+1}\big(\frac{c}{2(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial_\omega T(\omega)}{(z-\omega)} + \text{regular part}\big) \\ &= \frac{1}{2\pi i}\oint_0 d\omega\ \omega^{m+1}\big(\frac{c}{12}(n+1)n(n-1)\omega^{n-2} + 2(n+1)\omega^n T(\omega) + \omega^{n+1}\partial_\omega T(\omega)\big) \\ &= \frac{1}{2\pi i}\Big\{\oint_0 d\omega\ \Big(\frac{c(n+1)n(n-1)}{12}\omega^{m+n-1})\Big) - (m-n)\oint_0 d\omega\omega^{m+n+1}T(\omega)\Big\} \\ &= \frac{c}{12}n(n^2-1)\delta_{n+m,0} - (m-n)L_{n+m}\end{aligned}$$

4.4 Commutation relations in free boson

We have

$$\varphi = \varphi_0 + \frac{4\pi}{l}\pi t + i\sum_{n\neq 0}\frac{1}{n}\Big(a_n e^{-2\pi i n(t-x)/l} + \bar{a}_n e^{-2\pi n(t+x)/l}\Big)$$

on cylinder.

$$\Pi = \frac{\pi_0}{l} + \frac{1}{2l}\sum_{n\neq 0}\Big(a_n e^{-2\pi i n(t-x)/l} + \bar{a}_n e^{-2\pi i n(t+x)/l}\Big)$$

With $[\Pi,\Pi]=0$, we can get $[\pi_0, a_n]=0$ and $[\pi_0,\bar{a}_n]=0$. Furthermore, since $[\varphi,\varphi]=0$, we have $[\varphi_0, a_n] = [\varphi_0,\bar{a}_n] = 0$.

$$\begin{aligned}i\delta(x-y) = [\varphi(x,t), \Pi(y,t)] \quad &= \frac{1}{l}[\varphi_0,\pi_0] + \frac{i}{2l}\sum_{n\neq 0, m\neq 0}\frac{1}{n}\Big([a_n,\bar{a}_m]\exp(-2\pi i[(n+m)t - nx + my]/l) \\ &\qquad + [\bar{a}_n, a_m]\exp(-2\pi i[(n+m)t + nx - my]/l) \\ &\qquad + [a_n, a_m]\exp(-2\pi i[(n+m)t - nx - my]/l) \\ &\qquad + [\bar{a}_n,\bar{a}_m]\exp(-2\pi i[(n+m)t + nx + my]/l)\Big)\end{aligned}$$

Since t can be arbitrary, $[a_n, \bar{a}_m] = [\bar{a}_n, a_m] = 0$ when $n + m \neq 0$. Then let $t = 0$, we get

$$\begin{aligned}
i\delta(x-y) &= [\varphi(x,0), \Pi(y,0)] \\
&= \frac{1}{l}[\varphi_0, \pi_0] + \frac{i}{2l}\sum_{n\neq 0}\frac{1}{n}\Big([a_n, \bar{a}_{-n}]e^{-2\pi i(-nx-ny)/l} + [\bar{a}_n, a_{-n}]e^{-2\pi i(nx+ny)/l} + \text{other terms}\Big) \\
&= \frac{1}{l}[\varphi_0, \pi_0] + \frac{i}{l}\sum_{n\neq 0}\Big(\frac{2[a_n, \bar{a}_{-n}]}{n}\Big)e^{2\pi i(n(x+y))/l} + \text{other terms}
\end{aligned}$$

Since $e^{2\pi i(n(x+y))/l}$ is independent of $x-y$, then its coefficient is zero. Hence $[a_n, \bar{a}_{-n}] = 0$. Take integral of both left and right side, we can get $[\varphi_0, \pi_0] = i$ and $[a_n, a_{-n}] = [\bar{a}_n, \bar{a}_{-n}] = 1$.

4.5 Action of free fermion

$$S = \frac{1}{4\pi}\int d^2x \psi^\dagger \gamma^0(\gamma^0\partial_0\psi + \gamma^1\partial_1\psi)$$

But

$$\gamma^0(\gamma^0 + \gamma^1)\psi = \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix}\psi$$

Write ψ as $\begin{pmatrix}\varphi \\ \bar{\varphi}\end{pmatrix}$, then we have

$$\gamma^0(\gamma^0 + \gamma^1)\psi = \begin{pmatrix} 2\partial_{\bar{z}}\varphi \\ 2\partial_z\bar{\varphi}\end{pmatrix}$$

Hence

$$S = \frac{1}{2\pi}\int d^2x(\bar{\varphi}\partial_z\bar{\varphi} + \varphi\partial_{\bar{z}}\varphi)$$

4.6 TT OPE in free fermion

By derivative, we get

$$\begin{aligned}
\langle\psi(z), \partial_\omega\psi(\omega)\rangle &\sim \frac{1}{(z-\omega)^2} \\
\langle\partial_z\psi, \partial_\omega\psi\rangle &\sim \frac{-2}{(z-\omega)^3}
\end{aligned}$$

Hence

$$\begin{aligned}
T(z)\partial_\omega\psi(\omega) &= \frac{1}{2} :\psi(z)\partial_z\psi(z): \partial_\omega\psi(\omega) \\
&\sim -\frac{\psi(\omega)}{(z-\omega)^3} - \frac{1}{2}\frac{\partial_\omega\psi(\omega)}{(z-\omega)^2}
\end{aligned}$$

and

$$\begin{aligned}
T(z)T(\omega) &= \frac{1}{4} :\psi(z)\partial_z\psi(z)::\psi(\omega)\partial_\omega\psi(\omega) \\
&\sim \frac{1}{4}\Big\{-\frac{:\partial_z\psi(z)\partial_\omega\psi(\omega)}{z-\omega} + \frac{2:\psi(z)\psi(\omega):}{(z-\omega)^3} - \frac{:\psi(z)\partial_\omega\psi(\omega) + \partial_z\psi(z)\psi(\omega):}{(z-\omega)^2} + \frac{1}{(z-\omega)^4}\Big\} \\
&\sim \frac{1}{4}\Big\{\frac{2\partial_\omega T(\omega)}{z-\omega} + \frac{4T(\omega)}{(z-\omega)^2} + \frac{1}{(z-\omega)^4}\Big\}
\end{aligned}$$

5 Vertex operator and OPE

If we write $\varphi(z,\bar{z})$ into laurent series since φ is free boson, then we can find $\exp(ik\varphi)$ is product of infinite exponential components which are commutative. Hence the normal ordering has taylor expansion form

$$: \exp(ik\varphi(z,\bar{z})) := \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} : \varphi(z,\bar{z})^n :$$

To justify that $: \exp(ik\varphi) :$ is primary field, we calculate its OPE

$$\begin{aligned}
T(z) : \exp ik\varphi(\omega,\bar{\omega}) : &= -\frac{1}{2}\sum_{n=0}^{\infty} : \partial\varphi(z)\partial\varphi(z) :: \varphi(\omega,\bar{\omega})^n \\
&\sim -\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} n : \partial\varphi(z)\,\overbrace{\partial\varphi(z)\varphi(\omega,\bar{\omega})}\,\varphi(\omega,\bar{\omega})^{n-1} : \\
&- \frac{1}{2}\sum_{n=2}^{\infty} \frac{(ik)^n}{n!} n(n-1) : \overbrace{\partial\varphi(z)\,\overbrace{\partial\varphi(z)\varphi(\omega,\bar{\omega})}\,\varphi(\omega,\bar{\omega})}\,\varphi(\omega,\bar{\omega})^{n-2} : \\
&\sim \frac{ik\partial_\omega\varphi(\omega)}{z-\omega} : \exp(ik\varphi) : + \frac{k^2}{2(z-\omega)^2} : \exp(ik\varphi) :
\end{aligned}$$

This form implies that $: \exp(ik\varphi) :$ is primary field with conformal dimension $\frac{k^2}{2}$.

6 bc ghost system

$$\begin{aligned}
T(z)b(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :)b(\omega) \\
&2\frac{b(z)}{(z-\omega)^2} - \frac{\partial_z b(z)}{z-\omega}
\end{aligned}$$

Take Taylor expansion of $b(z)$ and $\partial_z b(z)$ around ω, we have

$$T(z)b(w) \sim 2\frac{b(\omega)}{(z-\omega)^2} + \frac{\partial_\omega b(\omega)}{z-\omega}$$

Hence the conformal dimension of b is 2.

Similarly, we have

$$\begin{aligned}
T(z)c(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :)c(\omega) \\
&\sim -\frac{c(z)}{(z-\omega)^2} + 2\frac{\partial_z c(z)}{z-\omega} \\
&\sim -\frac{c(\omega)}{(z-\omega)^2} + \frac{\partial_\omega c(\omega)}{z-\omega}
\end{aligned}$$

Hence conformal dimension of c is -1.

$$\begin{aligned}
T(z)T(\omega) = 4 &: b(z)\partial c(\omega) :: b(\omega)\partial c(\omega) : \\
&- 2 : c(z)\partial b(z) :: b(\omega)\partial c(\omega) : -2 : b(z)\partial c(z) :: c(\omega)\partial b(\omega) : \\
&+ c(z)\partial b(z) :: c(\omega)\partial b(\omega) :
\end{aligned}$$

We will calculate it term by term

$$
\begin{aligned}
&4:b(z)\partial c(z)::b(\omega)\partial c(\omega): \\
&\sim 4\Big(:\overbrace{b(z)\partial c(z)b(\omega)\partial c(\omega)}:+b(z)\overbrace{\partial c(z)b(\omega)}\partial c(\omega)+:\overbrace{b(z)\overbrace{\partial c(z)b(\omega)}\partial c(\omega)}:\Big) \\
&\sim \frac{4(-:\partial_z c(z)b(\omega):+:b(z)\partial_\omega c(\omega):)}{(z-\omega)^2}-\frac{4}{(z-\omega)^4} \\
&\sim -\frac{4}{(z-\omega)^4}+\frac{8b(\omega)\partial_\omega c(\omega)}{(z-\omega)^2}+\frac{-4:\partial_\omega^2 c(\omega)b(\omega):+4\partial_\omega b(z)\partial_\omega c(\omega)}{z-\omega}
\end{aligned}
$$

and

$$
\begin{aligned}
&2:c(z)\partial b(z)::b(\omega)\partial c(\omega): \\
&\sim 2\Big(-:\overbrace{c(z)\partial b(z)b(\omega)}\partial c(\omega)-:c(z)\overbrace{\partial b(z)b(\omega)\partial c(\omega))}-:\overbrace{\partial b(z)\overbrace{c(z)b(\omega)}\partial c(\omega)}\Big) \\
&\sim \frac{4}{(z-\omega)^4}+\frac{4:c(z)b(\omega):}{(z-\omega)^3}-\frac{2:\partial b(z)\partial c(\omega)}{(z-\omega)} \\
&\sim \frac{4}{(z-\omega)^4}+\frac{4:c(\omega)b(\omega):}{(z-\omega)^3}+\frac{4:\partial_\omega c(\omega)b(\omega):}{(z-\omega)^2}+\frac{2:\partial_\omega^2 c(\omega)b(\omega):-2:\partial_\omega b(\omega)\partial_\omega c(\omega):}{z-\omega}
\end{aligned}
$$

and symmetrically

$$
\begin{aligned}
&2:b(z)\partial c(z)::c(\omega)\partial b(\omega) \\
&\sim \frac{4}{(z-\omega)^4}+\frac{4:b(\omega)c(\omega):}{(z-\omega)^3}+\frac{4:\partial_\omega b(\omega)c(\omega):}{(z-\omega)^2}+\frac{2:\partial_\omega^2 b(\omega)c(\omega):-2:\partial_\omega c(\omega)\partial_\omega b(\omega):}{z-\omega}
\end{aligned}
$$

and

$$
\begin{aligned}
&c(z)\partial b(z)::c(\omega)\partial b(\omega): \\
&\sim \frac{2:c(\omega)\partial_\omega b(\omega)}{(z-\omega)^2}+\frac{-\partial_\omega^2 b(\omega)c(\omega)+\partial_\omega c(\omega)\partial_\omega b(\omega)}{z-\omega}-\frac{1}{(z-\omega)^4}
\end{aligned}
$$

Hence we have

$$
T(z)T(\omega)\sim -\frac{13}{(z-\omega)^4}+\frac{2T(\omega)}{(z-\omega)^2}+\frac{\partial T(\omega)}{z-\omega}
$$

So the central charge is equal to -26.

7 Schwarzian derivatives

Let $\omega=\frac{az+b}{cz+d}$, then

$$
\begin{aligned}
\frac{d\omega}{dz}&=\frac{ad-bc}{(cz+d)^2} \\
\frac{d^2\omega}{dz^2}&=\frac{-2c(ad-bc)}{(cz+d)^3} \\
\frac{d^3\omega}{dz^3}&=\frac{6c^2(ad-bc)}{(cz+d)^4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\{\omega,z\}&=\frac{6c^2}{(cz+d)^2}-\frac{3}{2}\Big(\frac{-2c}{cz+d}\Big)^2 \\
&=0
\end{aligned}
$$

Then

$$\Big(\frac{a\omega+b}{c\omega+d}\Big)'_z = \frac{ad-bc}{(c\omega+d)^2}\omega'_z$$
$$\Big(\frac{a\omega+b}{c\omega+d}\Big)''_z = \frac{-2c(ad-bc)}{(c\omega+d)^3}(\omega'_z)^2 + \frac{ad-bc}{(c\omega+d)^2}\omega''_z$$
$$\Big(\frac{a\omega+b}{c\omega+d}\Big)'''_z = \frac{6c^2(ad-bc)}{(c\omega+d)^4}(\omega'_z)^3 + 3\frac{-2c(ad-bc)}{(c\omega+d)^3}\omega''_z\omega'_z + \frac{ad-bc}{(c\omega+d)^2}\omega'''_z$$

Hence we have

$$\{\frac{a\omega+b}{c\omega+d}, z\} = \frac{6c^2}{(c\omega+d)^2}(\omega'_z)^2 + 3\frac{-2c}{c\omega+d}\omega''_z + \frac{\omega'''_z}{\omega'_z} - \frac{3}{2}\Big(\frac{-2c}{c\omega+d}\omega'_z + \frac{\omega''_z}{\omega'_z}\Big)^2$$

It is equal to $\{\omega, z\}$.

2d CFT

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8 Modular invariant

$$Z_R(z,\bar{z}) = \frac{1}{|\eta(\tau)|^2}\sum_{m,n} q^{\frac{1}{2}(\frac{m}{R}+\frac{Rn}{2})^2}\bar{q}^{\frac{1}{2}(\frac{m}{R}-\frac{Rn}{2})^2}$$

First, we compute the sum part. Let $z = x + iy$

$$\begin{aligned}
\sum_{m,n} &= \sum_{m.n} q^{\frac{1}{2}(\frac{m}{R}+\frac{Rn}{2})^2}\bar{q}^{\frac{1}{2}(\frac{m}{R}+\frac{Rn}{2})^2} \\
&= \sum_{m,n} \exp\{z\pi i(\frac{m}{R}+\frac{Rn}{2})^2 - \bar{z}\pi i(\frac{m}{R}-\frac{Rn}{2})^2\} \\
&= \sum_{m.n} \exp\{-\frac{2\pi y}{R^2}m^2 - \frac{\pi y R^2}{2}n^2 + 2\pi i x m n\} \\
&= \sum_{m,n} \exp\{-\frac{2\pi y}{R^2}(m - \frac{R^2 x i}{2y}n)^2\}\exp\{-\frac{\pi R^2 x^2}{2y}n^2 - \frac{\pi R^2 y}{2}n^2\}
\end{aligned}$$

Let $a = \frac{R^2}{2y}, b = \frac{\pi R^2 x}{y}n$ in Possion formula, then we have

$$\begin{aligned}
&= \frac{R}{\sqrt{2y}}\sum_{m,n}\exp\{-\frac{\pi R^2}{2y}m^2 + \frac{\pi R^2 x}{y}mn - \frac{\pi R^2 x^2}{2y}n^2 - \frac{\pi R^2}{2}n^2\} \\
&= \frac{R}{\sqrt{2y}}\sum_{m,n}\exp\{-\frac{\pi R}{2y}(m^2 - xmn + (xn)^2 + (yn)^2\} \\
&= \frac{R}{\sqrt{2y}}\sum_{m,n}\exp\{-\frac{\pi R}{2y}((m-xn)^2 + (yn)^2)\} \\
&= \frac{R}{\sqrt{2y}}\sum_{m,n}\exp\{-\frac{\pi R}{2y}|nz - m|^2\}
\end{aligned}$$

Hence when $z \mapsto -1/z$, the sum part becomes

$$\frac{R|z|}{\sqrt{2y}}\sum_{m,n}\exp\{-\frac{\pi R}{2y}|n\bar{z} - m|^2\} = \frac{R|z|}{\sqrt{2y}}\sum_{m,n}\exp\{-\frac{\pi R}{2y}|nz - m|^2\}$$

Since we have

$$\eta(-\frac{1}{z}) = \sqrt{-iz}\,\eta(z)$$

it norm is

$$|\eta(-\frac{1}{z})| = \sqrt{|z|}|\eta(z)|$$

Hence we can conclude that

$$Z_R(z,\bar{z}) = Z_R(-\frac{1}{z}, -\frac{1}{\bar{z}})$$

9 Modular transformation

We have

$$\begin{aligned}\gamma\cdot\tau &= \frac{a\tau+b}{c\tau+d}\\ &= \frac{(a\tau+b)(c\bar\tau+d)}{|c\tau+d|^2}\\ &= \frac{ac|\tau|^2+bc\bar\tau+ad\tau}{|c\tau+d|^2}\\ &= \frac{ac|\tau|^2+(ad+bc)\operatorname{Re}\tau+i(ad-bc)\operatorname{Im}\tau}{|c\tau+d|^2}\end{aligned}$$

Since $ad-bc=1$, we can conclude that

$$\operatorname{Im}(\gamma\cdot\tau)=\frac{\operatorname{Im}\tau}{|c\tau+d|^2}$$

In the upper-half plane, the gray region can be described as

$$\begin{aligned}-\frac{1}{2}\le\operatorname{Re}\tau\le\frac{1}{2}\\ |\tau|>1\end{aligned}$$

If S acts on the gray region, then we have $S(z)=-\frac{1}{z}=-\frac{\bar z}{|z|^2}$, hence it sends the region to the red region. And the blue region is the image of the gray origin under T. Finally, the transformation ST, maps to the green region.

10 Boson-Fermion correspondence

10.1 $:e^{i\varphi}:\cong\psi$, $:e^{-i\varphi}\cong\bar\psi$

Calculate the OPE of $:e^{i\varphi}:$ directly

$$\begin{aligned}:e^{i\varphi(z)}::e^{i\varphi(0)} &\sim \sum_{m,n,k}\frac{k!}{m!n!}\binom{m}{k}\binom{n}{k}(-\varphi(z)\varphi(0)):(i\varphi(0)^{m+n-k}):\\ &\sim \sum_{m,n,k}\frac{1}{k!}\ln^k(z)\frac{1}{(m-k)!}\frac{1}{(n-k)!}:i\varphi(0)^{m+n-k}:\\ &= e^{\ln z}:e^{2i\varphi(0)}:\\ &\sim 0\end{aligned}$$

The OPE of T_φ with vertex operators $:e^{i\varphi}$ and $:e^{-i\varphi}$ is given in last homework

$$T_\varphi(z):e^{i\varphi}:\sim\frac{\partial_z\exp(i\varphi)(0)}{z}+\frac{\exp(i\varphi)(0)}{2z^2}$$

$$T_\varphi(z):e^{-i\varphi}:\sim\frac{\partial_z\exp(-i\varphi)(0)}{z}+\frac{\exp(-i\varphi)(0)}{2z^2}$$

On the other hand, the OPE of T_ψ with ψ and $\bar\psi$ can be calculated as follows. First,

$$\begin{aligned}\partial\bar\psi(z)\psi(0) &\sim \frac{1}{2}(\partial\psi^1(z)-i\partial\psi^2(z))(\psi^1(0)+i\psi^2(0))\\ &\sim -\frac{1}{z^2}\end{aligned}$$

$$
\begin{aligned}
\bar{\psi}(z)\partial\psi(0) &\sim \frac{1}{2}(\psi^1(z) - i\psi^2(z))(\partial\psi^1(0) + i\partial\psi^2(0)) \\
&\sim \frac{1}{z^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\psi}(z)\psi(0) &\sim \frac{1}{2}(\psi^1(z) - i\psi^2(z))(\psi^1(0) + i\psi(0)) \\
&\sim \frac{1}{z}
\end{aligned}
$$

Hence

$$
\begin{aligned}
T_\varphi(z)\psi(0) =& -\frac{1}{2} : \psi\partial\bar{\psi} : \psi(0) - \frac{1}{2} : \bar{\psi}\partial\psi : \psi(0) \\
&\sim -\frac{1}{2}\psi(z)(-\frac{1}{z^2}) + \frac{1}{2}\frac{1}{z}\partial_z\psi(z) \\
&\sim \frac{\psi(0) + z\partial_z\psi(0)}{2z^2} + \frac{\partial_z\psi(0)}{2z} \\
&\sim \frac{\psi(0)}{2z^2} + \frac{\partial_z\psi(0)}{z}
\end{aligned}
$$

and

$$
\begin{aligned}
T_\varphi(z)\bar{\psi}(0) =& -\frac{1}{2} : \psi\partial\bar{\psi} : \bar{\psi}(0) - \frac{1}{2} : \bar{\psi}\partial\psi : \bar{\psi}(0) \\
&\sim \frac{1}{2}\partial\bar{\psi}(z)(\frac{1}{z}) - \frac{1}{2}\bar{\psi}(z)(-\frac{1}{z^2}) \\
&\sim \frac{\bar{\psi}(0) + z\partial_z\bar{\psi}(0)}{2z^2} + \frac{\partial_z\bar{\psi}(0)}{2z} \\
&\sim \frac{\bar{\psi}(0)}{2z^2} + \frac{\partial_z\bar{\psi}(0)}{z}
\end{aligned}
$$

10.2 $i\partial\varphi \cong \psi\bar{\psi}$

Also, as calculated before, the OPE of $i\partial\varphi$ is as follows

$$
T_\varphi(z)(i\partial\varphi(0)) \sim \frac{i\partial\varphi(0)}{z^2} + \frac{i\partial_z^2\varphi(0)}{z}
$$

$$
\begin{aligned}
T_\psi(z)\psi\bar{\psi}(0) =& -\frac{1}{2} : \psi(z)\partial\bar{\psi}(z) : \psi(0)\bar{\psi}(0) - \frac{1}{2} : \bar{\psi}(z)\partial\psi(z) : \psi(0)\bar{\psi}(0) \\
&\sim -\frac{1}{2} : \psi(z)\partial\bar{\psi}(z)\psi(0)\bar{\psi}(0) : -\frac{1}{2} : \psi(z)\partial\bar{\psi}(z)\psi(0)\bar{\psi}(0) : \\
&- \frac{1}{2} : \psi\partial\bar{\psi}\psi(0)\psi(0) : \\
&- \frac{1}{2} : \bar{\psi}(z)\partial\psi(z)\psi(0)\bar{\psi}(0 :) - \frac{1}{2} : \bar{\psi}(z)\partial\psi(z)\psi(0)\bar{\psi}(0) \\
&- \frac{1}{2} : \bar{\psi}(z)\partial\psi(z)\psi(0)\bar{\psi}(0) \\
&\sim \frac{\partial\bar{\psi}(0)\psi(0) - \partial\psi(0)\psi(0) + \partial\psi(0)\bar{\psi}(0) - \partial\bar{\psi}(0)\psi(0)}{2z} \\
&+ \frac{\bar{\psi}(0)\psi(0) - \psi(0)\bar{\psi}(0)}{2z^2} \\
&\sim \frac{\partial(\psi\bar{\psi})(0)}{z} + \frac{\psi\bar{\psi}(0)}{z^2}
\end{aligned}
$$

(Tips: I'm confused here. Is the i in $i\partial\varphi$ necessary? I failed to get it in Fermion side.)

10.3 $T_\varphi \cong T_\psi$

Finally, we compute the TT OPE for φ and ψ. We have

$$T_\varphi(z)T_\varphi(0) \sim \frac{1/2}{z^4} + \frac{2T_\varphi(0)}{z^2} + \frac{\partial T_\varphi(0)}{z}$$

In complex Fermion case, we have

$$\begin{aligned}
T_\psi(z)T_\psi(0) \sim \frac{1}{4}\Big[&: \psi(z)\partial\bar{\psi}(z) :: \psi(0)\partial\bar{\psi}(0) : + : \bar{\psi}\partial\psi(z) :: \psi(0)\partial\bar{\psi}(0) : \\
&+ : \psi(z)\partial\bar{\psi}(z) :: \bar{\psi}(0)\partial\bar{\psi}(0) : + : \bar{\psi}(z)\partial\psi(z) :: \bar{\psi}(0)\partial\psi(0) : \Big] \\
\sim \frac{1}{4}\Big[&: \psi(z)\partial\bar{\psi}(z)\psi(0)\partial\bar{\psi}(0) + : \psi(z)\partial\bar{\psi}(z)\psi(0)\partial\bar{\psi}(0) : + : \psi(z)\partial\bar{\psi}(z)\psi(0)\partial\bar{\psi}(0) \Big] \\
+ \frac{1}{4}\Big[&: \bar{\psi}(z)\partial\psi(z)\psi(0)\partial\bar{\psi}(0) + : \psi(\bar{z})\partial\psi(z)\psi(0)\partial\bar{\psi}(0) : + : \psi(\bar{z})\partial\psi(z)\psi(0)\partial\bar{\psi}(0) \Big] \\
+ \frac{1}{4}\Big[&: \psi(z)\partial\psi(\bar{z})\bar{\psi}(0)\partial\psi(0) + : \psi(z)\partial\bar{\psi}(z)\psi(\bar{0})\partial\psi(0) : + : \psi(z)\partial\bar{\psi}(z)\bar{\psi}(0)\partial\psi(0) \Big] \\
+ \frac{1}{4}\Big[&: \psi(\bar{z})\partial\psi(z)\psi(\bar{0})\partial\psi(0) + : \psi(\bar{z})\partial\psi(z)\bar{\psi}(0)\partial\psi(0) : + : \bar{\psi}(z)\partial\psi(z)\bar{\psi}(0)\partial\psi(0) \Big] \\
\sim \frac{1/2}{z^4} &+ \frac{2T_\psi(0)}{z^2} + \frac{\partial T_\psi(0)}{z}
\end{aligned}$$

11 Homework 7

11.1 Null states at level 3

With commutation relation of Virasoro algebra, we have

$$[L_1 L_{-3}]\big|\phi_I\rangle = 4L_{-2}\big|\varphi_I\rangle$$

$$\begin{aligned}[L_1, L_{-1}L_{-2}]\big|\phi_I\rangle &= \{[L_1, L_{-1}]L_{-2} + L_1[L_1, L_{-2}]\}\big|\phi_I\rangle \\ &= \{2L_0 L_{-2} + 3L_{-1}^2\}\big|\phi_I\rangle \\ &= \{2[L_0, L_{-2}] + 2L_{-2}L_0 + 3L_{-1}^2\}\big|\phi_I\rangle \\ &= \{2(2+h_I)L_{-2} + 3L_{-1}^2\}\big|\phi_I\rangle\end{aligned}$$

and

$$\begin{aligned}[L_1, L_{-1}^3]\big|\phi_I\rangle &= \{[L_1, L_{-1}]L_{-1}^2 + L_{-1}[L_1, L_{-1}]L_{-1} + L_{-1}^2[L_1, L_{-1}]\big|\phi_I\rangle \\ &= \{2L_0 L_{-1}^2 + 2L_{-1}L_0 L_{-1} + 2L_{-1}2L_0\}\big|\phi_I\rangle \\ &= \{6 + 6h_I\}L_{-1}^2\big|\phi_I\rangle\end{aligned}$$

Suppose the

$$\big|\chi_I\rangle = \alpha L_{-3} + \beta L_{-1}L_{-2} + L_{-1}^3\big|\phi_I\rangle$$

since $L_1\big|\chi_I\rangle = 0$, we get equations

$$4\alpha + 2(2+h_I)\beta = 0 \tag{2}$$

$$\beta + 2h_I + 2 = 0 \tag{3}$$

The solution of this system is

$$\beta = -2(h_I+1) \quad \alpha = (h_I+1)(h_I+2)$$

Now, we have

$$\big|\chi_I\rangle = \Big[(h_I+1)(h_I+2)L_{-3} - 2(h_I+1)L_{-1}L_{-2} + L_{-1}^3\Big]\big|\phi_I\rangle$$

Then, we write L_{-3}, L_{-1}, L_{-2} into differential operators as

$$\begin{aligned}\mathcal{L}_{-3} &= \sum_{i=1}^{N} \frac{2h_i}{(\omega_i-\omega)^3} - \frac{1}{(\omega_i-\omega)^2}\partial_{\omega_i} \\ \mathcal{L}_{-2} &= \sum_{i=1}^{N} \frac{h_i}{(\omega_i-\omega)^2} - \frac{1}{\omega_i-\omega}\partial_{\omega_i} \\ \mathcal{L}_{-1} &= -\sum_{i=1}^{N} \partial_{\omega_i}^3\end{aligned}$$

We have following differential equations

$$\Big[(h_I+1)(h_I+2)\mathcal{L}_{-3} - 2(h_I+1)\mathcal{L}_{-1}L_{-2} + \mathcal{L}_{-1}^3\Big]\langle\phi_I(\omega)\phi_{\bar{I}}(z)\rangle$$

In this case, we have

$$\begin{aligned}\mathcal{L}_{-3} &= \frac{2h_{\bar{I}}}{(z-\omega)^3} - \frac{1}{(z-\omega)^2}\partial_z \\ \mathcal{L}_{-2} &= \frac{h_{\bar{I}}}{(z-\omega)^2} - \frac{1}{z-\omega}\partial_z \\ \mathcal{L}_{-1} &= -\partial_z = \partial_\omega\end{aligned}$$

so we further have

$$\mathcal{L}_{-1}\mathcal{L}_{-2} = -\mathcal{L}_{-3} - \frac{1}{z-\omega}\partial_z^2$$

Take them into equation, we get

$$\left\{\frac{2(h_I+1)(h_I+4)h_{\bar{I}}}{(z-\omega)^3} - \frac{(h_I+1)(h_I+4)}{(z-\omega)^2}\partial_z + \frac{2(h_I+1)}{z-\omega}\partial_z^2 - \partial_z^3\right\}\langle\phi_I(\omega)\phi_{\bar{I}}(z)\rangle = 0$$

11.2 Minimal models

The formula (5.40) implies for minimal model $\mathcal{M}_{2,2n+1}$, the central charge is

$$\begin{aligned} c &= 1 - 6\frac{(2-2n-1)^2}{2(2n+1)} \\ &= -\frac{2(6n-1)(n-1)}{2n+1} \end{aligned}$$

And in this model, the formulas (5.44) and (5.45) becomes as follows for fusion rule $\phi_{(r,s)} \times \phi_{(a,b)}$,

$$k_{\max} = \begin{cases} a+r-1 & \text{if } 2 \le a+r \le 2n+1 \\ 2(2n+1)-(a+r-1) & \text{if } 2n+1 < a+r \le 4n \end{cases}$$

and

$$l_{\max} = 1 \quad (s,b) = (1,1)$$

So we have fusion rules

$$\begin{aligned} \phi_{(r,s)} \times \phi_{(a,b)} &= \phi_{(r,1)} \times \phi_{(a,1)} \\ &= \sum_{k=1+|r-a|,k+r+a=1 \mod 2}^{k_{\max}} \phi_{(k,1)} \end{aligned}$$

If $2 \le r+a \le 2n+1$, then we have

$$\sum_{k=1+|r-a|,k+r+a=1 \mod 2}^{r+a-1} \phi_{(k,1)} = \sum_{j=0}^{a-1} \phi_{(r-a+1+2j,1)}$$

If $2n+1 < r+a \le 4n$, then we have

$$\sum_{k=1+|r-a|,k+r+a=1 \mod 2}^{4n+1-(r+a)} \phi_{(k,1)} = \sum_{j=0}^{2n-r} \phi_{(r-a+1+2j,1)}$$

11.3 Star-triangle relation

To derive eq2, we have

$$\begin{aligned} 2\cosh(L\sigma_i + L(\sigma_j+\sigma_k)) &= 2(\cosh(L\sigma_i)\cosh(L(\sigma_j+\sigma_k)) + \sinh(L\sigma_i)\sinh(L(\sigma_j+\sigma_k))) \\ &= 2\cosh L(\cosh L\cosh L + \sigma_j\sigma_k\sinh L\sinh L) \\ &+ 2\sigma_i\sinh L(\sigma_j\sinh L\cosh L + \sigma_k\cosh L\sinh L) \\ &= 2\cosh^3 L + 2\cosh L\sinh^2 L\Big[\sigma_i\sigma_j + \sigma_i\sigma_k + \sigma_j\sigma_k\Big] \end{aligned}$$

To derive eq3, we first notice that

$$\exp(K[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i]) = \cosh(K[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i]) + \sinh(K[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i])$$

Similarly, we have

$$\begin{aligned}\cosh(K[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i]) &= \cosh^3 K + \cosh K \sinh^2 K \left[\sigma_i^2\sigma_j\sigma_k + \sigma_i\sigma_j^2\sigma_k + \sigma_i\sigma_j\sigma_k^2\right] \\ &= \cosh^3 K + \cosh K \sinh^2 K \left[\sigma_i\sigma_j + \sigma_i\sigma_k + \sigma_j\sigma_k\right]\end{aligned}$$

and

$$\begin{aligned}&\sinh(K[\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i]) \\ &= \sinh(K\sigma_i\sigma_j)\cosh(K[\sigma_j\sigma_k + \sigma_k\sigma_i]) + \cosh K \sinh(K[\sigma_j\sigma_k + \sigma_k\sigma_j]) \\ &= \sigma_i\sigma_j \sinh K(\cosh^2 K + \sigma_i\sigma_k^2\sigma_j \sinh^2 K) + \sigma_j\sigma_k \cosh^2 K \sinh K + \sigma_k\sigma_j \cosh^2 K \sinh K \\ &= \sinh^3 K + (\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_j) \sinh K \cosh^2 K\end{aligned}$$

sum the two equations, then we get eq3.

12 Week 8

12.1 Derivation

We have following equations

$$\partial_z F(z\bar{z}) = \bar{z}F'(z\bar{z}) \tag{4}$$
$$\partial_{\bar{z}} F(z\bar{z}) = zF'(z\bar{z}) \tag{5}$$

Combining the definition of $\dot{F}$, we have

$$\dot{F} = z\partial_z F(z\bar{z}) = \bar{z}\partial_{\bar{z}} F(z\bar{z}) \tag{6}$$

Since $F(z\bar{z}) = z^4\langle T(z,\bar{z})T(0,0)\rangle$, we have

$$\dot{F} = \bar{z}\partial_{\bar{z}}\big(z^4\langle T(z,\bar{z})T(0,0)\rangle\big) = z^4\bar{z}\langle\partial_{\bar{z}}T(z,\bar{z})T(0,0)\rangle \tag{7}$$

Similarly, for G, we have

$$\begin{aligned}
\frac{1}{4}\dot{G} &= \frac{1}{4}\big(z\partial_z G(z\bar{z})\big) \\
&= \frac{1}{4}\big(z\partial_z(z^3\bar{z}\langle\Theta(z,\bar{z})T(0,0)\rangle)\big) \\
&= \frac{3}{4}z^3\bar{z}\langle\Theta(z,\bar{z})T(0,0)\rangle + \frac{1}{4}z^4\bar{z}\langle\partial_z\Theta(z,\bar{z})T(0,0)\rangle \\
&= \frac{3}{4}G - z^4\bar{z}\langle\partial_{\bar{z}}T(z,\bar{z})T(0,0)\rangle \\
&= \frac{3}{4}G - \dot{F}
\end{aligned} \tag{8}$$

With the relation $\frac{1}{4}\langle\partial_z\Theta(z,\bar{z})T(0,0)\rangle = \langle T(z,\bar{z})T(0,0)\rangle$, we get

$$\frac{1}{4}\dot{G} = \frac{3}{4}G - \dot{F} \tag{9}$$

It implies $\dot{F} + \frac{1}{4}\dot{G} - \frac{3}{4}G = 0$. Furthermore, we have

$$\begin{aligned}
\dot{G} &= \bar{z}\partial_{\bar{z}}\big(z^3\bar{z}\langle T(z,\bar{z})\Theta(0,0)\rangle\big) \\
&= z^3\bar{z}\langle T(z,\bar{z})\Theta(0,0)\rangle + z^3\bar{z}^2\langle\partial_{\bar{z}}T(z,\bar{z})\Theta(0,0)\rangle \\
&= G - \frac{1}{4}z^3\bar{z}^2\langle\partial_z\Theta(z,\bar{z})\Theta(0,0)\rangle
\end{aligned} \tag{10}$$

and for H, we have

$$\begin{aligned}
\dot{H} &= z\partial_z\big(z^2\bar{z}^2\langle\Theta(z,\bar{z})\Theta(0,0)\rangle\big) \\
&= z\big(2z\bar{z}^2\langle\Theta(z,\bar{z})\Theta(0,0)\rangle + z^2\bar{z}^2\langle\partial_z\Theta(z,\bar{z})\Theta(0,0)\rangle\big) \\
&= 2H + z^3\bar{z}^2\langle\partial_z\Theta(z,\bar{z})\Theta(0,0)\rangle
\end{aligned} \tag{11}$$

Combining equation 10 and 11, we get

$$\dot{G} - G + \frac{1}{4}\dot{H} - \frac{1}{2}H = 0$$

By computation, we have

$$\frac{\partial}{\partial R}C(z\bar{z}, g) = C'(z\bar{z}, g)2R \tag{12}$$

It implies that

$$\begin{aligned} R\frac{\partial}{\partial R}C(z\bar{z},g) &= 2R^2C'(z\bar{z}) \\ &= 2z\bar{z}C'(z\bar{z},g) \\ &= 2\dot{C} = -\frac{3}{2}H \end{aligned} \tag{13}$$

When $R = 1$, we have

$$R\frac{\partial}{\partial R}C(z\bar{z},g) = -\frac{3}{2}\beta^a(g)\beta^b(g)\langle\mathcal{O}_a(x)\mathcal{O}_b(x)\rangle$$

since $\Theta(x) = \beta^a(g)\mathcal{O}_a(x)$. Hence we get

$$\beta^a\frac{\partial}{\partial g^a}C(g) = -\beta^a(g)\beta^b(g)\langle\mathcal{O}_a(x)\mathcal{O}_b(x)\rangle \tag{14}$$

12.2 β-functions

To compute the integral

$$\int_{a<|x_{12}|<a(1+\delta\lambda)}\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\frac{dx_1}{a^{2-h}}\frac{dx_2}{a^{2-h}}$$

With OPE $\mathcal{O}(x_1)$ and $\mathcal{O}(x_2)$, we have

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{C}{|x_{12}|^h}\langle\mathcal{O}(x_2)\rangle \tag{15}$$

Hence we have

$$\int_{a<|x_{12}|<a(1+\delta\lambda)}\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\frac{dx_1}{a^{2-h}}\frac{dx_2}{a^{2-h}} = \int_{a<|x_{12}|<a(1+\delta\lambda)}\frac{C}{|x_{12}|^h}\frac{dx_1}{a^{2-h}}\langle\mathcal{O}(x_2)\rangle\frac{dx_2}{a^{2-h}} \tag{16}$$

Since $|x_{12}| \sim a$ as δ infinitesimal, 16 becomes

$$\int_{a<|x_{12}|<a(1+\delta\lambda)}\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\frac{dx_1}{a^{2-h}}\frac{dx_2}{a^{2-h}} = a^{-2}C\int_{a<|x_{12}|<a(1+\delta\lambda)}dx_1\int\langle\mathcal{O}(x)\rangle\frac{dx}{a^{2-h}} \tag{17}$$

Since the area of $a < |x_{12}| < a(1+\delta\lambda$ is $\delta\lambda\pi a^2(\delta\lambda+2)$, we have

$$\begin{aligned} \int_{a<|x_{12}|<a(1+\delta\lambda)}\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle\frac{dx_1}{a^{2-h}}\frac{dx_2}{a^{2-h}} &= C\delta\lambda\pi(\delta\lambda+2)\int\langle\mathcal{O}(x)\rangle\frac{dx}{a^{2-h}} \\ &\sim 2\delta\lambda\pi C\int\langle\mathcal{O}(x)\rangle\frac{dx}{a^{2-h}} \end{aligned} \tag{18}$$

Hence the second term is

$$\delta\lambda\pi C\hat{g}^2\int\langle\mathcal{O}(x)\rangle\frac{dx}{a^{2-h}}$$

Therefore, we get deformation

$$\mathrm{Def}_{\hat{g}}(\lambda) = \hat{g} + \delta(2-h)\lambda\hat{g} - \delta\pi C\lambda\hat{g}^2 + O(\hat{g}^3)(\lambda) \tag{19}$$

Now we have

$$\frac{d\hat{g}}{d\lambda} = (2-h)\hat{g} - \pi C\hat{g}^2 + O(\hat{g}^3) \tag{20}$$

The β-function is

$$\beta(\hat{g}) = (2-h)\hat{g} - \pi C\hat{g}^2 + O(\hat{g}^3)$$

Hence near $\hat{g} = 0$, we have the picture of curve of β-function

To solve equation

$$\frac{d\hat{g}}{d\lambda} = -\pi C \hat{g}^2 \tag{21}$$

we separate variables

$$-\frac{1}{\hat{g}^2} d\hat{g} = \pi C \lambda d\lambda$$

Taking integrals of both sides, we get

$$\frac{1}{\hat{g}} = \pi C \lambda + K \tag{22}$$

where K is constant. When $\lambda = 0$, $\hat{g} = g_*$, so $K = \frac{1}{g^*}$. Therefore,

$$\hat{g}(\lambda) = \frac{g^*}{g^* \pi C \lambda + 1}$$

Since the β-function is

$$\beta(\hat{g}) = -\pi C \hat{g}^2$$

, we have

$C < 0$ $\mathcal{O}$ is marginal relevant

$C = 0$ $\mathcal{O}$ is exactly marginal

$C > 0$ $\mathcal{O}$ is marginal irrelevant