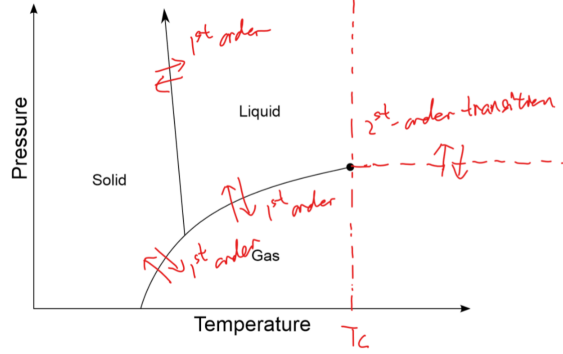


2d CFT

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Week 1

Exercies 0.0.1. The first order transitions and second order transitions show in the diagram



Exercies 0.0.2. By the homogeneous relation

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

we have

$$f(t, h) = t^{\frac{d}{y_t}} g(\alpha)$$

where $g(\alpha) = f(1, \alpha)$ and $\alpha = t^{-\frac{y_h}{y_t}} h$. It is easy to see that α is invariance under scaling transformation $x \rightarrow x/b$. Hence we have

$$C(t, 0) = -T \frac{\partial^2 f}{\partial T^2} \Big|_{h=0} = -\frac{1}{T_c} t^{\frac{d}{y_t}-2} g''(0)$$

$$M(t, 0) = -\frac{\partial f}{\partial B} \Big|_{h=0} = t^{\frac{d-y_h}{y_t}} g'(0)$$

$$\chi(t, 0) = \frac{\partial^2 f}{\partial B^2} \Big|_{h=0} = t^{(d-2y_h)/y_t} g''(0)$$

As function with single variable h , $\lim_{t \rightarrow 0} M(t, h) \sim h^{\frac{1}{\delta}}$, which implies that $g'(\alpha) \sim \alpha^{\frac{1}{\delta}}$ since α is linear function of h . Hence we have

$$\lim_{t \rightarrow 0} M = \lim_{t \rightarrow 0} t^{(d-y_h-\frac{y_h}{\delta})} h^{1/\delta}$$

since it is non-zero, we have $d - y_h - y_h \frac{1}{\delta} = 0$. Hence we have

$$\delta = \frac{y_h}{d - y_h}$$

Exercies 0.0.3. We have following relation

$$G_\sigma(\mathbf{r}; t, h) = t^{-2x_\sigma} G_\sigma\left(\frac{\mathbf{r}}{b}; b^{y_t} t, b^{y_h} h\right) \quad (1)$$

Let $h = 0, K = b^{y_t} t$,

$$G_\sigma(\mathbf{r}; t, 0) = t^{2x_\sigma/y_t} G_\sigma\left(\frac{\mathbf{r}}{K t^{-1/y_t}}; K, 0\right)$$

Since $G_\sigma(\mathbf{r}) \sim r^{-\tau} e^{-\frac{r}{\xi}}$, we have $\xi \sim t^{-1/y_t}$. It implies $\nu = 1/y_t$. With relation 1, we have

$$\chi(t, h) = \frac{1}{T} \int d^d \mathbf{r} G_\sigma(\mathbf{r}; t, h) = t^{d-2x_\sigma} \chi(b^{y_t} t, b^{y_h} h)$$

So $\gamma = (d - 2x_\sigma)/y_t$. But we have $\eta = 2x_\sigma + 2 - d$ for finite limit of $G(r)$ when $t \rightarrow 0$ and $h = 0$. Therefore, we get

$$\gamma = \nu(2 - \eta)$$

With scaling relations

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \alpha + \beta(1 + \delta) &= 2 \end{aligned}$$

and $\alpha = 2 - d\nu$, we have

$$\begin{aligned} \beta &= \frac{d\nu - 2\nu + \nu\eta}{2} \\ \delta &= \frac{d - \eta + 2}{d + \eta - 2} \end{aligned}$$

Exercies 0.0.4. By listed commutation relations, we have, for $r, s > 0$,

$$\begin{aligned} [D, J_{rs}] &= [D, L_{rs}] = \frac{i}{2} [D, [K_r, P_s]] \\ &= -\frac{i}{2} ([P_s, [D, K_r]] + [K_r, [P_s, D]]) \\ &= \frac{1}{2} [P_s, K_r] - \frac{1}{2} [K_r, P_s] \\ &= 0 \end{aligned}$$

For $r = -1, s = 0$, $[D, J_{rs}] = [D, D] = 0$. For $r = -1, s \neq 0$, $[D, J_{-1,s}] = [D, \frac{1}{2}(P_s - K_s)] = \frac{i}{2}(P_s + K_s)$. For $r = 0$, $[D, J_{0s}] = \frac{i}{2}(P_s - K_s)$. Hence (2,25) is satisfied when $(m, n) = (-1, 0)$.

If $(m, n) = (-1, n)$, then we have

$$[J_{mn}, J_{rs}] = \frac{1}{2} [P_n, J_{rs}] - \frac{1}{2} [K_n, J_{rs}]$$

With listed commutation relations, we can easily check it coincides with (2,25) respectively. Similarly check in the case of $(m, n) = (0, n)$.

2d CFT (Week 3)

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1 $SL_2(\mathbb{C})$

Exercies 1.0.1. We have $\det X = t^2 - (x^2 + y^2 + z^2)$. Since points in $\mathbb{R}^{1,3}$ can be written with Pauli matrix as base. Elements in $SO(1,3)$ can be viewed as action on $M_2(\mathbb{C})$ with form $P \mapsto PXP^*$, which preserve \det of X . We have exact sequence of groups as follows:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{C}) \xrightarrow{sp} SO(1,3) \longrightarrow 1$$

where sp is map $P \mapsto (X \mapsto PXP^*)$. Since for $P \in SL_2(\mathbb{C})$, $\det(PXP^*) = \det(X) = t^2 - (x^2 + y^2 + z^2)$, sp is well-defined. Hence $SO(1,3) \cong SL_2(\mathbb{C})/\mathbb{Z}_2$.

Exercies 1.0.2. •

$$z \mapsto \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}$$

- We have

$$\begin{aligned} (w_1 - w_3) &= \frac{(az_1 + b)(cz_3 + d) - (az_2 + b)(cz_4 + d)}{(cz_1 + d)(cz_3 + d)} \\ &= \frac{z_1 - z_3}{(cz_1 + d)(cz_3 + d)} \end{aligned}$$

Hence we have $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$.

2 Three-point function

Exercies 2.0.1. Let $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle = f(z_{12}, z_{23}, z_{13})$. Under scalar transformation $z_i \mapsto \lambda z_i$, we have

$$f(z_{12}, z_{23}, z_{13}) = \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13})$$

Therefore, f is with form

$$f(z_{12}, z_{23}, z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$$

where $a + b + c = h_1 + h_2 + h_3$. Then under comformal transformation $z_i \mapsto \frac{1}{z_i}$, we have

$$z_1^{-2h_1} z_2^{-2h_2} z_3^{-2h_3} \frac{(z_1 z_2)^a (z_2 z_3)^b (z_1 z_3)^c}{z_{12}^a z_{23}^b z_{13}^c} = \frac{1}{z_{12}^a z_{23}^b z_{13}^c}$$

Hence $a = h_1 + h_2 - h_3, b = h_2 + h_3 - h_1, c = h_1 + h_3 - h_2$.

3 Energy-momentum tensor

Exercies 3.0.1. •

$$T^{\mu\nu} = -\eta^{\mu\nu} \partial_k \varphi \partial^k \varphi + 2\partial^\mu \varphi \partial^\nu \varphi$$

- We have

$$\delta \sqrt{g} = -\frac{1}{2} \sqrt{g} g_{\mu\nu} \delta g^{\mu\nu}$$

Therefore,

$$\begin{aligned} \tilde{T}^{\mu\nu} &= \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{2} (-\delta_{\mu\nu} + 2) \partial_\mu \varphi \partial_\nu \varphi \end{aligned}$$

4 Derivations

4.1 Energy-momentum tensor in complex coordinate

Since

$$\begin{aligned}\partial_0\Phi &= \partial_z\Phi + \partial_{\bar{z}}\Phi \\ \partial_1\Phi &= i\partial_z\Phi - i\partial_{\bar{z}}\Phi\end{aligned}$$

we have

$$\begin{aligned}\partial_z\Phi &= \frac{1}{2}\partial_0\Phi - \frac{i}{2}\partial_1\Phi \\ \partial_{\bar{z}}\Phi &= \frac{1}{2}\partial_0\Phi + \frac{i}{2}\partial_1\Phi\end{aligned}$$

Also, since there are metric tensors in complex coordinates

$$g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

, we have $\partial^z\Phi = 2\partial_{\bar{z}}\Phi$ and $\partial^{\bar{z}}\Phi = 2\partial_z\Phi$. Therefore, from definition of energy-momentum tensor

$$T_{\alpha\beta} = -g_{\alpha\beta}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial^\alpha\Phi)}\partial_\beta\Phi$$

we get expression of them in complex coordinates

$$\begin{aligned}T_{zz} &= \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right) \\ T_{z\bar{z}} &= -\frac{1}{2}\mathcal{L} + \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right) \\ T_{\bar{z}z} &= -\frac{1}{2}\mathcal{L} + \frac{1}{4}\left(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\right)\end{aligned}$$

Hence

$$\begin{aligned}T_{zz} &= \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11})\end{aligned}$$

4.2 Schwarzian derivative

$$\tilde{T}(z + \epsilon(z)) = (1 + \partial_z\epsilon(z))^{-2} \left[T(z) - \frac{c}{12} \left(\frac{\partial_z^3\epsilon(z)}{1 + \partial_z\epsilon(z)} - \frac{2}{3} \frac{\partial_z^2\epsilon(z)}{1 + \partial_z\epsilon(z)} \right) \right]$$

Since

$$\frac{1}{1 + \partial_z\epsilon(z)} = 1 - \partial_z\epsilon(z) + (\partial_z\epsilon)^2 + \dots$$

we have

$$\begin{aligned}\tilde{T}(z + \epsilon(z)) &\approx T(z)(1 - 2\partial_z \epsilon(z)) - \frac{c}{12}(\partial_z^3 \epsilon(z) - \frac{2}{3}\partial_z^2 \epsilon(z)) \\ &\approx T(z) - 2\partial_z \epsilon(z)T(z) - \frac{c}{12}\partial_z^3 \epsilon(z)\end{aligned}$$

Hence

$$\tilde{T}(z + \epsilon(z)) - [\epsilon(z)\partial_z T(z) + T(z)] \approx -\frac{c}{12}\partial_z^3 \epsilon(z) - 2\partial_z \epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$$

It implies that $\delta_\epsilon(T(z)) = -\frac{c}{12}\partial_z^3 \epsilon(z) - 2\partial_z \epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$.

4.3 Virasoro algebra

The Laurent expansion of z^{n+1} around ω is

$$z^{n+1} = (z - \omega)^{n+1} + \binom{n+1}{1}\omega(z - \omega)^n + \dots + \binom{n+1}{i}\omega^i(z - \omega)^{n+1-i} + \dots + \omega^{n+1}$$

Hence we have following residues:

$$\begin{aligned}\text{Res}_\omega \frac{z^{n+1}}{(z - \omega)^4} &= 2\pi i \frac{(n+1)n(n-1)}{6} \omega^{n-2} \\ \text{Res}_\omega \frac{z^{n+1}}{(z - \omega)^2} &= 2\pi i (n+1) \omega^n \\ \text{Res}_\omega \frac{z^{n+1}}{z - \omega} &= 2\pi i \omega^{n+1}\end{aligned}$$

Hence we have

$$\begin{aligned}[L_n, L_m] &= \frac{1}{(2\pi i)^2} \oint_0 d\omega \omega^{m+1} \oint_\omega dz z^{n+1} \left(\frac{c}{2(z - \omega)^4} + \frac{2T(\omega)}{(z - \omega)^2} + \frac{\partial_\omega T(\omega)}{(z - \omega)} + \text{regular part} \right) \\ &= \frac{1}{2\pi i} \oint_0 d\omega \omega^{m+1} \left(\frac{c}{12}(n+1)n(n-1)\omega^{n-2} + 2(n+1)\omega^n T(\omega) + \omega^{n+1}\partial_\omega T(\omega) \right) \\ &= \frac{1}{2\pi i} \left\{ \oint_0 d\omega \left(\frac{c(n+1)n(n-1)}{12} \omega^{m+n-1} \right) - (m-n) \oint_0 d\omega \omega^{m+n+1} T(\omega) \right\} \\ &= \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} - (m-n) L_{n+m}\end{aligned}$$

4.4 Commutation relations in free boson

We have

$$\varphi = \varphi_0 + \frac{4\pi}{l} \pi t + i \sum_{n \neq 0} \frac{1}{n} \left(a_n e^{-2\pi i n(t-x)/l} + \bar{a}_n e^{-2\pi i n(t+x)/l} \right)$$

on cylinder.

$$\Pi = \frac{\pi_0}{l} + \frac{1}{2l} \sum_{n \neq 0} \left(a_n e^{-2\pi i n(t-x)/l} + \bar{a}_n e^{-2\pi i n(t+x)/l} \right)$$

With $[\Pi, \Pi] = 0$, we can get $[\pi_0, a_n] = 0$ and $[\pi_0, \bar{a}_n] = 0$. Furthermore, since $[\varphi, \varphi] = 0$, we have $[\varphi_0, a_n] = [\varphi_0, \bar{a}_n] = 0$.

$$\begin{aligned}i\delta(x-y) = [\varphi(x, t), \Pi(y, t)] &= \frac{1}{l} [\varphi_0, \pi_0] + \frac{i}{2l} \sum_{n \neq 0, m \neq 0} \frac{1}{n} \left([a_n, \bar{a}_m] \exp(-2\pi i [(n+m)t - nx + my]/l) \right. \\ &\quad + [\bar{a}_n, a_m] \exp(-2\pi i [(n+m)t + nx - my]/l) \\ &\quad + [a_n, a_m] \exp(-2\pi i [(n+m)t - nx - my]/l) \\ &\quad \left. + [\bar{a}_n, \bar{a}_m] \exp(-2\pi i [(n+m)t + nx + my]/l) \right)\end{aligned}$$

Since t can be arbitrary, $[a_n, \bar{a}_m] = [\bar{a}_n, a_m] = 0$ when $n + m \neq 0$. Then let $t = 0$, we get

$$\begin{aligned} i\delta(x-y) &= [\varphi(x, 0), \Pi(y, 0)] \\ &= \frac{1}{l}[\varphi_0, \pi_0] + \frac{i}{2l} \sum_{n \neq 0} \frac{1}{n} \left([a_n, \bar{a}_{-n}] e^{-2\pi i(-nx-ny)/l} + [\bar{a}_n, a_{-n}] e^{-2\pi i(nx+ny)/l} + \text{other terms} \right) \\ &= \frac{1}{l}[\varphi_0, \pi_0] + \frac{i}{l} \sum_{n \neq 0} \left(\frac{2[a_n, \bar{a}_{-n}]}{n} \right) e^{2\pi i(n(x+y))/l} + \text{other terms} \end{aligned}$$

Since $e^{2\pi i(n(x+y))/l}$ is independent of $x - y$, then its coefficient is zero. Hence $[a_n, \bar{a}_{-n}] = 0$. Take integral of both left and right side, we can get $[\varphi_0, \pi_0] = i$ and $[a_n, a_{-n}] = [\bar{a}_n, \bar{a}_{-n}] = 1$.

4.5 Action of free fermion

$$S = \frac{1}{4\pi} \int d^2x \psi^\dagger \gamma^0 (\gamma^0 \partial_0 \psi + \gamma^1 \partial_1 \psi)$$

But

$$\gamma^0 (\gamma^0 + \gamma^1) \psi = \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} \psi$$

Write ψ as $\begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix}$, then we have

$$\gamma^0 (\gamma^0 + \gamma^1) \psi = \begin{pmatrix} 2\partial_{\bar{z}}\varphi \\ 2\partial_z\bar{\varphi} \end{pmatrix}$$

Hence

$$S = \frac{1}{2\pi} \int d^2x (\bar{\varphi} \partial_z \varphi + \varphi \partial_{\bar{z}} \bar{\varphi})$$

4.6 TT OPE in free fermion

By derivative, we get

$$\begin{aligned} \langle \psi(z), \partial_\omega \psi(\omega) \rangle &\sim \frac{1}{(z-\omega)^2} \\ \langle \partial_z \psi, \partial_\omega \psi \rangle &\sim \frac{-2}{(z-\omega)^3} \end{aligned}$$

Hence

$$\begin{aligned} T(z) \partial_\omega \psi(\omega) &= \frac{1}{2} : \psi(z) \partial_z \psi(z) : \partial_\omega \psi(\omega) \\ &\sim -\frac{\psi(\omega)}{(z-\omega)^3} - \frac{1}{2} \frac{\partial_\omega \psi(\omega)}{(z-\omega)^2} \end{aligned}$$

and

$$\begin{aligned} T(z)T(\omega) &= \frac{1}{4} : \psi(z) \partial_z \psi(z) :: \psi(\omega) \partial_\omega \psi(\omega) \\ &\sim \frac{1}{4} \left\{ -\frac{: \partial_z \psi(z) \partial_\omega \psi(\omega) :}{z-\omega} + \frac{2 : \psi(z) \psi(\omega) :}{(z-\omega)^3} - \frac{: \psi(z) \partial_\omega \psi(\omega) + \partial_z \psi(z) \psi(\omega) :}{(z-\omega)^2} + \frac{1}{(z-\omega)^4} \right\} \\ &\sim \frac{1}{4} \left\{ \frac{2\partial_\omega T(\omega)}{z-\omega} + \frac{4T(\omega)}{(z-\omega)^2} + \frac{1}{(z-\omega)^4} \right\} \end{aligned}$$

5 Vertex operator and OPE

If we write $\varphi(z, \bar{z})$ into laurent series since φ is free boson, then we can find $\exp(ik\varphi)$ is product of infinite exponential components which are commutative. Hence the normal ordering has taylor expansion form

$$:\exp(ik\varphi(z, \bar{z})):= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} : \varphi(z, \bar{z})^n :$$

To justify that $:\exp(ik\varphi):$ is primary field, we calculate its OPE

$$\begin{aligned} T(z) : \exp ik\varphi(\omega, \bar{\omega}) : &= -\frac{1}{2} \sum_{n=0}^{\infty} : \partial\varphi(z) \partial\varphi(z) :: \varphi(\omega, \bar{\omega})^n \\ &\sim -\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} n : \partial\varphi(z) \overbrace{\partial\varphi(z) \varphi(\omega, \bar{\omega})}^{n-1} \varphi(\omega, \bar{\omega})^{n-1} : \\ &\quad -\frac{1}{2} \sum_{n=2}^{\infty} \frac{(ik)^n}{n!} n(n-1) : \partial\varphi(z) \overbrace{\partial\varphi(z) \varphi(\omega, \bar{\omega}) \varphi(\omega, \bar{\omega})}^{n-2} \varphi(\omega, \bar{\omega})^{n-2} : \\ &\sim \frac{ik\partial_{\omega}\varphi(\omega)}{z-\omega} : \exp(ik\varphi) : + \frac{k^2}{2(z-\omega)^2} : \exp(ik\varphi) : \end{aligned}$$

This form implies that $:\exp(ik\varphi):$ is primary field with conformal dimension $\frac{k^2}{2}$.

6 bc ghost system

$$\begin{aligned} T(z)b(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :) b(\omega) \\ &\quad 2\frac{b(z)}{(z-\omega)^2} - \frac{\partial_z b(z)}{z-\omega} \end{aligned}$$

Take Taylor expansion of $b(z)$ and $\partial_z b(z)$ around ω , we have

$$T(z)b(\omega) \sim 2\frac{b(\omega)}{(z-\omega)^2} + \frac{\partial_{\omega} b(\omega)}{z-\omega}$$

Hence the conformal dimension of b is 2.

Similarly, we have

$$\begin{aligned} T(z)c(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :) c(\omega) \\ &\sim -\frac{c(z)}{(z-\omega)^2} + 2\frac{\partial_z c(z)}{z-\omega} \\ &\sim -\frac{c(\omega)}{(z-\omega)^2} + \frac{\partial_{\omega} c(\omega)}{z-\omega} \end{aligned}$$

Hence conformal dimension of c is -1.

$$\begin{aligned} T(z)T(\omega) &= 4 : b(z)\partial c(\omega) :: b(\omega)\partial c(\omega) : \\ &\quad -2 : c(z)\partial b(z) :: b(\omega)\partial c(\omega) : -2 : b(z)\partial c(z) :: c(\omega)\partial b(\omega) : \\ &\quad + c(z)\partial b(z) :: c(\omega)\partial b(\omega) : \end{aligned}$$

We will calculate it term by term

$$\begin{aligned}
& 4 : b(z) \partial c(z) :: b(\omega) \partial c(\omega) : \\
& \sim 4 \left(: \overbrace{b(z) \partial c(z) b(\omega) \partial c(\omega)} : + b(z) \overbrace{\partial c(z) b(\omega)} \partial c(\omega) + : \overbrace{b(z) \partial c(z) b(\omega)} \partial c(\omega) : \right) \\
& \sim \frac{4(- : \partial_z c(z) b(\omega) : + : b(z) \partial_\omega c(\omega) :)}{(z - \omega)^2} - \frac{4}{(z - \omega)^4} \\
& \sim -\frac{4}{(z - \omega)^4} + \frac{8b(\omega) \partial_\omega c(\omega)}{(z - \omega)^2} + \frac{-4 : \partial_\omega^2 c(\omega) b(\omega) : + 4 \partial_\omega b(z) \partial_\omega c(\omega)}{z - \omega}
\end{aligned}$$

and

$$\begin{aligned}
& 2 : c(z) \partial b(z) :: b(\omega) \partial c(\omega) : \\
& \sim 2 \left(- : \overbrace{c(z) \partial b(z) b(\omega)} \partial c(\omega) - : c(z) \overbrace{\partial b(z) b(\omega) \partial c(\omega)} - : \overbrace{\partial b(z) c(z) b(\omega)} \partial c(\omega) \right) \\
& \sim \frac{4}{(z - \omega)^4} + \frac{4 : c(z) b(\omega) :}{(z - \omega)^3} - \frac{2 : \partial b(z) \partial c(\omega)}{(z - \omega)} \\
& \sim \frac{4}{(z - \omega)^4} + \frac{4 : c(\omega) b(\omega) :}{(z - \omega)^3} + \frac{4 : \partial_\omega c(\omega) b(\omega) :}{(z - \omega)^2} + \frac{2 : \partial_\omega^2 c(\omega) b(\omega) : - 2 : \partial_\omega b(\omega) \partial_\omega c(\omega) :}{z - \omega}
\end{aligned}$$

and symmetrically

$$\begin{aligned}
& 2 : b(z) \partial c(z) :: c(\omega) \partial b(\omega) : \\
& \sim \frac{4}{(z - \omega)^4} + \frac{4 : b(\omega) c(\omega) :}{(z - \omega)^3} + \frac{4 : \partial_\omega b(\omega) c(\omega) :}{(z - \omega)^2} + \frac{2 : \partial_\omega^2 b(\omega) c(\omega) : - 2 : \partial_\omega c(\omega) \partial_\omega b(\omega) :}{z - \omega}
\end{aligned}$$

and

$$\begin{aligned}
& c(z) \partial b(z) :: c(\omega) \partial b(\omega) : \\
& \sim \frac{2 : c(\omega) \partial_\omega b(\omega) :}{(z - \omega)^2} + \frac{-\partial_\omega^2 b(\omega) c(\omega) + \partial_\omega c(\omega) \partial_\omega b(\omega)}{z - \omega} - \frac{1}{(z - \omega)^4}
\end{aligned}$$

Hence we have

$$T(z)T(\omega) \sim -\frac{13}{(z - \omega)^4} + \frac{2T(\omega)}{(z - \omega)^2} + \frac{\partial T(\omega)}{z - \omega}$$

So the central charge is equal to -26 .

7 Schwarzian derivatives

Let $\omega = \frac{az+b}{cz+d}$, then

$$\begin{aligned}
\frac{d\omega}{dz} &= \frac{ad - bc}{(cz + d)^2} \\
\frac{d^2\omega}{dz^2} &= \frac{-2c(ad - bc)}{(cz + d)^3} \\
\frac{d^3\omega}{dz^3} &= \frac{6c^2(ad - bc)}{(cz + d)^4}
\end{aligned}$$

Hence

$$\begin{aligned}
\{\omega, z\} &= \frac{6c^2}{(cz + d)^2} - \frac{3}{2} \left(\frac{-2c}{cz + d} \right)^2 \\
&= 0
\end{aligned}$$

Then

$$\begin{aligned}\left(\frac{a\omega + b}{c\omega + d}\right)'_z &= \frac{ad - bc}{(c\omega + d)^2} \omega'_z \\ \left(\frac{a\omega + b}{c\omega + d}\right)''_z &= \frac{-2c(ad - bc)}{(c\omega + d)^3} (\omega'_z)^2 + \frac{ad - bc}{(c\omega + d)^2} \omega''_z \\ \left(\frac{a\omega + b}{c\omega + d}\right)'''_z &= \frac{6c^2(ad - bc)}{(c\omega + d)^4} (\omega'_z)^3 + 3 \frac{-2c(ad - bc)}{(c\omega + d)^3} \omega''_z \omega'_z + \frac{ad - bc}{(c\omega + d)^2} \omega'''_z\end{aligned}$$

Hence we have

$$\left\{\frac{a\omega + b}{c\omega + d}, z\right\} = \frac{6c^2}{(c\omega + d)^2} (\omega'_z)^2 + 3 \frac{-2c}{c\omega + d} \omega''_z + \frac{\omega'''_z}{\omega'_z} - \frac{3}{2} \left(\frac{-2c}{c\omega + d} \omega'_z + \frac{\omega''_z}{\omega'_z} \right)^2$$

It is equal to $\{\omega, z\}$.

8 Modular invariant

$$Z_R(z, \bar{z}) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n} q^{\frac{1}{2}(\frac{m}{R} + \frac{Rn}{2})^2} \bar{q}^{\frac{1}{2}(\frac{m}{R} - \frac{Rn}{2})^2}$$

First, we compute the sum part. Let $z = x + iy$

$$\begin{aligned} \sum_{m,n} &= \sum_{m,n} q^{\frac{1}{2}(\frac{m}{R} + \frac{Rn}{2})^2} \bar{q}^{\frac{1}{2}(\frac{m}{R} - \frac{Rn}{2})^2} \\ &= \sum_{m,n} \exp \left\{ z\pi i \left(\frac{m}{R} + \frac{Rn}{2} \right)^2 - \bar{z}\pi i \left(\frac{m}{R} - \frac{Rn}{2} \right)^2 \right\} \\ &= \sum_{m,n} \exp \left\{ -\frac{2\pi y}{R^2} m^2 - \frac{\pi y R^2}{2} n^2 + 2\pi i x m n \right\} \\ &= \sum_{m,n} \exp \left\{ -\frac{2\pi y}{R^2} \left(m - \frac{R^2 x i}{2y} n \right)^2 \right\} \exp \left\{ -\frac{\pi R^2 x^2}{2y} n^2 - \frac{\pi R^2 y}{2} n^2 \right\} \end{aligned}$$

Let $a = \frac{R^2}{2y}$, $b = \frac{\pi R^2 x}{y} n$ in Poisson formula, then we have

$$\begin{aligned} &= \frac{R}{\sqrt{2y}} \sum_{m,n} \exp \left\{ -\frac{\pi R^2}{2y} m^2 + \frac{\pi R^2 x}{y} m n - \frac{\pi R^2 x^2}{2y} n^2 - \frac{\pi R^2}{2} n^2 \right\} \\ &= \frac{R}{\sqrt{2y}} \sum_{m,n} \exp \left\{ -\frac{\pi R}{2y} (m^2 - x m n + (x n)^2 + (y n)^2) \right\} \\ &= \frac{R}{\sqrt{2y}} \sum_{m,n} \exp \left\{ -\frac{\pi R}{2y} ((m - x n)^2 + (y n)^2) \right\} \\ &= \frac{R}{\sqrt{2y}} \sum_{m,n} \exp \left\{ -\frac{\pi R}{2y} |n z - m|^2 \right\} \end{aligned}$$

Hence when $z \mapsto -1/z$, the sum part becomes

$$\frac{R|z|}{\sqrt{2y}} \sum_{m,n} \exp \left\{ -\frac{\pi R}{2y} |n \bar{z} - m|^2 \right\} = \frac{R|z|}{\sqrt{2y}} \sum_{m,n} \exp \left\{ -\frac{\pi R}{2y} |n z - m|^2 \right\}$$

Since we have

$$\eta\left(-\frac{1}{z}\right) = \sqrt{-iz} \eta(z)$$

it norm is

$$|\eta\left(-\frac{1}{z}\right)| = \sqrt{|z|} |\eta(z)|$$

Hence we can conclude that

$$Z_R(z, \bar{z}) = Z_R\left(-\frac{1}{z}, -\frac{1}{\bar{z}}\right)$$

9 Modular transformation

We have

$$\begin{aligned}
\gamma \cdot \tau &= \frac{a\tau + b}{c\tau + d} \\
&= \frac{(a\tau + b)(c\bar{\tau} + d)}{|c\tau + d|^2} \\
&= \frac{ac|\tau|^2 + bc\bar{\tau} + ad\tau}{|c\tau + d|^2} \\
&= \frac{ac|\tau|^2 + (ad + bc) \operatorname{Re} \tau + i(ad - bc) \operatorname{Im} \tau}{|c\tau + d|^2}
\end{aligned}$$

Since $ad - bc = 1$, we can conclude that

$$\operatorname{Im}(\gamma \cdot \tau) = \frac{\operatorname{Im} \tau}{|c\tau + d|^2}$$

In the upper-half plane, the gray region can be described as

$$\begin{aligned}
-\frac{1}{2} &\leq \operatorname{Re} \tau \leq \frac{1}{2} \\
|\tau| &> 1
\end{aligned}$$

If S acts on the gray region, then we have $S(z) = -\frac{1}{z} = -\frac{\bar{z}}{|z|^2}$, hence it sends the region to the red region. And the blue region is the image of the gray origin under T . Finally, the transformation ST , maps to the green region.

10 Boson-Fermion correspondence

10.1 : $e^{i\varphi} : \cong \psi$, $e^{-i\varphi} \cong \bar{\psi}$

Calculate the OPE of $:e^{i\varphi}:$ directly

$$\begin{aligned}
:e^{i\varphi(z)}::e^{i\varphi(0)} &\sim \sum_{m,n,k} \frac{k!}{m!n!} \binom{m}{k} \binom{n}{k} (-\overline{\varphi(z)}\varphi(0)) : (i\varphi(0))^{m+n-k} : \\
&\sim \sum_{m,n,k} \frac{1}{k!} \ln^k(z) \frac{1}{(m-k)!} \frac{1}{(n-k)!} : i\varphi(0)^{m+n-k} : \\
&= e^{\ln z} : e^{2i\varphi(0)} : \\
&\sim 0
\end{aligned}$$

The OPE of T_φ with vertex operators $:e^{i\varphi}$ and $:e^{-i\varphi}$ is given in last homework

$$\begin{aligned}
T_\varphi(z) : e^{i\varphi} : &\sim \frac{\partial_z \exp(i\varphi)(0)}{z} + \frac{\exp(i\varphi)(0)}{2z^2} \\
T_\varphi(z) : e^{-i\varphi} : &\sim \frac{\partial_z \exp(-i\varphi)(0)}{z} + \frac{\exp(-i\varphi)(0)}{2z^2}
\end{aligned}$$

On the other hand, the OPE of T_ψ with ψ and $\bar{\psi}$ can be calculated as follows. First,

$$\begin{aligned}
\partial\bar{\psi}(z)\psi(0) &\sim \frac{1}{2}(\partial\psi^1(z) - i\partial\psi^2(z))(\psi^1(0) + i\psi^2(0)) \\
&\sim -\frac{1}{z^2}
\end{aligned}$$

$$\begin{aligned}\bar{\psi}(z)\partial\psi(0) &\sim \frac{1}{2}(\psi^1(z) - i\psi^2(z))(\partial\psi^1(0) + i\partial\psi^2(0)) \\ &\sim \frac{1}{z^2}\end{aligned}$$

and

$$\begin{aligned}\bar{\psi}(z)\psi(0) &\sim \frac{1}{2}(\psi^1(z) - i\psi^2(z))(\psi^1(0) + i\psi(0)) \\ &\sim \frac{1}{z}\end{aligned}$$

Hence

$$\begin{aligned}T_\varphi(z)\psi(0) &= -\frac{1}{2} : \psi \overline{\partial\psi} : \psi(0) - \frac{1}{2} : \overline{\psi} \partial\psi : \psi(0) \\ &\sim -\frac{1}{2}\psi(z)\left(-\frac{1}{z^2}\right) + \frac{1}{2}\frac{1}{z}\partial_z\psi(z) \\ &\sim \frac{\psi(0) + z\partial_z\psi(0)}{2z^2} + \frac{\partial_z\psi(0)}{2z} \\ &\sim \frac{\psi(0)}{2z^2} + \frac{\partial_z\psi(0)}{z}\end{aligned}$$

and

$$\begin{aligned}T_\varphi(z)\bar{\psi}(0) &= -\frac{1}{2} : \overline{\psi \partial\psi} : \bar{\psi}(0) - \frac{1}{2} : \bar{\psi} \overline{\partial\psi} : \bar{\psi}(0) \\ &\sim \frac{1}{2}\partial\bar{\psi}(z)\left(\frac{1}{z}\right) - \frac{1}{2}\bar{\psi}(z)\left(-\frac{1}{z^2}\right) \\ &\sim \frac{\bar{\psi}(0) + z\partial_z\bar{\psi}(0)}{2z^2} + \frac{\partial_z\bar{\psi}(0)}{2z} \\ &\sim \frac{\bar{\psi}(0)}{2z^2} + \frac{\partial_z\bar{\psi}(0)}{z}\end{aligned}$$

10.2 $i\partial\varphi \cong \psi\bar{\psi}$

Also, as calculated before, the OPE of $i\partial\varphi$ is as follows

$$\begin{aligned}T_\varphi(z)(i\partial\varphi(0)) &\sim \frac{i\partial\varphi(0)}{z^2} + \frac{i\partial_z^2\varphi(0)}{z} \\ T_\psi(z)\psi\bar{\psi}(0) &= -\frac{1}{2} : \psi(z)\partial\bar{\psi}(z) : \psi(0)\bar{\psi}(0) - \frac{1}{2} : \bar{\psi}(z)\partial\psi(z) : \psi(0)\bar{\psi}(0) \\ &\sim -\frac{1}{2} : \psi(z)\partial\overline{\psi(z)\psi(0)\bar{\psi}(0)} : - \frac{1}{2} : \overline{\psi(z)\partial\bar{\psi}(z)\psi(0)\bar{\psi}(0)} : \\ &\quad - \frac{1}{2} : \overline{\psi\partial\bar{\psi}\psi(0)\bar{\psi}(0)} : \\ &\quad - \frac{1}{2} : \bar{\psi}(z)\overline{\partial\psi(z)\psi(0)\bar{\psi}(0)} : - \frac{1}{2} : \bar{\psi}(z)\partial\overline{\psi(z)\psi(0)\bar{\psi}(0)} : \\ &\quad - \frac{1}{2} : \overline{\bar{\psi}(z)\partial\psi(z)\psi(0)\bar{\psi}(0)} : \\ &\sim \frac{\partial\bar{\psi}(0)\psi(0) - \partial\psi(0)\bar{\psi}(0) + \partial\psi(0)\bar{\psi}(0) - \partial\bar{\psi}(0)\psi(0)}{2z} \\ &\quad + \frac{\bar{\psi}(0)\psi(0) - \psi(0)\bar{\psi}(0)}{2z^2} \\ &\sim \frac{\partial(\psi\bar{\psi})(0)}{z} + \frac{\psi\bar{\psi}(0)}{z^2}\end{aligned}$$

(Tips: I'm confused here. Is the i in $i\partial\varphi$ necessary? I failed to get it in Fermion side.)

10.3 $T_\varphi \cong T_\psi$

Finally, we compute the TT OPE for φ and ψ . We have

$$T_\varphi(z)T_\varphi(0) \sim \frac{1/2}{z^4} + \frac{2T_\varphi(0)}{z^2} + \frac{\partial T_\varphi(0)}{z}$$

In complex Fermion case, we have

$$\begin{aligned} T_\psi(z)T_\psi(0) &\sim \frac{1}{4} \left[: \psi(z)\partial\bar{\psi}(z) :: \psi(0)\partial\bar{\psi}(0) : + : \bar{\psi}\partial\psi(z) :: \psi(0)\partial\bar{\psi}(0) : \right. \\ &\quad \left. + : \psi(z)\partial\bar{\psi}(z) :: \bar{\psi}(0)\partial\bar{\psi}(0) : + : \bar{\psi}(z)\partial\psi(z) :: \bar{\psi}(0)\partial\psi(0) : \right] \\ &\sim \frac{1}{4} \left[: \overline{\psi(z)\partial\bar{\psi}(z)\psi(0)\partial\bar{\psi}(0)} : + : \overline{\psi(z)\partial\bar{\psi}(z)\psi(0)\partial\bar{\psi}(0)} : + : \overline{\psi(z)\partial\bar{\psi}(z)\psi(0)\partial\bar{\psi}(0)} : \right] \\ &\quad + \frac{1}{4} \left[: \overline{\bar{\psi}(z)\partial\psi(z)\psi(0)\partial\bar{\psi}(0)} : + : \overline{\bar{\psi}(z)\partial\psi(z)\psi(0)\partial\bar{\psi}(0)} : + : \overline{\bar{\psi}(z)\partial\psi(z)\psi(0)\partial\bar{\psi}(0)} : \right] \\ &\quad + \frac{1}{4} \left[: \overline{\psi(z)\partial\bar{\psi}(z)\bar{\psi}(0)\partial\psi(0)} : + : \overline{\psi(z)\partial\bar{\psi}(z)\bar{\psi}(0)\partial\psi(0)} : + : \overline{\psi(z)\partial\bar{\psi}(z)\bar{\psi}(0)\partial\psi(0)} : \right] \\ &\quad + \frac{1}{4} \left[: \overline{\bar{\psi}(z)\partial\psi(z)\bar{\psi}(0)\partial\psi(0)} : + : \overline{\bar{\psi}(z)\partial\psi(z)\bar{\psi}(0)\partial\psi(0)} : + : \overline{\bar{\psi}(z)\partial\psi(z)\bar{\psi}(0)\partial\psi(0)} : \right] \\ &\sim \frac{1/2}{z^4} + \frac{2T_\psi(0)}{z^2} + \frac{\partial T_\psi(0)}{z} \end{aligned}$$

11 Null states at level 3

With commutation relation of Virasoro algebra, we have

$$[L_1 L_{-3}]|\phi_I\rangle = 4L_{-2}|\varphi_I\rangle$$

$$\begin{aligned} [L_1, L_{-1} L_{-2}]|\phi_I\rangle &= \{[L_1, L_{-1}]L_{-2} + L_1[L_{-1}, L_{-2}]\}|\phi_I\rangle \\ &= \{2L_0 L_{-2} + 3L_{-1}^2\}|\phi_I\rangle \\ &= \{2[L_0, L_{-2}] + 2L_{-2}L_0 + 3L_{-1}^2\}|\phi_I\rangle \\ &= \{2(2 + h_I)L_{-2} + 3L_{-1}^2\}|\phi_I\rangle \end{aligned}$$

and

$$\begin{aligned} [L_1, L_{-1}^3]|\phi_I\rangle &= \{[L_1, L_{-1}]L_{-1}^2 + L_{-1}[L_1, L_{-1}]L_{-1} + L_{-1}^2[L_{-1}, L_{-1}]\}|\phi_I\rangle \\ &= \{2L_0 L_{-1}^2 + 2L_{-1}L_0 L_{-1} + 2L_{-1}^2 L_0\}|\phi_I\rangle \\ &= \{6 + 6h_I\}L_{-1}^2|\phi_I\rangle \end{aligned}$$

Suppose the

$$|\chi_I\rangle = \alpha L_{-3} + \beta L_{-1} L_{-2} + L_{-1}^3|\phi_I\rangle$$

since $L_1|\chi_I\rangle = 0$, we get equations

$$4\alpha + 2(2 + h_I)\beta = 0 \tag{2}$$

$$\beta + 2h_I + 2 = 0 \tag{3}$$

The solution of this system is

$$\beta = -2(h_I + 1) \quad \alpha = (h_I + 1)(h_I + 2)$$

Now, we have

$$|\chi_I\rangle = \left[(h_I + 1)(h_I + 2)L_{-3} - 2(h_I + 1)L_{-1}L_{-2} + L_{-1}^3 \right]|\phi_I\rangle$$

Then, we write L_{-3}, L_{-1}, L_{-2} into differential operators as

$$\begin{aligned}\mathcal{L}_{-3} &= \sum_{i=1}^N \frac{2h_i}{(\omega_i - \omega)^3} - \frac{1}{(\omega_i - \omega)^2} \partial_{\omega_i} \\ \mathcal{L}_{-2} &= \sum_{i=1}^N \frac{h_i}{(\omega_i - \omega)^2} - \frac{1}{\omega_i - \omega} \partial_{\omega_i} \\ \mathcal{L}_{-1} &= - \sum_{i=1}^N \partial_{\omega_i}^3\end{aligned}$$

We have following differential equations

$$\left[(h_I + 1)(h_I + 2)\mathcal{L}_{-3} - 2(h_I + 1)\mathcal{L}_{-1}L_{-2} + \mathcal{L}_{-1}^3 \right] \langle \phi_I(\omega) \phi_{\bar{I}}(z) \rangle$$

In this case, we have

$$\begin{aligned}\mathcal{L}_{-3} &= \frac{2h_{\bar{I}}}{(z - \omega)^3} - \frac{1}{(z - \omega)^2} \partial_z \\ \mathcal{L}_{-2} &= \frac{h_{\bar{I}}}{(z - \omega)^2} - \frac{1}{z - \omega} \partial_z \\ \mathcal{L}_{-1} &= -\partial_z = \partial_\omega\end{aligned}$$

so we further have

$$\mathcal{L}_{-1}\mathcal{L}_{-2} = -\mathcal{L}_{-3} - \frac{1}{z - \omega} \partial_z^2$$

Take them into equation, we get

$$\left\{ \frac{2(h_I + 1)(h_I + 4)h_{\bar{I}}}{(z - \omega)^3} - \frac{(h_I + 1)(h_I + 4)}{(z - \omega)^2} \partial_z + \frac{2(h_I + 1)}{z - \omega} \partial_z^2 - \partial_z^3 \right\} \langle \phi_I(\omega) \phi_{\bar{I}}(z) \rangle = 0$$

12 Minimal models

The formula (5.40) implies for minimal model $\mathcal{M}_{2,2n+1}$, the central charge is

$$\begin{aligned}c &= 1 - 6 \frac{(2 - 2n - 1)^2}{2(2n + 1)} \\ &= - \frac{2(6n - 1)(n - 1)}{2n + 1}\end{aligned}$$

And in this model, the formulas (5.44) and (5.45) becomes as follows for fusion rule $\phi_{(r,s)} \times \phi_{(a,b)}$,

$$k_{\max} = \begin{cases} a + r - 1 & \text{if } 2 \leq a + r \leq 2n + 1 \\ 2(2n + 1) - (a + r - 1) & \text{if } 2n + 1 < a + r \leq 4n \end{cases}$$

and

$$l_{\max} = 1 \quad (s, b) = (1, 1)$$

So we have fusion rules

$$\begin{aligned}\phi_{(r,s)} \times \phi_{(a,b)} &= \phi_{(r,1)} \times \phi_{(a,1)} \\ &= \sum_{k=1+|r-a|, k+r+a=1 \pmod{2}}^{k_{\max}} \phi_{(k,1)}\end{aligned}$$

If $2 \leq r + a \leq 2n + 1$, then we have

$$\sum_{k=1+|r-a|, k+r+a=1 \pmod{2}}^{r+a-1} \phi_{(k,1)} = \sum_{j=0}^{a-1} \phi_{(r-a+1+2j,1)}$$

If $2n + 1 < r + a \leq 4n$, then we have

$$\sum_{k=1+|r-a|, k+r+a=1 \pmod{2}}^{4n+1-(r+a)} \phi_{(k,1)} = \sum_{j=0}^{2n-r} \phi_{(r-a+1+2j,1)}$$

13 Star-triangle relation

To derive eq2, we have

$$\begin{aligned} 2 \cosh(L\sigma_i + L(\sigma_j + \sigma_k)) &= 2(\cosh(L\sigma_i) \cosh(L(\sigma_j + \sigma_k)) + \sinh(L\sigma_i) \sinh(L(\sigma_j + \sigma_k))) \\ &= 2 \cosh L (\cosh L \cosh L + \sigma_j \sigma_k \sinh L \sinh L) \\ &\quad + 2 \sigma_i \sinh L (\sigma_j \sinh L \cosh L + \sigma_k \cosh L \sinh L) \\ &= 2 \cosh^3 L + 2 \cosh L \sinh^2 L [\sigma_i \sigma_j + \sigma_i \sigma_k + \sigma_j \sigma_k] \end{aligned}$$