2d CFT

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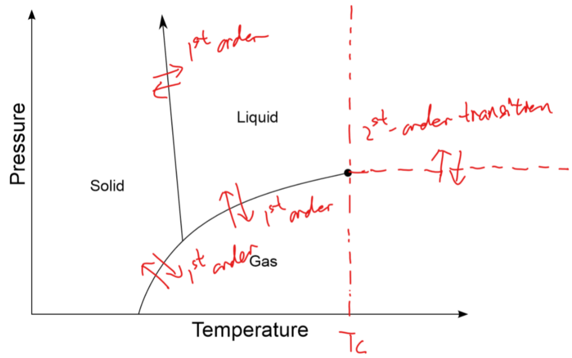
Week 1

Exercies 0.0.1. The first order transitions and second order transitions show in the diagram



Exercies 0.0.2. By the homogeneous relation

$$f(t,h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

we have

$$f(t,h) = t^{\frac{d}{y_t}} g(\alpha)$$

where $g(\alpha) = f(1,\alpha)$ and $\alpha = t^{-\frac{y_h}{y_t}} h$. It is easy to see that α is invariance under scaling transformation $x \to x/b$. Hence we have

$$C(t,0) = -T\frac{\partial^2 f}{\partial T^2}\Big|_{h=0} = -\frac{1}{T_c} t^{\frac{d}{y_t}-2} g''(0)$$
$$M(t,0) = -\frac{\partial f}{\partial B}\Big|_{h=0} = t^{\frac{d-y_h}{y_t}} g'(0)$$
$$\chi(t,0) = \frac{\partial^2 f}{\partial B^2}\Big|_{h=0} = t^{(d-2y_h)/y_t} g''(0)$$

As function with single variable h, $\lim_{t\to 0} M(t,h) \sim h^{\frac{1}{\delta}}$, which implies that $g'(\alpha) \sim \alpha^{\frac{1}{\delta}}$ since α is linear function of h. Hence we have

$$\lim_{t\to 0} M = \lim_{t\to 0} t^{(d-y_n-\frac{y_n}{\delta})} h^{1/\delta}$$

since it is non-zero, we have $d - y_n - y_n \frac{1}{\delta} = 0$. Hence we have

$$\delta = \frac{y_h}{d-y_h}$$

Exercies 0.0.3. We have following relation

$$G_\sigma(\mathbf{r};t,h) = t^{-2x_\sigma} G_\sigma(\frac{\mathbf{r}}{b}; b^{y_t} t, b^{y_h} h) \tag{1}$$

Let $h = 0, K = b^{y_t} t$,

$$G_\sigma(\mathbf{r};t,0) = t^{2x_\sigma/y_t} G_\sigma(\frac{\mathbf{r}}{Kt^{-1/y_t}}; K, 0)$$

Since $G_\sigma(\mathbf{r}) \sim r^{-\tau} e^{-\frac{r}{\xi}}$, we have $\xi \sim t^{-1/y_t}$. It implies $\nu = 1/y_t$. With relation 1, we have

$$\chi(t,h) = \frac{1}{T}\int d^d\mathbf{r} G_\sigma(\mathbf{r};t,h) = t^{d-2x_\sigma}\chi(b^{y_t}t, b^{y_h}h)$$

So $\gamma = (d-2x_\sigma)/y_t$. But we have $\eta = 2x_\sigma + 2 - d$ for finite limit of $G(r)$ when $t \to 0$ and $h = 0$. Therefore, we get

$$\gamma = \nu(2-\eta)$$

With scaling relations

$$\alpha + 2\beta + \gamma = 2$$
$$\alpha + \beta(1+\delta) = 2$$

and $\alpha = 2 - d\nu$, we have

$$\beta = \frac{d\nu - 2\nu + \nu\eta}{2}$$
$$\delta = \frac{d-\eta+2}{d+\eta-2}$$

Exercies 0.0.4. By listed commutation relations, we have, for $r, s > 0$,

$$\begin{aligned}[D, J_{rs}] &= [D, L_{rs}] = \frac{i}{2}[D,[K_r,P_s]] \\ &= -\frac{i}{2}\big([P_s,[D,K_r]] + [K_r,[P_s,D]]\big) \\ &= \frac{1}{2}[P_s,K_r] - \frac{1}{2}[K_r,P_s] \\ &= 0\end{aligned}$$

For $r = -1, s = 0$, $[D, J_{rs}] = [D, D] = 0$. For $r = -1, s \neq 0$, $[D, J_{-1,s}] = [D, \frac{1}{2}(P_s - K_s)] = \frac{i}{2}(P_s + K_s)$. For $r = 0$, $[D, J_{0s}] = \frac{i}{2}(P_s - K_s)$. Hence (2,25) is satisfied when $(m,n) = (-1,0)$.

If $(m,n) = (-1,n)$, then we have

$$[J_{mn}, J_{rs}] = \frac{1}{2}[P_n, J_{rs}] - \frac{1}{2}[K_n, J_{rs}]$$

With listed commutation relations, we can easily check it coincides with (2,25) respectively. Similarly check in the case of $(m,n) = (0,n)$.

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1 $SL_2(\mathbb{C})$

Exercies 1.0.1. We have $\det X = t^2 - (x^2 + y^2 + z^2)$. Since points in $\mathbb{R}^{1,3}$ can be written with Pauli matrix as base. Elements in $SO(1,3)$ can be viewed as action on $M_2(\mathbb{C})$ with form $P \mapsto PXP^*$, which preserve det of X. We have exact sequence of groups as follows:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{C}) \xrightarrow{sp} SO(1,3) \longrightarrow 1$$

where sp is map $P \mapsto (X \mapsto PXP^*)$. Since for $P \in SL_2(\mathbb{C})$, $\det(PXP*) = \det(X) = t^2 - (x^2 + y^2 + z^2)$, sp is well-defined. Hence $SO(1,3) \cong SL_2(\mathbb{C})/\mathbb{Z}_2$.

Exercies 1.0.2.

- $$z \mapsto \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}$$
- We have
$$\begin{aligned}(w_1 - w_3) =& \frac{(az_1 + b)(cz_3 + d) - (az_2 + b)(cz_4 + d)}{(cz_1 + d)(cz_3 + d)} \\ =& \frac{z_1 - z_3}{(cz_1 + d)(cz_3 + d)}\end{aligned}$$
  Hence we have $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$.

2 Three-point function

Exercies 2.0.1. Let $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = f(z_{12}, z_{23}, z_{13})$. Under scalar transformation $z_i \mapsto \lambda z_i$, we have

$$f(z_{12}, z_{23}, z_{13}) = \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13})$$

Therefore, f is with form

$$f(z_{12}, z_{23}, z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$$

where $a + b + c = h_1 + h_2 + h_3$. Then under comformal transformation $z_i \mapsto \frac{1}{z_i}$, we have

$${z_1}^{-2h_1} {z_2}^{-2h_2} {z_3}^{-2h_3} \frac{(z_1z_2)^a (z_2z_3)^b (z_1z_3)^c}{z_{12}^a z_{23}^b z_{13}^c} = \frac{1}{z_{12}^a z_{23}^b z_{13}^c}$$

Hence $a = h_1 + h_2 - h_3, b = h_2 + h_3 - h_1, c = h_1 + h_3 - h_2$.

3 Energy-momentum tensor

Exercies 3.0.1.

- $$T^{\mu\nu} = -\eta^{\mu\nu}\partial_k\varphi\partial^k\varphi + 2\partial^\mu\varphi\partial^\nu\varphi$$
- We have
$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}$$
  Therefore,
$$\begin{aligned}\tilde{T}^{\mu\nu} &= \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{2}(-\delta_{\mu\nu} + 2)\partial_\mu\varphi\partial_\nu\varphi\end{aligned}$$

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4 Derivations

4.1 Energy-momentum tensor in complex coordinate

Since

$$\partial_0\Phi = \partial_z\Phi + \partial_{\bar{z}}\Phi$$
$$\partial_1\Phi = i\partial_z\Phi - i\partial_{\bar{z}}\Phi$$

we have

$$\partial_z\Phi = \frac{1}{2}\partial_0\Phi - \frac{i}{2}\partial_1\Phi$$
$$\partial_{\bar{z}}\Phi = \frac{1}{2}\partial_0\Phi + \frac{i}{2}\partial_1\Phi$$

Also, since there are metric tensors in complex coordinates

$$g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad g_{\alpha\beta} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

, we have $\partial^z\Phi = 2\partial_{\bar{z}}$ and $\partial^{\bar{z}}\Phi = 2\partial_{\bar{z}}\Phi$. Therefore, from definition of energy-momentum tensor

$$T_{\alpha\beta} = -g_{\alpha\beta}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial^\alpha\Phi)}\partial_\beta\Phi$$

we get expression of them in complex coordinates

$$T_{zz} = \frac{1}{4}\big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\big)$$
$$T_{\bar{z}\bar{z}} = \frac{1}{4}\big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\big)$$
$$T_{z\bar{z}} = -\frac{1}{2}\mathcal{L} + \frac{1}{4}\big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\big)$$
$$T_{z\bar{z}} = -\frac{1}{2}\mathcal{L} + \frac{1}{4}\big(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi\big)$$

Hence

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})$$
$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11})$$
$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11})$$

4.2 Schwarzian derivative

$$\tilde{T}(z+\epsilon(z)) = (1+\partial_z\epsilon(z))^{-2}\Big[T(z) - \frac{c}{12}\big(\frac{\partial_z^3\epsilon(z)}{1+\partial_z\epsilon(z)} - \frac{2}{3}\frac{\partial_z^2\epsilon(z)}{1+\partial_z\epsilon(z)}\big)\Big]$$

Since

$$\frac{1}{1+\partial_z\epsilon(z)} = 1 - \partial_z\epsilon(z) + (\partial_z\epsilon)^2 + \cdots$$

we have

$$\begin{aligned}\tilde{T}(z+\epsilon(z)) &\approx T(z)(1-2\partial_z\epsilon(z)) - \frac{c}{12}(\partial_z^3\epsilon(z) - \frac{2}{3}\partial_z^2\epsilon(z)) \\ &\approx T(z) - 2\partial_z\epsilon(z)T(z) - \frac{c}{12}\partial_z^3\epsilon(z)\end{aligned}$$

Hence

$$\tilde{T}(z+\epsilon(z)) - \big[\epsilon(z)\partial_z T(z) + T(Z)\big] \approx -\frac{c}{12}\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$$

It implies that $\delta_\epsilon(T(z)) = -\frac{c}{12}\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\partial_z T(z)$.

4.3 Virasoro algebra

The Larrent expansion of z^{n+1} around ω is

$$z^{n+1} = (z-\omega)^{n+1} + \binom{n+1}{1}\omega(z-\omega)^n + \cdots + \binom{n+1}{i}\omega^i(z-\omega)^{n+1-i} + \cdots + \omega^{n+1}$$

Hence we have following residues:

$$\begin{aligned}\mathrm{Res}_\omega \frac{z^{n+1}}{(z-\omega)^4} &= 2\pi i\frac{(n+1)n(n-1)}{6}\omega^{n-2} \\ \mathrm{Res}_\omega \frac{z^{n+1}}{(z-\omega)^2} &= 2\pi i(n+1)\omega^n \\ \mathrm{Res}_\omega \frac{z^{n+1}}{z-\omega} &= 2\pi i\omega^{n+1}\end{aligned}$$

Hence we have

$$\begin{aligned}&[L_n, L_m] = \frac{1}{(2\pi i)^2}\oint_0 d\omega\ \omega^{m+1}\oint_\omega dz\ z^{n+1}\big(\frac{c}{2(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial_\omega T(\omega)}{(z-\omega)} + \text{regular part}\big) \\ &= \frac{1}{2\pi i}\oint_0 d\omega\ \omega^{m+1}\big(\frac{c}{12}(n+1)n(n-1)\omega^{n-2} + 2(n+1)\omega^n T(\omega) + \omega^{n+1}\partial_\omega T(\omega)\big) \\ &= \frac{1}{2\pi i}\Big\{\oint_0 d\omega\ \Big(\frac{c(n+1)n(n-1)}{12}\omega^{m+n-1})\Big) - (m-n)\oint_0 d\omega\omega^{m+n+1}T(\omega)\Big\} \\ &= \frac{c}{12}n(n^2-1)\delta_{n+m,0} - (m-n)L_{n+m}\end{aligned}$$

4.4 Commutation relations in free boson

4.5 Hamitonian in free boson

4.6 Action of free fermion

$$S = \frac{1}{4\pi}\int d^2x\psi^\dagger\gamma^0(\gamma^0\partial_0\psi + \gamma^1\partial_1\psi)$$

But

$$\gamma^0(\gamma^0+\gamma^1)\psi = \begin{pmatrix}\partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1\end{pmatrix}\psi$$

Write ψ as $\begin{pmatrix}\varphi \\ \bar{\varphi}\end{pmatrix}$, then we have

$$\gamma^0(\gamma^0+\gamma^1)\psi = \begin{pmatrix}2\partial_{\bar{z}}\varphi \\ 2\partial_z\bar{\varphi}\end{pmatrix}$$

Hence

$$S = \frac{1}{2\pi}\int d^2x(\bar{\varphi}\partial_z\bar{\varphi} + \varphi\partial_{\bar{z}}\varphi)$$

4.7 TT OPE in free fermion

By derivative, we get

$$\langle \psi(z), \partial_\omega \psi(\omega)\rangle \sim \frac{1}{(z-\omega)^2}$$
$$\langle \partial_z\psi, \partial_\omega \psi\rangle \sim \frac{-2}{(z-\omega)^3}$$

Hence

$$\begin{aligned} T(z)\partial_\omega\psi(\omega) &= \frac{1}{2} : \psi(z)\partial_z\psi(z) : \partial_\omega\psi(\omega) \\ &\sim -\frac{\psi(\omega)}{(z-\omega)^3} - \frac{1}{2}\frac{\partial_\omega\psi(\omega)}{(z-\omega)^2} \end{aligned}$$

and

$$\begin{aligned} T(z)T(\omega) &= \frac{1}{4} : \psi(z)\partial_z\psi(z) :: \psi(\omega)\partial_\omega\psi(\omega) \\ &\sim \frac{1}{4}\{ -\frac{:\partial_z\psi(z)\partial_\omega\psi(\omega)}{z-\omega} + \frac{2:\psi(z)\psi(\omega):}{(z-\omega)^3} - \frac{:\psi(z)\partial_\omega\psi(\omega) + \partial_z\psi(z)\psi(\omega):}{(z-\omega)^2} + \frac{1}{(z-\omega)^4}\} \\ &\sim \frac{1}{4}\{\frac{2\partial_\omega T(\omega)}{z-\omega} + \frac{4T(\omega)}{(z-\omega)^2} + \frac{1}{(z-\omega)^4}\} \end{aligned}$$

5 Vertex operator and OPE

If we write $\varphi(z,\bar{z})$ into laurent series since φ is free boson, then we can find $\exp(ik\varphi)$ is product of infinite exponential components which are commutative. Hence the normal ordering has taylor expansion form

$$: \exp(ik\varphi(z,\bar{z})) := \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} : \varphi(z,\bar{z})^n :$$

To justify that $:\exp(ik\varphi):$ is primary field, we calculate its OPE

$$\begin{aligned} T(z) : \exp ik\varphi(\omega,\bar{\omega}) : &= -\frac{1}{2}\sum_{n=0}^{\infty} : \partial\varphi(z)\partial\varphi(z) :: \varphi(\omega,\bar{\omega})^n \\ &\sim -\sum_{n=1}^{\infty}\frac{(ik)^n}{n!} n : \partial\varphi(z)\, \overbrace{\partial\varphi(z)\varphi(\omega,\bar{\omega})}\, \varphi(\omega,\bar{\omega})^{n-1} : \\ &- \frac{1}{2}\sum_{n=2}^{\infty}\frac{(ik)^n}{n!} n(n-1) : \overbrace{\partial\varphi(z)\, \overbrace{\partial\varphi(z)\varphi(\omega,\bar{\omega})}\, \varphi(\omega,\bar{\omega})}\, \varphi(\omega,\bar{\omega})^{n-2} : \\ &\sim \frac{ik\partial_\omega\varphi(\omega)}{z-\omega} : \exp(ik\varphi) : + \frac{k^2}{2(z-\omega)^2} : \exp(ik\varphi) : \end{aligned}$$

This form implies that $:\exp(ik\varphi):$ is primary field with conformal dimension $\frac{k^2}{2}$.

6 bc ghost system

$$\begin{aligned} T(z)b(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :)b(\omega) \\ &2\frac{b(z)}{(z-\omega)^2} - \frac{\partial_z b(z)}{z-\omega} \end{aligned}$$

Take Taylor expansion of $b(z)$ and $\partial_z b(z)$ around ω, we have

$$T(z)b(w) \sim 2\frac{b(\omega)}{(z-\omega)^2} + \frac{\partial_\omega b(\omega)}{z-\omega}$$

Hence the conformal dimension of b is 2.

Similarly, we have

$$\begin{aligned} T(z)c(\omega) &= (-2 : b(z)\partial c(z) : + : c(z)\partial b(z) :)c(\omega) \\ &\sim -\frac{c(z)}{(z-\omega)^2} + 2\frac{\partial_z c(z)}{z-\omega} \\ &\sim -\frac{c(\omega)}{(z-\omega)^2} + \frac{\partial_\omega c(\omega)}{z-\omega} \end{aligned}$$

Hence conformal dimension of c is -1.

$$\begin{aligned} T(z)T(\omega) = {}& 4 : b(z)\partial c(\omega) :: b(\omega)\partial c(\omega) : \\ & - 2 : c(z)\partial b(z) :: b(\omega)\partial c(\omega) : -2 : b(z)\partial c(z) :: c(\omega)\partial b(\omega) : \\ & + c(z)\partial b(z) :: c(\omega)\partial b(\omega) : \end{aligned}$$

We will calculate it term by term

$$\begin{aligned} & 4 : b(z)\partial c(z) :: b(\omega)\partial c(\omega) : \\ & \sim 4\Big(: \overbrace{b(z)\partial c(z) b(\omega)\partial c(\omega)} : + b(z)\overbrace{\partial c(z) b(\omega)}\partial c(\omega) + : \overbrace{b(z)\overbrace{\partial c(z) b(\omega)}\partial c(\omega)} : \Big) \\ & \sim \frac{4(- : \partial_z c(z) b(\omega) : + : b(z)\partial_\omega c(\omega) :)}{(z-\omega)^2} - \frac{4}{(z-\omega)^4} \\ & \sim -\frac{4}{(z-\omega)^4} + \frac{8b(\omega)\partial_\omega c(\omega)}{(z-\omega)^2} + \frac{-4 : \partial_\omega^2 c(\omega) b(\omega) : + 4\partial_\omega b(z)\partial_\omega c(\omega)}{z-\omega} \end{aligned}$$

and

$$\begin{aligned} & 2 : c(z)\partial b(z) :: b(\omega)\partial c(\omega) : \\ & \sim 2\Big(- : \overbrace{c(z)\partial b(z) b(\omega)}\partial c(\omega) - : c(z)\overbrace{\partial b(z) b(\omega)\partial c(\omega))} - : \overbrace{\partial b(z)\overbrace{c(z) b(\omega)}\partial c(\omega)} \Big) \\ & \sim \frac{4}{(z-\omega)^4} + \frac{4 : c(z) b(\omega) :}{(z-\omega)^3} - \frac{2 : \partial b(z)\partial c(\omega)}{(z-\omega)} \\ & \sim \frac{4}{(z-\omega)^4} + \frac{4 : c(\omega) b(\omega) :}{(z-\omega)^3} + \frac{4 : \partial_\omega c(\omega) b(\omega) :}{(z-\omega)^2} + \frac{2 : \partial_\omega^2 c(\omega) b(\omega) : -2 : \partial_\omega b(\omega)\partial_\omega c(\omega) :}{z-\omega} \end{aligned}$$

and symmetrically

$$\begin{aligned} & 2 : b(z)\partial c(z) :: c(\omega)\partial b(\omega) \\ & \sim \frac{4}{(z-\omega)^4} + \frac{4 : b(\omega) c(\omega) :}{(z-\omega)^3} + \frac{4 : \partial_\omega b(\omega) c(\omega) :}{(z-\omega)^2} + \frac{2 : \partial_\omega^2 b(\omega) c(\omega) : -2 : \partial_\omega c(\omega)\partial_\omega b(\omega) :}{z-\omega} \end{aligned}$$

and

$$\begin{aligned} & c(z)\partial b(z) :: c(\omega)\partial b(\omega) : \\ & \sim \frac{2 : c(\omega)\partial_\omega b(\omega)}{(z-\omega)^2} + \frac{-\partial_\omega^2 b(\omega) c(\omega) + \partial_\omega c(\omega)\partial_\omega b(\omega)}{z-\omega} - \frac{1}{(z-\omega)^4} \end{aligned}$$

Hence we have

$$T(z)T(\omega) \sim -\frac{13}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}$$