

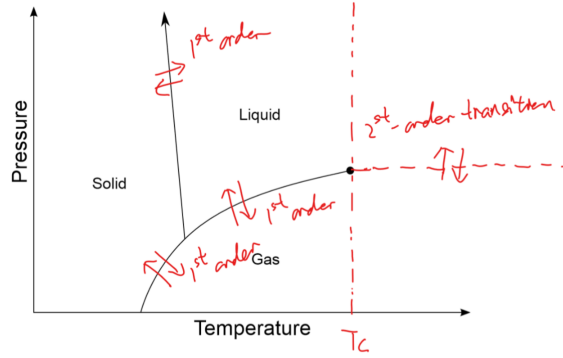
2d CFT

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Week 1

Exercies 0.0.1. The first order transitions and second order transitions show in the diagram



Exercies 0.0.2. By the homogeneous relation

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

we have

$$f(t, h) = t^{\frac{d}{y_t}} g(\alpha)$$

where $g(\alpha) = f(1, \alpha)$ and $\alpha = t^{-\frac{y_h}{y_t}} h$. It is easy to see that α is invariance under scaling transformation $x \rightarrow x/b$. Hence we have

$$C(t, 0) = -T \frac{\partial^2 f}{\partial T^2} \Big|_{h=0} = -\frac{1}{T_c} t^{\frac{d}{y_t}-2} g''(0)$$

$$M(t, 0) = -\frac{\partial f}{\partial B} \Big|_{h=0} = t^{\frac{d-y_h}{y_t}} g'(0)$$

$$\chi(t, 0) = \frac{\partial^2 f}{\partial B^2} \Big|_{h=0} = t^{(d-2y_h)/y_t} g''(0)$$

As function with single variable h , $\lim_{t \rightarrow 0} M(t, h) \sim h^{\frac{1}{\delta}}$, which implies that $g'(\alpha) \sim \alpha^{\frac{1}{\delta}}$ since α is linear function of h . Hence we have

$$\lim_{t \rightarrow 0} M = \lim_{t \rightarrow 0} t^{(d-y_h-\frac{y_h}{\delta})} h^{1/\delta}$$

since it is non-zero, we have $d - y_h - y_h \frac{1}{\delta} = 0$. Hence we have

$$\delta = \frac{y_h}{d - y_h}$$

Exercies 0.0.3. We have following relation

$$G_\sigma(\mathbf{r}; t, h) = t^{-2x_\sigma} G_\sigma\left(\frac{\mathbf{r}}{b}; b^{y_t} t, b^{y_h} h\right) \quad (1)$$

Let $h = 0, K = b^{y_t} t$,

$$G_\sigma(\mathbf{r}; t, 0) = t^{2x_\sigma/y_t} G_\sigma\left(\frac{\mathbf{r}}{K t^{-1/y_t}}; K, 0\right)$$

Since $G_\sigma(\mathbf{r}) \sim r^{-\tau} e^{-\frac{\tau}{\xi}}$, we have $\xi \sim t^{-1/y_t}$. It implies $\nu = 1/y_t$. With relation 1, we have

$$\chi(t, h) = \frac{1}{T} \int d^d \mathbf{r} G_\sigma(\mathbf{r}; t, h) = t^{d-2x_\sigma} \chi(b^{y_t} t, b^{y_h} h)$$

So $\gamma = (d - 2x_\sigma)/y_t$. But we have $\eta = 2x_\sigma + 2 - d$ for finite limit of $G(r)$ when $t \rightarrow 0$ and $h = 0$. Therefore, we get

$$\gamma = \nu(2 - \eta)$$

With scaling relations

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \alpha + \beta(1 + \delta) &= 2 \end{aligned}$$

and $\alpha = 2 - d\nu$, we have

$$\begin{aligned} \beta &= \frac{d\nu - 2\nu + \nu\eta}{2} \\ \delta &= \frac{d - \eta + 2}{d + \eta - 2} \end{aligned}$$

Exercies 0.0.4. By listed commutation relations, we have, for $r, s > 0$,

$$\begin{aligned} [D, J_{rs}] &= [D, L_{rs}] = \frac{i}{2} [D, [K_r, P_s]] \\ &= -\frac{i}{2} ([P_s, [D, K_r]] + [K_r, [P_s, D]]) \\ &= \frac{1}{2} [P_s, K_r] - \frac{1}{2} [K_r, P_s] \\ &= 0 \end{aligned}$$

For $r = -1, s = 0$, $[D, J_{rs}] = [D, D] = 0$. For $r = -1, s \neq 0$, $[D, J_{-1,s}] = [D, \frac{1}{2}(P_s - K_s)] = \frac{i}{2}(P_s + K_s)$. For $r = 0$, $[D, J_{0s}] = \frac{i}{2}(P_s - K_s)$. Hence (2,25) is satisfied when $(m, n) = (-1, 0)$.

If $(m, n) = (-1, n)$, then we have

$$[J_{mn}, J_{rs}] = \frac{1}{2} [P_n, J_{rs}] - \frac{1}{2} [K_n, J_{rs}]$$

With listed commutation relations, we can easily check it coincides with (2,25) respectively. Similarly check in the case of $(m, n) = (0, n)$.

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1 $SL_2(\mathbb{C})$

Exercies 1.0.1. We have $\det X = t^2 - (x^2 + y^2 + z^2)$. Since points in $\mathbb{R}^{1,3}$ can be written with Pauli matrix as base. Elements in $SO(1, 3)$ can be viewed as action on $M_2(\mathbb{C})$ with form $P \mapsto PXP^*$, which preserve \det of X . We have exact sequence of groups as follows:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{C}) \xrightarrow{sp} SO(1, 3) \longrightarrow 1$$

where sp is map $P \mapsto (X \mapsto PXP^*)$. Since for $P \in SL_2(\mathbb{C})$, $\det(PXP^*) = \det(X) = t^2 - (x^2 + y^2 + z^2)$, sp is well-defined. Hence $SO(1, 3) \cong SL_2(\mathbb{C})/\mathbb{Z}_2$.

Exercies 1.0.2. •

$$z \mapsto \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}$$

• We have

$$\begin{aligned} (w_1 - w_3) &= \frac{(az_1 + b)(cz_3 + d) - (az_2 + b)(cz_4 + d)}{(cz_1 + d)(cz_3 + d)} \\ &= \frac{z_1 - z_3}{(cz_1 + d)(cz_3 + d)} \end{aligned}$$

Hence we have $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$.

2 Three-point function

Exercies 2.0.1. Let $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle = f(z_{12}, z_{23}, z_{13})$. Under scalar transformation $z_i \mapsto \lambda z_i$, we have

$$f(z_{12}, z_{23}, z_{13}) = \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13})$$

Therefore, f is with form

$$f(z_{12}, z_{23}, z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$$

where $a + b + c = h_1 + h_2 + h_3$. Then under comformal transformation $z_i \mapsto \frac{1}{z_i}$, we have

$$z_1^{-2h_1} z_2^{-2h_2} z_3^{-2h_3} \frac{(z_1 z_2)^a (z_2 z_3)^b (z_1 z_3)^c}{z_{12}^a z_{23}^b z_{13}^c} = \frac{1}{z_{12}^a z_{23}^b z_{13}^c}$$

Hence $a = h_1 + h_2 - h_3, b = h_2 + h_3 - h_1, c = h_1 + h_3 - h_2$.

3 Energy-momentum tensor

Exercies 3.0.1. •

$$T^{\mu\nu} = -\eta^{\mu\nu} \partial_k \varphi \partial^k \varphi + 2\partial^\mu \varphi \partial^\nu \varphi$$

- We have

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}$$

Therefore,

$$\begin{aligned}\tilde{T}^{\mu\nu} &= \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{2}(-\delta_{\mu\nu} + 2)\partial_\mu\varphi\partial_\nu\varphi\end{aligned}$$