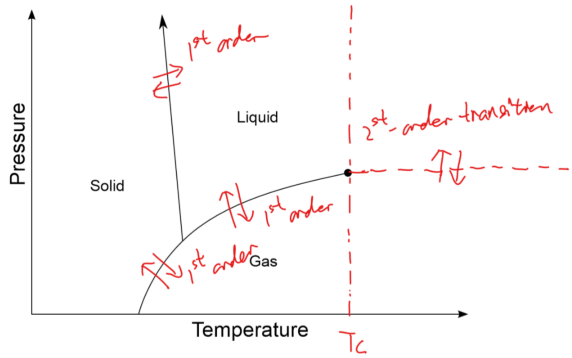
2d CFT

邹海涛
ID: 17210180015

Week 1

Exercies 0.0.1. The first order transitions and second order transitions show in the diagram



Exercies 0.0.2. By the homogeneous relation

$$f(t,h) = b^{-d} f(b^{y_t} t, b^{y_h} h)$$

we have

$$f(t,h) = t^{\frac{d}{y_t}} g(\alpha)$$

where $g(\alpha) = f(1,\alpha)$ and $\alpha = t^{-\frac{y_h}{y_t}} h$. It is easy to see that α is invariance under scaling transformation $x \to x/b$. Hence we have

$$C(t,0) = -T\frac{\partial^2 f}{\partial T^2}\Big|_{h=0} = -\frac{1}{T_c} t^{\frac{d}{y_t}-2} g''(0)$$
$$M(t,0) = -\frac{\partial f}{\partial B}\Big|_{h=0} = t^{\frac{d-y_h}{y_t}} g'(0)$$
$$\chi(t,0) = \frac{\partial^2 f}{\partial B^2}\Big|_{h=0} = t^{(d-2y_h)/y_t} g''(0)$$

As function with single variable h, $\lim_{t\to 0} M(t,h) \sim h^{\frac{1}{\delta}}$, which implies that $g'(\alpha) \sim \alpha^{\frac{1}{\delta}}$ since α is linear function of h. Hence we have

$$\lim_{t\to 0} M = \lim_{t\to 0} t^{(d-y_n-\frac{y_n}{\delta})} h^{1/\delta}$$

since it is non-zero, we have $d - y_n - y_n\frac{1}{\delta} = 0$. Hence we have

$$\delta = \frac{y_h}{d-y_h}$$

Exercies 0.0.3. We have following relation

$$G_\sigma(\mathbf{r}; t, h) = t^{-2x_\sigma} G_\sigma(\frac{\mathbf{r}}{b}; b^{y_t} t, b^{y_h} h) \tag{1}$$

Let $h = 0, K = b^{y_t} t$,

$$G_\sigma(\mathbf{r}; t, 0) = t^{2x_\sigma/y_t} G_\sigma(\frac{\mathbf{r}}{Kt^{-1/y_t}}; K, 0)$$

Since $G_\sigma(\mathbf{r}) \sim r^{-\tau} e^{-\frac{r}{\xi}}$, we have $\xi \sim t^{-1/y_t}$. It implies $\nu = 1/y_t$. With relation 1, we have

$$\chi(t, h) = \frac{1}{T} \int d^d\mathbf{r} G_\sigma(\mathbf{r}; t, h) = t^{d-2x_\sigma} \chi(b^{y_t} t, b^{y_h} h)$$

So $\gamma = (d - 2x_\sigma)/y_t$. But we have $\eta = 2x_\sigma + 2 - d$ for finite limit of $G(r)$ when $t \to 0$ and $h = 0$. Therefore, we get

$$\gamma = \nu(2 - \eta)$$

With scaling relations

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \alpha + \beta(1 + \delta) &= 2 \end{aligned}$$

and $\alpha = 2 - d\nu$, we have

$$\begin{aligned} \beta &= \frac{d\nu - 2\nu + \nu\eta}{2} \\ \delta &= \frac{d - \eta + 2}{d + \eta - 2} \end{aligned}$$

Exercies 0.0.4. By listed commutation relations, we have, for $r, s > 0$,

$$\begin{aligned} [D, J_{rs}] = [D, L_{rs}] &= \frac{i}{2}[D, [K_r, P_s]] \\ &= -\frac{i}{2}\big([P_s, [D, K_r]] + [K_r, [P_s, D]]\big) \\ &= \frac{1}{2}[P_s, K_r] - \frac{1}{2}[K_r, P_s] \\ &= 0 \end{aligned}$$

For $r = -1, s = 0$, $[D, J_{rs}] = [D, D] = 0$. For $r = -1, s \neq 0$, $[D, J_{-1,s}] = [D, \frac{1}{2}(P_s - K_s)] = \frac{i}{2}(P_s + K_s)$. For $r = 0$, $[D, J_{0s}] = \frac{i}{2}(P_s - K_s)$. Hence (2,25) is satisfied when $(m, n) = (-1, 0)$.

If $(m, n) = (-1, n)$, then we have

$$[J_{mn}, J_{rs}] = \frac{1}{2}[P_n, J_{rs}] - \frac{1}{2}[K_n, J_{rs}]$$

With listed commutation relations, we can easily check it coincides with (2,25) respectively. Similarly check in the case of $(m, n) = (0, n)$.

2d CFT

邹海涛
ID: 17210180015

1 $SL_2(\mathbb{C})$

Exercies 1.0.1. We have $\det X = t^2 - (x^2 + y^2 + z^2)$. Since points in $\mathbb{R}^{1,3}$ can be written with Pauli matrix as base. Elements in $SO(1,3)$ can be viewed as action on $M_2(\mathbb{C})$ with form $P \mapsto PXP^*$, which preserve det of X. We have exact sequence of groups as follows:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow SL_2(\mathbb{C}) \xrightarrow{sp} SO(1,3) \longrightarrow 1$$

where sp is map $P \mapsto (X \mapsto PXP^*)$. Since for $P \in SL_2(\mathbb{C})$, $\det(PXP*) = \det(X) = t^2 - (x^2 + y^2 + z^2)$, sp is well-defined. Hence $SO(1,3) \cong SL_2(\mathbb{C})/\mathbb{Z}_2$.

Exercies 1.0.2.

- $$z \mapsto \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}$$
- We have
$$\begin{aligned}(w_1 - w_3) =& \frac{(az_1 + b)(cz_3 + d) - (az_2 + b)(cz_4 + d)}{(cz_1 + d)(cz_3 + d)} \\ =& \frac{z_1 - z_3}{(cz_1 + d)(cz_3 + d)}\end{aligned}$$
Hence we have $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$.

2 Three-point function

Exercies 2.0.1. Let $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = f(z_{12}, z_{23}, z_{13})$. Under scalar transformation $z_i \mapsto \lambda z_i$, we have

$$f(z_{12}, z_{23}, z_{13}) = \lambda^{h_1+h_2+h_3} f(\lambda z_{12}, \lambda z_{23}, \lambda z_{13})$$

Therefore, f is with form

$$f(z_{12}, z_{23}, z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$$

where $a + b + c = h_1 + h_2 + h_3$. Then under comformal transformation $z_i \mapsto \frac{1}{z_i}$, we have

$${z_1}^{-2h_1}{z_2}^{-2h_2}{z_3}^{-2h_3}\frac{(z_1z_2)^a(z_2z_3)^b(z_1z_3)^c}{z_{12}^a z_{23}^b z_{13}^c} = \frac{1}{z_{12}^a z_{23}^b z_{13}^c}$$

Hence $a = h_1 + h_2 - h_3, b = h_2 + h_3 - h_1, c = h_1 + h_3 - h_2$.

3 Energy-momentum tensor

Exercies 3.0.1.

- $$T^{\mu\nu} = -\eta^{\mu\nu}\partial_k\varphi\partial^k\varphi + 2\partial^\mu\varphi\partial^\nu\varphi$$
- We have
$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}$$
Therefore,
$$\begin{aligned}\tilde{T}^{\mu\nu} &= \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{2}(-\delta_{\mu\nu} + 2)\partial_\mu\varphi\partial_\nu\varphi\end{aligned}$$

2d CFT

邹海涛
ID: 17210180015

4 Derivations

4.1 Energy-momentum tensor in complex coordinate

Since

$$\partial_0\Phi = \partial_z\Phi + \partial_{\bar{z}}\Phi$$
$$\partial_1\Phi = i\partial_z\Phi - i\partial_{\bar{z}}\Phi$$

we have

$$\partial_z\Phi = \frac{1}{2}\partial_0\Phi - \frac{i}{2}\partial_1\Phi$$
$$\partial_{\bar{z}}\Phi = \frac{1}{2}\partial_0\Phi + \frac{i}{2}\partial_1\Phi$$

Also, since there are metric tensors in complex coordinates

$$g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad g_{\alpha\beta}\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

, we have $\partial^z\Phi = 2\partial_{\bar{z}}$ and $\partial^{\bar{z}}\Phi = 2\partial_{\bar{z}}\Phi$. Therefore, from definition of energy-momentum tensor

$$T_{\alpha\beta} = -g_{\alpha\beta}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial^\alpha\Phi)}\partial_\beta\Phi$$

we get expression of them in complex coordinates

$$T_{zz} = \frac{1}{4}(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi)$$
$$T_{\bar{z}\bar{z}} = \frac{1}{4}(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi - \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi)$$
$$T_{z\bar{z}} = -\frac{1}{2}\mathcal{L} + \frac{1}{4}(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi - i\frac{\partial}{\partial_1\Phi}\partial_0\Phi + i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi)$$
$$T_{z\bar{z}} = -\frac{1}{2}\mathcal{L} + \frac{1}{4}(\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_0\Phi + i\frac{\partial}{\partial_1\Phi}\partial_0\Phi - i\frac{\partial\mathcal{L}}{\partial_0\Phi}\partial_1\Phi + \frac{\partial\mathcal{L}}{\partial_1\Phi}\partial_1\Phi)$$

Hence

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})$$
$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11})$$
$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11})$$

5 Schwarzian derivative

6 Virasoro algebra

The Larrent expansion of z^{n+1} around ω is

$$z^{n+1} = (z-\omega)^{n+1} + \binom{n+1}{1}\omega(z-\omega)^n + \cdots + \binom{n+1}{i}\omega^i(z-\omega)^{n+1-i} + \cdots + \omega^{n+1}$$

Hence we have following residues:

$$
\begin{aligned}
\mathrm{Res}_{\omega}\,\frac{z^{n+1}}{(z-\omega)^4} &= 2\pi i\frac{(n+1)n(n-1)}{6}\omega^{n-2}\\
\mathrm{Res}_{\omega}\,\frac{z^{n+1}}{(z-\omega)^2} &= 2\pi i(n+1)\omega^{n}\\
\mathrm{Res}_{\omega}\,\frac{z^{n+1}}{z-\omega} &= 2\pi i\omega^{n+1}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&[L_n, L_m] = \frac{1}{(2\pi i)^2}\oint_0 d\omega\ \omega^{m+1}\oint_\omega dz\ z^{n+1}\big(\frac{c}{2(z-\omega)^4}+\frac{2T(\omega)}{(z-\omega)^2}+\frac{\partial_\omega T(\omega)}{(z-\omega)}+\text{regular part}\big)\\
&= \frac{1}{2\pi i}\oint_0 d\omega\ \omega^{m+1}\big(\frac{c}{12}(n+1)n(n-1)\omega^{n-2}+2(n+1)\omega^{n}T(\omega)+\omega^{n+1}\partial_\omega T(\omega)\big)\\
&= \frac{1}{2\pi i}\Big\{\oint_0 d\omega\ \Big(\frac{c(n+1)n(n-1)}{12}\omega^{m+n-1})\Big) - (m-n)\oint_0 d\omega\omega^{m+n+1}T(\omega)\Big\}\\
&= \frac{c}{12}n(n^2-1)\delta_{n+m,0} - (m-n)L_{n+m}
\end{aligned}
$$