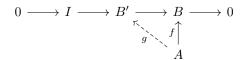
ALGEBRAIC GEOMETRY: PRIVATE NOTES

1. Deformation theory

First, we introduce essential algebraic notion in deformation theory called . Let k be algebraic closed field, let A be a finitely generated k-algebra such that Spec A is non-singular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally, suppose given a k-algebra homomorphism $f: A \to B$. Then there exists a k-algebra homomorphism $f: A \to B'$ making a commutative diagram

infinitesimal lifting property



We call this result the infinitesimal lifting property for A. We prove this result in several steps:

• First suppose that $g: A \to B'$ is a given homomorphism lifting f. If $g': A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, where I is canonical A-module since $I^2 = 0$.

Proof. Notice that I/I^2 can be viewed as sub k-algebra of $B \cong B'/I$ since I is ideal of B'. B is A-module given by $f: A \to B$, so I/I^2 is with natural A-module structure from B. Since $I^2 = 0$, $I \cong I/I^2$ is naturally A-module. Explicitly, A-module structure of I can be written as

$$A \otimes_B I \to I$$

 $(a,i) \mapsto f(a)\bar{i}$

where \bar{i} is the image of i in B. For all $a \in A$, we have

$$\pi(\theta(a)) = \pi(g - g')(a) = f(a) - f(a) = 0$$

It implies $\theta(a) \in I$. For $a, b \in A$, we have

$$\theta(ab) = (g - g')(ab) = g(a)g(b) - g'(a)g'(b)$$

= $g(a)(g'(b) - g'(b) + g(b)) - g'(a)g'(b)$
= $\theta(a)g'(b) + g(a)\theta(b)$

Under isomorphism $B \cong B'/I$,

$$\theta(a)g'(b) + g(a)\theta(b) = \theta(a) \cdot b + a \cdot \theta(b)$$

where \cdot is scalar product of I as A-module.

• Now let $P = k[x_1, \dots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h: P \to B'$ making a commutative diagram,

and show that h induces an A-linear map $\bar{h}: J/J^2 \to I$.

Proof. Since P is free k-algebra of rank n, it is projective object. Hence there is k-algebra homomorphism $h \colon P \to B'$ making such commutative diagram since in the diagram $B' \to B$ is epimorphism.

ullet Now use the hypothesis Spec A nonsingular and theorem 8.17 to show to obtain an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \to \operatorname{Hom}_A(\Omega_{A/k}, I) \to \operatorname{Hom}_P(\Omega_{P/k}, I) \to \operatorname{Hom}_A(J/J^2, I) \to 0$$

Let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\bar{h} \in \operatorname{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that h'(J) = 0. Thus h' induces the desired homomorphism $g: A \to B'$.

Proposition 1.0.1. Let X be a scheme of finite type over k algebraically closed. Suppose that for every morphism $F: Y \to X$ of a punctual scheme Y (meaning Y is the Spec of a local Artin ring), finite over k, and for every infinitesimal thickening $Y \subseteq Y'$ with ideal sheaf of square zero, there is a lifting $g: Y' \to X$. Then X is nonsingular.

Proof. content...

Definition 1.0.1. If X is a scheme over k, and A an Artin ring over k, we define a deformation of X over A to be a scheme X', flat over A, together with a closed immersion $i\colon X\hookrightarrow X'$ such that the induced map $i\times_k k\colon X\to X'\times_A k$ is an isomorphism. Two such deformations (X_1',i_1) and (X_2',i_2) are equivalent if there is an isomorphism $f\colon X_1'\to X_2'$ over A compatible with i_1 and i_2 , i.e., such that $i_2=f\circ i_1$.

2. Deformation theory with DG-Lie algebras