ALGEBRAIC GEOMETRY: PRIVATE NOTES

1. Deformation theory

First, we introduce essential algebraic notion in deformation theory called . Let k be algebraic closed field, let A be a finitely generated k-algebra such that $\operatorname{Spec} A$ is non-singular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally, suppose given a k-algebra homomorphism $f\colon A \to B$. Then there exists a k-algebra homomorphism $f\colon A \to B'$ making a commutative diagram

infinitesimal lifting property

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\
 & & & A \arrow[u, "f"] \arrow[ul, dashed, "g"] &
\end{tikzcd}$$

We call this result the infinitesimal lifting property for A. We prove this result in several steps:

- First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g'\colon A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, where I is canonical A-module since $I^2 = 0$.

  Proof. Notice that I/I^2 can be viewed as sub k-algebra of $B \cong B'/I$ since I is ideal of B'. B is A-module given by $f\colon A \to B$, so I/I^2 is with natural A-module structure from B. Since $I^2 = 0$, $I \cong I/I^2$ is naturally A-module. Explicitly, A-module structure of I can be written as
  $$\begin{aligned} A \otimes_B I &\to I \\ (a, i) &\mapsto f(a)\bar{i} \end{aligned}$$
  where $\bar{i}$ is the image of i in B. For all $a \in A$, we have
  $$\pi(\theta(a)) = \pi(g - g')(a) = f(a) - f(a) = 0$$
  It implies $\theta(a) \in I$. For $a, b \in A$, we have
  $$\begin{aligned} \theta(ab) &= (g - g')(ab) = g(a)g(b) - g'(a)g'(b) \\ &= g(a)(g'(b) - g'(b) + g(b)) - g'(a)g'(b) \\ &= \theta(a)g'(b) + g(a)\theta(b) \end{aligned}$$
  Under isomorphism $B \cong B'/I$,
  $$\theta(a)g'(b) + g(a)\theta(b) = \theta(a) \cdot b + a \cdot \theta(b)$$
  where $\cdot$ is scalar product of I as A-module. □

- Now let $P = k[x_1, \cdots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h\colon P \to B'$ making a commutative diagram,
  $$\begin{tikzcd}
  0 \arrow[r] & I \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\
  0 \arrow[r] & J \arrow[r] & P \arrow[r] \arrow[u, dashed, "h"] & A \arrow[r] \arrow[u, "f"] & 0
  \end{tikzcd}$$
  and show that h induces an A-linear map $\bar{h}\colon J/J^2 \to I$.

  Proof. Since P is free k-algebra of rank n, it is projective object. Hence there is k-algebra homomorphism $h\colon P \to B'$ making such commutative diagram since in the diagram $B' \to B$ is epimorphism. □

- Now use the hypothesis $\operatorname{Spec} A$ nonsingular and theorem 8.17 to show to obtain an exact sequence
  $$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

Show furthermore that applying the functor $\mathrm{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0$$

Let $\theta \in \mathrm{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\bar{h} \in \mathrm{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g\colon A \to B'$.

Proposition 1.0.1. *Let X be a scheme of finite type over k algebraically closed. Suppose that for every morphism $F\colon Y \to X$ of a punctual scheme Y (meaning Y is the* Spec *of a local Artin ring), finite over k, and for every infinitesimal thickening $Y \subseteq Y'$ with ideal sheaf of square zero, there is a lifting $g\colon Y' \to X$. Then X is nonsingular.*

Proof. content... □

Definition 1.0.1. If X is a scheme over k, and A an Artin ring over k, we define a deformation of X over A to be a scheme X', flat over A, together with a closed immersion $i\colon X \hookrightarrow X'$ such that the induced map $i \times_k k\colon X \to X' \times_A k$ is an isomorphism. Two such deformations (X_1', i_1) and (X_2', i_2) are equivalent if there is an isomorphism $f\colon X_1' \to X_2'$ over A compatible with i_1 and i_2, i.e., such that $i_2 = f \circ i_1$.

2. Deformation theory with dg-Lie algebras