

SEMINAR NOTES: COMMUTATIVE ALGEBRAS

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1. RINGS AND IDEALS

Definition 1.1. A **ring** R is a set with two maps (addition) $+: R \times R \rightarrow R$, (multiplication) $\times: R \times R \rightarrow R$ (denote $+(x, y)$ by $x + y$ and $\times(x, y)$ by $x \times y$) that satisfy following properties

- (1) R is an abelian group with respect to addition, its identity is denoted by 0;
- (2) R is a monoid with identity $1 \in R$ with respect to multiplication;
- (3) $z \times (x + y) = z \times x + z \times y$ and $(x + y) \times z = x \times z + y \times z$ for any given x, y, z .

We typically write xy for $x \times y$.

In a ring R , if $1 = 0$, then R has only one element, it is trivial and called **zero ring**. Denoted zero ring by 0.

Suppose R be a ring. R is commutative if for any $x, y \in R$, $xy = yx$. Rings mentioned in this notes will always be commutative other assumption.

Definition 1.2. Let A and B be two rings. 1_A and 1_B are their identities. A ring homomorphism from A to B is a map $f: A \rightarrow B$, which preserves both addition and multiplication structure, that means, for any $x, y \in A$

$$\begin{aligned}f(x + y) &= f(x) + f(y) \\f(xy) &= f(x)f(y) \\f(1_A) &= 1_B\end{aligned}$$

Suppose $f: A \rightarrow B$ be a ring homomorphism. We have $f(0_A) = f(1_A - 1_A) = f(1_A) - f(1_A) = 1_B - 1_B = 0_B$.

- (1) If f is surjective as map, then f is called surjective homomorphism.
- (2) If f is injective as map, then f is called injective homomorphism.

Definition 1.3. An **isomorphism** between two rings A and B is a ring homomorphism $f: A \rightarrow B$ such that there is another ring homomorphism $g: B \rightarrow A$ satisfying

$$f \circ g = \text{id}_B \quad g \circ f = \text{id}_A$$

Remark 1.1. f is isomorphism if and only if f is both surjective and injective as ring homomorphism.

Proof. If f is isomorphism, then $f(x) = f(y)$ implies $g(f(x)) = g(f(y))$. But $g \circ f = \text{id}_A$, so $x = y$. Hence f is injective. For any $b \in B$, $b = f \circ g(b)$ since $f \circ g = \text{id}_B$. Let $a = g(b)$, $b = f(a)$. That means f is surjective.

If f is both injective and surjective homomorphism, then we only need to check if f^{-1} is ring homomorphism. $f(f^{-1}(b_1 + b_2)) = b_1 + b_2 = f \circ f^{-1}(b_1) + f \circ f^{-1}(b_2) = f(f^{-1}(b_1) + f^{-1}(b_2))$. Since f is surjective, $f^{-1}(b_1 + b_2) = f^{-1}(b_1) + f^{-1}(b_2)$.

Similarly, $f^{-1}(b_1 b_2) = f^{-1}(b_1) f^{-1}(b_2)$. □

It is not always true in arbitrary category (**Top**, **Sch**/ k , **Mod** $_k$, etc).

If two rings are isomorphic, then we view them as same object in ring category.

Definition 1.4. Let R be a ring. We call $i : \tilde{R} \rightarrow R$ is a subring if i is injective ring homomorphism, written as $\tilde{R} \subset R$

Remark 1.2. The definition of subring in "Atiyah& MacDonald" is not exact since it doesn't require \tilde{R} to be even a ring.

Remark 1.3. $i(\tilde{R}) \simeq \tilde{R}$, so \tilde{R} can be viewed as $i(\tilde{R})$ whose elements are in R .

Definition 1.5. Let R be a ring, I be an additive subgroup of R . I is called an **ideal** of R if for any $r \in R$

$$rI := \{ra | a \in I\} \subset I$$

$$Ir := \{ar | a \in I\} \subset I$$

Since R is commutative, $Ir = rI$. We only need to check one of them. If I is ideal of R , then we denote the fact by $I \triangleleft R$.

An ideal $\mathfrak{p} \triangleleft R$ is called **prime ideal** if $xy \in \mathfrak{p}$ implies either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

An ideal $\mathfrak{m} \triangleleft R$ is called **maximal ideal** if $\mathfrak{m} \neq (1)$ and if there is no ideal I such that $\mathfrak{m} \subsetneq I \subsetneq (1)$.

$$\text{Ideal}(R) := \{\text{ideals of } R\}$$

Let $\varphi : A \rightarrow B$ be a ring homomorphism. Then there is induced map

$$\varphi^\# : \text{Ideal}(B) \rightarrow \text{Ideal}(A)$$

$$\mathfrak{b} \mapsto \varphi^{-1}(\mathfrak{b})$$

For any $x, y \in \varphi^{-1}(\mathfrak{b})$, $\varphi(x+y) = \varphi(x) + \varphi(y) \in \mathfrak{b}$, $\varphi(ax) = \varphi(a)\varphi(x) \in \mathfrak{b}$ implies that $ax \in \varphi^{-1}(\mathfrak{b})$. Hence $\varphi^{-1}(\mathfrak{b}) \in \text{Ideal}(A)$. Furthermore, it can be checked that $\varphi^\#$ is map from $\text{Spec } B$ to $\text{Spec } A$.

$$\ker \varphi := \{a \in A | \varphi(a) = 0\} = \varphi^{-1}((0))$$

If $a_0 \in \ker \varphi$, then for any $a \in A$, $\varphi(aa_0) = \varphi(a)\varphi(a_0) = 0$, so $aa_0 \in \ker \varphi$. Hence $\ker \varphi \in \text{Ideal}(A)$. Since 0 is contained in any ideals, $\varphi^\#(\mathfrak{b}) = \varphi^{-1}(\mathfrak{b}) \supset \ker \varphi$

Lemma 1.1. Let $I \triangleleft R$. Relation such that \sim_I on R defined as $x \sim_I y$ if and only if $x - y \in I$ is a equivalence relation.

Proof. (1) $x - x = 0 \in I \Rightarrow x \sim_I x$

(2) $x - y \in I \Rightarrow y - x = -(x - y) \in I \Rightarrow y \sim_I x$

(3) $x \sim_I y, y \sim_I z \Rightarrow x - y \in I, y - z \in I \Rightarrow x - z = (x - y) + (y - z) \in I \Rightarrow x \sim_I z$. □

Definition 1.6. Let I be a ring

$$R/I := (R / \sim_I, \times, +)$$

$$\bar{x} + \bar{y} = \overline{x + y}$$

$$\bar{x} \times \bar{y} = \overline{xy}$$

is called quotient ring of R by ideal I .

Remark 1.4. It is easy to check R/I is well defined

$$\varphi : R \rightarrow R/I$$

$$r \mapsto \bar{r}$$

$\varphi(r_1 + r_2) = \overline{r_1 + r_2} = \bar{r}_1 + \bar{r}_2 = \varphi(r_1) + \varphi(r_2)$, $\varphi(r_1 r_2) = \overline{r_1 r_2} = \bar{r}_1 \bar{r}_2 = \varphi(r_1) \varphi(r_2)$ and $\overline{1_R r} = 1_R \bar{r} = \bar{r}$, so $\varphi(1_R) = \bar{1}_R$ is identity of R/I . Hence φ is ring homomorphism.

FACT:

(1) $\ker \varphi = I$;

(2) φ is surjective;

(3) $\varphi^\#$ is injective. If $\varphi^\#(\bar{\alpha}) = \varphi^\#(\bar{\beta})$, then $\varphi^{-1}(\bar{\alpha}) = \varphi^{-1}(\bar{\beta})$. φ is surjective so $\bar{\alpha} = \bar{\beta}$.

- (4) If $\ker \varphi \subset I \triangleleft R$, then for any $\bar{i} \in \varphi(I)$, $\bar{r}\bar{i} = \overline{ri} = \varphi(ri)$ and $\varphi(I)$ is additive subgroup of R/I , $\varphi(I) \in \text{Ideal}(R/I)$. φ is surjective, so $I = \varphi^{-1}(\varphi(I)) = \varphi^\#(\varphi(I))$.

(3) and (4) implies following proposition.

Proposition 1.2. $\varphi^\#$ is one-to-one correspondence between $\text{Ideal}(B/I)$ and set of ideals contain I in R .

□

Definition 1.7. Let R be a ring.

- (1) $x \in R$ is called **zero divisor** if there is $r \in R, r \neq 0$ such that $rx = 0$.
- (2) $x \in R$ is called **nilpotent element** if $x^n = 0$ for some $n > 0$.
- (3) $x \in R$ is an **unit** of R if x has inverse under multiplication.
- (4) If R has no zero divisors except 0, then R is called **integral domain**.

Remark 1.5. A nilpotent element in a ring is always zero divisor since $xx^{n-1} = 0$. If x is a unit in R , then x is not a zero divisor. Conversely, it is not always true.

Definition 1.8. A **principal ideal** of R is an ideal that can be generated by one element, written as (x) , where x is the generator.

For simple example, $(3, 6) \triangleleft \mathbb{Z}$ is principal ideal generated by 6. R itself is also a principal ideal since it can be generated by 1, written as (1) .

Let $I_1 \triangleleft R, I_2 \triangleleft R$. We give following several constructions of ideals

$$I_1 \cdot I_2 = \{xy \in R | x \in I_1, y \in I_2\} \quad \prod_{i=1}^n I_i = \{x_1 x_2 \cdots x_n \in R | x_i \in I_i\}$$

$$I_1 + I_2 = \{x + y | x \in I_1, y \in I_2\} \quad \sum_{\alpha} I_{\alpha} = \left\{ \sum_{\alpha} x_{\alpha} | x_{\alpha} \in I_{\alpha} \text{ and only finite } x_{\alpha} \text{ are not zero} \right\}$$

$I_1 \cap I_2$ is obviously an ideal since $\forall x, y \in I_1 \cap I_2, r \in R, xr \in I_1 \cap I_2$ and $x + y \in I_1 \cap I_2$.

Examples 1.6. Let $A = \mathbb{Z}$, $(m), (n)$ two principal ideal generated by m and n .

$(m) + (n) = ((m, n))$ is generated by (m, n) , the g.c.d of m and n

$(m) \cdot (n) = (m \cdot n)$

$(m) \cap (n) = ([m, n])$ is generated by $[m, n]$, the l.c.d of m and n .

If $(m, n) = 1$, then $(m) + (n) = (1), (m)(n) = (m) \cap (n)$.

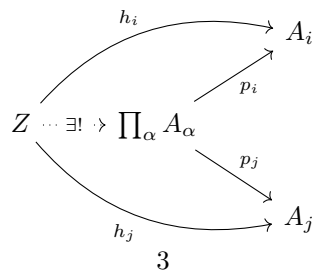
Let $I \triangleleft R, x \in R$. (x, I) is ideal generated by x and elements of I . Since $(x) + I$ is minimal ideal contains both x and elements of I , $(x) + I = (x, I)$.

Definition 1.9. If $I_1 \triangleleft R, I_2 \triangleleft R$. I_1 and I_2 are called **coprime** if $I_1 + I_2 = (1)$.

Proposition 1.3. If $I_1 \triangleleft R, I_2 \triangleleft R$ are coprime, then $I_1 \cdot I_2 = I_1 \cap I_2$.

Proof. By definition, $I_1 \cdot I_2 \subseteq I_1 \cap I_2$. Let $x \in I_1 \cap I_2$, x can be represented by $x = ar_1 + br_2$, where $a \in I_1, b \in I_2$. Hence $x \in I_1 \cdot I_2 = I_1 \cap I_2$. □

Definition 1.10. Let A_{α} be a family of rings. Their **direct product** is defined as object $\prod_{\alpha} A_{\alpha}$ in **Rings** satisfying following universal property



If α is finite, then elements of $\prod_{\alpha} A_{\alpha}$ can be written as (x_1, \dots, x_n) , $x_i \in A_i$ for some n .

$$\begin{aligned}(x_1, \dots, x_n) \cdot (x'_1, \dots, x'_n) &= (x_1 x'_1, \dots, x_n x'_n) \\ (x_1, \dots, x_n) + (x'_1, \dots, x'_n) &= (x_1 + x'_1, \dots, x_n + x'_n) \\ 1 &= (1_{A_1}, \dots, 1_{A_n})\end{aligned}$$

Let A be a ring and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals of A . Define a homomorphism

$$\phi : A \rightarrow \prod_{i=1}^n (A/\mathfrak{a}_i)$$

by rules $\phi(x) = (x + \mathfrak{a}_1, \dots, x + \mathfrak{a}_n)$.

Proposition 1.4 (??). (1) If $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$, then $\prod_i \mathfrak{a}_i = \bigcap_i \mathfrak{a}_i$;
 (2) ϕ is surjective $\Leftrightarrow \mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$;
 (3) ϕ is injective $\Leftrightarrow \bigcap_i \mathfrak{a}_i = (0)$.

Proof. (1) By 1.3 the case $n = 2$ is proved. Assume it is true when $n = k$. When $n = k + 1$, since \mathfrak{a}_i and \mathfrak{a}_{k+1} are coprime for $1 \leq i \leq k$, $\mathfrak{a}_i + \mathfrak{a}_{k+1} = (1)$. It implies that $x_i + y_i = 1$ for some $x_i \in \mathfrak{a}_i$, $y_i \in \mathfrak{a}_{k+1}$, $1 \leq i \leq k$

$$\prod_{i=1}^k x_i = 1 \text{ in } A/\mathfrak{a}_{k+1}$$

that means $\prod_{i=1}^k x_i + x_{k+1} = 1$ for some $x_n \in \mathfrak{a}_{k+1}$ in R . Hence $\prod_{i=1}^k \mathfrak{a}_i$ and \mathfrak{a}_{k+1} are coprime. Then

$$\prod_{i=1}^{k+1} \mathfrak{a}_i = \left(\prod_{i=1}^k \mathfrak{a}_i \right) \cdot \mathfrak{a}_{k+1} = \left(\bigcap_{i=1}^k \mathfrak{a}_i \right) \cap \mathfrak{a}_{k+1} = \bigcap_{i=1}^{k+1} \mathfrak{a}_i$$

by induction.

(2) If ϕ is surjective, then there exists $x \in A$ such that $\phi(x) = (\delta_1^i, \dots, \delta_n^i)$. Hence $x \equiv 1 \pmod{\mathfrak{a}_i}$, $x \equiv 0 \pmod{\mathfrak{a}_j}$ whenever $i \neq j$. So

$$(1 - x) + x = 1$$

where $1 - x \in \mathfrak{a}_i$, $x \in \mathfrak{a}_j$. Hence \mathfrak{a}_i and \mathfrak{a}_j are coprime.

Since $\prod_{i=1}^n (A/\mathfrak{a}_i)$ can be linearly represented by $(\delta_j^i)_{j=1}^n$, $1 \leq i \leq n$, it is enough to show for any $(\delta_j^i)_{j=1}^n$, there is $x_i \in R$ such that $\phi(x_i) = (\delta_j^i)_{j=1}^n$.

Since \mathfrak{a}_i and \mathfrak{a}_j are coprime for all $j \neq i$, there are equations $x_j + x_i = 1$, $x_j \in \mathfrak{a}_j$, $y_j \in \mathfrak{a}_i$

$$\prod_{j \neq i} x_j \equiv 0 \pmod{\mathfrak{a}_i}$$

$$\prod_{j \neq i} x_j = \prod_{j \neq i} (1 - y_j) \equiv 1 \pmod{\mathfrak{a}_j}$$

whenever $i \neq j$. Hence $\phi(\prod_{j \neq i} x_j) = (\delta_j^i)_{j=1}^n$.

(3) $\phi(x) = 0$ means that $x \in \mathfrak{a}_i$ for all $1 \leq i \leq n$. Hence it is equivalent to $x \in \bigcap_{i=1}^n \mathfrak{a}_i$. Hence ϕ is injective $\Leftrightarrow \ker \phi = (0) \Leftrightarrow \bigcap_{i=1}^n \mathfrak{a}_i = (0)$. □

Following are equivalent criteria for prime ideals and maximal ideals

Proposition 1.5. Let R be a ring.

- (1) $\mathfrak{p} \triangleleft R$ is prime ideal if and only if R/\mathfrak{p} is integral domain.
- (2) $\mathfrak{m} \triangleleft R$ is maximal ideal if and only if R/\mathfrak{m} is a field.

Proof. (1) Let $\mathfrak{p} \triangleleft R$ be a prime ideal. For any $x, y \in R$, $\bar{x}, \bar{y} = \bar{0}$ is equivalent to $xy \in \mathfrak{p}$. But $xy \in \mathfrak{p}$ implies that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, equivalently, $\bar{x} = 0$ or $\bar{y} = 0$. This shows that R/\mathfrak{p} is integral domain.

Conversely, if R/\mathfrak{p} is integral domain, then for any $x, y \in R$ such that $xy \in \mathfrak{p}$, $\bar{x}\bar{y} = \bar{0}$ in R/\mathfrak{p} , we have $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$. That means $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Hence we can conclude the equivalence.

- (2) Let $\mathfrak{m} \triangleleft R$ be a maximal ideal. If $\bar{x} \in R/\mathfrak{m}, \bar{x} \neq \bar{0}$, then $x \notin \mathfrak{m}$. Since \mathfrak{m} is maximal, $m \subsetneq (\mathfrak{m}, x) \subset (1)$ implies that $(\mathfrak{m}, x) = (1)$. That means, there exists $y \in R$ such that $xy + m = 1$ for some $m \in \mathfrak{m}$. Obviously, $y \notin \mathfrak{m}$, so $\bar{x}\bar{y} = \bar{1}$ in R/\mathfrak{m} . Hence each non-zero element in R/\mathfrak{m} is unit. Hence R/\mathfrak{m} is a field.

Conversely, if R/\mathfrak{m} is a field, then $\bar{x} \in R/\mathfrak{m}, \bar{x} \neq \bar{0}$ is unit. But $\bar{x} \neq \bar{0}$ is equivalent to $x \notin \mathfrak{m}$ and \bar{x} is unit R/\mathfrak{m} if and only if x is unit in R . So $(\mathfrak{m}, x) = (1)$ if $x \notin \mathfrak{m}$. Hence \mathfrak{m} is maximal. The proof is complete. \square

Theorem 1.6 (Krull's theorem). *If R is a ring and $R \neq 0$, then R has at least one maximal ideal.*

Proof. Since $R \neq 0$, $(0) \in \text{Ideal}(R)$ and $(0) \neq (1)$. We can order $\Sigma = \text{Ideal}(R) - \{(1)\}$ by inclusion ($I_1 \leq I_2$ iff $I_1 \subseteq I_2$). Suppose $\{I_\alpha\}$ be a chain in Σ , i.e. $\forall I_\alpha, I_\beta \in \{I_\alpha\}, I_\alpha \leq I_\beta$ or $I_\beta \leq I_\alpha$. Denote $\bigcup_\alpha I_\alpha$ by I . I is obviously an ideal and $1 \notin I$ since for each $\alpha, 1 \notin I_\alpha$. Hence $I \in \Sigma$ and I is upper bound of $\{I_\alpha\}$. By Zorn's lemma, Σ has at least one maximal element, it is an maximal ideal in R by definition. \square

Corollary 1.7. *Let R be ring. If $I \triangleleft R$ and $I \neq (1)$, then I is contained in one maximal ideal.*

Proof. $R/I \neq 0$. By Krull's theorem, R/I has at least one maximal ideal $\bar{\mathfrak{m}}$. Then $\varphi^\#(\bar{\mathfrak{m}})$ is maximal ideal which contain I since $\varphi^\#$ induce one-to-one correspondence between $\Sigma_{R/I}$ and set of non-trivial ideals which contain I and $\varphi^\#$ preserves order. \square

Corollary 1.8. *Any non-unit in R is contained in a maximal ideal.*

Definition 1.11. A ring with only one maximal ideal \mathfrak{m} is called a local ring with maximal ideal \mathfrak{m} . Suppose (R, \mathfrak{m}) be a local ring with maximal ideal \mathfrak{m} . R/\mathfrak{m} is called residue field of R .

Proposition 1.9 (??). (i) *Let A be a ring and $\mathfrak{m} \neq (1)$ and ideal of A such that every $x \in A - \mathfrak{m}$ is a unit in A . Then A is local ring and \mathfrak{m} its maximal ideal.*

(ii) *Let A be a ring and \mathfrak{m} a maximal ideal of A , such that every element of $1 + \mathfrak{m}$ is a unit. Then A is a local ring.*

Proof. (i) Since elements in $A - \mathfrak{m}$ are all units and every ideal not equal to (1) contains non-unit, all maximal ideals are contained in \mathfrak{m} . Hence \mathfrak{m} is maximal ideal and the only one.

(ii) Let $x \in A - \mathfrak{m}$. Since \mathfrak{m} is maximal, $(\mathfrak{m}, x) = (1)$. That means there exist $y \in A$ and $m \in \mathfrak{m}$ such that $xy + m = 1$. Hence $xy = 1 - m$ is unit by hypothesis so is x . Hence A is local ring by (i). \square

Definition 1.12. Let R be a ring.

$$\mathbf{Rad}(R) = \{r \in R | r \text{ is nilpotent}\}$$

is called nilradical of R or simply radical of R .

$$\mathbf{JRad}(R) = \{r \in R | \forall y \in R, 1 - ry \text{ is unit}\}$$

is called Jacobson radical of R .

Proposition 1.10. $\mathbf{Rad}(R)$ is intersection of all prime ideals of R ; $\mathbf{JRad}(R)$ is intersection of all maximal ideals of R .

Proof. If $\mathfrak{p} \triangleleft R$ is prime, then R/\mathfrak{p} is integral. Hence every $x \notin \mathfrak{p}$ is not nilpotent otherwise \bar{x} in R/\mathfrak{p} is also nilpotent. Hence

$$\mathbf{Rad}(R) \subseteq \bigcap_{\mathfrak{p} \triangleleft R \text{ is prime}} \mathfrak{p}$$

If x is not nilpotent, then we need to prove that there is prime ideal does not contain x . Let $S = \{1, x, x^2, \dots\}$ and Σ be the set of ideals that disjoint with S . Since $(0) \in \Sigma$ and Σ is ordered

by inclusion, by Zorn's lemma \sum has maximal element, denote it by \mathfrak{p} . We need to prove \mathfrak{p} is a prime ideal.

Let $a \notin \mathfrak{p}, b \notin \mathfrak{p}$. (a, \mathfrak{p}) and (b, \mathfrak{p}) are not elements in \sum since \mathfrak{p} is maximal. That means there exist $m, n \geq 0$ such that $x^m \in (a, \mathfrak{p}), x^n \in (b, \mathfrak{p})$. It implies

$$x^m = r_1 a + p_1, x^n = r_2 b + p_2 \quad r_1, r_2 \in R$$

Hence $x^{m+n} = r_1 r_2 ab + (r_2 b p_1 + r_1 a p_2 + p_1 p_2) \in (ab, \mathfrak{p})$. Hence $(ab, \mathfrak{p}) \in \sum$ and therefore $ab \in \mathfrak{p}$. Hence \mathfrak{p} is prime ideal and $x \in \mathfrak{p}$.

Let $x \in R$. If there is $y \in R$ such that $1 - xy$ is not unit, then there is a maximal \mathfrak{m} . Hence

$$x \notin \bigcap_{\mathfrak{m} \triangleleft R \text{ is maximal}} \mathfrak{m}$$

If $x \in \mathfrak{m}$ for some maximal ideal \mathfrak{m} , then $(x, \mathfrak{m}) = (1)$. That means $1 = rx + m$ for some $r \in R, m \in \mathfrak{m}$. Hence $1 - rx \in \mathfrak{m}$ is not unit. Hence $x \notin \mathbf{J}\text{Rad}(R)$. \square

Here we will introduce some essential facts about prime ideals that used frequently in algebraic geometry.

Proposition 1.11 (??). (i) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals and α be ideals contained in $\bigcup_{i=1}^n \mathfrak{p}_i$. Then $\alpha \subseteq \mathfrak{p}_i$ for some i .

(ii) Let $\alpha_1, \dots, \alpha_n$ be ideals and let \mathfrak{p} be a prime ideal which contains $\bigcap_{i=1}^n \alpha_i$. Then $\alpha_i \subseteq \mathfrak{p}$ for some i . In particular, if $\mathfrak{p} = \bigcap_{i=1}^n \alpha_i$, then $\mathfrak{p} = \alpha_i$ for some i .

Proof. (i) When $n = 1$, it is true obviously.

If it is true that $\alpha \not\subseteq \mathfrak{p}_i (1 \leq i \leq n)$ for some $n > 0$ can implies $\alpha \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$, then for given $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_{n+1}$, if $\alpha \not\subseteq \mathfrak{p}_i (1 \leq i \leq n+1)$, then $\alpha \not\subseteq \bigcup_{i=1}^{n+1} \mathfrak{p}_i$. So for each $1 \leq j \leq n$, there is $x_j \in \alpha$ such that $x_j \notin \mathfrak{p}_i$ whenever $i \neq j$.

Let

$$y = \sum_{j=1}^n x_1 x_2 \cdots \hat{x}_j \cdots x_n$$

Since \mathfrak{p}_j is prime, $x_1 x_2 \cdots \hat{x}_j \cdots x_n \notin \mathfrak{p}_j$. Hence $y \notin \mathfrak{p}_j$ for all $1 \leq j \leq n$. But $y \in \alpha$, hence $\alpha \not\subseteq \bigcup_{j=1}^n \mathfrak{p}_j$. By induction, it is true for all $n > 0$.

(ii) If $\alpha_i \not\subseteq \mathfrak{p}$ for all i , then there are $x_i \in \alpha_i$ such that $x_i \notin \mathfrak{p}$ for all i . Since \mathfrak{p} is prime, $x_1, x_2, \dots, x_n \notin \mathfrak{p}$. But $x_1 x_2 \cdots x_n \in \prod_{i=1}^n \alpha_i \subseteq \bigcap_{i=1}^n \alpha_i$. Hence $\bigcap_{i=1}^n \alpha_i \not\subseteq \mathfrak{p}$. Hence $\bigcap_{i=1}^n \alpha_i \subseteq \mathfrak{p} \Rightarrow \alpha_i \subseteq \mathfrak{p}$ for some i .

If $\bigcap_{i=1}^n \alpha_i = \mathfrak{p}$, then $\mathfrak{p} \subseteq \alpha_j$ for all j . But $\alpha_j \subseteq \mathfrak{p}$, so $\alpha_i = \mathfrak{p}$. \square

Let $I_1 \triangleleft R, I_2 \triangleleft R$.

$$(I_1 : I_2) := \{r \in R \mid rI_2 \subseteq I_1\}$$

Examples 1.7. 1) Let $I \triangleleft R$. $(0 : I) = ((0) : I) = \text{ann}(I)$ is called **annihilator** of I .

If $I = (x)$ is principal ideal, then $\text{ann}((x))$ is shortly denoted by $\text{ann}(x)$, called **annihilator of x** . If x is non-zero-divisor, then $\text{ann}(x) = 0$.

2) Let $R = \mathbb{Z}$, $I_1 = (m), I_2 = (n)$. $m = \prod_{i=1}^n p_i^{\alpha_i}, n = \prod_{i=1}^n p_i^{\beta_i}$ are prime decomposition by p_1, \dots, p_n . Let $\gamma_i = \max\{\alpha_i - \beta_i, 0\}$, then $(I_1 : I_2) = (\prod_{i=1}^n p_i^{\gamma_i})$ and $\prod_{i=1}^n p_i^{\gamma_i} = \frac{m}{(m, n)}$.

Definition 1.13. Let $I \triangleleft R$.

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n\}$$

is called radical ideal of I .

Remark 1.8. $\mathbf{Rad}(R) = \sqrt{(0)}$

Proposition 1.12. \sqrt{I} is the intersection of all primes ideals which contain I .

Proof.

$$\begin{aligned}
& x \in \sqrt{I} \\
& \Leftrightarrow \exists n > 0, x^n \in I \\
& \Leftrightarrow \exists n > 0, (\bar{x})^n \text{ in } R/I \\
& \Leftrightarrow \bar{x} \in \mathbf{Rad}(R/I) \text{ is intersection of prime ideals in } R/I \\
& \Leftrightarrow x = \varphi^{-1}(\bar{x}) \in \varphi^\#(\mathbf{Rad}(R/I)) \text{ is intersection of prime ideals which contain } I \text{ in } R
\end{aligned}$$

□

Proposition 1.13. $D = \text{set of zero-divisors of } A = \bigcup_{x \neq 0} \sqrt{(\text{ann}(x))}$

Proof. First, $\sqrt{D} = D$ since D is prime.

Next, $\sqrt{\bigcup_\alpha E_\alpha} = \bigcup_\alpha \sqrt{E_\alpha}$ for any family of subset of R . Hence $D = \sqrt{D} = \sqrt{\bigcup_{x \neq 0} \text{ann}(x)} = \bigcup_{x \neq 0} \sqrt{\text{ann}(x)}$. □

2. TOPOLOGICAL PROPERTIES OF PRIME SPECTRUM

Let R be a ring.

$$\text{Spec } R = \{\text{all prime ideals of } R\}$$

We can give $\text{Spec } R$ topology called Zariski topology, which is the origin of modern algebraic geometry. Closed sets of Zariski topology is defined with form

$$V(I) = \{\mathfrak{p} \in \text{Spec } R \mid I \subseteq \mathfrak{p}\}$$

where I is arbitrary ideal of R .

It is homework to check that these subsets of $\text{Spec } R$ satisfy closed set axioms

- (1) $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$
- (2) $\bigcap V(I) = V(\sum I)$
- (3) $\emptyset = V(1)$ and $\text{Spec } R = V(0)$

Examples 2.1. $\text{Spec } \mathbb{Z}$

$\text{Spec } k[x]$, where k is a algebraic closed field.

$$D(I) = \text{Spec } R \setminus V(I)$$

are used to denote the open sets of $\text{Spec } R$ in Zariski topology. In particular, if $I = (f)$ is principal ideal generated by x , then $D(I) = D(f)$ is called a principal open set.

Proposition 2.1. $V(I) = V(\sqrt{I}); D(I) = D(\sqrt{I})$

Proof. If I is contained in a prime ideal \mathfrak{p} , then $\forall x \in \sqrt{I}, \exists n > 0, x^n \in I$. So $x^n \in \mathfrak{p}$. Since \mathfrak{p} is prime, $x \cdot x^{n-1} \in \mathfrak{p}$ implies that $x \in \mathfrak{p}$ or $x^{n-1} \in \mathfrak{p}$, then by induction of n , we can conclude that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subseteq \mathfrak{p}$. Hence $V(I) = V(\sqrt{I})$. □

Proposition 2.2. *Principal open subsets of $\text{Spec } R$ form topology base of Zariski topology.*

Proof. (1) $\bigcup_{f \in I} D(f) = D(I)$ for any open subset $D(I)$ of $\text{Spec } R$.

- (2) For any ideal J , there is a principal ideal (f) such that $D_f \subseteq D(I)$. For example, we can choose any element f of I .

□

Proposition 2.3. *$\text{Spec } R$ is quasi-compact.*

Proof. It is sufficient to prove for any principal open cover $\bigcup_i D_{f_i}$.

$\bigcup_i D_{f_i} = \text{Spec } R$ means $\bigcap_i V(f_i) = V(1)$. Since $\bigcap_i V(f_i) = V(\sum_i (f_i))$, there are elements $r_i \in R$ such that

$$\sum_i r_i f_i = 1$$

and only finite a_i are not 0. Hence the correspondent finite D_{f_k} form a finite cover of $\text{Spec } R$. □

Proposition 2.4 (Separability of $\text{Spec } R$). *$\text{Spec } R$ is T_0 and T_1 as following figures show.*

Remark 2.2. $\text{Spec } R$ is not Hausdorff if $R \neq 0$. Recall that in a Hausdorff space, each single point is closed point, i.e. $\overline{\text{pt}} = \text{pt}$. $(0) \in D_f, \forall f \in R, f \neq 0$, since $(0) \subsetneq (f)$. Hence $\{(0)\}$ is dense in $\text{Spec } R$. Hence $\{(0)\}$ is not closed, it implies that $\text{Spec } R$ is not Hausdorff. (0) is called a **generic point** of $\text{Spec } R$.

Proposition 2.5. $\mathfrak{p} \in \text{Spec } R$ is closed if and only if \mathfrak{p} is maximal ideal.

Proof. Notice that

$$\overline{\{\mathfrak{p}\}} = \bigcap_{I \subseteq \mathfrak{p}} V(I) = V\left(\sum_{I \subseteq \mathfrak{p}} I\right) = V(\mathfrak{p})$$

Hence $\overline{\{\mathfrak{p}\}} = \{\mathfrak{p}\}$ implies \mathfrak{p} is the unique ideal which contains \mathfrak{p} . Since \mathfrak{p} is contained in a maximal ideal, and maximal ideals are all primes ideals, then \mathfrak{p} is maximal. \square

Suppose A, B be two rings. $x, y \in \varphi^{-1}(\mathfrak{p})$ means $\varphi(xy) \in \mathfrak{p} \triangleleft A$. Hence $\varphi(x) \in \mathfrak{p}$ or $\varphi(y) \in \mathfrak{p}$. Hence $x \in \varphi^{-1}(\mathfrak{p})$ or $y \in \varphi^{-1}(\mathfrak{p})$. By upper statement, we can conclude that

$$\begin{aligned} \varphi^* : \text{Spec } B &\rightarrow \text{Spec } A \\ \varphi^*(\mathfrak{p}) &:= \varphi^{-1}(\mathfrak{p}) \end{aligned}$$

is well defined.

Proposition 2.6. (1) φ^* is continuous under Zariski topology.

(2) φ^* is dominant if and only if $\ker \varphi \subseteq \text{Rad}(R) = \sqrt{(0)}$

Proof. (1) $(\varphi^*)^{-1}(V(f)) = \{\mathfrak{p} \in \text{Spec } B \mid f \in \varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})\}$. It is equal to $V(\varphi(f)) = \{\mathfrak{p} \in \text{Spec } B \mid \varphi(f) \in \mathfrak{p}\}$. Hence the inverse image of any closed subset is also closed. This means φ^* is continuous.

(2) Notice that

$$\varphi^*(\text{Spec } B) = \{\varphi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec } A\} \subseteq V\left(\bigcap_{\ker \varphi \subseteq \mathfrak{p}} \mathfrak{p}\right) = V(\sqrt{\ker \varphi}) \subseteq \text{Spec } A$$

If $\overline{\varphi^*(\text{Spec } B)} = \text{Spec } A$. then by upper fact, we know $\sqrt{\ker \varphi} = \text{Rad } A$ since they are all radical ideal. This implies $\ker \varphi \subseteq \text{Rad } A$

Conversely,

$$\overline{\varphi^*(\text{Spec } B)} = \overline{\varphi^*(V(0))} = V(\varphi^{-1}(0)) = V(\ker \varphi).$$

Hence $\ker \varphi \subseteq \text{Rad } A$ implies $\text{Spec } A \subseteq V(\ker \varphi)$. Hence $\overline{\varphi^*(\text{Spec } B)} = \text{Spec } A$. \square

In particular, if A is integral, then $\varphi^* : \text{Spec } B \rightarrow \text{Spec } A$ is dominant if and only if φ is injective.

Examples 2.3. Let k be a algebraic closed field,

$$X = \text{Spec } k[x_1, \dots, x_n]/(f_1) \quad \mathcal{O}_X = k[x_1, \dots, x_n]/(f_1)$$

$$Y = \text{Spec } k[x_1, \dots, x_n]/(f_2) \quad \mathcal{O}_Y = k[x_1, \dots, x_n]/(f_2)$$

where f_1, f_2 are irreducible polynomials over k . Denote K_X and K_Y the fraction field of \mathcal{O}_X and \mathcal{O}_Y , they are called **function fields** of X and Y . Obviously, \mathcal{O}_X and \mathcal{O}_Y are integral, hence by upper proposition, if $\varphi : Y \rightarrow X$ is dominant, then φ induce an embedding from \mathcal{O}_X to \mathcal{O}_Y . Furthermore, there is also an embedding from K_X to K_Y , which can be looked as a field extension. It is also used to give a definition of dimesions for reduced irreducible affine schemes