Calabi-Yau categories and Calabi-Yau algebras

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Abstract

These are short notes for Calabi-Yau algebras and Calabi-Yau categories.

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1 Introduction

TBA.

2 Preliminaries

2.1 Kan extension

Kan extension is an essential notion in category theory, especially in categorical homotopy theory. In this subsection, we will give a short review of Kan extension and some of its applications. We refer reader to [3] and [2] for more details on Kan extensions and model category.

Definition 2.1.1. Suppose $F : \mathcal{C} \to \mathcal{D}$ be a given functor. Let $K : \mathcal{C} \to \mathcal{E}$ be another functor. The **left Kan extension** of K along with F is a functor $\mathrm{Lan}_F K$ such that $\eta : K \Rightarrow \mathrm{Lan}_F K \circ F$ is universal i.e. if $\gamma : K \Rightarrow G \circ F$ is a natural transformation, then there exists an unique natural transformation $\delta : \mathrm{Lan}_F K \Rightarrow G$ such that $\gamma = (\delta F) \circ \eta$. In diagram viewpoint, a Kan extension means

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "K"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Downarrow\gamma"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "K"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "\mathrm{Lan}_F K"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\exists!\Downarrow"] & \end{tikzcd}$$

Dually, we can also defined **right Kan extension** $\mathrm{Ran}_K F$ of K along with F as following diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "K"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Uparrow\phi"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "K"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Uparrow\varepsilon"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"] \arrow[ur, bend right=60, "\mathrm{Ran}_F K"'] \arrow[ur, phantom, bend right=30, "\exists!\Uparrow"] & \end{tikzcd}$$

Unfortunately, Kan extension is not always exists in general case though the existence of Kan extension of some functor implies fantastic results.

In Quillen's homotopical algebra theory, Kan extension is used to construct total derived functors.

Definition 2.1.2. Suppose given localizations and functor

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[d, "\gamma_1"] & \mathcal{C}_2 \arrow[d, "\gamma_2"] \\ S_1^{-1}\mathcal{C}_1 & S_2^{-1}\mathcal{C}_2 \end{tikzcd}$$

The **left total derived functor** of F is right Kan extension of $\gamma_2 \circ F$ along with γ_1, denoted by $(\mathbf{L}F, \varepsilon)$, where ε is the natural transformation in definition.

Proposition 2.1.1 ([1])**:** Suppose given localizations and adjoint functors

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] \arrow[d, "\gamma_1"] & \mathcal{C}_2 \arrow[l, shift left, "G"] \arrow[d, "\gamma_2"] \\ S_1^{-1}\mathcal{C}_1 & S_2^{-1}\mathcal{C}_2 \end{tikzcd}$$

such that

1. S_1 contains all isomorphisms of $\mathcal{C}_1$. If f, g are maps of $\mathcal{C}_1$ such that gf is defined, then if any two of the maps f, g, gf are in S_1 so is the third.

2. A map f in $\mathcal{C}_2$ is in S_2 if and only if $Gf \in S_1$.

3. There exists a functor $R : \mathcal{C}_1 \to \mathcal{C}_1$ and a natural transformation $\xi R \to \mathrm{id}$ such that for all $X \in \mathbf{Ob}(\mathcal{C}_1)$ the maps $\xi : RX \to X$ and $\beta : RX \to GFRX$ are in S_1.

Then the left derived functor $\mathbf{L}F$ exists and is quasi-inverse to the functor $\widetilde{G} : S_2^{-1}\mathcal{C}_2 \to S_1^{-1}\mathcal{C}_1$ induced by G. In particular $\widetilde{G}$ and $\mathbf{L}F$ are equivalences of categories.

If M and N are two model categories, we can defined derived functor between them since they have natural localizations to their homotopy categories.

Corollary 2.1.1: If M and N are two model categories, then for any pair of Quillen adjunction (F, G), $\mathbf{L}F$ exists.

$$\begin{tikzcd}
\mathcal{M} \arrow[r, shift left, "F"] \arrow[d, "\gamma_1"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left, "G"] \arrow[d, "\gamma_2"] \\
\mathbf{Ho}\mathcal{M} \arrow[r, "\mathbf{L}F"] & \mathbf{Ho}\mathcal{N}
\end{tikzcd}$$

2.2 Enriched categories and enriched Yoneda lemma

Enriched category theory deals with categories whose morphism sets have additional structures. Always, the structure on the morphism sets makes original category taking more information.

Definition 2.2.1. Suppose $\mathcal{E}$ be a category and $\mathcal{S}$ be a monoidal category. $\mathcal{E}$ is called a enriched category over S if $\mathcal{C}(X,Y)$ is a object in S for all X, Y in $\mathcal{C}$ with following two families of morphisms

$$\circ_{(X,Y,Z)} : \mathcal{E}(Y,Z) \otimes \mathcal{E}(X,Y) \longrightarrow \mathcal{E}(X,Z) \qquad \forall X, Y, Z \in \mathbf{Ob}(\mathcal{E})$$

$$u_X : 1 \longrightarrow \mathcal{E}(X,X) \qquad \forall X \in \mathbf{Ob}(\mathcal{E})$$

These morphisms satisfy following properties

$$\begin{tikzcd}
(\mathcal{E}(C,D) \otimes \mathcal{E}(B,C)) \otimes \mathcal{E}(A,B) \arrow[rr, "\simeq"] \arrow[d, "\circ_{(B,C,D)} \otimes id"'] & & \mathcal{E}(C,D) \otimes (\mathcal{E}(B,C) \otimes \mathcal{E}(A,B)) \arrow[d, "id \otimes \circ_{(A,B,C)}"] \\
\mathcal{E}(B,D) \otimes \mathcal{E}(A,B) \arrow[dr, "\circ_{(A,B,D)}"] & & \mathcal{E}(C,D) \otimes \mathcal{E}(A,C) \arrow[dl, "\circ_{(A,C,D)}"'] \\
& \mathcal{E}(A,D) &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{E}(X,Y) \otimes 1 \arrow[d, "\simeq"] \arrow[dr, "u_X"] & \\
\mathcal{E}(Y,Y) \otimes \mathcal{E}(X,Y) \arrow[r, "\circ_{(X,Y,Y)}"] & \mathcal{E}(X,Y) & \mathcal{E}(X,Y) \otimes \mathcal{E}(X,X) \arrow[l, "\circ_{(X,X,Y)}"'] \\
& 1 \otimes \mathcal{E}(X,Y) \arrow[u, "\simeq"'] \arrow[ul, "u_Y"] &
\end{tikzcd}$$

Example 2.2.1: (∞, k)-categories can be defined as ∞-categories enriched over $(\infty, k-1)$-categories by induction.

Example 2.2.2: Let k be a field. A **differential graded category** over k is a category whose morphism sets are all non-negative chain complexes. By the mean, a differential graded category is an enriched category over category $\mathbf{Ch}^+(k)$, of non-negative chain complexes over field k. It is a essential notion in modern non-commutative geometry.

2.3 Some homological algebras

Derived category, a notion in homological algebra, is closely related to abelian category. In homotopical view point, a derived category of a abelian category is the localization of chain complexes category over this abelian category with quasi-isomorphisms.

Moreover, we need a notion, a generalization of derived category of an abelian category in classical homological algebra.

Definition 2.3.1. An additive category $\mathcal{C}$ is called a triangulated category if it is with following data:

(i) An additive auto-equivalence functor $T : \mathcal{C} \to \mathcal{C}$, the functor is called **shift functor**. For a object A in $\mathcal{C}$, its n-shift (i.e by n times actions of T on A) is denoted by $A[n]$ or s^nA, where n is a integer.

(ii) A class of **distinguished triangles**(also called **exact triangles**)

$$A \to B \to C \to A[1]$$

They satisfy following axioms

TR1 $A \xrightarrow{\mathbf{id}_A} A \to 0 \to A[1]$ is distinguished

TR2 any morphism $f : A \to B$ can be completed to a distinguished triangle;

TR3 a triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ is distinguished if and only if

$$B \xrightarrow{v} C \xrightarrow{w} A[1] \xrightarrow{-u[1]} B[1]$$

is a distinguished triangle;

TR4 Any commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[d, "g"] \\ A' \arrow[r, "u'"] & B' \end{tikzcd}$$

extends to a morphism of triangles(i.e a commutative diagram whose rows are distinguished triangles)

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & A'[1] \end{tikzcd}$$

TR5 Given three distinguished triangles

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{k} A[1] \qquad B \xrightarrow{v} C \xrightarrow{l} A' \xrightarrow{i} B[1] \qquad A \xrightarrow{v \circ u} C \to B' \to A[1]$$

there exist two morphisms $f : C' \to B', g : B' \to A'$ such that$(\mathbf{id}_A, v, f), (u, \mathbf{id}_C, g)$ define morphisms of the triangles and

$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{j[1] \circ i} C'[1]$$

is a distinguished triangle.

Definition 2.3.2. A commutative square

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & Y' \\ {\scriptstyle \alpha''}\downarrow & & \downarrow{\scriptstyle \beta'} \\ Y'' & \xrightarrow{\beta''} & Z \end{array}$$

is called homotopy cartesian square if there exists an distinguished triangle

$$X \xrightarrow{\begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix}} Y' \coprod Y'' \xrightarrow{\begin{pmatrix} \beta' & -\beta'' \end{pmatrix}} Z \xrightarrow{\gamma} X[1]$$

γ is called a differential of homotopy cartesian square.

Proposition 2.3.1: Diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & Y' \\ {\scriptstyle \alpha''}\downarrow & & \downarrow{\scriptstyle \beta'} \\ Y'' & \xrightarrow{\beta''} & Z \end{array}$$

is pull-back iff

$$0 \to X \xrightarrow{\begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix}} Y' \coprod Y'' \xrightarrow{\begin{pmatrix} \beta' & -\beta'' \end{pmatrix}} Z \to 0$$

is a distinguished triangle.

Definition 2.3.3. A subcategory of a triangulated category is called **triangulated subcategory** if each its morphism $A \to B$ can be extended to a distinguished triangle with morphisms in the subcategory.

3 Duality and Serre functor

3.1 Coherent sheaves category and quasi-coherent sheaves category

Lemma 3.1.1: Suppose A be a commutative ring, M be a A-module, then following sequence of A-module is exact

$$0 \to M \xrightarrow{\alpha} \bigoplus_{i=1}^{n} M_{g_i} \xrightarrow{\beta} \bigoplus_{i,j} M_{g_i g_j}$$

where

$$\alpha(m) = (\frac{m}{1}, \cdots, \frac{m}{1}) \qquad \beta(\frac{m_1}{g_1^{e_1}}, \cdots, \frac{m_n}{g_n^{e_n}}) = (\frac{m_i}{g_i^{e_i}} - \frac{m_j}{g_j^{e_j}})_{i,j}$$

Proof. Clearly, α is injective as A-module morphism and $\beta(\alpha(m)) = (\frac{m}{1} - \frac{m}{1})_{i,j} = 0$. So what is left to verify is that kernel of β is contained in the image of α, exactly, if $(\frac{m_i}{g_i^{e_i}} - \frac{m_j}{g_j^{e_j}})_{i,j} = 0$, then all $\frac{m_i}{g_i^{e_i}}$ equal to $\frac{m}{1}$ for some $m \in M$.

If

$$\frac{m_i}{g_i^{e_i}} - \frac{m_j}{g_j^{e_j}} = 0, \forall i, j = 1, \cdots, n$$

then there exist $n_{ij} \in \mathcal{N}$ such that

$$(g_i g_j)^{n_{ij}} (g_j^{e_j} m_i - g_i^{e_i} m_j) = 0 \tag{1}$$

Take N be the maximal one of n_{ij}, then

$$(g_i g_j)^N (g_j^{e_j} m_i - g_i^{e_i} m_j) = 0, \forall i, j \tag{2}$$

which implies that

$$g_i^N g_j^{N+e_j} m_i = g_j^N g_i^{N+e_i} m_j \tag{3}$$

But there exist a_i that $1 = \sum_{i=1}^n a_i g_i$. Let $M = \max\{N + e_i\}$, then

$$1 = (\sum_{i=1}^n a_i g_i)^{nM} = \sum_{i=1}^n \tilde{a}_i g_i^M = \sum_{i=1}^n \tilde{a}_i \tilde{g}_i^{k_i} g_i^{N+e_i} \tag{4}$$

Let $b_i = \tilde{a}_i \tilde{g}_i^{k_i}$. By equations 3 and 4, we have

$$\frac{m_i}{g_i^{e_i}} = \frac{n_i g_i^N}{g_i^{N+e_i}} = \sum_{j=1}^n \frac{b_j g_j^{N+e_j} m_i g_i^N}{g_i^{N+e_i}} = \sum_{j=1}^n \frac{b_j g_j^N m_j}{1}$$

Let $m = \sum_{j=1}^n b_j g_j^N m_j$, then

$$\frac{m_i}{g_i^{e_i}} = \frac{m}{1}, \forall i$$

Hence the given sequence is exact. $\square$

Definition 3.1.1. Suppose X be affine scheme Spec$A$, M be an A-module. Then there is a presheaf over module defined openset-wise as

$$\widetilde{M}(U) = \{s : U \to \coprod_{p \in U} M_p | s(p) \in M_p, s \text{ locally be} \frac{m}{f}, m \in M, f \in A\}$$

More precisely, $\widetilde{M}$ can be constructed on each standard open as

$$\widetilde{M}(D(f)) = M_f$$

Such presheaf $\widetilde{M}$ is called **associated presheaf** to M.

Remark: Presheaf $\widetilde{M}$ is actually a sheaf. We can check sheaf condition on each standard open set, which need to verify that following sequence exact

$$0 \to \widetilde{M}(D(f)) \xrightarrow{\oplus \rho_{f,g_i}} \bigoplus_{i=1}^n \widetilde{M}(D(g_i)) \xrightarrow{\oplus_{i,j} \rho_{g_i, g_i g_j} - \rho_{g_j, g_i g_j}} \bigoplus_{i,j} \widetilde{M}(D(g_i) \cap D_(g_j))$$

This sequence is equal to

$$0 \to M_f \xrightarrow{\alpha} \bigoplus_{i=1}^n M_{fg_i} \xrightarrow{\beta} \bigoplus_{i,j} M_{fg_ig_j}$$

where α and β are defined as in lemma 3.1.1. The result is clear from lemma 3.1.1 by changing M into M_f.

Definition 3.1.2. Let $X = \mathrm{Spec}A$. The associated sheaf to A(as A-module) $\widetilde{A}$ is called **structure sheaf** of X and denoted by $\mathcal{O}_X$. More generally, for any scheme X that is covered by affine open sets $\mathrm{Spec}A_i$, sheaf of X assigning each open set $\mathrm{Spec}A_i$ associated sheaf to A_i is also called structure sheaf of X.

Proposition 3.1.1: Let $X = \mathrm{Spec}A$, A be a commutative ring

(i)
$$(-)^{\sim} : A - \mathbf{Mod} \to \mathbf{Sh(X)}$$
is fully-faithful functor from A-modules category to sheaves category of affine scheme X.

(ii) $(\widetilde{M \otimes_A N}) \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, where the right tensor is tensor of sheaves over modules.

(iii) $(\widetilde{\oplus M_i}) \simeq \oplus \widetilde{M_i}$, where the right direct sum if direct sum of sheaves over modules.

Definition 3.1.3 (Hartshorne, Algebraic Geomtry)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules F is **quasi-coherent** if X can covered by open affine subsets $U_i = \mathrm{Spec}A_i$ such that for each i there is an A_i-module M_i with $F|_{U_i} \simeq \widetilde{M_i}$. We say that is **coherent** if furthermore each M_i can be taken to be a finitely generated A_i-module.

It is easy to see that definition 3.1.3 is about scheme. More generally, we can also define quasi-coherent sheaf and coherent sheaf on a arbitrary ringed space.

Definition 3.1.4 (Stack Project)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. Let F be a sheaf of $\mathcal{O}_X$-modules. We say that F is **quasi-coherent** sheaf of $\mathcal{O}_X$-modules if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $F|_U$ is isomorphic to the cokernel of map

$$\bigoplus_{j \in J} \mathcal{O}_U \to \bigoplus_{i \in I} \mathcal{O}_U$$

This means that X is covered by open sets U such that $F|_U$ has a presentation of form

$$\bigoplus_{j \in J} \mathcal{O}_U \to \bigoplus_{i \in I} \mathcal{O}_U \to F|_U \to 0$$

If quasi-coherent sheaf F satisfies following conditions:

(i) F is of finite type

(ii) for every open set $U \subset X$ and every finite collection $s_i \in F(U)$, the kernel of associated map $\bigoplus_i \mathcal{O}_X \to F|_U$ is of finite type

then F is called **coherent** sheaf.

Example 3.1.1: On any scheme X, the structure sheaf $\mathcal{O}_X$ is quasi-coherent and coherent.

Definition 3.1.5. Let B be a graded ring and let $X = \mathrm{Proj}B$. For any $n \in \mathbb{Z}$, we define that sheaf $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$. We call $\mathcal{O}_X(1)$ the twisting sheaf of Serre(or very ample sheaf). For any sheaf of $\mathcal{O}_X$-module, F, we denote by $F(n)$ the twisted $F \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

3.2 Differential forms sheaf

Definition 3.2.1. Let $\phi : R \to S$ be a ring morphism, in other words, S is a R-algebra defined by ϕ. Let M be a S-module. A R-derivation into M is a map $D : S \to M$ which satisfies

(i) $D(r_1 s_1 + r_2 s_2) = r_1 D(s_1) + r_2 D(s_2)$

(ii) $D(s_1 s_2) = s_1 D(s_2) + D(s_1) s_2$

Set of R-derivation into M is denoted by $\mathrm{Der}_R(S, M)$. It is also a S-module.

$$\mathrm{Der}_R(S, -) : \mathbf{S-Mod} \to \mathbf{S-Mod}$$

is a functor and it is corepresentable. $\Omega^1_{S/R}$ is the co-representing object of functor $\mathrm{Der}_R(S, -)$, i.e for any S-module M there exists a isomorphism

$$\alpha_M : \mathbf{S-Mod}(\Omega^1_{S/R}, M) \to \mathrm{Der}_R(S, M)$$

Let $d = \alpha_S^{-1}(\mathrm{id}_S)$. $(\Omega^1_{S/R}, d)$ is called **relative differential forms** of S over R(also called **Kahler differential**).

Definition 3.2.2 (Modules of differentials)**.** Let X be a topological space. Let $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings. Let F be a $\mathcal{O}_2$-module. A $\mathcal{O}_1$-derivation(ϕ-derivation) into F is a map $D : \mathcal{O}_2 \to F$ which is $\mathcal{O}_1$-linear and satisfies Leihbniz rule

$$D(ab) = aD(b) + D(a)b$$

It implies that $D(1) = D(1 \cdot 1) = 2D(1)$, so $D(1) = 0$ and for all $r \in \phi(\mathcal{O}_1)$, $D(r) = 0$.

Analogously, $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, -)$ is a representable functor and $\Omega^1_{\mathcal{O}_1/\mathcal{O}_2}$ is the representing object of $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, -)$. $\Omega^1_{\mathcal{O}_1/\mathcal{O}_2}$ is called **relative differential forms sheaf** of $\mathcal{O}_1$ over $\mathcal{O}_2$.

Actually, in scheme case, the relative differential forms sheaf is compatible as following construction.

Proposition 3.2.1: Let $f : X \to Y$ be a morphism of schemes. Then there exists a unique quasi-coherent sheaf $\Omega^1_{X/Y}$ on X such that for any affine open subset V of Y, any affine open subset U of $f^{-1}(V)$, and any $x \in U$ we have

$$\Omega^1_{X/Y}|_U = (\Omega^1_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)})^{\sim}$$

$\mathcal{O}_2[F]$ is sheaflication of presheaf $U \mapsto \mathcal{O}_2(U)[F(U)]$ where this denotes the free $\mathcal{O}_2$-module on the set $F(U)$. For $s \in F(U)$, $[s]$ is the corresponding section of $\mathcal{O}_2[F]$ over U. If F is a sheaf of $\mathcal{O}_2$-module, then there is a canonical map

$$c : \mathcal{O}_2[F] \to F$$

which on the presheaf level is given by rule $\sum f_s[S] \mapsto \sum f_s s$.

Let $\Omega^1_{\mathcal{O}_2/\mathcal{O}_1}$ be cokernel of following map.

$$\begin{aligned} M = \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \bigoplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \bigoplus \mathcal{O}_2[\mathcal{O}_1] &\to \mathcal{O}_2[\mathcal{O}_2] \\ [(a,b)] \oplus [(f,g)] \oplus [h] &\mapsto [a+b] - [a] - [b] \\ &\quad + [fg] - [f]g - f[g]) \\ &\quad + [\phi(h)] \end{aligned}$$

so $\Omega^1_{\mathcal{O}_2/\mathcal{O}_1}$ is quotient module of $\mathcal{O}_2[\mathcal{O}_2]$ by required relations in definition.

$$\begin{tikzcd} M \arrow[r] & \mathcal{O}_2[\mathcal{O}_2] \arrow[r] \arrow[d] & \Omega_{\mathcal{O}_2/\mathcal{O}_1} \arrow[r] \arrow[d, dashed] & 0 \\ & \mathcal{O}_2 \arrow[ur, "d"] \arrow[r, "D"] & F & \end{tikzcd}$$

3.3 Canonical sheaf and complex geometry

Canonical sheaf over a variety or a scheme is analogous to canonical bundle on complex manifold.

3.4 Serre-Grothendieck duality

In this subsection, we will give an introduction about Serre-Grothendieck duality and more general notion about Serre functor. This is main part of the section that help us make clear how can we define a Calabi-Yau structure in some category and how to define Calabi-Yau algebras in more general setting. Firstly, we give a definition of duality in a monoidal category. In the setting of basic linear algebra, we has notion of linear dual vector space for a given vector, which consists of all linear forms on it. Suppose V be a vector space over k, its linear dual is of form like

$$V^* = \hom(V, k)$$

More generally, we can also define 'duality structure' in a monoidal as mentioned in the beginning.

Definition 3.4.1. Let $\mathcal{C}$ be a monoidal category. A **duality datumn** in $\mathcal{C}$ consists of the following data:

(i) A pair of objects $(X, X^\vee) \in \mathcal{C}$

(ii) A pair of morphisms

$$c : 1 \to X \otimes X^\vee \qquad\qquad e : X^\vee \otimes X \to 1$$

where 1 denotes the unit object of $\mathcal{C}$.

These morphisms are required to satisfy the following condition: The compositions of these morphisms

$$X \xrightarrow{c\otimes id} X \otimes X^\vee \otimes X \xrightarrow{id\otimes e} X$$
$$X^\vee \xrightarrow{id\otimes c} X^\vee \otimes X \otimes X^\vee \xrightarrow{e\otimes id} X^\vee$$

are the identity on X and $X^\vee$, respectively.

Suppose A be an associative k-algebra, M be an A-module. The A-linear dual of M

$$\textbf{Mod-A}(M, A)$$

is denoted by M^*. Naturally, $(-)^*$ is a functor from category **Mod-A** to itself and also duality datumn on **Mod-A**.
Furthermore, we can consider the derived functor of $(-)^*$,

$$(-)^! : D^*(\textbf{Mod-A}) \to D^*(\textbf{Mod-A})$$

In particular, if $A = k$, M is an quadratic algebra over k, then $M^!$ is linear Koszul dual to M.(remained to check?)

3.5 Serre functors and Calabi-Yau categories

Given an equivalence of k-linear(enriched triangulated) categories $S : \mathcal{C} \to \mathcal{C}$, it is called a Serre functor if there is an isomorphism of bifucntors from $\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathbf{Ab}$.

$$\mathcal{C}(?, ?) \simeq \mathcal{T}(?, S(?))^\vee \circ \tau$$

where $\tau : \mathcal{T}^{\text{op}} \times \mathcal{T} \to \mathcal{T}^{\text{op}} \times \mathcal{T}$ is the bifunctor exchanging two components. In particular, for any two objects A, B in $\mathcal{T}$,

$$\mathcal{T}(A, B) \simeq \mathcal{T}(B, S(A))^\vee$$

Proposition 3.5.1: If there is a Serre functor on a k-linear category, it is unique up to natural isomorphism.

Proof. Suppose S_1 and S_2 both be Serre functors on category $\mathcal{T}$. Let A be any object of $\mathcal{T}$. then we have following isomorphisms by definition

$$\mathcal{T}(A, A) \simeq \mathcal{T}(A, S_1(A))^\vee \simeq \mathcal{T}(S_1(A), S_2(A))$$

Thus, the image of identity morphism id_A in $\mathcal{T}(S_1(A), S_2(A))$ is an isomorphism between $S_1(A)$ and $S_2(A)$, so it can be extended to a natural isomorphism between S_1 and S_2. □

Proposition 3.5.2: A Serre functor of triangulated categories is automatically a δ-functor.

Proof. TBA □

Proposition 3.5.3: Suppose $\mathcal{T}_1, \mathcal{T}_2$ be two triangulated categories with Serre functors S_1, S_2 individually. If $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ be a functor, then left adjoint functor ϕ^* exists if and only if right adjoint functor$\phi^!$ exists and

$$S_1 \circ \phi^! = \phi^* \circ S_2$$

Proof. TBA □

Definition 3.5.1 (Calabi-Yau category)**.** Suppose $\mathcal{A}$ be a triangulated category(enriched on category of finitely dimensional vector spaces) with Serre functor S. $\mathcal{A}$ is called **n-Calabi-Yau category** if Serre functor S is natural equivalent to shifted identity functor $\text{id}_{\mathcal{A}}[n]$ of $\mathcal{A}$, that is ,there is

$$\omega : S \simeq \text{id}_{\mathcal{A}}[n]$$

ω is called a **weak Calabi-Yau structure** with dimension n.

4 Calabi-Yau structures and examples

At the end of last section, we introduced **n-Calabi-Yau categories** and weak Calabi-Yau structures on them. Kontsenvich discussed the notion of Calabi-Yau category in setting of A_∞ in his well-known paper **??** and Victor Ginzburg defined Calabi-Yau algebras in another way for associative algberas, which is not equivalent to Kontsenvich's definition in general but some cases. Major missions for this section is to compare the difference between these two definitions and try to offer a more general treatment of Calabi-Yau structures.

We need to clarify what a weak Calabi-Yau structure really means before detailed disscution. Suppose $\omega : S \simeq \text{id}_{\mathcal{A}}[n]$ be a given weak n-Calabi-Yau structure on $\mathcal{A}$. Kontsenvich conjectured that if $\mathcal{A}$ is good enough then we can find a Hochschild class $[\gamma_{\mathcal{A}}] \in \text{HH}_n(\mathcal{A})$ such that the induced map $[\gamma_{\mathcal{A}} s^{-n}] : \mathcal{A}^![n] \to \mathcal{A}$ is an isomorphism in derived category $D(\mathcal{A}^e)$, which is proved by Ginzburg in **??**. Thus, Hochschild homologies contain all information about Calabi-Yau in our definition.

4.1 absolute Calabi-Yau structures

A weak Calabi-Yau structure is not strong enough to describe the 'Calabi-Yau properties' for a (DG) category. Now we introduce an enrichment suggested in **??**.

Definition 4.1.1. An **n-Calabi-Yau structure** on a DG-category $\mathcal{A}$ is a negative cyclic class $[\tilde{\gamma_{\mathcal{A}}}] \in \mathrm{HC}_n^-(\mathcal{A})$ which induces a weak n-Calabi-Yau structure on $\mathcal{A}$.

Definition 4.1.2. A **weak relative n-Calabi-Yau structure** on a DG functor $F : \mathcal{A} \to \mathcal{B}$ between homologically smooth DG categories is a relative Hochschild class $[\gamma] \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ such that the induced maps in $D(\mathcal{A}^e)$ and $D(\mathcal{B}^e)$ are both isomorphisms.

Definition 4.1.3. An **relative n-Calabi-Yau structure** on DG functor $F : \mathcal{A} \to \mathcal{B}$ is a relative negative cyclic class $[\tilde{\gamma}] \in \mathrm{HC}_n^-(\mathcal{A}, \mathcal{B})$ whose underlying relative Hochschild class $h([\tilde{\gamma}]) \in \mathrm{HH}_n(\mathcal{B}, \mathcal{A})$ is a weak relative n-Calabi-Yau structure on F.

4.2 relative Calabi-Yau structures

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