Motivic Homotopy Theory

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1 Construction of Unstable Homotopy Category

In order to refine the triangulated category of Voevodsky's motives and apply more tools from algebraic topology, we need well-behaved homotopy theory for schemes. Homotopy theory for schemes also allow us to construct motivic cohomology and algebraic K-theory which are looked as generalized cohomology theory defined with Brown representativity theorem.

1.1 Simplicial Methods for Algebraic Geometry

Let us firstly recall some simplicial homotopy theory. Δ is category consists of objects such as

$$[n] = 0 \to 1 \to 2 \to \cdots \to n$$

for all non-negative integer n. And morphisms are functions of sets preserving the order of arrows. The category of *simplicial sets* means the category of presheaves on Δ with values over **Sets**. It is denoted by **sSets**.

In category of simplicial sets, we have following canonical objects

$$\Delta[n] := \Delta(-, [n])$$

They are call *standard simplicial sets.*

Proposition 1.1.1. $\Delta([m],[n])$ *is generated by following morphisms*

$$d^i \colon [k] \to [k+1]$$

$$d^i(0 \to 1 \to 2 \to \cdots \to k) = \begin{cases} 1 \to 2 \to \cdots \to k+1 & \textit{if } i = 0 \\ 0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to k+1 & \textit{if } 1 \leq i \leq k \\ 0 \to 1 \to \cdots \to k & \textit{if } i = k+1 \end{cases}$$

and

$$s^i \colon [k] \to [k-1]$$

$$s^i(0 \to 1 \to 2 \to \cdots \to k) = \begin{cases} 0 \to 0 \to 1 \to \cdots \to k-1 & \textit{if } i = 0 \\ 0 \to \cdots \to i \to i \to \cdots \to k-1 & \textit{if } 1 \leq i \leq k-1 \end{cases}$$

where d^i is called co-face map and s^i is called co-degeneracy map.

Δ is small category. As diagram, Δ looks like

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & {[1]} \arrow[r, shift left=2, "d^1"] \arrow[r, "d^0" description] \arrow[r, shift right=2, "d^2"'] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} {[0]} & {[1]} \arrow[l, "s^0"'] & \cdots \arrow[l, shift right, "s^0"'] \arrow[l, shift left, "s^1"] \end{tikzcd}$$

By Yoneda lemma, for simplicial set $X \in \mathbf{sSets}$, we have

$$X([n]) \simeq \mathbf{sSets}(\Delta[n], X)$$

For convenience, we always denote $X([n])$ by X_n for simplicial set X, $d_i = X(d^i)$ and $s_i = X(s^i)$.

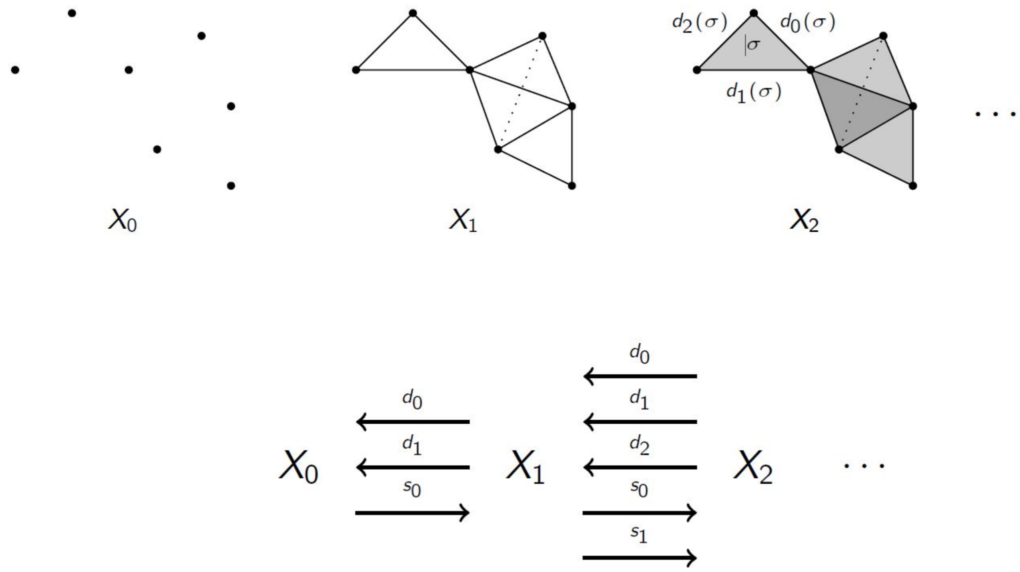
Let Δ^n be standard simplex in topological space category. Then we can defined topological space respect to given simplicial set X as follows

$$|X| := \operatorname*{colim}_{(\Delta_n \to X) \in \mathbf{sSets}/X} \Delta^n$$

Furthermore, $|-|$ is covariant functorial in $\mathbf{sSets}$ since co-representable $\mathbf{sSets}(\Delta_n, -)$ and colimit are functorial. $|-|$ is called *geometric realization functor.*

In classical simplicial homotopy theory, we can endow $\mathbf{sSets}$ with model structure where fibrations are Kan fibrations and weak equivalence are morphisms which induce isomorphisms between fibrant objects on homotopy groups (can be defined as homotopy groups of corresponding geometric realizations).

Figure 1: Idea of simplicial sets (from ncatlab)



1.1.1 Simplicial Objects

Let $\mathcal{C}$ be arbitrary category. We say X_* is a *simplicial object* on $\mathcal{C}$ if X_* is a presheaf $X_*\colon \Delta^{op} \to \mathcal{C}$ and $\mathbf{s}\mathcal{C}$ denotes the category of simplicial objects on $\mathcal{C}$. Hence simplicial sets are simplicial objects on **Sets**.

If $\mathcal{A}$ is an Abelian category, then we have following correspondence

Proposition 1.1.2 (Dold-Kan Correspondence)**.**

$$N\colon s\mathcal{A} \to \mathbf{Ch}_{\geq 0}(\mathcal{A})$$

is equivalence of categories and is also Quillen equivalence (i.e. Quillen functor induce equivalence between homotopy categories) with respect to canonical model structure on $s\mathcal{A}$ and projective model on $\mathbf{Ch}_{\geq 0}(\mathcal{A})$. N is functor of normalized complex and $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ is category of non-negative chain complexes. This equivalence is called Dold-Kan correspondence*.*

This proposition means that homological algebra over Abelian category in low bounded case is equivalent to homotopy theory for its simplicial objects. It is convenient to study homotopy theory for simplicial objects because we have geometric realization functor to make constructions more natural. So we can transplant properties from classical homotopy theory into homological algebra.

1.1.2 Simplicial Homotopy Theory for Presheaves and Sheafification

Suppose $\mathbf{PSh}^{\mathbf{Sets}}(\mathcal{C})$ be the category of presheaves of sets on $\mathcal{C}$. Then the associated category of simplicial objects $s\mathbf{PSh}^{\mathbf{Sets}}(\mathcal{C})$ is isomorphic to $\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ — the category of presheaves of simplicial sets on $\mathcal{C}$. Let $X_* \in s\mathbf{PSh}^{\mathbf{Sets}}(\mathcal{C})$

$$F_* \mapsto F$$

where

$$F(X)_n := F_n(X)\ \forall X \in \mathcal{C}$$

In this case, $s\mathbf{PSh}^{\mathbf{Sets}}(\mathcal{C})$ or $\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ is called category of simplicial presheaves of sets on $\mathcal{C}$.

$\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ can be endowed with model structure objectwise by model structure of **sSets**.

- $\alpha\colon F \Rightarrow G$ is weak-equivalence if and only if for all $X \in \mathcal{C}$, $\alpha_X : F(X) \to G(X)$ is weak-equivalence in **sSets**.
- $\beta\colon F \Rightarrow G$ is fibration if and only if for all $X \in \mathcal{C}$, β_X is Kan fibration.

Then the model structure of $\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ is fibrantly generated. This model structure on $\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ is called *global model structure.*

If $\mathcal{C}$ is a site with Grothendieck topology, we can define $\mathbf{Sh}_\tau^{\mathbf{sSets}}(\mathcal{C})$ be the category of τ-sheaves of sets on $\mathcal{C}$ and $\mathbf{PSh}_\tau^{\mathbf{sSets}}(\mathcal{C})$ the category of simplicial presheaves with model structure called *τ-local model structure* as follows.

Definition 1.1.1 (τ-local weak equivalence)**.** A *τ-local weak equivalence* is a morphism between presheaves $f\colon F \Rightarrow G$ such that

- The morphism induces isomorphism $\tilde{\pi}_0(f) : \tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ in $\mathbf{Sh}^{\mathbf{Sets}}(\mathcal{C})$;
- For all $U \in \mathcal{C}$, $\tilde{\pi}_n(f) : \tilde{\pi}_n(X, x) \to \tilde{\pi}_n(Y, f(x))$ is isomorphism on $\mathcal{C}/U$ for any choice of base point $x \in X(U)$.

The τ-local cofibrations are same as ordinary cofibrations of $\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ and τ-fibrations are morphisms in $\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ satisfy RLP for τ-acyclic cofibration (i.e. be both τ-cofibration and τ-weak equivalence). Jardine proved that these datum actually define a model structure.

In particular, if $\mathcal{C} = (Sm/S)_\tau$ is site of smooth S-schemes with Grothendieck topology τ, we denote $\mathbf{Spc}^{pre}(S)$ (resp. $\mathbf{Spc}_\tau^{pre}(S)$, resp. $\mathbf{Spc}_\tau(S)$) the category of simplicial presheaves (res. simplical presheaves with τ-local model structue, resp. simplicial τ-sheaves) on $\mathcal{C} = (Sm/S)_\tau$. $\mathbf{Spc}_\tau(S)$ is full subcategory of $\mathbf{Spc}_\tau^{pre}(S)$ and left adjoint functor s of inclusion exists. This fucntor is called is called *sheafification functor* under topology τ. Hence $\mathbf{Spc}_\tau(S)$ is model category. Furthermore, we have

Proposition 1.1.3. *The pair* (s, i) *of adjunction is a pair of Quillen equivalence.*

Hence we have $\mathbf{Ho}(\mathbf{Spc}_\tau^{pre}(S)) \simeq \mathbf{Ho}(\mathbf{Spc}_\tau(S))$, denoted by $\mathcal{H}_\tau(S)$.

1.1.3 Hypercovers and Bousfield localization

More generally, we can realize τ-local model structure on $\mathbf{PSh}_\tau^{\mathbf{sSets}}(\mathcal{C})$ as Bousfield localization with respect to τ-hypercovers.

Let $\mathcal{M}$ be a model category. A subcategory $\mathcal{S} \subseteq \mathcal{M}$ is called a *subcategory of weak equivalences* if all identity maps in $\mathcal{M}$ are in $\mathcal{S}$, $\mathcal{S}$ is closed under retracts and $\mathcal{S}$ satisfies the two out of three property.

Definition 1.1.2. Suppose $\mathcal{S}$ be a subcategory of weak equivalence. An object X of $\mathcal{M}$ is called $\mathcal{S}$-local if

- X is fibrant object in $\mathcal{M}$;
- For every morphism $Y \xrightarrow{f} Z \in \mathcal{S}$, the induced morphism

  $$f^* \colon [Z, X] \to [Y, X]$$

  is bijection, where $[-, -]$ is hom-set in homotopy category of $\mathcal{M}$.

In particular, if $\mathcal{M}$ is a simplicial model category (we omit the definition and the only things we need to know are (1) $\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C})$ is simplicial model category; (2) Hom-sets in this category are in **sSets**). Then $X \in \mathcal{M}$ is called $\mathcal{S}$-local if

- X is fibrant object in $\mathcal{M}$.
- For every morphism $Y \xrightarrow{f} Z \in \mathcal{S}$, the induced morphism

  $$f^* \colon \mathcal{M}(Z, X) \to \mathcal{M}(Y, X)$$

  is weak equivalence in **sSets**.

And a morphism $X \xrightarrow{f} Y \in \mathcal{M}$ if and only if for all $\mathcal{S}$-local object Z, the induced morphism

$$f^* \colon \mathcal{M}(Y, Z) \to \mathcal{M}(X, Z)$$

is weak equivalence in **sSets**.

Definition 1.1.3 (Bousfield localization)**.** Let $\mathbf{Fib}_{\mathcal{S}}$ be the class of morphisms in $\mathcal{M}$ satisfy RLP for $\mathcal{S} \cap \mathbf{Cof}$. If tuple $(S, \mathbf{Cof}, \mathbf{Fib}_{\mathcal{S}})$ forms a model structure on $\mathcal{M}$, then a $\mathcal{S}$-fibrant replacement functor $X \mapsto LX$ is called Bousfield localization with respect to $\mathcal{S}$ and the new model category is denoted by $L_{\mathcal{S}}\mathcal{M}$.

Theorem 1.1.4. *For category of simplical presheaves on essentially small category $\mathcal{C}$, the global model structure admits (left) Bousfield localization for all $\mathcal{S}$.*

Let $\mathcal{C}$ be a site with Grothendieck topology τ. Then define $\mathcal{W}_\tau$ be the smallest subcategory of weak equivalences contains class of τ-hypercovers $U_* \to X$. Then we have

Theorem 1.1.5. *The identity functor $L_{\mathcal{W}_\tau}\mathbf{PSh}^{\mathbf{sSets}}(\mathcal{C}) \to \mathbf{PSh}^{\mathbf{sSets}}_\tau(\mathcal{C})$ is Quillen equivalence.*