Positive Characteristic Geometry

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1. Witt Vector cohomology

Let p be a prime number. We then define the Witt polynomials ($\mathbb{Z}$-coefficient polynomials) respect to p as follows

$$W_0(x_0) := x_0$$
$$W_1(x_0, x_1) := x_0^p + px_1$$
$$\cdots$$
$$W_n(x_0, \cdots, x_n) := \sum_{i=0}^{n} p^i {x_i}^{p^{n-i}}$$

Let $x = (x_0, \cdots, x_n), y = (y_0, \cdots, y_n)$. From Serre's theorem [], there are polynomials $S_0, \cdots, S_n$, $P_0, \cdots, P_n$ with $2n+2$ variables satisfying

$$W_n(x) + W_n(y) = W_n(S_0(x,y), \cdots, S_n(x,y))$$
$$W_n(x) \cdot W_n(y) = W_n(P_0(x,y), \cdots, P_n(x,y))$$

For example,

$$S_0(x,y) = x_0 + y_0 \quad S_1(x,y) = x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0+y_0)^p}{p}$$
$$P_0(x,y) = x_0 y_0 \qquad P_1(x,y) = y_0^p x_1 + y_1 x_0^p + p x_1 y_1$$

Now, we define a new ring structure on n-couple for ring R, which is denoted by $\mathcal{W}_n(R)$. For all $X, Y in \mathcal{W}_n(R)$

$$X + Y = (S_0(X,Y), \cdots, S_{n-1}(X,Y))$$
$$X \cdot Y = (P_0(X,Y), \cdots, P_{n-1}(X,Y))$$

Let X be non-singular variety over perfect field k of characteristic $p > 0$ with dimension n. $\mathcal{W}_k$ is sheaf of Witt vectors of lenght k.

We have three operators

- Frobenius $F\colon \mathcal{W}_k\mathcal{O}_X \to \mathcal{W}_k\mathcal{O}_X$ locally defined by $(a_0, \cdots, a_{k-1}) \mapsto (a_0^p, \cdots, a_{k-1}^p)$;
- Verschieburg $V\colon \mathcal{W}_k\mathcal{O}_X \to \mathcal{W}_{k+1}\mathcal{O}_X$ locally defined by $(a_0, \cdots, a_{k-1}) \mapsto (0, \cdots, a_{k-1})$;
- Restriction $R\colon \mathcal{W}_{k+1}\mathcal{O}_X \to \mathcal{W}_k\mathcal{O}_X$ locally defined by $(a_0, \cdots, a_n) \mapsto (a_0, \cdots, a_{n-1})$

It is well-known that

$$RVF = RFV = FRV = p \cdot \mathrm{id}_{\mathcal{W}_k\mathcal{O}_X}$$

where the "$\cdot$" is multiplication in $\mathcal{W}_k\mathcal{O}_X$.

Let $\mathcal{W}(M) := \lim_k \mathcal{W}_{k+1}(M) \xrightarrow{R} \mathcal{W}_k(M)$ inverse limit of projective system induced by R. For example, since $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k\mathbb{Z}$, we have $\mathcal{W}(\mathbb{F}_p) = \mathbb{Z}_p$ the p-adic integer ring.

We have following two canonical exact sequence of $\mathcal{O}_X$-module

$$0 \to \mathcal{W}_{k-1}\mathcal{O}_X \xrightarrow{V} \mathcal{W}_k\mathcal{O}_X \xrightarrow{R^{k-1}} \mathcal{O}_X \to 0 \tag{1}$$
$$0 \to \mathcal{O}_X \to \mathcal{W}_k\mathcal{O}_X \xrightarrow{R} \mathcal{W}_{k-1}\mathcal{O}_X \to 0 \tag{2}$$

Witt vector cohomology is defined as

$$H^n(X, \mathcal{W}\mathcal{O}_X) = \lim_k H^n(X, \mathcal{W}_k\mathcal{O}_X)$$

in projective system $[R : H^n(X, \mathcal{W}_{k+1}\mathcal{O}_X) \to H^n(X, \mathcal{W}_k\mathcal{O}_X)]$.

2. Varieties in positive characteristic under Frobenius

Let X be a scheme over $\mathbb{F}_p$. F_X is called **absolute Frobenius morphism** of X which is induced by $f \mapsto f^p$ on affine covers. Frobenius map σ of field $\mathbb{F}_p$ induces endmorphism of Spec $\mathbb{F}_p$. It is also denoted by F_p. Furthermore, we have following base change

$$
\begin{tikzcd}
X \arrow[drr, bend left, "F_X"] \arrow[dr, dashed, "F"] \arrow[ddr, bend right, "f"'] & & \\
& X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& \mathrm{Spec}\ \mathbb{F}_p \arrow[r, "F_p"] & \mathrm{Spec}\ \mathbb{F}_p
\end{tikzcd} \tag{3}
$$

Since F_X and F_p are homeomorphisms on underlying topological space of spectrum Spec $\mathbb{F}_p$, F is also homeomorphism on underlying space. Hence it is easy to see that F_X is affine morphism of finite type. For all $\mathcal{O}_X$-module $\mathcal{F}$ on X, $(F_X)_*(\mathcal{F})$ is same to $\mathcal{F}$ as sheaves over abelian groups although the their $\mathcal{O}_X$-module structures are different. Let $B^i_X = \mathrm{Im}(d : \Omega^{i-1}_X \to \Omega^i_X)$, then we have following exact sequences from definitions of Frobenius.

$$
\mathcal{O}_X \xrightarrow{F_X} \mathcal{O}_X \xrightarrow{d} B^1_X \qquad \mathcal{O}_{X'} \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{F_*d} F_*B^1_X \tag{4}
$$

Example 2.1. Let $Y = \mathrm{Spec}\ A, X = \mathbb{A}^n_Y$. We have

$$
\begin{aligned}
\mathcal{O}_{X'} &= (A, F_A) \otimes_A A[x_1, x_2, \cdots, x_n] \\
&\cong A^p[x_1, x_2, \cdots, x_n]
\end{aligned}
$$

If k is perfect field, i.e. Frobenius map $\sigma : k \to k$ is automorphism, then absolute Frobenius morphism is automorphism of scheme Spec k. Hence the pull-back $X' \to X$ is also isomorphism and under isomorphism relative Frobenius F do same as absolute Frobenius F_X on X. So we also denote F_X by F in this case without confusion.

2.1. F-split.

Definition 2.2. The k-scheme of characteristic $p > 0$ is called Frobenius split if exact sequence from Cartier isomorphism

$$
0 \longrightarrow \mathcal{O}_{X'} \xrightarrow{F^\#} F_*\mathcal{O}_X \xrightarrow{d} F_*B^1_X \longrightarrow 0 \tag{5}
$$

is split exact.

If k is perfect, then we have absolute Cartier isomorphism C^{-1}_{abs}, which induce following exact sequences

$$
0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to F_*B^1_X \to 0 \tag{6}
$$

$$
0 \to F_*B^n_X \to F_*\Omega^n_X \xrightarrow{C_{\mathrm{abs}}} \Omega^n_X \to 0 \tag{7}
$$

Under assumption that k is perfect, X is Frobenius split if and only if 6 is split. This is because in this condition, base change of X along Frobenius morphism of field k is isomorphic to X. To simplify the notation, we denote $F_*B^i_X$ or $(F_X)_*B^i_X$ by (B^i_X).

Suppose ξ is corresponding extension class of 5 in $\mathrm{Ext}^1_{\mathcal{O}_{X'}}((B^1_X), \mathcal{O}_{X'})$. It follows that X is F-split if and only if $\xi = 0$. In particular, if extension group $\mathrm{Ext}^1_{\mathcal{O}_{X'}}((B^1_X), \mathcal{O}_{X'})$ vanishes, then 5 must be split. Writing it in cohomology form, we have $H^1(X', (B^1_X)^\vee) = \mathrm{Ext}^1_{\mathcal{O}_{X'}}((B^1_X), \mathcal{O}_{X'})$.

Lemma 2.1. *If X is F-split, then $H^i(X, B^1_X) = 0$ for all $i \geq 0$.*

Proof. Exact sequence 6 being splitting implies it induces isomorphism

$$
\mathcal{O}_X \oplus (B^1_X) \cong (F_X)_*\mathcal{O}_X
$$

Hence at cohomolgy level, we have

$$
H^i(X, \mathcal{O}_X) \oplus H^i(X, (B^1_X)) \cong H^i(X, (F_X)_*\mathcal{O}_X) \tag{8}
$$

while $H^i(X, (F_X)_*\mathcal{O}_X) = H^i(X, \mathcal{O}_X)$ for all $i \geq 0$, hence we have $\dim H^i(X, (B^1_X)) = 0$, so that $H^i(X, B^1_X)$ vanishes. $\square$

2.2. Slope Spectral Sequence. Slope spectral sequence is constructed in works [] of Luc Illusie on de Rham-Witt cohomology and comparasion to crystalline cohomology. It plays essential role in de Rham-Witt cohomology as Hodge-de Rham spectral sequence in de Rham cohomology. Actually, this spectral sequence gives explict descibtion of F-crystal structure on crystalline cohomology with de Rham-Witt cohomology.

Let X be smooth variety over perfect field k with characteristic $p > 0$. We have following de Rham-Witt complexes according to Illusie's works

$$\mathcal{W}\Omega_X^* := [\mathcal{W}\mathcal{O}_X \to \mathcal{W}\Omega_X^1 \to \cdots \to \mathcal{W}\Omega_X^n] \tag{9}$$

We have canonical filtration on $\mathcal{W}\Omega_X^*$ as Hodge filtration in Hodge theory such that

$$F^i\mathcal{W}\Omega_X^* = \mathcal{W}\Omega^{\geq i} = [0 \to \cdots \to 0 \to \mathcal{W}\Omega_X^i \cdots \to \mathcal{W}\Omega_X^n] \tag{10}$$

Filtration $F^i\mathcal{W}\Omega^*X$ induces spectral sequence

$$E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*) \tag{11}$$

which is called **slope spectral sequence** of X.Illusie proved that hypercohomology $H^n(X, \mathcal{W}\Omega_X^*)$ computes crystalline cohomology $H^n_{\text{crys}}(X/W)$ and $E_1^{i,j} \otimes K$ without p-torsion. Hence

Theorem 2.2 (Illusie). *If $H^j(X, \mathcal{W}\Omega_X^i)$ are all torsion-free, then the slope spectral sequence*

$$E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*) \tag{12}$$

degenerates in degree 1.

As name of this spectral sequence, it captures information of "slope" of F-crystal on $H^*_{\text{crys}}(X/W) \otimes K$. We have following corollary

Corollary 2.3 (Bloch). *For any i, canonical homomorphism $H^*(X, \mathcal{W}\Omega_X^{\geq i}) \hookrightarrow H^*_{crys}(X/W)$ and $H^*_{crys}(X/W) \twoheadrightarrow H^*(X, \mathcal{W}\Omega_X^{\leq i})$ induces isomorphism*

$$H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \xrightarrow{\sim} (H^*(X/W) \otimes K)_{\geq i} \tag{13}$$

$$(H^*_{crys}(X/W) \otimes K)_{[0,i[} \xrightarrow{\sim} H^*(X, \mathcal{W}\Omega_X^{\leq i}) \otimes K \tag{14}$$

Proof. We have following exact sequence

$$0 \to \mathcal{W}\Omega_X^{\geq i} \to \mathcal{W}\Omega_X^* \to \mathcal{W}\Omega_X^{\leq i-1} \to 0$$

Since connection map $0 = d_1 \otimes K : H^{*-1}(X, \mathcal{W}\Omega^{\leq i-1}) \to H^*(X, \mathcal{W}\Omega^{\geq i})$, upper exact sequence induces exact sequence of F-isocrystals

$$0 \to H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \to H^*_{\text{crys}}(X/W) \otimes K \to H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \otimes K \to 0$$

We have already known that $F(\operatorname{Im} H^*(X, \mathcal{W}\Omega_X^{\geq i})) \subseteq p^i H^*_{\text{crys}}(X/W)$. Hence the slope of $H^*(X, \mathcal{W}\Omega_X^{\geq i})$ is greater than i. On other hand,

$$H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \cong W_\sigma[[V]][F]/(FV - p^i, VF - p^i)$$

It implies that slope of $H^*(X, \mathcal{W}\Omega_X^{\leq i-1})$ strictly less than i. Hence we can conclude the proof. □

On other hand, we can describe Hodge polygon and Newton polygon explicitly. In fact, there is following theorem due to Ogus.

Theorem 2.4. *Let*

- $h^i = h^{i,n-i} = \dim_k H^{n-i}(X, \Omega_X^i)$
- $a_i = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i)/VH^i(X, \mathcal{W}\Omega_X^i)$
- $a'_i = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i)/FH^{n-i}(X, \mathcal{W}\Omega_X^i)$

then

$$a'_n = 0 \quad h^0 = a_0 \quad h^i = a'_{i-1} + a_i (1 \leq i \leq n)$$

and

$$Hdg(t) = (\sum_{i \leq t} h^i, \sum_{i \leq t} ih^i)$$

$$Nwt(t) = Hdg(t) + (a'_{t-1}, ta'_{t-1})$$

Corollary 2.5. *If $H^j(X, \mathcal{W}\Omega^i_X)$ is torsion-free for all i, j, then X being ordinary is equivalent to condition that $Nwt(t) = Hdg(t)$.*

Proof. TBA ∎

2.3. Ordinary Varieties.

Definition 2.3. Let X be a smooth, proper variety over field k with $\operatorname{char} k > 0$. We say X is **ordinary** if it satisfies

$$H^i(X, B^j_X) = 0 \qquad \text{for all } i \geq 0, j > 0 \tag{15}$$

where $B^j_X = d(\Omega^{j-1}_{X/k})$ is the sheaf of boundaries of algebraic de Rham complex in degree j.

Equivalently, $H^i(X', F_* B^i_X) =$ for all $i \geq 0, j > 0$ implies ordinarity of X [1]. where F is relative Frobenius morphism $F = F_{X/k} : X \to X'$.

More numerical description of ordinarity according to property 2.5:

If $H^q(X, W\Omega^r_X)$ are all torsion free, then X is ordinary if and only if for every q, Newton polygon defined by the slopes of action of Frobenius on $H^q_{\text{crys}}(X/W)$ is equal to Hodge polygon.

Here we list some properties of ordinary varieties.

Proposition 2.6. *Let X be variety over perfect field k. If X is ordinary and with trivial canonical bundle, then X is splitting.*

Proof. Homological algebra tells us that exact sequence

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to (B^1_X) \to 0$$

lies in extension group $\operatorname{Ext}^1((B^1_X), \mathcal{O}_X)$ as extension class. Denote this class as ξ. If $\xi = 0$ then this sequence splits. Hence if this extension group vanishes then it has no choice but splits. But we have

$$H^1(X, (B^1_X)^\vee) = \operatorname{Ext}^1((B^1_X), \mathcal{O}_X)$$

and

$$H^1(X, (B^1_X)^\vee) \cong H^{n-1}(X, B^1_X \otimes \omega_X)^* \cong H^{n-1}(X, B^1_X)^*.$$

Hence it actually vanishes if X is ordinary. ∎

Proposition 2.7. *If S is algebraic surface over positive characteristic perfect field, then S being F-split $\Rightarrow$ ordinary.*

Proof. As Cartier isomorphism is actually trace map in Grothendieck duality, we have following two perfect pairings:

$$F_* \mathcal{O}_X \otimes F_* \Omega^2_X \to \Omega^2_X \tag{16}$$

$$F_* \Omega^1_X \otimes F_* \Omega^1_X \to \Omega^2_X \tag{17}$$

The pairing 16 induces perfect pairings

$$(B^1_X) \otimes (B^2_X) \to \Omega^2_X \qquad \mathcal{O}_X \otimes \Omega^n_X \to \Omega^2_X \tag{18}$$

by interchange following exact sequences with exact functor $\operatorname{Hom}(-, \Omega^2_X) = \operatorname{Hom}(-, \mathcal{O}_X)$.

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to (B^1_X) \to 0$$
$$0 \to (B^2_X) \to F_* \Omega^2_X \to \Omega^2_X \to 0$$

Hence with lemma 2.1

$$H^i(X, (B^2_X)) \cong H^{n-i}(X, (B^2_X)^\vee \otimes \Omega^2_X)^* \cong H^{n-i}(X, B^1_X)^* = 0 \tag{19}$$

for all $0 \leq i \leq n$. This completes the proof. ∎

Corollary 2.8. *If X is K3 surface, then X being F-split $\Leftrightarrow$ ordinary.*

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[1] by Leray spectral sequence

Actually, we can genenralize for perfect pairings appearing in proof of 2.7. As in [], we use following notations

$$B_m\Omega^i_X := C^{-1}(B_{m-1}\Omega^i_X) \qquad B_1\Omega^i_X := (B^i_X) \qquad B_0\Omega^i_X = \underline{0} \tag{20}$$

$$Z_m\Omega^i_X := C^{-1}(Z_{m-1}\Omega^i_X) \qquad Z_1\Omega^i_X := (Z^i_X) \qquad Z_0\Omega^i_X = \Omega^i_X \tag{21}$$

Note that we have following inclusion sequence

$$\underline{0} = B_0\Omega^i_X \subset B_1\Omega^i_X \subset \cdots \subset B_m\Omega^i_X \subset \cdots Z_m\Omega^i_X \subset Z_{m-1}\Omega^i_X \subset \cdots \subset Z_0\Omega^i_X = \Omega^i_X \tag{22}$$

$B_m\Omega^i_X$ and $Z_m\Omega^i_X$ can be viewed as locally free submodules of $F^m_*(\Omega^i_X)$.This implies that

$$0 \to B_m\Omega^i_X \to F^m_*\Omega^i_X \to F^m_*\Omega^i_X/B_m\Omega^i_X \to 0 \tag{23}$$

$$0 \to Z_m\Omega^{n-i}_X \to F^m_*\Omega^{n-i}_X \to F^m_*\Omega^{n-i}_X/Z_m\Omega^{n-i}_X \to 0 \tag{24}$$

are exact. $\mathcal{H}om(-,\Omega^n_X)$ is exact since Ω^n_X is trivial. Applying it on 24, we get exact sequence

$$0 \to \mathcal{H}om(F^m_*\Omega^{n-i}_X/Z_m\Omega^{n-i}_X,\Omega^n_X) \to \mathcal{H}om(F^m_*\Omega^{n-i}_X,\Omega^n_X) \to \mathcal{H}om(Z_m\Omega^{n-i}_X,\Omega^n_X) \to 0 \tag{25}$$

Since the middle term is isomorphism to $F^m_*\Omega^i_X$ induced from perfect pairing

$$F^m_*\Omega^i_X \otimes F^m_*\Omega^{n-i}_X \to \Omega^n_X \tag{26}$$

we can get following two canonical perfect pairings

$$B_m\Omega^i_X \otimes F^m_*\Omega^{n-i}_X/Z_m\Omega^{n-i}_X \to \Omega^n_X \tag{27}$$

$$Z_m\Omega^{n-i}_X \otimes F^m_*\Omega^i_X/B_m\Omega^i_X \to \Omega^n_X \tag{28}$$

Note that we have canonical isomorphism $F^m_*\Omega^{n-i}/Z_m\Omega^{n-i}_X \cong B_m\Omega^{n-i+1}_X$ induced by differential $d:\Omega^{n-i}_X \to \Omega^{n-i+1}_X$. Hence 27 becomes $B_m\Omega^i_X \otimes B_m\Omega^{n-i+1}_X \to \Omega^n_X$.In particular, taking $i=1, n=2, m=1$ we get perfect pairing 18.

Notice that we have exact sequence

$$0 \to B_1\Omega^1_X \to B_{m+1}\Omega^1_X \xrightarrow{C} B_m\Omega^1_X \to 0$$

Hence we have surjective maps $C: H^n(X, B_{m+1}\Omega^1_X) \to H^n(X, B_m\Omega^1_X)$. So we get a projective system, then taking inverse limit

$$H^n(X, B\Omega^1_X) := \lim_m H^n(X, B_m\Omega^1_X)$$

Definition 2.4. The dimension of $H^n(X, B\Omega^1_X)$ is called b-number of X.

In next subsection, we will use these to give explicit formula for height of Calabi-Yau varieties over positive characteristic which is also called h-number in [].

2.4. Formal group law. In algebraic geometry, **formal group** is referred by group object in category of formal schemes. If $\mathcal{G}$ is a functor from Artin algebras to groups which is left exact, then $\mathcal{G}$ is representable by formal group. $\mathcal{G}$ is called a **formal group law**. Let

$$F^r_S(S) = \ker\{H^r(X \times S, \mathbb{G}_m) \to H^r(X, \mathbb{G}_m)\}$$

the cohomology is étale cohomology here. Theorem of Artin and Mazur say that this functor $F^r_S : \mathrm{Art}_k \to \mathrm{Ab}$ is pro-representable by formal group Φ_X if X is proper Calabi-Yau with dimension n over perfect field k. Moreover, we have that tangent space of Φ_X is isomorphic to $H^n(X, \mathcal{O}_X)$. Hence since Calabi-Yau variety is with geometric genus 1, e.g. $\dim_k H^n(X, \mathcal{O}_X) = 1$, Φ_X is formal group of dimension 1.

2.5. Height of Calabi-Yau varieties. Let X be Calabi-Yau variety over perfect field k with characteristic p and Φ_X be its formal Brauer group. Then we have following theorem

Theorem 2.9. *If X is of dimension n, then we have following formula for hegiht*

$$h(\Phi_X) = \min\left\{i \geq 1 : [F : H^n(W_i\mathcal{O}_X) \to H^n(W_i\mathcal{O}_X)] \neq 0\right\} \tag{29}$$

Before giving the proof of this theorem, we will prepare some properties of Witt vector cohomology of Calabi-Yau varieties.

Proposition 2.10. *Let X be Calabi-Yau variety over perfect field k of dimension n. We have*

(1) *$H^i(X,\mathcal{W}_j\mathcal{O}_X)=0$ for all $j>0, 1\le i\le n-1$. Furthermore, $H^i(X,\mathcal{W}\mathcal{O}_X)=0$;*
(2) *Pull-back of R on cohomology $R: H^n(X,\mathcal{W}_k\mathcal{O}_X)\to H^n(X,\mathcal{W}_k\mathcal{O}_X)$ is surjective with kernel $H^n(X,\mathcal{O}_X)$;*
(3) *Assume that for some $0<k\le n$ the map $F: H^k(X,\mathcal{W}_j\mathcal{O}_X)\to H^k(X,\mathcal{W}_j\mathcal{O}_X)$ vanishes, then for all $0\le i\le j$ the map $F\colon H^k(X,\mathcal{W}_i\mathcal{O}_X)\to H^k(X,\mathcal{W}_i\mathcal{O}_X)$ vanishes too. Moreover, for all $H^n(X,\mathcal{W}_i\mathcal{O}_X)$ is vector space over K;*
(4)
$$0\to H^n(X,\mathcal{W}_{i-1}\mathcal{O}_X)\xrightarrow{V} H^n(X,\mathcal{W}_i\mathcal{O}_X)\xrightarrow{R^{n-1}} H^n(X,\mathcal{O}_X)\to 0$$
and
$$0\to H^n(X,\mathcal{W}\mathcal{O}_X)\xrightarrow{V'} H^n(X,\mathcal{W}\mathcal{O}_X)\xrightarrow{R'} H^n(X,\mathcal{O}_X)\to 0$$
are both exact.

Proof. (1) Note that we have following exact sequence of sheaves for all $j\ge 1$
$$0\to \mathcal{W}_{j-1}\mathcal{O}_X\to \mathcal{W}_j\xrightarrow{R^{j-1}}\mathcal{O}_X\to 0$$
Hence we have exact sequence of groups
$$H^i(X,\mathcal{W}_{j-1}\mathcal{O}_X)\to H^i(X,\mathcal{W}_j\mathcal{O}_X)\to H^i(X,\mathcal{O}_X)$$
If $H^i(X,\mathcal{W}_{j-1})=0$ and $1\le i\le n-1$, then $H^i(X.\mathcal{W}_j\mathcal{O}_X)=0$ since two sides of this sequence are zero. Hence by induction, we complete the proof.

(2) We have following exact sequence
$$0\simeq H^{n-1}(X,\mathcal{W}_{k-1})\to H^n(X,\mathcal{O}_X)\to H^n(X,\mathcal{W}_k\mathcal{O}_X)\xrightarrow{R} H^n(X,\mathcal{W}_{k-1}\mathcal{O}_X)\to 0$$
It implies $H^n(X,\mathcal{O}_X)=\ker\left(H^n(\mathcal{W}_k\mathcal{O}_X)\xrightarrow{R} H^n(X,\mathcal{W}_{k-1}\mathcal{O}_X)\right)$.

(3) Note that following diagram commutes for all $j>i$
$$\begin{array}{ccc} H^n(X,\mathcal{W}_j\mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X,\mathcal{W}_i\mathcal{O}_X)\\ \downarrow F & & \downarrow F\\ H^n(X,\mathcal{W}_j\mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X,\mathcal{W}_i\mathcal{O}_X)\end{array}$$
Hence if left Frobenius map vanishes then so does right one.

□

Proof of theorem 2.9. Let $h=h(\Phi_X)$. If F is zero map on each $H^n(X,\mathcal{W}_i\mathcal{O}_X)$, then $H^n(X,\mathcal{W}\mathcal{O}_X)$ is k-vector space. But we have following exact sequence
$$0\to H^n(X,\mathcal{W}\mathcal{O}_X)\to H^n(X,\mathcal{W}\mathcal{O}_X)\to H^n(X,\mathcal{O}_X)\to 0$$
and $H^n(X,\mathcal{O}_X)$ is actually k-vector space. Hence $H^n(X,\mathcal{W}\mathcal{O}_X)$ is not of finite dimension as k-vector space. Since $h=\mathrm{rank}_{W(k)} H^n(X,\mathcal{W}\mathcal{O}_X)$, we can conclude that in this case Φ_X is of height ∞.

Notice that
$$H^n(X,\mathcal{W}\mathcal{O}_X)/VH^n(X,\mathcal{W}\mathcal{O}_X)\simeq H^n(X,\mathcal{O}_X)$$
as $W(k)$-module and $H^n(X,\mathcal{O}_X)$ is naturally k-vector space, they are isomorphic as k-vector space. Hence from equality
$$\mathrm{rank}_{W(k)}(D(\Phi_X))=\dim_k H^n(X,\mathcal{W}\mathcal{O}_X)/FH^n(X,\mathcal{W}\mathcal{O}_X)+\dim_k H^n(X,\mathcal{W}\mathcal{O}_X)/VH^n(X,\mathcal{W}\mathcal{O}_X)$$
we get
$$\dim_k H^n(X,\mathcal{W}\mathcal{O}_X)/FH^n(X,\mathcal{W}\mathcal{O}_X)=h-1$$
If $F: H^n(X,\mathcal{W}_i\mathcal{O}_X)\to H^n(X,\mathcal{W}_i\mathcal{O}_X)$ is zero ,then projection
$$[P_i]: H^n(X,\mathcal{W}\mathcal{O}_X)/FH^n(X,\mathcal{W}\mathcal{O}_X)\to H^n(X,\mathcal{W}_i\mathcal{O}_X)/FH^n(X,\mathcal{W}_i\mathcal{O}_X)$$
is well-defined and surjective since R,F commutes. As vector space, $H^n(X,\mathcal{W}_i\mathcal{O}_X)$ is of dimension i. This implies that we have inequality relation
$$h\ge i+1$$

Hence we get

$$h \geq \min\left\{i \geq 1 : [F : H^n(W_i\mathcal{O}_X) \to H^n(W_i\mathcal{O}_X)] \neq 0\right\}$$

Conversely, it is sufficient to prove $FH^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) = 0$.

Manin's result in [] implies that

$$H^n(X, \mathcal{W}\mathcal{O}_X) \cong D(\Phi_X) \cong \frac{W(k)[F,V]}{(F - V^{h-1})} \tag{30}$$

Hence $FH^n(X, \mathcal{W}\mathcal{O}_X) \cong V^{h-1}H^n(X, \mathcal{W}\mathcal{O}_X)$.

$$\begin{tikzcd} V^{h-1}H^n(X, \mathcal{W}\mathcal{O}_X) \arrow[r] \arrow[d, two heads] & H^n(X, \mathcal{W}\mathcal{O}_X) \arrow[d, two heads] \\ 0 = V^{h-1}H^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) \arrow[r] & H^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) \end{tikzcd}$$

commutes, this implies $FH^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) = 0$. □

We will call $h(\Phi_X)$ the h-number of X, denoted by $h(X)$.

Corollary 2.11. *For proper Calabi-Yau variety X over perfect field k, we have $b(X) = \dim_k H^n(X, \mathcal{W}\mathcal{O}_X)/FH^n(X$ Furthermore,*

$$\dim_k H^{n-1}(X, B_m\Omega^1_X) = \dim_k H^n(X, B_m\Omega^1_X) = \begin{cases} \min\{m, h-1\} & \text{if } h < \infty \\ m & \text{if } h = \infty \end{cases}$$

Proof. We have exact sequence

$$\mathcal{W}_m\mathcal{O}_X \xrightarrow{F} \mathcal{W}_m\mathcal{O}_X \xrightarrow{D_m} B_m\Omega^1_X \to 0$$

It induces

$$H^n(X, \mathcal{W}_m\mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W}_m\mathcal{O}_X) \to H^n(X, B_m\Omega^1_X) \to 0$$

Taking limit with m, we get

$$H^n(X, \mathcal{W}\mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W}\mathcal{O}_X) \to H^n(X, B\Omega^1_X) \to 0$$

exact. Hence $b(X) = h(X) - 1$. □

3. Eigenvalues of Frobenius on l-adic cohomology

3.1. Part I. Let α an eigenvalue of Frobenius acting on $H^i(V/\bar{k})$ where $H^*(V/\bar{k})$ is any one of the known cohomologies. Then by Poncaré duality $\frac{q^d}{\alpha}$ is an eigenvalue of Frobenius actiong on $H^{2d-i}(V/\bar{k})$. Thus, both α and $\frac{q^d}{\alpha}$ are algebraic integers. It follows that in the field $F = \mathbb{Q}(\alpha)$, α is a λ-adic unit for any prime λ which does not lie over $p = \operatorname{char} k$.

Since, by the Riemann hypothesis, one has that every complex absolute value of α is $q^{i/2}$, one is left with the intriguing question: What are the l-adic absolute values of α, for primes l lying over over p?

This is a question, which can by hindsight be seen to have had a long tradition — being related to Stickelberger's determination of the p-adic nature of Gauss sums and the Chevalley-Warning theorem which says that if N is the number of solution of a polynomial equation mod p.

It is reasonable to expect thta the study of the above question should use a p-adic cohomology, and it dose. To describe, briefly, the know results, we introduce the *Newton polygon* of a monotone increasing sequence $a_1, a_2, \cdots, a_m$ of non-negative rational numbers: by definition, it is the graph of the real-valued continuous piece-wise linear function $v_{a_1, a_2, \cdots a_m}$ on $[0, m]$ which takes the value 0 at 0 and whose derivate is the constant a_j on the interval $(j-1, j)$.

Now let $V/\mathcal{W}(k)$ be a smooth projective scheme. where $\mathcal{W}(k)$ is the ring of Witt vectors of k. Let V/k be its reduction to k through canonical morphism Spec $(k) \to$ Spec $(\mathcal{W}(k))$. Make the hypothesis that the de Rham cohomology of V/k is extremely well-behaved: namely, $H^q(V/\mathcal{W}(k), \Omega^p)$ is a free $\mathcal{W}(k)$-module (whose rank we shall denote $h^{p,q}$– the (p,q)-th Hodge number of V/k) for all p, q. Fix an integer r, and define the r-th Hodge polygon of V/k to be the Newton polygon of the sequence of integers.

$$\underbrace{0, 0, \cdots, 0}_{h^{0,r}}, \underbrace{1, 1, \cdots, 1}_{h^{1,r-1}}, \underbrace{2, 2, \cdots, 2,}_{h^{2,r-2}}, \cdots \underbrace{r, r, \cdots, r}_{h^{r,0}}$$

Define the r-th Newton polygon of V to be the Newton polygon of the sequence:

$$\mathrm{ord}_q\,\alpha_1, \mathrm{ord}_q\,\alpha_2, \cdots, \mathrm{ord}_q\,\alpha_m \quad (m = r\text{-th Betti number of } V)$$

where $\alpha_1, \cdots, \alpha_m \in \bar{K}$ are the complete set of eigenvalues (with multiplicities) of Frobenius acting on the r-dimensional p-adic cohomology of V/k; ord_q denotes the p-adic ord function normalized so that $\mathrm{ord}_q\, q = 1$; and the α_j are indexed in such a manner that the above sequence is monotone increasing.

What is known about the r-th Newton polygon of V/k is the following:

The r-th Newton polygon and the r-th Hodge polygon both end at the point $(m, rm/2)$ in the euclidean plane. The break-points (non-differentiable points) of these polygon occur at integral lattice points. The r-th Newton polygons lies in the closed region bounded by the r-th Hodge polygon, and the straight line joining $(0,0)$ and $(m, rm/2)$.

Note: One has a further symmetry in the geometry of the Newton polygon implied by the strong Lefschetz theorem, and also the existence of algebraic cohomology classes in dimension r implies that a part of the Newton polygon must have slope $r/2$.

General remarks: It was an important discovery of Dwork, that (in a certain, unobvious, sense) the eigenvalues of Frobenius vary p-adic analytically if one varies the variety V/k over a parameter space in characteristic p. A trace of this phenomenon may be seen in a theorem of Grothendieck: The Newton polygon rises under specialization of V/k.

3.2. Explicit description of Newton Polygon and Hodge Polygon. Let $H^n = H^n_{\mathrm{crys}}(X/W)$ be torsion free. Let

- $h^i = h^{i,n-i} = \dim_k H^{n-i}(X, \Omega^i_X)$
- $a_i = \dim H^{n-i}(X, \mathcal{W}\Omega^i_X)/VH^{n-i}(X, \mathcal{W}\Omega^i_X)$
- $a'_i = \dim H^{n-i}(X, \mathcal{W}\Omega^i_X)/FH^{n-i}(X, \mathcal{W}\Omega^i_X)$

then

$$h^0 = a_0 \quad h^i = a'_{i-1} + a_i (1 \le i \le n) \quad a'_n = 0$$

and

$$\mathrm{Hdg}(t) = \begin{cases} (0,0) & t = 0 \\ (\sum_{i \le t} h^i, \sum_{i \le t} ih^i) & 1 \le t \le n+1 \end{cases}$$

$$\mathrm{Nwt}(t) = \begin{cases} (0,0) & t = 0 \\ \mathrm{Hdg}(t) + (a'_{t-1}, ta'_{t-1}) & 1 \le t \le n+1 \end{cases}$$

Corollary 3.1. *If $H^j(X, \mathcal{W}\Omega^i_X)$ are all torsion-free, then X being ordinary is equivalent to $Nwt(t) = Hdg(t)$.*