

Positive Characteristic Geometry

Chow.

1.	Witt Vector cohomology	1
2.	Varieties in positive characteristic under Frobenius	1
2.1.	F-split	2
2.2.	Slope Spectral Sequence	3
2.3.	Ordinary Varieties	4
2.4.	Formal group law	5
2.5.	Height of Calabi-Yau varieties	5
3.	Eigenvalues of Frobenius on l -adic cohomology	7
3.1.	Part I	7
3.2.	Explicit description of Newton Polygon and Hodge Polygon	8

1. WITT VECTOR COHOMOLOGY

Let p be a prime number. We then define the Witt polynomials (\mathbb{Z} -coefficient polynomials) respect to p as follows

$$\begin{aligned} W_0(x_0) &:= x_0 \\ W_1(x_0, x_1) &:= x_0^p + px_1 \\ &\dots \\ W_n(x_0, \dots, x_n) &:= \sum_{i=0}^n p^i x_i^{p^{n-i}} \end{aligned}$$

Let $x = (x_0, \dots, x_n), y = (y_0, \dots, y_n)$. From Serre's theorem [], there are polynomials $S_0, \dots, S_n, P_0, \dots, P_n$ with $2n+2$ variables satisfying

$$\begin{aligned} W_n(x) + W_n(y) &= W_n(S_0(x, y), \dots, S_n(x, y)) \\ W_n(x) \cdot W_n(y) &= W_n(P_0(x, y), \dots, P_n(x, y)) \end{aligned}$$

For example,

$$\begin{aligned} S_0(x, y) &= x_0 + y_0 & S_1(x, y) &= x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0 + y_0)^p}{p} \\ P_0(x, y) &= x_0 y_0 & P_1(x, y) &= y_0^p x_1 + y_1 x_0^p + p x_1 y_1 \end{aligned}$$

Now, we define a new ring structure on n -couple for ring R , which is denoted by $\mathcal{W}_n(R)$. For all $X, Y \in \mathcal{W}_n(R)$

$$\begin{aligned} X + Y &= (S_0(X, Y), \dots, S_{n-1}(X, Y)) \\ X \cdot Y &= (P_0(X, Y), \dots, P_{n-1}(X, Y)) \end{aligned}$$

Let X be non-singular variety over perfect field k of characteristic $p > 0$ with dimension n . \mathcal{W}_k is sheaf of Witt vectors of length k .

We have three operators

- Frobenius $F: \mathcal{W}_k \mathcal{O}_X \rightarrow \mathcal{W}_k \mathcal{O}_X$ locally defined by $(a_0, \dots, a_{k-1}) \mapsto (a_0^p, \dots, a_{k-1}^p)$;
- Verschiebung $V: \mathcal{W}_k \mathcal{O}_X \rightarrow \mathcal{W}_{k+1} \mathcal{O}_X$ locally defined by $(a_0, \dots, a_{k-1}) \mapsto (0, \dots, a_{k-1})$;
- Restriction $R: \mathcal{W}_{k+1} \mathcal{O}_X \rightarrow \mathcal{W}_k \mathcal{O}_X$ locally defined by $(a_0, \dots, a_n) \mapsto (a_0, \dots, a_{n-1})$

It is well-known that

$$RVF = RFV = FRV = p \cdot \text{id}_{\mathcal{W}_k \mathcal{O}_X}$$

where the "·" is multiplication in $\mathcal{W}_k \mathcal{O}_X$.

Let $\mathcal{W}(M) := \lim_k \mathcal{W}_{k+1}(M) \xrightarrow{R} \mathcal{W}_k(M)$ inverse limit of projective system induced by R . For example, since $\mathcal{W}_k(\mathbb{F}_p) = \mathbb{Z}/p^k \mathbb{Z}$, we have $\mathcal{W}(\mathbb{F}_p) = \mathbb{Z}_p$ the p -adic integer ring.

We have following two canonical exact sequence of \mathcal{O}_X -module

$$\begin{aligned} (1) \quad & 0 \rightarrow \mathcal{W}_{k-1} \mathcal{O}_X \xrightarrow{V} \mathcal{W}_k \mathcal{O}_X \xrightarrow{R^{k-1}} \mathcal{O}_X \rightarrow 0 \\ (2) \quad & 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{W}_k \mathcal{O}_X \xrightarrow{R} \mathcal{W}_{k-1} \mathcal{O}_X \rightarrow 0 \end{aligned}$$

Witt vector cohomology is defined as

$$H^n(X, \mathcal{W} \mathcal{O}_X) = \lim_k H^n(X, \mathcal{W}_k \mathcal{O}_X)$$

in projective system $[R: H^n(X, \mathcal{W}_{k+1} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_k \mathcal{O}_X)]$.

2. VARIETIES IN POSITIVE CHARACTERISTIC UNDER FROBENIUS

Let X be a scheme over \mathbb{F}_p . F_X is called **absolute Frobenius morphism** of X which is induced by $f \mapsto f^p$ on affine covers. Frobenius map σ of field \mathbb{F}_p induces endmorphism of $\text{Spec } \mathbb{F}_p$. It is also denoted by F_p . Furthermore, we have following base change

$$(3) \quad \begin{array}{ccccc} & & F_X & & \\ & \curvearrowright & & \searrow & \\ X & & & & X \\ & \swarrow F & \longrightarrow & & \\ & X' & & & \\ & \downarrow f' & & & \downarrow f \\ & \text{Spec } \mathbb{F}_p & \xrightarrow{F_p} & \text{Spec } \mathbb{F}_p & \end{array}$$

Since F_X and F_p are homeomorphisms on underlying topological space of spectrum $\text{Spec } \mathbb{F}_p$, F is also homeomorphism on underlying space. Hence it is easy to see that F_X is affine morphism of finite type. For all \mathcal{O}_X -module \mathcal{F} on X , $(F_X)_*(\mathcal{F})$ is same to \mathcal{F} as sheaves over abelian groups although the their \mathcal{O}_X -module structures are different. Let $B_X^i = \text{Im}(d : \Omega_X^{i-1} \rightarrow \Omega_X^i)$, then we have following exact sequences from definitions of Frobenius.

$$(4) \quad \mathcal{O}_X \xrightarrow{F_X} \mathcal{O}_X \xrightarrow{d} B_X^1 \quad \mathcal{O}_{X'} \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{F_*d} F_*B_X^1$$

Example 2.1. Let $Y = \text{Spec } A, X = \mathbb{A}_Y^n$. We have

$$\begin{aligned} \mathcal{O}_{X'} &= (A, F_A) \otimes_A A[x_1, x_2, \dots, x_n] \\ &\cong A^p[x_1, x_2, \dots, x_n] \end{aligned}$$

If k is perfect field, i.e. Frobenius map $\sigma : k \rightarrow k$ is automorphism, then absolute Frobenius morphism is automorphism of scheme $\text{Spec } k$. Hence the pull-back $X' \rightarrow X$ is also isomorphism and under isomorphism relative Frobenius F do same as absolute Frobenius F_X on X . So we also denote F_X by F in this case without confusion.

2.1. F-split.

Definition 2.2. The k -scheme of characteristic $p > 0$ is called Frobenius split if exact sequence from Cartier isomorphism

$$(5) \quad 0 \longrightarrow \mathcal{O}_{X'} \xrightarrow{F^\#} F_*\mathcal{O}_X \xrightarrow{d} F_*B_X^1 \longrightarrow 0$$

is split exact.

If k is perfect, then we have absolute Cartier isomorphism C_{abs}^{-1} , which induce following exact sequences

$$(6) \quad 0 \rightarrow \mathcal{O}_X \rightarrow F_*\mathcal{O}_X \rightarrow F_*B_X^1 \rightarrow 0$$

$$(7) \quad 0 \rightarrow F_*B_X^n \rightarrow F_*\Omega_X^n \xrightarrow{C_{\text{abs}}} \Omega_X^n \rightarrow 0$$

Under assumption that k is perfect, X is Frobenius split if and only if 6 is split. This is because in this condition, base change of X along Frobenius morphism of field k is isomorphic to X . To simplify the notation, we denote $F_*B_X^i$ or $(F_X)_*B_X^i$ by (B_X^i) .

Suppose ξ is corresponding extension class of 5 in $\text{Ext}_{\mathcal{O}_{X'}}^1((B_X^1), \mathcal{O}_{X'})$. It follows that X is F -split if and only if $\xi = 0$. In particular, if extension group $\text{Ext}_{\mathcal{O}_{X'}}^1((B_X^1), \mathcal{O}_{X'})$ vanishes, then 5 must be split. Writing it in cohomology form, we have $H^1(X', (B_X^1)^\vee) = \text{Ext}_{\mathcal{O}_{X'}}^1((B_X^1), \mathcal{O}_{X'})$.

Lemma 2.1. If X is F -split, then $H^i(X, B_X^1) = 0$ for all $i \geq 0$.

Proof. Exact sequence 6 being splitting implies it induces isomorphism

$$\mathcal{O}_X \oplus (B_X^1) \cong (F_X)_*\mathcal{O}_X$$

Hence at cohomolgy level, we have

$$(8) \quad H^i(X, \mathcal{O}_X) \oplus H^i(X, (B_X^1)) \cong H^i(X, (F_X)_*\mathcal{O}_X)$$

while $H^i(X, (F_X)_*\mathcal{O}_X) = H^i(X, \mathcal{O}_X)$ for all $i \geq 0$, hence we have $\dim H^i(X, (B_X^1)) = 0$, so that $H^i(X, B_X^1)$ vanishes. \square

2.2. Slope Spectral Sequence. Slope spectral sequence is constructed in works [] of Luc Illusie on de Rham-Witt cohomology and comparasion to crystalline cohomology. It plays essential role in de Rham-Witt cohomology as Hodge-de Rham spectral sequence in de Rham cohomology. Actually, this spectral sequence gives explict descibtion of F -crystal structure on crystalline cohomology with de Rham-Witt cohomology.

Let X be smooth variety over perfect field k with characteristic $p > 0$. We have following de Rham-Witt complexes according to Illusie's works

$$(9) \quad \mathcal{W}\Omega_X^* := [\mathcal{W}\mathcal{O}_X \rightarrow \mathcal{W}\Omega_X^1 \rightarrow \cdots \rightarrow \mathcal{W}\Omega_X^n]$$

We have canonical filtration on $\mathcal{W}\Omega_X^*$ as Hodge filtration in Hodge theory such that

$$(10) \quad F^i \mathcal{W}\Omega_X^* = \mathcal{W}\Omega_X^{\geq i} = [0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathcal{W}\Omega_X^i \cdots \rightarrow \mathcal{W}\Omega_X^n]$$

Filtration $F^i \mathcal{W}\Omega^* X$ induces spectral sequence

$$(11) \quad E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*)$$

which is called **slope spectral sequence** of X . Illusie proved that hypercohomology $H^n(X, \mathcal{W}\Omega_X^*)$ computes crystalline cohomology $H_{\text{crys}}^n(X/W)$ and $E_1^{i,j} \otimes K$ without p -torsion. Hence

Theorem 2.2 (Illusie). *If $H^j(X, \mathcal{W}\Omega_X^i)$ are all torsion-free, then the slope spectral sequence*

$$(12) \quad E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*)$$

degenerates in degree 1.

As name of this spectral sequence, it captures information of "slope" of F -crystal on $H_{\text{crys}}^*(X/W) \otimes K$. We have following corollary

Corollary 2.3 (Bloch). *For any i , canonical homomorphism $H^*(X, \mathcal{W}\Omega_X^{\geq i}) \hookrightarrow H_{\text{crys}}^*(X/W)$ and $H_{\text{crys}}^*(X/W) \rightarrow H^*(X, \mathcal{W}\Omega_X^{\leq i})$ induces isomorphism*

$$(13) \quad H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \xrightarrow{\sim} (H^*(X/W) \otimes K)_{\geq i}$$

$$(14) \quad (H_{\text{crys}}^*(X/W) \otimes K)_{[0,i]} \xrightarrow{\sim} H^*(X, \mathcal{W}\Omega_X^{\leq i}) \otimes K$$

Proof. We have following exact sequence

$$0 \rightarrow \mathcal{W}\Omega_X^{\geq i} \rightarrow \mathcal{W}\Omega_X^* \rightarrow \mathcal{W}\Omega_X^{\leq i-1} \rightarrow 0$$

Since connection map $0 = d_1 \otimes K : H^{*-1}(X, \mathcal{W}\Omega_X^{\leq i-1}) \rightarrow H^*(X, \mathcal{W}\Omega_X^{\geq i})$, upper exact sequence induces exact sequence of F -isocrystals

$$0 \rightarrow H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \rightarrow H_{\text{crys}}^*(X/W) \otimes K \rightarrow H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \otimes K \rightarrow 0$$

We have already known that $F(\text{Im } H^*(X, \mathcal{W}\Omega_X^{\geq i})) \subseteq p^i H_{\text{crys}}^*(X/W)$. Hence the slope of $H^*(X, \mathcal{W}\Omega_X^{\geq i})$ is greater than i . On other hand,

$$H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \cong W_\sigma[[V]][F]/(FV - p^i, VF - p^i)$$

It implies that slope of $H^*(X, \mathcal{W}\Omega_X^{\leq i-1})$ strictly less than i . Hence we can conclude the proof. \square

On other hand, we can describe Hodge polygon and Newton polygon explicitly. In fact, there is following theorem due to Ogus.

Theorem 2.4. *Let*

- $h^i = h^{i,n-i} = \dim_k H^{n-i}(X, \Omega_X^i)$
- $a_i = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i) / VH^i(X, \mathcal{W}\Omega_X^i)$
- $a'_i = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i) / FH^{n-i}(X, \mathcal{W}\Omega_X^i)$

then

$$a'_n = 0 \quad h^0 = a_0 \quad h^i = a'_{i-1} + a_i (1 \leq i \leq n)$$

and

$$Hdg(t) = \left(\sum_{i \leq t} h^i, \sum_{i \leq t} ih^i \right)$$

$$Nwt(t) = Hdg(t) + (a'_{t-1}, ta'_{t-1})$$

Corollary 2.5. *If $H^j(X, \mathcal{W}\Omega_X^i)$ is torsion-free for all i, j , then X being ordinary is equivalent to condition that $Nwt(t) = Hdg(t)$.*

Proof. TBA □

2.3. Ordinary Varieties.

Definition 2.3. Let X be a smooth, proper variety over field k with $\text{char } k > 0$. We say X is **ordinary** if it satisfies

$$(15) \quad H^i(X, B_X^j) = 0 \quad \text{for all } i \geq 0, j > 0$$

where $B_X^j = d(\Omega_{X/k}^{j-1})$ is the sheaf of boundaries of algebraic de Rham complex in degree j .

Equivalently, $H^i(X', F_*B_X^i) = 0$ for all $i \geq 0, j > 0$ implies ordinarity of X ¹. where F is relative Frobenius morphism $F = F_{X/k} : X \rightarrow X'$.

More numerical description of ordinarity according to property 2.5:

If $H^q(X, \mathcal{W}\Omega_X^r)$ are all torsion free, then X is ordinary if and only if for every q , Newton polygon defined by the slopes of action of Frobenius on $H_{\text{crys}}^q(X/W)$ is equal to Hodge polygon.

Here we list some properties of ordinary varieties.

Proposition 2.6. *Let X be variety over perfect field k . If X is ordinary and with trivial canonical bundle, then X is splitting.*

Proof. Homological algebra tells us that exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow F_*\mathcal{O}_X \rightarrow (B_X^1) \rightarrow 0$$

lies in extension group $\text{Ext}^1((B_X^1), \mathcal{O}_X)$ as extension class. Denote this class as ξ . If $\xi = 0$ then this sequence splits. Hence if this extension group vanishes then it has no choice but splits. But we have

$$H^1(X, (B_X^1)^\vee) = \text{Ext}^1((B_X^1), \mathcal{O}_X)$$

and

$$H^1(X, (B_X^1)^\vee) \cong H^{n-1}(X, B_X^1 \otimes \omega_X)^* \cong H^{n-1}(X, B_X^1)^*.$$

Hence it actually vanishes if X is ordinary. □

Proposition 2.7. *If S is algebraic surface over positive characteristic perfect field, then S being F -split \Rightarrow ordinary.*

Proof. As Cartier isomorphism is actually trace map in Grothendieck duality, we have following two perfect pairings:

$$(16) \quad F_*\mathcal{O}_X \otimes F_*\Omega_X^2 \rightarrow \Omega_X^2$$

$$(17) \quad F_*\Omega_X^1 \otimes F_*\Omega_X^1 \rightarrow \Omega_X^2$$

The pairing 16 induces perfect pairings

$$(18) \quad (B_X^1) \otimes (B_X^2) \rightarrow \Omega_X^2 \quad \mathcal{O}_X \otimes \Omega_X^n \rightarrow \Omega_X^2$$

by interchange following exact sequences with exact functor $\text{Hom}(-, \Omega_X^2) = \text{Hom}(-, \mathcal{O}_X)$.

$$0 \rightarrow \mathcal{O}_X \rightarrow F_*\mathcal{O}_X \rightarrow (B_X^1) \rightarrow 0$$

$$0 \rightarrow (B_X^2) \rightarrow F_*\Omega_X^2 \rightarrow \Omega_X^2 \rightarrow 0$$

Hence with lemma 2.1

$$(19) \quad H^i(X, (B_X^2)) \cong H^{n-i}(X, (B_X^2)^\vee \otimes \Omega_X^2)^* \cong H^{n-i}(X, B_X^1)^* = 0$$

for all $0 \leq i \leq n$. This completes the proof. □

Corollary 2.8. *If X is K3 surface, then X being F -split \Leftrightarrow ordinary.*

¹by Leray spectral sequence

Actually, we can generalize for perfect pairings appearing in proof of 2.7. As in [], we use following notations

$$(20) \quad B_m \Omega_X^i := C^{-1}(B_{m-1} \Omega_X^i) \quad B_1 \Omega_X^i := (B_X^i) \quad B_0 \Omega_X^i = 0$$

$$(21) \quad Z_m \Omega_X^i := C^{-1}(Z_{m-1} \Omega_X^i) \quad Z_1 \Omega_X^i := (Z_X^i) \quad Z_0 \Omega_X^i = \Omega_X^i$$

Note that we have following inclusion sequence

$$(22) \quad 0 = B_0 \Omega_X^i \subset B_1 \Omega_X^i \subset \cdots \subset B_m \Omega_X^i \subset \cdots \subset Z_m \Omega_X^i \subset Z_{m-1} \Omega_X^i \subset \cdots \subset Z_0 \Omega_X^i = \Omega_X^i$$

$B_m \Omega_X^i$ and $Z_m \Omega_X^i$ can be viewed as locally free submodules of $F_*^m(\Omega_X^i)$. This implies that

$$(23) \quad 0 \rightarrow B_m \Omega_X^i \rightarrow F_*^m \Omega_X^i \rightarrow F_*^m \Omega_X^i / B_m \Omega_X^i \rightarrow 0$$

$$(24) \quad 0 \rightarrow Z_m \Omega_X^{n-i} \rightarrow F_*^m \Omega_X^{n-i} \rightarrow F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i} \rightarrow 0$$

are exact. $\mathcal{H}om(-, \Omega_X^n)$ is exact since Ω_X^n is trivial. Applying it on 24, we get exact sequence

$$(25) \quad 0 \rightarrow \mathcal{H}om(F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i}, \Omega_X^n) \rightarrow \mathcal{H}om(F_*^m \Omega_X^{n-i}, \Omega_X^n) \rightarrow \mathcal{H}om(Z_m \Omega_X^{n-i}, \Omega_X^n) \rightarrow 0$$

Since the middle term is isomorphism to $F_*^m \Omega_X^i$ induced from perfect pairing

$$(26) \quad F_*^m \Omega_X^i \otimes F_*^m \Omega_X^{n-i} \rightarrow \Omega_X^n$$

we can get following two canonical perfect pairings

$$(27) \quad B_m \Omega_X^i \otimes F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i} \rightarrow \Omega_X^n$$

$$(28) \quad Z_m \Omega_X^{n-i} \otimes F_*^m \Omega_X^i / B_m \Omega_X^i \rightarrow \Omega_X^n$$

Note that we have canonical isomorphism $F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i} \cong B_m \Omega_X^{n-i+1}$ induced by differential $d : \Omega_X^{n-i} \rightarrow \Omega_X^{n-i+1}$. Hence 27 becomes $B_m \Omega_X^i \otimes B_m \Omega_X^{n-i+1} \rightarrow \Omega_X^n$. In particular, taking $i = 1, n = 2, m = 1$ we get perfect pairing 18.

Notice that we have exact sequence

$$0 \rightarrow B_1 \Omega_X^1 \rightarrow B_{m+1} \Omega_X^1 \xrightarrow{C} B_m \Omega_X^1 \rightarrow 0$$

Hence we have surjective maps $C : H^n(X, B_{m+1} \Omega_X^1) \rightarrow H^n(X, B_m \Omega_X^1)$. So we get a projective system, then taking inverse limit

$$H^n(X, B \Omega_X^1) := \lim_m H^n(X, B_m \Omega_X^1)$$

Definition 2.4. The dimension of $H^n(X, B \Omega_X^1)$ is called b -number of X .

In next subsection, we will use these to give explicit formula for height of Calabi-Yau varieties over positive characteristic which is also called h -number in [].

2.4. Formal group law. In algebraic geometry, **formal group** is referred by group object in category of formal schemes. If \mathcal{G} is a functor from Artin algebras to groups which is left exact, then \mathcal{G} is representable by formal group. \mathcal{G} is called a **formal group law**. Let

$$F_S^r(S) = \ker \{H^r(X \times S, \mathbb{G}_m) \rightarrow H^r(X, \mathbb{G}_m)\}$$

the cohomology is étale cohomology here. Theorem of Artin and Mazur say that this functor $F_S^r : \text{Art}_k \rightarrow \text{Ab}$ is pro-representable by formal group Φ_X if X is proper Calabi-Yau with dimension n over perfect field k . Moreover, we have that tangent space of Φ_X is isomorphic to $H^n(X, \mathcal{O}_X)$. Hence since Calabi-Yau variety is with geometric genus 1, e.g. $\dim_k H^n(X, \mathcal{O}_X) = 1$, Φ_X is formal group of dimension 1.

2.5. Height of Calabi-Yau varieties. Let X be Calabi-Yau variety over perfect field k with characteristic p and Φ_X be its formal Brauer group. Then we have following theorem

Theorem 2.9. If X is of dimension n , then we have following formula for height

$$(29) \quad h(\Phi_X) = \min \{i \geq 1 : [F : H^n(W_i \mathcal{O}_X) \rightarrow H^n(W_i \mathcal{O}_X)] \neq 0\}$$

Before giving the proof of this theorem, we will prepare some properties of Witt vector cohomology of Calabi-Yau varieties.

Proposition 2.10. Let X be Calabi-Yau variety over perfect field k of dimension n . We have

- (1) $H^i(X, \mathcal{W}_j \mathcal{O}_X) = 0$ for all $j > 0, 1 \leq i \leq n-1$. Furthermore, $H^i(X, \mathcal{W} \mathcal{O}_X) = 0$;
- (2) Pull-back of R on cohomology $R : H^n(X, \mathcal{W}_k \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_k \mathcal{O}_X)$ is surjective with kernel $H^n(X, \mathcal{O}_X)$;
- (3) Assume that for some $0 < k \leq n$ the map $F : H^k(X, \mathcal{W}_j \mathcal{O}_X) \rightarrow H^k(X, \mathcal{W}_j \mathcal{O}_X)$ vanishes, then for all $0 \leq i \leq j$ the map $F : H^i(X, \mathcal{W}_i \mathcal{O}_X) \rightarrow H^i(X, \mathcal{W}_i \mathcal{O}_X)$ vanishes too. Moreover, for all $H^n(X, \mathcal{W}_i \mathcal{O}_X)$ is vector space over K ;
- (4)

$$0 \rightarrow H^n(X, \mathcal{W}_{i-1} \mathcal{O}_X) \xrightarrow{V} H^n(X, \mathcal{W}_i \mathcal{O}_X) \xrightarrow{R^{n-1}} H^n(X, \mathcal{O}_X) \rightarrow 0$$

and

$$0 \rightarrow H^n(X, \mathcal{W} \mathcal{O}_X) \xrightarrow{V'} H^n(X, \mathcal{W} \mathcal{O}_X) \xrightarrow{R'} H^n(X, \mathcal{O}_X) \rightarrow 0$$

are both exact.

Proof. (1) Note that we have following exact sequence of sheaves for all $j \geq 1$

$$0 \rightarrow \mathcal{W}_{j-1} \mathcal{O}_X \rightarrow \mathcal{W}_j \xrightarrow{R^{j-1}} \mathcal{O}_X \rightarrow 0$$

Hence we have exact sequence of groups

$$H^i(X, \mathcal{W}_{j-1} \mathcal{O}_X) \rightarrow H^i(X, \mathcal{W}_j \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$$

If $H^i(X, \mathcal{W}_{j-1}) = 0$ and $1 \leq i \leq n-1$, then $H^i(X, \mathcal{W}_j \mathcal{O}_X) = 0$ since two sides of this sequence are zero. Hence by induction, we complete the proof.

- (2) We have following exact sequence

$$0 \simeq H^{n-1}(X, \mathcal{W}_{k-1}) \rightarrow H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_k \mathcal{O}_X) \xrightarrow{R} H^n(X, \mathcal{W}_{k-1} \mathcal{O}_X) \rightarrow 0$$

It implies $H^n(X, \mathcal{O}_X) = \ker \left(H^n(\mathcal{W}_k \mathcal{O}_X) \xrightarrow{R} H^n(X, \mathcal{W}_{k-1} \mathcal{O}_X) \right)$.

- (3) Note that following diagram commutes for all $j > i$

$$\begin{array}{ccc} H^n(X, \mathcal{W}_j \mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X, \mathcal{W}_i \mathcal{O}_X) \\ \downarrow F & & \downarrow F \\ H^n(X, \mathcal{W}_j \mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X, \mathcal{W}_i \mathcal{O}_X) \end{array}$$

Hence if left Frobenius map vanishes then so does right one. □

Proof of theorem 2.9. Let $h = h(\Phi_X)$. If F is zero map on each $H^n(X, \mathcal{W}_i \mathcal{O}_X)$, then $H^n(X, \mathcal{W} \mathcal{O}_X)$ is k -vector space. But we have following exact sequence

$$0 \rightarrow H^n(X, \mathcal{W} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X) \rightarrow 0$$

and $H^n(X, \mathcal{O}_X)$ is actually k -vector space. Hence $H^n(X, \mathcal{W} \mathcal{O}_X)$ is not of finite dimension as k -vector space. Since $h = \text{rank}_{W(k)} H^n(X, \mathcal{W} \mathcal{O}_X)$, we can conclude that in this case Φ_X is of height ∞ .

Notice that

$$H^n(X, \mathcal{W} \mathcal{O}_X) / V H^n(X, \mathcal{W} \mathcal{O}_X) \simeq H^n(X, \mathcal{O}_X)$$

as $W(k)$ -module and $H^n(X, \mathcal{O}_X)$ is naturally k -vector space, they are isomorphic as k -vector space. Hence from equality

$$\text{rank}_{W(k)}(D(\Phi_X)) = \dim_k H^n(X, \mathcal{W} \mathcal{O}_X) / F H^n(X, \mathcal{W} \mathcal{O}_X) + \dim_k H^n(X, \mathcal{W} \mathcal{O}_X) / V H^n(X, \mathcal{W} \mathcal{O}_X)$$

we get

$$\dim_k H^n(X, \mathcal{W} \mathcal{O}_X) / F H^n(X, \mathcal{W} \mathcal{O}_X) = h - 1$$

If $F : H^n(X, \mathcal{W}_i \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_i \mathcal{O}_X)$ is zero, then projection

$$[P_i] : H^n(X, \mathcal{W} \mathcal{O}_X) / F H^n(X, \mathcal{W} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_i \mathcal{O}_X) / F H^n(X, \mathcal{W}_i \mathcal{O}_X)$$

is well-defined and surjective since R, F commutes. As vector space, $H^n(X, \mathcal{W}_i \mathcal{O}_X)$ is of dimension i . This implies that we have inequality relation

$$h \geq i + 1$$

Hence we get

$$h \geq \min \{i \geq 1 : [F : H^n(W_i \mathcal{O}_X) \rightarrow H^n(W_i \mathcal{O}_X)] \neq 0\}$$

Conversely, it is sufficient to prove $FH^n(X, \mathcal{W}_{h-1} \mathcal{O}_X) = 0$.

Manin's result in \square implies that

$$(30) \quad H^n(X, \mathcal{W} \mathcal{O}_X) \cong D(\Phi_X) \cong \frac{W(k)[F, V]}{(F - V^{h-1})}$$

Hence $FH^n(X, \mathcal{W} \mathcal{O}_X) \cong V^{h-1} H^n(X, \mathcal{W} \mathcal{O}_X)$.

$$\begin{array}{ccc} V^{h-1} H^n(X, \mathcal{W} \mathcal{O}_X) & \longrightarrow & H^n(X, \mathcal{W} \mathcal{O}_X) \\ \downarrow & & \downarrow \\ 0 = V^{h-1} H^n(X, \mathcal{W}_{h-1} \mathcal{O}_X) & \longrightarrow & H^n(X, \mathcal{W}_{h-1} \mathcal{O}_X) \end{array}$$

commutes, this implies $FH^n(X, \mathcal{W}_{h-1} \mathcal{O}_X) = 0$. \square

We will call $h(\Phi_X)$ the h -number of X , denoted by $h(X)$.

Corollary 2.11. *For proper Calabi-Yau variety X over perfect field k , we have $b(X) = \dim_k H^n(X, \mathcal{W} \mathcal{O}_X) / FH^n(X, \mathcal{W} \mathcal{O}_X)$. Furthermore,*

$$\dim_k H^{n-1}(X, B_m \Omega_X^1) = \dim_k H^n(X, B_m \Omega_X^1) = \begin{cases} \min\{m, h-1\} & \text{if } h < \infty \\ m & \text{if } h = \infty \end{cases}$$

Proof. We have exact sequence

$$\mathcal{W}_m \mathcal{O}_X \xrightarrow{F} \mathcal{W}_m \mathcal{O}_X \xrightarrow{D_m} B_m \Omega_X^1 \rightarrow 0$$

It induces

$$H^n(X, \mathcal{W}_m \mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W}_m \mathcal{O}_X) \rightarrow H^n(X, B_m \Omega_X^1) \rightarrow 0$$

Taking limit with m , we get

$$H^n(X, \mathcal{W} \mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W} \mathcal{O}_X) \rightarrow H^n(X, B \Omega_X^1) \rightarrow 0$$

exact. Hence $b(X) = h(X) - 1$. \square

3. EIGENVALUES OF FROBENIUS ON l -ADIC COHOMOLOGY

3.1. Part I. Let α an eigenvalue of Frobenius acting on $H^i(V/\bar{k})$ where $H^*(V/\bar{k})$ is any one of the known cohomologies. Then by Poncaré duality $\frac{q^d}{\alpha}$ is an eigenvalue of Frobenius acting on $H^{2d-i}(V/\bar{k})$. Thus, both α and $\frac{q^d}{\alpha}$ are algebraic integers. It follows that in the field $F = \mathbb{Q}(\alpha)$, α is a λ -adic unit for any prime λ which does not lie over $p = \text{char } k$.

Since, by the Riemann hypothesis, one has that every complex absolute value of α is $q^{i/2}$, one is left with the intriguing question: What are the l -adic absolute values of α , for primes l lying over p ?

This is a question, which can by hindsight be seen to have had a long tradition — being related to Stickelberger's determination of the p -adic nature of Gauss sums and the Chevalley-Warning theorem which says that if N is the number of solution of a polynomial equation mod p .

It is reasonable to expect that the study of the above question should use a p -adic cohomology, and it does. To describe, briefly, the known results, we introduce the *Newton polygon* of a monotone increasing sequence a_1, a_2, \dots, a_m of non-negative rational numbers: by definition, it is the graph of the real-valued continuous piece-wise linear function v_{a_1, a_2, \dots, a_m} on $[0, m]$ which takes the value 0 at 0 and whose derivate is the constant a_j on the interval $(j-1, j)$.

Now let $V/\mathcal{W}(k)$ be a smooth projective scheme. where $\mathcal{W}(k)$ is the ring of Witt vectors of k . Let V/k be its reduction to k through canonical morphism $\text{Spec } (k) \rightarrow \text{Spec } (\mathcal{W}(k))$. Make the hypothesis that the de Rham cohomology of V/k is extremely well-behaved: namely, $H^q(V/\mathcal{W}(k), \Omega^p)$ is a free $\mathcal{W}(k)$ -module (whose rank we shall denote $h^{p,q}$ — the (p, q) -th Hodge number of V/k) for all p, q . Fix an integer r , and define the r -th Hodge polygon of V/k to be the Newton polygon of the sequence of integers.

$$\underbrace{0, 0, \dots, 0}_{h^{0,r}}, \underbrace{1, 1, \dots, 1}_{h^{1,r-1}}, \underbrace{2, 2, \dots, 2}_{h^{2,r-2}}, \dots, \underbrace{r, r, \dots, r}_{h^{r,0}}$$

Define the r -th Newton polygon of V to be the Newton polygon of the sequence:

$$\text{ord}_q \alpha_1, \text{ord}_q \alpha_2, \dots, \text{ord}_q \alpha_m \quad (m = r\text{-th Betti number of } V)$$

where $\alpha_1, \dots, \alpha_m \in \bar{K}$ are the complete set of eigenvalues (with multiplicities) of Frobenius acting on the r -dimensional p -adic cohomology of V/k ; ord_q denotes the p -adic ord function normalized so that $\text{ord}_q q = 1$; and the α_j are indexed in such a manner that the above sequence is monotone increasing.

What is known about the r -th Newton polygon of V/k is the following:

The r -th Newton polygon and the r -th Hodge polygon both end at the point $(m, rm/2)$ in the euclidean plane. The break-points (non-differentiable points) of these polygon occur at integral lattice points. The r -th Newton polygons lies in the closed region bounded by the r -th Hodge polygon, and the straight line joining $(0, 0)$ and $(m, rm/2)$.

Note: One has a further symmetry in the geometry of the Newton polygon implied by the strong Lefschetz theorem, and also the existence of algebraic cohomology classes in dimension r implies that a part of the Newton polygon must have slope $r/2$.

General remarks: It was an important discovery of Dwork, that (in a certain, unobvious, sense) the eigenvalues of Frobenius vary p -adically if one varies the variety V/k over a parameter space in characteristic p . A trace of this phenomenon may be seen in a theorem of Grothendieck: The Newton polygon rises under specialization of V/k .

3.2. Explicit description of Newton Polygon and Hodge Polygon. Let $H^n = H_{\text{crys}}^n(X/W)$ be torsion free. Let

- $h^i = h^{i, n-i} = \dim_k H^{n-i}(X, \Omega_X^i)$
- $a_i = \dim H^{n-i}(X, \mathcal{W}\Omega_X^i)/VH^{n-i}(X, \mathcal{W}\Omega_X^i)$
- $a'_i = \dim H^{n-i}(X, \mathcal{W}\Omega_X^i)/FH^{n-i}(X, \mathcal{W}\Omega_X^i)$

then

$$h^0 = a_0 \quad h^i = a'_{i-1} + a_i \quad (1 \leq i \leq n) \quad a'_n = 0$$

and

$$\begin{aligned} \text{Hdg}(t) &= \begin{cases} (0, 0) & t = 0 \\ (\sum_{i \leq t} h^i, \sum_{i \leq t} ih^i) & 1 \leq t \leq n+1 \end{cases} \\ \text{Nwt}(t) &= \begin{cases} (0, 0) & t = 0 \\ \text{Hdg}(t) + (a'_{t-1}, ta'_{t-1}) & 1 \leq t \leq n+1 \end{cases} \end{aligned}$$

Corollary 3.1. *If $H^j(X, \mathcal{W}\Omega_X^i)$ are all torsion-free, then X being ordinary is equivalent to $\text{Nwt}(t) = \text{Hdg}(t)$.*