# Positive Characteristic Geometry

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## 1. Witt Vector cohomology

Let $p$ be a prime number. We then define the Witt polynomials ($\mathbb{Z}$-coefficient polynomials) respect to $p$ as follows

$$W_0(x_0) := x_0$$
$$W_1(x_0, x_1) := x_0^p + px_1$$
$$\cdots$$
$$W_n(x_0, \cdots, x_n) := \sum_{i=0}^{n} p^i {x_i}^{p^{n-i}}$$

Let $x = (x_0, \cdots, x_n), y = (y_0, \cdots, y_n)$. From Serre's theorem [], there are polynomials $S_0, \cdots, S_n$, $P_0, \cdots, P_n$ with $2n+2$ variables satisfying

$$W_n(x) + W_n(y) = W_n(S_0(x, y), \cdots, S_n(x, y))$$
$$W_n(x) \cdot W_n(y) = W_n(P_0(x, y), \cdots, P_n(x, y))$$

For example,

$$S_0(x, y) = x_0 + y_0 \quad S_1(x, y) = x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0 + y_0)^p}{p}$$
$$P_0(x, y) = x_0 y_0 \qquad P_1(x, y) = y_0^p x_1 + y_1 x_0^p + p x_1 y_1$$

Now, we define a new ring structure on $n$-couple for ring $R$, which is denoted by $\mathcal{W}_n(R)$. For all $X, Y in \mathcal{W}_n(R)$

$$X + Y = (S_0(X, Y), \cdots, S_{n-1}(X, Y))$$
$$X \cdot Y = (P_0(X, Y), \cdots, P_{n-1}(X, Y))$$

Let $X$ be non-singular variety over perfect field $k$ of characteristic $p > 0$ with dimension $n$. $\mathcal{W}_k$ is sheaf of Witt vectors of lenght $k$.

We have three operators

- Frobenius $F\colon \mathcal{W}_k\mathcal{O}_X \to \mathcal{W}_k\mathcal{O}_X$ locally defined by $(a_0, \cdots, a_{k-1}) \mapsto (a_0^p, \cdots, a_{k-1}^p)$;
- Verschieburg $V\colon \mathcal{W}_k\mathcal{O}_X \to \mathcal{W}_{k+1}\mathcal{O}_X$ locally defined by $(a_0, \cdots, a_{k-1}) \mapsto (0, \cdots, a_{k-1})$;
- Restriction $R\colon \mathcal{W}_{k+1}\mathcal{O}_X \to \mathcal{W}_k\mathcal{O}_X$ locally defined by $(a_0, \cdots, a_n) \mapsto (a_0, \cdots, a_{n-1})$

It is well-known that

$$RVF = RFV = FRV = p \cdot \mathrm{id}_{\mathcal{W}_k\mathcal{O}_X}$$

where the "$\cdot$" is multiplication in $\mathcal{W}_k\mathcal{O}_X$.

Let $\mathcal{W}(M) := \lim_k \mathcal{W}_{k+1}(M) \xrightarrow{R} \mathcal{W}_k(M)$ inverse limit of projective system induced by $R$. For example, since $W_k(\mathbb{F}_p) = \mathbb{Z}/p^k\mathbb{Z}$, we have $\mathcal{W}(\mathbb{F}_p) = \mathbb{Z}_p$ the $p$-adic integer ring.

We have following two canonical exact sequence of $\mathcal{O}_X$-module

$$0 \to \mathcal{W}_{k-1}\mathcal{O}_X \xrightarrow{V} \mathcal{W}_k\mathcal{O}_X \xrightarrow{R^{k-1}} \mathcal{O}_X \to 0 \tag{1}$$
$$0 \to \mathcal{O}_X \to \mathcal{W}_k\mathcal{O}_X \xrightarrow{R} \mathcal{W}_{k-1}\mathcal{O}_X \to 0 \tag{2}$$

Witt vector cohomology is defined as

$$H^n(X, \mathcal{W}\mathcal{O}_X) = \lim_k H^n(X, \mathcal{W}_k\mathcal{O}_X)$$

in projective system $[R : H^n(X, \mathcal{W}_{k+1}\mathcal{O}_X) \to H^n(X, \mathcal{W}_k\mathcal{O}_X)]$.

## 2. Varieties in positive characteristic under Frobenius

Let $X$ be a scheme over $\mathbb{F}_p$. $F_X$ is called **absolute Frobenius morphism** of $X$ which is induced by $f \mapsto f^p$ on affine covers. Frobenius map $\sigma$ of field $\mathbb{F}_p$ induces endmorphism of Spec $\mathbb{F}_p$. It is also denoted by $F_p$. Furthermore, we have following base change

$$
\begin{tikzcd}
X \arrow[drr, bend left, "F_X"] \arrow[dr, dashed, "F"] \arrow[ddr, bend right, "f"'] & & \\
& X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
& \mathrm{Spec}\ \mathbb{F}_p \arrow[r, "F_p"] & \mathrm{Spec}\ \mathbb{F}_p
\end{tikzcd} \tag{3}
$$

Since $F_X$ and $F_p$ are homeomorphisms on underlying topological space of spectrum Spec $\mathbb{F}_p$, $F$ is also homeomorphism on underlying space. Hence it is easy to see that $F_X$ is affine morphism of finite type. For all $\mathcal{O}_X$-module $\mathcal{F}$ on $X$, $(F_X)_*(\mathcal{F})$ is same to $\mathcal{F}$ as sheaves over abelian groups although the their $\mathcal{O}_X$-module structures are different. Let $B_X^i = \mathrm{Im}(d : \Omega_X^{i-1} \to \Omega_X^i)$, then we have following exact sequences from definitions of Frobenius.

$$
\mathcal{O}_X \xrightarrow{F_X} \mathcal{O}_X \xrightarrow{d} B_X^1 \qquad \mathcal{O}_{X'} \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{F_*d} F_*B_X^1 \tag{4}
$$

**Example 2.1.** Let $Y = \mathrm{Spec}\ A, X = \mathbb{A}_Y^n$. We have

$$
\begin{aligned}
\mathcal{O}_{X'} &= (A, F_A) \otimes_A A[x_1, x_2, \cdots, x_n] \\
&\cong A^p[x_1, x_2, \cdots, x_n]
\end{aligned}
$$

If $k$ is perfect field, i.e. Frobenius map $\sigma : k \to k$ is automorphism, then absolute Frobenius morphism is automorphism of scheme Spec $k$. Hence the pull-back $X' \to X$ is also isomorphism and under isomorphism relative Frobenius $F$ do same as absolute Frobenius $F_X$ on $X$. So we also denote $F_X$ by $F$ in this case without confusion.

## 2.1. F-split.

**Definition 2.2.** The $k$-scheme of characteristic $p > 0$ is called Frobenius split if exact sequence from Cartier isomorphism

$$
0 \longrightarrow \mathcal{O}_{X'} \xrightarrow{F^\#} F_*\mathcal{O}_X \xrightarrow{d} F_*B_X^1 \longrightarrow 0 \tag{5}
$$

is split exact.

If $k$ is perfect, then we have absolute Cartier isomorphism $C_{\mathrm{abs}}^{-1}$, which induce following exact sequences

$$
0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to F_*B_X^1 \to 0 \tag{6}
$$

$$
0 \to F_*B_X^n \to F_*\Omega_X^n \xrightarrow{C_{\mathrm{abs}}} \Omega_X^n \to 0 \tag{7}
$$

Under assumption that $k$ is perfect, $X$ is Frobenius split if and only if 6 is split. This is because in this condition, base change of $X$ along Frobenius morphism of field $k$ is isomorphic to $X$. To simplify the notation, we denote $F_*B_X^i$ or $(F_X)_*B_X^i$ by $(B_X^i)$.

Suppose $\xi$ is corresponding extension class of 5 in $\mathrm{Ext}^1_{\mathcal{O}_{X'}}((B_X^1), \mathcal{O}_{X'})$. It follows that $X$ is $F$-split if and only if $\xi = 0$. In particular, if extension group $\mathrm{Ext}^1_{\mathcal{O}_{X'}}((B_X^1), \mathcal{O}_{X'})$ vanishes, then 5 must be split. Writing it in cohomology form, we have $H^1(X', (B_X^1)^\vee) = \mathrm{Ext}^1_{\mathcal{O}_{X'}}((B_X^1), \mathcal{O}_{X'})$.

**Lemma 2.1.** *If $X$ is $F$-split, then $H^i(X, B_X^1) = 0$ for all $i \geq 0$.*

*Proof.* Exact sequence 6 being splitting implies it induces isomorphism

$$
\mathcal{O}_X \oplus (B_X^1) \cong (F_X)_*\mathcal{O}_X
$$

Hence at cohomolgy level, we have

$$
H^i(X, \mathcal{O}_X) \oplus H^i(X, (B_X^1)) \cong H^i(X, (F_X)_*\mathcal{O}_X) \tag{8}
$$

while $H^i(X, (F_X)_*\mathcal{O}_X) = H^i(X, \mathcal{O}_X)$ for all $i \geq 0$, hence we have $\dim H^i(X, (B_X^1)) = 0$, so that $H^i(X, B_X^1)$ vanishes. ∎

**2.2. Slope Spectral Sequence.** Slope spectral sequence is constructed in works [] of Luc Illusie on de Rham-Witt cohomology and comparasion to crystalline cohomology. It plays essential role in de Rham-Witt cohomology as Hodge-de Rham spectral sequence in de Rham cohomology. Actually, this spectral sequence gives explict descibtion of $F$-crystal structure on crystalline cohomology with de Rham-Witt cohomology.

Let $X$ be smooth variety over perfect field $k$ with characteristic $p > 0$. We have following de Rham-Witt complexes according to Illusie's works

$$\mathcal{W}\Omega_X^* := [\mathcal{W}\mathcal{O}_X \to \mathcal{W}\Omega_X^1 \to \cdots \to \mathcal{W}\Omega_X^n] \tag{9}$$

We have canonical filtration on $\mathcal{W}\Omega_X^*$ as Hodge filtration in Hodge theory such that

$$F^i\mathcal{W}\Omega_X^* = \mathcal{W}\Omega^{\geq i} = [0 \to \cdots \to 0 \to \mathcal{W}\Omega_X^i \cdots \to \mathcal{W}\Omega_X^n] \tag{10}$$

Filtration $F^i\mathcal{W}\Omega^*X$ induces spectral sequence

$$E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*) \tag{11}$$

which is called **slope spectral sequence** of $X$.Illusie proved that hypercohomology $H^n(X, \mathcal{W}\Omega_X^*)$ computes crystalline cohomology $H^n_{\text{crys}}(X/W)$ and $E_1^{i,j} \otimes K$ without $p$-torsion. Hence

**Theorem 2.2** (Illusie). *If $H^j(X, \mathcal{W}\Omega_X^i)$ are all torsion-free, then the slope spectral sequence*

$$E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*) \tag{12}$$

*degenerates in degree 1.*

As name of this spectral sequence, it captures information of "slope" of $F$-crystal on $H^*_{\text{crys}}(X/W) \otimes K$. We have following corollary

**Corollary 2.3** (Bloch). *For any $i$, canonical homomorphism $H^*(X, \mathcal{W}\Omega_X^{\geq i}) \hookrightarrow H^*_{crys}(X/W)$ and $H^*_{crys}(X/W) \twoheadrightarrow H^*(X, \mathcal{W}\Omega_X^{\leq i})$ induces isomorphism*

$$H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \xrightarrow{\sim} (H^*(X/W) \otimes K)_{\geq i} \tag{13}$$

$$(H^*_{crys}(X/W) \otimes K)_{[0,i[} \xrightarrow{\sim} H^*(X, \mathcal{W}\Omega_X^{\leq i}) \otimes K \tag{14}$$

*Proof.* We have following exact sequence

$$0 \to \mathcal{W}\Omega_X^{\geq i} \to \mathcal{W}\Omega_X^* \to \mathcal{W}\Omega_X^{\leq i-1} \to 0$$

Since connection map $0 = d_1 \otimes K : H^{*-1}(X, \mathcal{W}\Omega^{\leq i-1}) \to H^*(X, \mathcal{W}\Omega^{\geq i})$, upper exact sequence induces exact sequence of $F$-isocrystals

$$0 \to H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \to H^*_{\text{crys}}(X/W) \otimes K \to H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \otimes K \to 0$$

We have already known that $F(\operatorname{Im} H^*(X, \mathcal{W}\Omega_X^{\geq i})) \subseteq p^i H^*_{\text{crys}}(X/W)$. Hence the slope of $H^*(X, \mathcal{W}\Omega_X^{\geq i})$ is greater than $i$. On other hand,

$$H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \cong W_\sigma[[V]][F]/(FV - p^i, VF - p^i)$$

It implies that slope of $H^*(X, \mathcal{W}\Omega_X^{\leq i-1})$ strictly less than $i$. Hence we can conclude the proof. □

On other hand, we can describe Hodge polygon and Newton polygon explicitly. In fact, there is following theorem due to Ogus.

**Theorem 2.4.** *Let*

- $h^i = h^{i,n-i} = \dim_k H^{n-i}(X, \Omega_X^i)$
- $a_i = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i)/VH^i(X, \mathcal{W}\Omega_X^i)$
- $a_i' = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i)/FH^{n-i}(X, \mathcal{W}\Omega_X^i)$

*then*

$$a_n' = 0 \quad h^0 = a_0 \quad h^i = a_{i-1}' + a_i (1 \leq i \leq n)$$

*and*

$$Hdg(t) = (\sum_{i \leq t} h^i, \sum_{i \leq t} ih^i)$$

$$Nwt(t) = Hdg(t) + (a_{t-1}', ta_{t-1}')$$

**Corollary 2.5.** *If $H^j(X, \mathcal{W}\Omega^i_X)$ is torsion-free for all $i, j$, then $X$ being ordinary is equivalent to condition that $Nwt(t) = Hdg(t)$.*

*Proof.* TBA □

### 2.3. Ordinary Varieties.

**Definition 2.3.** Let $X$ be a smooth, proper variety over field $k$ with $\operatorname{char} k > 0$. We say $X$ is **ordinary** if it satisfies

$$H^i(X, B^j_X) = 0 \qquad \text{for all } i \geq 0, j > 0 \tag{15}$$

where $B^j_X = d(\Omega^{j-1}_{X/k})$ is the sheaf of boundaries of algebraic de Rham complex in degree $j$.

Equivalently, $H^i(X', F_* B^i_X) =$ for all $i \geq 0, j > 0$ implies ordinarity of $X$ [1]. where $F$ is relative Frobenius morphism $F = F_{X/k} : X \to X'$.

More numerical description of ordinarity according to property 2.5:

If $H^q(X, W\Omega^r_X)$ are all torsion free, then $X$ is ordinary if and only if for every $q$, Newton polygon defined by the slopes of action of Frobenius on $H^q_{\text{crys}}(X/W)$ is equal to Hodge polygon.

Here we list some properties of ordinary varieties.

**Proposition 2.6.** *Let $X$ be variety over perfect field $k$. If $X$ is ordinary and with trivial canonical bundle, then $X$ is splitting.*

*Proof.* Homological algebra tells us that exact sequence

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to (B^1_X) \to 0$$

lies in extension group $\operatorname{Ext}^1((B^1_X), \mathcal{O}_X)$ as extension class. Denote this class as $\xi$. If $\xi = 0$ then this sequence splits. Hence if this extension group vanishes then it has no choice but splits. But we have

$$H^1(X, (B^1_X)^\vee) = \operatorname{Ext}^1((B^1_X), \mathcal{O}_X)$$

and

$$H^1(X, (B^1_X)^\vee) \cong H^{n-1}(X, B^1_X \otimes \omega_X)^* \cong H^{n-1}(X, B^1_X)^*.$$

Hence it actually vanishes if $X$ is ordinary. □

**Proposition 2.7.** *If $S$ is algebraic surface over positive characteristic perfect field, then $S$ being $F$-split $\Rightarrow$ ordinary.*

*Proof.* As Cartier isomorphism is actually trace map in Grothendieck duality, we have following two perfect pairings:

$$F_* \mathcal{O}_X \otimes F_* \Omega^2_X \to \Omega^2_X \tag{16}$$

$$F_* \Omega^1_X \otimes F_* \Omega^1_X \to \Omega^2_X \tag{17}$$

The pairing 16 induces perfect pairings

$$(B^1_X) \otimes (B^2_X) \to \Omega^2_X \qquad \mathcal{O}_X \otimes \Omega^n_X \to \Omega^2_X \tag{18}$$

by interchange following exact sequences with exact functor $\operatorname{Hom}(-, \Omega^2_X) = \operatorname{Hom}(-, \mathcal{O}_X)$.

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to (B^1_X) \to 0$$
$$0 \to (B^2_X) \to F_* \Omega^2_X \to \Omega^2_X \to 0$$

Hence with lemma 2.1

$$H^i(X, (B^2_X)) \cong H^{n-i}(X, (B^2_X)^\vee \otimes \Omega^2_X)^* \cong H^{n-i}(X, B^1_X)^* = 0 \tag{19}$$

for all $0 \leq i \leq n$. This completes the proof. □

**Corollary 2.8.** *If $X$ is K3 surface, then $X$ being $F$-split $\Leftrightarrow$ ordinary.*

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[1] by Leray spectral sequence

Actually, we can genenralize for perfect pairings appearing in proof of 2.7. As in [], we use following notations

(20) $$B_m\Omega^i_X := C^{-1}(B_{m-1}\Omega^i_X) \qquad B_1\Omega^i_X := (B^i_X) \qquad B_0\Omega^i_X = \underline{0}$$

(21) $$Z_m\Omega^i_X := C^{-1}(Z_{m-1}\Omega^i_X) \qquad Z_1\Omega^i_X := (Z^i_X) \qquad Z_0\Omega^i_X = \Omega^i_X$$

Note that we have following inclusion sequence

(22) $$\underline{0} = B_0\Omega^i_X \subset B_1\Omega^i_X \subset \cdots \subset B_m\Omega^i_X \subset \cdots Z_m\Omega^i_X \subset Z_{m-1}\Omega^i_X \subset \cdots \subset Z_0\Omega^i_X = \Omega^i_X$$

$B_m\Omega^i_X$ and $Z_m\Omega^i_X$ can be viewed as locally free submodules of $F^m_*(\Omega^i_X)$.This implies that

(23) $$0 \to B_m\Omega^i_X \to F^m_*\Omega^i_X \to F^m_*\Omega^i_X/B_m\Omega^i_X \to 0$$

(24) $$0 \to Z_m\Omega^{n-i}_X \to F^m_*\Omega^{n-i}_X \to F^m_*\Omega^{n-i}_X/Z_m\Omega^{n-i}_X \to 0$$

are exact. $\mathcal{H}om(-,\Omega^n_X)$ is exact since $\Omega^n_X$ is trivial. Applying it on 24, we get exact sequence

(25) $$0 \to \mathcal{H}om(F^m_*\Omega^{n-i}_X/Z_m\Omega^{n-i}_X,\Omega^n_X) \to \mathcal{H}om(F^m_*\Omega^{n-i}_X,\Omega^n_X) \to \mathcal{H}om(Z_m\Omega^{n-i}_X,\Omega^n_X) \to 0$$

Since the middle term is isomorphism to $F^m_*\Omega^i_X$ induced from perfect pairing

(26) $$F^m_*\Omega^i_X \otimes F^m_*\Omega^{n-i}_X \to \Omega^n_X$$

we can get following two canonical perfect pairings

(27) $$B_m\Omega^i_X \otimes F^m_*\Omega^{n-i}_X/Z_m\Omega^{n-i}_X \to \Omega^n_X$$

(28) $$Z_m\Omega^{n-i}_X \otimes F^m_*\Omega^i_X/B_m\Omega^i_X \to \Omega^n_X$$

Note that we have canonical isomorphism $F^m_*\Omega^{n-i}/Z_m\Omega^{n-i}_X \cong B_m\Omega^{n-i+1}_X$ induced by differential $d : \Omega^{n-i}_X \to \Omega^{n-i+1}_X$. Hence 27 becomes $B_m\Omega^i_X \otimes B_m\Omega^{n-i+1}_X \to \Omega^n_X$.In particular, taking $i = 1, n = 2, m = 1$ we get perfect pairing 18.

Notice that we have exact sequence

$$0 \to B_1\Omega^1_X \to B_{m+1}\Omega^1_X \xrightarrow{C} B_m\Omega^1_X \to 0$$

Hence we have surjective maps $C : H^n(X, B_{m+1}\Omega^1_X) \to H^n(X, B_m\Omega^1_X)$. So we get a projective system, then taking inverse limit

$$H^n(X, B\Omega^1_X) := \lim_m H^n(X, B_m\Omega^1_X)$$

**Definition 2.4.** The dimension of $H^n(X, B\Omega^1_X)$ is called $b$-number of $X$.

In next subsection, we will use these to give explicit formula for height of Calabi-Yau varieties over positive characteristic which is also called $h$-number in [].

**2.4. Formal group law.** In algebraic geometry, **formal group** is referred by group object in category of formal schemes. If $\mathcal{G}$ is a functor from Artin algebras to groups which is left exact, then $\mathcal{G}$ is representable by formal group. $\mathcal{G}$ is called a **formal group law**. Let

$$F^r_S(S) = \ker\{H^r(X \times S, \mathbb{G}_m) \to H^r(X, \mathbb{G}_m)\}$$

the cohomology is étale cohomology here. Theorem of Artin and Mazur say that this functor $F^r_S$ : $\mathrm{Art}_k \to \mathrm{Ab}$ is pro-representable by formal group $\Phi_X$ if $X$ is proper Calabi-Yau with dimension $n$ over perfect field $k$. Moreover, we have that tangent space of $\Phi_X$ is isomorphic to $H^n(X, \mathcal{O}_X)$. Hence since Calabi-Yau variety is with geometric genus 1, e.g. $\dim_k H^n(X, \mathcal{O}_X) = 1$, $\Phi_X$ is formal group of dimension 1.

**2.5. Height of Calabi-Yau varieties.** Let $X$ be Calabi-Yau variety over perfect field $k$ with characteristic $p$ and $\Phi_X$ be its formal Brauer group. Then we have following theorem

**Theorem 2.9.** *If $X$ is of dimension $n$, then we have following formula for hegiht*

(29) $$h(\Phi_X) = \min\left\{i \geq 1 : [F : H^n(W_i\mathcal{O}_X) \to H^n(W_i\mathcal{O}_X)] \neq 0\right\}$$

Before giving the proof of this theorem, we will prepare some properties of Witt vector cohomology of Calabi-Yau varieties.

**Proposition 2.10.** *Let $X$ be Calabi-Yau variety over perfect field $k$ of dimension $n$. We have*

*(1) $H^i(X, \mathcal{W}_j\mathcal{O}_X) = 0$ for all $j > 0, 1 \le i \le n-1$. Furthermore, $H^i(X, \mathcal{W}\mathcal{O}_X) = 0$;*
*(2) Pull-back of $R$ on cohomology $R : H^n(X, \mathcal{W}_k\mathcal{O}_X) \to H^n(X, \mathcal{W}_k\mathcal{O}_X)$ is surjective with kernel $H^n(X, \mathcal{O}_X)$;*
*(3) Assume that for some $0 < k \le n$ the map $F : H^k(X, \mathcal{W}_j\mathcal{O}_X) \to H^k(X, \mathcal{W}_j\mathcal{O}_X)$ vanishes, then for all $0 \le i \le j$ the map $F\colon H^k(X, \mathcal{W}_i\mathcal{O}_X) \to H^k(X, \mathcal{W}_i\mathcal{O}_X)$ vanishes too. Moreover, for all $H^n(X, \mathcal{W}_i\mathcal{O}_X)$ is vector space over $K$;*
*(4)*

$$0 \to H^n(X, \mathcal{W}_{i-1}\mathcal{O}_X) \xrightarrow{V} H^n(X, \mathcal{W}_i\mathcal{O}_X) \xrightarrow{R^{n-1}} H^n(X, \mathcal{O}_X) \to 0$$

*and*

$$0 \to H^n(X, \mathcal{W}\mathcal{O}_X) \xrightarrow{V'} H^n(X, \mathcal{W}\mathcal{O}_X) \xrightarrow{R'} H^n(X, \mathcal{O}_X) \to 0$$

*are both exact.*

*Proof.* (1) Note that we have following exact sequence of sheaves for all $j \ge 1$

$$0 \to \mathcal{W}_{j-1}\mathcal{O}_X \to \mathcal{W}_j \xrightarrow{R^{j-1}} \mathcal{O}_X \to 0$$

Hence we have exact sequence of groups

$$H^i(X, \mathcal{W}_{j-1}\mathcal{O}_X) \to H^i(X, \mathcal{W}_j\mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$$

If $H^i(X, \mathcal{W}_{j-1}) = 0$ and $1 \le i \le n-1$, then $H^i(X.\mathcal{W}_j\mathcal{O}_X) = 0$ since two sides of this sequence are zero. Hence by induction, we complete the proof.

(2) We have following exact sequence

$$0 \simeq H^{n-1}(X, \mathcal{W}_{k-1}) \to H^n(X, \mathcal{O}_X) \to H^n(X, \mathcal{W}_k\mathcal{O}_X) \xrightarrow{R} H^n(X, \mathcal{W}_{k-1}\mathcal{O}_X) \to 0$$

It implies $H^n(X, \mathcal{O}_X) = \ker\left(H^n(\mathcal{W}_k\mathcal{O}_X) \xrightarrow{R} H^n(X, \mathcal{W}_{k-1}\mathcal{O}_X)\right)$.

(3) Note that following diagram commutes for all $j > i$

$$\begin{array}{ccc} H^n(X, \mathcal{W}_j\mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X, \mathcal{W}_i\mathcal{O}_X) \\ \downarrow F & & \downarrow F \\ H^n(X, \mathcal{W}_j\mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X, \mathcal{W}_i\mathcal{O}_X) \end{array}$$

Hence if left Frobenius map vanishes then so does right one.

□

*Proof of theorem 2.9.* Let $h = h(\Phi_X)$. If $F$ is zero map on each $H^n(X, \mathcal{W}_i\mathcal{O}_X)$, then $H^n(X, \mathcal{W}\mathcal{O}_X)$ is $k$-vector space. But we have following exact sequence

$$0 \to H^n(X, \mathcal{W}\mathcal{O}_X) \to H^n(X, \mathcal{W}\mathcal{O}_X) \to H^n(X, \mathcal{O}_X) \to 0$$

and $H^n(X, \mathcal{O}_X)$ is actually $k$-vector space. Hence $H^n(X, \mathcal{W}\mathcal{O}_X)$ is not of finite dimension as $k$-vector space. Since $h = \operatorname{rank}_{W(k)} H^n(X, \mathcal{W}\mathcal{O}_X)$, we can conclude that in this case $\Phi_X$ is of height $\infty$.

Notice that

$$H^n(X, \mathcal{W}\mathcal{O}_X)/VH^n(X, \mathcal{W}\mathcal{O}_X) \simeq H^n(X, \mathcal{O}_X)$$

as $W(k)$-module and $H^n(X, \mathcal{O}_X)$ is naturally $k$-vector space, they are isomorphic as $k$-vector space. Hence from equality

$$\operatorname{rank}_{W(k)}(D(\Phi_X)) = \dim_k H^n(X, \mathcal{W}\mathcal{O}_X)/FH^n(X, \mathcal{W}\mathcal{O}_X) + \dim_k H^n(X, \mathcal{W}\mathcal{O}_X)/VH^n(X, \mathcal{W}\mathcal{O}_X)$$

we get

$$\dim_k H^n(X, \mathcal{W}\mathcal{O}_X)/FH^n(X, \mathcal{W}\mathcal{O}_X) = h - 1$$

If $F : H^n(X, \mathcal{W}_i\mathcal{O}_X) \to H^n(X, \mathcal{W}_i\mathcal{O}_X)$ is zero ,then projection

$$[P_i] : H^n(X, \mathcal{W}\mathcal{O}_X)/FH^n(X, \mathcal{W}\mathcal{O}_X) \to H^n(X, \mathcal{W}_i\mathcal{O}_X)/FH^n(X, \mathcal{W}_i\mathcal{O}_X)$$

is well-defined and surjective since $R, F$ commutes. As vector space, $H^n(X, \mathcal{W}_i\mathcal{O}_X)$ is of dimension $i$. This implies that we have inequality relation

$$h \ge i + 1$$

Hence we get
$$h \geq \min\left\{i \geq 1 : [F : H^n(W_i\mathcal{O}_X) \to H^n(W_i\mathcal{O}_X)] \neq 0\right\}$$
Conversely, it is sufficient to prove $FH^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) = 0$.

Manin's result in [] implies that

$$H^n(X, \mathcal{W}\mathcal{O}_X) \cong D(\Phi_X) \cong \frac{W(k)[F,V]}{(F - V^{h-1})} \tag{30}$$

Hence $FH^n(X, \mathcal{W}\mathcal{O}_X) \cong V^{h-1}H^n(X, \mathcal{W}\mathcal{O}_X)$.

$$\begin{tikzcd}
V^{h-1}H^n(X, \mathcal{W}\mathcal{O}_X) \arrow[r] \arrow[d, two heads] & H^n(X, \mathcal{W}\mathcal{O}_X) \arrow[d, two heads] \\
0 = V^{h-1}H^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) \arrow[r] & H^n(X, \mathcal{W}_{h-1}\mathcal{O}_X)
\end{tikzcd}$$

commutes, this implies $FH^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) = 0$. □

We will call $h(\Phi_X)$ the $h$-number of $X$, denoted by $h(X)$.

**Corollary 2.11.** *For proper Calabi-Yau variety $X$ over perfect field $k$, we have $b(X) = \dim_k H^n(X, \mathcal{W}\mathcal{O}_X)/FH^n(X$ Furthermore,*

$$\dim_k H^{n-1}(X, B_m\Omega^1_X) = \dim_k H^n(X, B_m\Omega^1_X) = \begin{cases} \min\{m, h-1\} & \text{if } h < \infty \\ m & \text{if } h = \infty \end{cases}$$

*Proof.* We have exact sequence
$$\mathcal{W}_m\mathcal{O}_X \xrightarrow{F} \mathcal{W}_m\mathcal{O}_X \xrightarrow{D_m} B_m\Omega^1_X \to 0$$
It induces
$$H^n(X, \mathcal{W}_m\mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W}_m\mathcal{O}_X) \to H^n(X, B_m\Omega^1_X) \to 0$$
Taking limit with $m$, we get
$$H^n(X, \mathcal{W}\mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W}\mathcal{O}_X) \to H^n(X, B\Omega^1_X) \to 0$$
exact. Hence $b(X) = h(X) - 1$. □

## 3. Eigenvalues of Frobenius on $l$-adic cohomology

**3.1. Part I.** Let $\alpha$ an eigenvalue of Frobenius acting on $H^i(V/\bar{k})$ where $H^*(V/\bar{k})$ is any one of the known cohomologies. Then by Poncaré duality $\frac{q^d}{\alpha}$ is an eigenvalue of Frobenius actiong on $H^{2d-i}(V/\bar{k})$. Thus, both $\alpha$ and $\frac{q^d}{\alpha}$ are algebraic integers. It follows that in the field $F = \mathbb{Q}(\alpha)$, $\alpha$ is a $\lambda$-adic unit for any prime $\lambda$ which does not lie over $p = \operatorname{char} k$.

Since, by the Riemann hypothesis, one has that every complex absolute value of $\alpha$ is $q^{i/2}$, one is left with the intriguing question: What are the $l$-adic absolute values of $\alpha$, for primes $l$ lying over over $p$?

This is a question, which can by hindsight be seen to have had a long tradition — being related to Stickelberger's determination of the $p$-adic nature of Gauss sums and the Chevalley-Warning theorem which says that if $N$ is the number of solution of a polynomial equation mod $p$.

It is reasonable to expect thta the study of the above question should use a $p$-adic cohomology, and it dose. To describe, briefly, the know results, we introduce the *Newton polygon* of a monotone increasing sequence $a_1, a_2, \cdots, a_m$ of non-negative rational numbers: by definition, it is the graph of the real-valued continuous piece-wise linear function $v_{a_1,a_2,\cdots a_m}$ on $[0, m]$ which takes the value 0 at 0 and whose derivate is the constant $a_j$ on the interval $(j-1, j)$.

Now let $V/\mathcal{W}(k)$ be a smooth projective scheme. where $\mathcal{W}(k)$ is the ring of Witt vectors of $k$. Let $V/k$ be its reduction to $k$ through canonical morphism Spec $(k) \to$ Spec $(\mathcal{W}(k))$. Make the hypothesis that the de Rham cohomology of $V/k$ is extremely well-behaved: namely, $H^q(V/\mathcal{W}(k), \Omega^p)$ is a free $\mathcal{W}(k)$-module (whose rank we shall denote $h^{p,q}$– the $(p,q)$-th Hodge number of $V/k$) for all $p, q$. Fix an integer $r$, and define the $r$-th Hodge polygon of $V/k$ to be the Newton polygon of the sequence of integers.
$$\underbrace{0, 0, \cdots, 0}_{h^{0,r}}, \; \underbrace{1, 1, \cdots, 1}_{h^{1,r-1}}, \; \underbrace{2, 2, \cdots, 2,}_{h^{2,r-2}}, \; \cdots \; \underbrace{r, r, \cdots, r}_{h^{r,0}}$$

Define the $r$-th Newton polygon of $V$ to be the Newton polygon of the sequence:

$$\operatorname{ord}_q \alpha_1, \operatorname{ord}_q \alpha_2, \cdots, \operatorname{ord}_q \alpha_m \quad (m = r\text{-th Betti number of } V)$$

where $\alpha_1, \cdots, \alpha_m \in \bar{K}$ are the complete set of eigenvalues (with multiplicities) of Frobenius acting on the $r$-dimensional $p$-adic cohomology of $V/k$; $\operatorname{ord}_q$ denotes the $p$-adic ord function normalized so that $\operatorname{ord}_q q = 1$; and the $\alpha_j$ are indexed in such a manner that the above sequence is monotone increasing.

What is known about the $r$-th Newton polygon of $V/k$ is the following:

The $r$-th Newton polygon and the $r$-th Hodge polygon both end at the point $(m, rm/2)$ in the euclidean plane. The break-points (non-differentiable points) of these polygon occur at integral lattice points. The $r$-th Newton polygons lies in the closed region bounded by the $r$-th Hodge polygon, and the straight line joining $(0,0)$ and $(m, rm/2)$.

Note: One has a further symmetry in the geometry of the Newton polygon implied by the strong Lefschetz theorem, and also the existence of algebraic cohomology classes in dimension $r$ implies that a part of the Newton polygon must have slope $r/2$.

General remarks: It was an important discovery of Dwork, that (in a certain, unobvious, sense) the eigenvalues of Frobenius vary $p$-adic analytically if one varies the variety $V/k$ over a parameter space in characteristic $p$. A trace of this phenomenon may be seen in a theorem of Grothendieck: The Newton polygon rises under specialization of $V/k$.

### 3.2. Explicit description of Newton Polygon and Hodge Polygon.

Let $H^n = H^n_{\text{crys}}(X/W)$ be torsion free. Let

- $h^i = h^{i,n-i} = \dim_k H^{n-i}(X, \Omega^i_X)$
- $a_i = \dim H^{n-i}(X, \mathcal{W}\Omega^i_X)/V H^{n-i}(X, \mathcal{W}\Omega^i_X)$
- $a'_i = \dim H^{n-i}(X, \mathcal{W}\Omega^i_X)/F H^{n-i}(X, \mathcal{W}\Omega^i_X)$