

# Positive Characteristic Geometry

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## 1. WITT VECTOR COHOMOLOGY

Let  $p$  be a prime number. We then define the Witt polynomials ( $\mathbb{Z}$ -coefficient polynomials) respect to  $p$  as follows

$$\begin{aligned} W_0(x_0) &:= x_0 \\ W_1(x_0, x_1) &:= x_0^p + px_1 \\ &\dots \\ W_n(x_0, \dots, x_n) &:= \sum_{i=0}^n p^i x_i^{p^{n-i}} \end{aligned}$$

Let  $x = (x_0, \dots, x_n), y = (y_0, \dots, y_n)$ . From Serre's theorem [], there are polynomials  $S_0, \dots, S_n, P_0, \dots, P_n$  with  $2n + 2$  variables satisfying

$$\begin{aligned} W_n(x) + W_n(y) &= W_n(S_0(x, y), \dots, S_n(x, y)) \\ W_n(x) \cdot W_n(y) &= W_n(P_0(x, y), \dots, P_n(x, y)) \end{aligned}$$

For example,

$$\begin{aligned} S_0(x, y) &= x_0 + y_0 & S_1(x, y) &= x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0 + y_0)^p}{p} \\ P_0(x, y) &= x_0 y_0 & P_1(x, y) &= y_0^p x_1 + y_1 x_0^p + p x_1 y_1 \end{aligned}$$

Now, we define a new ring structure on  $n$ -couple for ring  $R$ , which is denoted by  $\mathcal{W}_n(R)$ . For all  $X, Y \in \mathcal{W}_n(R)$

$$\begin{aligned} X + Y &= (S_0(X, Y), \dots, S_{n-1}(X, Y)) \\ X \cdot Y &= (P_0(X, Y), \dots, P_{n-1}(X, Y)) \end{aligned}$$

Let  $X$  be non-singular variety over perfect field  $k$  of characteristic  $p > 0$  with dimension  $n$ .  $\mathcal{W}_k$  is sheaf of Witt vectors of length  $k$ .

We have three operators

- Frobenius  $F: \mathcal{W}_k \mathcal{O}_X \rightarrow \mathcal{W}_k \mathcal{O}_X$  locally defined by  $(a_0, \dots, a_{k-1}) \mapsto (a_0^p, \dots, a_{k-1}^p)$ ;
- Verschiebung  $V: \mathcal{W}_k \mathcal{O}_X \rightarrow \mathcal{W}_{k+1} \mathcal{O}_X$  locally defined by  $(a_0, \dots, a_{k-1}) \mapsto (0, \dots, a_{k-1})$ ;
- Restriction  $R: \mathcal{W}_{k+1} \mathcal{O}_X \rightarrow \mathcal{W}_k \mathcal{O}_X$  locally defined by  $(a_0, \dots, a_n) \mapsto (a_0, \dots, a_{n-1})$

It is well-known that

$$RVF = RFV = FRV = p \cdot \text{id}_{\mathcal{W}_k \mathcal{O}_X}$$

where the "·" is multiplication in  $\mathcal{W}_k \mathcal{O}_X$ .

Let  $\mathcal{W}(M) := \lim_k \mathcal{W}_{k+1}(M) \xrightarrow{R} \mathcal{W}_k(M)$  inverse limit of projective system induced by  $R$ . For example, since  $\mathcal{W}_k(\mathbb{F}_p) = \mathbb{Z}/p^k \mathbb{Z}$ , we have  $\mathcal{W}(\mathbb{F}_p) = \mathbb{Z}_p$  the  $p$ -adic integer ring.

We have following two canonical exact sequence of  $\mathcal{O}_X$ -module

$$\begin{aligned} (1) \quad & 0 \rightarrow \mathcal{W}_{k-1} \mathcal{O}_X \xrightarrow{V} \mathcal{W}_k \mathcal{O}_X \xrightarrow{R^{k-1}} \mathcal{O}_X \rightarrow 0 \\ (2) \quad & 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{W}_k \mathcal{O}_X \xrightarrow{R} \mathcal{W}_{k-1} \mathcal{O}_X \rightarrow 0 \end{aligned}$$

Witt vector cohomology is defined as

$$H^n(X, \mathcal{W} \mathcal{O}_X) = \lim_k H^n(X, \mathcal{W}_k \mathcal{O}_X)$$

in projective system  $[R: H^n(X, \mathcal{W}_{k+1} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_k \mathcal{O}_X)]$ .

## 2. VARIETIES IN POSITIVE CHARACTERISTIC UNDER FROBENIUS

Let  $X$  be a scheme over  $\mathbb{F}_p$ .  $F_X$  is called **absolute Frobenius morphism** of  $X$  which is induced by  $f \mapsto f^p$  on affine covers. Frobenius map  $\sigma$  of field  $\mathbb{F}_p$  induces endomorphism of  $\text{Spec } \mathbb{F}_p$ . It is also denoted by  $F_p$ . Furthermore, we have following base change

$$(3) \quad \begin{array}{ccccc} & & F_X & & \\ & \searrow & & \nearrow & \\ X & & & & X \\ & \swarrow F & \xrightarrow{\quad} & & \\ & X' & & & \\ & \downarrow f' & & & \downarrow f \\ & \text{Spec } \mathbb{F}_p & \xrightarrow{F_p} & \text{Spec } \mathbb{F}_p & \end{array}$$

Since  $F_X$  and  $F_p$  are homeomorphisms on underlying topological space of spectrum  $\text{Spec } \mathbb{F}_p$ ,  $F$  is also homeomorphism on underlying space. Hence it is easy to see that  $F_X$  is affine morphism of finite type. For all  $\mathcal{O}_X$ -module  $\mathcal{F}$  on  $X$ ,  $(F_X)_*(\mathcal{F})$  is same to  $\mathcal{F}$  as sheaves over abelian groups although the their  $\mathcal{O}_X$ -module structures are different. Let  $B_X^i = \text{Im}(d : \Omega_X^{i-1} \rightarrow \Omega_X^i)$ , then we have following exact sequences from definitions of Frobenius.

$$(4) \quad \mathcal{O}_X \xrightarrow{F_X} \mathcal{O}_X \xrightarrow{d} B_X^1 \quad \mathcal{O}_{X'} \xrightarrow{F} F_*\mathcal{O}_X \xrightarrow{F_*d} F_*B_X^1$$

*Examples 2.1.* Let  $Y = \text{Spec } A$ ,  $X = \mathbb{A}_Y^n$ . We have

$$\begin{aligned} \mathcal{O}_{X'} &= (A, F_A) \otimes_A A[x_1, x_2, \dots, x_n] \\ &\cong A^p[x_1, x_2, \dots, x_n] \end{aligned}$$

If  $k$  is perfect field, i.e. Frobenius map  $\sigma : k \rightarrow k$  is automorphism, then absolute Frobenius morphism is automorphism of scheme  $\text{Spec } k$ . Hence the pull-back  $X' \rightarrow X$  is also isomorphism and under isomorphism relative Frobenius  $F$  do same as absolute Frobenius  $F_X$  on  $X$ . So we also denote  $F_X$  by  $F$  in this case without confusion.

### 2.1. F-split.

**Definition 2.1.** The  $k$ -scheme of characteristic  $p > 0$  is called Frobenius split if exact sequence from Cartier isomorphism

$$(5) \quad 0 \longrightarrow \mathcal{O}_{X'} \xrightarrow{F^\#} F_*\mathcal{O}_X \xrightarrow{d} F_*B_X^1 \longrightarrow 0$$

is split exact.

If  $k$  is perfect, then we have absolute Cartier isomorphism  $C_{\text{abs}}^{-1}$ , which induce following exact sequences

$$(6) \quad 0 \rightarrow \mathcal{O}_X \rightarrow F_*\mathcal{O}_X \rightarrow F_*B_X^1 \rightarrow 0$$

$$(7) \quad 0 \rightarrow F_*B_X^n \rightarrow F_*\Omega_X^n \xrightarrow{C_{\text{abs}}} \Omega_X^n \rightarrow 0$$

Under assumption that  $k$  is perfect,  $X$  is Frobenius split if and only if **6** is split. This is because in this condition, base change of  $X$  along Frobenius morphism of field  $k$  is isomorphic to  $X$ . To simplify the notation, we denote  $F_*B_X^i$  or  $(F_X)_*B_X^i$  by  $(B_X^i)$ .

Suppose  $\xi$  is corresponding extension class of **5** in  $\text{Ext}_{\mathcal{O}_{X'}}^1((B_X^1), \mathcal{O}_{X'})$ . It follows that  $X$  is  $F$ -split if and only if  $\xi = 0$ . In particular, if extension group  $\text{Ext}_{\mathcal{O}_{X'}}^1((B_X^1), \mathcal{O}_{X'})$  vanishes, then **5** must be split. Writing it in cohomology form, we have  $H^1(X', (B_X^1)^\vee) = \text{Ext}_{\mathcal{O}_{X'}}^1((B_X^1), \mathcal{O}_{X'})$ .

**Lemma 2.1.** *If  $X$  is  $F$ -split, then  $H^i(X, B_X^1) = 0$  for all  $i \geq 0$ .*

*Proof.* Exact sequence **6** being splitting implies it induces isomorphism

$$\mathcal{O}_X \oplus (B_X^1) \cong (F_X)_*\mathcal{O}_X$$

Hence at cohomolgy level, we have

$$(8) \quad H^i(X, \mathcal{O}_X) \oplus H^i(X, (B_X^1)) \cong H^i(X, (F_X)_*\mathcal{O}_X)$$

while  $H^i(X, (F_X)_*\mathcal{O}_X) = H^i(X, \mathcal{O}_X)$  for all  $i \geq 0$ , hence we have  $\dim H^i(X, (B_X^1)) = 0$ , so that  $H^i(X, (B_X^1))$  vanishes.  $\square$

**2.2. Slope Spectral Sequence.** Slope spectral sequence is constructed in works [] of Luc Illusie on de Rham-Witt cohomology and comparasion to crystalline cohomology. It plays essential role in de Rham-Witt cohomology as Hodge-de Rham spectral sequence in de Rham cohomology. Actually, this spectral sequence gives explict descibtion of F-crystal structure on crystalline cohomology with de Rham-Witt cohomology.

Let  $X$  be smooth variety over perfect field  $k$  with characteristic  $p > 0$ . We have following de Rham-Witt complexes according to Illusie's works

$$(9) \quad \mathcal{W}\Omega_X^* := [\mathcal{W}\mathcal{O}_X \rightarrow \mathcal{W}\Omega_X^1 \rightarrow \cdots \rightarrow \mathcal{W}\Omega_X^n]$$

We have canonical filtration on  $\mathcal{W}\Omega_X^*$  as Hodge filtration in Hodge theory such that

$$(10) \quad F^i \mathcal{W}\Omega_X^* = \mathcal{W}\Omega_X^{\geq i} = [0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathcal{W}\Omega_X^i \cdots \rightarrow \mathcal{W}\Omega_X^n]$$

Filtration  $F^i \mathcal{W}\Omega_X^*$  induces spectral sequence

$$(11) \quad E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*)$$

which is called **slope spectral sequence** of  $X$ . Illusie proved that hypercohomology  $H^n(X, \mathcal{W}\Omega_X^*)$  computes crystalline cohomology  $H_{\text{crys}}^n(X/W)$  and  $E_1^{i,j} \otimes K$  without  $p$ -torsion. Hence

**Theorem 2.2** (Illusie). *If  $H^j(X, \mathcal{W}\Omega_X^i)$  are all torsion-free, then the slope spectral sequence*

$$(12) \quad E_1^{i,j} \cong H^j(X, \mathcal{W}\Omega_X^i) \Rightarrow \mathbb{H}^n(X, \mathcal{W}\Omega_X^*)$$

*degenerates in degree 1.*

As name of this spectral sequence, it captures information of "slope" of F-crystal on  $H_{\text{crys}}^*(X/W) \otimes K$ . We have following corollary

**Corollary 2.3** (Bloch). *For any  $i$ , canonical homomorphism  $H^*(X, \mathcal{W}\Omega_X^{\geq i}) \hookrightarrow H_{\text{crys}}^*(X/W)$  and  $H_{\text{crys}}^*(X/W) \rightarrow H^*(X, \mathcal{W}\Omega_X^{\leq i})$  induces isomorphism*

$$(13) \quad H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \xrightarrow{\sim} (H^*(X/W) \otimes K)_{\geq i}$$

$$(14) \quad (H_{\text{crys}}^*(X/W) \otimes K)_{[0,i]} \xrightarrow{\sim} H^*(X, \mathcal{W}\Omega_X^{\leq i}) \otimes K$$

*Proof.* We have following exact sequence

$$0 \rightarrow \mathcal{W}\Omega_X^{\geq i} \rightarrow \mathcal{W}\Omega_X^* \rightarrow \mathcal{W}\Omega_X^{\leq i-1} \rightarrow 0$$

Since connection map  $0 = d_1 \otimes K : H^{*-1}(X, \mathcal{W}\Omega_X^{\leq i-1}) \rightarrow H^*(X, \mathcal{W}\Omega_X^{\geq i})$ , upper exact sequence induces exact sequence of F-isocrystals

$$0 \rightarrow H^*(X, \mathcal{W}\Omega_X^{\geq i}) \otimes K \rightarrow H_{\text{crys}}^*(X/W) \otimes K \rightarrow H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \otimes K \rightarrow 0$$

We have already known that  $F(\text{Im } H^*(X, \mathcal{W}\Omega_X^{\geq i})) \subseteq p^i H_{\text{crys}}^*(X/W)$ . Hence the slope of  $H^*(X, \mathcal{W}\Omega_X^{\geq i})$  is greater than  $i$ . On other hand,

$$H^*(X, \mathcal{W}\Omega_X^{\leq i-1}) \cong W_\sigma[[V]][F]/(FV - p^i, VF - p^i)$$

It implies that slope of  $H^*(X, \mathcal{W}\Omega_X^{\leq i-1})$  strictly less than  $i$ . Hence we can conclude the proof.  $\square$

On other hand, we can describe Hodge polygon and Newton polygon explicitly. In fact, there is following theorem due to Ogus.

**Theorem 2.4.** *Let*

- $h^i = h^{i,n-i} = \dim_k H^{n-i}(X, \Omega_X^i)$
- $\alpha_i = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i) / V H^i(X, \mathcal{W}\Omega_X^i)$
- $\alpha'_i = \dim_k H^{n-i}(X, \mathcal{W}\Omega_X^i) / F H^{n-i}(X, \mathcal{W}\Omega_X^i)$

*then*

$$\alpha'_n = 0 \quad h^0 = \alpha_0 \quad h^i = \alpha'_{i-1} + \alpha_i \quad (1 \leq i \leq n)$$

*and*

$$\text{Hdg}(t) = \left( \sum_{i \leq t} h^i, \sum_{i \leq t} i h^i \right)$$

$$\text{Nwt}(t) = \text{Hdg}(t) + (\alpha'_{t-1}, t \alpha'_{t-1})$$

**Corollary 2.5.** *If  $H^i(X, \mathcal{W}\Omega_X^i)$  is torsion-free for all  $i, j$ , then  $X$  being ordinary is equivalent to condition that  $\text{Nwt}(t) = \text{Hdg}(t)$ .*

*Proof.* TBA □

### 2.3. Ordinary Varieties.

**Definition 2.2.** Let  $X$  be a smooth, proper variety over field  $k$  with  $\text{char } k > 0$ . We say  $X$  is **ordinary** if it satisfies

$$(15) \quad H^i(X, B_X^j) = 0 \quad \text{for all } i \geq 0, j > 0$$

where  $B_X^j = d(\Omega_{X/k}^{j-1})$  is the sheaf of boundaries of algebraic de Rham complex in degree  $j$ .

Equivalently,  $H^i(X', F_* B_X^i) = 0$  for all  $i \geq 0, j > 0$  implies ordinarity of  $X$ <sup>1</sup>. where  $F$  is relative Frobenius morphism  $F = F_{X/k} : X \rightarrow X'$ .

More numerical description of ordinarity according to property 2.5:

If  $H^q(X, \mathcal{W}\Omega_X^q)$  are all torsion free, then  $X$  is ordinary if and only if for every  $q$ , Newton polygon defined by the slopes of action of Frobenius on  $H_{\text{crys}}^q(X/W)$  is equal to Hodge polygon.

Here we list some properties of ordinary varieties.

**Proposition 2.6.** *Let  $X$  be variety over perfect field  $k$ . If  $X$  is ordinary and with trivial canonical bundle, then  $X$  is splitting.*

*Proof.* Homological algebra tells us that exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow F_* \mathcal{O}_X \rightarrow (B_X^1) \rightarrow 0$$

lies in extension group  $\text{Ext}^1((B_X^1), \mathcal{O}_X)$  as extension class. Denote this class as  $\xi$ . If  $\xi = 0$  then this sequence splits. Hence if this extension group vanishes then it has no choice but splits. But we have

$$H^1(X, (B_X^1)^\vee) = \text{Ext}^1((B_X^1), \mathcal{O}_X)$$

and

$$H^1(X, (B_X^1)^\vee) \cong H^{n-1}(X, B_X^1 \otimes \omega_X)^* \cong H^{n-1}(X, B_X^1)^*.$$

Hence it actually vanishes if  $X$  is ordinary. □

**Proposition 2.7.** *If  $S$  is algebraic surface over positive characteristic perfect field, then  $S$  being  $F$ -split  $\Rightarrow$  ordinary.*

*Proof.* As Cartier isomorphism is actually trace map in Grothendieck duality, we have following two perfect pairings:

$$(16) \quad F_* \mathcal{O}_X \otimes F_* \Omega_X^2 \rightarrow \Omega_X^2$$

$$(17) \quad F_* \Omega_X^1 \otimes F_* \Omega_X^1 \rightarrow \Omega_X^2$$

The pairing 16 induces perfect pairings

$$(18) \quad (B_X^1) \otimes (B_X^2) \rightarrow \Omega_X^2 \quad \mathcal{O}_X \otimes \Omega_X^n \rightarrow \Omega_X^2$$

by interchange following exact sequences with exact functor  $\text{Hom}(-, \Omega_X^2) = \text{Hom}(-, \mathcal{O}_X)$ .

$$0 \rightarrow \mathcal{O}_X \rightarrow F_* \mathcal{O}_X \rightarrow (B_X^1) \rightarrow 0$$

$$0 \rightarrow (B_X^2) \rightarrow F_* \Omega_X^2 \rightarrow \Omega_X^2 \rightarrow 0$$

Hence with lemma 2.1

$$(19) \quad H^i(X, (B_X^2)) \cong H^{n-i}(X, (B_X^2)^\vee \otimes \Omega_X^2)^* \cong H^{n-i}(X, B_X^1)^* = 0$$

for all  $0 \leq i \leq n$ . This completes the proof. □

**Corollary 2.8.** *If  $X$  is K3 surface, then  $X$  being  $F$ -split  $\Leftrightarrow$  ordinary.*

<sup>1</sup>by Leray spectral sequence

Actually, we can generalize for perfect pairings appearing in proof of 2.7. As in [], we use following notations

$$(20) \quad B_m \Omega_X^i := C^{-1}(B_{m-1} \Omega_X^i) \quad B_1 \Omega_X^i := (B_X^i) \quad B_0 \Omega_X^i = \underline{0}$$

$$(21) \quad Z_m \Omega_X^i := C^{-1}(Z_{m-1} \Omega_X^i) \quad Z_1 \Omega_X^i := (Z_X^i) \quad Z_0 \Omega_X^i = \Omega_X^i$$

Note that we have following inclusion sequence

$$(22) \quad \underline{0} = B_0 \Omega_X^i \subset B_1 \Omega_X^i \subset \cdots \subset B_m \Omega_X^i \subset \cdots \subset Z_m \Omega_X^i \subset Z_{m-1} \Omega_X^i \subset \cdots \subset Z_0 \Omega_X^i = \Omega_X^i$$

$B_m \Omega_X^i$  and  $Z_m \Omega_X^i$  can be viewed as locally free submodules of  $F_*^m(\Omega_X^i)$ . This implies that

$$(23) \quad 0 \rightarrow B_m \Omega_X^i \rightarrow F_*^m \Omega_X^i \rightarrow F_*^m \Omega_X^i / B_m \Omega_X^i \rightarrow 0$$

$$(24) \quad 0 \rightarrow Z_m \Omega_X^{n-i} \rightarrow F_*^m \Omega_X^{n-i} \rightarrow F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i} \rightarrow 0$$

are exact.  $\mathcal{H}om(-, \Omega_X^n)$  is exact since  $\Omega_X^n$  is trivial. Applying it on 24, we get exact sequence

$$(25) \quad 0 \rightarrow \mathcal{H}om(F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i}, \Omega_X^n) \rightarrow \mathcal{H}om(F_*^m \Omega_X^{n-i}, \Omega_X^n) \rightarrow \mathcal{H}om(Z_m \Omega_X^{n-i}, \Omega_X^n) \rightarrow 0$$

Since the middle term is isomorphism to  $F_*^m \Omega_X^i$  induced from perfect pairing

$$(26) \quad F_*^m \Omega_X^i \otimes F_*^m \Omega_X^{n-i} \rightarrow \Omega_X^n$$

we can get following two canonical perfect pairings

$$(27) \quad B_m \Omega_X^i \otimes F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i} \rightarrow \Omega_X^n$$

$$(28) \quad Z_m \Omega_X^{n-i} \otimes F_*^m \Omega_X^i / B_m \Omega_X^i \rightarrow \Omega_X^n$$

Note that we have canonical isomorphism  $F_*^m \Omega_X^{n-i} / Z_m \Omega_X^{n-i} \cong B_m \Omega_X^{n-i+1}$  induced by differential  $d : \Omega_X^{n-i} \rightarrow \Omega_X^{n-i+1}$ . Hence 27 becomes  $B_m \Omega_X^i \otimes B_m \Omega_X^{n-i+1} \rightarrow \Omega_X^n$ . In particular, taking  $i = 1, n = 2, m = 1$  we get perfect pairing 18.

Notice that we have exact sequence

$$0 \rightarrow B_1 \Omega_X^1 \rightarrow B_{m+1} \Omega_X^1 \xrightarrow{C} B_m \Omega_X^1 \rightarrow 0$$

Hence we have surjective maps  $C : H^n(X, B_{m+1} \Omega_X^1) \rightarrow H^n(X, B_m \Omega_X^1)$ . So we get a projective system, then taking inverse limit

$$H^n(X, B \Omega_X^1) := \lim_m H^n(X, B_m \Omega_X^1)$$

**Definition 2.3.** The dimension of  $H^n(X, B \Omega_X^1)$  is called b-number of  $X$ .

In next subsection, we will use these to give explicit formula for height of Calabi-Yau varieties over positive characteristic which is also called h-number in [].

**2.4. Formal group law.** In algebraic geometry, **formal group** is referred by group object in category of formal schemes. If  $\mathcal{G}$  is a functor from Artin algebras to groups which is left exact, then  $\mathcal{G}$  is representable by formal group.  $\mathcal{G}$  is called a **formal group law**. Let

$$F_S^r(S) = \ker \{H^r(X \times S, \mathbb{G}_m) \rightarrow H^r(X, \mathbb{G}_m)\}$$

the cohomology is étale cohomology here. Theorem of Artin and Mazur say that this functor  $F_S^r : \text{Art}_k \rightarrow \text{Ab}$  is pro-representable by formal group  $\Phi_X$  if  $X$  is proper Calabi-Yau with dimension  $n$  over perfect field  $k$ . Moreover, we have that tangent space of  $\Phi_X$  is isomorphic to  $H^n(X, \mathcal{O}_X)$ . Hence since Calabi-Yau variety is with geometric genus 1, e.g.  $\dim_k H^n(X, \mathcal{O}_X) = 1$ ,  $\Phi_X$  is formal group of dimension 1.

**2.5. Height of Calabi-Yau varieties.** Let  $X$  be Calabi-Yau variety over perfect field  $k$  with characteristic  $p$  and  $\Phi_X$  be its formal Brauer group. Then we have following theorem

**Theorem 2.9.** If  $X$  is of dimension  $n$ , then we have following formula for height

$$(29) \quad h(\Phi_X) = \min \{i \geq 1 : [F : H^n(W_i \mathcal{O}_X) \rightarrow H^n(W_i \mathcal{O}_X)] \neq 0\}$$

Before giving the proof of this theorem, we will prepare some properties of Witt vector cohomology of Calabi-Yau varieties.

**Proposition 2.10.** Let  $X$  be Calabi-Yau variety over perfect field  $k$  of dimension  $n$ . We have

$$(1) \quad H^i(X, W_j \mathcal{O}_X) = 0 \text{ for all } j > 0, 1 \leq i \leq n-1. \text{ Furthermore, } H^i(X, W \mathcal{O}_X) = 0;$$

- (2) Pull-back of  $R$  on cohomology  $R : H^n(X, \mathcal{W}_k \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_k \mathcal{O}_X)$  is surjective with kernel  $H^n(X, \mathcal{O}_X)$ ;  
(3) Assume that for some  $0 < k \leq n$  the map  $F : H^k(X, \mathcal{W}_j \mathcal{O}_X) \rightarrow H^k(X, \mathcal{W}_j \mathcal{O}_X)$  vanishes, then for all  $0 \leq i \leq j$  the map  $F : H^k(X, \mathcal{W}_i \mathcal{O}_X) \rightarrow H^k(X, \mathcal{W}_i \mathcal{O}_X)$  vanishes too. Moreover, for all  $H^n(X, \mathcal{W}_i \mathcal{O}_X)$  is vector space over  $K$ ;  
(4)

$$0 \rightarrow H^n(X, \mathcal{W}_{i-1} \mathcal{O}_X) \xrightarrow{V} H^n(X, \mathcal{W}_i \mathcal{O}_X) \xrightarrow{R^{n-1}} H^n(X, \mathcal{O}_X) \rightarrow 0$$

and

$$0 \rightarrow H^n(X, \mathcal{W} \mathcal{O}_X) \xrightarrow{V'} H^n(X, \mathcal{W} \mathcal{O}_X) \xrightarrow{R'} H^n(X, \mathcal{O}_X) \rightarrow 0$$

are both exact.

*Proof.* (1) Note that we have following exact sequence of sheaves for all  $j \geq 1$

$$0 \rightarrow \mathcal{W}_{j-1} \mathcal{O}_X \rightarrow \mathcal{W}_j \xrightarrow{R^{j-1}} \mathcal{O}_X \rightarrow 0$$

Hence we have exact sequence of groups

$$H^i(X, \mathcal{W}_{j-1} \mathcal{O}_X) \rightarrow H^i(X, \mathcal{W}_j \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$$

If  $H^i(X, \mathcal{W}_{j-1}) = 0$  and  $1 \leq i \leq n-1$ , then  $H^i(X, \mathcal{W}_j \mathcal{O}_X) = 0$  since two sides of this sequence are zero. Hence by induction, we complete the proof.

- (2) We have following exact sequence

$$0 \simeq H^{n-1}(X, \mathcal{W}_{k-1}) \rightarrow H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_k \mathcal{O}_X) \xrightarrow{R} H^n(X, \mathcal{W}_{k-1} \mathcal{O}_X) \rightarrow 0$$

It implies  $H^n(X, \mathcal{O}_X) = \ker \left( H^n(\mathcal{W}_k \mathcal{O}_X) \xrightarrow{R} H^n(X, \mathcal{W}_{k-1} \mathcal{O}_X) \right)$ .

- (3) Note that following diagram commutes for all  $j > i$

$$\begin{array}{ccc} H^n(X, \mathcal{W}_j \mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X, \mathcal{W}_i \mathcal{O}_X) \\ \downarrow F & & \downarrow F \\ H^n(X, \mathcal{W}_j \mathcal{O}_X) & \xrightarrow{R^{j-i}} & H^n(X, \mathcal{W}_i \mathcal{O}_X) \end{array}$$

Hence if left Frobenius map vanishes then so does right one. □

*Proof of theorem 2.9.* Let  $h = h(\Phi_X)$ . If  $F$  is zero map on each  $H^n(X, \mathcal{W}_i \mathcal{O}_X)$ , then  $H^n(X, \mathcal{W} \mathcal{O}_X)$  is  $k$ -vector space. But we have following exact sequence

$$0 \rightarrow H^n(X, \mathcal{W} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X) \rightarrow 0$$

and  $H^n(X, \mathcal{O}_X)$  is actually  $k$ -vector space. Hence  $H^n(X, \mathcal{W} \mathcal{O}_X)$  is not of finite dimension as  $k$ -vector space. Since  $h = \text{rank}_{W(k)} H^n(X, \mathcal{W} \mathcal{O}_X)$ , we can conclude that in this case  $\Phi_X$  is of height  $\infty$ .

Notice that

$$H^n(X, \mathcal{W} \mathcal{O}_X) / V H^n(X, \mathcal{W} \mathcal{O}_X) \simeq H^n(X, \mathcal{O}_X)$$

as  $W(k)$ -module and  $H^n(X, \mathcal{O}_X)$  is naturally  $k$ -vector space, they are isomorphic as  $k$ -vector space. Hence from equality

$$\text{rank}_{W(k)}(D(\Phi_X)) = \dim_k H^n(X, \mathcal{W} \mathcal{O}_X) / F H^n(X, \mathcal{W} \mathcal{O}_X) + \dim_k H^n(X, \mathcal{W} \mathcal{O}_X) / V H^n(X, \mathcal{W} \mathcal{O}_X)$$

we get

$$\dim_k H^n(X, \mathcal{W} \mathcal{O}_X) / F H^n(X, \mathcal{W} \mathcal{O}_X) = h - 1$$

If  $F : H^n(X, \mathcal{W}_i \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_i \mathcal{O}_X)$  is zero, then projection

$$[P_i] : H^n(X, \mathcal{W} \mathcal{O}_X) / F H^n(X, \mathcal{W} \mathcal{O}_X) \rightarrow H^n(X, \mathcal{W}_i \mathcal{O}_X) / F H^n(X, \mathcal{W}_i \mathcal{O}_X)$$

is well-defined and surjective since  $R, F$  commutes. As vector space,  $H^n(X, \mathcal{W}_i \mathcal{O}_X)$  is of dimension  $i$ . This implies that we have inequality relation

$$h \geq i + 1$$

Hence we get

$$h \geq \min \{ i \geq 1 : [F : H^n(\mathcal{W}_i \mathcal{O}_X) \rightarrow H^n(\mathcal{W}_i \mathcal{O}_X)] \neq 0 \}$$

Conversely, it is sufficient to prove  $F H^n(X, \mathcal{W}_{h-1} \mathcal{O}_X) = 0$ .

Manin's result in [1] implies that

$$(30) \quad H^n(X, \mathcal{W}\mathcal{O}_X) \cong D(\Phi_X) \cong \frac{W(k)[F, V]}{(F - V^{h-1})}$$

Hence  $FH^n(X, \mathcal{W}\mathcal{O}_X) \cong V^{h-1}H^n(X, \mathcal{W}\mathcal{O}_X)$ .

$$\begin{array}{ccc} V^{h-1}H^n(X, \mathcal{W}\mathcal{O}_X) & \longrightarrow & H^n(X, \mathcal{W}\mathcal{O}_X) \\ \downarrow & & \downarrow \\ 0 = V^{h-1}H^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) & \longrightarrow & H^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) \end{array}$$

commutes, this implies  $FH^n(X, \mathcal{W}_{h-1}\mathcal{O}_X) = 0$ . □

We will call  $h(\Phi_X)$  the  $h$ -number of  $X$ , denoted by  $h(X)$ .

**Corollary 2.11.** *For proper Calabi-Yau variety  $X$  over perfect field  $k$ , we have  $b(X) = \dim_k H^n(X, \mathcal{W}\mathcal{O}_X)/FH^n(X, \mathcal{W}\mathcal{O}_X)$ . Furthermore,*

$$\dim_k H^{n-1}(X, B_m\Omega_X^1) = \dim_k H^n(X, B_m\Omega_X^1) = \begin{cases} \min\{m, h-1\} & \text{if } h < \infty \\ m & \text{if } h = \infty \end{cases}$$

*Proof.* We have exact sequence

$$\mathcal{W}_m\mathcal{O}_X \xrightarrow{F} \mathcal{W}_m\mathcal{O}_X \xrightarrow{D_m} B_m\Omega_X^1 \rightarrow 0$$

It induces

$$H^n(X, \mathcal{W}_m\mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W}_m\mathcal{O}_X) \rightarrow H^n(X, B_m\Omega_X^1) \rightarrow 0$$

Taking limit with  $m$ , we get

$$H^n(X, \mathcal{W}\mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{W}\mathcal{O}_X) \rightarrow H^n(X, B\Omega_X^1) \rightarrow 0$$

exact. Hence  $b(X) = h(X) - 1$ . □