Simplicial Objects in Algebraic Topology

J.Peter May

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Chapter 1

Simplicial Objects and Homotopy

1.1 Definitions and examples

We introduce the concept of simplicial set and give several examples here. A categorical definition will be given in the next section.

Definition 1.1. A simplicial set K is a graded set indexed on the non-negative integers together with maps $d_i : K_q \to K_{q-1}$ and $s_i : K_q \to K_{q+1}$,$0 \leq i \leq q$, which satisfy the following identities:

(i) $d_i d_j = d_{j-1} d_i$ if i, j,

(ii) $s_i s_j = s_{j+1} s_i$ if $i \leq j$,

(iii) $d_i s_j = s_{j-1} d_i$ if $i < j$,
$d_j s_j = \text{identity} = d_{j+1} s_j$,
$d_i s_j = s_j d_{i-1}$ if $i > j+1$

The elements of K_q are called **q-simplices**. The d_i and s_j are called face and degeneracy operations. A simplex x is **degenerate** if $x = s_i y$ for some simplex y and degeneracy operator j_i; otherwise x is non-degenerate.

Definition 1.2. A simplicial map $f : K \to L$ is a map of degree zere of graded sets which commutes with the face and degeneracy operators; that is , f consists of $f_q : K_q \to L_q$ and

$$f_q d_i = d_i f_{q+1},$$
$$f_q s_i = s_i f_{q-1}.$$

Definition 1.3. A simplicial set k is said to satisfy the extension condition if for every collection of $n+1$ n-simplices $x_0, x_1, \ldots, x_{k-1}, \ldots, x_{n+1}$ which satisfy the compatibility condition $d_i x_j = d_{j-1} x_i, i < j, i \neq k, j \neq k$, there exists an $(n+1)$-simplex x such that $d_i x = x_i$ for $i \neq k$.

Example 1.4. We recall that a simplicial complex K is a set of finite subsets, called simplices, of a given set $\bar{K}$ subject to the condition that every non-empty subset of an element of K is itself an element of K. A simplicial set $\tilde{K}$ arises from K in the following manner. An n-simplex of $\tilde{K}$ is a sequence $(a_0, \ldots, a_n)$ of elements of $\bar{K}$ such that the set $\{a_0, \ldots, a_n\}$ is n m-simplex of K for some $m \leq n$. The face and degeneracy operators of $\tilde{K}$ are defined by:

$$d_i(a_0, \ldots, a_n) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

and

$$s_j(a_0, \ldots, d_n) = (a_0, \ldots, a_i, a_i, a_{i+1}, \ldots, a_n)$$

If the elements of $\bar{K}$ are ordered and we require $\tilde{K}$ to cnsist of those sequences $(a_0, \ldots, a_n)$ such that $(a_0, \ldots, a_n)$ such that $a_0 \leq a_1 \leq \cdots \leq a_n$ and $\{a_0, \ldots, a_n\}$ is an m-simplex of K for some $m \leq n$, then there will be exactly one non-degenerate n-simplex of $\tilde{K}$ for every n-simplex of K.

Example 1.5. Let $\Delta_n = \{(t_0, \ldots, t_n) | 0 \leq t_i \leq 1, \sum t_i = 1\} \subset R^{n+1}$. If X is a topological space, singular n-simplex of X is a continuous function $f : \Delta_n \to X$. The graded set $S(X)$, where $S_n(X)$ is the set of singular n-simplices of X, is called the total singular complex of X, $S(X)$ becomes a simplicial set if we define face and degeneracy operators by:

$$d_i f(t_0, \ldots, t_{n-1}) = f(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

and

$$s_i f(t_0, \ldots, t_{n+1}) = f(t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1})$$

The following elementary fact will later be used to show that $S(X)$ determines the homotopy groups of X.

Lemma 1.1. *$S(X)$ satisfies the extension condition.*

Proof. Since the union of any $n+1$ faces of Δ_{n+1} is a retract of Δ_{n+1}, any continuous function defined on such a union can be extended to Δ_{n+1}. □

Conventions 1.1. The word "complex"(unmodified) will always mean simplicial set. A complex which satisfies the extension condition will be called a Kan complex.

1.2 Simplicial objects in categories; homology

Recall that a category $\mathcal{C}$ is a class of object together with a family of disjoint sets $\mathrm{Hom}(A, B)$, one for each pair of objects, a function $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C), \alpha \times \beta \to \alpha\beta$, and an element $1_A \in \mathrm{Hom}(A, A)$, all subject to the conditions $\alpha(\beta\gamma)$ whenever either is defined and $\alpha \circ \mathrm{id}_A = \alpha = \mathrm{id}_B \circ \alpha, \alpha \in \mathrm{Hom}(A, B)$. The elements of $\mathrm{Hom}(A, B)$ are morphisms with domain A and range B. The opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ has an object A^{op} for each object A of $\mathcal{C}$ and a morphism $\alpha^{\mathrm{op}} \in \mathrm{Hom}(B^{\mathrm{op}}, A^{\mathrm{op}})$ for each morphism $\alpha \in \mathrm{Hom}(A, B)$; $\alpha^{\mathrm{op}}\beta^{\mathrm{op}}$ is defined and equal to $(\beta\alpha)^{\mathrm{op}}$ whenever $\beta\alpha$ is defined.

A convariant (resp., contravariant) functor $F : \mathcal{C} \to \mathcal{D}$ is a correspondence which assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$ and to each morphism $\alpha \in \mathrm{Hom}(A, B)$ a morphism $F(\alpha) \in \mathrm{Hom}(F(A), F(B))$ (resp., $F(\alpha) \in \mathrm{Hom}(F(B), F(A))$) subject to the conditions $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}, A \in \mathcal{C}$, and $F(\alpha\beta) = F(\alpha)F(\beta)$ (resp., $F(\alpha\beta) = F(\alpha)F(\beta)$) whenever $\alpha\beta$ is defined in $\mathcal{C}$. If $T : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is defined by $T(A) = A^{\mathrm{op}}$ and $T(\alpha) = \alpha^{\mathrm{op}}$, then T is a contravariant functor; any contravariant functor $F : \mathcal{C} \to \mathcal{D}$ may be considered as the covariant functor $TF : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ or $FT : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. If F and G are covariant (resp., contravariant) functors $\mathcal{C} \to \mathcal{D}$, a natural transformation $\lambda : F \Rightarrow G$ is a function which assigns to each object A of $\mathcal{C}$ a morphism $\lambda(A) \in \mathrm{Hom}(F(A), F(B))$ subject to the condition that if $\alpha \in \mathrm{Hom}(A, B)$, then $G(\alpha)\lambda(A) = \lambda(B)F(\alpha)$(resp., $G(a)\lambda(\beta) = \lambda(A)F(\alpha)$).

Now we define a category Δ^{op} as follows. The objects Δ_n of Δ^{op} are sequence of integers, $\Delta_n = (0, 1, \ldots, n), n \geq 0$. The morphisms of Δ^{op} are the monotonic maps $\mu : \Delta_n \to \Delta_m$, that is , the maps μ such that $\mu(i) \leq \mu(j)$ if $i < j$. Define morphisms $\delta_i : \Delta_{n-1} \to \Delta_n$ and $\sigma_i : \Delta_{n+1} \to \Delta_n, 0 \leq i \leq n$, by

$$\delta_i(j) = j \text{ if } j < i; \qquad \delta_i(j) = j + 1 \text{ if } j \geq i, \tag{1.1}$$

$$\sigma_i(j) = j \text{ if } j \leq i; \qquad \sigma_i(j) = j - 1 \text{ if } j > i. \tag{1.2}$$

Let $\mu \in \mathrm{Hom}(\Delta_n, \Delta_m)$, μ not an identity. Suppose $\imath_1, \ldots, i_s$, in reverse order, are the elements of Δ_m not in $\mu(\Delta_n)$ and $j_1, \ldots, j_{t'}$ in order, are the elements of Δ_n such that $\mu(j) = \mu(j+1)$. Then:

$$\mu = \delta_{i_1} \ldots \delta_{i_s} \sigma_{j_1} \ldots \sigma_{j_t}, \text{ where, } 0 \leq i_s < \cdots < i_1 \leq m, 0 \leq j_1 < \cdots < j_t < n, \text{and } n - t + s = m. \tag{1.3}$$

Further, the factorization of μ in the form 1.3 is unique. Having defined Δ^{op}, we can formulate

Definition 1.6. A simplicial object in a category $\mathcal{C}$ is a contravariant functor $F : \Delta^{\mathrm{op}} \to \mathcal{C}$. Such functors form the objects of a category $\mathcal{C}^s$, the elements of $F(\Delta_n)$ are called n-simplices, and the maps $d_i = F(\delta_i)$ and $s_j = F(\sigma_i)$ satisfy (i)-(iii) of 1.1. Any simplicial set K determines a contravariant functor $F : \Delta^{\mathrm{op}} \to \mathcal{C}$, where $\mathcal{C}$ is the category of sets, by the rules $F(\Delta_n) = K_n$ and

$$F(\mu) = s_{j_t} \dots s_{j_1} d_{i_s} \dots d_{i_1},$$

where μ is a morphism of Δ^{op} expressed in the form 1.3. Thus a simplicial set may be uniquely identified wiht a simplicial object in the category of sets. Analogously, we will speak of simplicial groups, simplicial modules, and so forth, depending on the choice of the category $\mathcal{C}$.

Remark 1.2. Let $\Delta = (\Delta^{\mathrm{op}})^{\mathrm{op}}$ denote the opposite category of Δ^{op}, $T : \Delta^{\mathrm{op}} \to \Delta$ the contravariant functor defined above. The category $\mathcal{C}^s$ could equally well be defined as that of covariant functors from Δ to $\mathcal{C}$.

Now suppose that $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor. By composition, F induces a covariant functor $F^s : \mathcal{C}^s \to \mathcal{D}^s$. In particular, suppose that $\mathcal{C}$ is the category of sets, $\mathcal{D}$ that of Abelian groups, and for $A \in \mathcal{C}$, $F(A)$ is the free Abelian group generated by A. Then if $K \in \mathcal{C}^s$, $F^s(K)$ may be given a structure of chain complex with differential d defined on $F^s(K)_n = F^s(K_n)$ by

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

We denote this chain complex by $C(K)$. If G is an Abelian group, we define the homology and cohomology of K with coefficients in G by

$$H_*(K; G) := H(C(K) \otimes G) \text{ and } H^*(K; G) := H(\mathrm{Hom}(C(K), G)).$$

In case $K = S(X)$, these are, of course, the singular homology and cohomology groups of the space X.

1.3 Homotopy of Kan complexes

Definition 1.7. Let K be a complex. Two n-simplices x and x' of K are homotopic, written $x \sim x'$, if $d_i x = d_i x', 0 \le i \le n$, and there exists a simplx $y \in K_{n+1}$ such that $d_n y = x, d_{n+1} y = x'$, and $d_i y = s_{n-1} d_i x = s_{n-1} d_i x', 0 \le i < n$. The simplex y is called a homotopy from x to x'.

Proposition 1.2. *If K is a Kan complex, then $\sim$ is an equivalence relation on the n-simplices of $K, n \ge 0$.*

Proof. The relation $\sim$ is relexive since

$$d_n s_n x = x = d_{n+1} s_n x$$

and $d_i s_n x = s_{n-1} d_i x, 0 \le i < n$. Suppose $x \sim x'$ and $x \sim x''$. We must prove $x' \sim x''$. Let y' satisfy

$$d_n y' = x, d_{n+1} y' = x', \text{and } d_i y' = s_{n-1} d_i x', i < n.$$

Let y'' satisfy

$$d_n y'' = x.d_{n+1} y'' = x'', \text{and } d_i y'' = s_{n-1} d_i x', i < n.$$

Then the $n+2$ $(n+1)$-simplices

$$d_0 s_n s_n x', \dots, d_{n-1} s_n s_n x', y', y''$$

are seen to satisfy the compatibility condition. Therefore there exists an $(n+2)$-simplex z such that $d_i z = d_i s_n s_n x', 0 \le i < n, d_n z = y'$, and $d_{n+1} z = y''$. It follows that

$$d_i d_{n+2} z = s_{n-1} d_i x', 0 \le i < n,$$

$d_n d_{n+2} z = x'$, and $d_{n+1} d_{n+2} z = x''$, hence $x' \sim x''$. $\square$

Definition 1.8. Let L be a subcomplex of K. Two n-simplices x and x', $n > 0$, are homotopic relative to L, written $x \sim x'$ rel L, if $d_i x = d_i x', 1 \leq i \leq n$, if $d_0 x \sim d_0 x'$ in L and a simplex $w \in K_{n+1}$ such that $d_0 w = y, d_n w - x, d_{n+1} w = x$'' and $d_i w = s_{n-1} d_i x = s_{n-1} d_i x', 1 \leq i < n$. The simplex w is called a relative homotopy from x to x'.

Proposition 1.3. *If L is a sub Kan complex of the Kan complex K, then $\sim$ rel L is an equivalence relative homotopy from x to x'.*

Proof. The relation $\sim$ rel L is reflexive since if $d_0 x \in L$, then $s_{n-1} d_0 x$ is a homotopy in L from $d_0 x$ to $d_0 x$, and if $w = s_n x$, then $d_i w = s_{n-1} d_i x, 0 \leq i < n$, and

$$d_n w = x = d_{n+1} w.$$

Suppose $x \sim x'$ rel L and $x \sim x''$ rel L. We must prove that $x' \sim x'$ rel L and $x \sim x''$ L . We must prove that $x' \sim x''$ rel L. Let y' and y'' be homotopies in L from $d_0 x$ to $d_0 x'$ and from $d_0 x$ to $d_0 x''$, and suppose w' and w'' are relative homotopies from x to x' and from x to x'' which satisfy $d_0 w' = y'$ and $d_0 w'' = y''$. As in the proof of 1.2, we may choose $z \in L_{n+1}$ such that

$$d_i z = d_i s_{n-1} s_{n-1} d_0 x', 0 \leq i < n-1$$
$$d_{n-1} z = y' \text{ and } d_n z = y''$$

Then $y = d_{n+1} z$ is a homotopy in L from $d_0 x'$ to $d_0 x''$. Now it is easy to see that the n+2 $(n+1)$-simplices

$$z, d_1 s_n s_n x', \ldots, d_{n-1} s_n s_n x', w', w''$$

satisfy the compatability condition so that there exists $v \in K_{n+2}$ such that $d_i v = d_i s_n s_n x', 1 \leq i < n, d_0 v = z, d_n v = w'$ and $d_{n+1} v = w''$. Let $w = d_{n+2} v$. Then $d_i w = s_{n-1} d_i x', 1 \leq i < n, d_0 w = y, d_n w = x'$, and $d_{n+1} w = x''$. □

Notations 1.3. Let K be a complex, $\phi \in K_0$. ϕ generates a subcomplex of K which has exactly one simplex $s_{n-1}, \ldots s_0 \phi$ in each dimension n. We will abuse notation by letting ϕ denote ambiguously either this subcomplex or any of its simplices. We call (K, ϕ) a Kan pair if K is a Kan complex. We call (K, L, ϕ) a Kan tripe if $\phi \in L_0$ and L is a sub Kan complex of the Kan complex K. Simplicial maps of pairs and triples are defined in the obvious manner.

Definition 1.9. Let (K, ϕ) be a Kan pair. Let $\tilde{K_n}, n \geq 0$, denote the set of all $x \in K_n$ which satisfy $d_i x = \phi, 0 \leq i \leq n$. The we define $\pi_n(K, \phi) = \tilde{K_n}/(\sim)$. $\pi_0(K, \phi)$ is called the set of path components of K. K is connected if $\pi_0(K, \phi) = \phi$(where we are letting ϕ denote its equivalence class). K is n-connected if $\pi_n(K, \phi) = \phi, 0 \leq i \leq n$. Let (K, L, ϕ) be a Kan triple. Let $\tilde{K}(L)_n, n \geq 1$, denote the set of all $x \in K_n$ which satisfy $d_0 x \in L_{n-1}$ and $d_i x = \phi, 1 \leq i \leq n$. Then we define

$$\pi_n(K, L, \phi) = \tilde{K}(L)_n/(\sim \text{ rel } L).$$

Note that $\pi_n(K, \phi, \phi) = \pi_n(K, \phi), n \geq 1$. Finally, we define $d : \pi_n(K, L, \phi) \to \pi_n(L, \phi), n \geq 1$, by $d[x] = [d_0 x]$, where $[x]$ denotes the homotopy class of x.

Theorem 1.4. *Let (K, L, ϕ) be a Kan triple. Then the sequence*

$$\cdots \to \pi_{n+1}(K, L, \phi) \xrightarrow{d} \pi_n(L, \phi) \xrightarrow{i} \pi_n(K, \phi) \xrightarrow{j} \pi_n(K, L, \phi) \to \ldots$$

of sets with distinguished elements ϕ is exact, where the maps i and j are induced by inclusion.

Proof. (i) $i \circ d = \phi$: Consider $i[d_0 x] = i \circ d[x], x \in K(L)_{n+1}$. The n+2 $(n+1)$-simplices $-, \phi, \ldots, \phi, x$ are compatible, hence we may choose $z \in K_{n+2}$ such that $d_i z = \phi, 1 \leq i \leq n+1, d_{n+2} z = x$. Then $d_i d_0 z = d_0 d_{i+1} z = \phi, 0 \leq i \leq n+1$, and $d_{n+1} d_0 z = d_0 x$, so that $d_0 x \sim \phi$ in K.

(ii) Image d = Kernel i: Let $i[y] = \phi, y \in \tilde{L}_n$. Then $y \sim \phi$ in K_n, say $d_i z\phi, 0 \leq i \leq n, d_{n+1}z = y$. The n+2 $(n+1)$-simplces $z, \phi, \ldots, \phi$ are compatible, hence there exists $w \in K_{n+2}$ with $d_0 w = z, d_i w = \phi, 1 \leq i \leq n+1$. $d_i d_{n+2} w = \phi, 1 \leq i \leq n+1$, and $d_0 d_{n+2} w = y$, so that $d[d_{n+2}w] = [y]$.

(iii) $d_j = \phi : d_j[x] = \phi$ since $d_0 x = \phi, x \in \tilde{K}_n$.

(iv) Image j = Kernel d: Let $d[x] = [d_0 x] = \phi, x \in \tilde{K}(L)_n$. There exits $z \in L_n$ such that $d_i z = \phi, 0 \leq i \leq n, d_n z = d_0 x$. The n+1 n-simplices $z, \phi, \ldots, -, x$ (where $-$ means $k = n$ in the compatibility condition) are compatible, say $d_0 y = z, d_i y = \phi, 1 \leq i \leq n-1, d_{n+1} y = x$. Thus $x \sim d_n y$ rel L. Since $d_i d_n y = \phi, 0 \leq i \leq n, [x] = j[d_n y]$.

(v) $j \circ i = \phi$: Consider $j \circ i[y], y \in \tilde{L}_n$. The n+1 n-simplices $-, \phi, \ldots, \phi, y$ are compatible in L, say $z \in L_{n+1}$ satisfies $d_i z = \phi, 1 \leq i \leq n, d_{n+1} z = y$. $d_i d_0 z = \phi, 0 \leq i \leq n$, hence $d_0 z$ is homotopy between ϕ and $\phi = d_0 y$, so z is a relative homotopy between ϕ and y.

(vi) Image i = Kernel j: Let $j[x] = \phi, x \in \tilde{K}_n$. Choose $w \in K_{n+1}$ such that $d_i w = \phi, 1 \leq i \leq n, d_{n+1} w = x$, and $d_0 w = z \in L_n$. The n+1 n-simplices $z, \phi, \ldots, \phi, -$ are compatible in L, say $d_0 v = z$ and $d_i v = \phi, 1 \leq i \leq n$. The n+2 $(n+1)$-simplices $s_{n-1}z, \phi, \ldots, \phi, v, w, -$ are compatible in K, say $d_0 t = s_{n-1} z, d_i t = \phi, 1 \leq i \leq n-1, d_n t = v, d_{n+1} t = w$. Then $d_{n+2} t$ is a homotopy $d_{n+1} v \sim x$, hence $[x] = i[d_{n+1} v]$.

□

1.4 The group structures

Definition 1.10. Let (K, ϕ) be a Kan pair. Wirte $x \in \alpha$ if x represent a. Suppose $\alpha, \beta \in \pi_n(K, \phi), n \leq 1$, and let $x \in \alpha, y \in \beta$. Then n+1 n-simplices $\phi, \ldots, \phi, x, -, y$ are compatible, say $d_i z = \phi, 0 \leq i < n-1, d_{n-1} z = x, d_{n+1} z = y$. We define $\alpha\beta = [d_n z]$.

Lemma 1.5. *$\alpha\beta$ is well defined.*

Proof. Suppose z' also satisfies $d_i z' = \phi, 0 \leq i < n-1, d_{n-1} z' = x$ and $d_{n+1} z' = y$. By the extension condition, there exists $w \in K_{n+2}$ such that $d_i w = \phi, 0 \leq i \leq n-2$,

$$d_{n-1} w = s_n d_{n-1} z, d_{n+1} w = z, \text{ and } d_{n+2} w = z'.$$

$d_n w$ is then a homotopy from $d_n z$ to $d_n z'$. Suppose $y \sim y'$, say $d_i w = \phi, 0 \leq i < n$, $d_n w = y', d_{n+1} w = y$. Choose z' such that $d_{n-1} z' = x, d_{n+1} z' = y'$, and $d_i z' = \phi, 0 \leq i < n-1$. By the extension condition, there exists $u \in K_{n+2}$ such that $d_j u = \phi$, $0 \leq i < n-1$ □