Variational Hodge conjecture II

1) algebraic de Rham cohomology
2) Gauss-Manin connection (algebraic version)
3). Statements of VHC.

Goal : The equivalence between VHCs.

(Grothendieck's variational Hodge conj)

$f: X \to S$ smooth, projective k-morphism, X, S sm / field k

S not finite type S is a sm connected finite type k-scheme.

$s \in S$ point $\alpha_s \in H^{2p}_{dR}(X_s/k(s))$ → algebraic   if α_s extends to a

g section of $H^{2p}_{dR}(X/S)$ $\Rightarrow$ $\alpha_{s'}$ is algebraic , $s' \in S$

α

$X_n = X \times \operatorname{Spec} k[t]/t^n$

$S = \operatorname{Spec} k[[t]]$ $X \to S$ sm, proj

(Katz) $H^{2p}_{dR}(X_1/k) \xrightarrow{\sim} H^{2p}_{dR}(X/S)^{\nabla^{GM}}$

$\rightsquigarrow$ $\alpha_1 \in H^{2p}_{dR}(X_1/k) \longrightarrow$ global section α in $H^{2p}_{dR}(X/S)$, horizontal

$\overset{VHC}{\rightsquigarrow}$ α_n is algebraic class $H^{2p}_{dR}(X_n/k)$

$\overset{\text{Zariski dense } X}{\rightsquigarrow}$ $\alpha \in H^{2p}_{dR}(X/k)$ algebraic   In general, it is not true.

X^{an}
$\downarrow f$ analytic morphism
S^{an} ← simply connected

monodromy $\Rightarrow$ $R^{2p}f_*\mathbb{Q}(p)$ Not true!

So X, S quasi-projective k-variety !

k a field char $= 0$

Statement 2 $\quad Z_s \in K_0(X_s)$ TFAE

1) $ch^{dR}(Z_s)$ extends to. $H^*_{dR}(X/\underline{k})$

2). $\exists\ Z \in K_0(X)_{\mathbb{Q}} \to K_0(X_s)_{\mathbb{Q}}$ $\quad ch^{dR}(Z|_s) = ch^{dR}(Z_s)$

Prop. Gro 66 $\Leftrightarrow$ Statement 2.

1) $CH^*(X)_{\mathbb{Q}} \xleftarrow{\sim} K_0(X)_{\mathbb{Q}}$ $\quad$ Grothendieck-Riemann-Roch + support filtration (γ-filtration on $K_0(X)_{\mathbb{Q}}$)

$ch(Z) \cap [X] \leftarrow\!\shortmid Z$

algebraic $\Leftrightarrow$ $\alpha = ch(Z)$

2). $\alpha_s = ch(Z_s)_i \in H^{2i}_{dR}(X_s/_{k(s)})$ and α_s extends to $H_{dR}(X/k)$.

Gro 66 holds $\Rightarrow$ $\alpha_{s'}$ is algebraic for any s'

$s' = \eta$ $\quad \alpha_\eta$ is algebraic

Process as before $\Rightarrow$ $\tilde{\alpha} \in H^{2i}_{dR}(X/k)$ algebraic

$\Rightarrow$ $\tilde{\alpha} = ch^{dR}(Z)$ $\quad$ Z satisfies Statement 2, (2)

$\Leftarrow$ $ch(Z) = \alpha$ is what we want.

$\square$

(Infinitesimal variational Hodge).

$S = \operatorname{Spec} k[[t]]$ $\quad X/_S$ is a sm. proj S-scheme. $Z_1 \in K_0(X_1)_{\mathbb{Q}}$

TFAE

1). $\Phi^{-1} \circ ch^{dR}(Z_1) \in \underbrace{\bigoplus_i H^{2i}_{dR}(X/S)^{\nabla} \cap F^i_{Hodge} H^{2i}_{dR}(X/S)}$ (compatible $\underset{\sim}{\text{flat}}$)

2) Z_ϕ extends to some $Z \in K_0(X)$ $\quad ch^{dR}(Z|_\phi) = ch^{dR}(Z_1)$

$$\underline{H^{2i}_{fl}(X/S)} := H^{2i}_{dR}(X/S)^{\nabla} \cap F^{i}_{Hodge} H^{2i}_{dR}(X/S)$$

Thm. Variational Hodge conj / all field k $\iff$ Infinitesimal Hodge / all field k.

proof

S in VHC finite type sc $\Rightarrow$ is non-trivial

Assume S is an affine curve.

$$\begin{tikzcd} & \hat{X} = X \times_S \hat{S} \arrow[r] \arrow[d, "\hat{f}"] & X \arrow[d] \\ \mathrm{Spec}(\hat{\mathcal{O}}_{S,s}) \arrow[r, equal] & \hat{S} \arrow[r] & S \end{tikzcd}$$

$\hat{\mathcal{O}}_{S,s} = k[[t]]$ as S is smooth

GFGA, classical Grothendieck Existence thm

$D^b_{coh}(X) \simeq D^b_{coh}(\hat{X})$.

$$\begin{tikzcd} \left(R^{2i}f_*\Omega^\bullet_{X/S}\right)^\wedge \arrow[r, "\sim"] \arrow[d, "(\nabla^{GM})^\wedge"] \arrow[dr, phantom, "\circlearrowright"] & R^{2i}\hat{f}_*\Omega^\bullet_{\hat{X}/\hat{S}} \arrow[d, "\nabla^{GM}_{\hat{f}}"] \\ \left(R^{2i}f_*\Omega^\bullet_{X/S}\right)^\wedge \otimes \hat{\Omega}^1_{S/k} \arrow[r] & R^{2i}\hat{f}_*\Omega^\bullet_{\hat{X}/\hat{S}} \otimes \underline{\Omega^1_{\hat{S}/k}} \end{tikzcd}$$

Infinitesimal

finite $k[[t]]$-module

$\Omega^1_{k[[t]]/k}$ is not finite dimension as k-space

$$\underbrace{k[[t]] = \varprojlim_n k[t]/t^n}$$

$$0 \to \underline{(t^n)/(t^{n+1})} \to \Omega^1_{S_{n+1}/k} \to \Omega^1_{S_n/k} \to 0$$

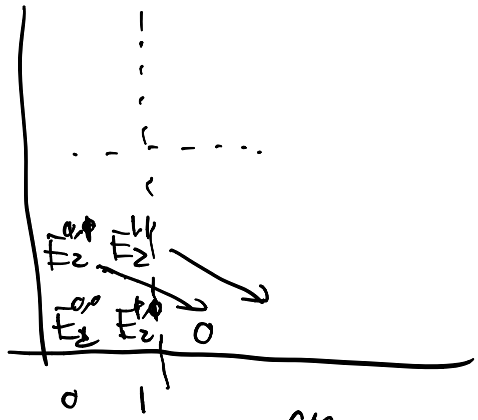
Infinitesimal $\Longrightarrow$ VHC

S affine curve

Leray SS of de Rham

$$E_2^{p,q} = R^p\Gamma(S, \Omega^\bullet_{S/k} \otimes \mathcal{H}^q_{dR}(X/S)) \Rightarrow \underline{H^{p+q}_{dR}(X/k)}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & S \arrow[d, "s"] \\ & \operatorname{Spec} k \end{tikzcd}$$



$$\rightsquigarrow \ \to H^q_{dR}(X/k) \to H^q_{dR}(X/S) \xrightarrow{\nabla^{GM}} H^q_{dR}(X/S) \otimes \Omega^1_{S/k} \to \cdots$$

$\rightsquigarrow$ flat section of $H^q_{dR}(X/S)$ extends to $H^q_{dR}(X/k)$

by IVHC $\rightsquigarrow$ if α_s algebraic $\quad \alpha_s = \alpha_1$ algebraic

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & X \arrow[d] & X_1 \arrow[l] \arrow[d] \\ \hat{S} & S \arrow[l] & k \arrow[l] \end{tikzcd}$$

X_1 is special fiber of $\hat{X}/\hat{S}$ over k

$\alpha_s = \alpha_1 \in H^{2p}_{dR}(X_1/k)$

algebraic $\quad ch(Z_1) = \alpha_1$

$\overset{\text{extends}}{\rightsquigarrow} \alpha_1 \qquad \underset{\alpha_s}{\nabla(\alpha_1)} = 0 \ \rightsquigarrow \ \hat{\alpha} \in H^i_{dR}(\hat{X}/k)$ algebraic

$\hat{Z} \in K_0(\hat{X})_{\mathbb{Q}}, \ \hat{\alpha} = ch^{dR}(\hat{Z})$

<u>FACT</u> $(Sch/S)^{op}_e \to Sets$

$U \mapsto K_0(X_U)$

is locally of finite presentation (i.e. $\varinjlim K_0(X_{A_i}) \xrightarrow{\sim} K_0(\varprojlim X_{A_i})$)

$\{A_i\}$ affine schemes filter <u>inverse system</u>

(Weibel K-book chapt II, I)

Artin approximation $\rightsquigarrow$

$$\begin{tikzcd} & S' \arrow[d] \\ s \in & S \end{tikzcd} \quad \exists \ \alpha' \in K_0(X_{S'}), \text{ s.t. } \alpha'|_{X_1} = \hat{\alpha}|_{X_1}$$

$\hat{\alpha} \in K_0(\hat{X} = X \times_S \hat{S})$

$\rightsquigarrow$ $ch(\alpha'|_{X_1}) = ch(\alpha_1)$ satisfies requirement of VHC

Let $Z = \oplus Rf'_* [ch(\alpha')] \in K_0(X)$, $f' : X' \to X$, étale morphism
(de Rham cohomology same)

Z is the algebraic cycle represents α.

VHC $\Rightarrow$ IHC

"□"

Idea: $\underline{\text{GAGA}}$ ✓ BEK
✓ formal scheme
$\underline{\text{GFGA}}$ $\Rightarrow$ "VHC $\Rightarrow$ IHC" $\overset{?}{=}$

Deligne $\underline{\text{Leray SS}}$ for $Rf_*^{2p} \mathbb{C}\mathbb{Q}$ "replace it by Leray SS for de Rham
Gauss-Manin algebraic.

□

VHC $\Leftrightarrow$ Infinitesimal Hodge

$\underline{\text{BEK 14}}$ "deformational part" + "algebraicization part"

1) $\exists \{Z_i\} \in \{K_0(X_i)\}$ "Inf deformation of cycles"

2) $\underline{\text{IHC condition (2) holds}}$ $\bigoplus_i H_{f_1}^{2i}(X/S)$

Bloch prove VHC for $Z_1 \in K_0(X_1)$ Z_1 is "semiregular" $\underline{1977}$
recommend to read

Problem: $\varprojlim K_0(X_n) \xleftarrow{\text{restriction}} K_0(X)$ is not surjective if $P_g(X) > 1$

$\varprojlim \mathrm{Pic}(X_n) \twoheadleftarrow K_0(X)$ is surjective

Part II Deligne's principle B about absolute Hodge classes

Thm: α_s algebraic $\rightsquigarrow$ α_s is (absolute) Hodge

$\alpha_{s'}$ algebraic $\rightsquigarrow$ $\alpha_{s'}$ is (absolute) Hodge

by Deligne's global invariant cycle thm

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \bar{X} \arrow[dl] \\ S & X_s \arrow[u] \end{tikzcd} \quad \bar{X} \text{ is sm proj } \mathbb{C}\text{-varieties}$$

i_s^*: $\overset{\text{care}}{H_B^{2i}(\bar{X})_{\mathbb{Q}}} \to H_B^{2i}(X_s)_{\mathbb{Q}}$ image $\cong H_B^{2i}(X_s)_{\mathbb{Q}}^{\pi_1(S,s)}$

$H_B^{2i}(X_s)_{\mathbb{Q}}^{\pi_1(S,s)}$ sub Hodge structure of $H_B^{2i}(X_s)_{\mathbb{Q}}$.

$\underbrace{H_B^{2i}(X_s)_{\mathbb{Q}}^{\pi_1(S,s)} \cong (\ker i_s^*)^{\perp}}$ $\perp$ is $\underbrace{\text{}}_{\text{respect to}}$ polarization on $\bar{X}$

$\rightsquigarrow$ we conclude it. $\square$

FACT: absolute is because Gauss-Manin connection is compatible with conjugation. $\sigma: \mathbb{C} \to \mathbb{C}$

Corollary: Hodge conj $\Rightarrow$ VHC conj

$S(\mathbb{C}) \subset S$ dense

$\rightsquigarrow$ all points of S. $\square$

$\rightsquigarrow$ In special cases: VHC conj $\Rightarrow$ Hodge conj

Abelian schemes $\begin{array}{c} A \\ \downarrow \\ \underline{S} \end{array}$ Hodge

Ref 1. "Notes on absolute Hodge classes" Charles, Schnell

2. Stacks project "algebraic de Rh..."

3. Bloch, Esnault, Kerz

2 papers.

deformation Gauss-Manin

3. Bloch, Katz