NOTES ON ∞-CATEGORIES

ZOU HAITAO

ABSTRACT. This is a short notes for ∞-categories and its applications.

1. ∞-CATEGORY

In this section, we will give the definition of ∞-category given by Jacob Lurie, which is defined with notion of quasi-category. We assume that readers have already learnt some simplicial homotopy theory and model category.

Definition 1.0.1. Suppose $\mathcal{C}$ be a simplicial set. $\mathcal{C}$ is called a ∞-category if it is a weak Kan complex i.e there is following extensions for each positive integers n and $0 < i < n$.

$$\begin{tikzcd} \wedge_i^n \arrow[r, "f_n"] \arrow[d, "i"'] & \mathcal{C} \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

Example 1.0.1: The nerve of a locally small category is a ∞-category since is a Kan complex so is a weak Kan complex.

An **object** of ∞-category $\mathcal{C}$ is a simplicial sets morphism $\Delta[0] \to \mathcal{C}$ or equivalently a 0-simplic of $\mathcal{C}$. A **1-morphism** of $\mathcal{C}$ is morphism $\Delta[1] \to \mathcal{C}$ or equivalently an 1-simplic of $\mathcal{C}$

$$x \to y$$

It is not very hard to verify that for any ∞-category, there is small category $\mathcal{C}'$ whose nerve is isomorhic to it [see 1] so any classical locally small category can be viewed as an ∞-category with respect to its nerve and ∞-category can be viewed as generalization of ordinary category from some viewpoint.

1.1. **Objects, morphisms and mapping spaces.**

As in classical category theory, we can alse define **initial object**, **terminal object** and **zero object** in the setting of ∞-category. But this generalization is not obvious which relies on definition of mapping spaces.

Suppose $\mathcal{C}$ be an ∞-category. We now define an object $\mathrm{Map}_{\mathcal{C}}(X, Y)$ for any two objects X, Y in $\mathcal{C}$, which plays the role of morphism calss in ordinary category theory.

First, we need to recall the homotopy category of a simplicial set. Suppose S be a simplicial set, we make a simplicial category $C[S]$ associated to S. Since $C[S]$ has model structure and all objects are fibrant and cofibrant, localization of weak equivalences is homotopy category, denoted by hS.

The simplicial category $C[\Delta^n]$ is defined as follows

- Ob : Objects of $[n]$
- Mor :

$$\mathrm{Map}_{C[\Delta^n]}(i, j) = \begin{cases} \emptyset & \text{if } j < i \\ N(P_{ij}) & \text{if } i \leq j \end{cases}$$

$P_{ij} = \{I \subset [n] \big| (i, j \in I) \wedge (\forall k \in I)[i \leq k \leq j]\}$ is a partial order set so can be viewed as a category.

Definition 1.1.1. Let $\mathcal{C}$ be a ∞-category.

(1) an initial object of $\mathcal{C}$ is an object which is initial object in associated homotopy category $h\mathcal{C}$ or equivalently mapping space $\mathrm{Map}_{\mathcal{C}}(0, Y)$ is weak contractible for any object $Y \in \mathcal{C}$.
(2) a final object of $\mathcal{C}$ is an object which is final object in associated homotopy category $h\mathcal{C}$ or equivalently mapping space $\mathrm{Map}_{\mathcal{C}}(X, 0)$ is weak contractible for any object $X \in \mathcal{C}$.
(3) a object is called **zero object** if it is both final and initial.

Definition 1.1.2. An ∞-category is **pointed** if it is with zero object.

1.2. ∞**-groupoids.** In ordinary category theory, groupoid is an important notion which is generalization of group. In ∞-category version, groupoids are not just a generalization of groups but essential role in homotopy theory.

Informally, an ∞-groupoid is an ∞-category with all k-morphisms are invertable.

Definition 1.2.1. An ∞-category $\mathcal{G}$ is called ∞-groupoid if it is a Kan complex.

Proposition 1.2.1 (Joyal)**:** Suppose $\mathcal{G}$ be an ∞-category, then following conditions are equivalent

(1) $\mathcal{G}$ is an ∞-groupoid
(2) $\mathcal{G}$ satisfies extension condition for all horn inclusions $\wedge^n_n$
(3) $\mathcal{G}$ satisfies extension condition for all horn inclusions $\wedge^n_0$
(4) homotopy category $h\mathcal{G}$ is a groupoid.

1.3. **Stable ∞-categories.**

Proposition 1.3.1: Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable iff the following conditions are satisfied

(1) The ∞-category $\mathcal{C}$ admits finite limits and colimits
(2) A square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

is pull-back if and only if is push-out.

REFERENCES

[1] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.