Algebraic Geometry

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1 Week 1

Exercies 1.1. Any nonempty open subset of an irreducible topological space is dense and irreducible.

Let X be an irreducible space and $U \hookrightarrow X$ be an nonempty open subset of X. Let $V_1 = X\backslash U$ and $V_2 = \bar{U}$. Then we have

$$V_1 \cup V_2 \supseteq (X\backslash U) \cup U = X$$

Since X is irreducible and V_1, V_2 are closed subsets, we have $V_1 = \emptyset$ or $V_2 = X$. That means $\bar{U} = X$. Hence U is dense open subset of X. We can further prove that X is irreducible if any nonempty open subset of X is dense. Otherwise, X is reducible, then $X = V_1 \cup V_2$ where V_1 and V_2 are non-trivial closed subset of X. Then $(X\backslash V_1) \cap (X\backslash V_2) = \emptyset$. It implies $X\backslash V_1$ is non-empty open subset of X which is not dense. Hence we can conclude that any nonempty open subset of irreducible space X is irreducible because its open subsets are all dense in X, also in itself. ♣

Exercies 1.2. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$ and assume that $Y \subsetneq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Suppose H be irreducible. It means ideal of H is prime ideal (f) of $k[x_1, \cdots, x_n]$. Let $Y \cap H = V_1 \cup \cdots \cup V_k$ be the irreducible components decomposition and ideal of V_i is $\mathfrak{p}_i$. Hence we have

$$I(V \cap H) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$$

Since $Y \subsetneq H$, we have $f \notin I(Y)$. Hence the minimal prime ideals of $I(Y) + (f)$ is with height $n - k + 1$ by Krull principal ideal theorem, since $I(Y)$ is of height $n - r$. We claim that p_i is a minimal ideal which contains $I(Y) + (f)$. Otherwise, let $\mathfrak{p}$ be the minimal prime ideal satisfying $I(Y) + (f) \subseteq \mathfrak{p} \subsetneq \mathfrak{p}_i$. Then $V_i \subsetneq Z(\mathfrak{p})$, a irreducible closed subset of X. It contradicts to the fact that V_i is irreducible component of $V \cap H$. Hence $ht(\mathfrak{p}_i) = n - r + 1$ for all $1 \leq i \leq k$. It implies

$$\dim V_i = \dim A(V_i) = \dim k[x_1, \cdots, x_n] - ht(\mathfrak{p}_i) = r - 1$$

♣

Exercies 1.3. Let $\alpha \subseteq k[x_1, \cdots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible components of $Z(\alpha)$ has dimension $\geq n - r$.

Let $Z(\alpha) = V_1 \cup \cdots \cup V_k$ be the decomposition of irreducible components, where $I(V_i) = \mathfrak{p}_i$ is prime ideal of $k[x_1, \cdots, x_n]$. It implies $\alpha \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$. So $\alpha \subseteq \mathfrak{p}_i$ for each i. Hence $ht(\mathfrak{p}_i) \leq r$ by Krull principal ideal theorem. Therefore, the dimension of V_i is greater than $n - r$. ♣

2 Week2

Exercies 2.1. Prove following statments

- If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

- If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
- For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
- If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
- For any subset $Y \subset \mathbb{P}^n$, $Z(I(Y)) = \bar{Y}$.

Let $x \in Z(T_2)$, we have $f(x) = 0$ for all $f \in Y_2$. Since all $g \in T_1$ are all in Y_2, we have $g(x) = 0$. Hence $g \in Z(T_1)$. If $g \in I(Y_2)$, then g vanishes on all Y_2, so on all Y_1. Hence $g \in I(Y_1)$. This also implies that both $I(Y_1)$ and $I(Y_2)$ contain $I(Y_1 \cup Y_2)$ since $Y_i \subseteq Y_1 \cup Y_2$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then f vanishes on both Y_1 and Y_2, so $f \in I(V_1 \cup V_2)$ by definition.

If$f \in \sqrt{\mathfrak{a}}$, then there exists $n \geq 1$ such that $f^k \in \mathfrak{a}$. It implies every homogeneous part vanishes on $Z(\mathfrak{a})$. Let $f = f_1 + \cdots + f_n$ be the homogeneous decomposition of f. Then the homogeneous part of f^k with degree nk is f_n^k, so $f_n^k(P) = 0$ for all $P \in Z(\mathfrak{a})$. Therefore, $f_n(P) = 0$. By induction, we can conclude that $f_i(P) = 0$ for all i. Hence $f \in I(Z(\mathfrak{a}))$. Conversely, if $f \in I(Z(\mathfrak{a}))$. By homogeneous Nullstellensatz, we have $f_i^{r_i} \in \mathfrak{a}$. Let $r = r_1 + \cdots r_n$, then $f^r \in \mathfrak{a}$. Hence $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Since $Y \subseteq Z(I(Y))$ and $Z(I(Y))$ is closed, we have $\bar{Y} \subseteq Z(I(Y))$. There is homogeneous ideal $\mathfrak{a}$ such that $\bar{Y} = Z(\mathfrak{a})$. From $Y \subseteq Z(\mathfrak{a})$, we have $I(Y) \subseteq I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. Hence $\bar{Y} = Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) \subseteq Z(I(Y))$. We now conclude that $Z(I(Y)) = \bar{Y}$. ♣

Exercies 2.2. a) There is a $1-1$ inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of S not equal to S_+ does not occur in this correspondence, it is sometimes called the irrelevant maximal of S.

b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

c) Show that $\mathbb{P}^n$ itself is irreducible.

a) If $\mathfrak{a}$ is radical homogeneous ideal of S such that $Z(\mathfrak{a}) \neq 0$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$. If $Z(\mathfrak{a}) = 0$, then $I(Z(\mathfrak{a})) = I(\emptyset) = S$. $Z(\mathfrak{a}) = 0$ implies $\mathfrak{a} = S$ or S_+. By assumption, S_+ is not in the correspondence, so $\mathfrak{a} = S$. Hence $I \circ Z$ is identity functor. Similarly, $Z \circ I$ is also identity. With previous exercise, this correspondence is inclusion-reversing.

b) If Y is irreducible, then for all $x, y \in I(Y)$, we can let $Y_1 = Z(x) \cap Y$ and $Y_2 = Z(y) \cap Y$. Since $Y_1 \cup Y_2 = (Z(x) \cap Y) \cup (Z(y) \cap Y) = Z(xy) \cap Y = Y$, $Y_1 = Y$ or $Y_2 = Y$ by irreducible condition. It implies that $Z(x) = Y$ or $Z(y) = Y$. So $x \in I(Y)$ or $y \in I(Y)$. Conversely, suppose $I(Y)$ is prime. However, for any closed cover $Y_1 \cup Y_2 = Y$, we have $I(Y) = I(Y_1) \cap I(Y_2)$, therefore $I(Y_1) = I(Y)$ or $I(Y_2) = I(Y)$. Hence $Y_1 = Y$ or $Y_2 = Y$ since they are closed.

c) $\mathbb{P}^n$ is algebraic set corresponding to radical homogeneous ideal (0). It is prime ideal since $k[x_0, x_1 \cdots x_n]$ is integral domain. So $\mathbb{P}^n$ is irreducible from previous statement.

♣

Exercies 2.3. If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.

Y is projective variety, so let $Y = Z(\mathfrak{p}) \subseteq \mathbb{P}^n$ for some prime homogeneous ideal $\mathfrak{p}$. Hence any descending chain of closed subset of Y corresponds to a descending chain of radical homogeneous ideal containing $\mathfrak{p}$ with the same length. However, S_+ is prime homogeneous ideal of S which contains any non-zero ideal of S but doesn't correspond to a algebraic set. Hence the $\dim S(Y) > \dim Y$. Since radical ideals contains $\mathfrak{p}$ except S_+ also correspond to closed subsets of Y, $\dim Y \geq \dim S(Y) - 1$. Hence $\dim S(Y) = \dim Y + 1$. ♣

3 Week 3

Exercies 3.1. A regular function on projective variety is continuous map (view k as affine line $\mathbb{A}^1$).

Let $Y \subseteq P^n$ be a projective variety, then Y can be covered by affine varieties $U_i = Y \cap A_i^n$, where A_i^n are canonical affine coverings of $\mathbb{P}^n$. If $f\colon Y \to k$ be a regular function, then its restrictions $f_{U_i}\colon U_i \to k$ are continuous maps, and since U_i are all open in Y, f itself is continuous on Y. ♣

Exercies 3.2. Let $\varphi : \mathbb{A}^1 \to C \hookrightarrow \mathbb{A}^2$ be curve defined as $t \mapsto (t^2, t^3)$. Obviously, φ is $1-1$ correspondence. Prove φ is not isomorphism between varieties.

Suppose the coordinate ring of $\mathbb{A}_k^2$ be $k[x, y]$. Then we have coordinate ring $A(C) \cong k[x, y]/(y^2 - x^3)$. From definition of φ, we can write down its pull-back on coordinate rings

$$\begin{aligned} \varphi^*\colon A(C) &\to k[t] \\ f &\mapsto f \circ \varphi \end{aligned}$$

If φ is isomorphism, then φ^* is isomorphism between coordinate ring. Therefore it is also bijection between regular functions. Since t is regular function on $\mathbb{A}_k^1$, it must have preimage f such that $\varphi^*(f) = t$. This means that $f(t^2, t^3) = t$. f is regular on C, so it is also regular at point $(0, 0)$. Nearby $(0, 0)$, f can be written as

$$\frac{\alpha(x, y)}{\beta(x, y)}$$

where α, β are polynomials in $k[x, y]$ and $\beta(0, 0) \neq 0$, $\frac{\alpha(t^2,t^3)}{\beta(t^2,t^3)} = t$. It is impossible since $\alpha(t^2, t^3) = t\beta(t^2, t^3)$ cannot have term of degree 1. Hence we can conclude φ can not be an isomorphism otherwise it will induce isomorphism between coordinate ring.

♣

Exercies 3.3. Let $S = Z(y_0 y_2 - y_1^2)$ be the surface in $\mathbb{P}_k^2$ with the coordinates $(y_0 : y_1 : y_2)$. Let $\mathbb{P}_k^1$ be projective line with coordinate ring $k[x_0, x_1]$. Consider morphism

$$\begin{aligned} \varphi : \mathbb{P}_k^1 &\to S \subset \mathbb{P}_k^2 \\ (x_0 : x_1) &\mapsto (x_0^2 : x_0 x_1 : x_1^2) \end{aligned}$$

and show that φ is isomorphism.

It is well-defined regular morphism since $(x_0^2)(x_1^2) - (x_0 x_1)^2 = 0$ and with polynomial in each component. We can see that φ is bijection and φ^{-1} is defined as

$$\varphi^{-1}\colon (y_0 : y_1 : y_2) = \begin{cases} (y_0 : y_1) & \text{if } y_0 \neq 0 \\ (y_1 : y_2) & \text{if } y_2 \neq 0 \end{cases}$$

It is well-defined since we have $(y_0 : y_1) = (\frac{y_0}{y_1} : 1) = (\frac{y_1}{y_2} : 1) = (y_1 : y_2)$ when neither y_0 or y_2 are equal to 0. Moreover, it is regular map because it is regular on open subsets $S \cap \{y_0 \neq 0\}$ and $S \cap \{y_2 \neq 0\}$. With the fact that $\varphi \circ \varphi^{-1} = id_S$, $\varphi^{-1} \circ \varphi = id_{\mathbb{P}_k^1}$, we can conclude that φ is isomorphism.
♣

4 Week 4

Exercies 4.1. Let Y be an affine variety in $\mathbb{A}_k^n$. Show that $K(Y) \cong K(\mathcal{O}_{Y,p})$ for all $p \in Y$.

Since Y is affine variety, we have $\mathcal{O}_{Y,p} \cong A(Y)_{m_p}$ for all point $p \in Y$. Therefore, $K(\mathcal{O}_{Y,p})$ is fraction field of local ring $A(Y)_{m_p}$. Moreover, $K(Y)$ is fraction field of coordinate ring $A(Y)$. Hence we have following commutative localization diagram

$$\begin{tikzcd} A(Y) \arrow[r, "l_{m_p}"] \arrow[d, "l"] & A(Y)_{m_p} \arrow[d, "l'"] \\ K(A(Y)) & K(A(Y)_{m_p}) \end{tikzcd}$$

l_{m_p} maps non-zero divisor of $A(Y)$ to non-zero divisor, so $l' \circ l_{m_p}$ maps non-zero divisors to units. Therefore, with universal property of localization, there is unique homomorphism $g\colon K(A(Y)) \to K(A(Y)_{m_p})$ making this diagram commute. Also, with universal property of l_{m_p}, there is unique morphism h making following diagram commute

$$\begin{tikzcd} A(Y) \arrow[r, "l_{m_p}"] \arrow[d, "l"] & A(Y)_{m_p} \arrow[dl, "h"] \\ K(A(Y)) & \end{tikzcd}$$

Now we get a new localization diagram

$$\begin{tikzcd} & A(Y)_{m_p} \arrow[dl, "h"'] \arrow[d, "l'"] \\ K(A(Y)) & K(A(Y)_{m_p}) \arrow[l, dashed, "\exists! f"] \end{tikzcd}$$

Hence we can conclude that $K(A(Y)) \cong K(A(Y)_{m_p})$, which implies that $K(Y) \cong K(\mathcal{O}_{Y,p})$ for all $p \in Y$. ♣

Exercies 4.2. Prove that for any integer $0 \le i \le n$

$$K(\mathbb{P}_k^n) \cong k(x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i)$$

For any variety Y, if U is open subvariety of Y, then by definition $K(Y) \cong K(U)$. More precisely, we can send $\langle f, V\rangle$ to $\langle f, V \cap U\rangle$ and it is an well-defined isomorphism between function fields. Therefore, for any $0 \le i \le n$, we have $K(A_i) \cong K(\mathbb{P}_k^n)$, where A_i is affine cover which is isomorphic to $\mathbb{A}_k^n$ with coordinate ring $k[x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i]$. With the conclusion in last exercise, we can conclude that

$$K(\mathbb{P}_k^n) \cong K(A_i) \cong k(x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i)$$

Exercies 4.3. Equation $x_0^2 + x_1^2 + x_2^2 = 0$ defines a conic $X \hookrightarrow \mathbb{P}_k^2$. Find $t \in K(X)$ such that $K(X) \cong k(t)$ is a transcendental extension of k with degree 1.

$K(X)$ is equal to function field of affine open subset defined by

$$(\frac{x_1}{x_0})^2 + (\frac{x_2}{x_0})^2 = -1 \tag{1}$$

Consider linear transform

$$y_1 = \frac{x_1}{x_0} + i\frac{x_2}{x_0} \qquad y_2 = \frac{x_1}{x_0} - i\frac{x_2}{x_0} \tag{2}$$

Then the equation 4 becomes

$$y_1 y_2 = -1 \tag{3}$$

Hence $K(X) = k(y_1, y_2) = k(y_1)$ since $y_2 = -1/y_1$. Therefore y_1 is the required t. ♣

5 Week5

Exercies 5.1. Suppose $S = Z(xy - zw) \subset \mathbb{P}^3_k = \operatorname{Proj} k[x, y, z, w]$. Let $H \subset \mathbb{P}^3_k$ the hyperplane in $\mathbb{P}^3_k$ defined by $x = 0$. It is isomorphic to $\mathbb{P}^2_k$.

$$\begin{aligned}\varphi\colon S &\dashrightarrow H\\ (x : y : z : w) &\mapsto (0 : y : z - x : w)\end{aligned}$$

Prove that φ is birational map.

Proof. Actually, we define its rational inverse as

$$\begin{aligned}\varphi^{-1}\colon H &\dashrightarrow S\\ (0 : s_1 : s_2 : s_3) &\mapsto (\frac{s_2 s_3}{s_1 - s_3} : s_1 : \frac{s_1 s_2}{s_1 - s_3} : s_3)\end{aligned}$$

It is defined on H on the open subset $\{s_1 \neq s_3\} \cap H$ and $\varphi^{-1}(H) \cap (S \cap A_i) \neq \emptyset$ for all canonical affine covers A_i of $\mathbb{P}^3_k$. It means φ^{-1} is well-defined dominant rational map on H. Furthermore, its compositions with φ are all identities as rational maps since

$$\varphi((\frac{s_2 s_3}{s_1 - s_3} : s_1 : \frac{s_1 s_2}{s_1 - s_3} : s_3)) = (0 : s_1 : \frac{s_2(s_1 - s_3)}{s_1 - s_3} : s_3) = (0 : s_1 : s_2 : s_3)$$

and

$$\begin{aligned}\varphi^{-1}((0 : y : z - x : w)) &= (\frac{(z - x)w}{y - w} : y : \frac{y(z - x)}{y - w} : w)\\ &= (\frac{xy - xw}{y - w} : y : \frac{yz - zw}{y - w} : w)\\ &= (x : y : z : w)\end{aligned}$$

♣

Exercies 5.2. If $Y, Z \subset \mathbb{A}^2_k$ are two distinct curves given by equations $f = 0, g = 0$, and if $p \in Y \cap Z$, we define the intersection multiplicity $(Y \cdot Z)_p$ at point p to be the length of the $\mathcal{O}_p$-module $\mathcal{O}_p/(f, g)$. Show that

- $(Y \cdot Z)_p$ is finite and $(Y \cdot Z)_p \geq \mu_p(Y)\mu_p(Z)$
- If $p \in Y$ show that for almost all lines l through p, we have $(l \cdot Y)_p = \mu_p(Y)$.
- If Y is a curve of degree d in $\mathbb{P}^2_k$, and if l is a line in $\mathbb{P}^2_k$, $l \neq Y$, show that $(l \cdot Y) = d$. Here we define $(l \cdot Y) = \sum_p (l \cdot Y)_p$ taken over all points $p \in l \cap Y$, where $(l \cdot Y)_p$ is defined using a suitable affine cover of $\mathbb{P}^2_k$.

Proof.
- To show $(Y \cdot Z)_p$ is finite, we just need to prove $\mathcal{O}_p$-module $\mathcal{O}_p/(f, g)$ is of finite length. This is equivalent to prove that $\mathcal{O}_p/(f, g)$ is both Artinian and Noetherian module. Since $\mathcal{O}$ is localization of Noether ring $k[x, y]$, it is also Noether. The natural morphism $\mathcal{O}_p \twoheadrightarrow \mathcal{O}_p/(f, g)$ implies that $\mathcal{O}_p/(f, g)$ is finitely generated $\mathcal{O}_p$-module, so it is Noetherian module. Furthermore, $\mathcal{O}_p/(f, g)$ is of Krull dimension 0, because of $\dim \mathcal{O}_p = 2$ together with Krull's principal ideal theorem. Therefore $\mathcal{O}_p/(f, g)$ is Artinian ring since $\mathcal{O}_p/(f, g)$ is Noether ring and it is also Artinian module as $\mathcal{O}_p$-module. Suppose Y and Z intersect at $(0, 0)$.

- Make a linear change of the coordinates so that $p=(0,0)$. Then $\mathcal{O}_p = k[x,y]_{(x,y)} \cong k[x,y]$. Therefore, any line through p can be defined by equation $ax+by=0$. If $b\neq 0$, then $y=-\frac{a}{b}x$. In this case, $\mathcal{O}_p/(f, ax+by) \cong k[x]/(f(x,-a/bx))$. Let $\mu_p(Y)=r$ and f is of degree d, then f can be written in the form
$$f(x,-\frac{a}{b}x) = c_0x^r + c_1x^{r+1} + \cdots + c_{d-r}x^d = c_0x^r(1+\cdots+c_{d-r}x^{d-r})$$
$1+\cdots c_{d-r}x^{d-r}$ can be viewed as polynomial of $\frac{a}{b}$ with coefficients in $k[x]$, so there are only finite a such that $1+\cdots c_{r-d}x^{r-d}=0$. Hence $\mathcal{O}_p/(f, ax+by) \cong k[x]/(c_0x^r)$ since c_0x^r and $1+\cdots c_{d-r}x^{d-r}$ are coprime except finite number of a. Therefore, length of $\mathcal{O}_p/(f,ax+by)=r$. If $b=0$, it determines only line in $\mathbb{A}^2_k$. Hence we reach the conclusion.

- Y is curve in $\mathbb{P}^2_k$, so Y is determined by homogeneous polynomial f. Let $A_i=\{x_i\neq 0\}\cap\mathbb{P}^2$ be the three affine covers of $\mathbb{P}^2_k$. Under linear transformation, we can make l through $(0:0:1)$, hence equation of l can be written as
$$ax+by=0$$
In this case l can be covered by $A_1\cup A_3$ if $b\neq 0$. Then
$$(l\cdot Y) = \sum_{p\in A_1\cap l\cap Y}(l\cdot Y)_p + (l\cdot Y)_{(0:0:1)} \tag{4}$$
Suppose Y is determined by homogeneous polynomial f. On A_1, the $Y\cap l$ is determined by
$$f(1,-\frac{b}{a}, z/x)=0$$
Let $F(t)=f(1,-\frac{1}{b},t)$ is of degree r. Then the first term of right part of 4 is equal to r. With last exercise, we know $(l\cdot Y)_{(0:0:1)} = \mu_{(0:0:1)}(Y) = d-r$. Hence, we can conclude that $(l\cdot Y)=d$

♣

6 Week6

Exercies 6.1. Let Y be the curve $y^2=x^3-x$ in $\mathbb{A}^2_k$, and assume that the characteristic of the base field k is not 2. We can find that Y is nonsingular and $A=A(Y)$ is integral closed domain. Furthermore, $k[x]\subset K=K(Y)$ is a polynomial subring and A is the integral closure of $k[x]$ in K. There is an automorphism $\sigma: A\to A$ which sends y to $-y$ and leaves x fixed. We define the norm N of A which maps a to $N(a)=a\cdot\sigma(a)$ for any a in A.

Using this norm, show that the units in A are precisely the non-zero elements of k. Show that x and y are irreducible elements of A. Show that A is not a UFD.

Exercies 6.2. We know that if X is a quasi-projective curve and $\varphi\colon X\backslash P\to Y$ where $P\in X$ and Y is a projective variety, there is a unique morphism $\bar{\varphi}\colon X\to Y$ extending φ. But it is not true when $\dim X\geq 2$.

Let $\varphi\colon \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ be the Cremona transformation which is defined on $\mathbb{P}^2_k-\{(1:0:0),(0:1:0),(0:0:1)\}$ and sends $(x:y:z)$ to $(yz:xz:xy)$, show that φ can not extend to a morphism from $\mathbb{P}^2_k$ to $\mathbb{P}^2_k$.

Proof. If φ can be extended to a morphism $\tilde{\varphi}$ on $\mathbb{P}_k^2$. Consider the embedding of projective line $i_1\colon \mathbb{P}_k^1 \hookrightarrow \mathbb{P}_k^2$ by sending $(x:y)$ to $(x:y:0)$. Then points $(1:0:0)$ and $(0:1:0)$ are in $i_1(\mathbb{P}_k^1)$. If $p \in \mathbb{P}_k^1$ is not equal to $(1:0)$ or $(0:1)$, then $\tilde{\varphi}\circ i_1(p) = (0:0:xy) = (0:0:1)$. Hence $\mathbb{P}_k^1 - (1:0) - (0:1) \subseteq S = (\tilde{\varphi}\circ i_1)^{-1}((0:0:1))$. It implies

$$\mathbb{P}_k^1 = \overline{\mathbb{P}_k^1 - (1:0) - (0:1)} \subseteq S \subseteq \mathbb{P}_k^1$$

Therefore, $S = \mathbb{P}_k^1$. This means $\tilde{\varphi}((1:0:0)) = \tilde{\varphi}((0:1:0)) = (0:0:1)$. However, if we take another embedding i_2 of projective line into $\mathbb{P}_k^2$,sending $(x:y)$ to $(x:0:y)$, then we can get $\tilde{\varphi}((1:0:0)) = \tilde{\varphi}((0:0:1)) = (0:1:0)$. So we cannot define the value of $(1:0:0)$ properly. Hence we can conclude that φ cannot extend to a morphism on $\mathbb{P}_k^2$. ♣

Exercies 6.3. Think $\mathbb{P}_k^1$ as $\mathbb{A}_k^1 \cup \infty$. Then we define a fractional linear transformation of $\mathbb{P}_k^1$ by sending $x \mapsto (ax+b)/(cx+d)$, for $a,b,c,d \in k$ and $ad-bc \neq 0$

- Show that a fractional linear transformation induces an automorphism of $\mathbb{P}_k^1$ (i.e., an isomorphism of $\mathbb{P}_k^1$ with itself). We denote the group of all these fractional linear transformations by $\mathrm{PGL}(1)$.
- Let $\mathrm{Aut}(\mathbb{P}_k^1)$ denote the group of all automorphisms of $\mathbb{P}_k^1$. Show that $\mathrm{Aut}(k(x))$, the group pf k-automorphisms of the fractional field $k(x)$.
- Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $\mathrm{PGL}(1) \to \mathrm{Aut}(\mathbb{P}_k^1)$ is an isomorphism.

Proof.

- Let $\infty = (1:0) \in \mathbb{P}_k^1$. Then fractional linear transformation $x \mapsto (ax+b)/(bx+d)$ can be written in coordinates as

$$(x:y) \mapsto (ax+by : cx+dy)$$

  It is well-defined regular morphism since its each component is polynomial of x,y and $(ax+by : cx+dy) \neq (0:0)$ since $ad-bc \neq 0$. Take the inverse matrix of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  We get the inverse of the fractional linear transformation

$$(m:n) \mapsto (\frac{dm-bn}{ad-bc} : \frac{-cm+an}{ad-bc})$$

  Hence it is automorphism on $\mathbb{P}_k^1$.

- First, we note that function field $K(\mathbb{P}_k^1) \cong K(\mathbb{A}_k^1) \cong k(x)$. Taking function field of variety is functional, hence automorphism of $\mathbb{P}_k^1$ induces automorphism of $k(x)$ and it is k-linear since it actually induces morphism of k-algebras. Conversely, any k-automorphism φ of $k(x)$ is determined by the image $\varphi(x)$. Let $f = \varphi(x) \in k(x)$, then we define $\phi\colon \mathbb{P}_k^1 \to \mathbb{P}_k^1$ as follows

$$\phi((x:y)) = \begin{cases} (f(x/y):1) & \text{if } y \neq 0 \\ (1:f(y/x)) & \text{if } x \neq 0 \end{cases}$$

  Or equivalently $\phi(x:y) = (f(x):f(y))$. Since $f \in k(x)$ and φ has inverse, ϕ is well-defined automorphism of $\mathbb{P}_k^1$. Furthermore, we have $K(\phi) = \varphi$, which is induced isomorphism by $\mathbb{P}_k^1$-automorphism on its function field. Hence we can conclude that $\mathrm{Aut}(\mathbb{P}_k^1) \cong \mathrm{Aut}(k(x))$.

- Since $f \in k(x)$, it can be written as

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{b_0 + b_1 x + \cdots + b_n x^n}$$

Let $y = f(x)$. Because φ is isomorphism, $f(x)$ must have form

$$f(x) = \frac{a_0 + a_1 x}{b_0 + b_1 x}$$

Since φ is determined by $f(x)$, there is one-to-one correspondence between $\mathrm{PGL}(1)$ and $\mathrm{Aut}(k(x))$ and it is isomorphism of groups.

♣

7 Week 8

Exercies 7.1.
- Find the degree of the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ where $N = \binom{n+d}{n} - 1$.
- Find the degree of the Serge embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$ where $N = r + s + rs$.

Proof. The d-uple embedding is defined as

$$\rho_d \colon P = (a_0 : \cdots : a_n) \mapsto (M_0(a) : \cdots : M_N(a))$$

where M_i are all monomial of degree d with variables $a_0, \cdots, a_n$. Let $S(\mathbb{P}^N) = k[y_0, \cdots, y_N]$ and $S(\mathbb{P}^n) = k[x_0, \cdots, x_n]$. ρ_d induce k-algebra homomorphism

$$\varrho_d \colon y_i \mapsto M_i(x_0, \cdots, x_n) = x_0^{i_0} \cdots x_n^{i_n}$$

Suppose kernel of this morphism is $\mathfrak{a}$. It is homogeneous prime ideal corresponds to image of ρ_d. Hence the coordinate ring of image M of ρ_d is $S(M) = k[y_0, \cdots, y_N]/\mathfrak{a} \cong \operatorname{Im} \varrho_d$. It means that

$$S(M) = \bigoplus_{l=0} S(M)_l \cong \bigoplus_{l=0} k[x_0, \cdots, x_n]_{ld}$$

where $S(M)_l \cong k[x_0, \cdots, x_n]_{ld}$ is $\binom{ld+n}{n}$ dimensional vector space. Hence the degree M is d^n.

Similarly as d-uple embedding, the ideal of image of Segre embedding is kernel of

$$\begin{aligned} \theta \colon k[z_{00}, \cdots, z_{ij}, \cdots, z_{rs}] &\to k[x_0, \cdots, x_r; y_0, \cdots, y_s] \\ z_{ij} &\mapsto x_i y_j \end{aligned}$$

We denote its kernel by $\mathfrak{b}$, hence $S(X) \cong k[z_{00}, \cdots, z_{ij}, \cdots, z_{rs}]/\mathfrak{b} \cong \operatorname{Im} \theta$. Hence

$$S(X) = \bigoplus_{j=0} S(X)_j \cong \bigoplus_{j=0} k[x_0 y_0, \cdots, x_r y_s]_j \cong \bigoplus_{j=0} k[x_0, \cdots, x_r]_j \times k[y_0, \cdots, y_s]_j$$

where $x_i y_j$ is of grade 1. Hence $\dim_k S(X)_j = \binom{j+r}{j}\binom{j+s}{j}$. ♣

Exercies 7.2. Show that an algebraic set Y of pure dimension r (i.e., every irreducible component of Y has dimension r) has degree 1 if and only if Y is a linear variety.

Proof. If Y is linear variety of codimenision 1 in $\mathbb{P}^n$, then Y is variety in $\mathbb{P}^n$ determined by single linear equation, so it is just hyperplane in $\mathbb{P}^n$. Then we assume that all linear varieties of codimension k is of degree 1. For some linear variety Y^{k+1} of codimension $k+1$, it intersection of a hyperplane H and a linear variety Y^k of codimension k, so with theorem 7.7, we have

$$i(Y^k, H; Y^{k+1}) \deg Y^{k+1} = \deg Y^k \deg H = 1$$

this means that $i(Y^k, H; Y^{k+1}) = \deg Y^{k+1} = 1$. Hence a linear variety is of degree 1. Conversely, let $Y^1 = Z(I)$ be some algebraic set in $\mathbb{P}^n$ of pure codimension 1 and $n \geq 2$. Assume $Y = \cup_i Y_i$ it the decomposition of irreducible components of Y. For each Y_i, we can choose two distinct points P, Q and a hyperplane H in $\mathbb{P}^n$ which contains this two points. The theorem 7.7 implies that the degree of Y_i is greater than 2 if Y_i is not contained in H. So if Y_i is of degree 1, then Y_i is contained in a hyperplane in $\mathbb{P}^n$. Hence Y_i is just this hyperplane since they are irreducible and of same dimension $n-1$. So each components of Y^1 is hyperplane in $\mathbb{P}^n$, hence Y^1 itself is irreducible hyperplane since every two hyperplane intersects with each other. If $n = 1$, then Y is just single point of degree 1 or $\mathbb{P}^1$ itself, of course it is linear. Now, by induction, we can assume it is true for algebraic set of pure codimension k with degree 1 that it is linear. For a algebraic set Y^{k+1} in projective space $\mathbb{P}^n$. Suppose Y_i^{k+1} is one of its irreducible components. We can also choose two distinct points P, Q in Y_i^{k+1} and it also implies Y_i^{k+1} is contained in a hyperplane H. Take an isomorphism $H \cong \mathbb{P}^{n-1}$, then Y_i^{k+1} can be viewed as variety in $\mathbb{P}^{n-1}$ with codimension k of degree 1 (since all mentioned embeddings are closed embeddings). Hence it is linear variety. We have already known that degree of algebraic set is sum the degrees of its irreducible components. Hence Y^{k+1} is irreducible, and furthermore it is linear variety. ♣

Exercies 7.3. Let Y^r be a r-dimensional variety of degree 2. Show that Y is contained in a linear subspace L of dimension $r+1$ in $\mathbb{P}^n$. Thus Y is isomorphic to a quadratic hypersurface in $\mathbb{P}^{r+1}$.

Proof. Choose non-singular point $x \in Y^r$. Then the union of all lines through x and some other points in Y^r is an $r+1$-dimensional variety of degree 1, denoted by P. Therefore, combine statement of previous exercise, we can conclude that P is linear.

Or we can choose three distinct points in Y^r, then we can find a hyperplane H in $\mathbb{P}^n$ that contains such three points. Since Y^r is of degree 2, then Y^r musted be contained in H. With linear transformation $H \cong \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$. Hence Y^r is variety in $\mathbb{P}^{n-1}$ with degree 2. Repeating this process, we can closed embed Y^r in $\mathbb{P}^{r+1} \hookrightarrow \mathbb{P}^n$ with linear transformation. Therefore Y^r is contained in a $r+1$-dimensional linear subspace of $\mathbb{P}^n$. Hence P is linear subspace of $\mathbb{P}^n$ of dimension $r+1$ which contains Y^r. This implies that Y^r is hypersurface of some $\mathbb{P}^{r+1} \hookrightarrow \mathbb{P}^n$ with degree 2. It is defined by polynomial of degree 2. Therefore we can conclude that it is quadratic in $\mathbb{P}^{r+1}$. ♣

8 Week 9

Exercies 8.1. Let $P = (0,0)$, $C = (x^2+y^2)^2 + 3x^2y - y^3$ and $D = (x^2+y^2)^3 - 4x^2y^2$, directly calculate the intersection number $i(P, C \cap D)$.

Proof.
- We need to compute ideal $((x^2+y^2)^2 + 3x^2y - y^3, (x^2+y^2)^3 - 4x^2y^2)$ in local ring $k[x,y]_{(x,y)} \cong k[x,y]$. See the attached draft.
- Let $f = (x^2+y^2)^2 + 3x^2y - y^3$ and $g = (x^2+y^2)^3 - 4x^2y^2$. Homogenizing them, we get

$$F = x^4 + 2x^2y^2 + y^4 + 2x^2yz - y^3z$$
$$G = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 4x^2y^2z^2$$

After linear transformation $(x:y:z) \mapsto (z:x:y)$, the equation of C and D in $\mathbb{P}^2$ becomes

$$\tilde{F} = y^4 + 2y^2z^2 + z^4 + 2y^2zx - y^3x$$
$$\tilde{G} = y^6 + 3y^4z^2 + 3y^2z^4 + z^6 - 4y^2z^2x^2$$

Consider them as polynomial in $k[x,y][z]$. Compute their resultant with Mathematica

$$\mathrm{Res}(\tilde{F}, \tilde{G}, z) = 256x^{10}y^{14} - 768x^9y^{15} + 224x^8y^{16} + 432x^7y^{17} + 125x^6y^{18}$$

♣

Hence $i(p, C \cap D)$ is degree of y in this resultant, it is equal to 14.

Exercies 8.2. Suppose char $k \neq 2$ and let X be a nonsingular projective curve of genus1, then for any effective divisor D on X we have $l(D) = \deg D$ and thus for $P \in X$ a point we have $l(P) = 1$. There is no element in $K(X)$ has one pole exactly at P. Hence, $L(2P) = (1, x) \subseteq K(X)$, the bracket here is linear span with given base, where P is two order point of $x \in K(X)$ and $L(3P) = (1, x, y) \subseteq K(X)$ where P is three order pole of y.

- Show $L(4P) = (1, x, y, x^2)$ and $L(5P) = (1, x, y, x^2, xy)$.

  Proof. For two degree terms x^2, xy, y^2, we have

  $$\mathrm{ord}_p(x^2) = 4 \quad \mathrm{ord}_p(xy) = 2 + 3 = 5 \quad \mathrm{ord}_p(y^2) = 2 \times 3 = 6$$

  Hence $x^2 \in L(4P)$ and others are not. It also implies $L(5P) = (1, x, y, x^2, xy)$. ♣

- Derive the Weierstrass equation for genus 1 curves.

  Proof. For three degree terms x^3, x^2y, xy^2, y^3, we have

  $$\mathrm{ord}_P(x^3) = 6 \quad \mathrm{ord}_P(x^2y) = 7 \quad \mathrm{ord}_P(xy^2) = 8 \quad \mathrm{ord}_P(y^3) = 9$$

  hence we have $(1, x, y, x^2, y^2, xy, x^3) \subseteq L(6P)$. While Riemann-Roch tells us $L(6P) \leq 6$, these generators are linear dependent. Therefore we have

  $$y^2 + b_0xy + b_1y = a_3x^3 + a_2x^2 + a_1x + a_0$$

  with suitable linear transformation we can assume

  $$y^2 = x^3 + ax + b$$

  This is Weierstrass equation for genus 1 curve. Since we have assume that X is non-singular,

  $$x^3 + ax + b = 0$$

  should has three distinct roots. Otherwise, if $x^3 + ax + b = (x - \alpha)^2(x - \beta)$, then the Jacobian of $y^2 - x^2 - ax - b$ at point $(\alpha, 0)$ vanishes. Therefore let

  $$y^2 = \prod_{i=1}^{3}(x - \tau_i)$$

  hence with embedding $(x, y) \mapsto (1 : x : y)$, X is birational to a cubic curve in $\mathbb{P}^2$ defined by

  $$zy^2 = x^3 + axz^2 + bz^3$$

  ♣