

# Algebraic Geometry

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## 1 Week 1

**Exercies 1.0.1.** Any nonempty open subset of an irreducible topological space is dense and irreducible.

Let  $X$  be an irreducible space and  $U \hookrightarrow X$  be a nonempty open subset of  $X$ . Let  $V_1 = X \setminus U$  and  $V_2 = \bar{U}$ . Then we have

$$V_1 \cup V_2 \supseteq (X \setminus U) \cup U = X$$

Since  $X$  is irreducible and  $V_1, V_2$  are closed subsets, we have  $V_1 = \emptyset$  or  $V_2 = X$ . That means  $\bar{U} = X$ . Hence  $U$  is dense open subset of  $X$ . We can further prove that  $X$  is irreducible if any nonempty open subset of  $X$  is dense. Otherwise,  $X$  is reducible, then  $X = V_1 \cup V_2$  where  $V_1$  and  $V_2$  are non-trivial closed subset of  $X$ . Then  $(X \setminus V_1) \cap (X \setminus V_2) = \emptyset$ . It implies  $X \setminus V_1$  is non-empty open subset of  $X$  which is not dense. Hence we can conclude that any nonempty open subset of irreducible space  $X$  is irreducible because its open subsets are all dense in  $X$ , also in itself. ♣

**Exercies 1.0.2.** Let  $Y$  be an affine variety of dimension  $r$  in  $\mathbb{A}^n$ . Let  $H$  be a hypersurface in  $\mathbb{A}^n$  and assume that  $Y \not\subseteq H$ . Then every irreducible component of  $Y \cap H$  has dimension  $r - 1$ .

Suppose  $H$  be irreducible. It means ideal of  $H$  is prime ideal  $(f)$  of  $k[x_1, \dots, x_n]$ . Let  $Y \cap H = V_1 \cup \dots \cup V_k$  be the irreducible components decomposition and ideal of  $V_i$  is  $\mathfrak{p}_i$ . Hence we have

$$I(V \cap H) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$$

Since  $Y \not\subseteq H$ , we have  $f \notin I(Y)$ . Hence the minimal prime ideals of  $I(Y) + (f)$  is with height  $n - k + 1$  by Krull principal ideal theorem, since  $I(Y)$  is of height  $n - r$ . We claim that  $\mathfrak{p}_i$  is a minimal ideal which contains  $I(Y) + (f)$ . Otherwise, let  $\mathfrak{p}$  be the minimal prime ideal satisfying  $I(Y) + (f) \subseteq \mathfrak{p} \subsetneq \mathfrak{p}_i$ . Then  $V_i \subsetneq Z(\mathfrak{p})$ , a irreducible closed subset of  $X$ . It contradicts to the fact that  $V_i$  is irreducible component of  $V \cap H$ . Hence  $ht(\mathfrak{p}_i) = n - r + 1$  for all  $1 \leq i \leq k$ . It implies

$$\dim V_i = \dim A(V_i) = \dim k[x_1, \dots, x_n] - ht(\mathfrak{p}_i) = r - 1$$

♣

**Exercies 1.0.3.** Let  $\alpha \subseteq k[x_1, \dots, x_n]$  be an ideal which can be generated by  $r$  elements. Then every irreducible components of  $Z(\alpha)$  has dimension  $\geq n - r$ .

Let  $Z(\alpha) = V_1 \cup \dots \cup V_k$  be the decomposition of irreducible components, where  $I(V_i) = \mathfrak{p}_i$  is prime ideal of  $k[x_1, \dots, x_n]$ . It implies  $\alpha \subseteq \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$ . So  $\alpha \subseteq \mathfrak{p}_i$  for each  $i$ . Hence  $ht(\mathfrak{p}_i) \leq r$  by Krull principal ideal theorem. Therefore, the dimension of  $V_i$  is greater than  $n - r$ . ♣

## 2 Week2

**Exercies 2.0.1.** Prove following statments

- If  $T_1 \subseteq T_2$  are subsets of  $S^h$ , then  $Z(T_1) \supseteq Z(T_2)$ .
- If  $Y_1 \subseteq Y_2$  are subsets of  $\mathbb{P}^n$ , then  $I(Y_1) \supseteq I(Y_2)$ .
- For any two subsets  $Y_1, Y_2$  of  $\mathbb{P}^n$ ,  $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ .
- If  $\mathfrak{a} \subseteq S$  is a homogeneous ideal with  $Z(\mathfrak{a}) \neq \emptyset$ , then  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ .
- For any subset  $Y \subset \mathbb{P}^n$ ,  $Z(I(Y)) = \bar{Y}$ .

Let  $x \in Z(T_2)$ , we have  $f(x) = 0$  for all  $f \in Y_2$ . Since all  $g \in T_1$  are all in  $Y_2$ , we have  $g(x) = 0$ . Hence  $g \in Z(T_1)$ . If  $g \in I(Y_2)$ , then  $g$  vanishes on all  $Y_2$ , so on all  $Y_1$ . Hence  $g \in I(Y_1)$ . This also implies that both  $I(Y_1)$  and  $I(Y_2)$  contain  $I(Y_1 \cup Y_2)$  since  $Y_i \subseteq Y_1 \cup Y_2$ . Conversely, if  $f \in I(Y_1) \cap I(Y_2)$ , then  $f$  vanishes on both  $Y_1$  and  $Y_2$ , so  $f \in I(Y_1 \cup Y_2)$  by definition.

If  $f \in \sqrt{\mathfrak{a}}$ , then there exists  $n \geq 1$  such that  $f^n \in \mathfrak{a}$ . It implies every homogeneous part vanishes on  $Z(\mathfrak{a})$ . Let  $f = f_1 + \cdots + f_n$  be the homogeneous decomposition of  $f$ . Then the homogeneous part of  $f^n$  with degree  $nk$  is  $f_n^k$ , so  $f_n^k(P) = 0$  for all  $P \in Z(\mathfrak{a})$ . Therefore,  $f_n(P) = 0$ . By induction, we can conclude that  $f_i(P) = 0$  for all  $i$ . Hence  $f \in I(Z(\mathfrak{a}))$ . Conversely, if  $f \in I(Z(\mathfrak{a}))$ . By homogeneous Nullstellensatz, we have  $f_i^{r_i} \in \mathfrak{a}$ . Let  $r = r_1 + \cdots + r_n$ , then  $f^r \in \mathfrak{a}$ . Hence  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ .

Since  $Y \subseteq Z(I(Y))$  and  $Z(I(Y))$  is closed, we have  $\bar{Y} \subseteq Z(I(Y))$ . There is homogeneous ideal  $\mathfrak{a}$  such that  $\bar{Y} = Z(\mathfrak{a})$ . From  $Y \subseteq Z(\mathfrak{a})$ , we have  $I(Y) \subseteq I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ . Hence  $\bar{Y} = Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) \subseteq Z(I(Y))$ . We now conclude that  $Z(I(Y)) = \bar{Y}$ . ♣

**Exercies 2.0.2.** a) There is a 1 – 1 inclusion-reversing correspondence between algebraic sets in  $\mathbb{P}^n$ , and homogeneous radical ideals of  $S$  not equal to  $S_+$  does not occur in this correspondence, it is sometimes called the irrelevant maximal of  $S$ .

b) An algebraic set  $Y \subseteq \mathbb{P}^n$  is irreducible if and only if  $I(Y)$  is a prime ideal.

c) Show that  $\mathbb{P}^n$  itself is irreducible.

a) If  $\mathfrak{a}$  is radical homogeneous ideal of  $S$  such that  $Z(\mathfrak{a}) \neq \emptyset$ , then  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$ . If  $Z(\mathfrak{a}) = \emptyset$ , then  $I(Z(\mathfrak{a})) = I(\emptyset) = S$ .  $Z(\mathfrak{a}) = \emptyset$  implies  $\mathfrak{a} = S$  or  $S_+$ . By assumption,  $S_+$  is not in the correspondence, so  $\mathfrak{a} = S$ . Hence  $I \circ Z$  is identity functor. Similarly,  $Z \circ I$  is also identity. With previous exercise, this correspondence is inclusion-reversing.

b) If  $Y$  is irreducible, then for all  $x, y \in I(Y)$ , we can let  $Y_1 = Z(x) \cap Y$  and  $Y_2 = Z(y) \cap Y$ . Since  $Y_1 \cup Y_2 = (Z(x) \cap Y) \cup (Z(y) \cap Y) = Z(xy) \cap Y = Y$ ,  $Y_1 = Y$  or  $Y_2 = Y$  by irreducible condition. It implies that  $Z(x) = Y$  or  $Z(y) = Y$ . So  $x \in I(Y)$  or  $y \in I(Y)$ . Conversely, suppose  $I(Y)$  is prime. However, for any closed cover  $Y_1 \cup Y_2 = Y$ , we have  $I(Y) = I(Y_1) \cap I(Y_2)$ , therefore  $I(Y_1) = I(Y)$  or  $I(Y_2) = I(Y)$ . Hence  $Y_1 = Y$  or  $Y_2 = Y$  since they are closed.

c)  $\mathbb{P}^n$  is algebraic set corresponding to radical homogeneous ideal  $(0)$ . It is prime ideal since  $k[x_0, x_1, \dots, x_n]$  is integral domain. So  $\mathbb{P}^n$  is irreducible from previous statement. ♣

**Exercies 2.0.3.** If  $Y$  is a projective variety with homogeneous coordinate ring  $S(Y)$ , show that  $\dim S(Y) = \dim Y + 1$ .

$Y$  is projective variety, so let  $Y = Z(\mathfrak{p}) \subseteq \mathbb{P}^n$  for some prime homogeneous ideal  $\mathfrak{p}$ . Hence any descending chain of closed subset of  $Y$  corresponds to a descending chain of radical homogeneous ideal containing  $\mathfrak{p}$  with the same length. However,  $S_+$  is prime homogeneous ideal of  $S$  which contains any non-zero ideal of  $S$  but doesn't correspond to a algebraic set. Hence the  $\dim S(Y) > \dim Y$ . Since radical ideals contains  $\mathfrak{p}$  except  $S_+$  also correspond to closed subsets of  $Y$ ,  $\dim Y \geq \dim S(Y) - 1$ . Hence  $\dim S(Y) = \dim Y + 1$ . ♣

### 3 Week 3

**Exercies 3.0.1.** A regular function on projective variety is continuous map (view  $k$  as affine line  $\mathbb{A}^1$ ).

Let  $Y \subseteq \mathbb{P}^n$  be a projective variety, then  $Y$  can be covered by affine varieties  $U_i = Y \cap A_i^n$ , where  $A_i^n$  are canonical affine coverings of  $\mathbb{P}^n$ . If  $f: Y \rightarrow k$  be a regular function, then its restrictions  $f_{U_i}: U_i \rightarrow k$  are continuous map, and since  $U_i$  are all open in  $Y$ ,  $f$  itself is continuous on  $Y$ . ♣

**Exercies 3.0.2.** Let  $\varphi: \mathbb{A}^1 \rightarrow C \hookrightarrow \mathbb{A}^2$  be curve defined as  $t \mapsto (t^2, t^3)$ . Obviously,  $\varphi$  is 1 – 1 correspondence. Prove  $\varphi$  is not isomorphism between varieties.

We have coordinate ring  $A(C) \cong k[x, y]/(y^2 - x^3)$ . It is not isomorphic to  $k[t]$ . Hence we can conclude  $\varphi$  can not be an isomorphism otherwise it will induce isomorphism between coordinate ring. ♣

**Exercies 3.0.3.** Let  $S = Z(y_0y_2 - y_1^2)$  be the surface in  $\mathbb{P}_k^2$  with the coordinates  $(y_0 : y_1 : y_2)$ . Let  $\mathbb{P}_k^1$  be projective line with coordinate ring  $k[x_0, x_1]$ . Consider morphism

$$\begin{aligned}\varphi: \mathbb{P}_k^1 &\rightarrow \mathbb{P}_k^2 \\ (x_0 : x_1) &\mapsto (x_0^2 : x_0x_1 : x_1^2)\end{aligned}$$

and show that  $\varphi$  is isomorphism.