

Algebraic Geometry

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1 Week 1

Exercies 1.0.1. Any nonempty open subset of an irreducible topological space is dense and irreducible.

Let X be an irreducible space and $U \hookrightarrow X$ be a nonempty open subset of X . Let $V_1 = X \setminus U$ and $V_2 = \bar{U}$. Then we have

$$V_1 \cup V_2 \supseteq (X \setminus U) \cup U = X$$

Since X is irreducible and V_1, V_2 are closed subsets, we have $V_1 = \emptyset$ or $V_2 = X$. That means $\bar{U} = X$. Hence U is dense open subset of X . We can further prove that X is irreducible if any nonempty open subset of X is dense. Otherwise, X is reducible, then $X = V_1 \cup V_2$ where V_1 and V_2 are non-trivial closed subset of X . Then $(X \setminus V_1) \cap (X \setminus V_2) = \emptyset$. It implies $X \setminus V_1$ is non-empty open subset of X which is not dense. Hence we can conclude that any nonempty open subset of irreducible space X is irreducible because its open subsets are all dense in X , also in itself. ♣

Exercies 1.0.2. Let Y be an affine variety of dimension r in \mathbb{A}^n . Let H be a hypersurface in \mathbb{A}^n and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Suppose H be irreducible. It means ideal of H is prime ideal (f) of $k[x_1, \dots, x_n]$. Let $Y \cap H = V_1 \cup \dots \cup V_k$ be the irreducible components decomposition and ideal of V_i is \mathfrak{p}_i . Hence we have

$$I(V \cap H) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$$

Since $Y \not\subseteq H$, we have $f \notin I(Y)$. Hence the minimal prime ideals of $I(Y) + (f)$ is with height $n - k + 1$ by Krull principal ideal theorem, since $I(Y)$ is of height $n - r$. We claim that \mathfrak{p}_i is a minimal ideal which contains $I(Y) + (f)$. Otherwise, let \mathfrak{p} be the minimal prime ideal satisfying $I(Y) + (f) \subseteq \mathfrak{p} \subsetneq \mathfrak{p}_i$. Then $V_i \subsetneq Z(\mathfrak{p})$, a irreducible closed subset of X . It contradicts to the fact that V_i is irreducible component of $V \cap H$. Hence $ht(\mathfrak{p}_i) = n - r + 1$ for all $1 \leq i \leq k$. It implies

$$\dim V_i = \dim A(V_i) = \dim k[x_1, \dots, x_n] - ht(\mathfrak{p}_i) = r - 1$$

♣

Exercies 1.0.3. Let $\alpha \subseteq k[x_1, \dots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible components of $Z(\alpha)$ has dimension $\geq n - r$.

Let $Z(\alpha) = V_1 \cup \dots \cup V_k$ be the decomposition of irreducible components, where $I(V_i) = \mathfrak{p}_i$ is prime ideal of $k[x_1, \dots, x_n]$. It implies $\alpha \subseteq \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$. So $\alpha \subseteq \mathfrak{p}_i$ for each i . Hence $ht(\mathfrak{p}_i) \leq r$ by Krull principal ideal theorem. Therefore, the dimension of V_i is greater than $n - r$. ♣

2 Week2

Exercies 2.0.1. Prove following statments

- If $T_1 \subseteq T_2$ are subsets of S^h , then $Z(T_1) \supseteq Z(T_2)$.
- If $Y_1 \subseteq Y_2$ are subsets of \mathbb{P}^n , then $I(Y_1) \supseteq I(Y_2)$.
- For any two subsets Y_1, Y_2 of \mathbb{P}^n , $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
- If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
- For any subset $Y \subset \mathbb{P}^n$, $Z(I(Y)) = \bar{Y}$.

Let $x \in Z(T_2)$, we have $f(x) = 0$ for all $f \in Y_2$. Since all $g \in T_1$ are all in Y_2 , we have $g(x) = 0$. Hence $g \in Z(T_1)$. If $g \in I(Y_2)$, then g vanishes on all Y_2 , so on all Y_1 . Hence $g \in I(Y_1)$. This also implies that both $I(Y_1)$ and $I(Y_2)$ contain $I(Y_1 \cup Y_2)$ since $Y_i \subseteq Y_1 \cup Y_2$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then f vanishes on both Y_1 and Y_2 , so $f \in I(Y_1 \cup Y_2)$ by definition.

If $f \in \sqrt{\mathfrak{a}}$, then there exists $n \geq 1$ such that $f^n \in \mathfrak{a}$. It implies every homogeneous part vanishes on $Z(\mathfrak{a})$. Let $f = f_1 + \cdots + f_n$ be the homogeneous decomposition of f . Then the homogeneous part of f^n with degree nk is f_n^k , so $f_n^k(P) = 0$ for all $P \in Z(\mathfrak{a})$. Therefore, $f_n(P) = 0$. By induction, we can conclude that $f_i(P) = 0$ for all i . Hence $f \in I(Z(\mathfrak{a}))$. Conversely, if $f \in I(Z(\mathfrak{a}))$. By homogeneous Nullstellensatz, we have $f_i^{r_i} \in \mathfrak{a}$. Let $r = r_1 + \cdots + r_n$, then $f^r \in \mathfrak{a}$. Hence $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Since $Y \subseteq Z(I(Y))$ and $Z(I(Y))$ is closed, we have $\bar{Y} \subseteq Z(I(Y))$. There is homogeneous ideal \mathfrak{a} such that $\bar{Y} = Z(\mathfrak{a})$. From $Y \subseteq Z(\mathfrak{a})$, we have $I(Y) \subseteq I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. Hence $\bar{Y} = Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) \subseteq Z(I(Y))$. We now conclude that $Z(I(Y)) = \bar{Y}$. ♣

Exercies 2.0.2. a) There is a 1 – 1 inclusion-reversing correspondence between algebraic sets in \mathbb{P}^n , and homogeneous radical ideals of S not equal to S_+ does not occur in this correspondence, it is sometimes called the irrelevant maximal of S .

b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

c) Show that \mathbb{P}^n itself is irreducible.

a) If \mathfrak{a} is radical homogeneous ideal of S such that $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$. If $Z(\mathfrak{a}) = \emptyset$, then $I(Z(\mathfrak{a})) = I(\emptyset) = S$. $Z(\mathfrak{a}) = \emptyset$ implies $\mathfrak{a} = S$ or S_+ . By assumption, S_+ is not in the correspondence, so $\mathfrak{a} = S$. Hence $I \circ Z$ is identity functor. Similarly, $Z \circ I$ is also identity. With previous exercise, this correspondence is inclusion-reversing.

b) If Y is irreducible, then for all $x, y \in I(Y)$, we can let $Y_1 = Z(x) \cap Y$ and $Y_2 = Z(y) \cap Y$. Since $Y_1 \cup Y_2 = (Z(x) \cap Y) \cup (Z(y) \cap Y) = Z(xy) \cap Y = Y$, $Y_1 = Y$ or $Y_2 = Y$ by irreducible condition. It implies that $Z(x) = Y$ or $Z(y) = Y$. So $x \in I(Y)$ or $y \in I(Y)$. Conversely, suppose $I(Y)$ is prime. However, for any closed cover $Y_1 \cup Y_2 = Y$, we have $I(Y) = I(Y_1) \cap I(Y_2)$, therefore $I(Y_1) = I(Y)$ or $I(Y_2) = I(Y)$. Hence $Y_1 = Y$ or $Y_2 = Y$ since they are closed.

c) \mathbb{P}^n is algebraic set corresponding to radical homogeneous ideal (0) . It is prime ideal since $k[x_0, x_1, \dots, x_n]$ is integral domain. So \mathbb{P}^n is irreducible from previous statement. ♣

Exercies 2.0.3. If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.

Y is projective variety, so let $Y = Z(\mathfrak{p}) \subseteq \mathbb{P}^n$ for some prime homogeneous ideal \mathfrak{p} . Hence any descending chain of closed subset of Y corresponds to a descending chain of radical homogeneous ideal containing \mathfrak{p} with the same length. However, S_+ is prime homogeneous ideal of S which contains any non-zero ideal of S but doesn't correspond to a algebraic set. Hence the $\dim S(Y) > \dim Y$. Since radical ideals contains \mathfrak{p} except S_+ also correspond to closed subsets of Y , $\dim Y \geq \dim S(Y) - 1$. Hence $\dim S(Y) = \dim Y + 1$. ♣

3 Week 3

Exercies 3.0.1. A regular function on projective variety is continuous map (view k as affine line \mathbb{A}^1).

Let $Y \subseteq \mathbb{P}^n$ be a projective variety, then Y can be covered by affine varieties $U_i = Y \cap A_i^n$, where A_i^n are canonical affine coverings of \mathbb{P}^n . If $f: Y \rightarrow k$ be a regular function, then its restrictions $f_{U_i}: U_i \rightarrow k$ are continuous map, and since U_i are all open in Y , f itself is continuous on Y . ♣

Exercies 3.0.2. Let $\varphi: \mathbb{A}^1 \rightarrow C \hookrightarrow \mathbb{A}^2$ be curve defined as $t \mapsto (t^2, t^3)$. Obviously, φ is 1 – 1 correspondence. Prove φ is not isomorphism between varieties.

Suppose the coordinate ring of \mathbb{A}_k^2 be $k[x, y]$. Then we have coordinate ring $A(C) \cong k[x, y]/(y^2 - x^3)$. From definition of φ , we can write down its pull-back on coordinate rings

$$\begin{aligned}\varphi^*: A(C) &\rightarrow k[t] \\ f &\mapsto f \circ \varphi\end{aligned}$$

If φ is isomorphism, then φ^* is isomorphism between coordinate ring. Therefore it is also bijection between regular functions. Since t is regular function on \mathbb{A}_k^1 , it must have preimage f such that $\varphi^*(f) = t$. This means that $f(t^2, t^3) = t$. f is regular on C , so it is also regular at point $(0, 0)$. Nearby $(0, 0)$, f can be written as

$$\frac{\alpha(x, y)}{\beta(x, y)}$$

where α, β are polynomials in $k[x, y]$ and $\beta(0, 0) \neq 0$, $\frac{\alpha(t^2, t^3)}{\beta(t^2, t^3)} = t$. It is impossible. Hence we can conclude φ can not be an isomorphism otherwise it will induce isomorphism between coordinate ring. ♣

Exercies 3.0.3. Let $S = Z(y_0 y_2 - y_1^2)$ be the surface in \mathbb{P}_k^2 with the coordinates $(y_0 : y_1 : y_2)$. Let \mathbb{P}_k^1 be projective line with coordinate ring $k[x_0, x_1]$. Consider morphism

$$\begin{aligned}\varphi: \mathbb{P}_k^1 &\rightarrow S \subset \mathbb{P}_k^2 \\ (x_0 : x_1) &\mapsto (x_0^2 : x_0 x_1 : x_1^2)\end{aligned}$$

and show that φ is isomorphism.

It is well-defined regular morphism since $(x_0^2)(x_1^2) - (x_0 x_1)^2 = 0$ and with polynomial in each component. We can see that φ is bijection and φ^{-1} is defined as

$$\varphi^{-1}: (y_0 : y_1 : y_2) = \begin{cases} (y_0 : y_1) & \text{if } y_2 \neq 0 \\ (y_1 : y_2) & \text{if } y_0 \neq 0 \end{cases}$$

It is well-defined regular morphism since if y_0 and y_2 are neither equal to 0 then $(y_0 : y_1) = (\frac{y_0}{y_1} : 1) = (\frac{y_1}{y_2} : 1) = (y_1 : y_2)$. And we have $\varphi \circ \varphi^{-1} = id_S$, $\varphi^{-1} \circ \varphi = id_{\mathbb{P}_k^1}$.

Exercies 3.0.4. Let Y be an affine variety in \mathbb{A}_k^n . Show that $K(Y) \cong K(\mathcal{O}_{Y,p})$ for all $p \in Y$.

Exercies 3.0.5. Prove that for any integer $0 \leq i \leq n$

$$K(\mathbb{P}_k^n) \cong k(x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i)$$

Exercies 3.0.6. Equation $x_0^2 + x_1^2 + x_2^2 = 0$ defines a conic $X \hookrightarrow \mathbb{P}_k^2$. Find $t \in K(X)$ such that $K(X) \cong k(t)$ is a transcendental extension of k with degree 1.