Algebraic Geometry

邹海涛
ID: 17210180015

1 Week 1

Exercies 1.1. Any nonempty open subset of an irreducible topological space is dense and irreducible.

Let X be an irreducible space and $U \hookrightarrow X$ be an nonempty open subset of X. Let $V_1 = X \backslash U$ and $V_2 = \bar{U}$. Then we have

$$V_1 \cup V_2 \supseteq (X \backslash U) \cup U = X$$

Since X is irreducible and V_1, V_2 are closed subsets, we have $V_1 = \emptyset$ or $V_2 = X$. That means $\bar{U} = X$. Hence U is dense open subset of X. We can further prove that X is irreducible if any nonempty open subset of X is dense. Otherwise, X is reducible, then $X = V_1 \cup V_2$ where V_1 and V_2 are non-trivial closed subset of X. Then $(X \backslash V_1) \cap (X \backslash V_2) = \emptyset$. It implies $X \backslash V_1$ is non-empty open subset of X which is not dense. Hence we can conclude that any nonempty open subset of irreducible space X is irreducible because its open subsets are all dense in X, also in itself. ♣

Exercies 1.2. Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$ and assume that $Y \subsetneq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Suppose H be irreducible. It means ideal of H is prime ideal (f) of $k[x_1, \cdots, x_n]$. Let $Y \cap H = V_1 \cup \cdots \cup V_k$ be the irreducible components decomposition and ideal of V_i is $\mathfrak{p}_i$. Hence we have

$$I(V \cap H) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$$

Since $Y \subsetneq H$, we have $f \notin I(Y)$. Hence the minimal prime ideals of $I(Y) + (f)$ is with height $n - k + 1$ by Krull principal ideal theorem, since $I(Y)$ is of height $n - r$. We claim that p_i is a minimal ideal which contains $I(Y) + (f)$. Otherwise, let $\mathfrak{p}$ be the minimal prime ideal satisfying $I(Y) + (f) \subseteq \mathfrak{p} \subsetneq \mathfrak{p}_i$. Then $V_i \subsetneq Z(\mathfrak{p})$, a irreducible closed subset of X. It contradicts to the fact that V_i is irreducible component of $V \cap H$. Hence $ht(\mathfrak{p}_i) = n - r + 1$ for all $1 \leq i \leq k$. It implies

$$\dim V_i = \dim A(V_i) = \dim k[x_1, \cdots, x_n] - ht(\mathfrak{p}_i) = r - 1$$

♣

Exercies 1.3. Let $\alpha \subseteq k[x_1, \cdots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible components of $Z(\alpha)$ has dimension $\geq n - r$.

Let $Z(\alpha) = V_1 \cup \cdots \cup V_k$ be the decomposition of irreducible components, where $I(V_i) = \mathfrak{p}_i$ is prime ideal of $k[x_1, \cdots, x_n]$. It implies $\alpha \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$. So $\alpha \subseteq \mathfrak{p}_i$ for each i. Hence $ht(\mathfrak{p}_i) \leq r$ by Krull principal ideal theorem. Therefore, the dimension of V_i is greater than $n - r$. ♣

2 Week2

Exercies 2.1. Prove following statments

- If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

- If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
- For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
- If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
- For any subset $Y \subset \mathbb{P}^n$, $Z(I(Y)) = \bar{Y}$.

Let $x \in Z(T_2)$, we have $f(x) = 0$ for all $f \in Y_2$. Since all $g \in T_1$ are all in Y_2, we have $g(x) = 0$. Hence $g \in Z(T_1)$. If $g \in I(Y_2)$, then g vanishs on all Y_2, so on all Y_1. Hence $g \in I(Y_1)$. This also implies that both $I(Y_1)$ and $I(Y_2)$ contain $I(Y_1 \cup Y_2)$ since $Y_i \subseteq Y_1 \cup Y_2$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then f vanishes on both Y_1 and Y_2, so $f \in I(V_1 \cup V_2)$ by definition.

If$f \in \sqrt{\mathfrak{a}}$, then there exists $n \geq 1$ such that $f^k \in \mathfrak{a}$. It implies every homogeneous part vanishes on $Z(\mathfrak{a})$. Let $f = f_1 + \cdots + f_n$ be the homogeneous decomposition of f. Then the homogeneous part of f^k with degree nk is f_n^k, so $f_n^k(P) = 0$ for all $P \in Z(\mathfrak{a})$. Therefore, $f_n(P) = 0$. By induction, we can conclude that $f_i(P) = 0$ for all i. Hence $f \in I(Z(\mathfrak{a}))$. Conversely, if $f \in I(Z(\mathfrak{a}))$. By homogeneous Nullstellensatz, we have $f_i^{r_i} \in \mathfrak{a}$. Let $r = r_1 + \cdots r_n$, then $f^r \in \mathfrak{a}$. Hence $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Since $Y \subseteq Z(I(Y))$ and $Z(I(Y))$ is closed, we have $\bar{Y} \subseteq Z(I(Y))$. There is homogeneous ideal $\mathfrak{a}$ such that $\bar{Y} = Z(\mathfrak{a})$. From $Y \subseteq Z(\mathfrak{a})$, we have $I(Y) \subseteq I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. Hence $\bar{Y} = Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) \subseteq Z(I(Y))$. We now conclude that $Z(I(Y)) = \bar{Y}$. ♣

Exercies 2.2. a) There is a $1-1$ inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of S not equal to S_+ does not occur in this correspondence, it is sometimes called the irrelevant maximal of S.

b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

c) Show that $\mathbb{P}^n$ itself is irreducible.

a) If $\mathfrak{a}$ is radical homogeneous ideal of S such that $Z(\mathfrak{a}) \neq 0$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$. If $Z(\mathfrak{a}) = 0$, then $I(Z(\mathfrak{a})) = I(\emptyset) = S$. $Z(\mathfrak{a}) = 0$ implies $\mathfrak{a} = S$ or S_+. By assumption, S_+ is not in the correspondence, so $\mathfrak{a} = S$. Hence $I \circ Z$ is identity functor. Similarly, $Z \circ I$ is also identity. With previous exercise, this correspondence is inclusion-reversing.

b) If Y is irreducible, then for all $x, y \in I(Y)$, we can let $Y_1 = Z(x) \cap Y$ and $Y_2 = Z(y) \cap Y$. Since $Y_1 \cup Y_2 = (Z(x) \cap Y) \cup (Z(y) \cap Y) = Z(xy) \cap Y = Y$, $Y_1 = Y$ or $Y_2 = Y$ by irreducible condition. It implies that $Z(x) = Y$ or $Z(y) = Y$. So $x \in I(Y)$ or $y \in I(Y)$. Conversely, suppose $I(Y)$ is prime. However, for any closed cover $Y_1 \cup Y_2 = Y$, we have $I(Y) = I(Y_1) \cap I(Y_2)$, therefore $I(Y_1) = I(Y)$ or $I(Y_2) = I(Y)$. Hence $Y_1 = Y$ or $Y_2 = Y$ since they are closed.

c) $\mathbb{P}^n$ is algebraic set corresponding to radical homogeneous ideal (0). It is prime ideal since $k[x_0, x_1 \cdots x_n]$ is integral domain. So $\mathbb{P}^n$ is irreducible from previous statement.

♣

Exercies 2.3. If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.

Y is projective variety, so let $Y = Z(\mathfrak{p}) \subseteq \mathbb{P}^n$ for some prime homogeneous ideal $\mathfrak{p}$. Hence any descending chain of closed subset of Y corresponds to a descending chain of radical homogeneous ideal containing $\mathfrak{p}$ with the same length. However, S_+ is prime homogeneous ideal of S which contains any non-zero ideal of S but doesn't correspond to a algebraic set. Hence the $\dim S(Y) > \dim Y$. Since radical ideals contains $\mathfrak{p}$ except S_+ also correspond to closed subsets of Y, $\dim Y \geq \dim S(Y) - 1$. Hence $\dim S(Y) = \dim Y + 1$. ♣

3 Week 3

Exercies 3.1. A regular function on projective variety is continuous map (view k as affine line $\mathbb{A}^1$).

Let $Y \subseteq P^n$ be a projective variety, then Y can be covered by affine varieties $U_i = Y \cap A_i^n$, where A_i^n are canonical affine coverings of $\mathbb{P}^n$. If $f\colon Y \to k$ be a regular function, then its restrictions $f_{U_i}\colon U_i \to k$ are continuous map, and since U_i are all open in Y, f itself is continuous on Y. ♣

Exercies 3.2. Let $\varphi : \mathbb{A}^1 \to C \hookrightarrow \mathbb{A}^2$ be curve defined as $t \mapsto (t^2, t^3)$. Obviously, φ is $1-1$ correspondence. Prove φ is not isomorphism between varieties.

Suppose the coordinate ring of $\mathbb{A}_k^2$ be $k[x, y]$. Then we have coordinate ring $A(C) \cong k[x, y]/(y^2 - x^3)$. From definition of φ, we can write down its pull-back on coordinate rings

$$\begin{aligned} \varphi^*\colon A(C) &\to k[t] \\ f &\mapsto f \circ \varphi \end{aligned}$$

If φ is isomorphism, then φ^* is isomorphism between coordinate ring. Therefore it is also bijection between regular functions. Since t is regular function on $\mathbb{A}_k^1$, it must have preimage f such that $\varphi^*(f) = t$. This means that $f(t^2, t^3) = t$. f is regular on C, so it is also regular at point $(0, 0)$. Nearby $(0, 0)$, f can be written as

$$\frac{\alpha(x, y)}{\beta(x, y)}$$

where α, β are polynomials in $k[x, y]$ and $\beta(0, 0) \neq 0$, $\frac{\alpha(t^2,t^3)}{\beta(t^2,t^3)} = t$. It is impossible. Hence we can conclude φ can not be an isomorphism otherwise it will induce isomorphism between coordinate ring. ♣

Exercies 3.3. Let $S = Z(y_0y_2 - y_1^2)$ be the surface in $\mathbb{P}_k^2$ with the coordinates $(y_0 : y_1 : y_2)$. Let $\mathbb{P}_k^1$ be projective line with coordinate ring $k[x_0, x_1]$. Consider morphism

$$\begin{aligned} \varphi : \mathbb{P}_k^1 &\to S \subset \mathbb{P}_k^2 \\ (x_0 : x_1) &\mapsto (x_0^2 : x_0x_1 : x_1^2) \end{aligned}$$

and show that φ is isomorphism.

It is well-defined regular morphism since $(x_0^2)(x_1^2) - (x_0x_1)^2 = 0$ and with polynomial in each component. We can see that φ is bijection and φ^{-1} is defined as

$$\varphi^{-1}\colon (y_0 : y_1 : y_2) = \begin{cases} (y_0 : y_1) & \text{if } y_0 \neq 0 \\ (y_1 : y_2) & \text{if } y_2 \neq 0 \end{cases}$$

It is well-defined regular morphism since if y_0 and y_2 are neither equal to 0 then $(y_0 : y_1) = (\frac{y_0}{y_1} : 1) = (\frac{y_1}{y_2} : 1) = (y_1 : y_2)$. And we have $\varphi \circ \varphi^{-1} = id_S, \varphi^{-1} \circ \varphi = id_{\mathbb{P}_k^1}$. ♣

4 Week 4

Exercies 4.1. Let Y be an affine variety in $\mathbb{A}_k^n$. Show that $K(Y) \cong K(\mathcal{O}_{Y,p})$ for all $p \in Y$.

Since Y is affine variety, we have $\mathcal{O}_{Y,p} \cong A(Y)_{m_p}$ for all point $p \in Y$. Therefore, $K(\mathcal{O}_{Y,p})$ is fraction field of local ring $A(Y)_{m_p}$. Moreover, $K(Y)$ is fraction field of coordinate ring $A(Y)$. Hence we have following commutative localization diagram

$$\begin{tikzcd} A(Y) \arrow[r, "l_{m_p}"] \arrow[d, "l"] & A(Y)_{m_p} \arrow[d, "l'"] \\ K(A(Y)) & K(A(Y)_{m_p}) \end{tikzcd}$$

l_{m_p} maps non-zero divisor of $A(Y)$ to non-zero divisor, so $l' \circ l_{m_p}$ maps non-zero divisors to units. Therefore, with universal property of localization, there is unique homomorphism $g\colon K(A(Y)) \to K(A(Y)_{m_p})$ making this diagram commute. Also, with universal property of l_{m_p}, there is unique morphism h making following diagram commute

$$\begin{tikzcd} A(Y) \arrow[r, "l_{m_p}"] \arrow[d, "l"] & A(Y)_{m_p} \arrow[dl, "h"] \\ K(A(Y)) & \end{tikzcd}$$

Now we get a new localization diagram

$$\begin{tikzcd} & A(Y)_{m_p} \arrow[dl, "h"'] \arrow[d, "l'"] \\ K(A(Y)) & K(A(Y)_{m_p}) \arrow[l, dashed, "\exists! f"] \end{tikzcd}$$

Hence we can conclude that $K(A(Y)) \cong K(A(Y)_{m_p})$, which implies that $K(Y) \cong K(\mathcal{O}_{Y,p})$ for all $p \in Y$. ♣

Exercies 4.2. Prove that for any integer $0 \leq i \leq n$

$$K(\mathbb{P}_k^n) \cong k(x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i)$$

For any variety Y, if U is open subvariety of Y, then by definition $K(Y) \cong K(U)$. More precisely, we can send $\langle f, V \rangle$ to $\langle f, V \cap U \rangle$ and it is an well-defined isomorphism between function fields. Therefore, for any $0 \leq i \leq n$, we have $K(A_i) \cong K(\mathbb{P}_k^n)$, where A_i is affine cover which is isomorphic to $\mathbb{A}_k^n$ with coordinate ring $k[x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i]$. With the conclusion in last exercise, we can conclude that

$$K(\mathbb{P}_k^n) \cong K(A_i) \cong k(x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i)$$

Exercies 4.3. Equation $x_0^2 + x_1^2 + x_2^2 = 0$ defines a conic $X \hookrightarrow \mathbb{P}_k^2$. Find $t \in K(X)$ such that $K(X) \cong k(t)$ is a transcendental extension of k with degree 1.

$K(X)$ is equal to function field of affine open subset defined by

$$(\frac{x_1}{x_0})^2 + (\frac{x_2}{x_0})^2 = -1 \tag{1}$$

Consider linear transform

$$y_1 = \frac{x_1}{x_0} + i\frac{x_2}{x_0} \qquad y_2 = \frac{x_1}{x_0} - i\frac{x_2}{x_0} \tag{2}$$

Then the equation 1 becomes

$$y_1 y_2 = -1 \tag{3}$$

Hence $K(X) = k(y_1, y_2) = k(y_1)$ since $y_2 = -1/y_1$. Therefore y_1 is the required t. ♣