# Algebraic Geometry

**邹海涛**
ID: 17210180015

## 1 Week 1

**Exercies 1.1.** Any nonempty open subset of an irreducible topological space is dense and irreducible.

Let $X$ be an irreducible space and $U \hookrightarrow X$ be an nonempty open subset of $X$. Let $V_1 = X \backslash U$ and $V_2 = \bar{U}$. Then we have

$$V_1 \cup V_2 \supseteq (X \backslash U) \cup U = X$$

Since $X$ is irreducible and $V_1, V_2$ are closed subsets, we have $V_1 = \emptyset$ or $V_2 = X$. That means $\bar{U} = X$. Hence $U$ is dense open subset of $X$. We can further prove that $X$ is irreducible if any nonempty open subset of $X$ is dense. Otherwise, $X$ is reducible, then $X = V_1 \cup V_2$ where $V_1$ and $V_2$ are non-trivial closed subset of $X$. Then $(X \backslash V_1) \cap (X \backslash V_2) = \emptyset$. It implies $X \backslash V_1$ is non-empty open subset of $X$ which is not dense. Hence we can conclude that any nonempty open subset of irreducible space $X$ is irreducible because its open subsets are all dense in $X$, also in itself. ♣

**Exercies 1.2.** Let $Y$ be an affine variety of dimension $r$ in $\mathbb{A}^n$. Let $H$ be a hypersurface in $\mathbb{A}^n$ and assume that $Y \subsetneq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$.

Suppose $H$ be irreducible. It means ideal of $H$ is prime ideal $(f)$ of $k[x_1, \cdots, x_n]$. Let $Y \cap H = V_1 \cup \cdots \cup V_k$ be the irreducible components decomposition and ideal of $V_i$ is $\mathfrak{p}_i$. Hence we have

$$I(V \cap H) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$$

Since $Y \subsetneq H$, we have $f \notin I(Y)$. Hence the minimal prime ideals of $I(Y) + (f)$ is with height $n - k + 1$ by Krull principal ideal theorem, since $I(Y)$ is of height $n - r$. We claim that $p_i$ is a minimal ideal which contains $I(Y) + (f)$. Otherwise, let $\mathfrak{p}$ be the minimal prime ideal satisfying $I(Y) + (f) \subseteq \mathfrak{p} \subsetneq \mathfrak{p}_i$. Then $V_i \subsetneq Z(\mathfrak{p})$, a irreducible closed subset of $X$. It contradicts to the fact that $V_i$ is irreducible component of $V \cap H$. Hence $ht(\mathfrak{p}_i) = n - r + 1$ for all $1 \leq i \leq k$. It implies

$$\dim V_i = \dim A(V_i) = \dim k[x_1, \cdots, x_n] - ht(\mathfrak{p}_i) = r - 1$$

♣

**Exercies 1.3.** Let $\alpha \subseteq k[x_1, \cdots, x_n]$ be an ideal which can be generated by $r$ elements. Then every irreducible components of $Z(\alpha)$ has dimension $\geq n - r$.

Let $Z(\alpha) = V_1 \cup \cdots \cup V_k$ be the decomposition of irreducible components, where $I(V_i) = \mathfrak{p}_i$ is prime ideal of $k[x_1, \cdots, x_n]$. It implies $\alpha \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$. So $\alpha \subseteq \mathfrak{p}_i$ for each $i$. Hence $ht(\mathfrak{p}_i) \leq r$ by Krull principal ideal theorem. Therefore, the dimension of $V_i$ is greater than $n - r$. ♣

## 2 Week2

**Exercies 2.1.** Prove following statments

- If $T_1 \subseteq T_2$ are subsets of $S^h$, then $Z(T_1) \supseteq Z(T_2)$.

- If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
- For any two subsets $Y_1, Y_2$ of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
- If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
- For any subset $Y \subset \mathbb{P}^n$, $Z(I(Y)) = \bar{Y}$.

Let $x \in Z(T_2)$, we have $f(x) = 0$ for all $f \in Y_2$. Since all $g \in T_1$ are all in $Y_2$, we have $g(x) = 0$. Hence $g \in Z(T_1)$. If $g \in I(Y_2)$, then $g$ vanishes on all $Y_2$, so on all $Y_1$. Hence $g \in I(Y_1)$. This also implies that both $I(Y_1)$ and $I(Y_2)$ contain $I(Y_1 \cup Y_2)$ since $Y_i \subseteq Y_1 \cup Y_2$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then $f$ vanishes on both $Y_1$ and $Y_2$, so $f \in I(V_1 \cup V_2)$ by definition.

If$f \in \sqrt{\mathfrak{a}}$, then there exists $n \geq 1$ such that $f^k \in \mathfrak{a}$. It implies every homogeneous part vanishes on $Z(\mathfrak{a})$. Let $f = f_1 + \cdots + f_n$ be the homogeneous decomposition of $f$. Then the homogeneous part of $f^k$ with degree $nk$ is $f_n^k$, so $f_n^k(P) = 0$ for all $P \in Z(\mathfrak{a})$. Therefore, $f_n(P) = 0$. By induction, we can conclude that $f_i(P) = 0$ for all $i$. Hence $f \in I(Z(\mathfrak{a}))$. Conversely, if $f \in I(Z(\mathfrak{a}))$. By homogeneous Nullstellensatz, we have $f_i^{r_i} \in \mathfrak{a}$. Let $r = r_1 + \cdots r_n$, then $f^r \in \mathfrak{a}$. Hence $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Since $Y \subseteq Z(I(Y))$ and $Z(I(Y))$ is closed, we have $\bar{Y} \subseteq Z(I(Y))$. There is homogeneous ideal $\mathfrak{a}$ such that $\bar{Y} = Z(\mathfrak{a})$. From $Y \subseteq Z(\mathfrak{a})$, we have $I(Y) \subseteq I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. Hence $\bar{Y} = Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}}) \subseteq Z(I(Y))$. We now conclude that $Z(I(Y)) = \bar{Y}$. ♣

**Exercies 2.2.** a) There is a $1-1$ inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of $S$ not equal to $S_+$ does not occur in this correspondence, it is sometimes called the irrelevant maximal of $S$.

b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

c) Show that $\mathbb{P}^n$ itself is irreducible.

a) If $\mathfrak{a}$ is radical homogeneous ideal of $S$ such that $Z(\mathfrak{a}) \neq 0$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$. If $Z(\mathfrak{a}) = 0$, then $I(Z(\mathfrak{a})) = I(\emptyset) = S$. $Z(\mathfrak{a}) = 0$ implies $\mathfrak{a} = S$ or $S_+$. By assumption, $S_+$ is not in the correspondence, so $\mathfrak{a} = S$. Hence $I \circ Z$ is identity functor. Similarly, $Z \circ I$ is also identity. With previous exercise, this correspondence is inclusion-reversing.

b) If $Y$ is irreducible, then for all $x, y \in I(Y)$, we can let $Y_1 = Z(x) \cap Y$ and $Y_2 = Z(y) \cap Y$. Since $Y_1 \cup Y_2 = (Z(x) \cap Y) \cup (Z(y) \cap Y) = Z(xy) \cap Y = Y$, $Y_1 = Y$ or $Y_2 = Y$ by irreducible condition. It implies that $Z(x) = Y$ or $Z(y) = Y$. So $x \in I(Y)$ or $y \in I(Y)$. Conversely, suppose $I(Y)$ is prime. However, for any closed cover $Y_1 \cup Y_2 = Y$, we have $I(Y) = I(Y_1) \cap I(Y_2)$, therefore $I(Y_1) = I(Y)$ or $I(Y_2) = I(Y)$. Hence $Y_1 = Y$ or $Y_2 = Y$ since they are closed.

c) $\mathbb{P}^n$ is algebraic set corresponding to radical homogeneous ideal $(0)$. It is prime ideal since $k[x_0, x_1 \cdots x_n]$ is integral domain. So $\mathbb{P}^n$ is irreducible from previous statement.

♣

**Exercies 2.3.** If $Y$ is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.

$Y$ is projective variety, so let $Y = Z(\mathfrak{p}) \subseteq \mathbb{P}^n$ for some prime homogeneous ideal $\mathfrak{p}$. Hence any descending chain of closed subset of $Y$ corresponds to a descending chain of radical homogeneous ideal containing $\mathfrak{p}$ with the same length. However, $S_+$ is prime homogeneous ideal of $S$ which contains any non-zero ideal of $S$ but doesn't correspond to a algebraic set. Hence the $\dim S(Y) > \dim Y$. Since radical ideals contains $\mathfrak{p}$ except $S_+$ also correspond to closed subsets of $Y$, $\dim Y \geq \dim S(Y) - 1$. Hence $\dim S(Y) = \dim Y + 1$. ♣

# 3 Week 3

**Exercies 3.1.** A regular function on projective variety is continuous map (view $k$ as affine line $\mathbb{A}^1$).

Let $Y \subseteq P^n$ be a projective variety, then $Y$ can be covered by affine varieties $U_i = Y \cap A_i^n$, where $A_i^n$ are canonical affine coverings of $\mathbb{P}^n$. If $f\colon Y \to k$ be a regular function, then its restrictions $f_{U_i}\colon U_i \to k$ are continuous maps, and since $U_i$ are all open in $Y$, $f$ itself is continuous on $Y$. ♣

**Exercies 3.2.** Let $\varphi : \mathbb{A}^1 \to C \hookrightarrow \mathbb{A}^2$ be curve defined as $t \mapsto (t^2, t^3)$. Obviously, $\varphi$ is $1-1$ correspondence. Prove $\varphi$ is not isomorphism between varieties.

Suppose the coordinate ring of $\mathbb{A}_k^2$ be $k[x,y]$. Then we have coordinate ring $A(C) \cong k[x,y]/(y^2 - x^3)$. From definition of $\varphi$, we can write down its pull-back on coordinate rings

$$\begin{aligned}\varphi^*\colon A(C) &\to k[t]\\ f &\mapsto f\circ\varphi\end{aligned}$$

If $\varphi$ is isomorphism, then $\varphi^*$ is isomorphism between coordinate ring. Therefore it is also bijection between regular functions. Since $t$ is regular function on $\mathbb{A}_k^1$, it must have preimage $f$ such that $\varphi^*(f) = t$. This means that $f(t^2, t^3) = t$. $f$ is regular on $C$, so it is also regular at point $(0,0)$. Nearby $(0,0)$, $f$ can be written as

$$\frac{\alpha(x,y)}{\beta(x,y)}$$

where $\alpha, \beta$ are polynomials in $k[x,y]$ and $\beta(0,0) \neq 0$, $\frac{\alpha(t^2,t^3)}{\beta(t^2,t^3)} = t$. It is impossible since $\alpha(t^2,t^3) = t\beta(t^2,t^3)$ cannot have term of degree 1. Hence we can conclude $\varphi$ can not be an isomorphism otherwise it will induce isomorphism between coordinate ring.

♣

**Exercies 3.3.** Let $S = Z(y_0y_2 - y_1^2)$ be the surface in $\mathbb{P}_k^2$ with the coordinates $(y_0 : y_1 : y_2)$. Let $\mathbb{P}_k^1$ be projective line with coordinate ring $k[x_0, x_1]$. Consider morphism

$$\begin{aligned}\varphi : \mathbb{P}_k^1 &\to S \subset \mathbb{P}_k^2\\ (x_0 : x_1) &\mapsto (x_0^2 : x_0x_1 : x_1^2)\end{aligned}$$

and show that $\varphi$ is isomorphism.

It is well-defined regular morphism since $(x_0^2)(x_1^2) - (x_0x_1)^2 = 0$ and with polynomial in each component. We can see that $\varphi$ is bijection and $\varphi^{-1}$ is defined as

$$\varphi^{-1}\colon (y_0 : y_1 : y_2) = \begin{cases}(y_0 : y_1) & \text{if } y_0 \neq 0\\ (y_1 : y_2) & \text{if } y_2 \neq 0\end{cases}$$

It is well-defined since we have $(y_0 : y_1) = (\frac{y_0}{y_1} : 1) = (\frac{y_1}{y_2} : 1) = (y_1 : y_2)$ when neither $y_0$ or $y_2$ are equal to 0. Moreover, it is regular map because it is regular on open subsets $S \cap \{y_0 \neq 0\}$ and $S \cap \{y_2 \neq 0\}$. With the fact that $\varphi\circ\varphi^{-1} = id_S, \varphi^{-1}\circ\varphi = id_{\mathbb{P}_k^1}$, we can conclude that $\varphi$ is isomorphism. ♣

# 4 Week 4

**Exercies 4.1.** Let $Y$ be an affine variety in $\mathbb{A}_k^n$. Show that $K(Y) \cong K(\mathcal{O}_{Y,p})$ for all $p \in Y$.

Since $Y$ is affine variety, we have $\mathcal{O}_{Y,p} \cong A(Y)_{m_p}$ for all point $p \in Y$. Therefore, $K(\mathcal{O}_{Y,p})$ is fraction field of local ring $A(Y)_{m_p}$. Moreover, $K(Y)$ is fraction field of coordinate ring $A(Y)$. Hence we have following commutative localization diagram

$$\begin{tikzcd} A(Y) \arrow[r, "l_{m_p}"] \arrow[d, "l"] & A(Y)_{m_p} \arrow[d, "l'"] \\ K(A(Y)) & K(A(Y)_{m_p}) \end{tikzcd}$$

$l_{m_p}$ maps non-zero divisor of $A(Y)$ to non-zero divisor, so $l' \circ l_{m_p}$ maps non-zero divisors to units. Therefore, with universal property of localization, there is unique homomorphism $g\colon K(A(Y)) \to K(A(Y)_{m_p})$ making this diagram commute. Also, with universal property of $l_{m_p}$, there is unique morphism $h$ making following diagram commute

$$\begin{tikzcd} A(Y) \arrow[r, "l_{m_p}"] \arrow[d, "l"] & A(Y)_{m_p} \arrow[dl, "h"] \\ K(A(Y)) & \end{tikzcd}$$

Now we get a new localization diagram

$$\begin{tikzcd} & A(Y)_{m_p} \arrow[dl, "h"'] \arrow[d, "l'"] \\ K(A(Y)) & K(A(Y)_{m_p}) \arrow[l, dashed, "\exists! f"] \end{tikzcd}$$

Hence we can conclude that $K(A(Y)) \cong K(A(Y)_{m_p})$, which implies that $K(Y) \cong K(\mathcal{O}_{Y,p})$ for all $p \in Y$. ♣

**Exercies 4.2.** Prove that for any integer $0 \leq i \leq n$

$$K(\mathbb{P}_k^n) \cong k(x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i)$$

For any variety $Y$, if $U$ is open subvariety of $Y$, then by definition $K(Y) \cong K(U)$. More precisely, we can send $\langle f, V \rangle$ to $\langle f, V \cap U \rangle$ and it is an well-defined isomorphism between function fields. Therefore, for any $0 \leq i \leq n$, we have $K(A_i) \cong K(\mathbb{P}_k^n)$, where $A_i$ is affine cover which is isomorphic to $\mathbb{A}_k^n$ with coordinate ring $k[x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i]$. With the conclusion in last exercise, we can conclude that

$$K(\mathbb{P}_k^n) \cong K(A_i) \cong k(x_0/x_i, \cdots, \widehat{x_i/x_i}, \cdots, x_n/x_i)$$

**Exercies 4.3.** Equation $x_0^2 + x_1^2 + x_2^2 = 0$ defines a conic $X \hookrightarrow \mathbb{P}_k^2$. Find $t \in K(X)$ such that $K(X) \cong k(t)$ is a transcendental extension of $k$ with degree 1.

$K(X)$ is equal to function field of affine open subset defined by

$$(\frac{x_1}{x_0})^2 + (\frac{x_2}{x_0})^2 = -1 \tag{1}$$

Consider linear transform

$$y_1 = \frac{x_1}{x_0} + i\frac{x_2}{x_0} \qquad y_2 = \frac{x_1}{x_0} - i\frac{x_2}{x_0} \tag{2}$$

Then the equation 4 becomes

$$y_1 y_2 = -1 \tag{3}$$

Hence $K(X) = k(y_1, y_2) = k(y_1)$ since $y_2 = -1/y_1$. Therefore $y_1$ is the required $t$. ♣

# 5 Week5

**Exercies 5.1.** Suppose $S = Z(xy - zw) \subset \mathbb{P}_k^3 = \operatorname{Proj} k[x,y,z,w]$. Let $H \subset \mathbb{P}_k^3$ the hyperplane in $\mathbb{P}_k^3$ defined by $x = 0$. It is isomorphic to $\mathbb{P}_k^2$.

$$\begin{aligned} \varphi\colon S &\dashrightarrow H \\ (x:y:z:w) &\mapsto (0:y:z-x:w) \end{aligned}$$

Prove that $\varphi$ is birational map.

*Proof.* Actually, we define its rational inverse as

$$\begin{aligned} \varphi^{-1}\colon H &\dashrightarrow S \\ (0:s_1:s_2:s_3) &\mapsto (\frac{s_2s_3}{s_1-s_3}:s_1:\frac{s_1s_2}{s_1-s_3}:s_3) \end{aligned}$$

It is defined on $H$ on the open subset $\{s_1 \neq s_3\} \cap H$ and $\varphi^{-1}(H) \cap (S \cap A_i) \neq \emptyset$ for all canonical affine covers $A_i$ of $\mathbb{P}_k^3$. It means $\varphi^{-1}$ is well-defined dominant rational map on $H$. Furthermore, its compositions with $\varphi$ are all identities as rational maps since

$$\varphi((\frac{s_2s_3}{s_1-s_3}:s_1:\frac{s_1s_2}{s_1-s_3}:s_3)) = (0:s_1:\frac{s_2(s_1-s_3)}{s_1-s_3}:s_3) = (0:s_1:s_2:s_3)$$

and

$$\begin{aligned} \varphi^{-1}((0:y:z-x:w)) &= (\frac{(z-x)w}{y-w}:y:\frac{y(z-x)}{y-w}:w) \\ &= (\frac{xy-xw}{y-w}:y:\frac{yz-zw}{y-w}:w) \\ &= (x:y:z:w) \end{aligned}$$

♣

**Exercies 5.2.** If $Y, Z \subset \mathbb{A}_k^2$ are two distinct curves given by equations $f = 0, g = 0$, and if $p \in Y \cap Z$, we define the intersection multiplicity $(Y \cdot Z)_p$ at point $p$ to be the length of the $\mathcal{O}_p$-module $\mathcal{O}_p/(f,g)$. Show that

- $(Y \cdot Z)_p$ is finite and $(Y \cdot Z)_p \geq \mu_p(Y)\mu_p(Z)$
- If $p \in Y$ show that for almost all lines $l$ through $p$, we have $(l \cdot Y)_p = \mu_p(Y)$.
- If $Y$ is a curve of degree $d$ in $\mathbb{P}_k^2$, and if $l$ is a line in $\mathbb{P}_k^2$, $l \neq Y$, show that $(l \cdot Y) = d$. Here we define $(l \cdot Y) = \sum_p (l \cdot Y)_p$ taken over all points $p \in l \cap Y$, where $(l \cdot Y)_p$ is defined using a suitable affine cover of $\mathbb{P}_k^2$.

*Proof.*
- To show $(Y \cdot Z)_p$ is finite, we just need to prove $\mathcal{O}_p$-module $\mathcal{O}_p/(f,g)$ is of finite length. This is equivalent to prove that $\mathcal{O}_p/(f,g)$ is both Artinian and Noetherian module. Since $\mathcal{O}$ is localization of Noether ring $k[x,y]$, it is also Noether. The natural morphism $\mathcal{O}_p \twoheadrightarrow \mathcal{O}_p/(f,g)$ implies that $\mathcal{O}_p/(f,g)$ is finitely generated $\mathcal{O}_p$-module, so it is Noetherian module. Furthermore, $\mathcal{O}_p/(f,g)$ is of Krull dimension 0, because of $\dim \mathcal{O}_p = 2$ together with Krull's principal ideal theorem. Therefore $\mathcal{O}_p/(f,g)$ is Artinian ring since $\mathcal{O}_p/(f,g)$ is Noether ring and it is also Artinian module as $\mathcal{O}_p$-module. Suppose $Y$ and $Z$ intersect at $(0,0)$.

- Make a linear change of the coordinates so that $p = (0,0)$. Then $\mathcal{O}_p = k[x,y]_{(x,y)} \cong k[x,y]$. Therefore, any line through $p$ can be defined by equation $ax + by = 0$. If $b \neq 0$, then $y = -\frac{a}{b}x$. In this case, $\mathcal{O}_p/(f, ax+by) \cong k[x]/(f(x, -a/bx))$. Let $\mu_p(Y) = r$ and $f$ is of degree $d$, then $f$ can be written in the form

$$f(x, -\frac{a}{b}x) = c_0x^r + c_1x^{r+1} + \cdots + c_{d-r}x^d = c_0x^r(1 + \cdots + c_{d-r}x^{d-r})$$

  $1 + \cdots c_{d-r}x^{d-r}$ can be viewed as polynomial of $\frac{a}{b}$ with coefficients in $k[x]$, so there are only finite $a$ such that $1 + \cdots c_{r-d}x^{r-d} = 0$. Hence $\mathcal{O}_p/(f, ax+by) \cong k[x]/(c_0x^r)$ since $c_0x^r$ and $1+\cdots c_{d-r}x^{d-r}$ are coprime except finite number of $a$. Therefore, length of $\mathcal{O}_p/(f, ax+by) = r$. If $b = 0$, it determines only line in $\mathbb{A}_k^2$. Hence we reach the conclusion.

- $Y$ is curve in $\mathbb{P}_k^2$, so $Y$ is determined by homogeneous polynomial $f$. Let $A_i = \{x_i \neq 0\} \cap \mathbb{P}^2$ be the three affine covers of $\mathbb{P}_k^2$. Under linear transformation, we can make $l$ through $(0:0:1)$, hence equation of $l$ can be written as

$$ax + by = 0$$

  In this case $l$ can be covered by $A_1 \cup A_3$ if $b \neq 0$. Then

$$(l \cdot Y) = \sum_{p \in A_1 \cap l \cap Y} (l \cdot Y)_p + (l \cdot Y)_{(0:0:1)} \tag{4}$$

  Suppose $Y$ is determined by homogeneous polynomial $f$. On $A_1$, the $Y \cap l$ is determined by

$$f(1, -\frac{b}{a}, z/x) = 0$$

  Let $F(t) = f(1, -\frac{1}{b}, t)$ is of degree $r$. Then the first term of right part of 4 is equal to $r$. With last exercise, we know $(l \cdot Y)_{(0:0:1)} = \mu_{(0:0:1)}(Y) = d - r$. Hence, we can conclude that $(l \cdot Y) = d$

♣

# 6 Week6

**Exercies 6.1.** Let $Y$ be the curve $y^2 = x^3 - x$ in $\mathbb{A}_k^2$, and assume that the characteristic of the base field $k$ is not 2. We can find that $Y$ is nonsingular and $A = A(Y)$ is integral closed domain. Furthermore, $k[x] \subset K = K(Y)$ is a polynomial subring and $A$ is the integral closure of $k[x]$ in $K$. There is an automorphism $\sigma : A \to A$ which sends $y$ to $-y$ and leaves $x$ fixed. We define the norm $N$ of $A$ which maps $a$ to $N(a) = a \cdot \sigma(a)$ for any $a$ in $A$.

Using this norm, show that the units in $A$ are precisely the non-zero elements of $k$. Show that $x$ and $y$ are irreducible elements of $A$. Show that $A$ is not a UFD.

**Exercies 6.2.** We know that if $X$ is a quasi-projective curve and $\varphi \colon X \backslash P \to Y$ where $P \in X$ and $Y$ is a projective variety, there is a unique morphism $\bar{\varphi} \colon X \to Y$ extending $\varphi$. But it is not true when $\dim X \geq 2$.

Let $\varphi \colon \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ be the Cremona transformation which is defined on $\mathbb{P}_k^2 - \{(1:0:0), (0:1:0), (0:0:1)\}$ and sends $(x:y:z)$ to $(yz:xz:xy)$, show that $\varphi$ can not extend to a morphism from $\mathbb{P}_k^2$ to $\mathbb{P}_k^2$.

*Proof.* If $\varphi$ can be extended to a morphism $\tilde{\varphi}$ on $\mathbb{P}^2_k$. Consider the embedding of projective line $i_1\colon \mathbb{P}^1_k \hookrightarrow \mathbb{P}^2_k$ by sending $(x:y)$ to $(x:y:0)$. Then points $(1:0:0)$ and $(0:1:0)$ are in $i_1(\mathbb{P}^1_k)$. If $p \in \mathbb{P}^1_k$ is not equal to $(1:0)$ or $(0:1)$, then $\tilde{\varphi} \circ i_1(p) = (0:0:xy) = (0:0:1)$. Hence $\mathbb{P}^1_k - (1:0) - (0:1) \subseteq S = (\tilde{\varphi} \circ i_1)^{-1}((0:0:1))$. It implies

$$\mathbb{P}^1_k = \overline{\mathbb{P}^1_k - (1:0) - (0:1)} \subseteq S \subseteq \mathbb{P}^1_k$$

Therefore, $S = \mathbb{P}^1_k$. This means $\tilde{\varphi}((1:0:0)) = \tilde{\varphi}((0:1:0)) = (0:0:1)$. However, if we take another embedding $i_2$ of projective line into $\mathbb{P}^2_k$,sending $(x:y)$ to $(x:0:y)$, then we can get $\tilde{\varphi}((1:0:0)) = \tilde{\varphi}((0:0:1)) = (0:1:0)$. So we cannot define the value of $(1:0:0)$ properly. Hence we can conclude that $\varphi$ cannot extend to a morphism on $\mathbb{P}^2_k$. ♣

**Exercies 6.3.** Think $\mathbb{P}^1_k$ as $\mathbb{A}^1_k \cup \infty$. Then we define a fractional linear transformation of $\mathbb{P}^1_k$ by sending $x \mapsto (ax+b)/(cx+d)$, for $a,b,c,d \in k$ and $ad - bc \neq 0$

- Show that a fractional linear transformation induces an automorphism of $\mathbb{P}^1_k$ (i.e., an isomorphism of $\mathbb{P}^1_k$ with itself). We denote the group of all these fractional linear transformations by $\mathrm{PGL}(1)$.
- Let $\mathrm{Aut}(\mathbb{P}^1_k)$ denote the group of all automorphisms of $\mathbb{P}^1_k$. Show that $\mathrm{Aut}(k(x))$, the group pf $k$-automorphisms of the fractional field $k(x)$.
- Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $\mathrm{PGL}(1) \to \mathrm{Aut}(\mathbb{P}^1_k)$ is an isomorphism.

*Proof.*

- Let $\infty = (1:0) \in \mathbb{P}^1_k$. Then fractional linear transformation $x \mapsto (ax+b)/(bx+d)$ can be written in coordinates as

  $$(x:y) \mapsto (ax+by : cx+dy)$$

  It is well-defined regular morphism since its each component is polynomial of $x, y$ and $(ax+by : cx+dy) \neq (0:0)$ since $ad - bc \neq 0$. Take the inverse matrix of

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  We get the inverse of the fractional linear transformation

  $$(m:n) \mapsto (\frac{dm - bn}{ad - bc} : \frac{-cm + an}{ad - bc})$$

  Hence it is automorphism on $\mathbb{P}^1_k$.

- First, we note that function field $K(\mathbb{P}^1_k) \cong K(\mathbb{A}^1_k) \cong k(x)$. Taking function field of variety is functional, hence automorphism of $\mathbb{P}^1_k$ induces automorphism of $k(x)$ and it is $k$-linear since it actually induces morphism of $k$-algebras. Conversely, any $k$-automorphism $\varphi$ of $k(x)$ is determined by the image $\varphi(x)$. Let $f = \varphi(x) \in k(x)$, then we define $\phi\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ as follows

  $$\phi((x:y)) = \begin{cases} (f(x/y):1) & \text{if } y \neq 0 \\ (1:f(y/x)) & \text{if } x \neq 0 \end{cases}$$

  Or equivalently $\phi(x:y) = (f(x):f(y))$. Since $f \in k(x)$ and $\varphi$ has inverse, $\phi$ is well-defined automorphism of $\mathbb{P}^1_k$. Furthermore, we have $K(\phi) = \varphi$, which is induced isomorphism by $\mathbb{P}^1_k$-automorphism on its function field. Hence we can conclude that $\mathrm{Aut}(\mathbb{P}^1_k) \cong \mathrm{Aut}(k(x))$.

- Since $f \in k(x)$, it can be written as

  $$f(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{b_0 + b_1 x + \cdots + b_n x^n}$$

  Let $y = f(x)$. Because $\varphi$ is isomorphism, $f(x)$ must have form

  $$f(x) = \frac{a_0 + a_1 x}{b_0 + b_1 x}$$

  Since $\varphi$ is determined by $f(x)$, there is one-to-one correspondence between $\mathrm{PGL}(1)$ and $\mathrm{Aut}(k(x))$ and it is isomorphism of groups.

♣

# 7 Week 8

**Exercies 7.1.**

- Find the degree of the $d$-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ where $N = \binom{n+d}{n} - 1$.
- Find the degree of the Serge embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$ where $N = r + s + rs$.

*Proof.* The $d$-uple embedding is defined as

$$\rho_d \colon P = (a_0 : \cdots : a_n) \mapsto (M_0(a) : \cdots : M_N(a))$$

where $M_i$ are all monomial of degree $d$ with variables $a_0, \cdots, a_n$. Let $S(\mathbb{P}^N) = k[y_0, \cdots, y_N]$ and $S(\mathbb{P}^n) = k[x_0, \cdots, x_n]$. $\rho_d$ induce $k$-algebra homomorphism

$$\varrho_d \colon y_i \mapsto M_i(x_0, \cdots, x_n) = x_0^{i_0} \cdots x_n^{i_n}$$

Suppose kernel of this morphism is $\mathfrak{a}$. It is homogeneous prime ideal corresponds to image of $\rho_d$. Hence the coordinate ring of image $M$ of $\rho_d$ is $S(M) = k[y_0, \cdots, y_N]/\mathfrak{a} \cong \operatorname{Im} \varrho_d$. It means that

$$S(M) = \bigoplus_{l=0} S(M)_l \cong \bigoplus_{l=0} k[x_0, \cdots, x_n]_{ld}$$

where $S(M)_l \cong k[x_0, \cdots, x_n]_{ld}$ is $\binom{ld+n}{n}$ dimensional vector space. Hence the degree $M$ is $d^n$.

Similarly as $d$-uple embedding, the ideal of image of Segre embedding is kernel of

$$\begin{aligned} \theta \colon k[z_{00}, \cdots, z_{ij}, \cdots, z_{rs}] &\to k[x_0, \cdots, x_r; y_0, \cdots, y_s] \\ z_{ij} &\mapsto x_i y_j \end{aligned}$$

We denote its kernel by $\mathfrak{b}$, hence $S(X) \cong k[z_{00}, \cdots, z_{ij}, \cdots, z_{rs}]/\mathfrak{b} \cong \operatorname{Im} \theta$. Hence

$$S(X) = \bigoplus_{j=0} S(X)_j \cong \bigoplus_{j=0} k[x_0 y_0, \cdots, x_r y_s]_j \cong \bigoplus_{j=0} k[x_0, \cdots, x_r]_j \times k[y_0, \cdots, y_s]_j$$

where $x_i y_j$ is of grade 1. Hence $\dim_k S(X)_j = \binom{j+r}{j}\binom{j+s}{j}$. ♣

**Exercies 7.2.** Show that an algebraic set $Y$ of pure dimension $r$ (i.e., every irreducible component of $Y$ has dimension $r$) has degree 1 if and only if $Y$ is a linear variety.

*Proof.* If $Y$ is linear variety of codimenision 1 in $\mathbb{P}^n$, then $Y$ is variety in $\mathbb{P}^n$ determined by single linear equation, so it is just hyperplane in $\mathbb{P}^n$. Then we assume that all linear varieties of codimension $k$ is of degree 1. For some linear variety $Y^{k+1}$ of codimension $k+1$, it intersection of a hyperplane $H$ and a linear variety $Y^k$ of codimension $k$, so with theorem 7.7, we have

$$i(Y^k, H; Y^{k+1}) \deg Y^{k+1} = \deg Y^k \deg H = 1$$

this means that $i(Y^k, H; Y^{k+1}) = \deg Y^{k+1} = 1$. Hence a linear variety is of degree 1. Conversely, let $Y^1 = Z(I)$ be some algebraic set in $\mathbb{P}^n$ of pure codimension 1 and $n \geq 2$. Assume $Y = \cup_i Y_i$ it the decomposition of irreducible components of $Y$. For each $Y_i$, we can choose two distinct points $P, Q$ and a hyperplane $H$ in $\mathbb{P}^n$ which contains this two points. The theorem 7.7 implies that the degree of $Y_i$ is greater than 2 if $Y_i$ is not contained in $H$. So if $Y_i$ is of degree 1, then $Y_i$ is contained in a hyperplane in $\mathbb{P}^n$. Hence $Y_i$ is just this hyperplane since they are irreducible and of same dimension $n-1$. So each components of $Y^1$ is hyperplane in $\mathbb{P}^n$, hence $Y^1$ itself is irreducible hyperplane since every two hyperplane intersects with each other. If $n = 1$, then $Y$ is just single point of degree 1 or $\mathbb{P}^1$ itself, of course it is linear. Now, by induction, we can assume it is true for algebraic set of pure codimension $k$ with degree 1 that it is linear. For a algebraic set $Y^{k+1}$ in projective space $\mathbb{P}^n$. Suppose $Y_i^{k+1}$ is one of its irreducible components. We can also choose two distinct points $P, Q$ in $Y_i^{k+1}$ and it also implies $Y_i^{k+1}$ is contained in a hyperplane $H$. Take an isomorphism $H \cong \mathbb{P}^{n-1}$, then $Y_i^{k+1}$ can be viewed as variety in $\mathbb{P}^{n-1}$ with codimension $k$ of degree 1 ( since all mentioned embeddings are closed embeddings). Hence it is linear variety. We have already known that degree of algebraic set is sum the degrees of its irreducible components. Hence $Y^{k+1}$ is irreducible, and furthermore it is linear variety. ♣

**Exercies 7.3.** Let $Y^r$ be a $r$-dimensional variety of degree 2. Show that $Y$ is contained in a linear subspace $L$ of dimension $r+1$ in $\mathbb{P}^n$. Thus $Y$ is isomorphic to a quadratic hypersurface in $\mathbb{P}^{r+1}$.

*Proof.* Choose non-singular point $x \in Y^r$. Then the union of all lines through $x$ and some other points in $Y^r$ is an $r+1$-dimensional variety of degree 1, denoted by $P$. Therefore, combine statement of previous exercise, we can conclude that $P$ is linear.

Or we can choose three distinct points in $Y^r$, then we can find a hyperplane $H$ in $\mathbb{P}^n$ that contains such three points. Since $Y^r$ is of degree 2, then $Y^r$ musted be contained in $H$. With linear transformation $H \cong \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$. Hence $Y^r$ is variety in $\mathbb{P}^{n-1}$ with degree 2. Repeating this process, we can closed embed $Y^r$ in $\mathbb{P}^{r+1} \hookrightarrow \mathbb{P}^n$ with linear transformation. Therefore $Y^r$ is contained in a $r+1$-dimensional linear subspace of $\mathbb{P}^n$. Hence $P$ is linear subspace of $\mathbb{P}^n$ of dimension $r+1$ which contains $Y^r$. This implies that $Y^r$ is hypersurface of some $\mathbb{P}^{r+1} \hookrightarrow \mathbb{P}^n$ with degree 2. It is defined by polynomial of degree 2. Therefore we can conclude that it is quadratic in $\mathbb{P}^{r+1}$. ♣

# 8 Week 9

**Exercies 8.1.** Let $P = (0,0)$, $C = (x^2+y^2)^2 + 3x^2y - y^3$ and $D = (x^2+y^2)^3 - 4x^2y^2$, directly calculate the intersection number $i(P, C \cap D)$.

*Proof.* We need to compute ideal $((x^2+y^2)^2+3x^2y-y^3, (x^2+y^2)^3-4x^2y^2)$ in local ring $k[x,y]_{(x,y)} \cong k[x,y]$. ♣

**Exercies 8.2.** Suppose $\operatorname{char} k \neq 2$ and let $X$ be a nonsingular projective curve of genus1, then for any effective divisor $D$ on $X$ we have $l(D) = \deg D$ and thus for $P \in X$ a point we have $l(P) = 1$. There is no element in $K(X)$ has one pole exactly at $P$. Hence, $L(2P) = (1, x) \subseteq K(X)$, the bracket here is linear span with given base, where $P$ is two order point of $x \in K(X)$ and $L(3P) = (1, x, y) \subseteq K(X)$ where $P$ is three order pole of $y$.

- Show $L(4P) = (1, x, y, x^2)$ and $L(5P) = (1, x, y, x^2, xy)$.

  *Proof.* For two degree terms $x^2, xy, y^2$, we have

  $$\operatorname{ord}_p(x^2) = 4 \quad \operatorname{ord}_p(xy) = 2+3 = 5 \quad \operatorname{ord}_p(y^2) = 2 \times 3 = 6$$

  Hence $x^2 \in L(4P)$ and others are not. It also implies $L(5P) = (1, x, y, x^2, xy)$. ♣

- Derive the Weierstrass equation for genus 1 curves.

  *Proof.* For three degree terms $x^3, x^2y, xy^2, y^3$, we have

  $$\operatorname{ord}_P(x^3) = 6 \quad \operatorname{ord}_P(x^2y) = 7 \quad \operatorname{ord}_P(xy^2) = 8 \quad \operatorname{ord}_P(y^3) = 9$$

hence we have $(1, x, y, x^2, y^2, xy, x^3) \subseteq L(6P)$. While Riemann-Roch tells us $L(6P) \leq 6$, these generators are linear dependent. Therefore we have

$$y^2 + b_0xy + b_1y = a_3x^3 + a_2x^2 + a_1x + a_0$$

with suitable linear transformation we can assume

$$y^2 = x^3 + ax + b$$

This is Weierstrass equation for genus 1 curve. Since we have assume that $X$ is non-singular,

$$x^3 + ax + b = 0$$

should has three distinct roots. Otherwise, if $x^3 + ax + b = (x - \alpha)^2(x - \beta)$, then the Jacobian of $y^2 - x^2 - ax - b$ at point $(\alpha, 0)$ vanishes. Therefore let

$$y^2 = \prod_{i=1}^{3}(x - \tau_i)$$

hence with embedding $(x, y) \mapsto (1 : x : y)$, $X$ is birational to a cubic curve in $\mathbb{P}^2$ defined by

$$zy^2 = x^3 + axz^2 + bz^3$$

♣