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Introduction to Feedback Control and Simulation

Workshop

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Department of Mechanical Engineering
Massachusetts Institute of Technology

Outline

- System Modeling
  - ODE functions for system dynamics
  - Laplace transformation and transfer function
- System Characterization
  - Step response
  - Frequency response and bode plot
- Closed Loop Control Algorithm
  - Closed loop control overview
  - PID controller
  - Loop shaping design
- System Identification Methods
  - Frequency response identification

System Modeling

- Using physical laws, the behavior of certain systems can be described using ordinary differential equation(s)
- First and second order ODEs are easy to solve and correspond to many physical models such as RLC circuit, mass-spring-damper model for mechanical systems as shown below

$$m\ddot{x} + c\dot{x} + kx = u$$

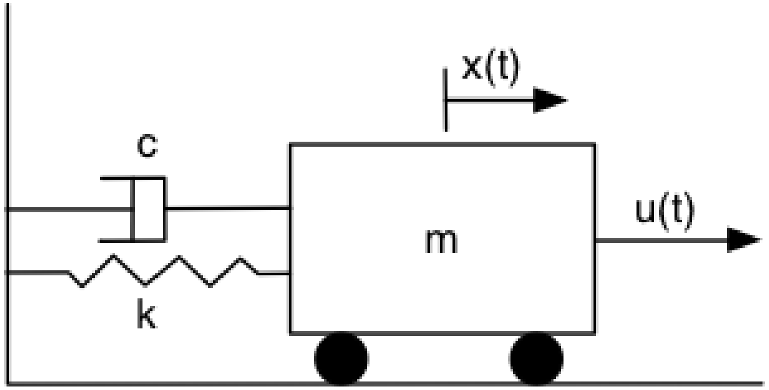
x : position of mass [m] at time t [s]

m : mass [kg]

c : viscous damping coefficient [N s / m]

k : spring constant [N / m]

u : force input [N]



Laplace Transformation

- Laplace transformation simplifies the analysis process for ODES
- Easy representation
- Solving with partial fraction expansion and lookup tables
- Definition

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

First Derivative	$\frac{df(t)}{dt} \overset{L}{\longleftrightarrow} sF(s) - f(0^-)$
Second Derivative	$\frac{d^2 f(t)}{dt^2} \overset{L}{\longleftrightarrow} s^2 F(s) - sf(0^-) - \dot{f}(0^-)$

Common Laplace Transform Pairs

Time Domain Function		Laplace Domain Function
Name	Definition*	
Unit Impulse	$\delta(t)$	1
Unit Step	$\gamma(t)$ †	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Parabola	t^2	$\frac{2}{s^3}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Asymptotic Exponential	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
Dual Exponential	$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
Asymptotic Dual Exponential	$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$
Time multiplied Exponential	te^{-at}	$\frac{1}{(s+a)^2}$
Sine	$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
Cosine	$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$
Decaying Sine	$e^{-at}\sin(\omega_d t)$	$\frac{\omega_d}{(s+a)^2+\omega_d^2}$
Decaying Cosine	$e^{-at}\cos(\omega_d t)$	$\frac{s+a}{(s+a)^2+\omega_d^2}$
Generic Oscillatory Decay	$e^{-at}\left[B\cos(\omega_d t)+\frac{C-aB}{\omega_d}\sin(\omega_d t)\right]$	$\frac{Bs+C}{(s+a)^2+\omega_d^2}$
Prototype Second Order Lowpass, underdamped	$\frac{\omega_0}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)$	$\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$
Prototype Second Order Lowpass, underdamped - Step Response	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t+\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_0^2}{s(s^2+2\zeta\omega_0 s+\omega_0^2)}$

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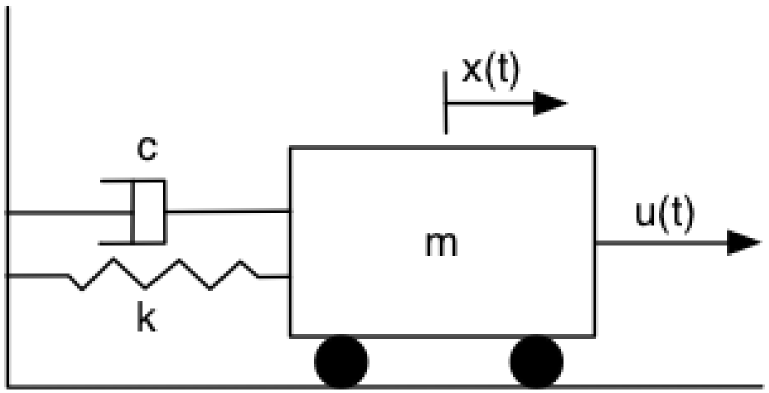
Transfer Function

Transfer Function

- Transfer function ignore initial conditions and represent the system response with zero initial conditions
- Transfer function can be cascaded into block diagram form and multiplied to represent complex systems

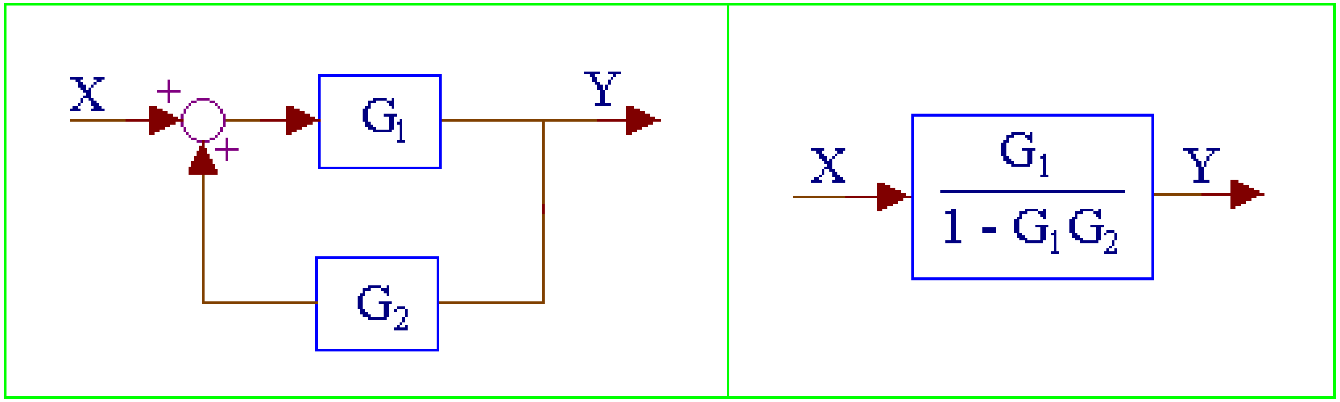
$$m\ddot{x} + c\dot{x} + kx = u$$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + cs + k}$$



Block Diagram Representation

$$Y = G_1 X + G_2 G_1 Y = \frac{G_1}{1 - G_1 G_2} X$$

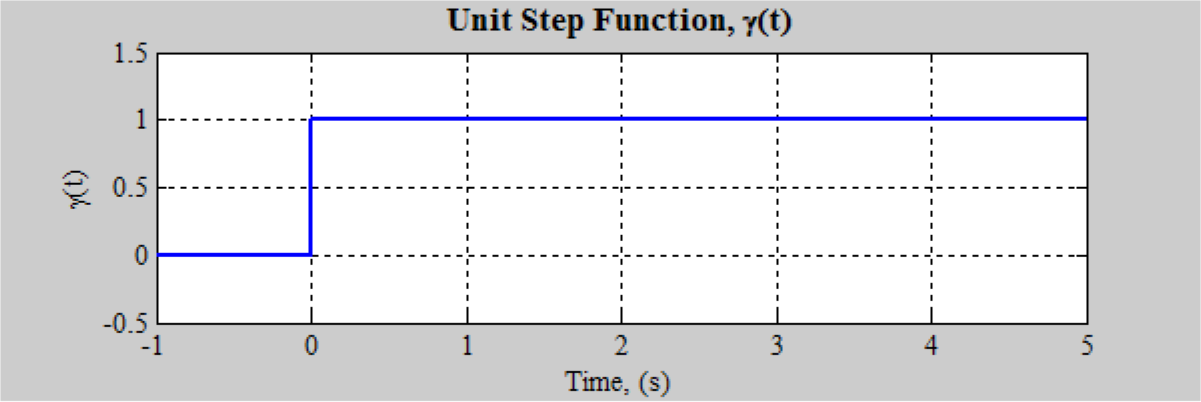


Step Response

- Definition: System response to a unit step function input with zero initial conditions

$$\gamma(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$\Gamma(s) = \frac{1}{s}$$



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1^{st} Order System Step Response Example

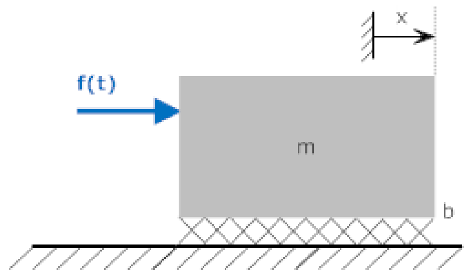
$$m\dot{v} + bv = f(t)$$

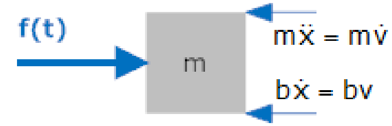
$$msV(s) + bV(s) = F(s)$$

$$\frac{V(s)}{F(s)} = H(s) = \frac{1}{ms+b} = \frac{1/m}{s+b/m}$$

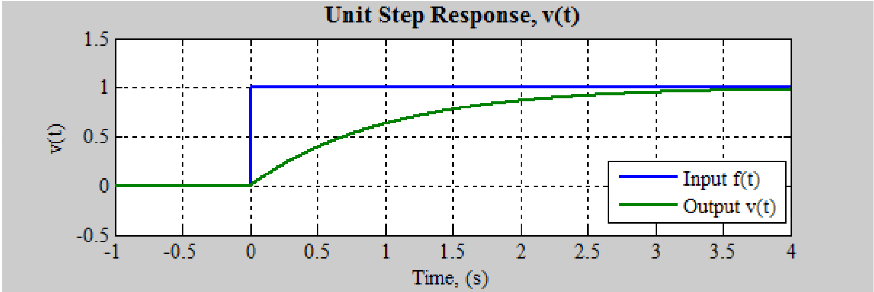
$$V(s) = F(s)H(s) = \frac{1}{s}\frac{1/m}{s+b/m}$$

$$v(t) = \frac{1}{b}\left(1 - e^{-(b/m)t}\right)$$





$v(t) = 1 - e^{-t}$



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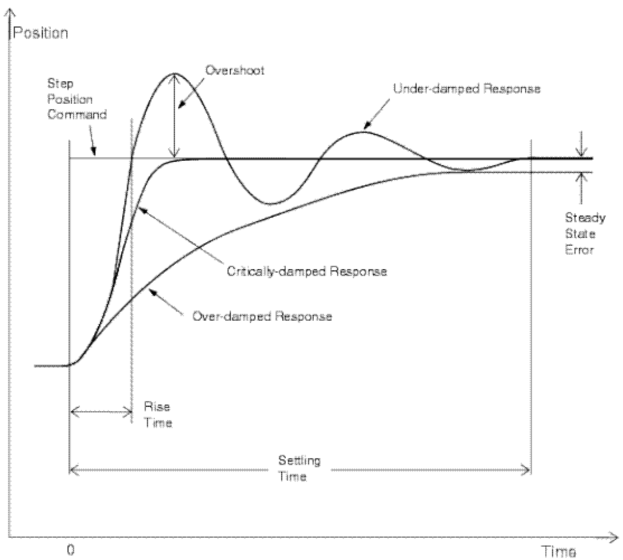
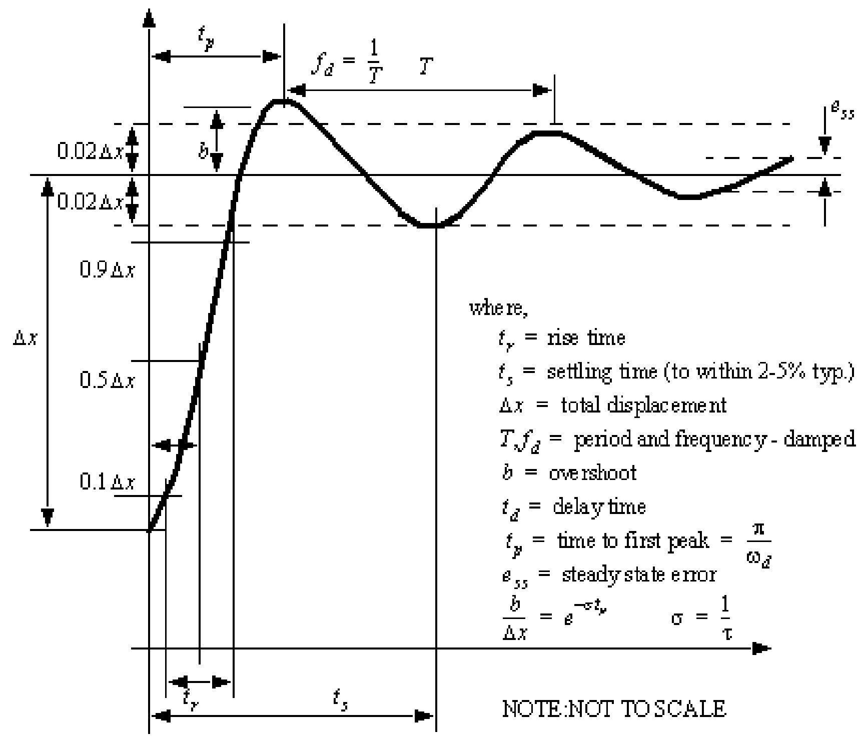
2st Order System Step Response Example

$$Y(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

$$y(t) = \left[1 - \frac{1}{\sqrt{1-\zeta^2}} \exp\left(-\zeta\omega_n t\right) \sin\left(\omega_d t + \operatorname{atan2}\left(\sqrt{1-\zeta^2}, \zeta\right)\right)\right] U_s(t)$$





where,

t_r = rise time

t_s = settling time (to within 2-5% typ.)

Δx = total displacement

T, f_d = period and frequency - damped

b = overshoot

t_d = delay time

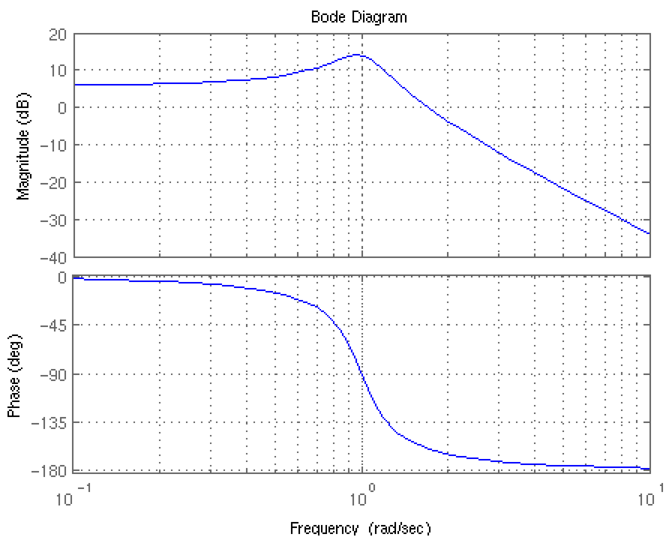
t_p = time to first peak = $\frac{\pi}{\omega_d}$

e_{ss} = steady state error

$\frac{b}{\Delta x} = e^{-\sigma t_p}$ $\quad \sigma = \frac{1}{\tau}$

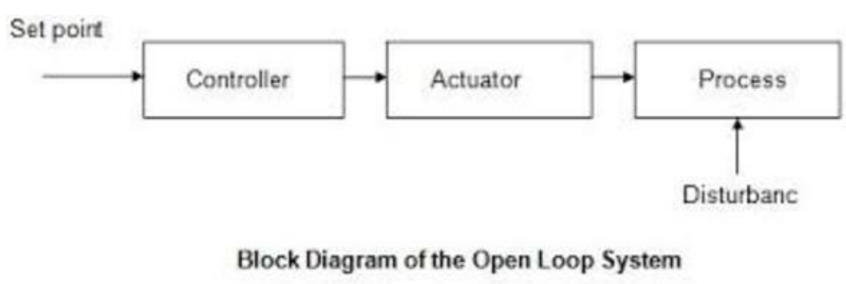
Frequency Response

- Definition: Frequency response characterize the output amplitude and phase with sinusoidal input
- Bode plot is used to give frequency response information
- Bode plot gain and phase can be calculated by substituting the s variable with jω and compute the complex number amplitude and phase
- Bode plot plays an important role in classical control theory

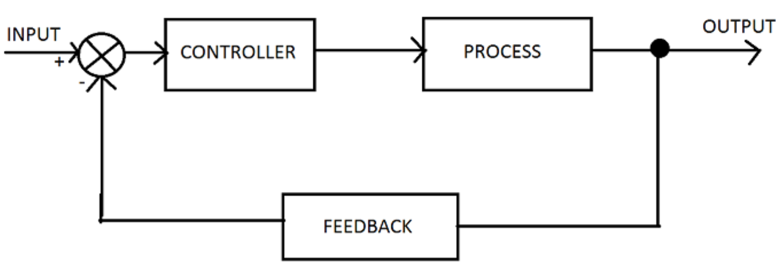


Closed Loop Control Overview

- Open loop controller utilize physical model of system
- Closed loop controller requires sensor to provide feedback signal
- Closed loop controller is more robust to external disturbance



Open loop system



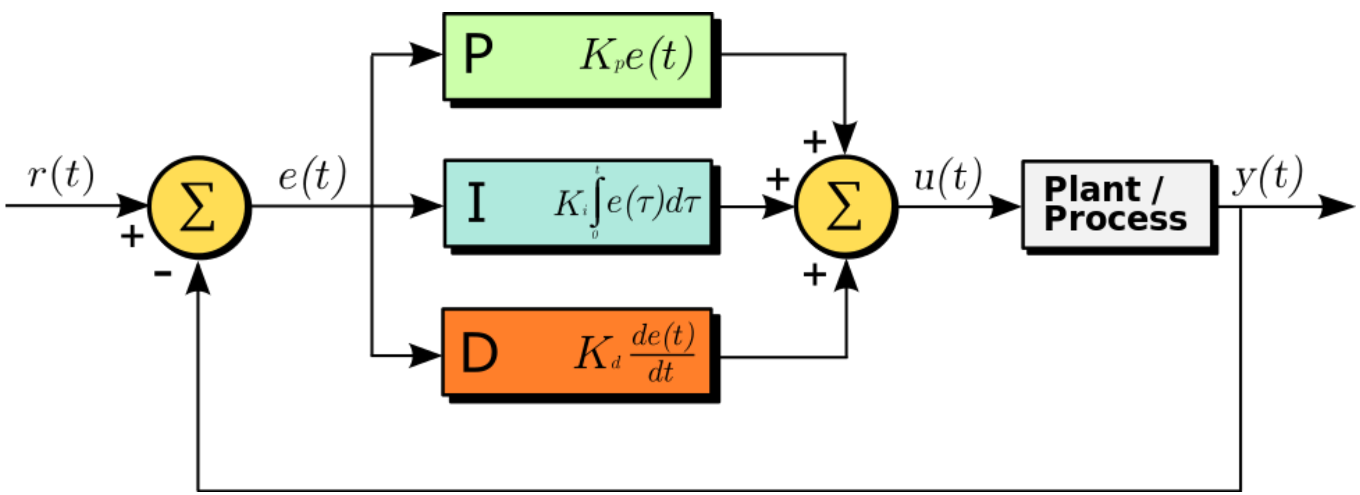
Closed loop system

Control Algorithm Overview

- Poles are roots of the denominator of the transfer function
- Zeros are roots of the numerator of the transfer function
- Closed loop control aims to stabilize the system and place the dominant poles at the desired location
- Common closed loop control tools go as the following
  - PID controller
  - Root-locus design
  - Loop shaping design and Nyquist plot
  - State space model
- Advanced control techniques
  - Adaptive control and learning
  - Robust control
  - Non-linear system control

PID Controller

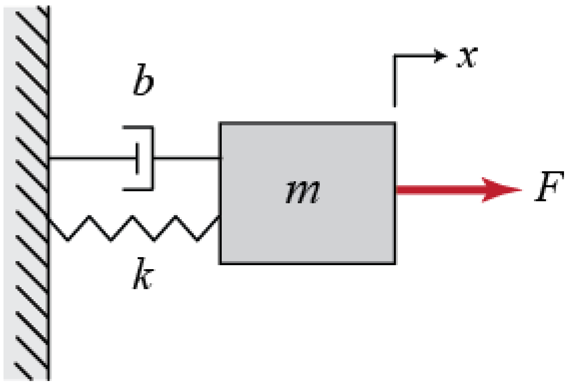
- Proportional, Integral and Derivative Controller



PID Controller

- When you are designing a PID controller for a given system, follow the steps shown below to obtain a desired response.
  - Obtain an open-loop response and determine what needs to be improved
  - Add a proportional control to improve the rise time
  - Add a derivative control to improve the overshoot
  - Add an integral control to eliminate the steady-state error
  - Adjust each of Kp, Ki, and Kd until you obtain a desired overall response. You can always refer to the table shown in this "PID Tutorial" page to find out which controller controls what characteristics.
- Not all PID terms are necessary

PID Controller Design Example

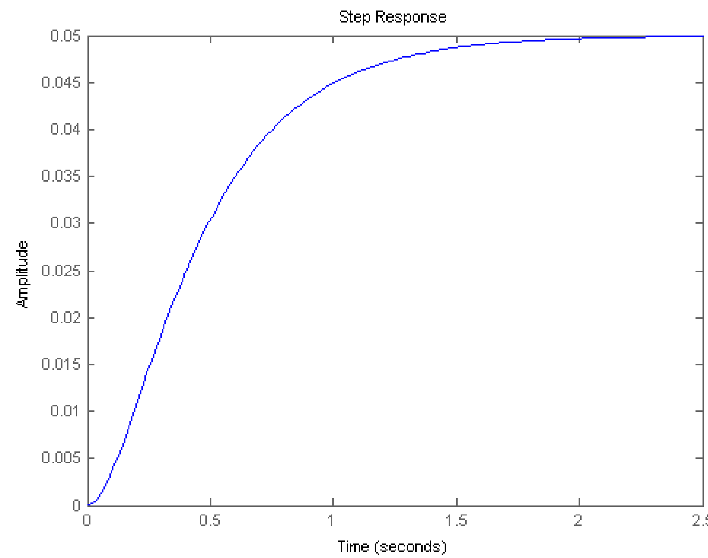


$$M\ddot{x} + b\dot{x} + kx = F$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + bs + k}$$

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 10s + 20}$$

```
M = 1 kg
b = 10 N s/m
k = 20 N/m
F = 1 N
```



System Step Response

```
s = tf('s');
P = 1/(s^2 + 10*s + 20);
step(P)
```

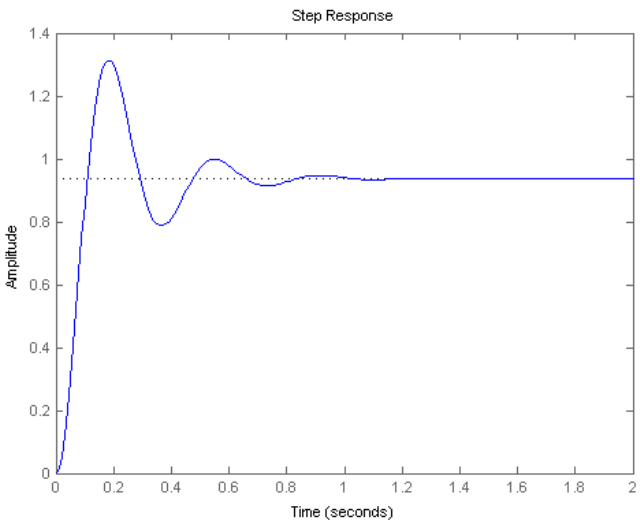
P Controller Example

$$\frac{X(s)}{F(s)} = \frac{K_p}{s^2 + 10s + (20 + K_p)}$$

- The proportional controller reduced both the rise time and the steady-state error, increased the overshoot, and decreased the settling time by small amount.

```
Kp = 300;
C = pid(Kp)
T = feedback(C*P,1)

t = 0:0.01:2;
step(T,t)
```



P Controller Step Response

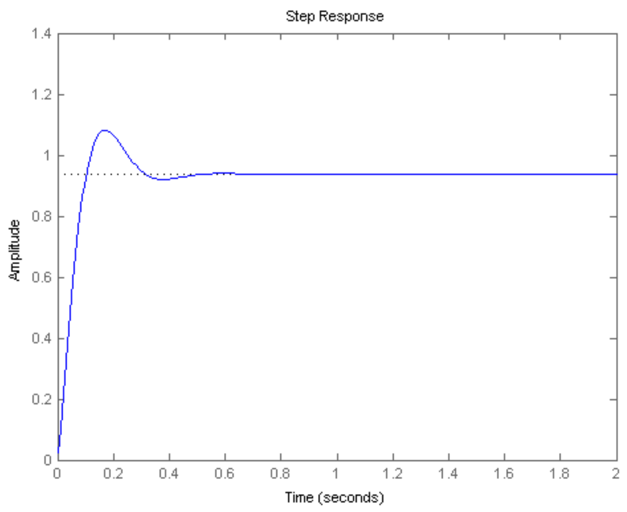
PD Controller Example

$$\frac{X(s)}{F(s)} = \frac{K_d s + K_p}{s^2 + (10 + K_d)s + (20 + K_p)}$$

- The derivative controller reduced both the overshoot and the settling time, and had a small effect on the rise time and the steady-state error.

```
Kp = 300;
Kd = 10;
C = pid(Kp,0,Kd)
T = feedback(C*P,1)

t = 0:0.01:2;
step(T,t)
```



PD Controller Step Response

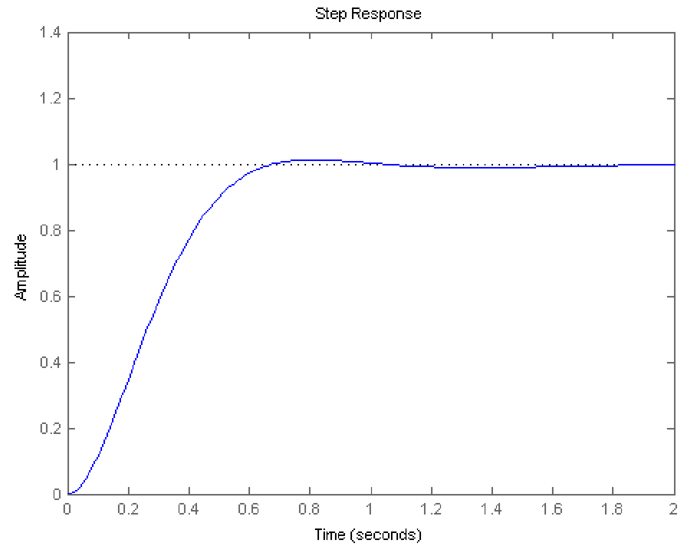
PI Controller Example

$$\frac{X(s)}{F(s)} = \frac{K_p s + K_i}{s^3 + 10s^2 + (20 + K_p s + K_i)}$$

- The derivative controller reduced both the overshoot and the settling time, and had a small effect on the rise time and the steady-state error.

```
Kp = 30;
Ki = 70;
C = pid(Kp,Ki)
T = feedback(C*P,1)

t = 0:0.01:2;
step(T,t)
```



PI Controller Step Response

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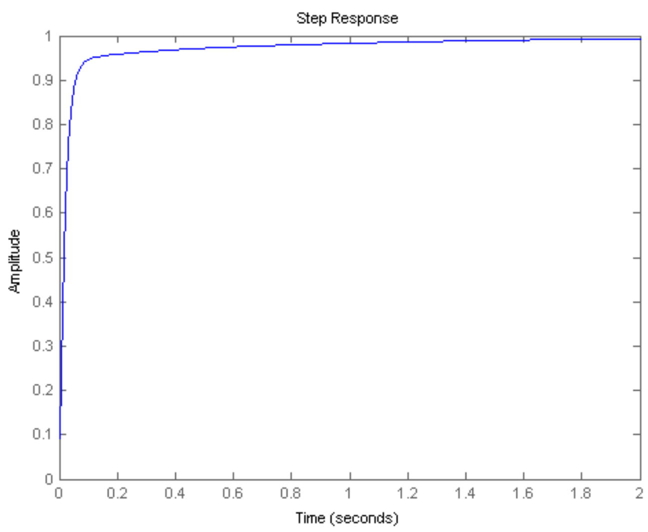
PID Controller Example

$$\frac{X(s)}{F(s)} = \frac{K_d s^2 + K_p s + K_i}{s^3 + (10 + K_d)s^2 + (20 + K_p)s + K_i}$$

- We have obtained a closed-loop system with no overshoot, fast rise time, and no steady-state error.

```
Kp = 350;
Ki = 300;
Kd = 50;
C = pid(Kp,Ki,Kd)
T = feedback(C*P,1);

t = 0:0.01:2;
step(T,t)
```



PID Controller Step Response

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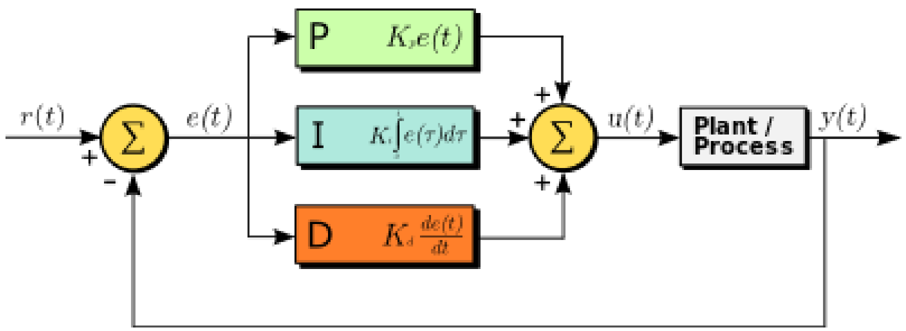
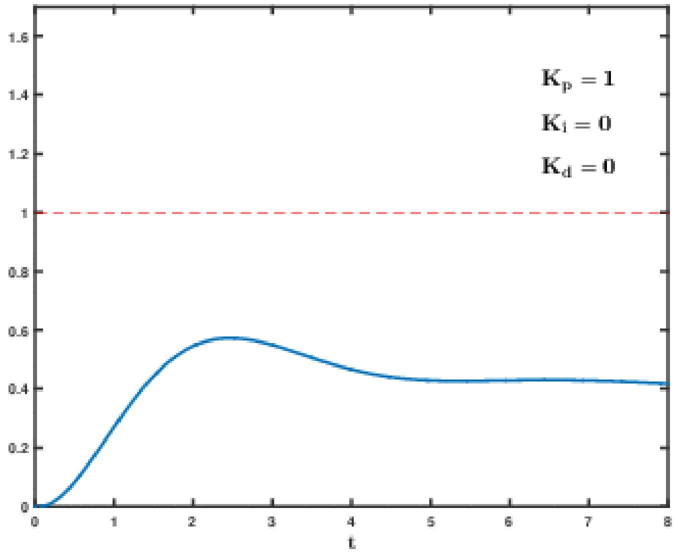
PID Controller Summary

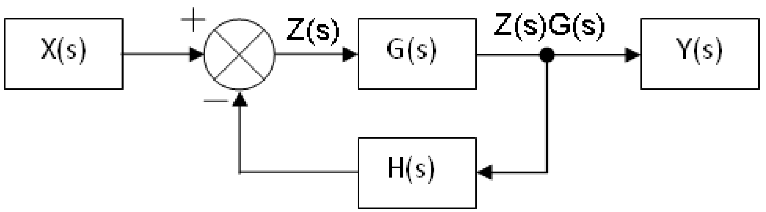




Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	Decrease	Increase	Small change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor change	Decrease	Decrease	No effect in theory	Improve if K_d small

Loop Shaping Design

- Closed loop transfer function can be derived directly from block diagram
- Loop transfer function is different from closed loop transfer function as it just captures the feedback loop (in this case G(s)H(s) is the loop transfer function)

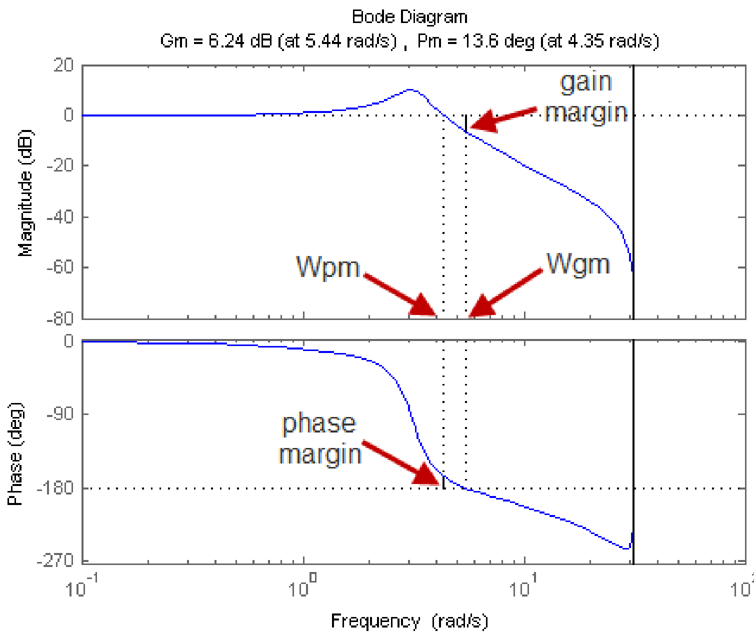


Using this figure we write:

$$Y(s) = Z(s)G(s)$$
$$Z(s) = X(s) - Y(s)H(s)$$
$$X(s) = Z(s) + Y(s)H(s)$$
$$X(s) = Z(s) + Z(s)G(s)H(s)$$
$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{Z(s)G(s)}{Z(s) + Z(s)G(s)H(s)}$$
$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Loop Shaping Design

- Phase Margin and Gain Margin definition
- Design for target unit gain cross over frequency with reasonable phase margin (30 to 60 degree)
- Phase margin = 100 damping ratio approximately
- Closed loop system behavior related to loop transfer function



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Phase Margin and Damping Ratio

- Loop transfer function

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

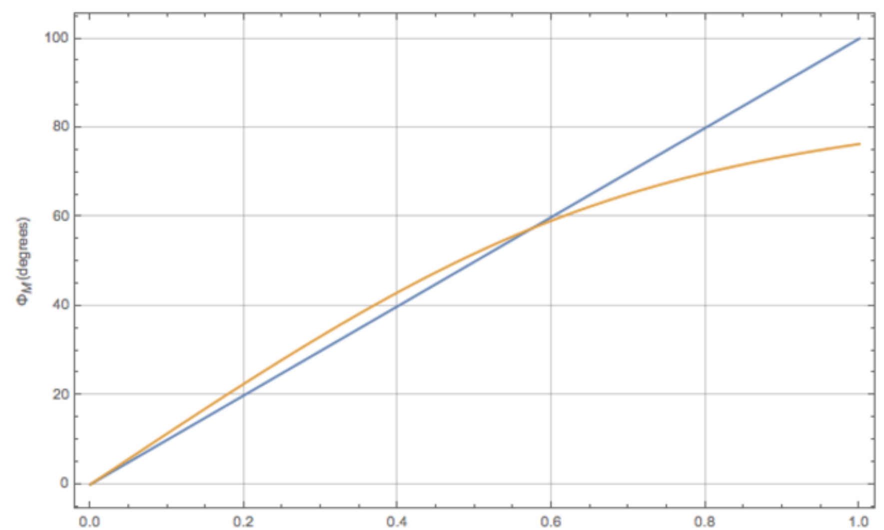
- Closed loop transfer function

$$T(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

- Phase Margin

$$\Phi_M = 90 - \tan^{-1}\frac{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}{2\zeta}$$

$$= \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$



$100\,\zeta$

$\tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}}\right)$

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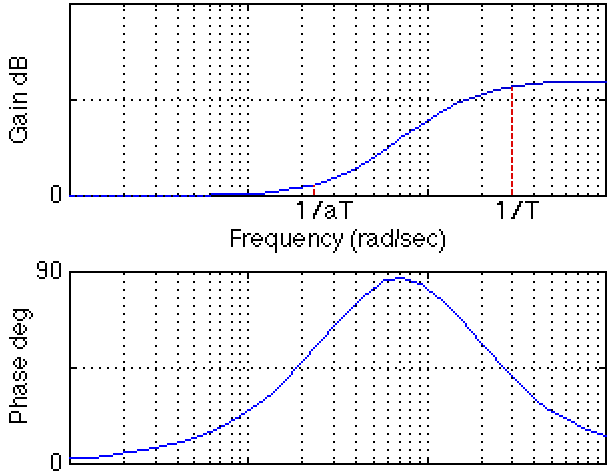
Lead Compensator

- Derivative term in PID controller is hard to implement and subject to noise
- A lead compensator is a good substitution for derivative control term
- Lead compensator helps to improve transient response

$$C(s) = \frac{1 + aTs}{1 + Ts}$$

$$\omega_m = \frac{1}{T\sqrt{a}}$$

$$\sin\phi = \frac{a-1}{a+1}$$



Lead Compensator Example

- Design a lead compensator to have the following character
  - Crossover frequency at 1000 rad/s
  - 55° phase lead

$$C(s) = \frac{1 + aTs}{1 + Ts}$$

$$\omega_m = \frac{1}{T\sqrt{a}}$$

$$\sin\phi = \frac{a-1}{a+1}$$

$$\phi_{max} = \sin^{-1}\left(\frac{\alpha-1}{\alpha+1}\right) = 55^\circ \text{ for } \alpha = 10$$

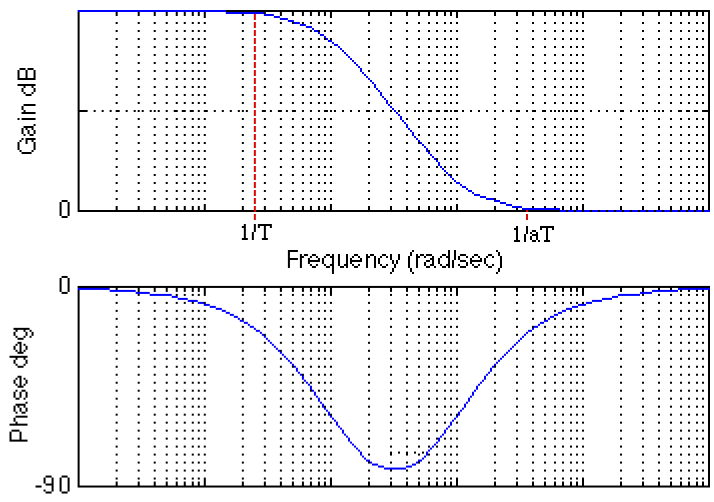
$$\omega_c = 1000 \qquad \frac{1}{\sqrt{\alpha}\,\tau} = 1000 \Rightarrow \tau = \frac{1}{\sqrt{\alpha}\cdot 1000} \Rightarrow \boxed{\tau = 0.316 \text{ msec}}$$

Lag Compensator

- Lag compensator is similar to integral control and improves the steady state behavior of the system by reducing the steady state tracking error
- Not used as often as lead compensator since PI controller can in many case be implemented easily

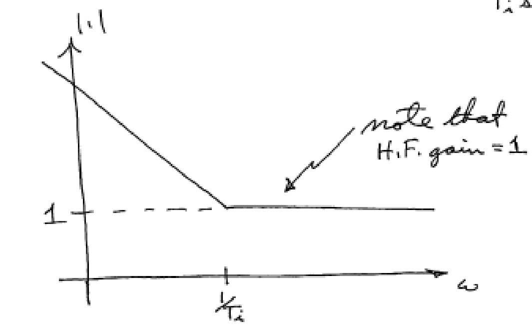
$$C(s) = \frac{1}{a}\left(\frac{1+aTs}{1+Ts}\right) \qquad [a<1]$$

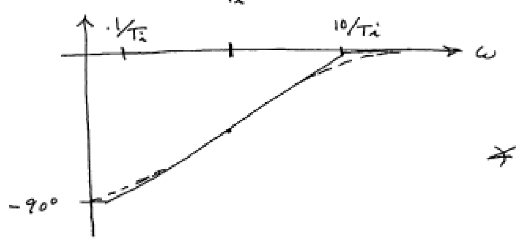
$$\omega_m = \frac{1}{T\sqrt{a}}$$



Lag Compensator

lag compensator $\quad \dfrac{T_i s + 1}{T_i s}$





$$\angle = -90^\circ + \underbrace{\tan^{-1} T_i \omega}_{\text{need } 80^\circ \text{ at } \omega = \omega_c \text{ to only lose } -10^\circ \text{ in } \phi_m}$$

$\Rightarrow \tan^{-1} T_i \omega_c = 80^\circ$

$\Rightarrow T_i = \dfrac{1}{\omega_c} \cdot \tan 80^\circ = \dfrac{5.7}{\omega_c}$

$\Rightarrow \boxed{T_i = 5.7 \text{ msec}}$

Because we have lag
H.F. gain = 1,
K is unchanged

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Conclusion

- Start with system physical model (differential equations)
- Linearize about operation point if needed (Taylor series)
- Convert to frequency domain (Laplace transformation)
- Obtain step response with overshoot and rise/settling time
- Draw Bode plot with phase and gain margin (replace s with jω)
- Draw closed loop control block diagram
- Specify desired system response parameter
- Apply PID control to obtain desired behavior
  - Use proportional control to decrease rise time
  - Use integral control to eliminate steady state error
  - Use derivative control to improve transient response (reduce overshoot)
- Do more detailed analysis with other control algorithm if needed

Homework Problem

- Design a PI controller with lead compensator to meet the specified requirement

- Use Matlab to simulate the system behavior and controller performance and record the plots

Thank You!