CHAPTER 3

[Part 4]

PROBABILITY THEORY

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Probability Theory

- The theory of probability plays a crucial role in making inferences.
- Probability measures how likely something is to occur.
- Probability is the ratio of the number of favourable cases to the total number of cases, assuming that all of the various cases are equally possible.

Terminology

Experiment

- A process to yield an outcome.
- Example: Rolling a dice.

Sample space

- A set of all possible outcomes from an experiment.
- For a dice, that would be values 1 to 6.

Element

- An item in the sample space.

Event

- An outcome or combination outcomes from an experiment.
- If you rolled a 4 on the dice, the event is the 4.

How to Compute Probability

- We assign a probability weight $P(x)$ to each element of the sample space.
- Weight represents what we believe to be relative likelihood of that outcome.
- There are two rules in assigning weight:
  - i) The weight must be non-negative numbers.
  - ii) The sum of the weights of all the elements in a sample space must be 1.

- Let E be an event.
- The probability of *E*, *P(E) is*

$$P(E) = \sum_{x \in E} P(x)$$

- We read this as "*P(E)* equals the sum, over all x such that x is in *E*, of *P*(x)"

Probability Axioms

Let S be a sample space. A probability function P from the set of all events in S to the set of real numbers satisfies the following three axioms:

For all events A and B in S,

1. $0 \leq P(A) \leq 1$
2. $P(\emptyset) = 0$ and $P(S) = 1$
3. If A and B are disjoint, the $P(A \cup B) = P(A) + P(B)$

A function P that satisfies these axioms is called a probability distribution or a probability measure.

Complementary Probabilities

- The complement of an event A in a sample space S is the set of all outcomes in S except those in A.

$$P(A') = 1 - P(A)$$

- **Example:**

  Let the probability for getting a prize in a lucky draw is 0.075. Thus, the probability of *not* getting a prize in a lucky draw is 1 - 0.075 = 0.925.

The Uniform Probability Distribution

- The probability of an event occurring is:

$$P(E) = \frac{|E|}{|S|}$$

where:

- **E** is the set of desired events.
- **S** is the set of all possible events.
- Note that $0 \le |E| \le |S|$:
  - Thus, the probability will always between 0 and 1.
  - An event that will never happen has probability 0.
  - An event that will always happen has probability 1.

Example 1

A coin is flipped four times and the outcome for each flip is recorded.

i) List all the possible outcomes in the sample space.

Sample = head , tail

ii) Find the event (E) that contain only the outcomes in which 1 tails appears.

Solution

- The list of all possible outcomes in the sample space. [heads(H), tails (T)]

HHHH	HHHT	HHTH	HTHH
THHH	HHTT	HTTH	HTHT
THHT	TTHH	THTH	HTTT
TTHT	TTTH	THTT	TTTT

- The event E that contains only the outcomes in which 1 tails appears.

E={ HHHT, HHTH, HTHH, THHH }

Example 2

dice = 1, 2, 3, 4, 5, 6

Sample Space = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

Two fair dice are rolled.

Find the event (E) that the sum of the numbers on the dice is 7.

E = { (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) }

example

- The number on top face of each die is 1,2,3,4,5,6.

- Let A={1,2,3,4,5,6}

- The sample space is,

  $$S=\{ (a, b) \in A\times A \mid a, b \in A \}$$

example

- The event E that the sum of the numbers on the dice is 7 is,

E = {(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)}

Example 3

$A = T, H$

$B = 1, 2, 3, 4, 5, 6$

$S = \{(a, b) \in A \times B \mid a \in A, b \in B\}$

Suppose that a coin is flipped and a dice is rolled.

$S = \{(T,1), (T,2), (T,3), (T,4), (T,5), (T,6), (H,1), (H,2), (H,3), (H,4), (H,5), (H,6)\}$

i) List the members of the sample space.

ii) List the members of the event "the coin shows a head and the dice shows a number less than 4".

$E = \{(H,1), (H,2), (H,3)\}$

Solution

- The sample space,

  $(H,1)$ $(H,2)$ $(H,3)$ $(H,4)$ $(H,5)$ $(H,6)$

  $(T,1)$ $(T,2)$ $(T,3)$ $(T,4)$ $(T,5)$ $(T,6)$

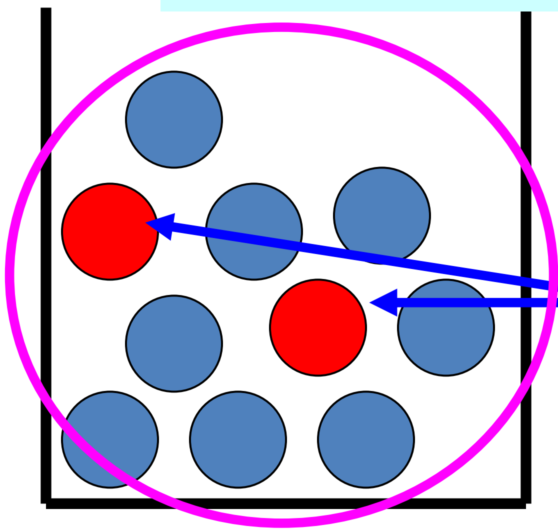
- The members of the event "the coin shows a head and the die shows a number less than 4,

  $(H,1)$ $(H,2)$ $(H,3)$

Example 4

- Question: What is the probability of picking a red marble out of a bowl with 2 red and 8 blue?

$$p(\text{red}) = \frac{\text{outcomes in which red occurs}}{\text{total number of outcomes}}$$



There are 2 outcomes that are red

There are 10 total possible outcomes

$$p(\text{red}) = 2 \ / \ 10 = 0.2$$

Example 5

Sample space $= 6 \cdot 6 = 36$, Event $= \{(6,4), (4,6), (5,5)\} = 3$

Two fair dice are rolled. What is the probability that the sum of the numbers on the dice is 10?

$$P(E) = \frac{3}{36} = \frac{1}{12}$$

Solution

- The size of sample space is,

  6.6 = 36

- 3 possible ways to obtain the sum of 10,

  (4,6), (5,5), (6,4)

- The probability is, $\dfrac{3}{36} = \dfrac{1}{12}$

Example 6

Five microprocessors are randomly selected from a lot of 1000 microprocessors among which 20 are defective. Find the probability of obtaining no defective microprocessors.

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Solution

- There are C(1000,5) ways to select 5 microprocessors among 1000.

- There are C(980,5) ways to select 5 good microprocessors since there are 1000-20=980 good microprocessors .

$^{980}C_5$

- The probability is, $\dfrac{C(980,5)}{C(1000,5)} = 0.903735781$

Example 7

There are exactly 3 red balls in a bucket of 15 balls. If we choose 4 balls at random, what is the probability that we do not choose a red ball?

ways to choose 4 balls $= {}^{15}C_4$

ways to choose other than red $= {}^{12}C_4$

$$P(E) = \frac{{}^{12}C_4}{{}^{15}C_4}$$

$$= \frac{33}{91}$$

Solution

- There are C(15,4) ways to select 4 balls among 15.
- If we do not choose a red ball, there are C(12,4) ways to select 4 balls among the remaining 12 balls.
- The probability is, $\dfrac{C(12,4)}{C(15,4)} = \dfrac{495}{1365} = \dfrac{33}{91}$

Example 8

- There are 5 red balls and 4 white balls in a box.

ways to chooce 2 red ball $= 5c_2$

ways to chooce 4 ball $= 4c_4$

ways to chooce 2 white ball $= 4c_2$

- 4 balls are selected at random from these balls.

$$P(E) = \frac{5c_2 \times 4c_2}{9c_4} = \frac{10}{21}$$

- Find the probability that 2 of the selected balls will be red and 2 will be white.

Solution

- The size of sample space is C(9,4)
- Select 2 red balls, C(5,2) ways
- Select 2 white balls, C(4,2) ways
- The probability is, $\frac{C(5,2).C(4,2)}{C(9,4)} = \frac{(10)(6)}{126} = \frac{10}{21}$

Example 9

Suppose that a die is biased (or loaded) so that the number 2 through 6 are equally likely to appear, but that 1 appears three times as likely as any other number to appear.

Solution

- To model this situation, we should have
  $P(2) = P(3) = P(4) = P(5) = P(6)$
- and
  $P(1) = 3P(2)$

$$\frac{P(1)}{3} = P(2) \qquad \frac{P(1)}{3} = \frac{1}{8}$$

$$P(1) = \frac{3}{8}$$

- Since
  $P(1)+ P(2)+ P(3)+ P(4)+ P(5)+ P(6) =1$
  $3P(2)+P(2)+ P(2)+ P(2)+ P(2)+ P(2)=1$
  $8P(2) =1$

$$P(2) = \frac{1}{8}$$

Solution

- Therefore,

  $P(2) = 1/8$

  $P(2) = P(3) = P(4) = P(5) = P(6) = 1/8$

- and $P(1) = 3/8$

Example 10

Refer to Example 9, what is the probability to obtain an **odd** number?

Solution:

$P(1) + P(3) + P(5) = 3/8 + 1/8 + 1/8 = 5/8$

Example 11

- 5 microprocessors are randomly selected from a lot of 1000 microprocessors among which 20 are defective.
- What is the probability of obtaining at least one defective microprocessor?

ways to selected randomly 5 mp = $^{1000}C_5$ / ways to selected at least one defective

$= 1 -$ ways to not selected any defective (exp 6)

$= 1 - 0.9037$

$= 0.0963$

Solution

- In previous example, we found that the probability of obtaining no defective microprocessor is 0.903735781

- The probability of obtaining at least one defective microprocessor is,

  1 - 0.903735781 = 0.096264219

Example 12

- What is the probability that if a fair coin is tossed 6 times you will get
  - Less than 2 heads

    $< 2H$

  - At least 2 heads

    $\geq 2H$

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Solution

- The number of possible outcome is:

  $2\times2\times2\times2\times2\times2 = 2^6 = 64$

- Let A be the event that less than 2 H's are observed,

  A= { TTTTTT, HTTTTT, THTTTT, TTHTTT, TTTHTT, TTTTHT, TTTTTH }

  Then, $P(A) = 7/64$

- Let B be the event that at least 2 H's are observed,

  $P(B) = 1- (7/64) = 57/64$

Probability Theory

- Let A and B be events.
- The event $A \cup B$ represents the event A or B (or both)
- The event $A \cap B$ represents the event A and B.

Example 13

- Among a group of students, some take discrete structure and some take numerical methods. A student is selected at random.
- Let *A* be the event "the student takes discrete structure"
- Let *B* be the event "the student takes numerical methods"

Example 12

- Then $A \cup B$ is the event "the student takes discrete structure or numerical methods or both"

- $A \cap B$ is the event "the student takes discrete structure and numerical methods"

Probability of a General Union of Two Events

If S is any sample space and A and B are any events in S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 13

In a math class of 30 students, 17 are boys and 13 are girls. On a unit test, 4 boys and 5 girls made an A grade. If a student is chosen at random from the class, what is the probability of choosing a girl or an A student?

Solution:

Probabilities: P(girl or A)= P(girl) + P(A) - P(girl and A)

$$= \frac{13}{30} + \frac{9}{30} - \frac{5}{30}$$

$$= \frac{17}{30}$$

Example 14

Two fair dice are rolled. What is the probability of getting doubles (2 dice showing the same number) or sum of 6?

Solution

- Let *A* denote the event "get doubles"

- Doubles can be obtained in 6 ways,

  (1,1), (2,2), (3,3), (4,4), (5,5), (6,6)

- P(*A*) = 6/36 = 1/6

Solution

- Let *B* denote the event "get a sum of 6".
- Sum of 6 can be obtained in 5 ways (1,5), (2,4), (3,3), (4,2), (5,1)
- P(*B*) = 5/36

Solution

- The event $A \cap B$ is "get doubles and get a sum of 6"

- Only 1 way, (3,3)

- $P(A \cap B) = 1/36$

Solution

- The probability of getting doubles or a sum of 6 is,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{6} + \frac{5}{36} - \frac{1}{36} = \frac{5}{18}$$

Example 15

- Suppose that a student is selected at random among 90 students, where 35 are over 100 pounds, 20 are boys, and 15 are over 100 pounds and boys.

- What is the probability that the student selected is over 100 pounds or a boy?

Solution

- Let A denote the event "the student is over 100 pounds".

  $P(A)=35/90$

- Let B denote the event "the student is a boy".

  $P(A)=20/90$

Solution

- $P(A \cap B) = 15/90$

- The probability that the student selected is over 100 pounds or a boy,

$$P(A \cup B) = \frac{35}{90} + \frac{20}{90} - \frac{15}{90} = \frac{40}{90}$$

Mutually Exclusive Events

- Two events are mutually exclusive if they cannot occur at the same time.

- Events A and B are mutually exclusive if

$$A \cap B = \emptyset$$

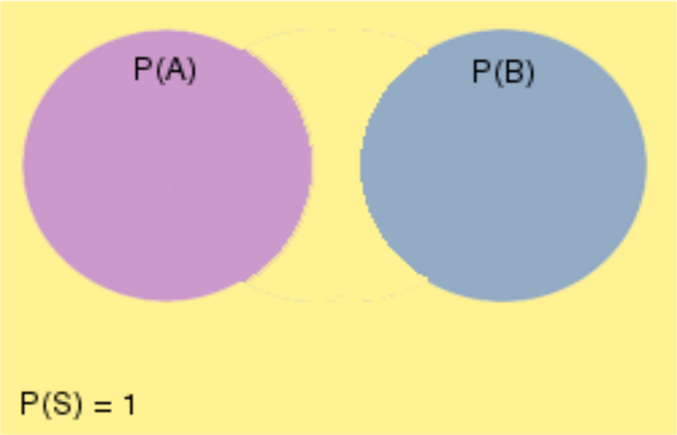
- If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

Mutually Exclusive Events (cont.)

Mutually Exclusive Events

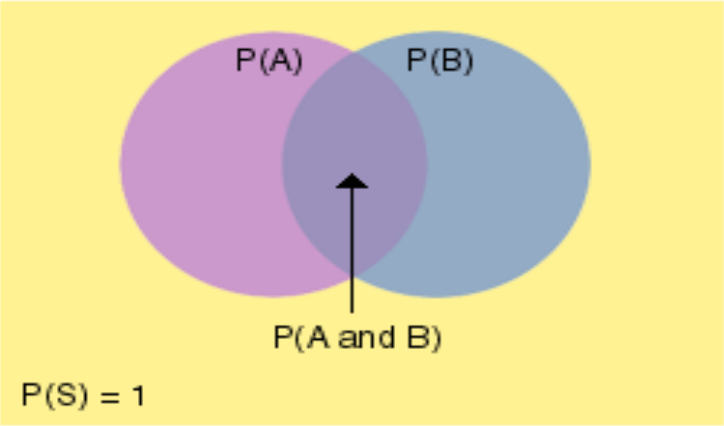
Two events are mutually exclusive if they cannot occur at the same time (i.e., they have no outcomes in common).



In the Venn Diagram above, the probabilities of events A and B are represented by two disjoint sets (i.e., they have no elements in common).

Non-Mutually Exclusive Events

Two events are non-mutually exclusive if they have one or more outcomes in common.



In the Venn Diagram above, the probabilities of events A and B are represented by two intersecting sets (i.e., they have some elements in common).

Note: In each Venn diagram above, the sample space of the experiment is represented by S, with P(S) = 1.

Example 15

Experiment 1: A single card is chosen at random from a standard deck of 52 playing cards. What is the probability of choosing a 5 or a king?

Possibilities: 1. The card chosen can be a 5.
2. The card chosen can be a king.

Experiment 2: A single card is chosen at random from a standard deck of 52 playing cards. What is the probability of choosing a club or a king?

Possibilities: 1. The card chosen can be a club.
2. The card chosen can be a king.
3. The card chosen can be a king and a club (i.e., the king of clubs).

- In Experiment 1, the card chosen can be a five or a King, but not both at the same time. These events are mutually exclusive.

- In Experiment 2, the card chosen can be a club, or a King, or both at the same time. These events are not mutually exclusive.

Example 17

- 2 fair dice are rolled.
- Find the probability of getting doubles or the sum of 5?

Solution

- Let A denote the event "get doubles"
- Let B denote the event "get the sum of 5"
- A and B are mutually exclusive.

  (you cannot get doubles and the sum of 5 simultaneously)

Solution

- A ={(1,1) (2,2) (3,3) (4,4) (5,5) (6,6)}

  $P(A) = 6/36 = 1/6$

- B ={(1,4) (2,3) (3,2) (4,1)}

  $P(B) = 4/36 = 1/9$

- The probability of getting doubles or the sum of 5 is,

  $P(A \cup B) = 1/6 + 1/9 = 5/18$

Example 17

A glass jar contains 1 **RED**, 3 **GREEN**, 2 **BLUE** and 4 **YELLOW** marbles. If a single marble is chosen at random from the jar, what is the probability that it is **YELLOW** or **GREEN** ?

Solution:

Probabilities:

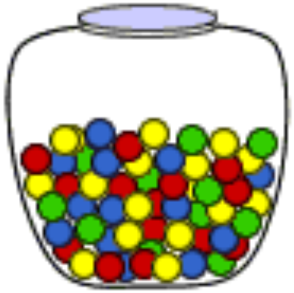
$$P(\text{yellow}) = \frac{4}{10}$$

$$P(\text{green}) = \frac{3}{10}$$

$$P(\text{yellow or green}) = P(\text{yellow}) + P(\text{green})$$

$$= \frac{4}{10} + \frac{3}{10}$$

$$= \frac{7}{10}$$



Conditional Probability

- Let *A* and *B* be events, and assume that P(*B*)>0.

- The conditional probability of *A* given *B* is,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Definition: The **conditional probability** of an event **A** in relationship to an event **B** is the probability that event **A** occurs given that event **B** has already occurred. The notation for conditional probability is **P(A|B)** [pronounced as ***the probability of event A given B***].

Example 18

Question: Suppose that we roll 2 fair dice. What is the probability of getting a sum of 10?

Solution:

Sample space

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

$A \cap B = \{(5,5)\}$

- The probability of getting a sum of 10 is 1/12.

$P(A \cap B) = 1$

Example 19

Question: Refer to **Example 18**, what is the probability of getting a sum of 10 given that at least one die is 5?

Solution:

- Let A denote the event "getting a sum of 10".
- Let B denote the event "at least one die is 5".
- Thus, $P(A \cap B) = 1/36$ and $P(B) = 11/36$
- The probability of getting a sum of 10 given that at least 1 die is 5 is,

$$P\left(A \middle| B\right) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Example 20

Suppose that we roll 2 fair dice.

Find the probability of **getting a sum of 7, given that the digit in the first die is greater than in the second.**

Solution

- The sample space *S* consists of $6 \times 6 = 36$ outcomes.

- Let A be the event "the sum of digits of the 2 dice is 7"

A={ (6,1), (5,2), (4,3), (3,4), (2,5), (1,6) }

Solution

- Let B be the event "the digit in the first die is greater than the second"

B={ (6,1), (6,2), (6,3), (6,4), (6,5), (5,1), (5,2), (5,3), (5,4), (4,1), (4,2), (4,3), (3,1), (3,2), (2,1) }

P(B)=15/36

Solution

- Let C be the event "the sum of digits of the 2 dice is 7 but the digit in the first die is greater than the second"

C={ (6,1), (5,2), (4,3) } = $A \cap B$

P($A \cap B$)= 3/36

Solution

- The probability of getting a sum of 7, given that the digit in the first die is greater than in the second is,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{(3/36)}{(15/36)} = \frac{1}{5}$$

Example 21

- Weather records show that the probability of high barometric pressure is 0.80 and the probability of rain and high barometric pressure is 0.10. $P(B) = 0.80$

  $P(A \cap B) = 0.10$

- What is the probability of rain given high barometric pressure?

Solution

- *R* denotes the event “rain” and *H* denotes the event “high barometric pressure”

- The probability of rain given high barometric pressure is

$$P(R \mid H) = \frac{P(R \cap H)}{P(H)} = \frac{0.1}{0.8} = 0.125$$

Chain Rule for Conditional Probability

- The chain rule for conditional probability with n events is as follows:

$$\boxed{P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1)\ldots P(A_n|A_{n-1}, A_{n-2}, \ldots A_1)}$$

Example 22

Mr. Basyir needs two students to help him with a science demonstration for his class of 18 girls and 12 boys. He randomly choose one student who comes to the front of the room. He then chooses a second student from those still seated. What is the probability that both students chosen are girls?

Solution:

$$P(\text{Girl1 and Girl2}) = P(\text{Girl1}) \text{ and } P\left(\text{Girl2} \middle| \text{Girl1}\right)$$

$$= \frac{18}{30} \times \frac{17}{29} = \frac{306}{870} = \frac{51}{145}$$

Example 23(a)

In a factory there are 100 units of a certain product, 5 of which are defective. We pick three units from the 100 units at random. What is the probability that none of them are defective?

Solution:

Let A_i as the event that i-th chosen unit is not defective.

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 | A_1)P(A_3 | A_2, A_1)$$

$$P(A_1) = \frac{95}{100}; \quad P(A_2 | A_1) = \frac{94}{99}; \quad P(A_3 | A_2, A_1) = \frac{93}{98}$$

$$\therefore P(A_1 \cap A_2 \cap A_3) = \frac{95}{100} \times \frac{94}{99} \times \frac{93}{98} = 0.8560$$

Example 23(b)

In a shipment of 20 computers, 3 are defective. Three computers are randomly selected and tested. What is the probability that all three are defective if the first and second ones are not replaced after being tested?

Solution:

Probabilities: P(3 defectives) =

$$\frac{3}{20} \cdot \frac{2}{19} \cdot \frac{1}{18} = \frac{6}{6840} = \frac{1}{1140}$$

Representing Conditional Probabilities with a Tree Diagram

Example:

In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{4}$. If it's rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{8}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is 0.25. You pick a random day.

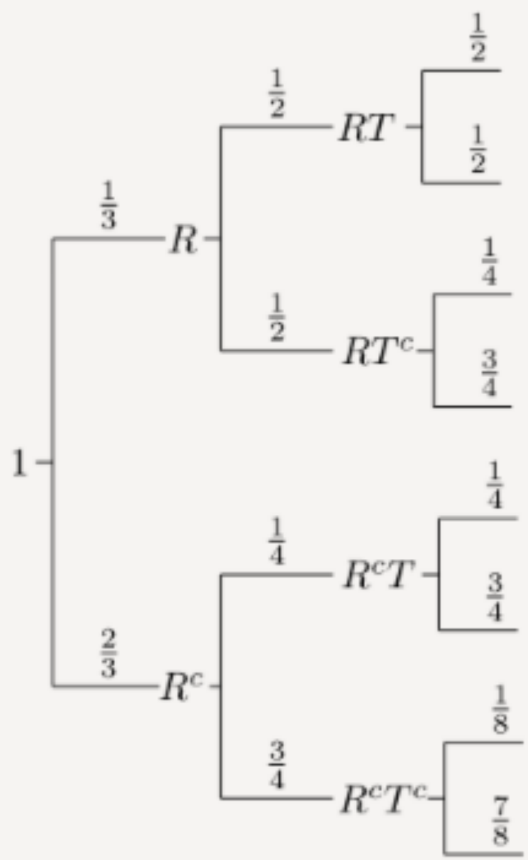
a. What is the probability that it's not raining and there is heavy traffic and I am not late?
b. What is the probability that I am late?
c. Given that I arrived late at work, what is the probability that it rained that day?

Solutions

Let, R = event of rainy,

T = event of heavy traffic,

L = event of I'm late to work



$RTL \to P(RTL) = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{12}$

$RTL^c \to P(RTL^c) = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{12}$

$RT^cL \to P(RT^cL) = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{24}$

$RT^cL^c \to P(RT^cL^c) = \frac{1}{3} \times \frac{1}{2} \times \frac{3}{4} = \frac{3}{24} = \frac{1}{8}$

$R^cTL \to P(R^cTL) = \frac{2}{3} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{24}$

$R^cTL^c \to P(R^cTL^c) = \frac{2}{3} \times \frac{1}{4} \times \frac{3}{4} = \frac{1}{8}$

$R^cT^cL \to P(R^cT^cL) = \frac{2}{3} \times \frac{3}{4} \times \frac{1}{8} = \frac{1}{16}$

$R^cT^cL^c \to P(R^cT^cL^c) = \frac{2}{3} \times \frac{3}{4} \times \frac{7}{8} = \frac{7}{16}$

Solutions (cont'd)

a) What is the probability that it's not raining and there is heavy traffic and I'm not late?

$$P(R^c \cap T \cap L^c) = P(R^c)P(T \mid R^c)P(L^c \mid T, R^c)$$

$$= \frac{2}{3} \times \frac{1}{4} \times \frac{3}{4} = \frac{1}{8}$$

b) What is the probability that I'm late?

$$P(L) = P(R,T,L) + P(R,T^c,L) + P(R^c,T,L) + P(R^c,T^c,L)$$

$$= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} = \frac{11}{48}$$

Solutions (cont'd)

c) Given that I arrived late at work, what is the probability that it's rained that day?

We need to find $P(R|L)$ using $P\left(R|L\right)=\dfrac{P(R\cap L)}{P(L)}$

$$P(R\cap L)=P(R,T,L)+P(R,T^c,L)=\frac{1}{12}+\frac{1}{24}=\frac{1}{8}$$

$$\therefore P\left(R|L\right)=\frac{P(R\cap L)}{P(L)}=\frac{\frac{1}{8}}{\frac{11}{48}}=\frac{6}{11}$$

Law of Total Probability

- If B_1 and B_2 are disjoint events with $P(B_1) + P(B_2) = 1$, then for any event E,

$$\boxed{\begin{aligned} P(E) &= P(E \cap B_1) + P(E \cap B_2) \\ &= P(E|B_1)P(B_1) + P(E|B_2)P(B_2) \end{aligned}}$$

- More generally, if $B_1, B_2, \ldots, B_k$ are disjoint events with $P(B_1) + P(B_2) + \ldots + P(B_k) = 1$, then for any event E,

$$\boxed{\begin{aligned} P(E) &= P(E \cap B_1) + P(E \cap B_2) + \ldots. + P(E \cap B_k) \\ &= P(E|B_1)P(B_1) + P(E|B_2)P(B_2) + \ldots + P(E|B_k)P(B_k) \end{aligned}}$$

Example 24

Paediatric department researcher examines the medical records of toddlers that came to a particular paediatric clinic. He found that 20% of them came for flu treatment and 10% of mothers of the toddler that having flu are also having flu. 30% of the mothers that came to the clinic are found having flu.

a) What is the probability of the toddler having flu given that the mother having flu.
b) What if the probability of the toddler having flu given that the mother is not having flu.

Solution

a) What is the probability of the toddler having flu given that the mother having flu.

Let, T : event of toddler having flu; M: event of mother having flu

$$P(T)=0.2;\quad P(M\cap T)=0.1; P(M)=0.3$$

$$P(T\mid M)=\frac{P(M\cap T)}{P(M)}=\frac{0.1}{0.3}=0.33$$

a) What if the probability of the toddler having flu given that the mother is not having flu.

According to the law of total probability:

$$P(T)=P(T\cap M)+P\left(T\cap M'\right); \text{ where } P(M)+P(M')=1$$

$$0.2=0.1+P\left(T\cap M'\right)$$

$$.:P\left(T\cap M'\right)=0.2-0.1=0.1;\ P(M')=1-P(M)=1-0.3=0.7$$

$$.:P(T\mid M')=\frac{P(T\cap M')}{P(M')}=\frac{0.1}{0.7}=0.143$$

Bayes' Theorem

Suppose that a sample space ***S*** is a union of mutually disjoint events $B_1, B_2, B_{3,} ...., B_n$, suppose ***A*** is an event in ***S***, and suppose ***A*** and all the ***B_k*** have nonzero probabilities, where k is an integer with $1 \leq k \leq n$. Then

$$P\left(B_k \middle| A\right) = \frac{P\left(A \middle| B_k\right) P(B_k)}{\sum_{i=1}^{n} P\left(A \middle| B_i\right) P(B_i)}$$

Example 25

- At the telemarketing firm, Foo, Raqib and Lee make calls.
- The table shows the percentage of call each caller makes and the percentage of persons who are annoyed and hang up on each caller.

	caller		
	Foo	Raqib	Lee
% of calls	40	25	35
% of hang-ups	20	55	30

Example 25 (cont.)

- Let A denote the event "Foo made the call".
- Let B denote the event "Raqib made the call".
- Let C denote the event "Lee made the call".
- Let H denote the event "the caller hung up".
- Find
  ① $P(A)$, $P(B)$, $P(C)$
  ② $P(H|A)$, $P(H|B)$, $P(H|C)$
  ③ $P(A|H)$, $P(B|H)$, $P(C|H)$
  ④ $P(H)$

Solution

- Since Foo made 40% of the calls,

  $P(A) = 0.40$

- Similarly, from the table we obtain

  $P(B) = 0.25$

  $P(C) = 0.35$

- Given that Foo made the call, the table shows that 20% of the persons hung up

  $P(H|A) = 0.20$

- Similarly,

  $P(H|B) = 0.55$

  $P(H|C) = 0.30$

Solution (cont.)

- To compute P(A|H), we use Bayes' Theorem:

$$P(A \mid H) = \frac{P(H \mid A)P(A)}{P(H \mid A)P(A) + P(H \mid B)P(B) + P(H \mid C)P(C)}$$

$$= \frac{(0.2)(0.4)}{(0.2)(0.4) + (0.55)(0.25) + (0.3)(0.35)}$$

$$= 0.248$$

Solution (cont.)

- To compute P(B|H), we use Bayes' Theorem:

$$P(B \mid H) = \frac{P(H \mid B)P(B)}{P(H \mid A)P(A) + P(H \mid B)P(B) + P(H \mid C)P(C)}$$

$$= \frac{(0.55)(0.25)}{(0.2)(0.4) + (0.55)(0.25) + (0.3)(0.35)}$$

$$= 0.426$$

Solution (cont.)

- To compute P(C|H), we use Bayes' Theorem:

$$P(C\mid H)=\frac{P(H\mid C)P(C)}{P(H\mid A)P(A)+P(H\mid B)P(B)+P(H\mid C)P(C)}$$

$$=\frac{(0.3)(0.35)}{(0.2)(0.4)+(0.55)(0.25)+(0.3)(0.35)}$$

$$=0.326$$

Solution (cont.)

- Compute P(H)

$$
\begin{aligned}
P(H) &= P(H \mid A)P(A) + P(H \mid B)P(B) + P(H \mid C)P(C) \\
&= (0.2)(0.4) + (0.55)(0.25) + (0.3)(0.35) \\
&= 0.3225
\end{aligned}
$$

Example 26

The ELISA test is used to detect antibodies in blood and can indicate the presence of the HIV virus. Approximately 15% of the patients at one clinic have the HIV virus. Among those that have the HIV virus, approximately 95% test positive on the ELISA test. Among those that do not have the HIV virus, approximately 2% test positive on the ELISA test. **Find the probability that a patient has the HIV virus if the ELISA test is positive.**

Solution

- Let H denote the event "patient has the HIV virus".
- Let T denote the event "patient does not have the HIV virus".
- Thus,

$$P(H) = 0.15$$

$$P(T) = 0.85$$

- Let *Pos* denote the feature "tests positive"

$$P(Pos|H) = 0.95$$

$$P(Pos|T) = 0.02$$

Solution (cont.)

- The probability that a patient has the HIV virus if the ELISA test is positive is

$$P(H \mid Pos) = \frac{P(Pos \mid H)P(H)}{P(Pos \mid H)P(H) + P(Pos \mid T)P(T)}$$

$$= \frac{(0.95)(0.15)}{(0.95)(0.15) + (0.02)(0.85)}$$

$$= 0.893$$

Example 27

- Hana, Amir and Dani write a program that schedule tasks for manufacturing toys.
- The table shows the percentage of code written by each person and the percentage of buggy code for each person.

	Coder		
	Hana	Amir	Dani
% of code	30	45	25
% of bugs	3	2	5

- Given that a bug was found, **find the probability that it was in the program code written by Dani.**

Solution

- Let
  - H denotes the event of “code written by Hana”
  - A denotes the event of “code written by Amir”
  - D denotes the event of “code written by Dani”
  - B denotes the event of “a bug found in code”
- Since Hana wrote 30% of the code

  $P(H) = 0.3$

- Similarly,

  $P(A) = 0.45$

  $P(D) = 0.25$

Solution (cont.)

- If Hana wrote the code, the table shows that 3% of bugs was found. Thus,

  P(B|H) = 0.03

- Similarly,

  P(B|A) = 0.02

  P(B|D) = 0.05

- The probability that a bug was found in the code written is

$$
\begin{aligned}
P(B) &= P(B \mid H)P(H) + P(B \mid A)P(A) + P(B \mid D)P(D) \\
&= (0.03)(0.3) + (0.02)(0.45) + (0.05)(0.25) \\
&= 0.0305
\end{aligned}
$$

Solution (cont.)

- If a bug was found, the probability that it was in the code written by Dani is

$$P(D \mid B) = \frac{P(B \mid D)P(D)}{P(B)}$$

$$= \frac{(0.05)(0.25)}{0.0305}$$

$$= 0.4098$$

Independent Events

- If the probability of event A does not depend on event B in the sense that $P(A|B)=P(A)$, we say that A and B are independent events.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$P(A \cap B) = P(A).P(B)$$

Independent Events (cont.)

- To find the probability of two independent events that occur in sequence, find the probability of each event occurring separately, and then **multiply** the probabilities.
- This **multiplication rule** is defined symbolically below.

Multiplication Rule 1: When two events, A and B, are independent, the probability of both occurring is: $P(A \text{ and } B) = P(A) \cdot P(B)$

- Note that multiplication is represented by AND.

Independent Events (cont.)

Some other examples of independent events are:

- Landing on heads after tossing a coin **AND** rolling a 5 on a single 6-sided die.
- Choosing a marble from a jar **AND** landing on heads after tossing a coin.
- Choosing a 3 from a deck of cards, replacing it, **AND** then choosing an ace as the second card.
- Rolling a 4 on a single 6-sided die, **AND** then rolling a 1 on a second roll of the die.

Example 27

A coin is loaded so that the probability of heads is 0.6. Suppose the coin is tossed twice. Although the probability of heads is greater than the probability of tails, there is no reason to believe that whether the coin lands heads or tails on one toss will affect whether it lands heads or tails on other toss. Thus it is reasonable to assume that the results of the tosses are independent.

i) What is the probability of obtaining **two heads**?
ii) What is the probability of obtaining **one head**?
iii) What is the probability of obtaining **no head**?
iv) What is the probability of obtaining **at least one head**?

Solution

- The sample space, ***S*** consists of four outcomes: { HH, HT, TH, TT}, which are not equally likely
- Let ***A*** denote the event "obtain head on the first toss".
- Let ***B*** denote the event "obtain head on the second toss".
- Then, P(***A***)=P(***B***) = 0.6, it is to be assumed that ***A*** and ***B*** is independent.

Solution (cont.)

i) What is the probability of obtaining **two heads**?

$$P(\text{Two heads}) = P\left(A \cap B\right)$$
$$= P(A) \bullet P(B)$$
$$=(0.6)(0.6)$$
$$= 0.36 = 36\%$$

Solution (cont.)

ii) What is the probability of obtaining **one head**?

$$P(One\ \text{head}) = P\big((A \cap B') \cup (A' \cap B)\big)$$

$$= P(A) \bullet P(B') + P(A') \bullet P(B)$$

$$= (0.6)(1 - 0.6) + (1 - 0.6)(0.6)$$

$$= (0.6)(0.4) + (0.4)(0.6)$$

$$= 0.48 = 48\%$$

Solution (cont.)

iii) What is the probability of obtaining **no head**?

$$
\begin{aligned}
P(no\ \text{head}) &= P\left(A' \cap B'\right) \\
&= P(A') \bullet P(B') \\
&= (1-0.6)(1-0.6) \\
&= (0.4)(0.4) \\
&= 0.16 = 16\%
\end{aligned}
$$

Solution (cont.)

iv) What is the probability of obtaining **at least one head**?

$$P(\geq 1 \text{ head}) = P(\textit{one} \text{ head}) + P(\textit{two} \text{ heads})$$
$$= (0.48) + (0.36)$$
$$= 0.84 = 84\%$$

Or,

$$P(\geq 1 \text{ head}) = 1 - P(\textit{no} \text{ head})$$
$$= 1 - (0.16)$$
$$= 0.84 = 84\%$$

Example 28

A dresser drawer contains one pair of socks with each of the following colors: blue, brown, red, white and black. Each pair is folded together in a matching set. You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?

Solution:

Probabilities:

$$P(\text{red}) = \frac{1}{5}$$

$$P(\text{red and red}) = P(\text{red}) \cdot P(\text{red})$$

$$= \frac{1}{5} \cdot \frac{1}{5}$$

$$= \frac{1}{25}$$

Example 29

A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and an eight?

Probabilities:

$$P(\text{jack}) = \frac{4}{52}$$

$$P(8) = \frac{4}{52}$$

$$\begin{aligned} P(\text{jack and } 8) &= P(\text{jack}) \cdot P(8) \\ &= \frac{4}{52} \cdot \frac{4}{52} \\ &= \frac{16}{2704} \\ &= \frac{1}{169} \end{aligned}$$

Example 30

Farid and Azie take a final examination in discrete structure. The probability that Farid passes is 0.70 and the probability that Azie passes is 0.95. Assume that the events “Farid passes” and “Azie passes” are independent. Find the probability that Farid or Azie, or both, passes the final exam.

Solution

- Let *F denotes the event “Farid pass the final exam”*
- Let *A denotes the event “ Azie pass the final exam”*

➔We are asked to compute $P(F \cup A)$

- $P(F \cup A) = P(F) + P(A) - P(F \cap A)$
- $P(F \cap A) = P(F) \bullet P(A) = (0.70)(0.95) = 0.665$

- $P(F \cup A) = P(F) + P(A) - P(F \cap A)$

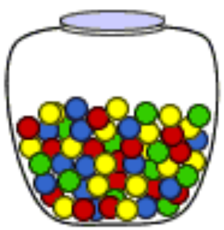
  $= 0.70 + 0.95 - 0.665$

  $= 0.985$

Example 30

A jar contains 3 Red, 5 Green, 2 Blue and 6 Yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a Green and a Yellow marble?

$$P(Green) = \frac{5}{16}; \qquad P(Yellow) = \frac{6}{16}$$



Solution:

$$P(Green \text{ and } Yellow) = P(Green) \bullet P(Yellow)$$

$$= \frac{5}{16} \bullet \frac{6}{16}$$

$$= \frac{30}{256}$$

Example 31

- Halim and Aina take a final examination in Fortran.
- The probability that Halim passes is 0.85, and the probability that Aina passes is 0.70.
- Assume that the events "Halim passes the final exam" and "Aina passes the final exam" are independent.

- Find the probability that Halim does not pass.
- Find the probability that both pass.
- Find the probability that both fail.
- Find the probability that at least one passes.

Solution

- Let *H denotes the event "Halim pass the exam"*
- Let *A denotes the event " Aina pass the exam"*
- Given, $P(H) = 0.85$; $P(A) = 0.70$

i) Probability Halim does not pass the exam:

$$P(H') = 1 - P(H) = 1 - 0.85 = 0.15$$

ii) Probability that both pass:

$$P(H \cap A) = P(H) \bullet P(A) = (0.85)(0.70) = 0.595$$

iii) Probability that both fail:

$$P(H' \cap A') = P(H') \bullet P(A') = (1-0.85)(1-0.70) = (0.15)(0.3) = 0.045$$

iv) Probability that at least one passes:

$$P(\text{at least one pass}) = 1 - P(H' \cap A') = 1 - (0.045) = 0.955$$