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Chapter 1

Part 4

Quantifiers & Proof Technique

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QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.
- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include **statements with the variables**.

QUANTIFIERS

$A = P(x)$

- Let *P(x)* is a statement with variable x and *A* is a set.
- *P* a **propositional function** or also known as **predicate** if for each x in *A*, *P(x)* is a proposition.
- Set *A* is the **domain of discourse** of *P*.
- Domain of discourse -> the particular domain of the variable in a propositional function.

QUANTIFIERS

- A **predicate** is a statement that contains variables.

Example:

$$P(x) : x > 3$$

$$Q(x,y) : x = y + 3$$

$$R(x,y,z) : x + y = z$$

Example

$x^2 + 4x$ is an odd integer

(domain of discourse is set of positive numbers).

$x^2 - x - 6 = 0$

(domain of discourse is set of real numbers).

x is rated as Research University in Malaysia

(domain of discourse is set of university in Malaysia).

QUANTIFIERS

A predicate becomes a proposition if the variable(s) contained is(are)

- **Assigned specific value(s)**
- **Quantified**

Example

- $P(x) : x > 3$.

  What are the truth values of $P(4)$ and $P(2)$?

  True false

- $Q(x,y) : x = y + 3$. (multiple variables)

  What are the truth values of $Q(1,2)$ and $Q(3,0)$?

  false true

Propositional functions

- Let P(x) : "x is a multiple of 5"
  - For what values of x is P(x) true?

    $x = 5n$, n can be any integer

- Let P(x) : x+1 > x
  - For what values of x is P(x) true?

    $x \geq 1$

- Let P(x) : x + 3 =0
  - For what values of x is P(x) true?

    $x = -3$

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QUANTIFIERS

$\forall$ = for every

$\exists$ = there exist / for at least one

$\neg \forall$ = not all

$\neg \exists$ = none

- Two types of quantifiers:
  - **Universal**
  - **Existential**

exp: all values are true or at least one values is false (counter example)

(if all true, the statement are true)

(if at least one values is false, the statement are false)

exp: $A = p(n)$

$p(n)$ is true if at least one n inside A is true

if exist $\neg$ in any universal or Existential quantifiers, it will change from universal into existential or vice versa

exp:

$\neg(\forall n\, P(n)) : \exists n \neg P(n)$

$\neg(\exists n\, P(n)) : \forall n \neg P(n)$

QUANTIFIERS

- Let A be a propositional function with domain of discourse B. The statement

  for every x, $A(x)$

  is **universally quantified statement**

- Symbol ∀ called a **universal quantifier** is used "**for every**".
- Can be read as "**for all**", "**for any**".

Universal quantifiers

Represented by an upside-down A: $\forall$

- It means "for all"

Example

Let P(x) = x+1 > x

We can state the following:

- $\forall x\, P(x)$
- English translation: "for all values of x, P(x) is true"
- English translation: "for all values of x, x+1>x is true"

QUANTIFIERS

- The statement can be written as

$$\forall x\, A(x)$$

- Above statement is true if $A(x)$ is true for every x in B

  (**false if $A(x)$ is false for at least one x in B**).

- OR In order to prove that a universal quantification is **true,**

  it must be shown for **ALL cases**

  In order to prove that a universal quantification is **false**, it must be shown to be false for only **ONE case**

- A value x in the domain of discourse that makes the statement $A(x)$ *false* is called a **counterexample** to the statement.

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Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \geq 0)$$

Domain of discourse is the set of real numbers.

- **This statement is true** because for every real number x, it is true that the square of x is positive or zero.

Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \leq 9)$$

  Domain of discourse is a set B = {1, 2, 3, 4}

- When $x = 4$, the statement produce false value.
- Thus, the above statement is false and the **counterexample is 4**.

QUANTIFIERS

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in School of Computing. It will be hard to show that every element in the set is *true*.

Use existential quantifier!!

QUANTIFIERS

- Let *A* be a propositional function with domain of discourse *B*. The statement

  There exist x, $A(x)$

  is **existentially quantified statement**

- Symbol ∃ called an **existential quantifier** is used "**there exist**".
- Can be read as "**for some**", "**for at least one**".

QUANTIFIERS

- The statement can be written as

$$\exists x \, A(x)$$

- Above statement is true if $A(x)$ is true for at least one x in B (**false if every x in B makes the statement $A(x)$ false**).

- **Just find one x that makes $A(x)$ true!**

Example

- Let the existentially quantified statement is

$$\exists x \quad \left(\frac{x}{x^2+1} = \frac{2}{5} \right)$$

Domain of discourse is the set of real numbers.

- **Statement is true** because it is **possible to find at least one real number *x*** to make the proposition true.
- For example, if $x = 2$, we obtain the true proposition as below

$$\left(\frac{x}{x^2+1} = \frac{2}{5} \right) = \left(\frac{2}{2^2+1} = \frac{2}{5} \right)$$

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Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg\ (\forall x\ P(x))\ ;\ \ \exists x\ \neg\ P(x)$$

$$\neg\ \ (\exists x\ P(x))\ ;\ \ \forall x\ \neg\ P(x)$$

Example

- Let $P(x)$: x is taking Discrete Structure course with the domain of discourse is the set of all students.

  $\forall x\, P(x)$
  All students are taking Discrete Structure course.

  $\exists x\, P(x)$
  There is some students who are taking Discrete Structure course.

Example

$$\neg\ (\exists x\ P(x))\ ;\ \ \forall x\ \neg\ P(x)$$

$\neg\exists x\ P(x)$
None of the students are taking Discrete Structure course.

$\forall x\ \neg\ P(x)$
All students are not taking Discrete Structure course.

Example

$$\neg (\forall x\ P(x))\ ;\ \exists x \neg P(x)$$

$\neg \forall x\ P(x)$

Not all students are taking Discrete Structure course.

$\exists x\ \neg P(x)$

There is some students who are not taking Discrete Structure course

Translating from English

- Consider "**For every student in this class, that student has studied calculus**"
- Rephrased: "**For every student x in this class, x has studied calculus**"
  - Let C(x) be "**x has studied calculus**"
  - Let S(x) be "**x is a student in this class**"

$$\forall x\, C(x)$$

  - True if the universe of discourse is **all students in this class**

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Translating from English

- What about if the universe of discourse is all students (or all people?)

  - $\forall x\,(S(x) \wedge C(x))$ X
    - This is wrong! Why? (because this statement says that all people are students in this class and have studied calculus)

  - $\forall x\,(S(x) \rightarrow C(x))$ √

C(x) : "x has studied calculus"
S(x) : "x is a student in this class"

Translating from English

- Consider:
  - “**Some students have visited Mexico**”
  - Rephrasing: “**There exists a student who has visited Mexico**”

- Let:
  - S(x) be “**x is a student**”
  - M(x) be “**x has visited Mexico**”

    $$\exists x\ M(x)$$

  - True if the universe of discourse is all students

Translating from English

- What about if the universe of discourse is all people?

$$\exists x\,(S(x) \land M(x))$$

$\exists x\,(S(x) \rightarrow M(x))$ **This is wrong! Why?**

suppose someone is not student= F->T or F->F, both make the statement true (**refer to truth table $p \rightarrow q$**)

S(x) :"x is a student"
M(x): "x has visited Mexico"

Translating from English

- Consider "**Every student in this class has visited Canada or Mexico**"
- Let, S(x) be "**x is a student in this class**"
  M(x) be "**x has visited Mexico**"
  C(x) be "**x has visited Canada**"

$$\forall x\ (M(x) \vee C(x))$$

(**When the universe of discourse is all students in this class**)

$$\forall x\ (S(x) \rightarrow (M(x) \vee C(x))$$

(**When the universe of discourse is all people or all students**)

Proof Techniques

Mathematical systems consists:

- **Axioms**: assumed to be true.
- **Definitions**: used to create new concepts.
- **Undefined terms**: some terms that are not explicitly defined.
- **Theorem**
  - Statement that can be shown to be true (under certain conditions)
  - Typically stated in one of three ways:
    - As Facts
    - As Implications
    - As Bi-implications

Proof Techniques (Study blk)

Direct Proof (Direct Method)

- Proof of those theorems that can be expressed in the form $\forall x\,(P(x) \rightarrow Q(x))$, *D* is the domain of discourse.
- Select a particular, but arbitrarily chosen, member a of the domain *D.*
- Show that the statement $P(a) \rightarrow Q(a)$ is true. (Assume that $P(a)$ is true).
- Show that $Q(a)$ is true.
- By the rule of Universal Generalization (UG),

  $\forall x\,(P(x) \rightarrow Q(x))$ is true.

Example

"For all integer *x*, if *x* is odd, then x^2 is odd"

Or $P(x)$: x is an odd integer

$Q(x)$: x^2 is an odd integer

$$\forall x(P(x) \rightarrow Q(x))$$

The domain of discourse is set *Z* of all integer.

Can verify the theorem for certain value of *x*.

$x=3$, $x^2=9$; odd

Or show that the square of an odd number is an odd number

Rephrased: "**if *n* is odd, then n^2 is odd**"

Example

- a is an odd integer

$$\Rightarrow a = 2n + 1$$ *for some integer **n***

$$\Rightarrow a^2 = (2n+1)^2$$

$$\Rightarrow a^2 = 4n^2 + 4n + 1$$

$$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$$

$$\Rightarrow a^2 = 2m + 1$$ *where **m= 2n²** + 2n is an integer*

$$\Rightarrow a^2$$ *is an odd integer*

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Proof Techniques

Indirect Proof

- The implication $p \rightarrow q$ is **equivalent** to the implication **($\neg q \rightarrow \neg p$) (contrapositive)**
- Therefore, in order to show that $p \rightarrow q$ is true, one can also **show that the implication ($\neg q \rightarrow \neg p$) is true**.
- To show that ($\neg q \rightarrow \neg p$) is true, assume that the negation of q is true and prove that the negation of p is true.

Example

P(n) **:** ***n*****²+3 is an odd number**

Q(n) **:** ***n*** **is even number**

$$\forall n(P(n) \to Q(n))$$

$$P(n) \to Q(n) \equiv \neg Q(n) \to \neg P(n)$$

¬ *Q(n)* is true , *n* is not even (*n* is odd), so *n=2k+1*

$$n^2 + 3 = (2k+1)^2 + 3$$
$$= 4k^2 + 4k + 1 + 3$$
$$= 4k^2 + 4k + 4$$
$$= 2(2k^2 + 2k + 2)$$

Example

$$n^2 + 3 = (2k+1)^2 + 3$$
$$= 4k^2 + 4k + 1 + 3$$
$$= 4k^2 + 4k + 4$$
$$= 2(2k^2 + 2k + 2)$$

$$t = 2k^2 + 2k + 2$$
$$n^2 + 3 = 2t$$

$\Rightarrow$ *t is integer*

n^2+3 is an even integer, thus ¬ *P(n)* is true

Which to use

- When do you use a direct proof versus an indirect proof?
- If it’s not clear from the problem, try direct first, then indirect second
  - If indirect fails, try the other proofs

Example

Prove that "**if n is an integer and *n^3+5* is odd, then *n* is even**"

Via direct proof

- $n^3+5 = 2k+1$ for some integer *k* (definition of odd numbers)
- $n^3 = 2k - 4$
- $n = \sqrt[3]{2k-4}$
- Umm…

So direct proof didn't work out.

Next up: indirect proof

Example

Prove that "**if n is an integer and *n^3+5* is odd, then *n* is even**"

Via indirect proof

- Contrapositive: If *n* is odd, then n^3+5 is even
- Assume *n* is odd, and show that n^3+5 is even
- n=2k+1 for some integer k (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it is even

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Proof Techniques

Proof by Contradiction

Assume that the hypothesis is true and that the conclusion is false and then, arrive at a contradiction.

Proposition "**if *P* then *Q***"
Proof. **Suppose *P* and *~Q***

Since we have a contradiction, it must be that Q is true

Example

Prove that "**there are infinitely many prime numbers**".

Proof:

- Assume there are not infinitely many prime numbers, therefore they are can be listed, i.e. $p_1, p_2, \ldots, p_n$
- Consider the number $q = p_1 \times p_2 \times \ldots \times p_n + 1$.
- q is either prime or not divisible, but not listed above. Therefore, q is a prime. However, it was not listed.
- **Contradiction**! Therefore, there are infinitely many primes numbers.

Example

- For all real numbers x and y, **if $x+y \geq 2$, then either $x \geq 1$ or $y \geq 1$**.

Proof

- Suppose that the conclusion is false. Then

  $x < 1$ and $y < 1$

  Add these inequalities, $x+y < 1+1 = 2$ (**$x+y < 2$**)
- **Contradiction**
- Thus we conclude that the statement is true.

Example

Suppose $a \in \mathbb{Z}$. **If a^2 is even, then a is even**

Proof

- Contradiction: Suppose a^2 is even and a is not even.
- Then a^2 is even, and a is odd
- Let, $a = 2c + 1$ (odd)

$$a^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1 \qquad \text{(odd)}$$

- **Contradiction**
- Thus we conclude that the statement is true.

Thank You

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