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INSPIRING CREATIVE AND INNOVATIVE MINDS

## Chapter 2 (Part 1)

## Relations



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## Relations



## Relations

- A (binary) relation  $R$  from a set  $X$  to a set  $Y$  is a subset of the Cartesian product  $X \times Y$ .
- If  $(x, y) \in R$ , we write

$x R y$  ( $x$  is related to  $y$ )

Domain = Set of all 1st element in the relation  
Range = Set of all 2nd element in the relation

(Binary) relation from  $X$  to  $Y$ , where  $x \in X$ ,  $y \in Y$ ,  
 $(x, y) \in X \times Y$  and  $R \subseteq X \times Y$

$$x R y \Leftrightarrow (x, y) \in R$$



## Example

- $A = \{ 1, 2, 3, 4 \}$ ,  $B = \{ p, q, r \}$
- $R = \{ (1, q), (2, r), (3, q), (4, p) \}$
- $R \subseteq A \times B$
- $R$  is the relation from  $A$  to  $B$

$1 R q$

$3 \not R p$

$A \times B$

$(1, p), (1, q), (1, r),$   
 $(2, p), (2, q), (2, r),$   
 $(3, p), (3, q), (3, r),$   
 $(4, p), (4, q), (4, r)$



## Example

$$aRb \leftrightarrow a-b \in \mathbb{Z}^{\text{even}}$$

- *Finite set:  $A = \{ 1, 2 \}$ ,  $B = \{ 1, 2, 3 \}$*

$$R = \{ (1, 1), (2, 2), (1, 3) \}$$

- *Infinite set:  $A = \mathbb{Z}$  and  $B = \mathbb{Z}$*

$$R = \{ \dots (-3, -1), (-2, 2), (1, 3), \dots \}$$

(note:  $\mathbb{Z}$  is set of integers)



## Example

- $A = \{ \text{New Delhi, Ottawa, London, Paris, Washington} \}$
- $B = \{ \text{Canada, England, India, France, United States} \}$
- Let  $x \in A, y \in B$ . Define the relation between  $x$  and  $y$  by  
“ $x$  is the capital of  $y$ ”  $(x, y) \in R$
- $R = \{ (\text{New Delhi, India}), (\text{Ottawa, Canada}), (\text{London, England}), (\text{Paris, France}), (\text{Washington, United States}) \}$



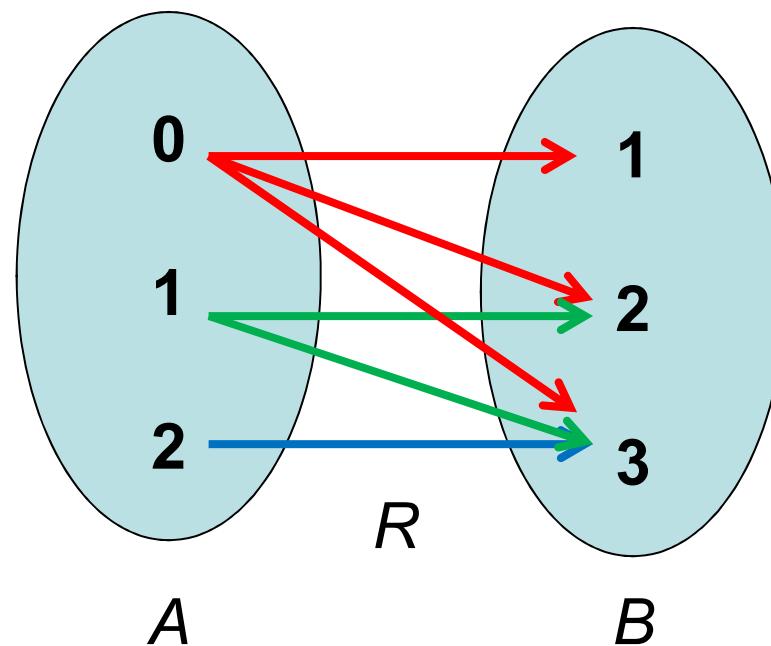
## Example

- “less than” relation from  $A=\{0, 1, 2\}$  to  $B=\{1, 2, 3\}$
- Traditional notation:  
 $0 < 1, 0 < 2, 0 < 3, 1 < 2, 1 < 3, 2 < 3$
- Set notation  
$$A \times B = \{ (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) \}$$
$$R = \{ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) \}$$

## example

$$R = \{ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) \}$$

- Arrow diagrams





## Domain and Range

- Let  $R$ , a relation from  $A$  to  $B$ .
- The set,  $\{a \in A \mid (a, b) \in R \text{ for some } b \in B\}$  is called the **domain** of  $R$ .
- The set,  $\{b \in B \mid (a, b) \in R \text{ for some } a \in A\}$  is called the **range** of  $R$ .
- In case  $A=B$ , we call  $R$  a(binary) **relation on  $A$** .



## Example

- Let  $R$  be a relation on  $X = \{1, 2, 3, 4\}$  defined by  $(x,y) \in R$  if  $x \leq y$ , and  $x, y \in X$ . (x, y) \in R
- Then,  $1 \leq 1, 1 \leq 2, 1 \leq 3, 1 \leq 4, 2 \leq 2, 2 \leq 3, 2 \leq 4, 3 \leq 3, 3 \leq 4, 4 \leq 4$   
$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$
- The domain and range of  $R$  are both equal to  $X$ .



## Example

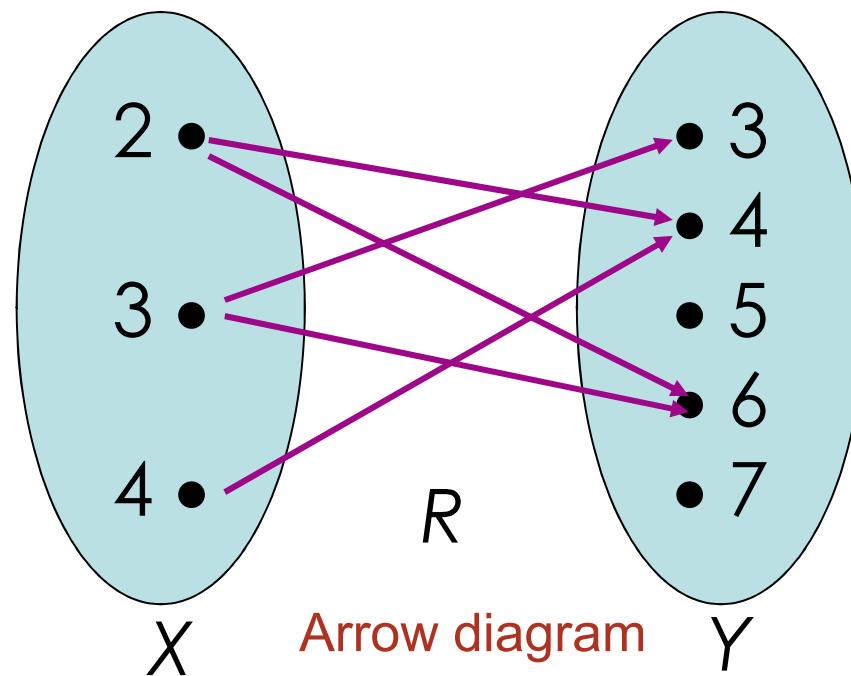
- Let  $X = \{ 2, 3, 4 \}$  and  $Y = \{ 3, 4, 5, 6, 7 \}$   
If we define a relation  $R$  from  $X$  to  $Y$  by,  
 $(x,y) \in R$  if  $x$  divides  $y$  (with zero remainder)
- We obtain,  
$$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$$

The domain of  $R$  is  $\{2,3,4\}$

The range of  $R$  is  $\{3,4,6\}$

## example

$$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$$



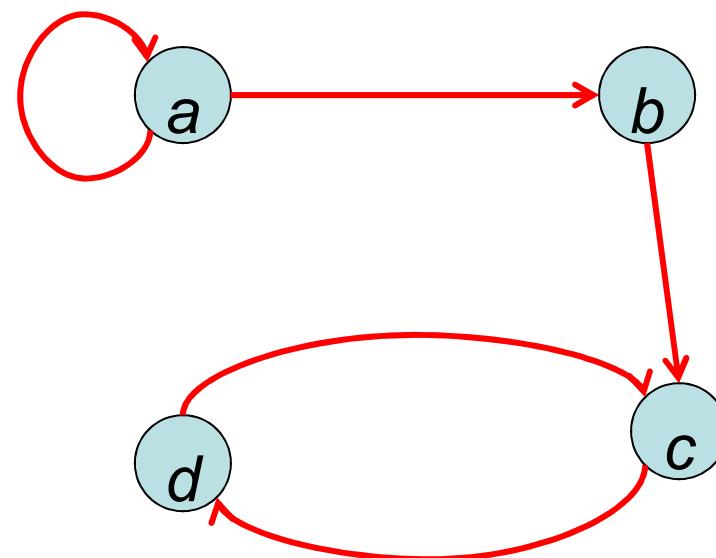


## Digraph

- An informative way to picture a relation on a set is to draw its digraph.
- Let  $R$  be a relation on a finite set  $A$ .
- Draw dots (**vertices**) to represent the elements of  $A$ .
- If the element  $(a,b) \in R$ , draw an arrow (called a **directed edge**) from  $a$  to  $b$  .

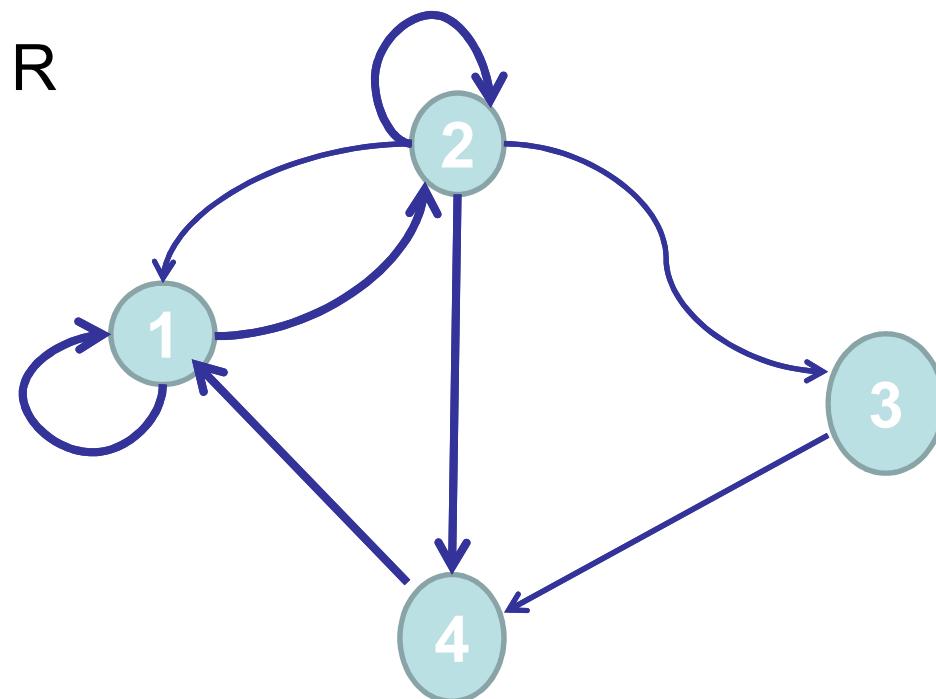
## example

- The relation  $R$  on  $A = \{a, b, c, d\}$ ,  
 $R = \{(a, a), (a, b), (c, d), (d, c), (b, c)\}$



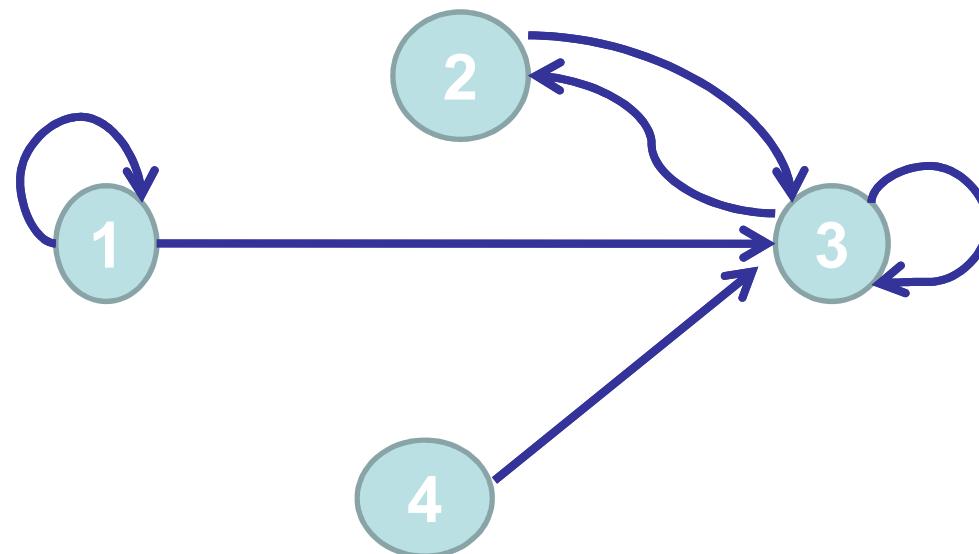
## example

- Let,  $A = \{ 1,2,3,4\}$  and  $R = \{ (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)\}$
- Draw the digraph of  $R$



## Example

Find the relation determined by digraph below.



- Since  $a R b$  if and only if there is an edge from  $a$  to  $b$ , so  
$$R = \{ (1,1), (1,3), (2,3), (3,2), (3,3), (4,3) \}$$



## Matrices of Relations

- A matrix is a convenient way to represent a relation  $R$  from  $A$  to  $B$ .
- Label the rows with the elements of  $A$  (in some arbitrary order)
- Label the columns with the elements of  $B$  (in some arbitrary order)



## Matrices of Relations

- Let  $A=\{a_1, a_2, \dots, a_n\}$  and  $B=\{b_1, b_2, \dots, b_p\}$  be finite nonempty sets.
- Let  $R$  be a relation from  $A$  into  $B$ .
- Let  $M_R = [m_{ij}]_{n \times p}$  be the Boolean  $n \times p$  matrix, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$



## Matrices of Relations

$$M_R = \begin{bmatrix} m_{11} & m_{12} & \dots & \dots & m_{1p} \\ m_{21} & m_{22} & \dots & \dots & m_{2p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & \dots & m_{np} \end{bmatrix}$$



## example

- Let  $A = \{1, 3, 5\}$  and  $B = \{1, 2\}$
- Let  $R$  be a relation from  $A$  to  $B$  and  $R = \{(1,1), (3,2), (5,1)\}$
- Then the matrix represent  $R$  is

$$\begin{matrix} & & 1 & 2 \\ & 1 & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 3 & & & \\ 5 & & & \end{matrix}$$



## example

- The relation,

$$R = \{ (1,b), (1,d), (2,c), (3,c), (3,b), (4,a) \}$$

from,  $X = \{ 1, 2, 3, 4 \}$  to  $Y = \{ a, b, c, d \}$

|   | <i>a</i> | <i>b</i> | <i>c</i> | <i>d</i> |
|---|----------|----------|----------|----------|
| 1 | 0        | 1        | 0        | 1        |
| 2 | 0        | 0        | 1        | 0        |
| 3 | 0        | 1        | 1        | 0        |
| 4 | 1        | 0        | 0        | 0        |

or

|   | <i>d</i> | <i>b</i> | <i>a</i> | <i>c</i> |
|---|----------|----------|----------|----------|
| 2 | 0        | 0        | 0        | 1        |
| 3 | 0        | 1        | 0        | 1        |
| 4 | 0        | 0        | 1        | 0        |
| 1 | 1        | 1        | 0        | 0        |



## example

- The matrix of the relation  $R$  from  $\{ 2, 3, 4 \}$  to  $\{ 5, 6, 7, 8 \}$  defined by

$x R y$  if  $x$  divides  $y$

|   |   |   |   |   |
|---|---|---|---|---|
|   | 5 | 6 | 7 | 8 |
| 2 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 1 |

## example

- Let  $A = \{ a, b, c, d \}$
- Let  $R$  be a relation on  $A$ .
- $R = \{ (a,a), (b,b), (c,c), (d,d), (b,c), (c,b) \}$

|     | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $a$ | 1   | 0   | 0   | 0   |
| $b$ | 0   | 1   | 1   | 0   |
| $c$ | 0   | 1   | 1   | 0   |
| $d$ | 0   | 0   | 0   | 1   |



## In Degree and Out Degree

- If  $R$  is a relation on a set  $A$  and  $a \in A$ , then the **in-degree** of  $a$  (relative to relation  $R$ ) is the number of  $b \in A$  such that  $(b, a) \in R$ .
- The **out degree** of  $a$  is the number of  $b \in A$  such that  $(a, b) \in R$



## In Degree and Out Degree

- Meaning that, in terms of the digraph of  $R$ , is that the in-degree of a vertex is  
“the number of edges terminating at the vertex”
- The out-degree of a vertex is  
“ the number of edges leaving the vertex”



## Example

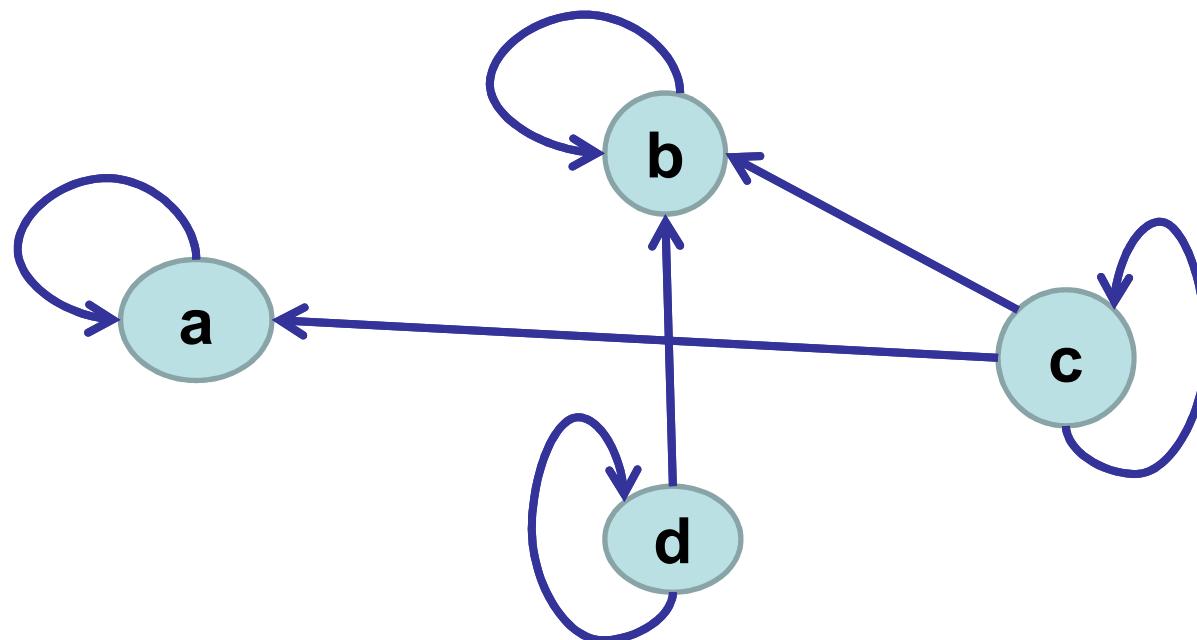
- Let  $A = \{a, b, c, d\}$ , and let  $R$  be the relation on  $A$  that has the matrix (given below)

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Construct the digraph of  $R$ , and list in-degrees and out-degrees of all vertices.

## Example

|            | a | b | c | d |
|------------|---|---|---|---|
| In-degree  | 2 | 3 | 1 | 1 |
| Out-degree | 1 | 1 | 3 | 2 |





## Reflexive Relations

■ A relation  $R$  on a set  $X$  is called **reflexive** if  $(x,x) \in R$  for every  $x \in X$ .

■ That is, if  $xRx$  for all  $x \in X$ .

**( $R$  is reflexive if every element  $x \in X$  is related to itself)**



## Example

- The relation  $R$  on  $X = \{1, 2, 3, 4\}$  defined by  $(x,y) \in R$  if  $x \leq y$ ,  $x, y \in X$  is reflexive because for each element  $x \in X$ ,  $(x,x) \in R$
- $(1,1), (2,2), (3,3), (4,4)$  are each in  $R$ .



## example

- The relation,

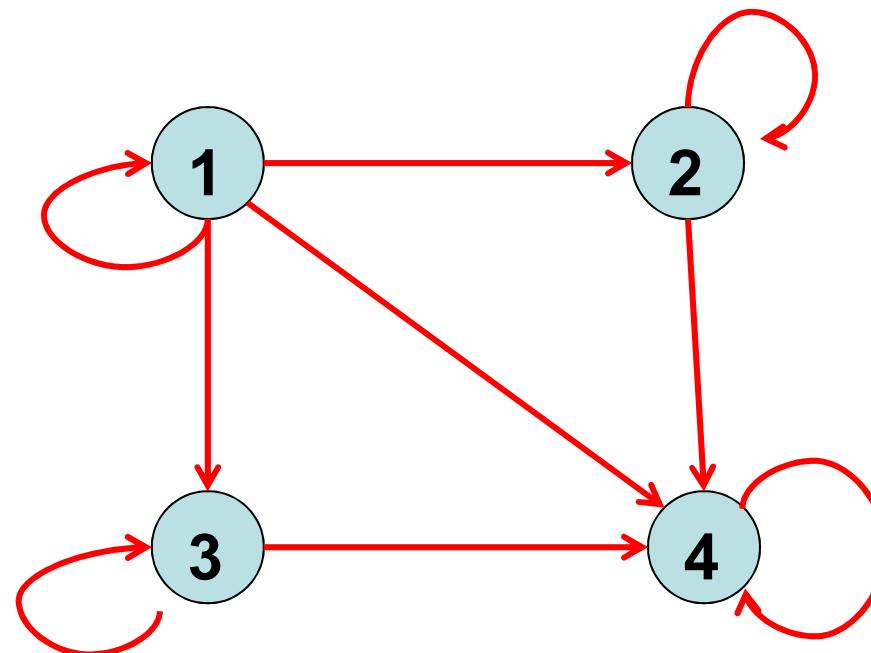
$$R = \{ (a,a), (b,c), (c,b), (d,d) \}$$

on  $X=\{a, b, c, d\}$  is not reflexive.

- For example,  $b \in X$ , but  $(b,b) \notin R$

## Reflexive Relations

- The digraph of a reflexive relation has a loop at every vertex.
- example





### ■ Irreflexive

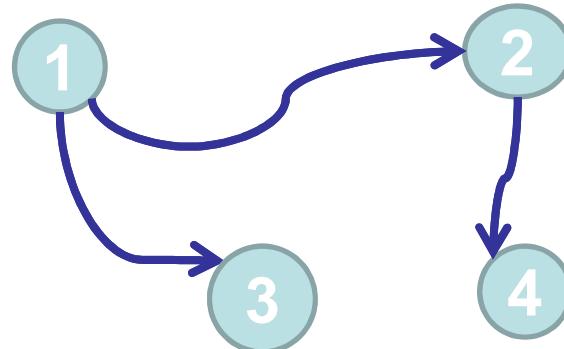
- A relation  $R$  on a set  $A$  is **irreflexive** if  $xR\bar{x}$  or  $(x,x)\notin R$ ;  $\forall x:x\in X$

### ■ Not Reflexive

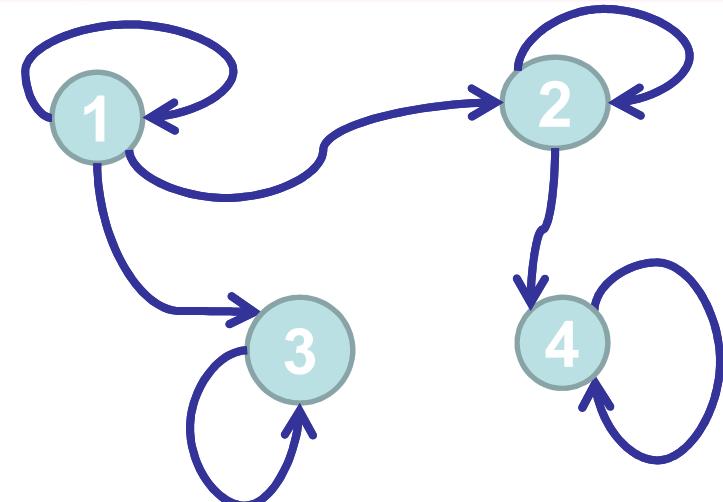
- A relation  $R$  is **not reflexive** if at least one pair of  $(x,x)\notin R$ ,  $\forall x:x\in X$

## example

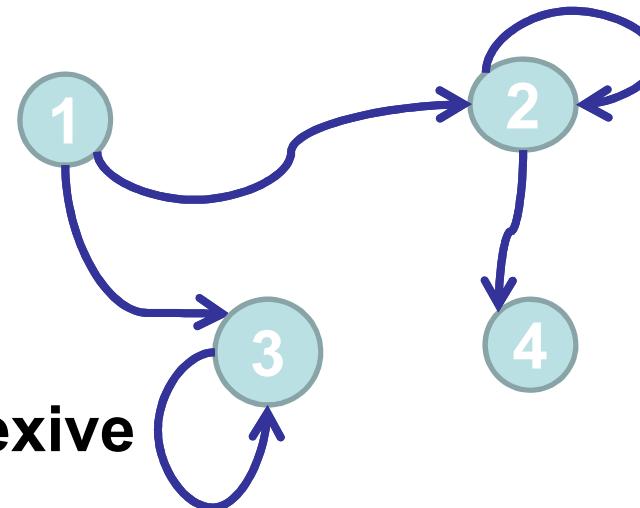
**Reflexive**



**Irreflexive**



**Not reflexive**





## Example

- Consider the following relations on the set  $\{1, 2, 3\}$

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3)\} \quad \text{= Reflexive}$$
$$R_2 = \{(1,1), (1,3), (2,2), (3,1)\} \text{ no } (3,3) \quad \text{= not reflexive}$$
$$R_3 = \{(2,3)\} \text{ no } (1,1), (2,2), (3,3) \quad \text{= irreflexive}$$
$$R_4 = \{(1,1)\} \quad \text{= not reflexive}$$

Which of them are reflexive?



## Reflexive Relations

- The relation  $R$  is reflexive if and only if the matrix of relation has 1's on the main diagonal.
- example

|          | <i>a</i> | <i>b</i> | <i>c</i> | <i>d</i> |
|----------|----------|----------|----------|----------|
| <i>a</i> | 1        | 0        | 0        | 0        |
| <i>b</i> | 0        | 1        | 1        | 0        |
| <i>c</i> | 0        | 1        | 1        | 0        |
| <i>d</i> | 0        | 0        | 0        | 1        |



## Reflexive Relations

The relation  $R$  is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

- example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$



## example

- The relation  $R$  is **not reflexive**.

|     | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $a$ | 1   | 0   | 0   | 0   |
| $b$ | 0   | 0   | 1   | 0   |
| $c$ | 0   | 1   | 1   | 0   |
| $d$ | 0   | 0   | 0   | 1   |

$b \in X$   
 $(b, b) \notin R$



## Symmetric Relations

- A relation  $R$  on a set  $X$  is called symmetric if for all  $x, y \in X$ , if  $(x, y) \in R$ , then  $(y, x) \in R$ .

$$\forall x, y \in X, (x, y) \in R \rightarrow (y, x) \in R$$

- Let  $M$  be the matrix of relation  $R$ .  
The relation  $R$  is symmetric if and only if for all  $i$  and  $j$ , the  $ij$ th entry of  $M$  is equal to the  $ji$ th entry of  $M$ .



## Symmetric Relations

- The matrix of relation  $M_R$  is symmetric if  $M_R = M_R^T$
- Example

$$M_R = \begin{array}{cccc} & a & b & c & d \\ a & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 1 \end{array} = M_R^T$$



## Symmetric Relations

- The digraph of a symmetric relation has the property that whenever there is a directed edge from  $v$  to  $w$ , there is also a directed edge from  $w$  to  $v$ .





## example

- The relation  $R = \{ (a,a), (b,c), (c,b), (d,d) \}$  on  $X = \{ a, b, c, d \}$

$(b,c) \in R$   
 $(c,b) \in R$

|          | <i>a</i> | <i>b</i> | <i>c</i> | <i>d</i> |
|----------|----------|----------|----------|----------|
| <i>a</i> | 1        | 0        | 0        | 0        |
| <i>b</i> | 0        | 0        | 1        | 0        |
| <i>c</i> | 0        | 1        | 0        | 0        |
| <i>d</i> | 0        | 0        | 0        | 1        |

symmetric



## example

- The relation  $R$  on  $X = \{ 1, 2, 3, 4 \}$ , defined by  
 $(x,y) \in R \quad \text{if } x \leq y, x,y \in X$

$(2,3) \in R$   
 $(3,2) \notin R$

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 1 | 1 |
| 3 | 0 | 0 | 1 | 1 |
| 4 | 0 | 0 | 0 | 1 |

not symmetric



## Antisymmetric Relations

- A relation,  $R$  on a set  $X$  is called antisymmetric, if for all  $x, y \in X$ , if  $(x, y) \in R$  and  $x \neq y$ , then  $(y, x) \notin R$ .
- A relation  $R$  on set  $X$  is antisymmetric if  $x \neq y$ , whenever  $xRy$ , then  $yR\cancel{x}$ . In other word if whenever  $xRy$ , then  $yRx$  then it implies that  $x=y$

$$\forall x, y \in A, (x, y) \in R \wedge x \neq y \rightarrow (y, x) \notin R$$

Or

$$\forall x, y \in A, (x, y) \in R \wedge (y, x) \in R \rightarrow x = y$$



## Antisymmetric Relations

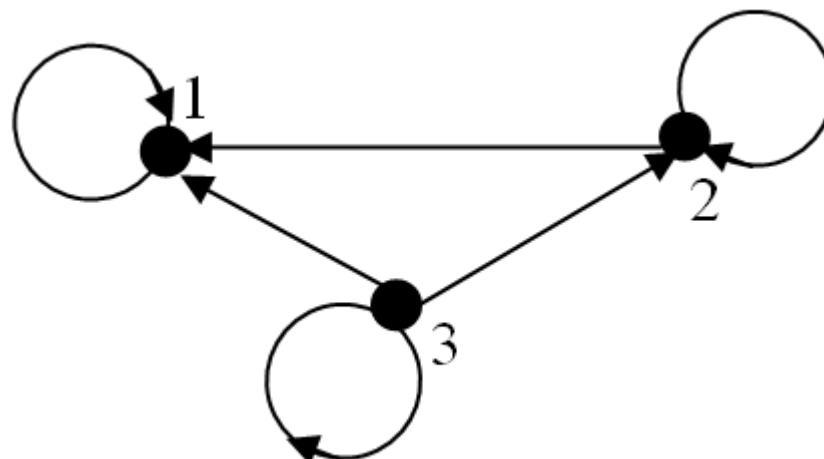
- Matrix  $M_R = [M_{ij}]$  of an antisymmetric relation  $R$  satisfies the property that if  $i \neq j$ , then  $m_{ij}=0$  or  $m_{ji}=0$ .
- If  $R$  is antisymmetric relation, then for different vertices  $i$  and  $j$  there cannot be an edge from vertex  $i$  to vertex  $j$  and an edge from vertex  $j$  to vertex  $i$
- At least one directed relation and one way

## Example

- Let  $R$  be a relation on  $A = \{1, 2, 3\}$  defined as  $(a, b) \in R$  if  $a \geq b$ ,  $a, b \in A$  is an antisymmetric relation because for all  $a, b \in A$ ,  $(a, b) \in R$  and  $a \neq b$ , then  $(b, a) \notin R$ , for example

$(3, 2) \in R$  but  $(2, 3) \notin R$

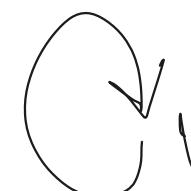
$(3, 3) \in R$  and  $(3, 3) \in R$  implies  $a = b$



## example

- The relation  $R$  on  $X = \{ 1, 2, 3, 4 \}$  defined by,

$(x,y) \in R$  if  $x \leq y, x,y \in X$



$(1,2) \in R$   
 $(2,1) \notin R$

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 1 | 1 |
| 3 | 0 | 0 | 1 | 1 |
| 4 | 0 | 0 | 0 | 1 |

antisymmetric



## example

- The relation  $R = \{ (a,a), (b,c), (c,b), (d,d) \}$  on  $X = \{ a, b, c, d \}$

$(b,c) \in R$   
 $(c,b) \in R$

|     | $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|-----|
| $a$ | 1   | 0   | 0   | 0   |
| $b$ | 0   | 0   | 1   | 0   |
| $c$ | 0   | 1   | 0   | 0   |
| $d$ | 0   | 0   | 0   | 1   |

not antisymmetric



## example

- The relation

$R = \{ (a,a), (b,b), (c,c) \}$   
on  $X = \{ a, b, c \}$

- $R$  has no members of the form  $(x,y)$  with  $x \neq y$ , then  $R$  is **antisymmetric**

|     |     |     |     |
|-----|-----|-----|-----|
|     | $a$ | $b$ | $c$ |
| $a$ | 1   | 0   | 0   |
| $b$ | 0   | 1   | 0   |
| $c$ | 0   | 0   | 1   |



## example

- Antisymmetric
- Reflexive
- Symmetric

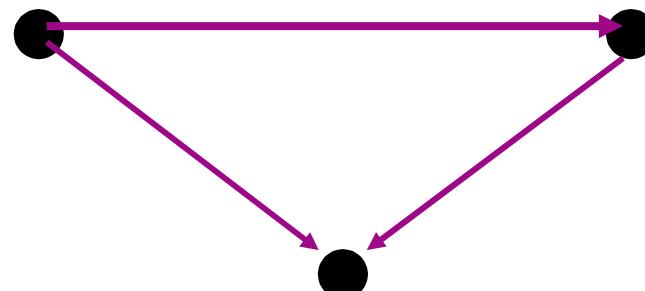
|     | $a$ | $b$ | $c$ |
|-----|-----|-----|-----|
| $a$ | 1   | 0   | 0   |
| $b$ | 0   | 1   | 0   |
| $c$ | 0   | 0   | 1   |

- “Antisymmetric” is not the same as “not symmetric”



## Antisymmetric Relations

- The digraph of an antisymmetric relation has at most one directed edge between each pair of vertices.
- Example





A relation  $R$  on set  $A$  is asymmetric if whenever  $aRb$ , then  $b \not Ra$ .

$$\forall x, y \in X, (x, y) \in R \rightarrow (y, x) \notin R$$

In this sense, a relation is asymmetric if and only if it is both antisymmetric and irreflexive.



- The matrix  $M_R = [m_{ij}]$  of an asymmetric relation  $R$  satisfies the property that
  - If  $m_{ij} = 1$  then  $m_{ji} = 0$
  - $m_{ii} = 0$  for all  $i$  (the main diagonal of matrix  $M_R$  consists entirely of 0's or otherwise)
- If  $R$  is asymmetric relation, then the digraph of  $R$  cannot simultaneously have an edge from vertex  $i$  to vertex  $j$  and an edge from vertex  $j$  to vertex  $i$
- All edges are “one way street”



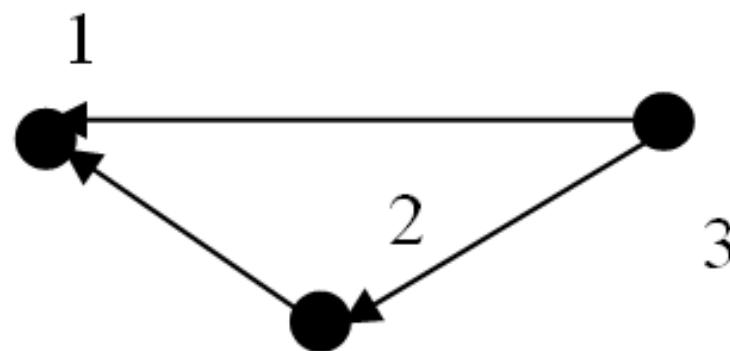
## Example

- Let  $R$  be the relation on  $A = \{1, 2, 3\}$  defined by  $(a, b) \in R$  if  $a > b$ ,  $a, b \in A$  is an asymmetric relation because,

$(2, 1) \in R$  but  $(1, 2) \notin R$

$(3, 1) \in R$  but  $(1, 3) \notin R$

$(3, 2) \in R$  but  $(2, 3) \notin R$





## Not Symmetric

- Let  $R$  be a relation on a set  $A$ .
- Then  $R$  is called ***not symmetric***, if for all  $a, b \in A$ , if  $(a, b) \in R$ , there exist  $(b, a) \notin R$ .

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$



## Not Symmetric and Not Antisymmetric

- Let  $R$  be a relation on a set  $A$ . Then  $R$  is called **not symmetric** and **not antisymmetric**, if for all  $a, b \in A$ , if  $(a, b) \in R$ , there exist  $(b, a) \in R$  and if  $(a, b) \in R$ , there exist  $(b, a) \in R$ .

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

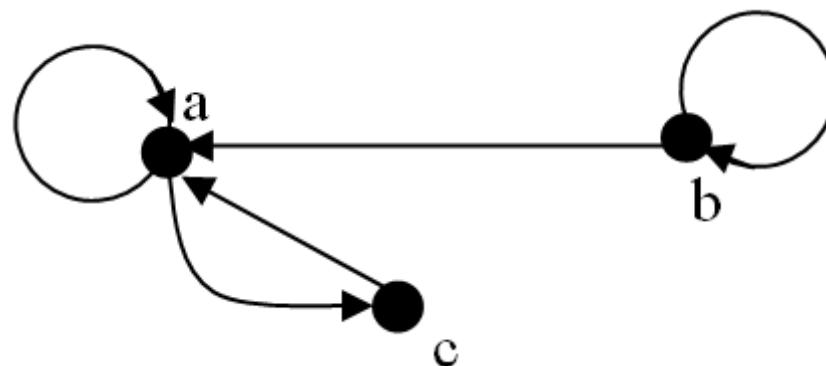
AND

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

## Example

- Relation  $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$  on  $A = \{a, b, c\}$  is not symmetric and not antisymmetric relation because there is,

$(a, c), (c, a) \in R$  and also  $(b, a) \in R$  but  $(a, b) \notin R$





## Example

1. Let  $A=\mathbb{Z}$ , the set of integers and let  $R=\{(a,b) \in A \times A \mid a < b\}$ .  
So that  $R$  is the relation “less than”.  
Is  $R$  symmetric, asymmetric or antisymmetric?
2. Let  $A=\{1,2,3,4\}$  and let  $R = \{(1,2), (2,2), (3,4), (4,1)\}$   
Determine whether  $R$  symmetric, asymmetric or antisymmetric.



## Example

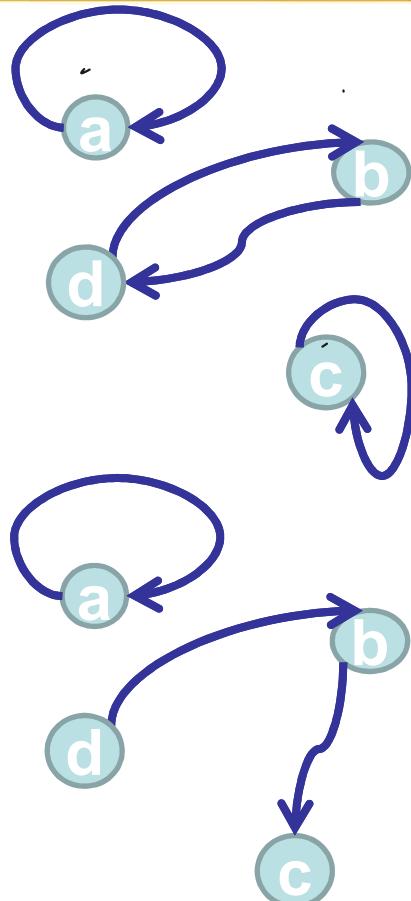
### Question 1

- **Symmetric** : If  $a < b$ , then it is not true that  $b < a$ , so  $R$  is not symmetric
- **Assymmetric** : If  $a < b$  then  $b > a$  ( $b$  is greater than  $a$ ), so  $R$  is assymmetric
- **Antisymmetric** : If  $a \neq b$ , then either  $a > b$  or  $b > a$ , so  $R$  is antisymmetric

### Question 2

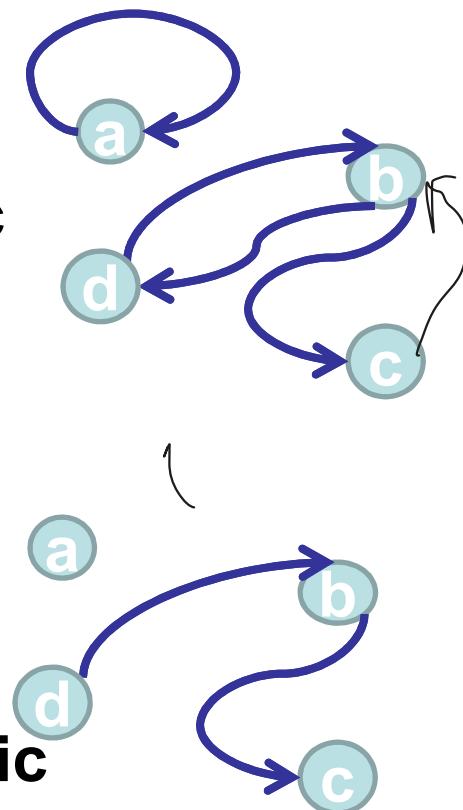
- $R$  is not symmetric since  $(1,2) \in R$ , but  $(2,1) \notin R$
- $R$  is not assymmetric , since  $(2,2) \in R$
- $R$  is antisymmetric, since  $a \neq b$ , either  $(a,b) \in R$  or  $(b,a) \in R$

## Summary on Symmetric



**Symmetric**

**Not Symmetric**



**Antisymmetric**

**Asymmetric**



## Transitive Relations

- A relation  $R$  on a set  $X$  is called transitive if for all  $x,y,z \in X$ ,  
if  $(x,y)$  and  $(y,z) \in R$  then  $(x,z) \in R$
- It is often convenient to say what it means for a relation to be not transitive.
- A relation  $R$  on  $X$  is **not transitive** if there exists  $x, y$ , and  $z$  in  $X$  so that  $xRy$  and  $yRz$ , but  $xRz$ . If such  $x, y$ , and  $z$  do not exist, then  $R$  is transitive.



## Example

- The relation  $R = \{ (a,a), (b,c), (c,b), (d,d) \}$  on  $X = \{ a, b, c, d \}$  is not transitive.
- $(b,c)$  and  $(c,b) \in R$ , but  $(b,b) \notin R$ .



## Transitive Relations

- Let  $M_R$  be the matrix of relation R. Let,

$$M_R \otimes M_R = N. \quad (\otimes \text{ Boolean product})$$

$$M_R = [m_{ij}] \text{ and } N = [n_{ij}]$$

The relation  $R$  is **transitive** if and only if the following is true:

$$\forall i \forall j, \text{ if } (n_{ij} = 1) \text{ then } (m_{ij} = 1)$$

The relation  $R$  is **not transitive** if and only if:

$$\exists i \exists j \ (n_{ij} = 1) \wedge (m_{ij} = 0)$$



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## Boolean Algebra

| + | 1 | 0 |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |

| . | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 0 | 0 |

INSPIRING CREATIVE AND INNOVATIVE MINDS



## Example

Let  $R$  be a relation on  $A=\{1,2,3\}$  is defined by  $(a,b) \in R$  if  $a \leq b$ ,  $a,b \in A$ . Find  $R$ . Is  $R$  a transitive relation?

**Solution:**

$$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

$R$  is a transitive relation because

$$(1,2) \text{ and } (2,2) \in R, (1,2) \in R$$

$$(1,2) \text{ and } (2,3) \in R, (1,3) \in R$$

$$(1,3) \text{ and } (3,3) \in R, (1,3) \in R$$

$$(2,2) \text{ and } (2,3) \in R, (2,3) \in R$$

$$(2,3) \text{ and } (3,3) \in R, (2,3) \in R$$



## example

The matrix of relation  $M_R$ ,

$$M_R = \begin{bmatrix} & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$\forall i \forall j, \text{ if } (n_{ij} = 1)$   
 $\text{then } (m_{ij} = 1)$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\nearrow 1 = 1$   
 $\circ = 0$   
 $\nwarrow M$

Note that,  $(1,2)$  and  $(2,3) \in R$ ,  $(1,3) \in R$



## Example

- Consider the following relations on the set  $\{1, 2, 3\}$

$$R1 = \{ (1,1), (1,2), (2,3) \}$$

$$R2 = \{ (1,2), (2,3), (1,3) \}$$

- Which of them is transitive?



## Example

$$R1 = \{ (1,1), (1,2), (2,3) \}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\forall i \forall j$ , if  $(n_{ij} = 1)$  then  $(m_{ij} = 1)$

$(n_{13} = 1) \wedge (m_{13} = 0)$

$(1,2)$  and  $(2,3) \in R$ ,  $(1,3) \notin R$

**Not transitive**



## Example

$$R2 = \{ (1,2), (2,3), (1,3) \}$$

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$\forall i \forall j$ , if  $(n_{ij} = 1)$  then  $(m_{ij} = 1)$

$(n_{13} = 1)$  then  $(m_{13} = 1)$

$(1,2)$  and  $(2,3) \in R$ ,  $(1,3) \in R$

$R$  is a transitive relation



## Example

The relation  $R$  on  $A=\{a,b,c,d\}$  is  $R=\{(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)\}$  is not transitive. The matrix of relation  $M_R$ ,

$$M_R = \begin{bmatrix} & a & b & c & d \\ a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{bmatrix}$$

$(n_{12} = 1) \wedge (m_{12} = 0)$

The product of boolean,

$$\begin{bmatrix} & a & b & c & d \\ a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} & a & b & c & d \\ a & 1 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & a & b & c & d \\ a & 1 & 1 & 1 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that,  $(a,c)$  and  $(c,b) \in R$ ,  $(a,b) \notin R$

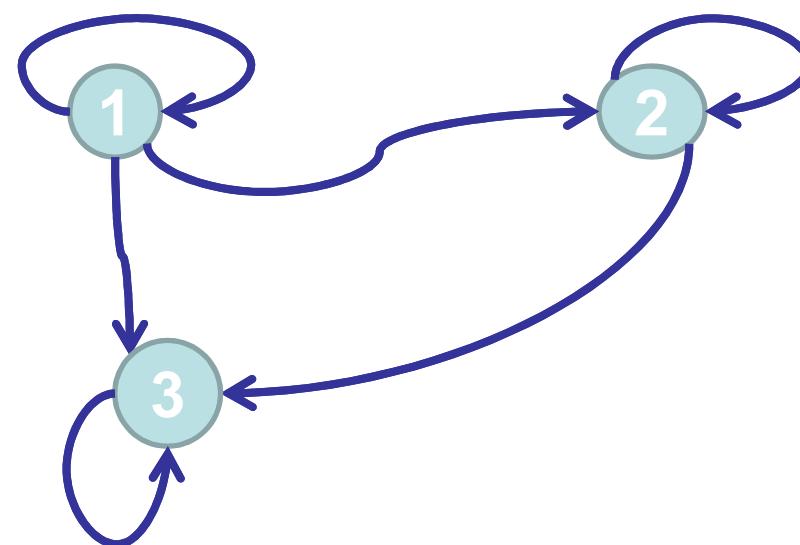


## Transitive Relations

In the digraph of  $R$ ,  $R$  is a transitive relation if and only if there is a directed edge from one vertex  $a$  to another vertex  $b$ , and if there exists a directed edge from vertex  $b$  to vertex  $c$ , then there must exist a directed edge from  $a$  to  $c$

$$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

The diagram:





## Equivalence Relations

- A relation  $R$  that is reflexive, symmetric and transitive on a set  $X$  is called an equivalence relation on  $X$ .



## Example

Let  $R=\{(1,1), (1,3), (2,2), (3,1), (3,3)\}$  on  $\{1,2,3\}$ , the matrix of the relation  $M_R$ ,

$$M_R = \begin{bmatrix} & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive



## Example

Let  $R=\{(1,1), (1,3), (2,2), (3,1), (3,3)\}$  on  $\{1,2,3\}$ , the matrix of the relation  $M_R$ ,

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive



## example

The transpose matrix  $M_R$ ,  $M_R^T$  is equal to  $M_R$ , so  $R$  is symmetric

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix} \quad M_R^T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

The product of Boolean show that the matrix is transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$



## Example

- The relation,  $R=\{ (1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5) \}$   
on  $\{ 1,2,3,4,5 \}$

- Reflexive?
- Symmetric?
- Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$



## example

### ■ Reflexive?

|   |   |   |   |   |
|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 |

Reflexive ✓



## example

### ■ Symmetric?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Symmetric ✓



## example

### ■ Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$



## example

- Reflexive
- Symmetric
- Transitive



Equivalence relation

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$



## example

- The relation  $R$  on  $X=\{ 1, 2, 3, 4 \}$ , defined by  
$$(x,y) \in R \quad \text{if } x \leq y, x,y \in X$$
- Not symmetric
  - $(2,3) \in R$  but  $(3,2) \notin R$
- $R$  is not equivalence relation on  $X$ .



## Partial Orders

- A relation,  $R$  on a set  $X$  is called a **partial order** if  $R$  is **reflexive, antisymmetric, and transitive**.



## Example

Let  $R$  be a relation on a set  $A=\{1,2,3\}$  defined by  $(a,b) \in R$  if  $a \leq b$ ,  $a, b \in R$ .

$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

$R$  is reflexive, antisymmetric and transitive.

So  $R$  is a partial order relation.



## Example

- The relation  $R$  defined on the positive integers by  
$$(x,y) \in R \quad \text{if } x \text{ divides } y \text{ (evenly)}$$

is reflexive, antisymmetric and transitive

- $R$  is a partial order.



## Example

- The relation  $R$  defined on the positive integers by  
$$(x,y) \in R \text{ if } x \text{ divides } y \text{ (evenly)}$$

is reflexive, antisymmetric and transitive

- $R$  is a partial order.