Numerical Options Pricing

Ting-Wei Su, Chien-Hao Wu, Wan-Yun Yang

Department of Mathematics, National Chiao Tung University

Jan 8, 2018

What is a Derivative?

A derivative is an instrument whose value depends on the values of other more basic underlying variables.

Call and Put Options:

1. A *call* option gives the owner the right to *buy* an asset by a certain date for a certain price.
2. A *put* option gives the owner the right to *sell* an asset by a certain date for a certain price.

   The stipulated price is called the *exercise price* or the *strike price*; the date in the contract is known as the *expiration date* or the *maturity time*.

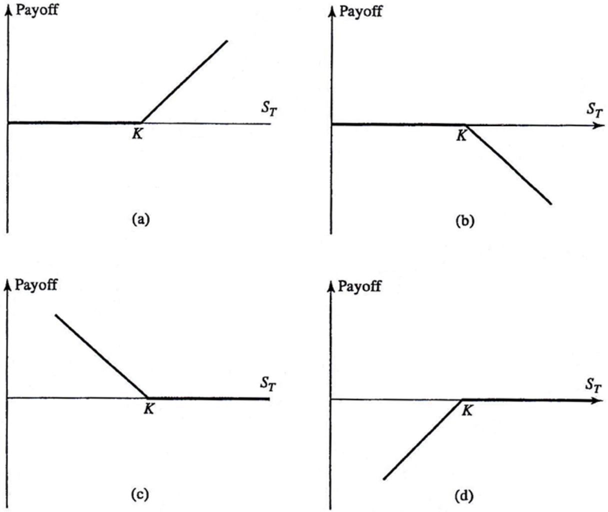

Figure 9.5 Payoffs from positions in European options: (a) long call, (b) short call, (c) long put, (d) short put. Strike price = K; price of asset at maturity = S_T

- European options can only be exercised at maturity time.
- American options, on the other hand, can be exercised at any time during its life.

These give us four basic options (vanilla options):

- European Call
- European Put
- American Call
- American Put

Determination of the Option Price:

1. Law of Supply and Demand gives the equilibrium price.
2. No riskless arbitrage opportunities exist in the market.
3. Efficient Market Hypothesis (EMH)

The Black-Scholes Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$$

- V: the value of the option considered
- S_0: the current stock price
- K: the strike price or the exercise price
- T: the maturity time or the expiration time
- σ: the volatility of the stock price
- r: the risk-free interest rate
- q: the expected dividend yield

We assume $q = 0$ throughout the talk, i.e., we only consider non-dividend-paying stocks. In short, we seek to solve

$$V(S, t; S_0, K, T, \sigma, r)$$

under some boundary conditions (to be returned in the second part).

Dimensional Analysis:

We take the dimension on both sides of the equation

$$\frac{[V]}{[T]} + [\sigma^2][S]^2\frac{[V]}{[S]^2} + [r][S]\frac{[V]}{[S]} - [r][V] = 0.$$

It reduces to

$$[\sigma^2] + [r] = \frac{1}{[T]}.$$

For this equation to hold, we must have

$$[\sigma^2] = \frac{1}{[T]}, \ [r] = \frac{1}{[T]}.$$

These two relations play important roles in deriving the transformation from the Black-Scholes equation to the standard heat equation.

Discretization Schemes

1. Heat-equation approach with Crank-Nicolson method
2. Direct Forward-Time Central-Space (FTCS) method
3. Direct Crank-Nicolson method

Heat-Equation Approach

Motivation:

- It's a one-dimensional heat equation in disguise.
- It's a backward (reverse) diffusion equation (see the coefficient).
- The present value can only be solved from the future payoff.

What are the Correct Boundary Conditions?

Boundary conditions for European Call:

$$V(0,t)=0,\ lim_{S\to\infty}V(S,t)\sim S,$$

$$V(S,T)=max(S-K,0).$$

Boundary conditions for European Put:

$$V(0,t)=Ke^{-r(T-t)},\ lim_{S\to\infty}V(S,t)\sim 0,$$

$$V(S,T)=max(K-S,0).$$

Boundary conditions for American Call:

$$V(0,t) = 0,\ lim_{S\to\infty} V(S,t) \sim S,$$

$$V(S,T) = max(S-K,0).$$

Boundary conditions for American Put:

$$V(0,t) = K,\ lim_{S\to\infty} V(S,t) \sim 0,$$

$$V(S,T) = max(K-S,0).$$

One important thing to note is that some conditions here only exist in mathematical sense.

How to Truncate the Conditions at Infinity?

Let lowercase letters c and p be the values of European call and put, and the upperecase letter C and P be the values of American call and put.

$$lim_{S\to\infty} c(S,t) \sim S \sim S_{max} - Ke^{-r(T-t)}$$

$$lim_{S\to\infty} C(S,t) \sim S \sim S_{max} - K$$

The option becomes approximately a forward contract with forward price equal to the strike price K and the stock price is S_{max} at each time t.

From Black-Scholes Equation to Heat Equation

Consider the following change of variables

$$S = Ke^x,\ t = T - \tau / \frac{1}{2}\sigma^2,\ V = Kv(x, \tau)$$

This results in the equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \tag{1}$$

where $k = r/\frac{1}{2}\sigma^2$. The initial condition becomes

$$v(x, 0) = max(e^x - 1, 0).$$

Notice that, in particular, we only have *one* dimensionless parameter $k = r/\frac{1}{2}\sigma^2$, although there are *four* dimensional parameters, K, T, σ^2 and r, in the original problem.

Equation (1) now looks more like a standard heat equation, but we still need a little work to transform it into our final goal by putting

$$v = e^{\alpha x + \beta \tau} u(x, \tau),$$

for some α and β to be determined.

Calculation yields

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

We can eliminate the u term and the $\frac{\partial u}{\partial x}$ term by choosing

$$\alpha = -\frac{1}{2}(k-1), \ \beta = -\frac{1}{4}(k+1)^2.$$

The suitable choice is then found to be

$$v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau).$$

After the transformation, we obtain the following

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \text{ for } -\infty < x < \infty, \ \tau > 0,$$

where

$$u(x,0) = u_0(x) = max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0). \tag{2}$$

We can then apply any finite difference method to solve it.

Forward-Time Central-Space Method

We directly apply the FTCS method to the original Black-Scholes equation by flipping the time horizon.

Let $\Delta t = \frac{T}{N}$ and $\Delta S = \frac{S_{max}}{M}$. Denote V_j^n as the approximated value for $v(S_j, t_n)$. Then the discretization scheme yields

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{(\Delta S)^2} + rj\Delta S \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta S} - rV_j^{n+1} = 0$$

$$\forall\, j = 1, 2, ..., M-1, \text{and } n = 0, 1, ..., N-1.$$

It can be rearranged to

$$V_j^n = (\alpha_j + \beta_j)V_{j+1}^{n+1} + (1 - r\Delta t - 2\alpha_j)V_j^{n+1} + (\alpha_j - \beta_j)V_{j-1}^{n+1},$$

where $\alpha_j = \frac{1}{2}\sigma^2 j^2 \Delta t$, and $\beta_j = \frac{1}{2} rj\Delta t$.

The local truncation error (LTE) τ_j^n is characterized by the approximation from grid points at $t = n + 1$.

Proposition

The LTE is given by the formula

$$\tau_j^n = -\frac{\Delta t}{2}(v_{tt})_j^{n+1} + \frac{(\Delta S)^2}{24}\left[2jr(v_{SSS})_j^{n+1} + j^2\sigma^2(v_{SSSS})_j^{n+1}\right] + O(\Delta t^2 + (\Delta S)^2),$$

$$\forall\, j = 1, 2, ..., M-1, \text{ and } n = 0, 1, ..., N-1.$$

If, in addition, $|v_{tt}| \leq M_1$, $|v_{SSS}| \leq M_2$ and $|v_{SSSS}| \leq M_3$, we can rewrite the formula as

$$||\tau||_\infty \leq \frac{\Delta t}{2}\left[M_1 + \frac{1}{12\gamma}(2rMM_2 + \sigma^2 M^2 M_3)\right],$$

where $\gamma = \frac{\Delta t}{(\Delta S)^2}$. If γ is fixed, then $||\tau||_\infty \to 0$ as $\Delta t \to 0$ yielding consistency.

Proposition

Let $e_j^n = V_j^n - v_j^n$. *Then*

$$||e^1||_\infty \leq \frac{[1-(1-r\Delta t)^N]}{r} \cdot ||\tau||_\infty$$

provided the following stability constraint holds.

$$N > T(r+\sigma^2 M^2) \text{ or equivalently } \Delta t(r + S_{max}^2 \frac{\sigma^2}{(\Delta S)^2}) < 1$$

Notice that, we use the backward time horizon, e^1 corresponds to the error at the original time $t = 0$ (the final time).

Letting $\Delta t \to 0$, $||e^1||_\infty \leq \frac{(1-e^{-rT})}{r} \lim_{\Delta t \to 0} ||\tau||_\infty \to 0$ yielding convergence.

Crank-Nicolson Method

Similar to what we did in the FTCS method, consider the following scheme using Crank-Nicolson method on the original equation.

$$\begin{bmatrix} \alpha_j - \beta_j \\ -1 - 2\alpha_j - \frac{1}{2}r\Delta t \\ \alpha_j + \beta_j \end{bmatrix}^T \begin{bmatrix} V_{j-1}^n \\ V_j^n \\ V_{j+1}^n \end{bmatrix} = - \begin{bmatrix} \alpha_j - \beta_j \\ 1 - 2\alpha_j - \frac{1}{2}r\Delta t \\ \alpha_j + \beta_j \end{bmatrix}^T \begin{bmatrix} V_{j-1}^{n+1} \\ V_j^{n+1} \\ V_{j+1}^{n+1} \end{bmatrix},$$

where $\alpha_j = \frac{1}{4}\sigma^2 j^2 \Delta t$ and $\beta_j = rj\Delta t$ for $n = 0, 1, ..., N-1$ and $j = 1, 2, ..., M-1$.

In matrix form, this results in a tridiagonal system. We use the Thomas algorithm to solve the system.

Comparison between different methods will be given in the third part.

Numerical Results

Consider the parameters $S_0 = 20$, $K = 21$, $r = 0.1$, $\sigma = 0.3$, $T = 4/12$, $S_{max} = 100$. We present the numerical results for European and American calls and puts in the following table.

	FTCS	Crank-Nicolson	Heat-Equation
European Call	1.23933221058	1.23922170202	1.23853825051
European Put	1.55086614288	1.55079742335	1.54922423526
American Call	1.23933221058	1.23922170202	1.23853825051
American Put	1.66210959214	1.66168684003	1.65072594226

Table: Numerical Results for Option Values

The exact solution for the European call and put are given by

$$c(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

and

$$p(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where

$$d_1 = \frac{ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

Using the parameters in the previous slide, the true value of $c(S_0, 0)$ is 1.240753218068958, and the true value of $p(S_0, 0)$ is 1.552291328191084, which accord with our numerical results.

Unfortunately, there is no exact (explicit) formula for American and other more complex exotic options since quite a lot of them are time-dependent or path-dependent options.

Finite difference yields its advantages throughout these areas. For this reason, we will use the binomial tree approximation to obtain a (nearly) exact solution for American-style options.

The binomial tree approximation will be introduced if time permits.

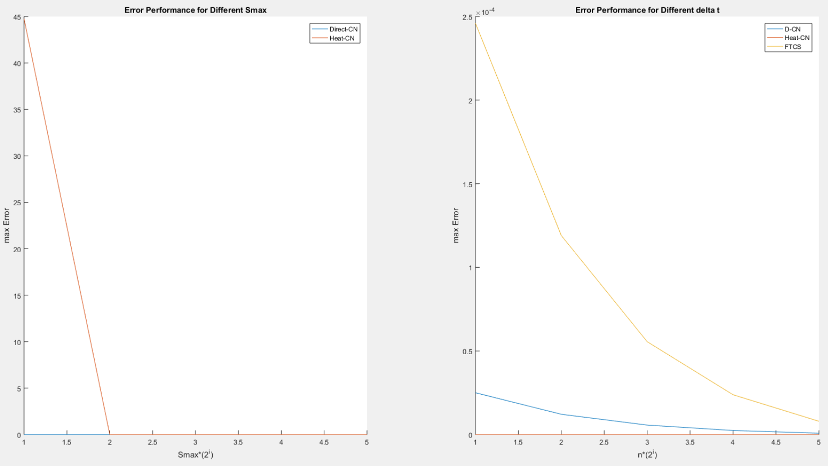

Figure: European Call

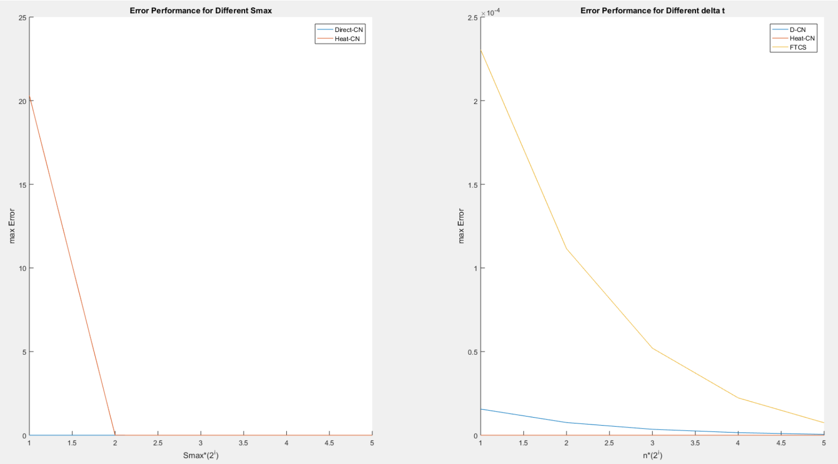

Figure: European Put

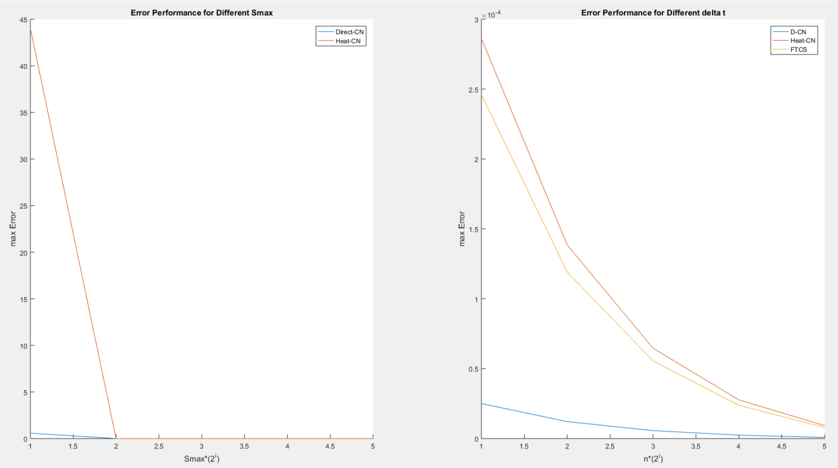

Figure: American Call

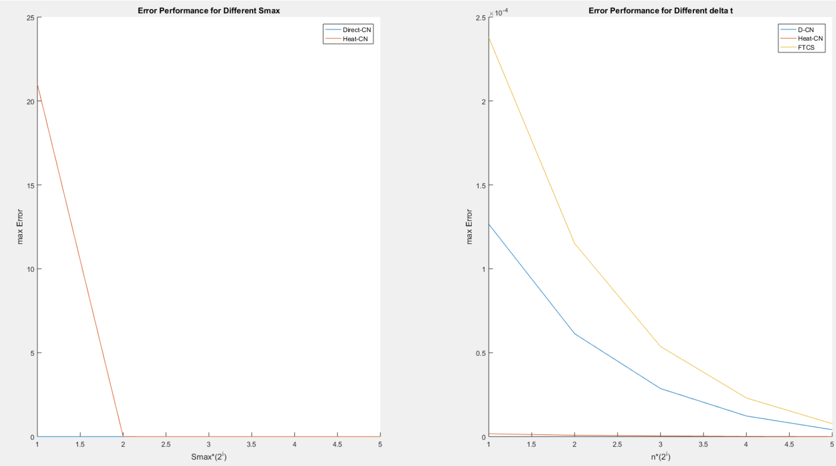

Figure: American Put

The Performance Ranking for Each Case:

1. European Call: Heat-CN $>$ D-CN $>$ FTCS
2. European Put: Heat-CN $>$ D-CN $>$ FTCS
3. American Call: D-CN $>$ FTCS $>$ Heat-CN
4. American Put: Heat-CN $>$ D-CN $>$ FTCS

Although it looks like that the Heat-CN method performs the best, there is one drawback for it: the uniform grid cannot be preserved after we transform the grid back causing the grid to be stretched out exponentially.

The Convergence Analysis:

	$N = 100, \ M = 100$	$N = 200, \ M = 200$	$N = 400, \ M = 400$
Direct CN	1.36107E-02	3.41084E-03	8.71198E-04
		1.9965	1.9690
Heat-CN	1.21221E-02	8.67617E-03	1.20609E-02
		0.4825	-0.4752
FTCS	1.29325E-02	3.20270E-03	7.988443E-04
		2.0136	2.0033

Table: Convergence Analysis for Options Pricing

Thanks for your attention