

Journey to Riemannian Geometry

wanderer1dd

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1	Christoffel Symbol	

$$\nabla_{\partial u^i} \vec{v} = \left(\frac{\partial v_k}{\partial u^i} + v^j \Gamma_{ij}^k \right) \vec{e}_k \quad (1)$$

Note, $\frac{\partial}{\partial u^i}(\vec{e}_j) = \frac{\partial^2}{\partial u^i \partial u^j}$.

$$\Gamma_{ij}^k = \frac{1}{2} \mathfrak{g}^{kl} \left(\frac{\partial g_{li}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right) \quad (2)$$

2 Levi-Civita Connection

Covariant Derivative (Levi Civita Connection):

Bilinearity

$$1. \quad \nabla_{a\vec{w}+b\vec{v}}\vec{t} = a\nabla_{\vec{w}}\vec{t} + b\nabla_{\vec{v}}\vec{t} \quad (3)$$

$$2. \quad \nabla_{\vec{t}}(\vec{w} + \vec{v}) = \nabla_{\vec{t}}\vec{w} + \nabla_{\vec{t}}\vec{v} \quad (4)$$

Leibniz Rule

$$3. \quad \nabla_{\vec{t}}(a\vec{v}) = (\nabla_{\vec{t}}a)\vec{v} + a(\nabla_{\vec{t}}\vec{v}) \quad (5)$$

Generalized Directional Derivative for Scalar Field

$$4. \quad \nabla_{\partial_i}(a) = \frac{\partial a}{\partial u^i} \quad (6)$$

Metric Compatibility: When we parallel transport two vectors, their dot product remains the same

$$5. \quad \nabla_{\vec{w}}(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \nabla_{\vec{w}}(\vec{v}) + \vec{v} \cdot \nabla_{\vec{w}}(\vec{u}) \quad (7)$$

Torsion-Free

$$6. \quad \nabla_{\vec{v}}\vec{w} = \nabla_{\vec{w}}\vec{v} \quad \square \quad (8)$$

$$\partial_k g_{ij} = \Gamma_{ik}^l g_{jl} + \Gamma_{kj}^l g_{il} \quad (9)$$

A derivative operator ∇ (covariant derivative) is a map that takes a smooth tensor field of type (k, l) to a smooth tensor field of type $(k, l + 1)$ satisfying property 1-4.

A Riemannian manifold is a manifold with a metric.

The Fundamental Theorem of Riemannian Geometry:

There is a unique covariant derivative satisfying property 5-6, called Levi-Civita Connection.

Geodesics: A curve resulting from parallel transporting a vector along itself.

$$\nabla_{\vec{v}} \vec{v} = 0 \quad (10)$$

$$\left(\frac{d^2 u^k}{d\lambda^2} + \frac{du^i}{d\lambda} \frac{du^j}{d\lambda} \Gamma_{ij}^k \right) \vec{e}_k = \vec{0} \quad (11)$$

3 Riemann Curvature Tensor

Riemann Curvature Tensor:

$$R(\vec{u}, \vec{v})(\vec{w}) = \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{[\vec{u}, \vec{v}]} \vec{w} \quad (12)$$

Riemann curvature tensor is multi-linear. Thus,

$$\begin{aligned} R(\vec{u}, \vec{v})(\vec{w}) &= u^i v^j w^k R(\vec{e}_i, \vec{e}_j)(\vec{e}_k) \\ &= u^i v^j w^k R_{kij}^m \vec{e}_m \end{aligned} \quad (13)$$

$$R_{cab}^d = \partial_a \Gamma_{bc}^d - \partial_b \Gamma_{ac}^d + \Gamma_{bc}^i \Gamma_{ai}^d - \Gamma_{ac}^j \Gamma_{bj}^d \quad (14)$$

4 Ricci Tensor

$$R_{\alpha\beta} = R_{\alpha\rho\beta}^{\rho} \quad (15)$$

Ricci tensor components are symmetric

$$R_{\alpha\beta} = R_{\beta\alpha} \quad (16)$$

Ricci Curvature:

$$Ric(\vec{V}, \vec{V}) = V^{\alpha} V^{\beta} R_{\alpha\beta} \quad (17)$$

Ricci Scalar:

$$R = \mathfrak{g}^{\alpha\beta} R_{\alpha\beta} \quad (18)$$

The second order derivative of volume change moving along geodesics has a term proportional to the volume, with the Ricci tensor as the coefficient of proportionality.

$$\frac{d^2V}{d\lambda^2} = -R_{xz}v^xv^z(V) + \textit{terms of volume change due to seperation of geodesics}. \quad (19)$$

Ricci scalar is the primary term for the volume difference ratio of hypersphere with radius r compared to that of hypersphere in flat space. D is the dimension of the manifold.

$$\frac{V(r)}{V_{flat}(r)} = 1 - \frac{R}{6(D+2)}r^2 + O(r^4) \quad (20)$$

5 Einstein's Equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (21)$$