Journey to Riemannian Geometry

wanderer1dd

September 2023

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1 Christoffel Symbol

$$\nabla_{\partial u^i}\vec{v} = (\frac{\partial v_k}{\partial u^i} + v_j \Gamma^k_{ij})\vec{e_k} \tag{1}$$

Note, $\frac{\partial}{\partial u^i}(\vec{e_j}) = \frac{\partial^2}{\partial u^i \partial u^j}$.

$$\Gamma^k_{ij} = \frac{1}{2}\mathfrak{g}^{kl}(\frac{\partial g_{li}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}) \tag{2}$$

2 Levi-Civita Connection

Covariant Derivative (Levi Civita Connection):

Bilinearity

$$1. \qquad \nabla_{a\vec{w}+b\vec{v}}\vec{t} = a\nabla_{\vec{w}}\vec{t} + b\nabla_{\vec{v}}\vec{t} \tag{3}$$

$$2. \qquad \nabla_{\vec{t}}(\vec{w}+\vec{v}) = \nabla_{\vec{t}}\vec{w} + \nabla_{\vec{t}}\vec{v} \tag{4}$$

Leibniz Rule

$$3. \qquad \nabla_{\vec{t}}(a\vec{v}) = (\nabla_{\vec{t}}\vec{a})\vec{v} + a(\nabla_{\vec{t}}\vec{v}) \tag{5}$$

Generalized Directional Derivative for Scalar Field

$$4. \qquad \nabla_{\partial_i}(a) = \frac{\partial a}{\partial u^i} \tag{6}$$

Metric Compatibility: When we parallel transport two vectors, their dot product remains the same

$$5. \qquad \nabla_{\vec{w}}(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\nabla_{\vec{w}}(\vec{v}) + \vec{v}\cdot\nabla_{\vec{w}}(\vec{u}) \tag{7}$$

Torsion-Free

$$6. \qquad \nabla_{\vec{v}}\vec{w} = \nabla_{\vec{w}}\vec{v} \quad \square \tag{8}$$

$$\partial_k g_{ij} = \Gamma^l_{ik} g_{jl} + \Gamma^l_{kj} g_{il} \tag{9}$$

A derivative operator ∇ (covariant derivative) is a map that takes a smooth tensor field of type (k, l) to a smooth tensor field of type $(k, l+1)$ satisfying property 1-4.

A Riemannian manifold is a manifold with a metric.

The Fundamental Theorem of Riemannian Geometry:

There is a unique covariant derivative satisfying property 5-6, called Levi-Civita Connection.

Geodesics: A curve resulting from parallel transporting a vector along itself.

$$\nabla_{\vec{v}} \vec{v} = 0 \tag{10}$$

$$\left(\frac{d^2 u^k}{d\lambda^2} + \frac{du^i}{d\lambda}\frac{du^j}{d\lambda}\Gamma^k_{ij}\right)\vec{e_k} = \vec{0} \tag{11}$$

3 Riemann Curvature Tensor

Riemann Curvature Tensor:

$$R(\vec{u},\vec{v})(\vec{w}) = \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{[\vec{u},\vec{v}]}\vec{w} \quad (12)$$

Riemann curvature tensor is multi-linear. Thus,

$$\begin{aligned} R(\vec{u},\vec{v})(\vec{w}) &= u^i v^j w^k R(\vec{e_i},\vec{e_j})(\vec{e_k}) \\ &= u^i v^j w^k R^m_{kij}\vec{e_m} \end{aligned} \quad (13)$$

$$R^d_{cab} = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^i_{bc}\Gamma^d_{ai} - \Gamma^j_{ac}\Gamma^d_{bj} \quad (14)$$

4 Ricci Tensor

$$R_{\alpha\beta} = R^\rho_{\alpha\rho\beta} \quad (15)$$

Ricci tensor components are symmetric

$$R_{\alpha\beta} = R_{\beta\alpha} \quad (16)$$

Ricci Curvature:

$$Ric(\vec{V},\vec{V}) = V^\alpha V^\beta R_{\alpha\beta} \quad (17)$$

Ricci Scalar:

$$R = \mathfrak{g}^{\alpha\beta} R_{\alpha\beta} \tag{18}$$

The second order derivative of volume change moving along geodesics has a term proportional to the volume, with the Ricci tensor as the coefficient of portionality.

$$\frac{d^2V}{d\lambda^2} = -R_{xz}v^x v^z(V)\ +\ terms\ of\ volume \\ change\ due\ to\ seperation\ of\ geodesics. \tag{19}$$

Ricci scalar is the primary term for the volume difference ratio of hypersphere with radius r compared to that of hypersphere in flat space. D is the dimension of the manifold.

$$\frac{V(r)}{V_{flat}(r)} = 1 - \frac{R}{6(D+2)}r^2 + O(r^4) \tag{20}$$

5 Eistein's Equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{21}$$