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Capstone Project

Quantum Speed Limits

Author:
Zeng Wenxin

Supervisor:
Prof. Chau,
Hoi Fung

Department of Physics
Faculty of Science

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Abstract

A quantum speed limit (QSL) is the minimum time required for a quantum system to evolve between two distinguishable states. In this work, I review the derivation and generalisation of the Mandelstam–Tamm (MT) and Margolus–Levitin (ML) QSLs under a time-independent Hamiltonian. I afterwards modify and enhance the Lee-Chau (LC) and Luo-Zhang (LZ) QSLs for any order, respectively, which then are tight for any fidelity. Thereby, I combine them together as a new optimal bound.

1 Introduction

Quantum evolution is governed by the Schrödinger equation, which implies itself continuous and reversible in Hilbert space. A **quantum speed limit (QSL)** is a fundamental limit on the minimum time for a quantum system to evolve between two distinguishable states.

There are three defining elements for a specific evolution: the initial state $|a\rangle$, the final state $|b\rangle$, and the **time-independent** system Hamiltonian $\mathbf{H}$. Incidentally, one may ask how distinguishable is one state from the other? However, instead of measuring the distinction, we measure the closeness of two quantum states. Conventionally, the closeness is described by fidelity, meaning the probability that one state is identified as the other. For pure states, the fidelity is simply the squared module of the inner product between the two kets, represented as $|\langle a|b\rangle|^2$.

By determining different elements, there are variant scenarios in the field [1]: One might specify all three of $|a\rangle$, $|b\rangle$, and H and seek to bound the time required to reach the final state $|b\rangle$ from $|a\rangle$. Furthermore, one can seek the time required to cycle through a sequence of quantum states. One can also suppose both the initial and final states $|a\rangle$ and $|b\rangle$ are given and ask for the Hamiltonian that accomplishes this state transition in minimum time, which is the quantum brachistochrone problem. What has been extensively investigated is the minimum time required for the evolution with respect to some fidelity because the QSLs expressed by various inequalities have implications for other fields such as quantum computation, quantum metrology, and control of quantum systems.

Since $\mathbf{H}$ is the generator of the dynamical evolution and defines the system energy spectrum, we expect that the answer to the question is closely related to the energy statistics of the system. As a matter of fact, the two most fundamental inequalities, the Margolus–Levitin (ML) and the Mandelstam–Tamm (MT) QSLs, are depending on the system's average energy $E = \langle \mathbf{H} \rangle$ and energy spread $\Delta E = \sqrt{\langle \mathbf{H}^2 \rangle - \langle \mathbf{H} \rangle^2}$, respectively, defined with ground state energy $E_0 = 0$.

In the following section, I will review the derivation of the MT and ML QSLs and their extension to more general energy statistics. Then, I will attempt to refine the prior work by integrating former ideas.

2 The Prior Work

2.1 The Mandelstam-Tamm QSL[2]

For an arbitrary ket $|\rangle$ and its two Hermitian observables $\mathbf{A}, \mathbf{B}$, the expectation values of $\mathbf{A}, \mathbf{B}$ with respect to state $|\rangle$ are $\langle \mathbf{A} \rangle$, $\langle \mathbf{B} \rangle$, respectively. Readily, we can compute the spreads of $\mathbf{A}$ and $\mathbf{B}$ [3]:

$$\begin{cases} \Delta A = \sqrt{\left\langle \left(\mathbf{A} - \langle \mathbf{A} \rangle\right)^2 \right\rangle} = \sqrt{\left\langle \left(\mathbf{\Delta A}\right)^2 \right\rangle}, \\ \Delta B = \sqrt{\left\langle \left(\mathbf{B} - \langle \mathbf{B} \rangle\right)^2 \right\rangle} = \sqrt{\left\langle \left(\mathbf{\Delta B}\right)^2 \right\rangle}. \end{cases} \tag{1}$$

Now, we note that $(\langle \alpha | + \lambda^* \langle \beta |) \cdot (|\alpha\rangle + \lambda |\beta\rangle) \geq 0$, where λ can be any complex number. This inequality must hold when λ is set equal to $-\langle \beta | \alpha \rangle / \langle \beta | \beta \rangle$:

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle - \left| \langle \alpha | \beta \rangle \right|^2 \geq 0, \tag{2}$$

which is the Schwarz inequality.

Substituting $|\alpha\rangle = \mathbf{\Delta A}|\rangle$ and $|\beta\rangle = \mathbf{\Delta B}|\rangle$ into (2), we obtain

$$\left\langle \left(\mathbf{\Delta A}\right)^2 \right\rangle \left\langle \left(\mathbf{\Delta B}\right)^2 \right\rangle \geq \left| \langle \mathbf{\Delta A \Delta B} \rangle \right|^2. \tag{3}$$

Expanding $\mathbf{\Delta A \Delta B}$, we have

$$\begin{aligned} \mathbf{\Delta A \Delta B} &= \frac{1}{2} \left(\mathbf{\Delta A \Delta B} - \mathbf{\Delta B \Delta A} \right) \\ &\quad + \frac{1}{2} \left(\mathbf{\Delta A \Delta B} + \mathbf{\Delta B \Delta A} \right) \\ &= \frac{1}{2} \left[\mathbf{\Delta A}, \mathbf{\Delta B} \right] + \frac{1}{2} \left\{ \mathbf{\Delta A}, \mathbf{\Delta B} \right\} \\ &= \frac{1}{2} \left[\mathbf{A}, \mathbf{B} \right] + \frac{1}{2} \left\{ \mathbf{\Delta A}, \mathbf{\Delta B} \right\}. \end{aligned}$$

The commutator $[\mathbf{A}, \mathbf{B}]$ is clearly anti-Hermitian

$$\left[\mathbf{A}, \mathbf{B} \right]^\dagger = \left(\mathbf{AB} - \mathbf{BA} \right)^\dagger = \mathbf{BA} - \mathbf{AB} = - \left[\mathbf{A}, \mathbf{B} \right],$$

which means itself purely imaginary. In contrast, the anti-commutator $\{\mathbf{A}, \mathbf{B}\}$ is obviously Hermitian and purely real. Consequently, we have

$$\begin{aligned}
|\langle \mathbf{\Delta A \Delta B} \rangle|^2 &= \left| \frac{1}{2} \langle [\mathbf{A}, \mathbf{B}] \rangle + \frac{1}{2} \langle \{ \mathbf{\Delta A}, \mathbf{\Delta B} \} \rangle \right|^2 \\
&= \frac{1}{4} |\langle [\mathbf{A}, \mathbf{B}] \rangle|^2 + \frac{1}{4} |\langle \{ \mathbf{\Delta A}, \mathbf{\Delta B} \} \rangle|^2 \\
&\geq \frac{1}{4} |\langle [\mathbf{A}, \mathbf{B}] \rangle|^2 .
\end{aligned} \tag{4}$$

From (3) and (4), we finally derive the uncertainty inequality:

$$\left\langle (\mathbf{\Delta A})^2 \right\rangle \left\langle (\mathbf{\Delta B})^2 \right\rangle \geq \frac{1}{4} |\langle [\mathbf{A}, \mathbf{B}] \rangle|^2 .$$

Taking the square root of both sides and recalling (1), we have

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [\mathbf{A}, \mathbf{B}] \rangle| . \tag{5}$$

Setting $\mathbf{B}$ to be the time-independent system Hamiltonian $\mathbf{H}$, the commutator in (5) is, according to the Heisenberg form of the Schrödinger equation,

$$[\mathbf{A},\ \mathbf{H}] = -i\hbar \frac{\partial}{\partial t} \mathbf{A}. \tag{6}$$

Let $\mathbf{A} = |\psi_t\rangle\langle\psi_t|$ be the projector onto the system state $|\psi_t\rangle$ at time t. Then $\langle \mathbf{A} \rangle = \langle \psi_0 | \mathbf{A} | \psi_0 \rangle$ is the fidelity $F(t) = | \langle \psi_0 | \psi_t \rangle |^2$ between $|\psi_t\rangle$ and the initial state $|\psi_0\rangle$. Now substituting (6) into (5) yields

$$\Delta A \Delta H \geq \frac{1}{2} |\langle [\mathbf{A}, \mathbf{H}] \rangle| = \frac{1}{2} \left| -i\hbar \left\langle \frac{\partial}{\partial t} \mathbf{A} \right\rangle \right| = \frac{\hbar}{2} \left| \frac{\partial}{\partial t} \langle \mathbf{A} \rangle \right| ,$$

i.e.

$$\Delta A \Delta E \geq \frac{\hbar}{2} \left| \frac{\partial}{\partial t} \langle \mathbf{A} \rangle \right| . \tag{7}$$

Since $\Delta A = \sqrt{\langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2} = \sqrt{\langle \mathbf{A} \rangle - \langle \mathbf{A} \rangle^2} = \sqrt{F(t)[1 - F(t)]}$, we then from (7) obtain

$$\frac{2\Delta E}{\hbar} \geq \frac{1}{\sqrt{F(t)\,(1 - F(t))}} \left| \frac{\partial}{\partial t} F(t) \right| . \tag{8}$$

Integrate each side of (8) over the time interval $[0, t_\epsilon]$, where t_ϵ is the first $t > 0$ satisfying $F(t) = \epsilon$. Note that ΔE is time-independent and $F(t)$ is monotonically decreasing over $[0, t_\epsilon]$, so we unveil that

$$\begin{aligned}\int_0^{t_\epsilon} \frac{2\Delta E}{\hbar} dt = \frac{2\Delta E}{\hbar} t_\epsilon \geq & \int_0^{t_\epsilon} \frac{1}{\sqrt{F(t)\left[1 - F(t)\right]}} \left[-\frac{\partial}{\partial t} F(t)\right] dt \\ = & \int_0^{t_\epsilon} \frac{-1}{\sqrt{F(t)\left[1 - F(t)\right]}} d\left[F(t)\right] \\ \xlongequal{F(t)=\xi} & \int_1^{\epsilon} \frac{-1}{\sqrt{\xi\left(1 - \xi\right)}} d\xi \\ = & \left[-2 \arcsin \sqrt{\xi}\right]_1^{\epsilon} \\ = & 2 \arccos \sqrt{\epsilon}.\end{aligned}$$

Consequently, we can obtain **the Mandelstam-Tamm (MT) inequality**,

$$t_\epsilon \geq \frac{\hbar}{\Delta E} \beta_2(\epsilon) \quad \text{or} \quad t_\epsilon \geq \frac{\pi\hbar}{2} \frac{\beta(\epsilon)}{\Delta E}, \tag{9}$$

where $\beta_2(\epsilon) = (\pi/2)\beta(\epsilon) = \arccos \sqrt{\epsilon}$. For time $t = t_\perp$ such that $F(t) = |\langle \psi_0 | \psi_t \rangle|^2 = 0$, **the basic MT inequality**

$$t_\perp \geq \frac{\pi}{2} \frac{\hbar}{\Delta E} \tag{10}$$

follows.

2.2 The Margolus–Levitin QSL[4]

Given the initial state $|\psi_0\rangle = \sum_{n=0}^{N-1} c_n |E_n\rangle$, the state of an autonomous system with time-independent Hamiltonian $\mathbf{H} = \sum_{n=0}^{N-1} E_n |E_n\rangle \langle E_n|$ at time t is

$$|\psi_t\rangle = \mathrm{e}^{-i \frac{\mathbf{H}}{\hbar} t} |\psi_0\rangle \tag{11}$$

with

$$\langle \psi_0 | \psi_t \rangle = \left\langle \psi_0 \left| \mathrm{e}^{-i \frac{\mathbf{H}}{\hbar} t} \right| \psi_0 \right\rangle = \sum_n |c_n|^2 \mathrm{e}^{-i \frac{E_n}{\hbar} t}. \tag{12}$$

To derive the ML inequality, we rewrite $\langle \psi_0 | \psi_t \rangle$ as $\langle \psi_0 | \psi_t \rangle = \mathfrak{Re}\langle \psi_0 | \psi_t \rangle + i\mathfrak{Im}\langle \psi_0 | \psi_t \rangle$, where i is the imaginary unit, and use a preliminary lemma $\cos\theta \geq 1 - \frac{2}{\pi}(\theta + \sin\theta)$, valid for $\theta \geq 0$, to find for the real and imaginary parts of $\langle \psi_0 | \psi_t \rangle$ that

$$\begin{aligned}\mathfrak{Re} \langle \psi_0 | \psi_t \rangle = & \sum_n |c_n|^2 \cos \frac{E_n}{\hbar} t \\ \geq & \sum_n |c_n|^2 \left(1 - \frac{2E_n t}{\pi\hbar} - \frac{2}{\pi} \sin \frac{E_n t}{\hbar}\right) \\ = & 1 - \frac{2Et}{\pi\hbar} - \frac{2}{\pi} \mathfrak{Im} \langle \psi_0 | \psi_t \rangle.\end{aligned}$$

We have $\langle\psi_0|\psi_t\rangle = 0$ if and only if $\mathfrak{Re}\langle\psi_0|\psi_t\rangle = \mathfrak{Im}\langle\psi_0|\psi_t\rangle = 0$, so for time $t = t_\perp$ such that $F(t) = |\langle\psi_0|\psi_t\rangle|^2 = 0$, **the basic Margolus–Levitin inequality**

$$t_\perp \geq \frac{\pi}{2}\frac{\hbar}{E} \tag{13}$$

follows. By analogy with the MT inequality, we conjecture that there exists a fidelity factor such that for any $\epsilon \in [0,1]$, the following inequality holds:

$$t_\epsilon \geq \frac{\pi\hbar}{2}\frac{\alpha(\epsilon)}{E} \quad \text{or} \quad t_\epsilon \geq \frac{\hbar}{E}\alpha_1(\epsilon), \tag{14}$$

where $\alpha(\epsilon) = (2/\pi)\alpha_1(\epsilon)$.

To determine $\alpha(\epsilon)$, Giovannetti et al. [5]: (i) gave a lower bound $\alpha_<(\epsilon)$ for it, (ii) gave an upper bound $\alpha_>(\epsilon)$ for it, and then (iii) showed numerically that these two bounds coincide, thus providing an estimation of $\alpha(\epsilon)$.

(i) A lower bound for $\alpha(\epsilon)$ can be constructed by the following class of inequalities for $q \geq 0$:

$$\cos x + q\sin x \geq 1 - ax, \tag{15}$$

which is applicable for $x \geq 0$ and where a is a function of q preserving the tangency between the lower $1 - ax$ and the upper $\cos x + q\sin x$.

We now observe that if the fidelity $F(t_\epsilon) = \epsilon$, then $\langle\psi_0|\psi_{t_\epsilon}\rangle = \sqrt{\epsilon}e^{i\theta}$, i.e. [5],

$$\begin{cases} \displaystyle\sum_n |c_n|^2 \cos\frac{E_n}{\hbar}t_\epsilon = \sqrt{\epsilon}\cos\theta, \\ \displaystyle\sum_n |c_n|^2 \sin\frac{E_n}{\hbar}t_\epsilon = -\sqrt{\epsilon}\sin\theta, \end{cases} \tag{16}$$

with $\theta \in [0, 2\pi]$.

Back to (15), since we assume zero ground state energy, all energy levels are positive and we can replace x with $E_n t_\epsilon/\hbar$ and multiply each side by $|c_n|^2$. Summing on n and employing (16), we obtain the inequality

$$\sqrt{\epsilon}\,(\cos\theta + q\sin\theta) \geq 1 - a\frac{Et_\epsilon}{\hbar},$$

where $E = \langle\mathbf{H}\rangle = \sum_n |c_n|^2 E_n$. After multiplying each side by $2/\pi$, we rearrange the inequality into

$$\frac{2Et_\epsilon}{\pi\hbar} \geq \frac{2}{\pi a}\left[1 - \sqrt{\epsilon}\,(\cos\theta + q\sin\theta)\right],$$

which implies

$$\alpha(\epsilon) \geq \frac{2}{\pi a}\left[1 - \sqrt{\epsilon}\,(\cos\theta + q\sin\theta)\right]. \tag{17}$$

Since, for any given θ, (17) must be valid for all $q \geq 0$, then the following lower bound for $\alpha(\epsilon)$ can be obtained

$$\alpha(\epsilon) \geq \alpha_<(\epsilon) \equiv \min_\theta\left\{\max_q\left\{\frac{2}{\pi a}\left[1 - \sqrt{\epsilon}\,(\cos\theta + q\sin\theta)\right]\right\}\right\}. \tag{18}$$

(*ii*) To provide an upper bound $\alpha_>(\epsilon)$ for $\alpha(\epsilon)$, we now consider the following family of two-level states, [5]

$$|E_\xi\rangle = \sqrt{1-\xi^2}\,|E_0\rangle + \xi\,|E_1\rangle\,, \tag{19}$$

where $\xi \in [0,1]$, and $|E_0\rangle$ and $|E_1\rangle$ are eigenstates of the Hamiltonian with eigenenergy 0 and E_1 respectively. For this family of states, one can rewrite (16) as

$$\begin{cases} \sqrt{\epsilon}\cos\theta = 1-\xi^2+\xi^2\cos\dfrac{E_1 t_\epsilon}{\hbar} \\ -\sqrt{\epsilon}\sin\theta = \xi^2\sin\dfrac{E_1 t_\epsilon}{\hbar} \end{cases}. \tag{20}$$

Solving this dynamical evolution of the state $|E_\xi\rangle$, one can show that the first time t for which $F(t)=\epsilon$ is given by

$$t_\epsilon = \frac{\hbar}{E_1}\arccos\left[\frac{\epsilon-1+2\xi^2(1-\xi^2)}{2\xi^2(1-\xi^2)}\right]. \tag{21}$$

Since the average energy of $|E_\xi\rangle$ is $E=\xi^2 E_1$, each side multiplied by $2\xi^2/\pi$, (21) can be transposed into

$$\frac{2Et_\epsilon}{\pi\hbar} = \frac{2\xi^2}{\pi}\arccos\left[\frac{\epsilon-1+2\xi^2(1-\xi^2)}{2\xi^2(1-\xi^2)}\right],$$

which gives the upper bound for $\alpha(\epsilon)$ after the right-hand side is minimised as follows:

$$\alpha(\epsilon) \le \alpha_>(\epsilon) \equiv \min_{\xi^2}\left\{\frac{2\xi^2}{\pi}\arccos\left[\frac{\epsilon-1+2\xi^2(1-\xi^2)}{2\xi^2(1-\xi^2)}\right]\right\}. \tag{22}$$

(*iii*) I reproduced the numerical demonstration of $\alpha_<(\epsilon)=\alpha_>(\epsilon)$ via Python 3.8, thus offering an estimation of $\alpha(\epsilon)$. (See Appendix A.)

After deriving $\alpha_<(\epsilon)$ and $\alpha_>(\epsilon)$, explicitly expressed by (18) and (22), Giovannetti et al. [5] compared them for some random values of ϵ, as shown in Figure 1. Since all the values are compatible with zero, one can conclude that $\alpha_<(\epsilon)=\alpha_>(\epsilon)=\alpha(\epsilon)$. The curves of $\alpha_<(\epsilon)$ and $\alpha_>(\epsilon)$, as an estimation of $\alpha(\epsilon)$, along with the curve of the difference $\alpha_>(\epsilon)-\alpha_<(\epsilon)$, are plotted in Figure 2.

2.3 The Luo-Zhang QSL[6]

Recalling (11) and (12), we use a lemma $\cos x + \frac{2p}{\pi}\sin x \ge 1-2(\frac{x}{\pi})^p$, valid for any $x \ge 0$ and $p>0$, to derive the inequality

$$\begin{aligned} \mathfrak{Re}\,\langle\psi_0|\psi_t\rangle - \frac{2p}{\pi}\mathfrak{Im}\,\langle\psi_0|\psi_t\rangle =& \sum_n |c_n|^2\left(\cos\frac{E_n t}{\hbar}+\frac{2p}{\pi}\sin\frac{E_n t}{\hbar}\right) \\ \ge& \sum_n |c_n|^2\left[1-2\left(\frac{E_n t}{\pi\hbar}\right)^p\right] \\ =& 1-\frac{2}{\pi^p\hbar^p}t^p\,\langle\mathbf{H}^p\rangle\,. \end{aligned} \tag{23}$$

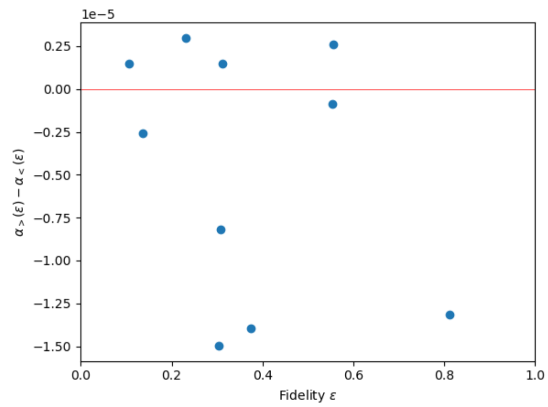


Figure 1: $\alpha_>(\epsilon) - \alpha_<(\epsilon)$

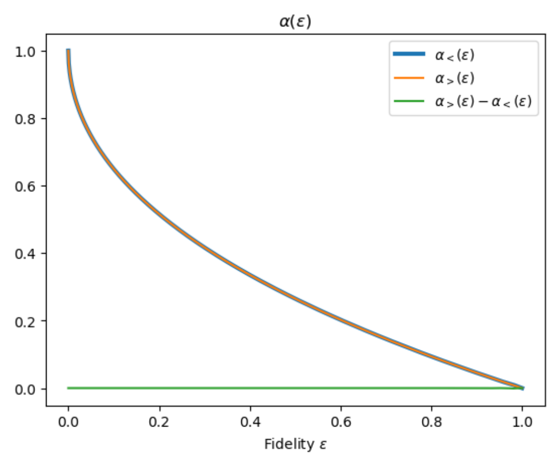


Figure 2: $\alpha(\epsilon)$

Now put $F(x,y) = x - \frac{2p}{\pi}y,\ x,y \in R$, and consider maximising $F(x,y)$ subject to the condition $x^2+y^2 = |\langle\psi_0|\psi_t\rangle|^2$. Clearly,

$$\max_{x^2+y^2=|\langle\psi_0|\psi_t\rangle|^2} F(x,y) = \max_{-|\langle\psi_0|\psi_t\rangle|\le y\le|\langle\psi_0|\psi_t\rangle|} \left\{ \sqrt{|\langle\psi_0|\psi_t\rangle|^2 - y^2} - \frac{2p}{\pi}y \right\}.$$

It is obvious that the maximum value

$$\max_{-|\langle\psi_0|\psi_t\rangle|\le y\le|\langle\psi_0|\psi_t\rangle|} \left\{ \sqrt{|\langle\psi_0|\psi_t\rangle|^2 - y^2} - \frac{2p}{\pi}y \right\} = |\langle\psi_0|\psi_t\rangle| \sqrt{1+\frac{4p^2}{\pi^2}} \tag{24}$$

is achieved at $x = |\langle\psi_0|\psi_t\rangle|\pi/\sqrt{4p^2+\pi^2}$, $y = -|\langle\psi_0|\psi_t\rangle|2p/\sqrt{4p^2+\pi^2}$. Consequently, from (23) and (24) we have

$$|\langle\psi_0|\psi_t\rangle| \sqrt{1+\frac{4p^2}{\pi^2}} \ge \mathfrak{Re}\,\langle\psi_0|\psi_t\rangle - \frac{2p}{\pi}\mathfrak{Im}\,\langle\psi_0|\psi_t\rangle \ge 1 - \frac{2}{\pi^p\hbar^p}t^p\,\langle\mathbf{H}^p\rangle\,,$$

which directly yields

$$|\langle\psi_0|\psi_t\rangle| \ge \frac{1-2\frac{\langle\mathbf{H}^p\rangle}{\pi^p\hbar^p}t^p}{\sqrt{1+\frac{4p^2}{\pi^2}}} \tag{25}$$

for $\forall t\ge 0$. In particular, for time $t = t_\epsilon$ such that

$$\sqrt{\epsilon} \ge \frac{1-2\frac{\langle\mathbf{H}^p\rangle}{\pi^p\hbar^p}t_\epsilon^p}{\sqrt{1+\frac{4p^2}{\pi^2}}},$$

it holds for any $0 < p \le \frac{\pi}{2}\sqrt{\frac{1}{\epsilon}-1}$ that

$$t_\epsilon \ge \frac{h}{2}\left[\frac{1-\sqrt{\epsilon\left(1+\frac{4p^2}{\pi^2}\right)}}{2\,\langle\mathbf{H}^p\rangle}\right]^{1/p}, \tag{26}$$

which is an extension of the ML inequality, called **the Luo-Zhang (LZ) QSL**.

2.4 The Lee-Chau QSL[7]

Recalling again (11) and (12), for the first time the fidelity falls to ϵ, $t = t_\epsilon$, we have

$$\sqrt{\epsilon} = |\langle\psi_0|\psi_{t_\epsilon}\rangle| \ge \mathfrak{Re}\,\langle\psi_0|\psi_{t_\epsilon}\rangle = \mathfrak{Re}\left\langle e^{-i\frac{\mathbf{H}}{\hbar}t_\epsilon}\right\rangle = \left\langle \cos\frac{\mathbf{H}}{\hbar}t_\epsilon \right\rangle. \tag{27}$$

Defining

$$A_p = \sup_{x\ge 0}\frac{1-\cos x}{x^p}, \tag{28}$$

then, for $p \in (0, 2]$,

$$\cos x \geq 1 - A_p \, |x|^p \tag{29}$$

holds for all x. From (27) and (29), we have

$$\sqrt{\epsilon} \geq \left\langle cos \frac{\mathbf{H}}{\hbar} t_\epsilon \right\rangle \geq \left\langle 1 - A_p \left| \frac{\mathbf{H}}{\hbar} t_\epsilon \right|^p \right\rangle \geq 1 - A_p \left(\frac{t_\epsilon}{\hbar} \right)^p \langle |\mathbf{H}|^p \rangle \, . \tag{30}$$

Since the reference energy for $\mathbf{H}$ has no physical meaning and can be shifted by any chosen value without invalidating (30). Choosing the reference energy E_p^* and denoting $M_p^* = \min_{E^*} \{ \langle |\mathbf{H} - E^*|^p \rangle \} = \langle |\mathbf{H} - E_p^*|^p \rangle$, we obtain

$$\sqrt{\epsilon} \geq 1 - A_p \left(\frac{t_\epsilon}{\hbar} \right)^p M_p^* . \tag{31}$$

Making t_ϵ the subject, this can be rearranged into

$$t_\epsilon \geq \hbar \left(\frac{1 - \sqrt{\epsilon}}{A_p M_p^*} \right)^{\frac{1}{p}} , \tag{32}$$

which is **the Lee-Chau (LC) QSL** valid for any $p \in (0, 2]$ for which $M_p^* = \langle \left| \mathbf{H} - E_p^* \right|^p \rangle$ exists.

3 My Work

3.1 Inspiration

The previous section reviews the prior work and establishes the scope of the project. The established QSLs based on either energy moments or energy central moments, however, remain cases of (p, ϵ) pairs of a considerable and unsatisfying region to be tightened. Furthermore, it is expected that stronger time bounds on evolutions of specific quantum systems can be found, given some metrics of their energy spectra.

The common approach for deriving a time limit involves two parts: first, applying mathematical inequalities to angular distances of some evolution $\frac{E_n}{\hbar} t$, which feature the eigenstates of a Hamiltonian, to establish a lower bound on the evolution time; and second, observing existing evolutions, such as those in two-level systems, to determine an upper bound on the evolution time. Once the upper and lower bounds coincide, implying the infimum of the evolution time, the quantum speed limit sought in this study can be consequently concluded.

For energy moments, Giovannetti et al. [5] proposed a class of mathematical inequalities and an instructive numerical method to derive the ML QSL. Later, Luo and Zhang [6] estimated the evolution time of a quantum state in terms of a moment of any order. Since the left-hand side of the inequality Luo and Zhang took to initiate their work is a special case of that used by Giovannetti et al., while Lee and Chau [7] provided a general form of the right-hand side, it is possible that combining the two general parts could produce two curves that are closer to each other, potentially yielding stronger bounds.

Regarding energy central moments, as pointed out by Frey [1], the LC QSL is not tight for $p = 2$, leading to speculation that a new parametric family of QSLs exists that can bridge the gap between the tight MT QSL and a new tight QSL. By utilising the properties of convex or concave functions, one can construct inequalities between moments of different orders, which directly modifies the LC QSL into the new QSL, which is not useful for large p's in practice, however. On the other hand, similar to the case of ML QSL, it is unnecessary for the lower line in terms of $|x|^p$ to intersect with the turning point of the upper trigonometric curve at its vertex. Removing this limitation, a class of inequalities then generates the same new QSL.

Building upon the aforementioned refinement, Prof. Chau compiles moments of negative energies and positive energies separately and constructively puts forward a series of novel bounds, which are potentially stronger than existing QSLs for many states. Thereby, I will seek to elucidate the derivation of two of these bounds in this article.

3.2 QSLs Depending Upon Energy Central Moments

3.2.1 The Modified LC QSL

Define $f(Y) = Y^\lambda$, where $Y \geq 0$ and $\lambda \in (0, 1]$. Taking the second derivative of $f(Y)$, one will find that $f^{''}(Y) = \lambda(\lambda - 1)Y^{\lambda-2} \leq 0$. This concavity leads to $\langle f(Y)\rangle \leq f(\langle Y\rangle)$ [8], i.e. $\langle Y\rangle^\lambda \geq \langle Y^\lambda\rangle$. For $Y = |\mathbf{H} - E^*_\nu|^\nu$, where $\nu = p/\lambda \in (0, +\infty)$, the inequality turns into

$$\langle|\mathbf{H} - E^*_\nu|^\nu\rangle^\lambda \geq \left\langle|\mathbf{H} - E^*_\nu|^{\nu\lambda}\right\rangle. \tag{33}$$

Since $M^*_p = \min_{E^*}\{\langle|\mathbf{H} - E^*|^p\rangle\}$, $\langle|\mathbf{H} - E^*_\nu|^{\nu\lambda}\rangle \geq \langle|\mathbf{H} - E^*_{\nu\lambda}|^{\nu\lambda}\rangle$. Consequently, we have

$$(M^*_\nu)^\lambda \geq M^*_p. \tag{34}$$

Combining (34) with the LC QSL (31), it yields

$$\sqrt{\epsilon} \geq 1 - A_p \left(\frac{t_\epsilon}{\hbar}\right)^p (M^*_\nu)^\lambda. \tag{35}$$

Upon transposition of (35), one can make t_ϵ the subject and obtain

$$t_\epsilon \geq \hbar \left(\frac{1 - \sqrt{\epsilon}}{A_p}\right)^{\frac{1}{p}} (M^*_\nu)^{-\frac{1}{\nu}}. \tag{36}$$

Since p can be any positive value not greater than ν, one can maximise the fidelity factor $[(1 - \sqrt{\epsilon})/A_p]^{1/p}$ over $p \in (0, \nu]$ for any ϵ. Denoting

$$\beta_\nu(\epsilon) = \max_{p\in(0,\nu]} \left(\frac{1 - \sqrt{\epsilon}}{A_p}\right)^{\frac{1}{p}},$$

the LC QSL is now **modified** to

$$t_\epsilon \geq \hbar \frac{\beta_\nu(\epsilon)}{(M_\nu^*)^{1/\nu}}, \tag{37}$$

holding for all positive ν.

3.2.2 The Fidelity Factor of the Second Order $\beta_2(\epsilon)$

If $p > 2$, $A_p \to +\infty$, and the fidelity factor $[(1-\sqrt{\epsilon})/A_p]^{1/p} \to 0^+$. It is evident that $\beta_\nu(\epsilon) = \beta_2(\epsilon)$ for $\forall \nu > 2$. Meanwhile, for $a > b > 0$, $\beta_a(\epsilon) \geq \beta_b(\epsilon)$. Hence, $\beta_2(\epsilon) = \max_{\nu>0}\{\beta_\nu(\epsilon)\}$, and it is instructive to derive $\beta_2(\epsilon)$. Define

$$B_p(x) = \frac{\sin\frac{x}{2}}{x^{p/2}},$$

for $x \in [0, +\infty)$ and $p \in (0, 2)$, and the supremum of its absolute value is

$$\sup_{x\geq 0}\left|\frac{\sin\frac{x}{2}}{x^{p/2}}\right| = \max\{\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{x^{p/2}},\ \sup_{x\in(0,\pi)}\frac{\sin\frac{x}{2}}{x^{p/2}},\ \pi^{-\frac{p}{2}},\ \sup_{x\in(\pi,+\infty)}\left|\frac{\sin\frac{x}{2}}{x^{p/2}}\right|\}. \tag{38}$$

For $\forall x_2 \in (\pi, +\infty)$, $\exists x_1 \in (0, \pi)$ satisfying

$$\left|\frac{\sin\frac{x_2}{2}}{x_2^{p/2}}\right| \leq \frac{1}{x_2^{p/2}} = \frac{(x_1/x_2)^{p/2}}{x_1^{p/2}} \leq \frac{\sin\frac{x_1}{2}}{x_1^{p/2}}, \tag{39}$$

i.e.

$$\sup_{x\in(\pi,+\infty)}\left|\frac{\sin\frac{x}{2}}{x^{p/2}}\right| \leq \frac{\sin\frac{x_1}{2}}{x_1^{p/2}} \leq \sup_{x\in(0,\pi)}\frac{\sin\frac{x}{2}}{x^{p/2}}. \tag{40}$$

At the same time,

$$\lim_{x\to 0^+}\frac{\sin\frac{x}{2}}{x^{p/2}} = \lim_{x\to 0^+} 2^{-\frac{p}{2}}\left(\frac{x}{2}\right)^{1-\frac{p}{2}} = 0 < \pi^{-\frac{p}{2}}. \tag{41}$$

Hence,

$$\sup_{x\geq 0}|B_p(x)| = \max\{\sup_{x\in(0,\pi)}\frac{\sin\frac{x}{2}}{x^{p/2}},\ \pi^{-\frac{p}{2}}\} = \sup_{x\in(0,\pi]}\frac{\sin\frac{x}{2}}{x^{p/2}}, \tag{42}$$

and

$$\begin{aligned} A_p &= \sup_{x\geq 0}\frac{1-\cos x}{x^p} = \sup_{x\geq 0}\frac{2\sin^2\frac{x}{2}}{x^p} \\ &= \sup_{x\geq 0}\{\ 2\left(\frac{\sin\frac{x}{2}}{x^{p/2}}\right)^2\} = 2\left(\sup_{x\geq 0}|B(x)|\right)^2 \\ &= 2\left(\sup_{x\in(0,\pi]}\frac{\sin\frac{x}{2}}{x^{p/2}}\right)^2. \end{aligned} \tag{43}$$

Now, taking the first and second derivatives of B_p with respect to x, it follows that

$$\begin{aligned}\frac{\partial B_p}{\partial x} &= \frac{\frac{1}{2}x^{p/2}\cos\frac{x}{2} - \frac{p}{2}x^{p/2-1}\sin\frac{x}{2}}{x^p} \\ &= \frac{x^{-p/2}}{2}\left[\cos\frac{x}{2} - \frac{p\sin\frac{x}{2}}{x}\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 B_p}{\partial x^2} &= \frac{-px^{-p/2-1}}{4}\left[\cos\frac{x}{2} - \frac{p\sin\frac{x}{2}}{x}\right] - \frac{x^{-p/2-1}}{4}\left[x\sin\frac{x}{2} + p\cos\frac{x}{2} - \frac{2p\sin\frac{x}{2}}{x}\right] \\ &= \frac{-(p+2)x^{-p/2-1}}{4}\left[\cos\frac{x}{2} - \frac{p\sin\frac{x}{2}}{x}\right] - \frac{x^{-p/2-1}}{4}\left[x\sin\frac{x}{2} + (p-2)\cos\frac{x}{2}\right].\end{aligned}$$

For $\forall p \in (0,2)$ and $x \in (0,\pi]$, where B_p may have extrema, $\frac{\partial B_p}{\partial x} = 0$. Since $\frac{x^{-p/2}}{2} > 0$, $\cos\frac{x}{2} - \frac{p\sin\frac{x}{2}}{x} = 0$.

For $x = \pi$, $\cos\frac{x}{2} - \frac{p\sin\frac{x}{2}}{x} = -\frac{p}{\pi} < 0$, where B_p does not have an extremum and $\lim_{x\to\pi^-} B_p(x) \geq B_p(\pi)$. Thus, $\sup_{x\in(0,\pi]} \frac{\sin\frac{x}{2}}{x^{p/2}} = \sup_{x\in(0,\pi)} \frac{\sin\frac{x}{2}}{x^{p/2}}$.

For $x \in (0,\pi)$, where $\frac{\partial B_p}{\partial x} = 0$, $p = x\cot\frac{x}{2}$, and therefore, $\frac{\partial^2 B_p}{\partial x^2}$ is reduced to

$$\begin{aligned}\frac{\partial^2 B_p}{\partial x^2} &= -\frac{x^{-p/2-1}}{4}\left[xsin\frac{x}{2} + (xcot\frac{x}{2} - 2)\cos\frac{x}{2}\right] \\ &= \frac{x^{-p/2-1}}{4}\left[\frac{\sin x - x}{\sin\frac{x}{2}}\right] < 0.\end{aligned} \tag{44}$$

Hence, for $x \in (0,\pi]$, where $\partial B_p/\partial x = 0$, $B_p(x)$ has relative maxima. For $\forall p \in (0,2)$, there is one and only one $x_p \in (0,\pi)$ satisfying $p = x_p\cot\frac{x_p}{2}$, i.e. for $\forall p \in (0,2)$ and $x \in (0,\pi]$, there is an only $x = x_p \in (0,\pi)$ maximising $B_p(x)$.

For $\forall p \in (0,2)$, since

$$A_p = 2\left(\sup_{x\in(0,\pi]} \frac{\sin\frac{x}{2}}{x^{p/2}}\right)^2 = 2\left[B_p(x_p)\right]^2,$$

then

$$p = x_p\cot\frac{x_p}{2}. \tag{45}$$

Now define

$$f_y(x_p) = \left(\frac{y}{A_p}\right)^{\frac{1}{p}} = x_p\left(\frac{y}{2\sin^2\frac{x_p}{2}}\right)^{\frac{1}{x_p\cot(x_p/2)}}, \tag{46}$$

for $y \in (0,1]$ and $x_p \in (0,\pi)$. Consequently,

$$\max_{p\in(0,2)}\left(\frac{y}{A_p}\right)^{\frac{1}{p}} = \max_{x_p\in(0,\pi)} f_y(x_p). \tag{47}$$

Similarly, taking the first and second derivatives of f_y with respect to x_p, it follows that[1]

$$\begin{aligned}
\frac{\partial f_y}{\partial x_p} &= \left(\frac{y}{2\sin^2\frac{x_p}{2}}\right)^{\frac{\tan\frac{x_p}{2}}{x_p}} \\
&+ x_p \left(\frac{y}{2\sin^2\frac{x_p}{2}}\right)^{\frac{\tan\frac{x_p}{2}}{x_p}} \left[\frac{\frac{1}{2}x_p\sec^2\frac{x_p}{2} - \tan\frac{x_p}{2}}{x_p^2}\, ln\,\frac{y}{2\sin^2\frac{x_p}{2}} + \frac{\tan\frac{x_p}{2}}{x_p}\frac{2\sin^2\frac{x_p}{2}}{y}\frac{-2y\cos\frac{x_p}{2}}{4\sin^3\frac{x_p}{2}}\right] \\
&= \left(\frac{y}{2\sin^2\frac{x_p}{2}}\right)^{\frac{\tan\frac{x_p}{2}}{x_p}} \frac{x_p - \sin x_p}{x_p^2(1+\cos x_p)}\, ln\,\frac{y}{2\sin^2\frac{x_p}{2}}. \\
\frac{\partial^2 f_y}{\partial x_p^2} &= \left(\frac{y}{2\sin^2\frac{x_p}{2}}\right)^{\frac{\tan\frac{x_p}{2}}{x_p}} \left\{\frac{1}{x_p}\left[\frac{x_p - \sin x_p}{x_p^2(1+\cos x_p)}\, ln\,\frac{y}{2\sin^2\frac{x_p}{2}}\right]^2 - \frac{1}{x_p}\frac{x_p - \sin x_p}{x_p^2(1+\cos x_p)}\, ln\,\frac{y}{2\sin^2\frac{x_p}{2}}\right. \\
&+ ln\,\frac{y}{2\sin^2\frac{x_p}{2}}\frac{(x_p^2+2)\sin x_p - 2x_p + \sin 2x_p - 2x_p\cos x_p}{x_p^3(1+\cos x_p)^2} \\
&\left.+ \frac{x_p - \sin x_p}{x_p^2(1+\cos x_p)}\frac{2\sin^2\frac{x_p}{2}}{y}\frac{-2y\cos\frac{x_p}{2}}{4\sin^3\frac{x_p}{2}}\right\}.
\end{aligned}$$

Where $f_y(x_p)$ has extrema, $\frac{\partial f_y}{\partial x_p} = 0$. Since $\left(\frac{y}{2\sin^2\frac{x_p}{2}}\right)^{[\tan(x_p/2)]/x_p}$, $\frac{x_p - \sin x_p}{x_p^2(1+\cos x_p)} > 0$, then $ln\,\frac{y}{2\sin^2(x_p/2)} = 0$, i.e. $\frac{y}{2\sin^2(x_p/2)} = 1$. Where $\frac{\partial f_y}{\partial x_p} = 0$, $\frac{\partial^2 f_y}{\partial x_p^2}$ is reduced to

$$\frac{\partial^2 f_y}{\partial x_p^2} = \frac{\cot\frac{x_p}{2}(\sin x_p - x_p)}{x_p^2(1+\cos x_p)}. \tag{48}$$

For $x_p \in (0,\pi)$, $\frac{\cot(x_p/2)(\sin x_p - x_p)}{x_p^2(1+\cos x_p)} < 0$, and thus, $\frac{\partial^2 f_y}{\partial x_p^2} < 0$, i.e. for $x_p \in (0,\pi)$, $f_y(x_p)$ has relative maxima, where $\frac{y}{2\sin^2(x_p/2)} = 1$. For $\forall y \in (0,1]$ and $x_p \in (0,\pi)$, we can find one and only one $x_y = \arccos(1-y)$, satisfying $\frac{y}{2\sin^2(x_p/2)} = 1$ and maximising $f_y(x_p)$. Accordingly, for $y \in (0,1]$,

$$\begin{aligned}
\max_{p\in(0,2)} \left(\frac{y}{A_p}\right)^{\frac{1}{p}} &= \max_{x_p\in(0,\pi)} x_p \left(\frac{y}{2\sin^2\frac{x_p}{2}}\right)^{\frac{1}{x_p\cot\frac{x_p}{2}}} \\
&= x_y \cdot 1^{\frac{1}{x_y\cot\frac{x_y}{2}}} \\
&= \arccos(1-y).
\end{aligned} \tag{49}$$

[1]It was negligent of me to take derivatives directly. Prof. Chau kindly reminded me that it would be much more convenient to take derivatives after taking logarithms. However, the original work is here retained.

In this case, those p's maximising f_y with respect to different y's are recast as

$$\begin{aligned} p &= x_y \cot \frac{x_y}{2} \\ &= \arccos(1-y) \cot \frac{\arccos(1-y)}{2} \\ &= \sqrt{\frac{2-y}{y}} \arccos(1-y). \end{aligned} \tag{50}$$

For $p = 2$, $A_p = \frac{1}{2}$, and then

$$\left(\frac{y}{A_p}\right)^{\frac{1}{p}} = \sqrt{2y} \leq \arccos(1-y). \tag{51}$$

Hence,

$$\max_{p\in(0,2]} \left(\frac{y}{A_p}\right)^{\frac{1}{p}} = \max_{p\in(0,2)} \left(\frac{y}{A_p}\right)^{\frac{1}{p}} = \arccos(1-y) \tag{52}$$

is in force for $y \in (0, 1]$.

For $y = 1 - \sqrt{\epsilon}$, where $\epsilon \in [0, 1)$, $\beta_2(\epsilon)$ and p can be rewritten as follows:

$$\beta_2(\epsilon) = \max_{p\in(0,2]} \left(\frac{1-\sqrt{\epsilon}}{A_p}\right)^{\frac{1}{p}} = \arccos\sqrt{\epsilon}, \tag{53}$$

and

$$p = \sqrt{\frac{1+\sqrt{\epsilon}}{1-\sqrt{\epsilon}}} \arccos\sqrt{\epsilon}. \tag{54}$$

3.2.3 The Strengthened MT QSL and the Optimised Bound

Raising both sides of (34) to the power of $-\frac{1}{p}$, one can find that

$$(M_\nu^*)^{-\frac{1}{\nu}} \leq (M_p^*)^{-\frac{1}{p}}, \tag{55}$$

where $0 < p < \nu$.

Define

$$\mu(\epsilon) = \sqrt{\frac{1+\sqrt{\epsilon}}{1-\sqrt{\epsilon}}} \arccos\sqrt{\epsilon}$$

for $\epsilon \in [0, 1)$. Since $\mu(\epsilon) < 2$, according to (55), there is a relation between $[M^*_{\mu(\epsilon)}]^{-1/\mu(\epsilon)}$ and $1/\Delta E$

$$[M^*_{\mu(\epsilon)}]^{-\frac{1}{\mu(\epsilon)}} \geq (M_2^*)^{-\frac{1}{2}} = \frac{1}{\Delta E}. \tag{56}$$

Simultaneously, $\beta_{\mu(\epsilon)}(\epsilon) = \beta_2(\epsilon) = \arccos\sqrt{\epsilon}$ for $\forall\epsilon \in [0, 1)$. Thus, for $\nu = \mu(\epsilon)$, the modified LC QSL (37) can be rewritten as

$$t_\epsilon \geq \hbar \frac{\arccos\sqrt{\epsilon}}{[M^*_{\mu(\epsilon)}]^{1/\mu(\epsilon)}}, \tag{57}$$

which is **the strengthened MT QSL**, tight for $\forall\epsilon \in [0,1)$. For all $\nu \geq \mu(\epsilon)$, (37) can be expressed as

$$t_\epsilon \geq \hbar\frac{\arccos\sqrt{\epsilon}}{(M_\nu^*)^{1/\nu}},$$

which is still tight, but impractical.

For a specific state, the lower bound is $t_\epsilon \geq \hbar\beta_\nu(M_\nu^*)^{-1/\nu}$. For $\nu \leq \mu(\epsilon)$, as the value of ν increases, $(M_\nu^*)^{-1/\nu}$ decreases while β_ν increases. It is promising that an optimised positive ν will give rise to the maximised bound

$$t_\epsilon \geq \sup_{\nu\leq\mu(\epsilon)} \frac{\hbar\beta_\nu}{(M_\nu^*)^{1/\nu}}. \tag{58}$$

3.2.4 Tightness

For $\nu \leq \mu(\epsilon)$, Lee and Chau [7] have discovered a class of initial states saturating the lower bound

$$|\Psi_0\rangle = \sqrt{\frac{\gamma}{2}}\,|-E_1\rangle + \sqrt{1-\gamma}\,|0\rangle + \sqrt{\frac{\gamma}{2}}\,|E_1\rangle\,, \tag{59}$$

where $|-E_1\rangle$, $|0\rangle$, and $|E_1\rangle$ are eigenstates of the system Hamiltonian, with corresponding eigenenergies of $-E_1$, 0, and E_1, respectively, and $\gamma = (1-\sqrt{\epsilon})/A_p x_c^p$, where $A_p = (1-\cos x_c)/x_c^p$.

In cases where $\nu \geq \mu(\epsilon)$, γ in (59) is reduced to 1 because $\cos x_c$ should be as much as $\sqrt{\epsilon}$. As a result, $|\Psi_0\rangle$ simply becomes a balanced two-level state, and $(M_\nu^*)^{1/\nu}$ of $|\Psi_0\rangle$ stays fixed at E_1. The modified LC QSL (37) now gives the lower bound $t_\epsilon \geq (\hbar/E_1)\arccos\sqrt{\epsilon}$. The exact evolution time for such a balanced two-level state to achieve any fidelity ϵ is given by $t_\epsilon = (\hbar/E_1)\arccos\sqrt{\epsilon}$, which matches the lower bound, hence confirming the tightness of the modified LC QSL.

3.2.5 An Alternative Derivation

Refining A_p as

$$A_p = \sup_{x\geq 0}\frac{C-\cos x}{x^p}, \tag{60}$$

where $C \in [0,1]$ satisfying

$$\cos x \geq C - A_p\,|x|^p \tag{61}$$

for all $x \in (-\infty,+\infty)$ and $p \in (0,+\infty)$. In this case, the lower line $C - A_p|x|^p$, with its vertex at $x = 0$ settling at C, is now required to be tangent to the upper $\cos x$ somewhere $|x| \in (0,\pi)$ for all $p \geq 2$. Applying the inequality (61) to the quantum system as done in (30) and (31), it subsequently yields a lower bound

$$t_\epsilon \geq \hbar\left(\frac{C-\sqrt{\epsilon}}{A_p}\right)^{\frac{1}{p}}(M_p^*)^{-\frac{1}{p}} \tag{62}$$

for all $p > 0$. Utilising the inequality based on concavity (34), the lower bound is reformulated into

$$t_\epsilon \geq \hbar \left(\frac{C - \sqrt{\epsilon}}{A_p} \right)^{\frac{1}{p}} (M_\nu^*)^{-\frac{1}{\nu}}. \tag{63}$$

Maximising the fidelity factor, the bound is recast as

$$t_\epsilon \geq \frac{\hbar}{(M_\nu^*)^{1/\nu}} \max_{p \in (0,\nu]} \left(\frac{C - \sqrt{\epsilon}}{A_p} \right)^{\frac{1}{p}}. \tag{64}$$

Nonetheless, the maximised fidelity factor results in the same value as $\beta_\nu(\epsilon)$, proving that the bound is equivalent to the modified LC QSL (37). It was expected that the bound would not be stronger than the LC QSL because $C - A_p|x|^p$ is not closer to $\cos x$ than $1 - A_p|x|^p$. By chance, this derivation provides geometric insight into the tightness of the LC QSL. For $p \geq \mu(\epsilon)$, the optimised fidelity factor eliminates the intersection between the upper and lower curves at $x = 0$, and the left two points of tangency suggest that only two-level systems can saturate the bound.

3.3 QSLs Depending Upon Energy Moments

3.3.1 The lower bound

We now deliberate a new family of inequalities for $q \geq 0$:

$$\cos x + q \sin x \geq 1 - a x^p, \tag{65}$$

which holds for $x \geq 0$ and where a, which is a function of q for a given p, modulates the lower line $1 - ax^p$ to be tangent to the upper trigonometric curve $\cos x + q \sin x$.

If the fidelity $F(t_\epsilon) = \epsilon$, then $\langle \psi_0 | \psi_{t_\epsilon} \rangle = \sqrt{\epsilon} e^{i\theta}$ (16). Upon the application of (65) to such states, we employ (16) in the left-hand side of the inequality, and arrive at

$$\sqrt{\epsilon} \left(\cos\theta + q \sin\theta \right) \geq 1 - a \langle E^p \rangle \left(\frac{t_\epsilon}{\hbar} \right)^p, \tag{66}$$

where $\langle E^p \rangle = \sum_n |c_n|^2 E_n^p$ with the assumption of zero ground state energy $E_0 = 0$.

The left-hand side is equal to $\sqrt{\epsilon}\sqrt{q^2+1}\cos(\theta - \arctan q)$, with the maximum $\sqrt{\epsilon}\sqrt{q^2+1}$ at $\theta = \arctan q$. Hence, inequality turns into

$$\sqrt{\epsilon}\sqrt{q^2+1} \geq 1 - a \langle E^p \rangle \left(\frac{t_\epsilon}{\hbar} \right)^p,$$

equivalent to

$$t_\epsilon \geq \hbar \left(\frac{1 - \sqrt{\epsilon}\sqrt{q^2+1}}{a} \right)^{\frac{1}{p}} \langle E^p \rangle^{-\frac{1}{p}}, \tag{67}$$

which is **the lower bound** we have been seeking for.

3.3.2 Optimisation of the Factor

To date, the lower bound provided remains far from practical use, as the values of q and a are irrelevant to the quantum evolution and still need to be determined. To determine these two parameters, we should review the original inequality (65). Suppose that the upper curve and the lower line are tangent to each other at $x = x_t \in [x_0, \pi]$, where

$$\begin{cases} 0 < x_0 < \pi \text{ and } x_0 \cot \dfrac{x_0}{2} = p, & \text{if } 0 < p < 2, \\ \qquad\qquad \text{or } x_0 = 0, & \text{if } p \geq 2, \end{cases}$$

and there hold the relations:

$$\begin{cases} \cos x_t + q \sin x_t = 1 - a x_t^p, \\ -\sin x_t + q \cos x_t = -a p x_t^{p-1}, \end{cases} \tag{68}$$

leading to parametric equations for q and a in terms of x_t and p

$$\begin{cases} q = \dfrac{p(1 - \cos x_t) - x_t \sin x_t}{p \sin x_t - x_t \cos x_t}, \\ a = \dfrac{x_t^{1-p}(1 - \cos x_t)}{p \sin x_t - x_t \cos x_t}. \end{cases} \tag{69}$$

Substituting (69) into (67), we can also write the lower bound in terms of x_t and p

$$t_\epsilon \geq \hbar \left(\frac{p \sin x_t - x_t \cos x_t - \sqrt{x_t^2 - 2p x_t \sin x_t - 2p^2 \cos x_t + 2p^2}}{x_t^{1-p}(1 - \cos x_t) \langle E^p \rangle} \right)^{\frac{1}{p}}. \tag{70}$$

Denoting the fidelity factor to the power of p

$$\frac{p \sin x_t - x_t \cos x_t - \sqrt{x_t^2 - 2p x_t \sin x_t - 2p^2 \cos x_t + 2p^2}}{x_t^{1-p}(1 - \cos x_t)}$$

by α^p and taking derivative of it on x_t, we then acquire

$$\begin{aligned} \frac{\partial \alpha^p}{\partial x_t} = \left(\frac{1}{x_t^{1-p}(1 - \cos x_t)} \right)^2 \Bigg\{ & (p \cos x_t - \cos x_t + x_t \sin x_t) x_t^{1-p}(1 - \cos x_t) \\ & + (p \sin x_t - x_t \cos x_t) \Big[(p-1) x^{-p}(1 - \cos x_t) - x_t^{1-p} \sin x_t \Big] \\ & - \sqrt{\epsilon} \Bigg[\frac{x_t - p x_t \cos x_t - p \sin x_t + p^2 \sin x_t}{\sqrt{x_t^2 - 2p x_t \sin x_t - 2p^2 \cos x_t + 2p^2}} x_t^{1-p}(1 - \cos x_t) \\ + \sqrt{x_t^2 - 2p x_t \sin x_t - 2p^2 \cos x_t + 2p^2} & \Big[(p-1) x^{-p}(1 - \cos x_t) - x_t^{1-p} \sin x_t \Big] \Bigg] \Bigg\}. \end{aligned}$$

When $\partial\alpha^p/\partial x_t = 0$, separating variables yields a parametric equation for $\sqrt{\epsilon}$ in terms of x_t and p. Upon simplification of the equation, we obtain

$$\sqrt{\epsilon} = \frac{\cot\frac{x_t}{2}\sqrt{x_t^2 - 2px_t\sin x_t - 2p^2\cos x_t + 2p^2}}{2p - x_t\cot\frac{x_t}{2}}. \tag{71}$$

After inserting (71) into **the factor** α^p, we finally arrive at

$$\alpha^p = \frac{x_t^p(p - x_t\cot\frac{x_t}{2})}{2p - x_t\cot\frac{x_t}{2}}, \tag{72}$$

and **the lower bound** becomes

$$t_\epsilon \geq \hbar x_t\left(\frac{p - x_t\cot\frac{x_t}{2}}{2p - x_t\cot\frac{x_t}{2}}\right)^{\frac{1}{p}}\langle E^p\rangle^{-\frac{1}{p}}. \tag{73}$$

For a specific state, the lower bound is $t_\epsilon \geq \hbar(\alpha^p)^{1/p}\langle E^p\rangle^{-1/p}$. As the value of p increases, it can be proved that $(\alpha^p)^{1/p}$ increases while $\langle E^p\rangle^{-1/p}$ decreases. In consequence, one can maximise the bound with an optimised p

$$t_\epsilon \geq \sup_{p>0}\frac{\hbar\alpha_p}{\langle E^p\rangle^{1/p}}, \tag{74}$$

where $\alpha_p = (\alpha^p)^{1/p}$.

3.3.3 Tightness

At this stage, we have derived a practical form of the lower bound, and we will demonstrate evolutions of existing states saturating the bound to show its tightness. We now recall the family of two-level states $|E_\xi\rangle = \sqrt{1-\xi^2}|E_0\rangle + \xi|E_1\rangle$ (19). From (20), one can express ξ^2 in terms of ϵ and $\cos\theta$

$$\xi^2 = \frac{\epsilon + 1 - 2\sqrt{\epsilon}\cos\theta}{2 - 2\sqrt{\epsilon}\cos\theta}. \tag{75}$$

Reminded that the left-hand side of (66) reaches its maximum at $\theta = \arctan q$, hence one can construct $\cos\theta = 1/\sqrt{q^2+1}$. Plugging this and (71), along with q in terms of x_t and p, into (75), one can attain

$$\xi^2 = \frac{p - x_t\cot\frac{x_t}{2}}{2p - x_t\cot\frac{x_t}{2}}. \tag{76}$$

Plugging (71) again and (76) into the equation for the evolution time (21), one can calculate the evolution time of $|E_\xi\rangle$

$$t_\epsilon = \frac{\hbar}{E_1}x_t \tag{77}$$

Applying (73) to $|E_\xi\rangle$, $\langle E^p\rangle^{1/p} = \xi^{2/p}E_1$, and the lower bound is transformed into

$$t_\epsilon \geq \frac{\hbar}{E_1}\left(\frac{\alpha^p}{\xi^2}\right)^{\frac{1}{p}}, \tag{78}$$

which meets the precise evolution time (77).

It is worth noting that Giovannetti et al. [5] numerically demonstrated such tightness for $p = 1$, disregarding the presence of numerical instability. Fortuitously, the analytical proof of the tightness for $\forall p > 0$ is now presented above, thereby designating the lower bound as a QSL.

3.3.4 Alternative Forms

In practice, QSLs in terms of ϵ are more useful than those in terms of p, which is from mathematical inequalities regardless of the initial and final states.

After squaring and transposing (71), one will obtain a quadratic equation for p

$$4\left(\epsilon - \cos^2\frac{x_t}{2}\right)p^2 - 4x_t\cot\frac{x_t}{2}\left(\epsilon - \cos^2\frac{x_t}{2}\right)p + (\epsilon - 1)\,x_t^2\cot^2\frac{x_t}{2} = 0 \tag{79}$$

with a valid root

$$p = \frac{x_t}{2}\cot\frac{x_t}{2}\left(1 + \sqrt{\frac{1-\cos^2\frac{x_t}{2}}{\epsilon - \cos^2\frac{x_t}{2}}}\right), \tag{80}$$

which is **the parametric equation for** p in terms of ϵ and x_t, where $x_t \in (2\arccos\sqrt{\epsilon}, \pi]$.

Putting (80) into (72), one can reach **an alternative form of** α^p

$$\alpha^p = \frac{x_t^p}{2}\left(1 - \sqrt{\frac{\epsilon - \cos^2\frac{x_t}{2}}{1-\cos^2\frac{x_t}{2}}}\right), \tag{81}$$

where p is defined as (80). ξ^2 of states $|E_\xi\rangle$ in (19), whose evolutions saturate the bound, can be converted to

$$\xi^2 = \frac{1}{2}\left(1 - \sqrt{\frac{\epsilon - \cos^2\frac{x_t}{2}}{1-\cos^2\frac{x_t}{2}}}\right). \tag{82}$$

For $\epsilon = 0$, x_t is fixed at π, and hence, p in (80) is undefined and, in fact, can take on any positive value. Under this condition, **the maximised bound** (74) is reduced to

$$t_\perp \geq \sup_{p>0}\frac{\pi\hbar}{\left(2\langle E^p\rangle\right)^{1/p}}, \tag{83}$$

which has been noted by Fu et al. [9].

As $x_t \to (2\arccos\sqrt{\epsilon})^+$, p tends to $+\infty$, and α_p approaches $(2\arccos\sqrt{\epsilon})^-$. As a consequence, $\langle E^p\rangle^{1/p} \to E_{\max}$ [1], and **the QSL** is then rephrased as

$$t_\epsilon \geq \frac{2\hbar\arccos\sqrt{\epsilon}}{E_{\max}}, \tag{84}$$

which encompasses the bound on the minimum orthogonalisation time $t_\perp \geq (\pi\hbar)/E_{\max}$ revealed by Levitin and Toffoli [10] as a special case.

Taking $x_t = 2\arccos\sqrt{\Lambda\epsilon}$, where $\Lambda \in [0,1)$, it generates **another alternative form of** α^p

$$\alpha^p = \frac{2^p \arccos^p \sqrt{\Lambda\epsilon}}{2}\left[1 - \sqrt{\frac{(1-\Lambda)\epsilon}{1-\Lambda\epsilon}}\right], \tag{85}$$

where

$$p = \arccos\sqrt{\Lambda\epsilon}\left(\frac{1}{\sqrt{\frac{1}{\Lambda\epsilon}-1}} + \frac{1}{\sqrt{\frac{1}{\Lambda}-1}}\right). \tag{86}$$

In this scenario, given a specific ϵ, one has the flexibility to select an appropriate Λ to determine the corresponding value of p, followed by the opportunity to maximise the bound.

3.4 Potentially Stronger Lower Bounds

3.4.1 Bound I

We now try to compile moments of negative energies and positive energies separately. To achieve this, let us first consider the following class of inequalities for $q \geq 0$:

$$\begin{cases} \cos x + q\sin x \geq 1 - a\,|x|^p\,, & \text{if } x \geq 0, \\ \cos x + q\sin x \geq 1 - b\,|x|^p\,, & \text{if } x < 0, \end{cases} \tag{87}$$

which holds for all x and where a and b, which are both functions of q for a specific $p \in (0,1]$, regulate the lower lines $1 - a|x|^p$ and $1 - b|x|^p$, respectively, to be tangent to the upper trigonometric curve $\cos x + q\sin x$.

For a time-independent system Hamiltonian, we denote negative eigenenergies by $E_{<m}$ with probability amplitudes $|c_{<m}|^2$, and zero- and positive eigenenergies by $E_{\geq n}$ with probability amplitudes $|c_{\geq n}|^2$. With respect to a state $|\Psi_i\rangle = \sum_m c_{<m}|E_{<m}\rangle + \sum_n c_{\geq n}|E_{\geq n}\rangle$, where the normalisation $\sum_m |c_{<m}|^2 + \sum_n |c_{\geq n}|^2 = 1$ holds, we apply (87) to the state and employ (16). An inequality then emerges as follows

$$\sqrt{\epsilon}\,(\cos\theta + q\sin\theta) \geq 1 - (a\,\langle|E_{\geq}|^p\rangle + b\,\langle|E_{<}|^p\rangle)\left(\frac{t_\epsilon}{\hbar}\right)^p, \tag{88}$$

where $\langle|E_{<}|^p\rangle = \sum_m |c_{<m}|^2|E_{<m}|^p$ and $\langle|E_{\geq}|^p\rangle = \sum_n |c_{\geq n}|^2|E_{\geq n}|^p$. After maximising the left-hand side of (88), we then rearrange the equation to isolate t_ϵ on one side

$$t_\epsilon \geq \hbar\left(\frac{1-\sqrt{\epsilon}\sqrt{q^2+1}}{a\,\langle|E_{\geq}|^p\rangle + b\,\langle|E_{<}|^p\rangle}\right)^{\frac{1}{p}}, \tag{89}$$

which is a new lower bound on quantum evolution, depending on new metrics. The existence of three points of tangency between the lower lines and the upper trigonometric curve, as illustrated by (87), displays potential tightness for certain three-level states and two-level states. However, this tightness is yet to be manifested.

3.4.2 Bound II

Given $p \in (0, 2]$, we modify the class of inequalities (87) into

$$\begin{cases} \cos x + q \sin x \geq 1 - a\,|x|^p\,, & \text{if } x \geq 0, \\ \cos x + q \sin x \geq 1 - b\,|x|^{\frac{p}{2}}\,, & \text{if } x < 0, \end{cases} \tag{90}$$

for all x. Still, a and b are both functions of q that regulate the lower lines to be tangent to the upper trigonometric curve.

Applying the modified inequalities to $|\Psi_i\rangle$, we achieve a quadratic inequality equation with respect to $(t_\epsilon/\hbar)^{p/2}$

$$a\left\langle |E_{\geq}|^p \right\rangle \left(\frac{t_\epsilon}{\hbar}\right)^p - b\left\langle |E_<|^{\frac{p}{2}} \right\rangle \left(\frac{t_\epsilon}{\hbar}\right)^{\frac{p}{2}} - \left(1 - \sqrt{\epsilon}\sqrt{q^2+1}\right) \geq 0 \tag{91}$$

with a valid root

$$\left(\frac{t_\epsilon}{\hbar}\right)^{\frac{p}{2}} \geq \frac{b\langle |E_<|^{\frac{p}{2}}\rangle + \sqrt{b^2\langle |E_<|^{\frac{p}{2}}\rangle^2 + 4a\left\langle |E_{\geq}|^p\right\rangle (1 - \sqrt{\epsilon}\sqrt{q^2+1})}}{2a\left\langle |E_{\geq}|^p\right\rangle},$$

directly establishing another bound on t_ϵ

$$t_\epsilon \geq \hbar\left[\frac{b\langle |E_<|^{\frac{p}{2}}\rangle + \sqrt{b^2\langle |E_<|^{\frac{p}{2}}\rangle^2 + 4a\left\langle |E_{\geq}|^p\right\rangle (1 - \sqrt{\epsilon}\sqrt{q^2+1})}}{2a\left\langle |E_{\geq}|^p\right\rangle}\right]^{\frac{2}{p}}. \tag{92}$$

Since $1 - a|x|^p$ could be closer to $\cos x + q \sin x$ if $p > 1$, this bound shows promise in being stronger than Bound I (89) for many states. Yet, the relations between a, b, q, and p are currently obscure, limiting us from exploiting both Bound I and II. Additionally, further investigation is also needed to confirm the tightness of Bound II.

4 Comparison

Consider the following three classes of initial states:

$$|\Phi_{a,n}\rangle = \sum_{k=-n}^{n} \frac{1}{\sqrt{2n+1}}\,|k\mathcal{E}\rangle\,,$$

$$|\Phi_{b,n}\rangle = c_{b,0}\,|0\rangle + c_{b,n}\left(|n\mathcal{E}\rangle + |-n\mathcal{E}\rangle\right) + \sum_{k=1}^{n-1} c_{b,k}\left(|\lambda_{b,k} n\mathcal{E}\rangle + |-\lambda_{b,k} n\mathcal{E}\rangle\right),$$

where $\{c_{b,k}\}$ is a sequence of random numbers and $|c_{b,0}|^2 + 2\sum_{k=1}^{n} |c_{b,k}|^2 = 1$, and $\{\lambda_{b,k}\}$ is a sequence of random numbers ranging in $(0,1)$ and arranged in ascending order, and

$$|\Phi_{c,n}\rangle = c_{c,0}\,|0\rangle + c_{c,2n}\,|2n\mathcal{E}\rangle + \sum_{k=1}^{2n-1} c_{c,k}\,|\lambda_{c,k}2n\mathcal{E}\rangle\,,$$

where $\{c_{c,k}\}$ is a sequence of random numbers and $\sum_{k=0}^{2n} |c_{c,k}|^2 = 1$, and $\{\lambda_{c,k}\}$ is a sequence of random numbers ranging in $(0,1)$ and arranged in ascending order. One can compare the time bounds provided by different QSLs with the actual evolution time.

As plotted in Figure 3, the strengthened MT QSL is stronger than the original one predictably. The QSLs depending on energy moments of different orders are more powerful than the ML QSL as a special case of the former ones, as plotted in Figure 4. (See Appendix B for your reference.)

5 Conclusions

Regarding QSLs depending on energy central moments, I modified the LC QSL, bridging the gap between the original one and the MT QSL. Relying on the exploitation of the property of the fidelity factor $\beta_\nu(\epsilon)$, I confirmed that the MT QSL, as a special case of the modified LC QSL, is weaker than the original LC QSL, constraining the range of ν for optimising the time bound.

Concerning QSLs depending on energy moments, I extended the applicability to include any (p, ϵ) pair, allowing for the optimisation of the time bound for more states. Additionally, by demonstrating the tightness of such QSLs, I analytically substantiated the identity of the upper bound $\alpha_>$ (22) and the lower bound $\alpha_<$ (18).

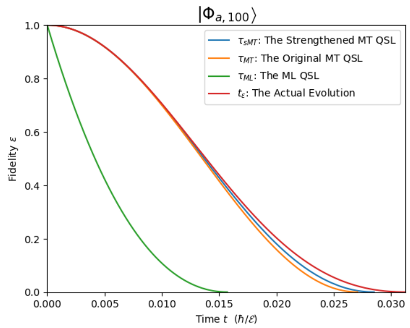
At this juncture, setting aside Bounds I and II, we can combine (74) and (58) and establish a new bound on the minimum evolution time

$$t_\epsilon \geq \mathcal{T}_\epsilon \equiv \max\left\{\sup_{p>0} \frac{\hbar\alpha_p}{\langle E^p\rangle^{1/p}},\ \sup_{\nu\leq\mu(\epsilon)} \frac{\hbar\beta_\nu}{(M_\nu^*)^{1/\nu}}\right\}, \tag{93}$$

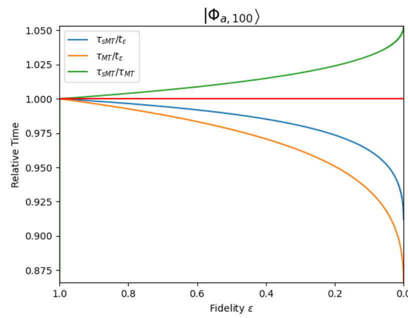
which must be more powerful than

$$\mathcal{T}_\epsilon(E, \Delta E) \equiv \max\left\{\alpha(\epsilon)\frac{\pi\hbar}{2E},\ \beta(\epsilon)\frac{\pi\hbar}{2\Delta E}\right\}$$

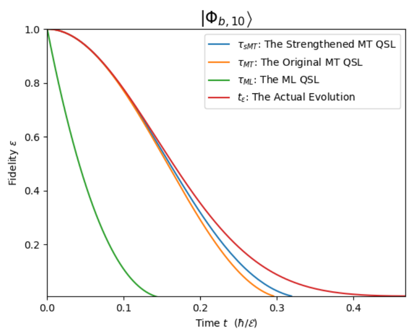
as defined in [5].



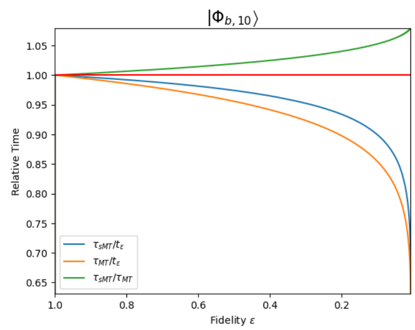
(a)



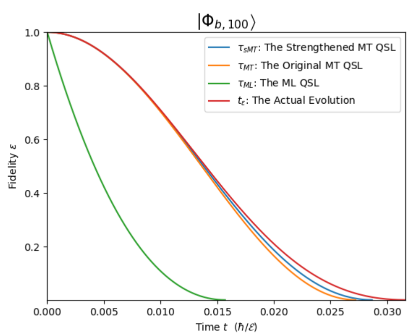
(b)



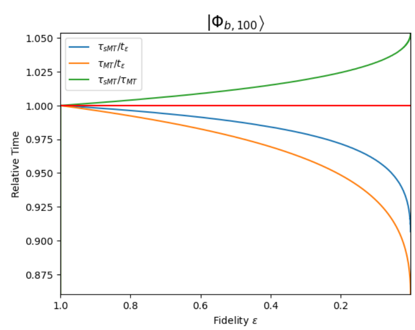
(c)



(d)

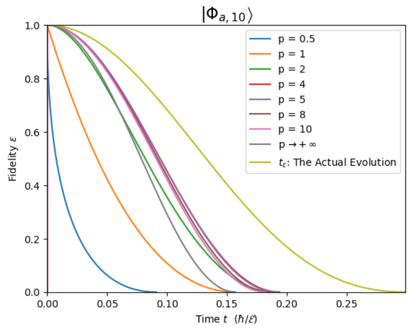


(e)

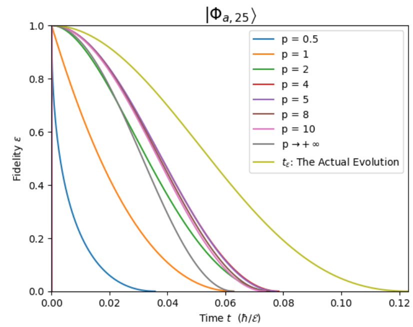


(f)

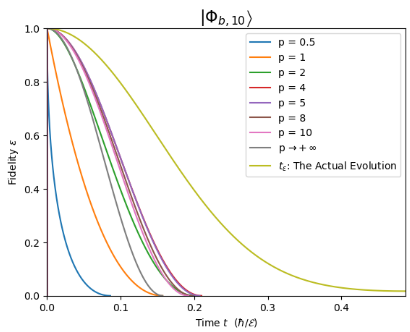
Figure 3: The Strengthened MT QSL



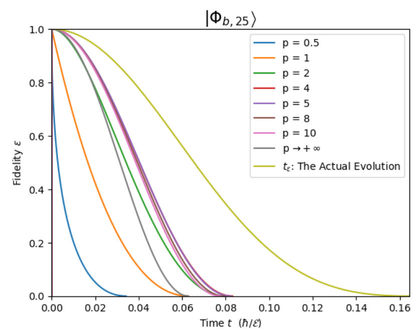
(a)



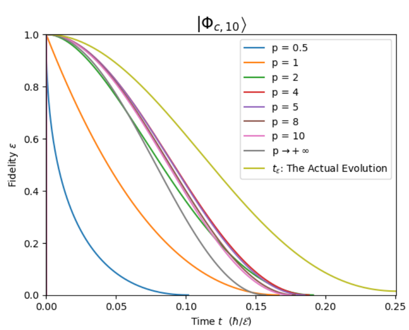
(b)



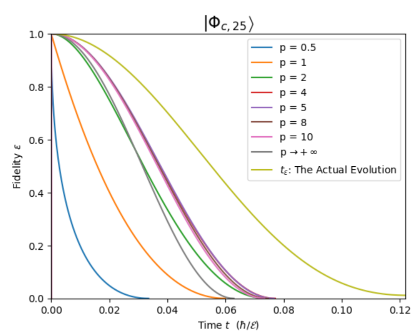
(c)



(d)



(e)



(f)

Figure 4: The QSLs Depending Upon Energy Moments of Different Orders

Acknowledgements

I here dedicate my work to the memory of my late grandfather. I am forever grateful for his guidance and influence in my life.

I would also like to express my heartfelt appreciation to my parents for their unwavering support, my family for their enduring love, my supervisor, Prof. Chau, for his encouragement and invaluable mentorship, and all my friends and classmates for their constant companionship.

Their presence makes the world a romantic place!

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A The Derivation of $\alpha_>$ and $\alpha_<$

SEE THE NEXT PAGE.

ALPHA

December 2, 2022

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
from multiprocess import Pool
from tqdm import tqdm
```

[2]:
```
epsilon=np.arange(1001)/1000
```

alpha_up

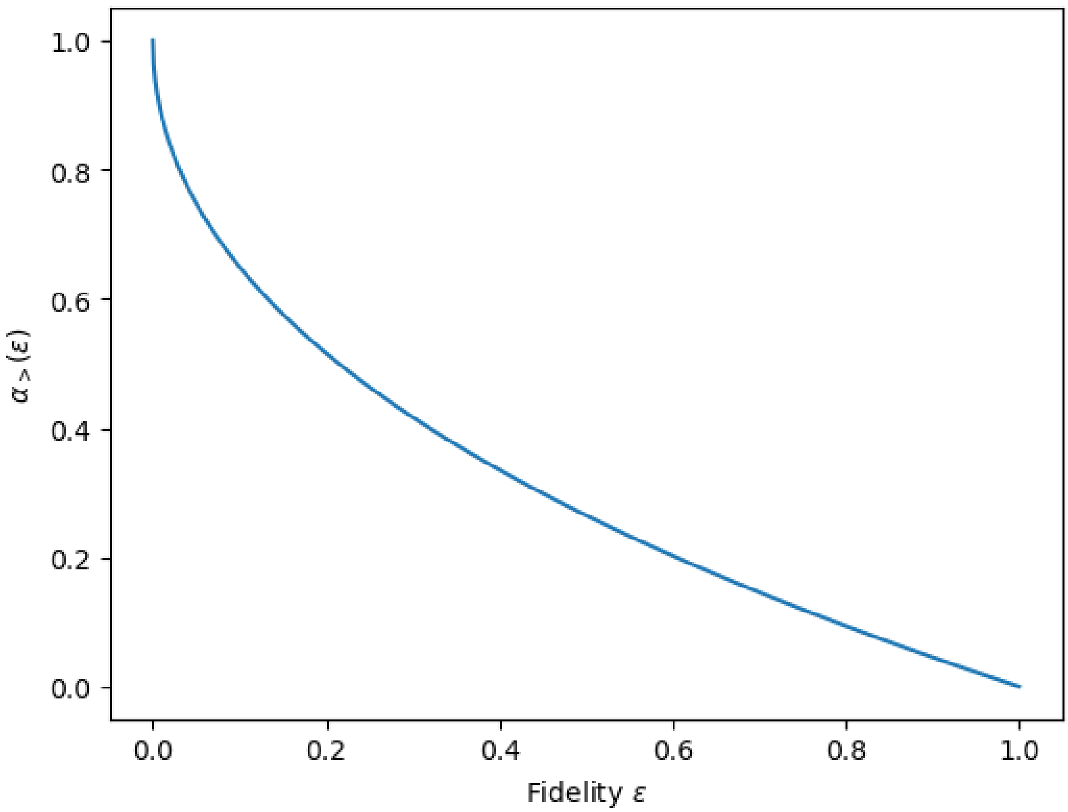
[3]:
```
def alpha_up_epsilon(epsilon):
    z=np.arange(1001)/1000
    alpha_up_z=np.arange(1001,dtype=float)
    for j in range(1001):
        if 2*z[j]*(1-z[j])==0:
            alpha_up_z[j]=1
        elif ((epsilon-1+2*z[j]*(1-z[j]))/(2*z[j]*(1-z[j])))**2<=1:
            alpha_up_z[j]=(2*z[j]/np.pi)*np.arccos((epsilon-1+2*z[j]*(1-z[j]))/
 ↪(2*z[j]*(1-z[j])))
        else:
            alpha_up_z[j]=1
    return min(alpha_up_z)
```

[4]:
```
P=Pool(8)
res=[P.apply_async(alpha_up_epsilon,(ep, )) for ep in epsilon]
answer=[ep.get() for ep in tqdm(res)]
alpha_up=answer
```

```
100%|                    | 1001/1001 [00:00<00:00, 1724.33it/s]
```

[5]:
```
plt.figure()
plt.ylabel(r"$\alpha_>(\epsilon)$")
plt.xlabel(r"Fidelity $\epsilon$")
plt.plot(epsilon,alpha_up)
```

[5]:
```
[<matplotlib.lines.Line2D at 0x7fce300a5cd0>]
```



alpha_down

[6]:
```
def alpha_down_epsilon(epsilon):
    xt=np.linspace(0,4,401)
    theta=np.linspace(0,2*np.pi,600)
    alpha_down_th=np.arange(600,dtype=float)
    alpha_down_xt=np.arange(401,dtype=float)
    for j in range(600):
        for k in range(401):
            denominator=np.sin(xt[k])-xt[k]*np.cos(xt[k])
            if denominator==0:
                alpha_down_xt[k]=0
            else:
                a=(1-np.cos(xt[k]))/denominator
                q=(1-np.cos(xt[k])-xt[k]*np.sin(xt[k]))/denominator
                if a==0:
                    alpha_down_xt[k]=0
                else:
                    alpha_down_xt[k]=2/np.pi/a*(1-np.sqrt(epsilon)*(np.
↪cos(theta[j])-q*np.sin(theta[j])))
        alpha_down_th[j]=max(alpha_down_xt)
```

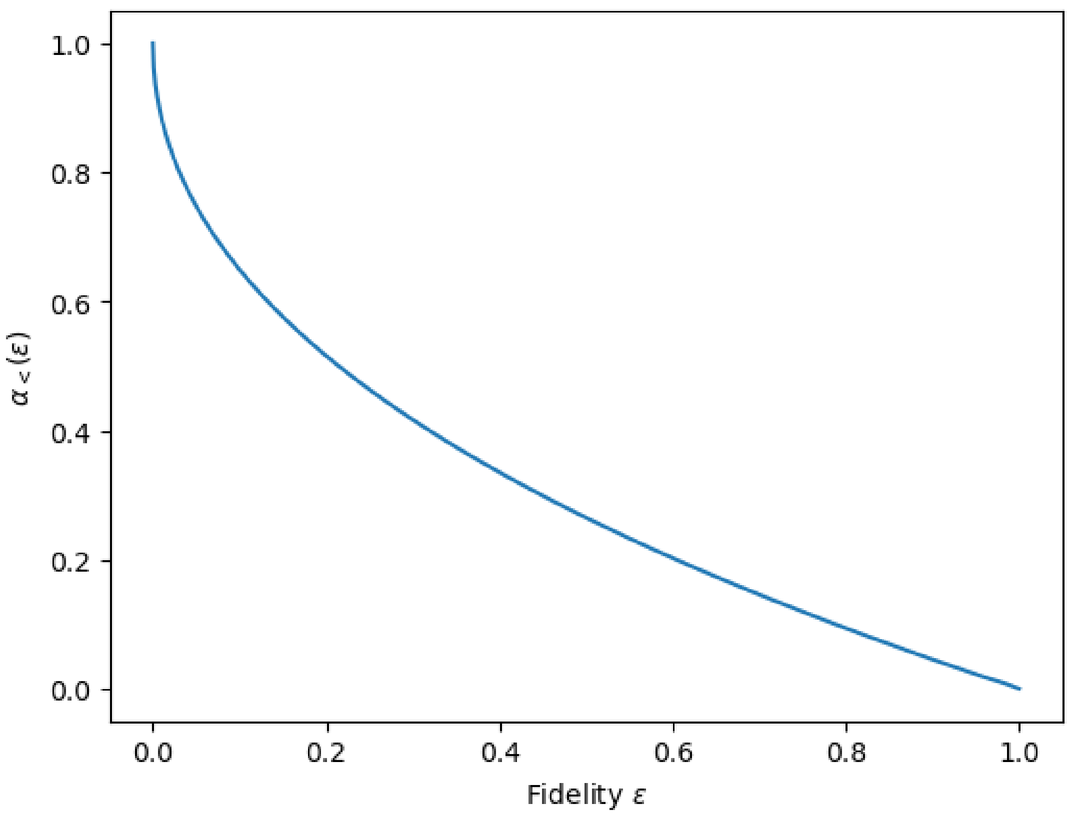
```
    return min(alpha_down_th)
```

[7]:
```
P=Pool(8)
res=[P.apply_async(alpha_down_epsilon,(ep, )) for ep in epsilon]
answer=[ep.get() for ep in tqdm(res)]
alpha_down=answer
```

```
100%|                         | 1001/1001 [32:23<00:00,  1.94s/it]
```

[8]:
```
plt.figure()
plt.ylabel(r"$\alpha_<(\epsilon)$")
plt.xlabel(r"Fidelity $\epsilon$")
plt.plot(epsilon,alpha_down)
```

[8]:
```
[<matplotlib.lines.Line2D at 0x7fce30028b50>]
```

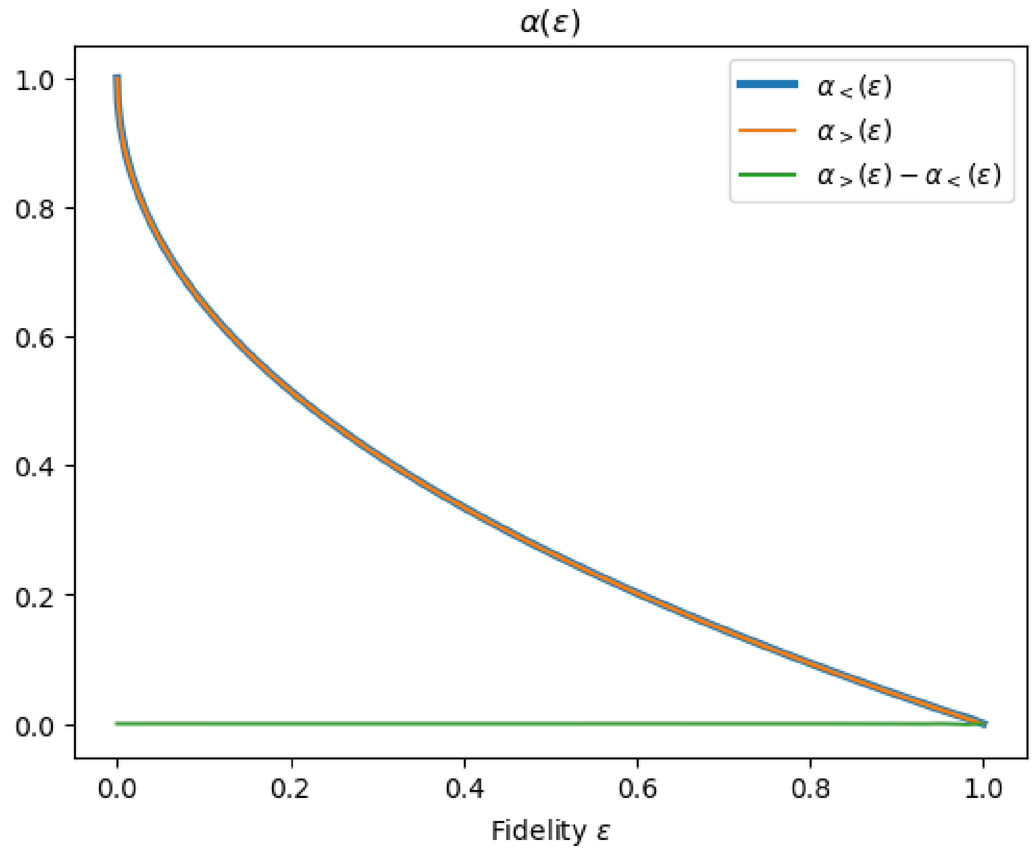


alpha_error

$\alpha(\epsilon)$

[10]:
```
plt.figure()
plt.title(r"$\alpha(\epsilon)$")
plt.xlabel(r"Fidelity $\epsilon$")
plt.plot(epsilon,alpha_down,label=r"$\alpha_<(\epsilon)$",linewidth=3)
plt.plot(epsilon,alpha_up,label=r"$\alpha_>(\epsilon)$")
plt.plot(epsilon,alpha_error,label=r"$\alpha_>(\epsilon)-\alpha_<(\epsilon)$")
plt.legend()
```

[10]: <matplotlib.legend.Legend at 0x7fce3009ca30>



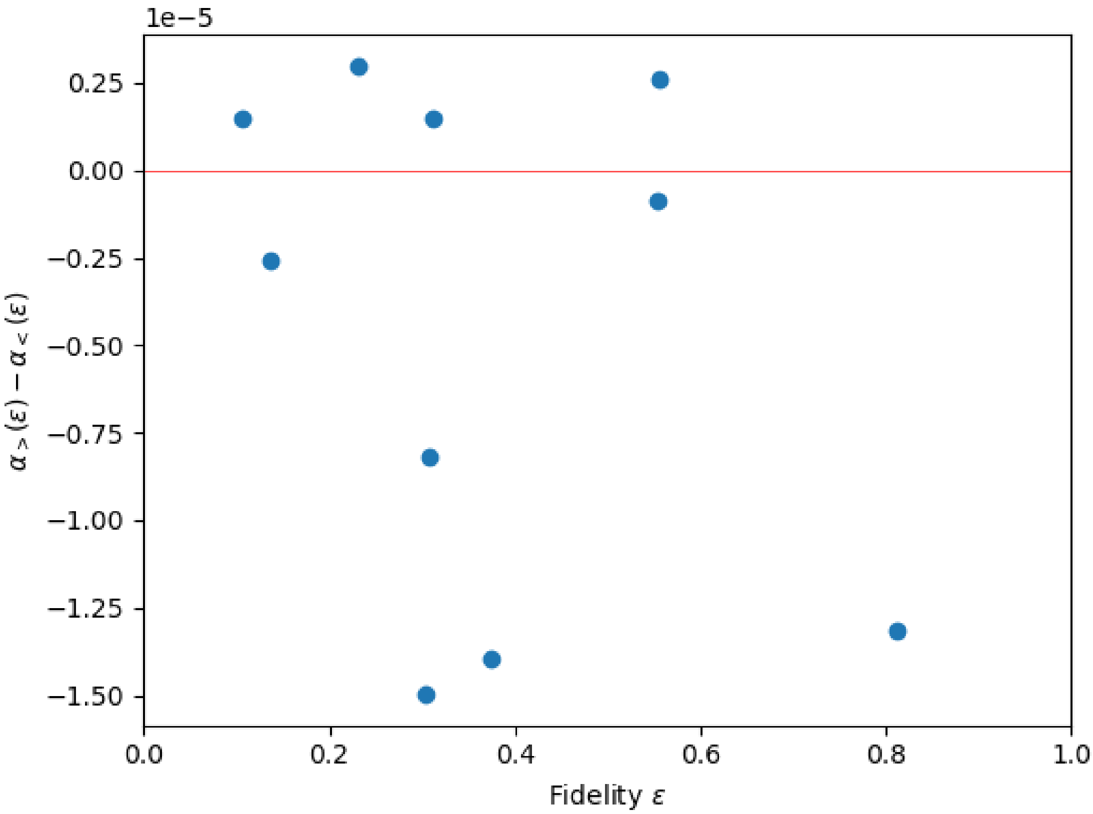
alpha_error

[11]:
```
import random
```

[52]:
```
idx_list=random.sample(range(1001),10)
plt.figure()
plt.ylabel(r"$\alpha_>(\epsilon)-\alpha_<(\epsilon)$")
plt.xlabel(r"Fidelity $\epsilon$")
plt.xlim(0,1)
plt.hlines(y=0,xmin=0,xmax=1,color="red",linewidth=0.5)
```

```
plt.scatter(epsilon[idx_list],alpha_error[idx_list])
```

[52]: <matplotlib.collections.PathCollection at 0x7fce5167d2e0>



B Comparison Between Different QSLs

SEE THE NEXT PAGE.

Comparison

May 24, 2023

[1]:
```
import numpy as np
from scipy.linalg import expm
import matplotlib.pyplot as plt
from scipy.optimize import fsolve
```

Energy central moments for balanced systems

[2]:
```
def H_p(E, w, p):
    E_stat = 0
    E_av = 0
    for i in range(len(E)):
        E_av += w[i] * E[i]
    for i in range(len(E)):
        E_stat += w[i] * (abs(E[i]-E_av) ** p)
    return E_stat ** (1/p)
```

Energy moments with zero ground state energy

[3]:
```
def E_p(E, w, p):
    E_stat = 0
    for i in range(len(E)):
        E_stat += w[i] * (abs(E[i] - E[-1]) ** p)
    return E_stat ** (1/p)
```

$\mu(\epsilon)$

[4]:
```
n = 1001
epsilon = np.linspace(0, 1, n)
mu = np.linspace(0, 2, n)
for i in range(n-1):
    mu[i] = np.sqrt((1 + np.sqrt(epsilon[i])) / (1 - np.sqrt(epsilon[i]))) * np.
 ↪arccos(np.sqrt(epsilon[i]))
def mu_ep(epsilon):
    return mu[round(epsilon * 1000)]
```

x_0 for mybound

[6]:
```
def equation(x, p):
    return x * np.cos(x/2) - p * np.sin(x/2)
def solve_equation(p, tolerance):
```

```
    x0 = p * np.pi
    x_p, = fsolve(equation, x0, args=(p,), xtol=tolerance)
    x_p += tolerance
    return x_p
def x_p(pp):
    if pp >= 2:
        x_p = 0
    if ((pp > 0) and (pp < 2)):
        x_p = solve_equation(pp, 5e-8)
    return x_p
p = np.linspace(0.001, 2, 2000)
x_pp = np.linspace(0.001, 2, 2000)
for i in range(2000):
    x_pp[i] = solve_equation((i+1)/1000, 5e-8)
def x_0(pp):
    x_00 = 1e-9
    if ((pp > 0) and (pp < 2)):
        x_00 = x_pp[round(pp * 1000 - 1)]
    return x_00
```

System Hamiltonian and inital states

[7]:
```
def H_n(n):
    return np.diag(np.arange(n, -n-1, -1))
def initial_state_n(n):
    return np.full(2*n+1, np.sqrt(1/(2*n+1)))
def t_list_n(n):
    return np.linspace(0, np.pi*2/(2*n+1), 1001)
```

Fidelity over time

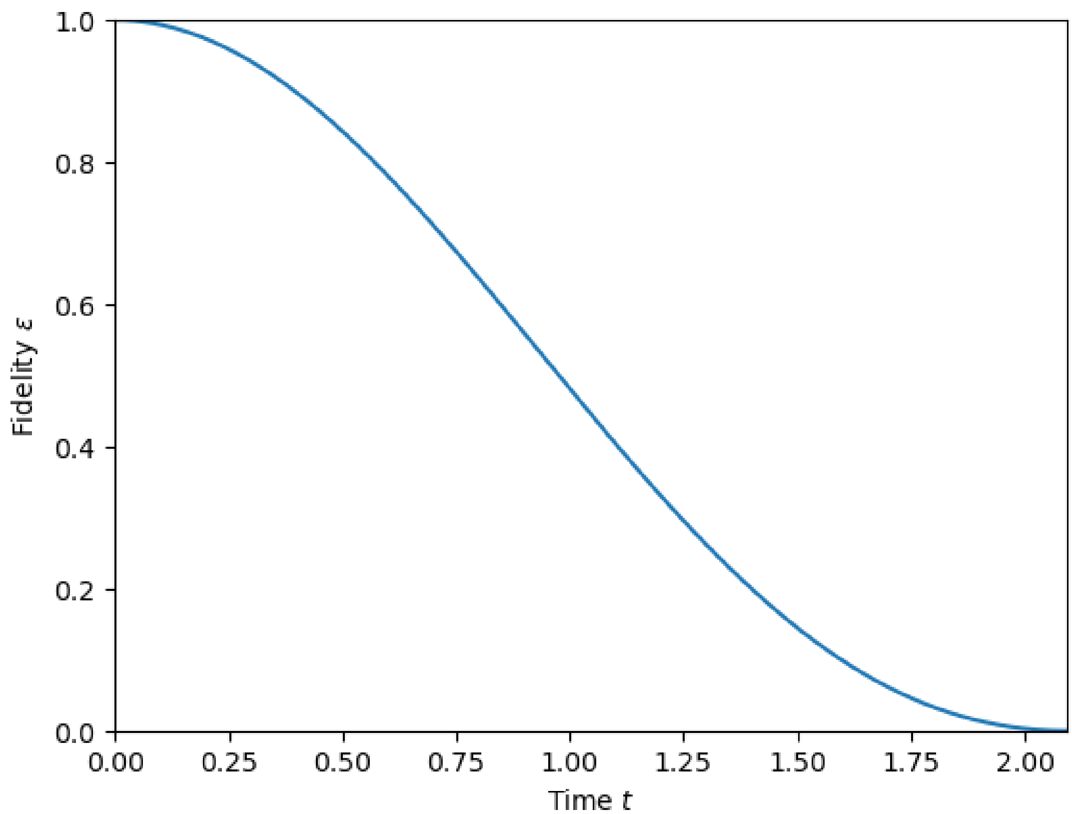
[8]:
```
def fidelity(H, initial_state, t_list):
    fidelity = np.linspace(0, 1, 1001)
    for i in range(1001):
        evolved_state = expm(-1j * H * t_list[i]) @ initial_state
        fidelity[i] = np.abs(np.abs(np.vdot(initial_state, evolved_state))**2)
    return fidelity
```

[9]:
```
def epsilon_t(n):
    H = H_n(n)
    initial_state = initial_state_n(n)
    time_list = t_list_n(n)
    return H, initial_state, time_list, fidelity(H, initial_state, time_list),␣
 ↪np.pi*2/(2*n+1)
```

[10]:
```
H, initial_state, t_list, fidelity_list, x_lim = epsilon_t(1)
```

[11]:
```
plt.figure()
plt.ylabel(r"Fidelity $\epsilon$")
plt.xlabel(r"Time $t$")
plt.ylim(0, 1)
plt.xlim(0, x_lim)
plt.plot(t_list, fidelity_list)
```

[11]: [<matplotlib.lines.Line2D at 0x7fb4087feac0>]



Generate random balanced Hamiltonians and initial states

[12]:
```
def generate_diagonal_matrix(n):
    diag_elements = np.random.rand(n-1) * n
    diag_elements.sort()
    diagonal_matrix = np.diag([n] + list(diag_elements[::-1]) + [0] +
 ↪list(-diag_elements) + [-n])
    return diagonal_matrix
def generate_symmetric_array(n):
    random_nums = np.random.rand(n)
    symmetric_array = np.concatenate((random_nums[::-1], np.random.rand(1),
 ↪random_nums))
```

```
    symmetric_array /= np.sum(symmetric_array)
    return np.sqrt(symmetric_array)
def t_list_random(n):
    return np.linspace(0, np.pi*4/(2*n+1), 1001)
```

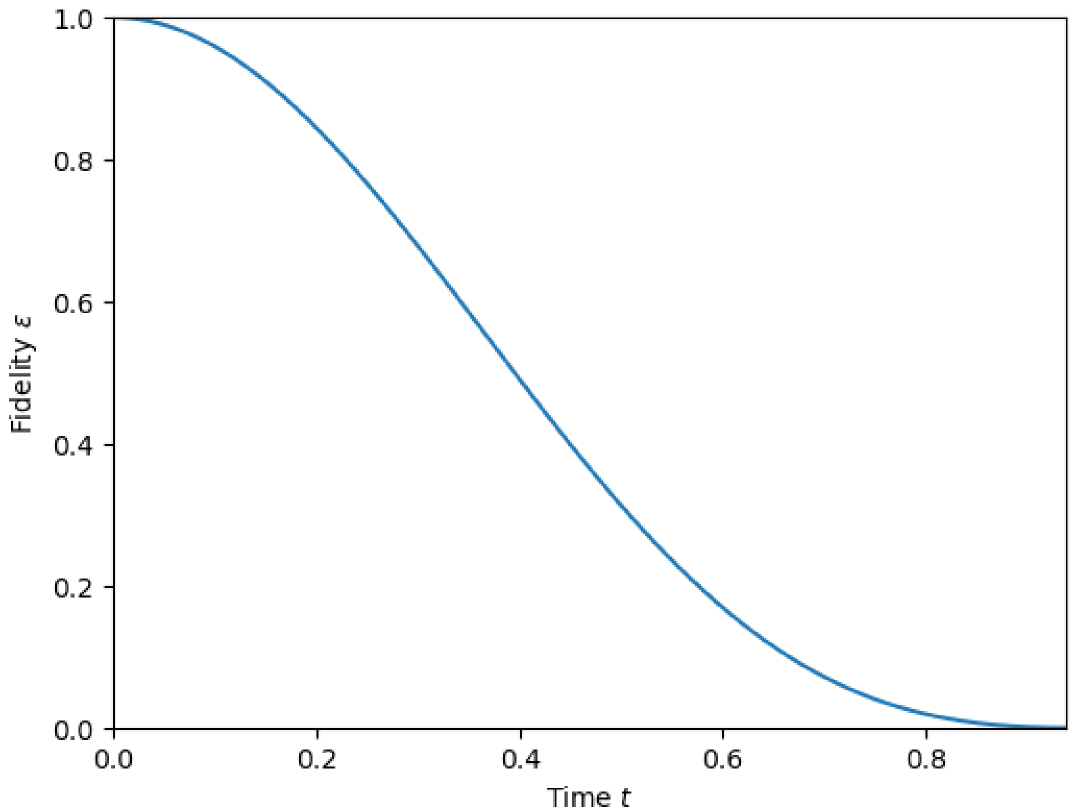
[13]:
```
def random_balanced_state_evol(n):
    H_random_n = generate_diagonal_matrix(n)
    state_random_n = generate_symmetric_array(n)
    time_list = t_list_random(n)
    return H_random_n, state_random_n, time_list, fidelity(H_random_n,␣
 ↪state_random_n, time_list), np.pi*2/(2*n+1)
```

Extract the decreasing part of the curve of the fidelity

[14]:
```
def decreasing_half(arr):
    n = len(arr)
    if n < 2:
        return arr

    end_index = 1
    for i in range(1, n):
        if arr[i] <= arr[i-1]:
            end_index = i
        else:
            break

    return arr[:end_index], end_index
```

[15]:
```
H_random, state_random, t_list, fidelity_list, x_lim =␣
 ↪random_balanced_state_evol(3)
fidelity_list, end_idx = decreasing_half(fidelity_list)
t_list = t_list[:end_idx]
plt.figure()
plt.ylabel(r"Fidelity $\epsilon$")
plt.xlabel(r"Time $t$")
plt.ylim(0, 1)
plt.xlim(0, t_list[-1])
plt.plot(t_list, fidelity_list)
```

[15]:
```
[<matplotlib.lines.Line2D at 0x7fb41b478940>]
```



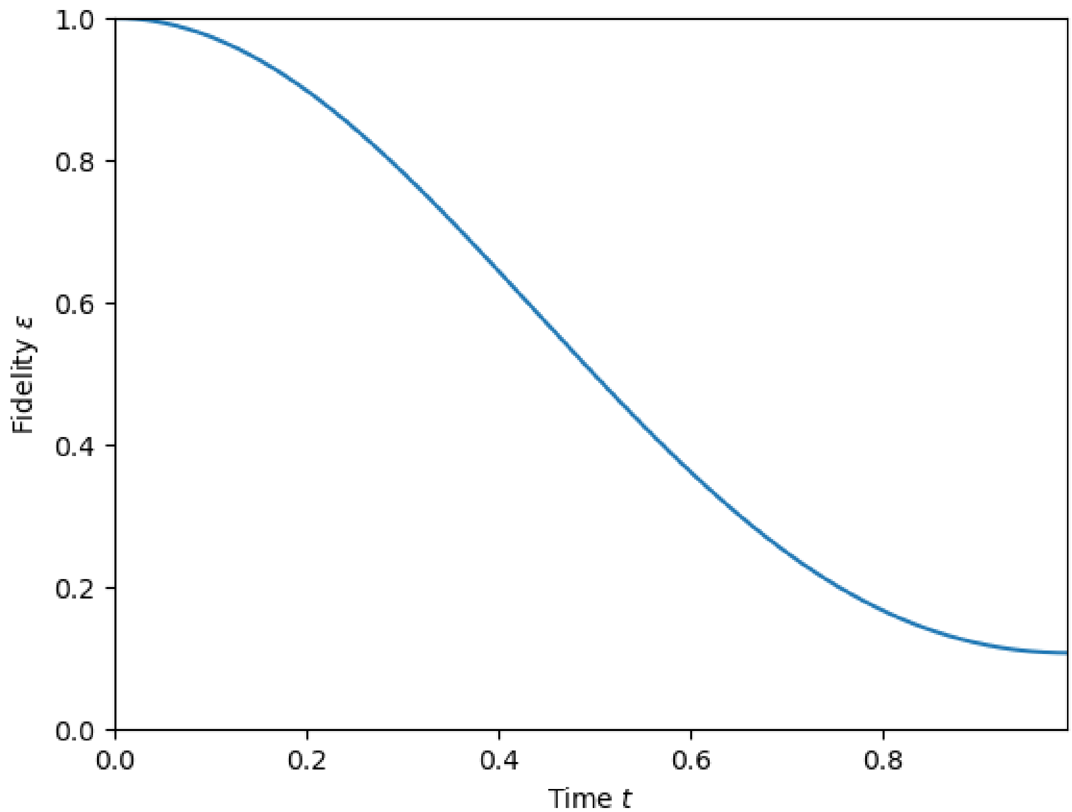
Generate randam energy-unbalanced systems and states

[16]:
```
def generate_unbalanced_matrix(n):
    diag_elements = np.random.rand(2*n-1) * 2*n
    diag_elements.sort()
    diagonal_matrix = np.diag([2*n] + list(diag_elements) + [0])
    return diagonal_matrix
def generate_unbalanced_array(n):
    random_nums = np.random.rand(2*n+1)
    random_nums /= np.sum(random_nums)
    return np.sqrt(random_nums)
def t_list_unbalanced(n):
    return np.linspace(0, np.pi*4/(2*n+1), 1001)
```

[17]:
```
def random_unbalanced_state_evol(n):
    H_random_n = generate_unbalanced_matrix(n)
    state_random_n = generate_unbalanced_array(n)
    time_list = t_list_unbalanced(n)
    return H_random_n, state_random_n, time_list, fidelity(H_random_n,
 ↪state_random_n, time_list), np.pi*2/(2*n+1)
```

```
[18]: H_unbalanced, state_unbalanced, t_list, fidelity_list, x_lim =␣
        ↪random_unbalanced_state_evol(3)
      fidelity_list, end_idx = decreasing_half(fidelity_list)
      t_list = t_list[:end_idx]
      plt.figure()
      plt.ylabel(r"Fidelity $\epsilon$")
      plt.xlabel(r"Time $t$")
      plt.ylim(0, 1)
      plt.xlim(0, t_list[-1])
      plt.plot(t_list, fidelity_list)
```

```
[18]: [<matplotlib.lines.Line2D at 0x7fb408b0abb0>]
```



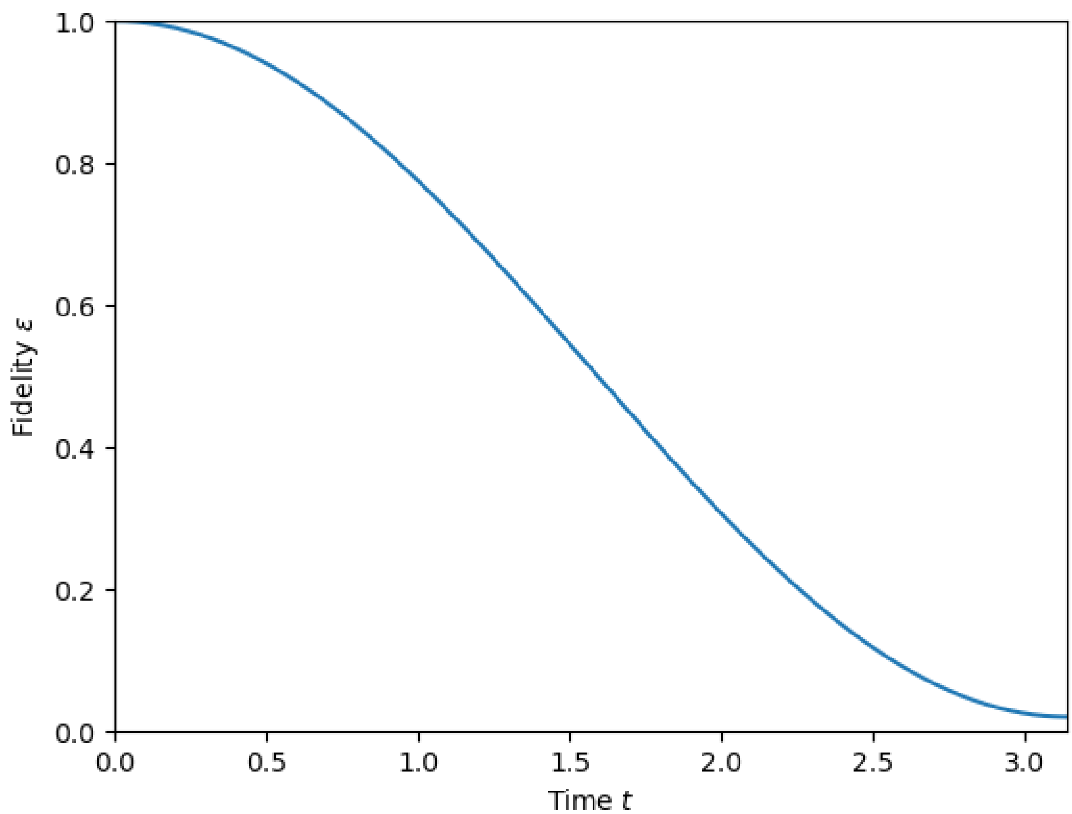
two-level

```
[19]: def two_level():
          two_level_hamil = np.diag([1, 0])
          random_nums = np.random.rand(2)
          two_level_state = np.sqrt(random_nums / np.sum(random_nums))
          time_list = np.linspace(0, np.pi, 1001)
```

```
    return two_level_hamil, two_level_state, time_list,␣
↪fidelity(two_level_hamil, two_level_state, time_list), np.pi*2
```

[20]:
```
system2, state2, t_list2, fidelity_list2, x_lim2 = two_level()
#fidelity_list, end_idx = decreasing_half(fidelity_list)
#t_list = t_list[:end_idx]
plt.figure()
plt.ylabel(r"Fidelity $\epsilon$")
plt.xlabel(r"Time $t$")
plt.ylim(0, 1)
plt.xlim(0, t_list2[-1])
plt.plot(t_list2, fidelity_list2)
```

[20]: [<matplotlib.lines.Line2D at 0x7fb3e8103610>]



Comparison between the strengthened and the original MT QSL

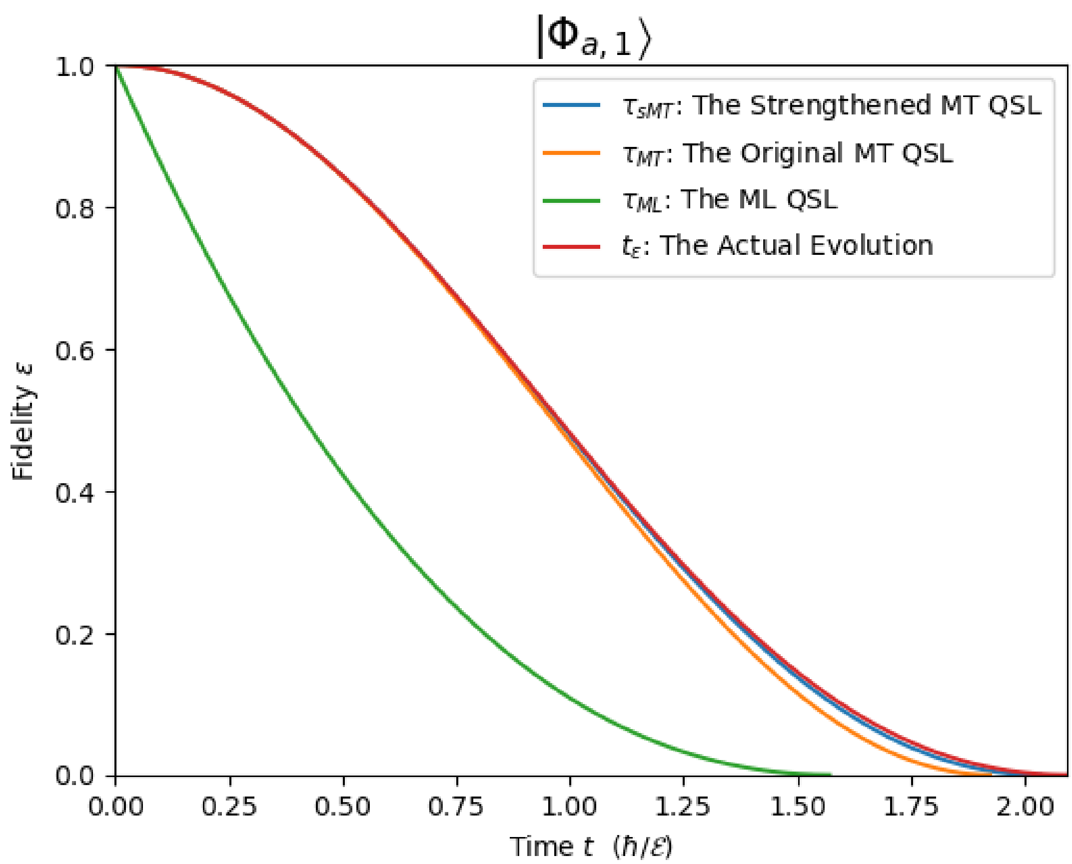
[21]:
```
def comparison_mt_ml(n):
    H, initial_state, t_list, fidelity_list, x_lim = epsilon_t(n)
    #MT
    E = np.diag(H)
```

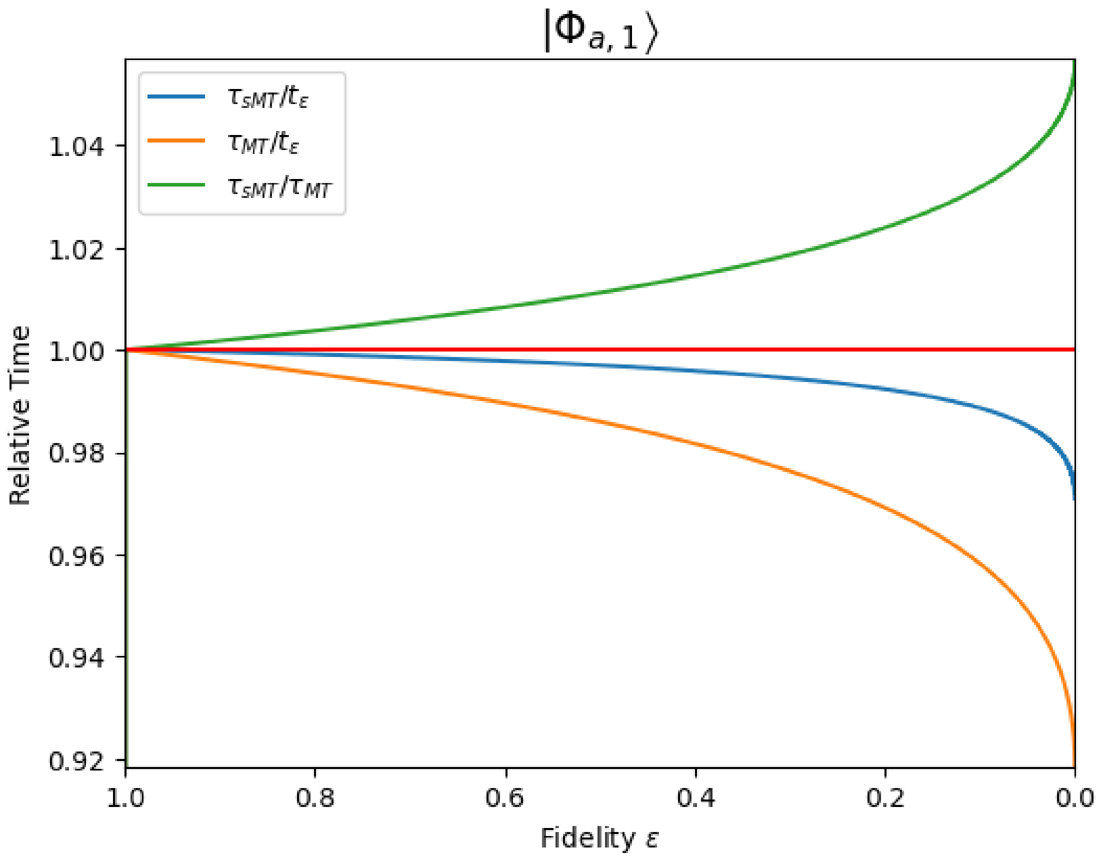
```python
    w = initial_state ** 2
    tau_eMT_list = np.linspace(0, 1, 1001)
    eMT_vs_actual = np.linspace(0, 1, 1001)
    tau_MT_list = np.linspace(0, 1, 1001)
    MT_vs_actual = np.linspace(0, 1, 1001)
    eMT_vs_MT = np.linspace(0, 1, 1001)
    for i in range(1001):
        tau_eMT_list[i] = np.arccos(np.sqrt(fidelity_list[i]))/ H_p(E, w, mu_ep(fidelity_list[i]))
        tau_MT_list[i] = np.arccos(np.sqrt(fidelity_list[i]))/ H_p(E, w, 2)
    for i in range(1, 1001):
        eMT_vs_actual[i] = tau_eMT_list[i] / t_list[i]
        MT_vs_actual[i] = tau_MT_list[i] / t_list[i]
        eMT_vs_MT[i] = tau_eMT_list[i] / tau_MT_list[i]
    #ML
    x_list = np.linspace(x_0(1), np.pi, 1001)
    fidelity_ml = np.linspace(0, 1, 1001)
    tau_ML_list = np.linspace(0, 1, 1001)
    for i in range(1001):
        fidelity_ml[i] = (x_list[i]**2 - 2*x_list[i]*np.sin(x_list[i]) - 2*np.cos(x_list[i]) + 2) / \
                (np.tan(x_list[i]/2)**2 * (2 - x_list[i]/np.tan(x_list[i]/2))**2)
        tau_ML_list[i] = x_list[i] * (1 - x_list[i]/np.tan(x_list[i]/2)) / (2 - x_list[i]/np.tan(x_list[i]/2)) / \
                E_p(E, w, 1)
    plt.figure()
    plt.ylabel(r"Fidelity $\epsilon$")
    plt.xlabel(r"Time $t\ \ (\hbar/\mathcal{E})$")
    plt.ylim(0, 1)
    plt.xlim(0, x_lim)
    plt.title(r"$\left|\Phi_{a,%.1d}\right\rangle$"%n, fontsize = 16)
    plt.plot(tau_eMT_list, fidelity_list, label = r"$\tau_{sMT}$: The Strengthened MT QSL")
    plt.plot(tau_MT_list, fidelity_list, label = r"$\tau_{MT}$: The Original MT QSL")
    plt.plot(tau_ML_list, fidelity_ml, label = r"$\tau_{ML}$: The ML QSL")
    plt.plot(t_list, fidelity_list, label = r"$t_{\epsilon}$: The Actual Evolution")
    plt.legend()
    plt.figure()
    plt.xlabel(r"Fidelity $\epsilon$")
    plt.ylabel(r"Relative Time")
    plt.ylim(min(eMT_vs_actual[-1], MT_vs_actual[-1]), eMT_vs_MT[-1])
    plt.xlim(1, 0)
    plt.title(r"$\left|\Phi_{a,%.1d}\right\rangle$"%n, fontsize = 16)
```

```
    plt.plot(fidelity_list, eMT_vs_actual, label = r"$\tau_{sMT}/t_\epsilon$")
    plt.plot(fidelity_list, MT_vs_actual, label = r"$\tau_{MT}/t_\epsilon$")
    plt.plot(fidelity_list, eMT_vs_MT, label = r"$\tau_{sMT}/\tau_{MT}$")
    plt.axhline(y = 1, color='r', linestyle='-')
    plt.legend()
```

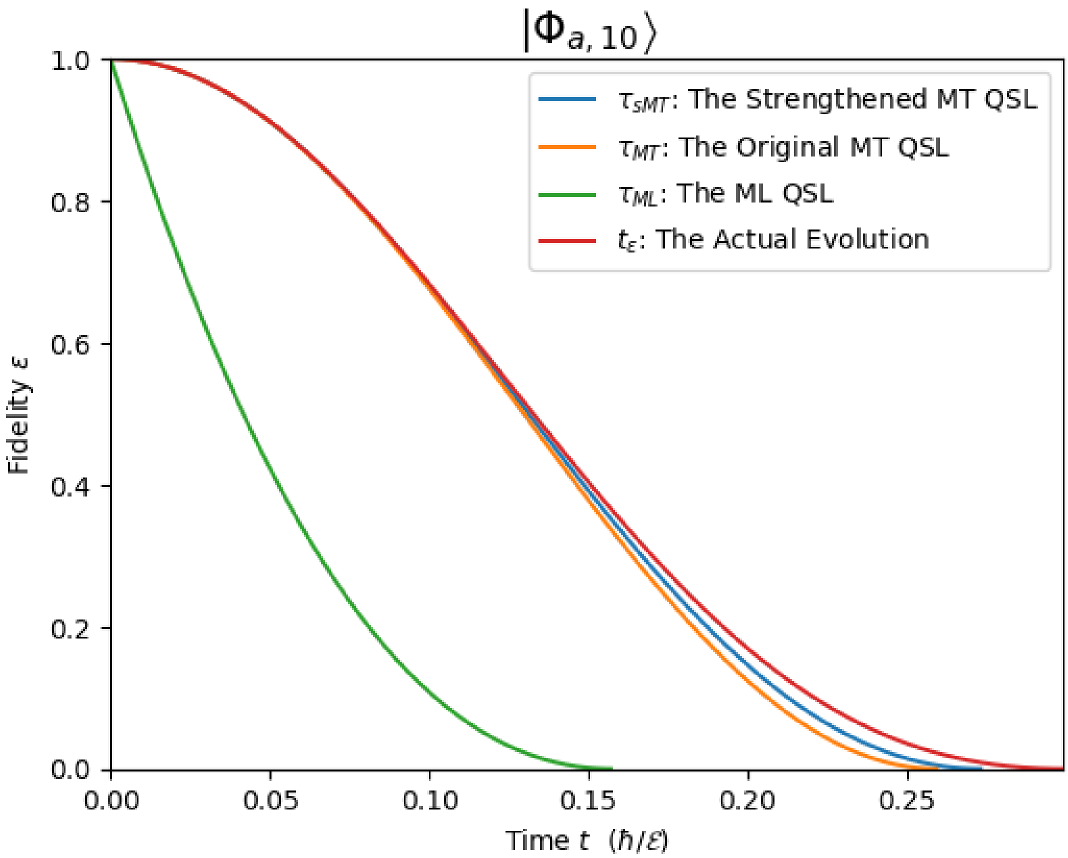
[22]:
```
def comparison_mt_ml_n(nlist):
    for i in range(len(nlist)):
        comparison_mt_ml(nlist[i])
```

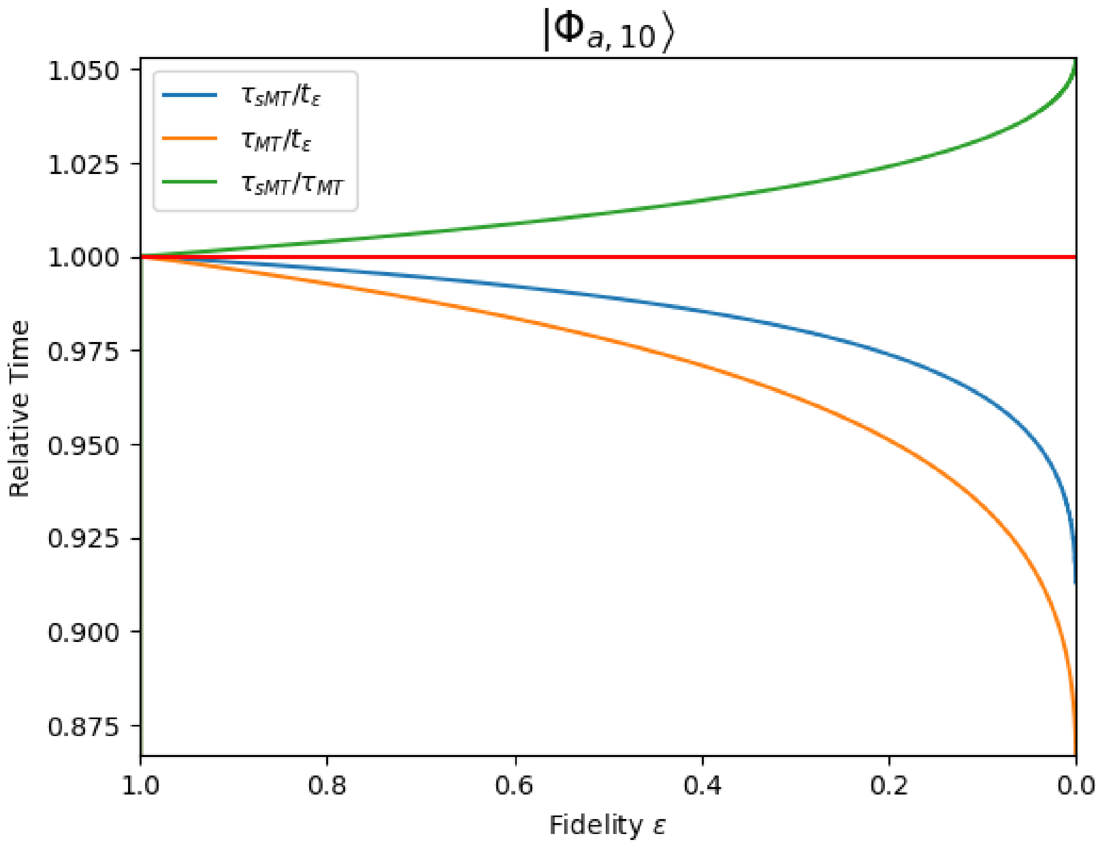
[23]:
```
comparison_mt_ml_n([1, 10, 100])
```



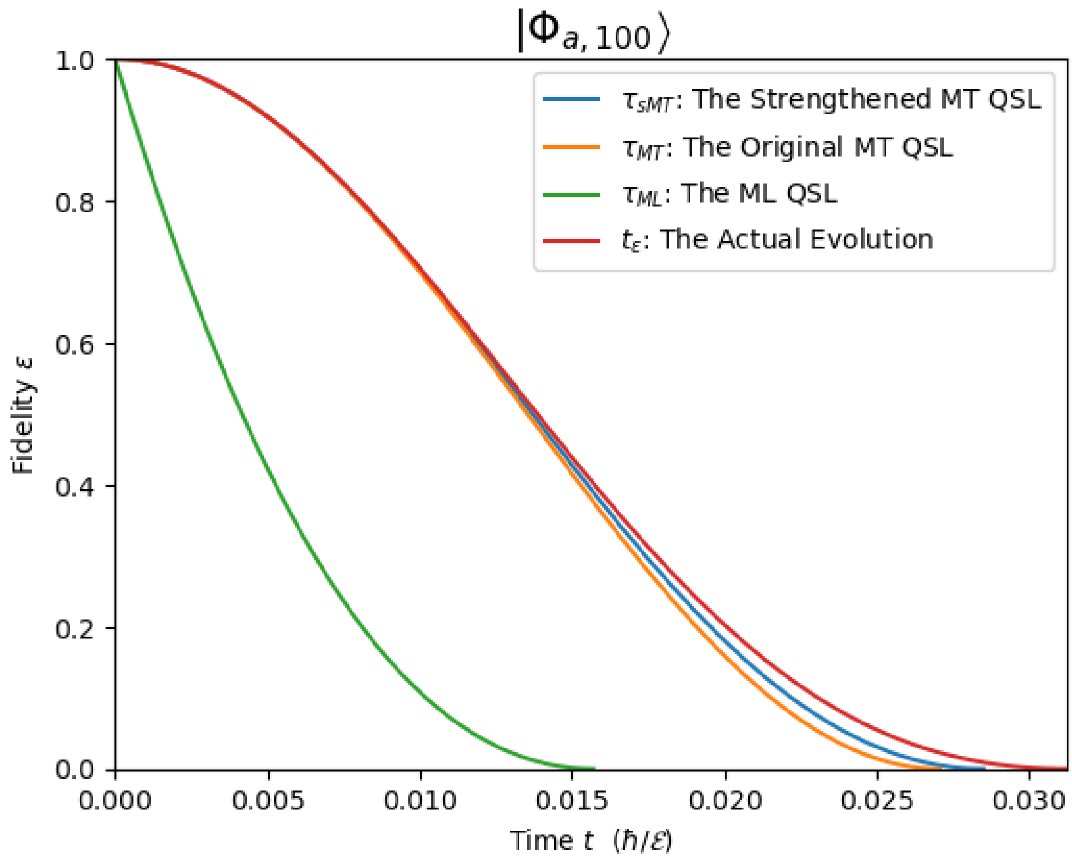


$|\Phi_{a,1}\rangle$
τ_{sMT}/t_ε
τ_{MT}/t_ε
τ_{sMT}/τ_{MT}
Relative Time
1.04
1.02
1.00
0.98
0.96
0.94
0.92
1.0
0.8
0.6
0.4
0.2
0.0
Fidelity ε

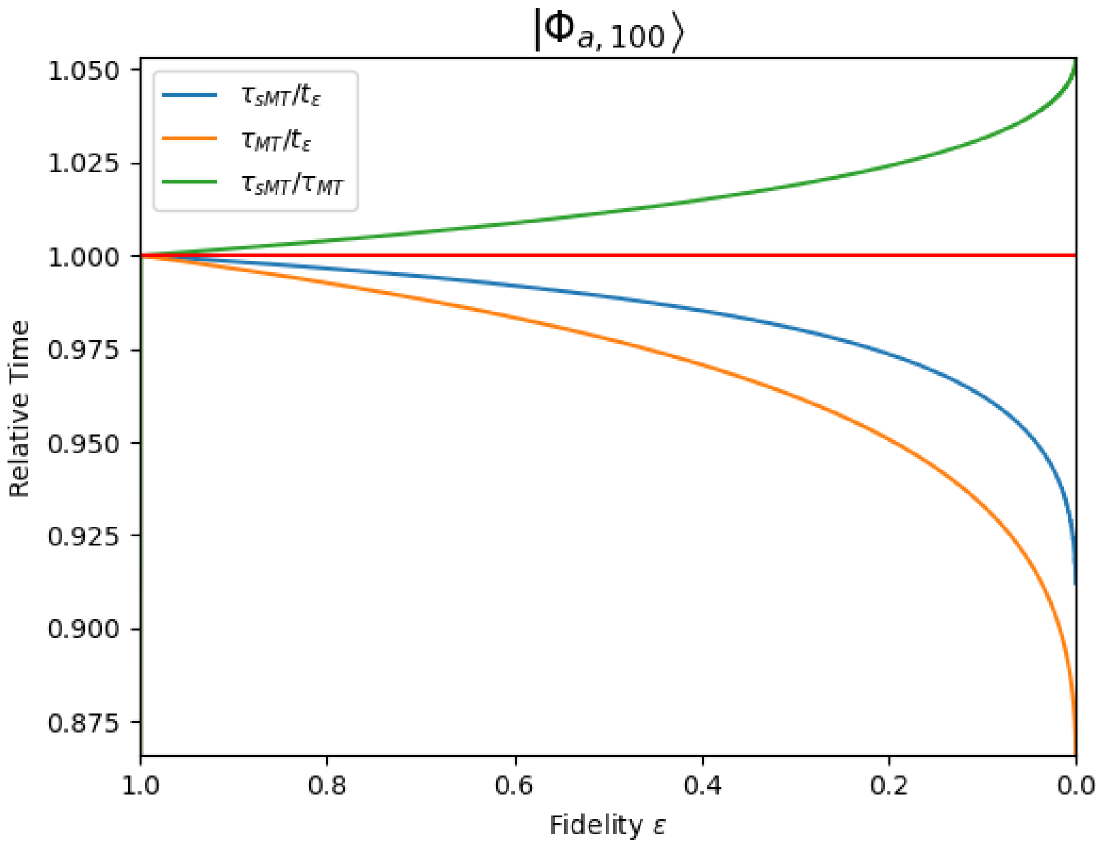

$|\Phi_{a,10}\rangle$
τ_{sMT}: The Strengthened MT QSL
τ_{MT}: The Original MT QSL
τ_{ML}: The ML QSL
t_ε: The Actual Evolution
Fidelity ε
Time t $(\hbar/\mathcal{E})$
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.05
0.10
0.15
0.20
0.25



$|\Phi_{a,10}\rangle$
τ_{sMT}/t_ε
τ_{MT}/t_ε
τ_{sMT}/τ_{MT}
1.050
1.025
1.000
0.975
0.950
0.925
0.900
0.875
Relative Time
1.0
0.8
0.6
0.4
0.2
0.0
Fidelity ε



$|\Phi_{a,100}\rangle$
τ_{sMT}: The Strengthened MT QSL
τ_{MT}: The Original MT QSL
τ_{ML}: The ML QSL
t_{ε}: The Actual Evolution
Fidelity ε
Time t $(\hbar/\mathcal{E})$
1.0
0.8
0.6
0.4
0.2
0.0
0.000
0.005
0.010
0.015
0.020
0.025
0.030



Comparison for random states

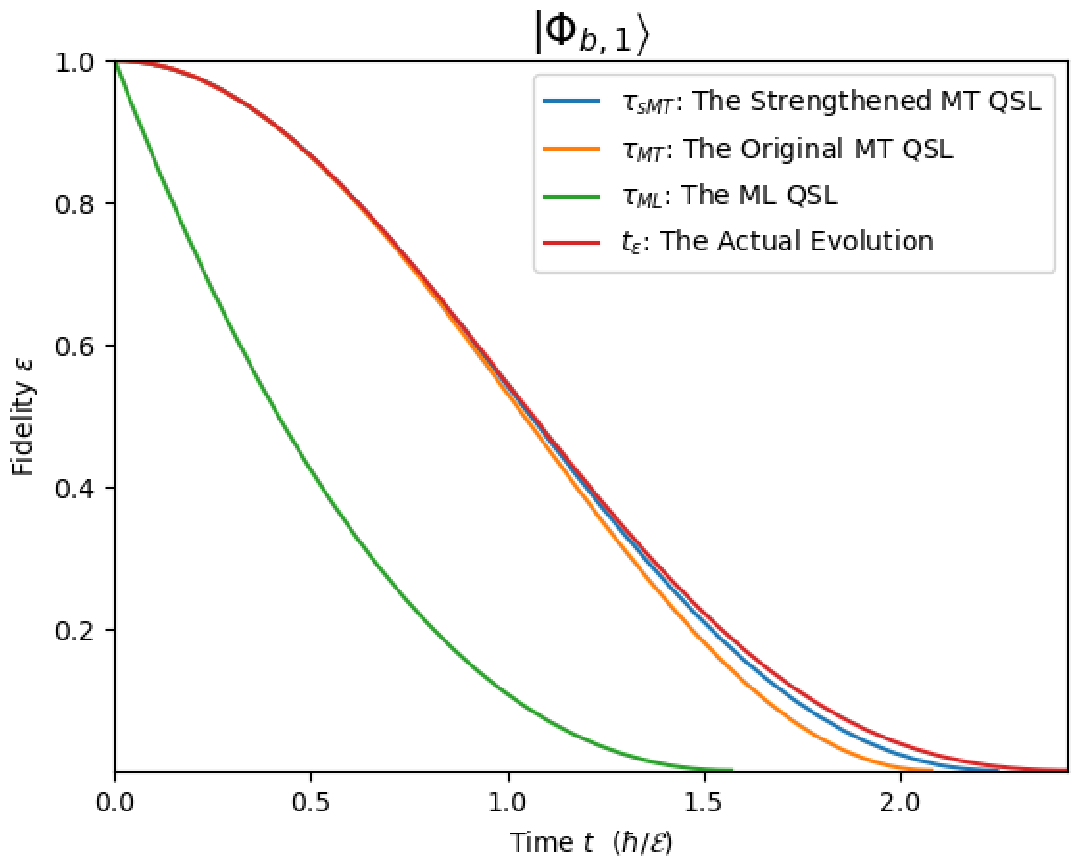
[24]:
```
def comparison_random(n):
    H_random, state_random, t_list_random, fidelity_list_random, x_lim =␣
↪random_balanced_state_evol(n)
    fidelity_list_random, end_idx = decreasing_half(fidelity_list_random)
    t_list_random = t_list_random[:end_idx]
    #MT
    E = np.diag(H_random)
    w = state_random ** 2
    tau_eMT_list = np.linspace(0, 1, 1001)[:end_idx]
    eMT_vs_actual = np.linspace(0, 1, 1001)[:end_idx]
    tau_MT_list = np.linspace(0, 1, 1001)[:end_idx]
    MT_vs_actual = np.linspace(0, 1, 1001)[:end_idx]
    eMT_vs_MT = np.linspace(0, 1, 1001)[:end_idx]
    for i in range(end_idx):
        tau_eMT_list[i] = np.arccos(np.sqrt(fidelity_list_random[i]))/ H_p(E,␣
↪w, mu_ep(fidelity_list_random[i]))
        tau_MT_list[i] = np.arccos(np.sqrt(fidelity_list_random[i]))/ H_p(E, w,␣
↪2)
```

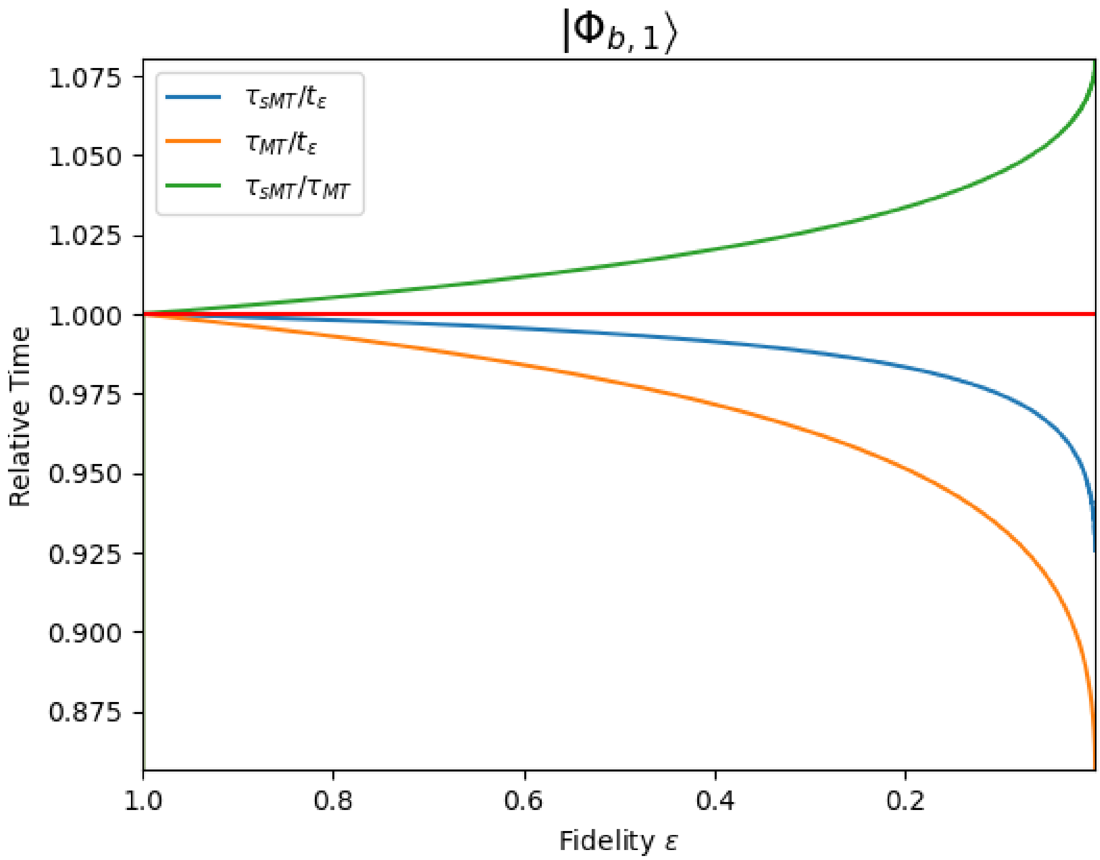
```
    for i in range(1, end_idx):
        eMT_vs_actual[i] = tau_eMT_list[i] / t_list_random[i]
        MT_vs_actual[i] = tau_MT_list[i] / t_list_random[i]
        eMT_vs_MT[i] = tau_eMT_list[i] / tau_MT_list[i]
    #ML
    x_list = np.linspace(x_0(1), np.pi, 1001)
    fidelity_ml = np.linspace(0, 1, 1001)
    tau_ML_list = np.linspace(0, 1, 1001)
    for i in range(1001):
        fidelity_ml[i] = (x_list[i]**2 - 2*x_list[i]*np.sin(x_list[i]) - 2*np.
↪cos(x_list[i]) + 2) / \
                (np.tan(x_list[i]/2)**2 * (2 - x_list[i]/np.tan(x_list[i]/
↪2))**2)
        tau_ML_list[i] = x_list[i] * (1 - x_list[i]/np.tan(x_list[i]/2)) / (2 -␣
↪x_list[i]/np.tan(x_list[i]/2)) / \
                E_p(E, w, 1)
    plt.figure()
    plt.ylabel(r"Fidelity $\epsilon$")
    plt.xlabel(r"Time $t\ \ (\hbar/\mathcal{E})$")
    plt.ylim(fidelity_list_random[-1], 1)
    plt.xlim(0, t_list_random[-1])
    plt.title(r"$\left|\Phi_{b,%.1d}\right\rangle$"%n, fontsize = 16)
    plt.plot(tau_eMT_list, fidelity_list_random, label = r"$\tau_{sMT}$: The␣
↪Strengthened MT QSL")
    plt.plot(tau_MT_list, fidelity_list_random, label = r"$\tau_{MT}$: The␣
↪Original MT QSL")
    plt.plot(tau_ML_list, fidelity_ml, label = r"$\tau_{ML}$: The ML QSL")
    plt.plot(t_list_random, fidelity_list_random, label = r"$t_{\epsilon}$: The␣
↪Actual Evolution")
    plt.legend()
    plt.figure()
    plt.xlabel(r"Fidelity $\epsilon$")
    plt.ylabel(r"Relative Time")
    plt.ylim(min(eMT_vs_actual[-1], MT_vs_actual[-1]), eMT_vs_MT[-1])
    plt.xlim(1, fidelity_list_random[-1])
    plt.title(r"$\left|\Phi_{b,%.1d}\right\rangle$"%n, fontsize = 16)
    plt.plot(fidelity_list_random, eMT_vs_actual, label = r"$\tau_{sMT}/
↪t_\epsilon$")
    plt.plot(fidelity_list_random, MT_vs_actual, label = r"$\tau_{MT}/
↪t_\epsilon$")
    plt.plot(fidelity_list_random, eMT_vs_MT, label = r"$\tau_{sMT}/\tau_{MT}$")
    plt.axhline(y = 1, color='r', linestyle='-')
    plt.legend()
```

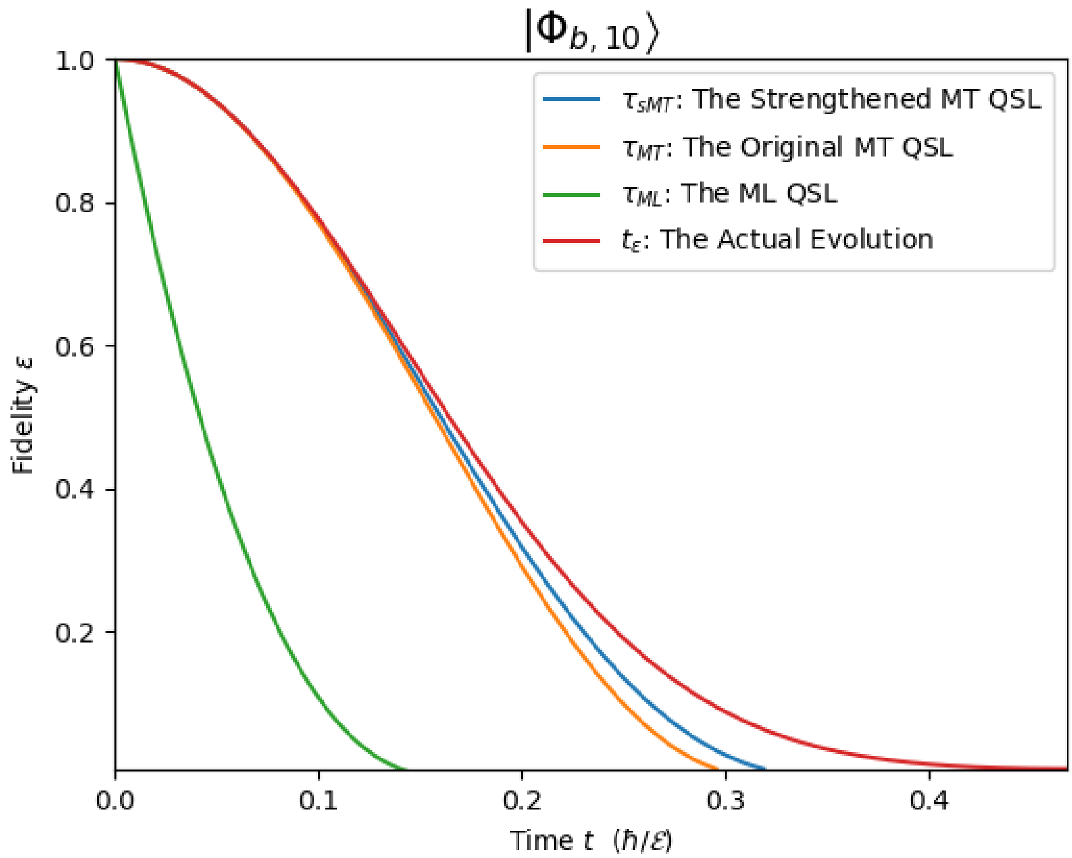
[25]:
```
def comparison_random_nlist(nlist):
    for i in range(len(nlist)):
```

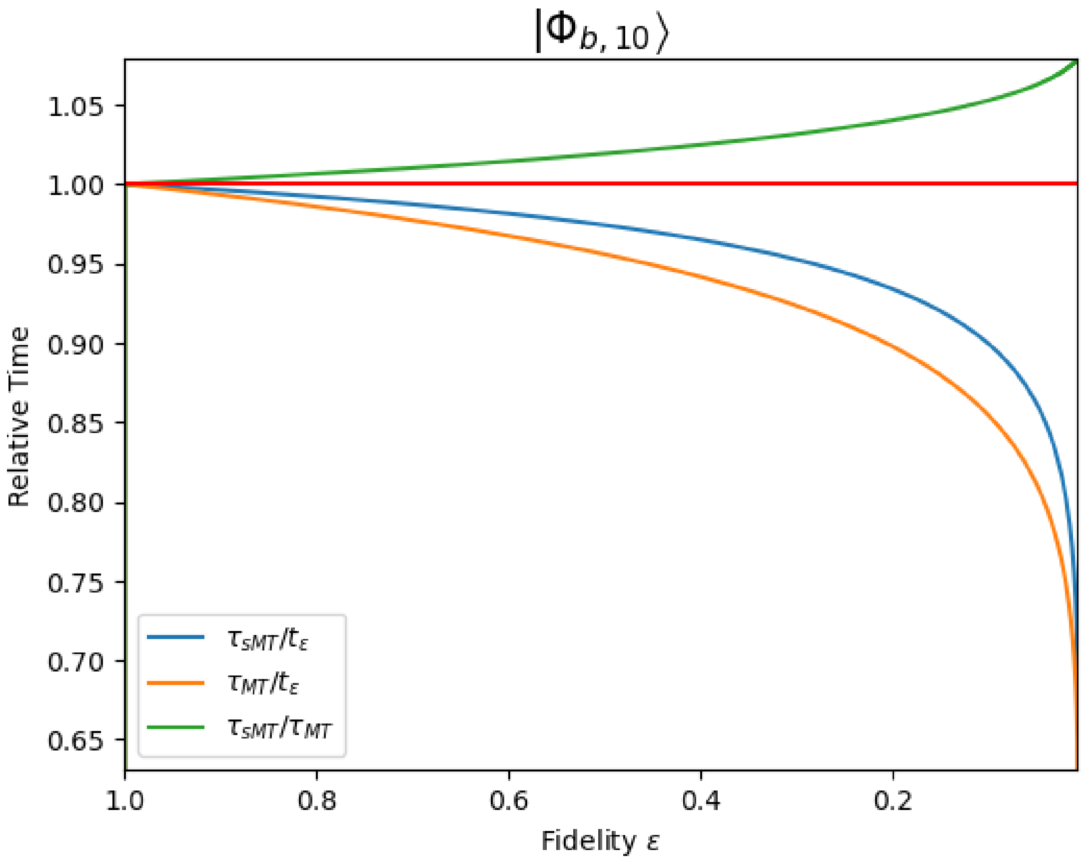
```
        comparison_random(nlist[i])
```

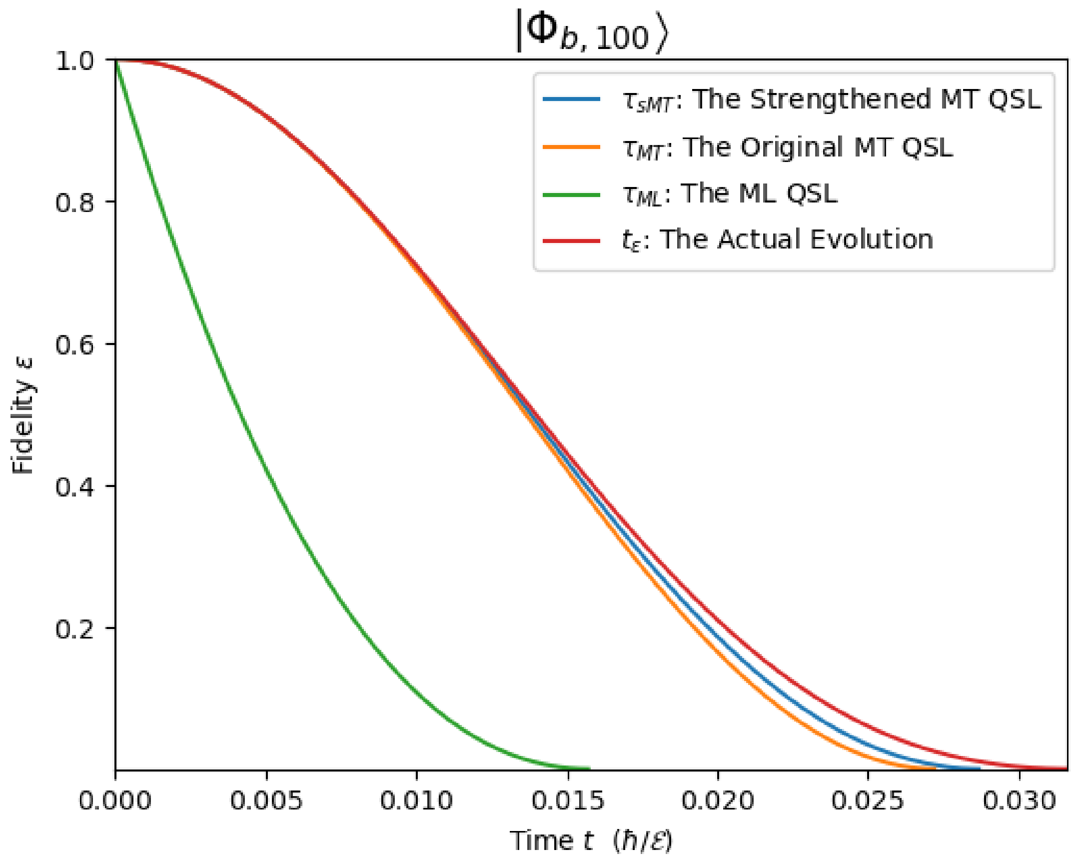
[27]:
```
comparison_random_nlist([1, 10, 100])
```



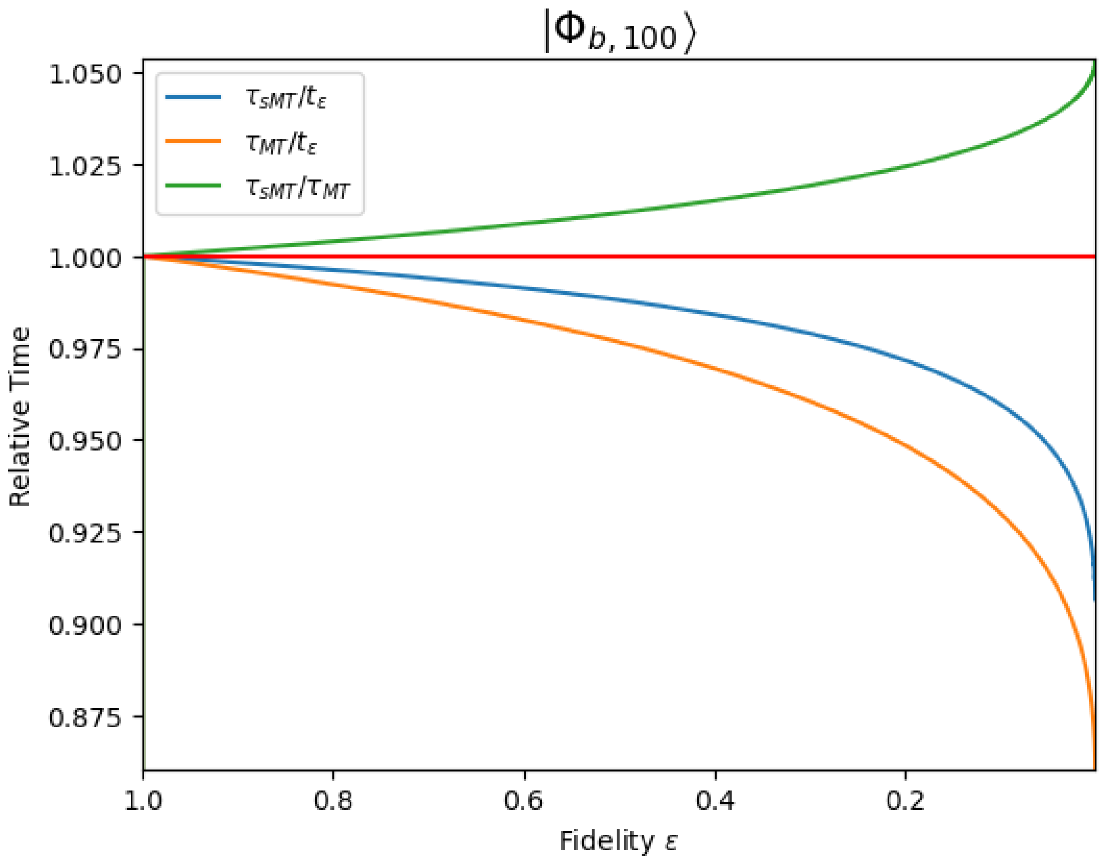

$|\Phi_{b,1}\rangle$
τ_{sMT}/t_ε
τ_{MT}/t_ε
τ_{sMT}/τ_{MT}
Relative Time
1.075
1.050
1.025
1.000
0.975
0.950
0.925
0.900
0.875
1.0
0.8
0.6
0.4
0.2
Fidelity ε


$|\Phi_{b,10}\rangle$
τ_{sMT}: The Strengthened MT QSL
τ_{MT}: The Original MT QSL
τ_{ML}: The ML QSL
t_ε: The Actual Evolution
Fidelity ε
Time t $(\hbar/\mathcal{E})$
1.0
0.8
0.6
0.4
0.2
0.0
0.1
0.2
0.3
0.4


$|\Phi_{b,10}\rangle$
Relative Time
Fidelity ε
τ_{sMT}/t_ε
τ_{MT}/t_ε
τ_{sMT}/τ_{MT}
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
1.0
0.8
0.6
0.4
0.2



$|\Phi_{b,100}\rangle$
τ_{sMT}: The Strengthened MT QSL
τ_{MT}: The Original MT QSL
τ_{ML}: The ML QSL
t_ϵ: The Actual Evolution
Fidelity ϵ
Time t $(\hbar/\mathcal{E})$
1.0
0.8
0.6
0.4
0.2
0.000
0.005
0.010
0.015
0.020
0.025
0.030



Bounds discovered by me, depending on moments of balanced energy of different orders

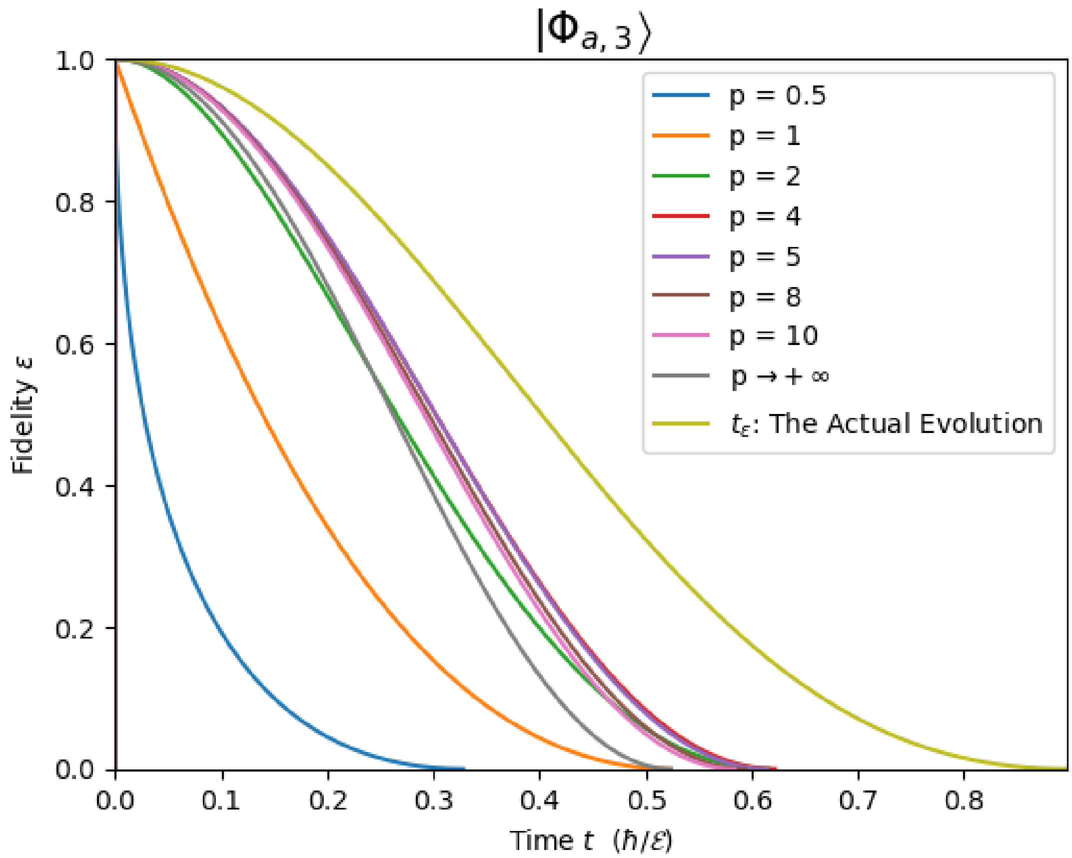
[28]:
```python
def bound(E, w, p):
    x_list = np.linspace(x_0(p), np.pi, 1001)
    fidelity_ml = np.linspace(0, 1, 1001)
    tau_ML_list = np.linspace(0, 1, 1001)
    for i in range(1001):
        fidelity_ml[i] = (x_list[i]**2 - 2*p*x_list[i]*np.sin(x_list[i]) - 2*(p**2)*np.cos(x_list[i]) + 2*(p**2)) / \
                (np.tan(x_list[i]/2)**2 * (2*p - x_list[i]/np.tan(x_list[i]/2))**2)
        tau_ML_list[i] = x_list[i] * \
                ((p - x_list[i]/np.tan(x_list[i]/2)) / (2*p - x_list[i]/np.tan(x_list[i]/2)))**(1/p) / \
                E_p(E, w, p)
    return fidelity_ml, tau_ML_list
def bound_inf(E):
    fidelity_ml = np.linspace(0, 1, 1001)
    tau_ML_list = np.linspace(0, 1, 1001)
```

```
    for i in range(1001):
        tau_ML_list[i] = 2 * np.arccos(np.sqrt(fidelity_ml[i])) /␣
 ↪abs(E[0]-E[-1])
    return fidelity_ml, tau_ML_list
def mybound(n, p, sMT_plot = 0):
    H, initial_state, t_list, fidelity_list, x_lim = epsilon_t(n)
    E = np.diag(H)
    w = initial_state ** 2

    #the strengthened MT
    tau_eMT_list = np.linspace(0, 1, 1001)
    for i in range(1001):
        tau_eMT_list[i] = np.arccos(np.sqrt(fidelity_list[i]))/ H_p(E, w,␣
 ↪mu_ep(fidelity_list[i]))
    #my bound
    fidelity_ml_infty, tau_ML_list_infty = bound_inf(E)
    fidelity_ml_p = [np.linspace(0, 1, 1001) for _ in range(len(p))]
    tau_ML_list_p = [np.linspace(0, 1, 1001) for _ in range(len(p))]
    plt.figure()
    plt.ylabel(r"Fidelity $\epsilon$")
    plt.xlabel(r"Time $t\ \ (\hbar/\mathcal{E})$")
    plt.ylim(0, 1)
    plt.xlim(0, x_lim)
    plt.title(r"$\left|\Phi_{a,%.1d}\right\rangle$"%n, fontsize = 16)
    for i in range(len(p)):
        fidelity_ml_p[i], tau_ML_list_p[i] = bound(E, w, p[i])
        plt.plot(tau_ML_list_p[i], fidelity_ml_p[i], label = r"p = "+str(p[i]))
    plt.plot(tau_ML_list_infty, fidelity_ml_infty, label = r"p $\to+\infty$")
    if sMT_plot:
        plt.plot(tau_eMT_list, fidelity_list, label = r"$\tau_{sMT}$: The␣
 ↪Strengthened MT QSL")
    plt.plot(t_list, fidelity_list, label = r"$t_{\epsilon}$: The Actual␣
 ↪Evolution")
    plt.legend()


    #random
    H_random, state_random, t_list_random, fidelity_list_random, x_lim =␣
 ↪random_balanced_state_evol(n)
    fidelity_list_random_end, end_idx = decreasing_half(fidelity_list_random)
    t_list_random_end = t_list_random[:end_idx]
    E_random = np.diag(H_random)
    w_random = state_random ** 2
    #the strengthened MT for random states
    tau_eMT_list_random = np.linspace(0, 1, 1001)
    fidelity_eMT_random = np.linspace(0, 1, 1001)
    for i in range(1001):
```

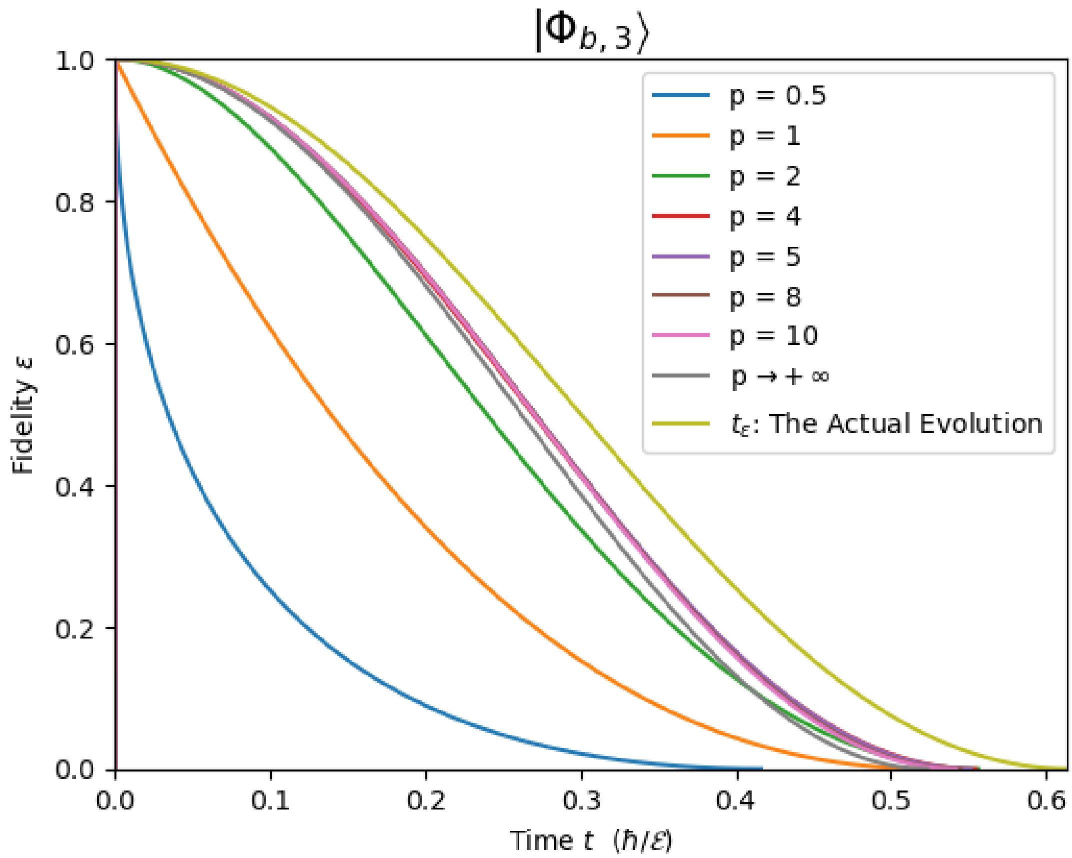
```
        tau_eMT_list_random[i] = np.arccos(np.sqrt(fidelity_eMT_random[i]))/␣
↪H_p(E_random, w_random, mu_ep(fidelity_eMT_random[i]))
    #my bound
    fidelity_random_infty, tau_random_list_infty = bound_inf(E_random)
    fidelity_ml_p_random = [np.linspace(0, 1, 1001) for _ in range(len(p))]
    tau_ML_list_p_random = [np.linspace(0, 1, 1001) for _ in range(len(p))]
    plt.figure()
    plt.ylabel(r"Fidelity $\epsilon$")
    plt.xlabel(r"Time $t\ \ (\hbar/\mathcal{E})$")
    plt.ylim(0, 1)
    plt.xlim(0, t_list_random_end[-1])
    plt.title(r"$\left|\Phi_{b,%.1d}\right\rangle$"%n, fontsize = 16)
    for i in range(len(p)):
        fidelity_ml_p_random[i], tau_ML_list_p_random[i] = bound(E_random,␣
↪w_random, p[i])
        plt.plot(tau_ML_list_p_random[i], fidelity_ml_p_random[i], label = r"p␣
↪= "+str(p[i]))
    plt.plot(tau_random_list_infty, fidelity_random_infty, label = r"p␣
↪$\to+\infty$")
    if sMT_plot:
        plt.plot(tau_eMT_list_random, fidelity_eMT_random, label =␣
↪r"$\tau_{sMT}$: The Strengthened MT QSL")
    plt.plot(t_list_random_end, fidelity_list_random_end, label =␣
↪r"$t_{\epsilon}$: The Actual Evolution")
    plt.legend()


    #unbalanced
    H_unbalanced, state_unbalanced, t_list_unbalanced,␣
↪fidelity_list_unbalanced, x_lim = random_unbalanced_state_evol(n)
    fidelity_list_unbalanced_end, end_idx_unbalanced =␣
↪decreasing_half(fidelity_list_unbalanced)
    t_list_unbalanced_end = t_list_unbalanced[:end_idx_unbalanced]
    E_unbalanced = np.diag(H_unbalanced)
    w_unbalanced = state_unbalanced ** 2
    #the strengthened MT for unbalanced states
    tau_eMT_list_unbalanced = np.linspace(0, 1, 1001)
    fidelity_eMT_unbalanced = np.linspace(0, 1, 1001)
    for i in range(1001):
        tau_eMT_list_unbalanced[i] = np.arccos(np.
↪sqrt(fidelity_eMT_unbalanced[i]))/ H_p(E_unbalanced, w_unbalanced,␣
↪mu_ep(fidelity_eMT_unbalanced[i]))
    #my bound
    fidelity_unbalanced_infty, tau_unbalanced_list_infty =␣
↪bound_inf(E_unbalanced)
    fidelity_ml_p_unbalanced = [np.linspace(0, 1, 1001) for _ in range(len(p))]
```

```
    tau_ML_list_p_unbalanced = [np.linspace(0, 1, 1001) for _ in range(len(p))]
    plt.figure()
    plt.ylabel(r"Fidelity $\epsilon$")
    plt.xlabel(r"Time $t\ \ (\hbar/\mathcal{E})$")
    plt.ylim(0, 1)
    plt.xlim(0, t_list_unbalanced_end[-1])
    plt.title(r"$\left|\Phi_{c,%.1d}\right\rangle$"%n, fontsize = 16)
    for i in range(len(p)):
        fidelity_ml_p_unbalanced[i], tau_ML_list_p_unbalanced[i] =␣
↪bound(E_unbalanced, w_unbalanced, p[i])
        plt.plot(tau_ML_list_p_unbalanced[i], fidelity_ml_p_unbalanced[i],␣
↪label = r"p = "+str(p[i]))
    plt.plot(tau_unbalanced_list_infty, fidelity_unbalanced_infty, label = r"p␣
↪$\to+\infty$")
    if sMT_plot:
        plt.plot(tau_eMT_list_unbalanced, fidelity_eMT_unbalanced, label =␣
↪r"$\tau_{sMT}^*$: The Strengthened MT QSL$^*$")
    plt.plot(t_list_unbalanced_end, fidelity_list_unbalanced_end, label =␣
↪r"$t_{\epsilon}$: The Actual Evolution")
    plt.legend()
```

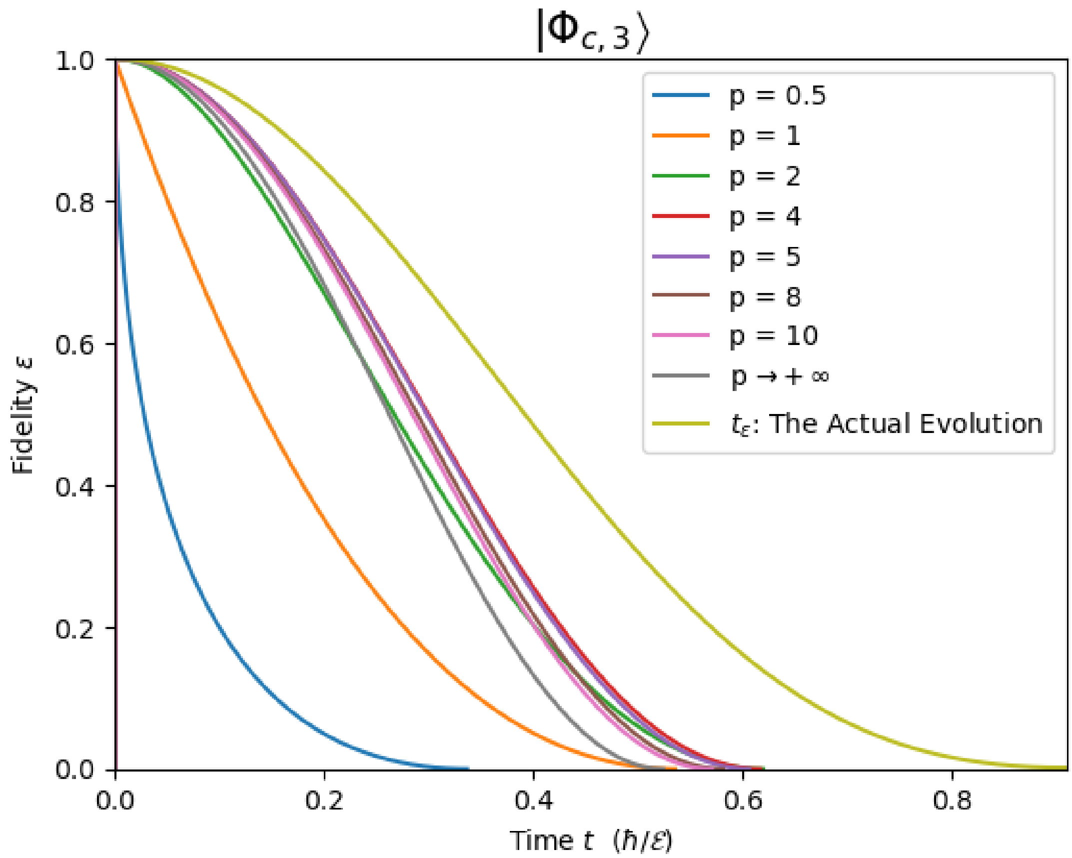
[29]:
```
mybound(3, [0.5, 1, 2, 4, 5, 8, 10])
```



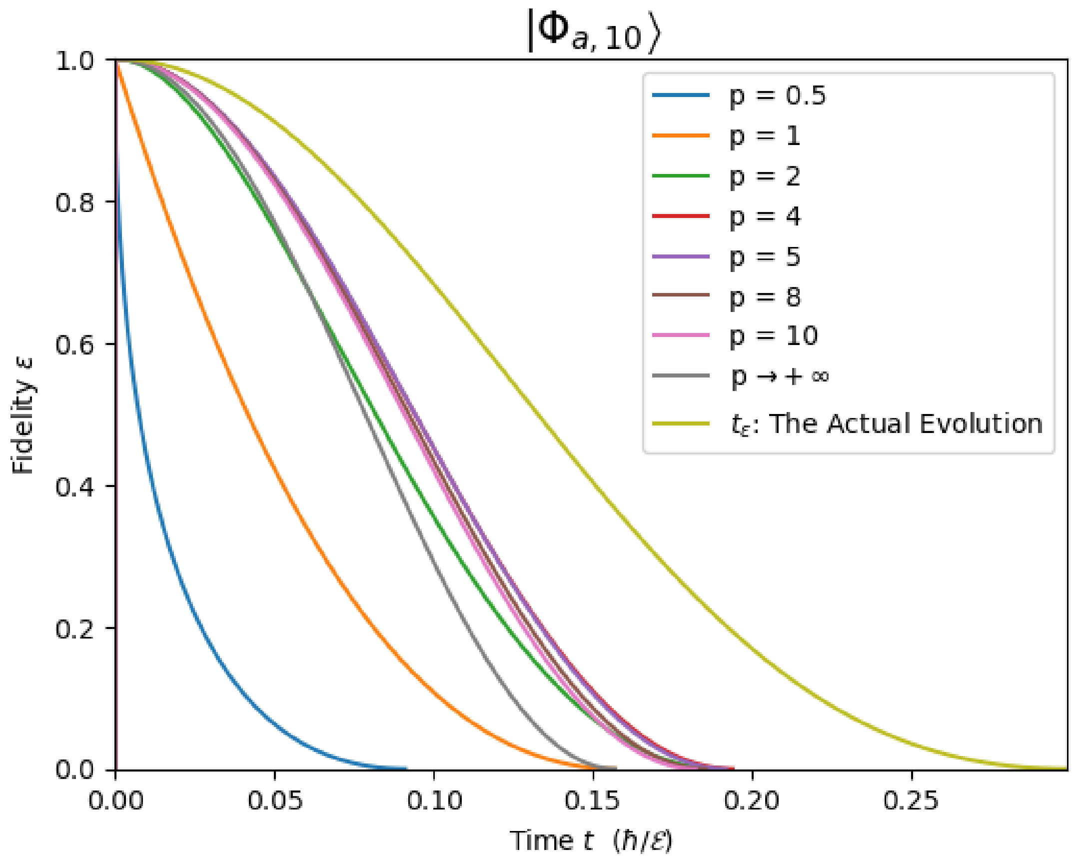
$|\Phi_{a,3}\rangle$
1.0
0.8
0.6
0.4
0.2
0.0
Fidelity ε
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Time t $(\hbar/\mathcal{E})$
p = 0.5
p = 1
p = 2
p = 4
p = 5
p = 8
p = 10
p $\to +\infty$
t_ε: The Actual Evolution



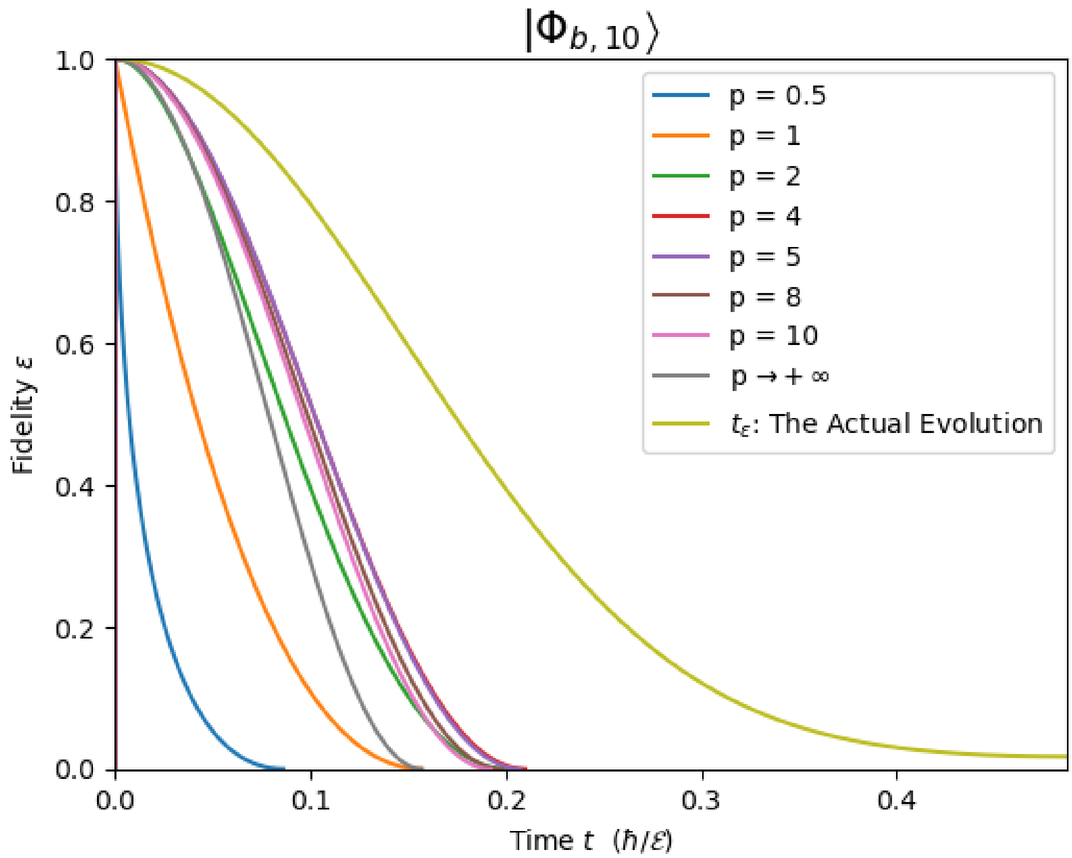
$|\Phi_{b,3}\rangle$
1.0
0.8
0.6
0.4
0.2
0.0
Fidelity ε
0.0
0.1
0.2
0.3
0.4
0.5
0.6
Time t $(\hbar/\mathcal{E})$
p = 0.5
p = 1
p = 2
p = 4
p = 5
p = 8
p = 10
p $\to +\infty$
t_ε: The Actual Evolution



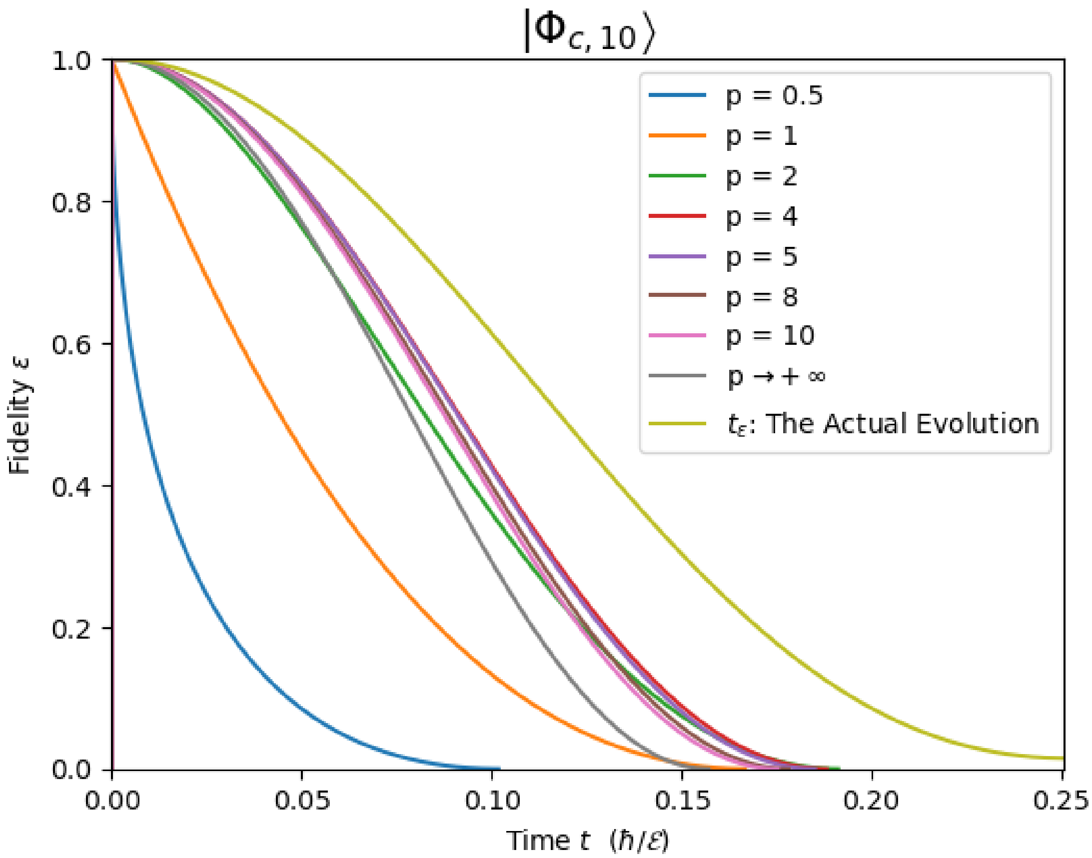
[30]:
```
mybound(10, [0.5, 1, 2, 4, 5, 8, 10])
```



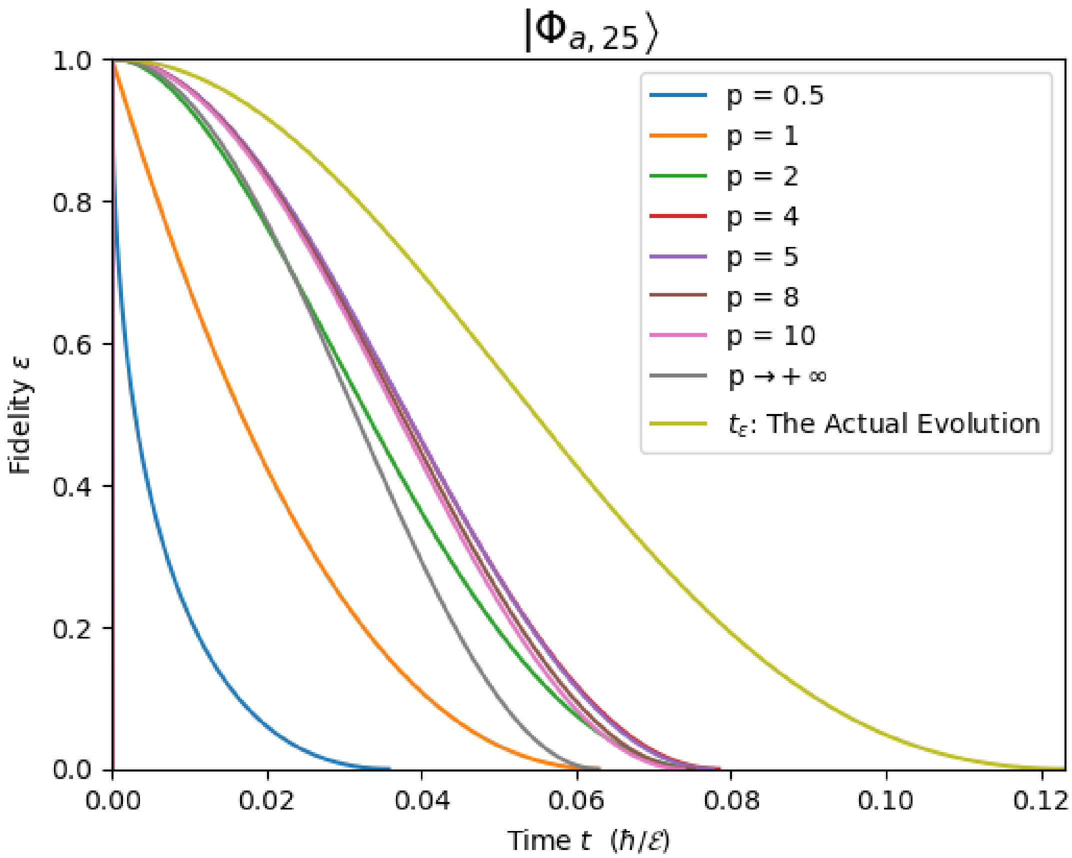
$|\Phi_{a,10}\rangle$
1.0
0.8
0.6
0.4
0.2
0.0
Fidelity ε
0.00
0.05
0.10
0.15
0.20
0.25
Time t $(\hbar/\mathcal{E})$
p = 0.5
p = 1
p = 2
p = 4
p = 5
p = 8
p = 10
p $\rightarrow + \infty$
t_ε: The Actual Evolution



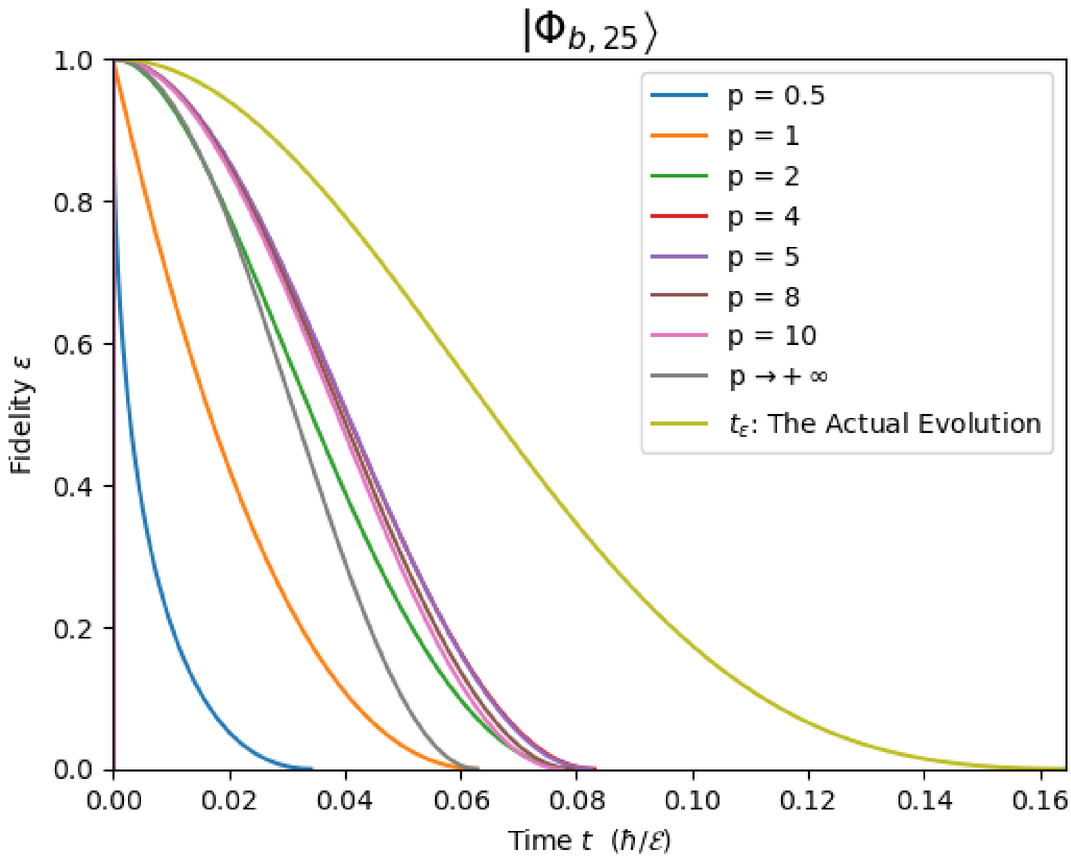
$|\Phi_{b,10}\rangle$
1.0
0.8
0.6
0.4
0.2
0.0
Fidelity ε
0.0
0.1
0.2
0.3
0.4
Time t $(\hbar/\mathcal{E})$
p = 0.5
p = 1
p = 2
p = 4
p = 5
p = 8
p = 10
$p \to +\infty$
t_ε: The Actual Evolution



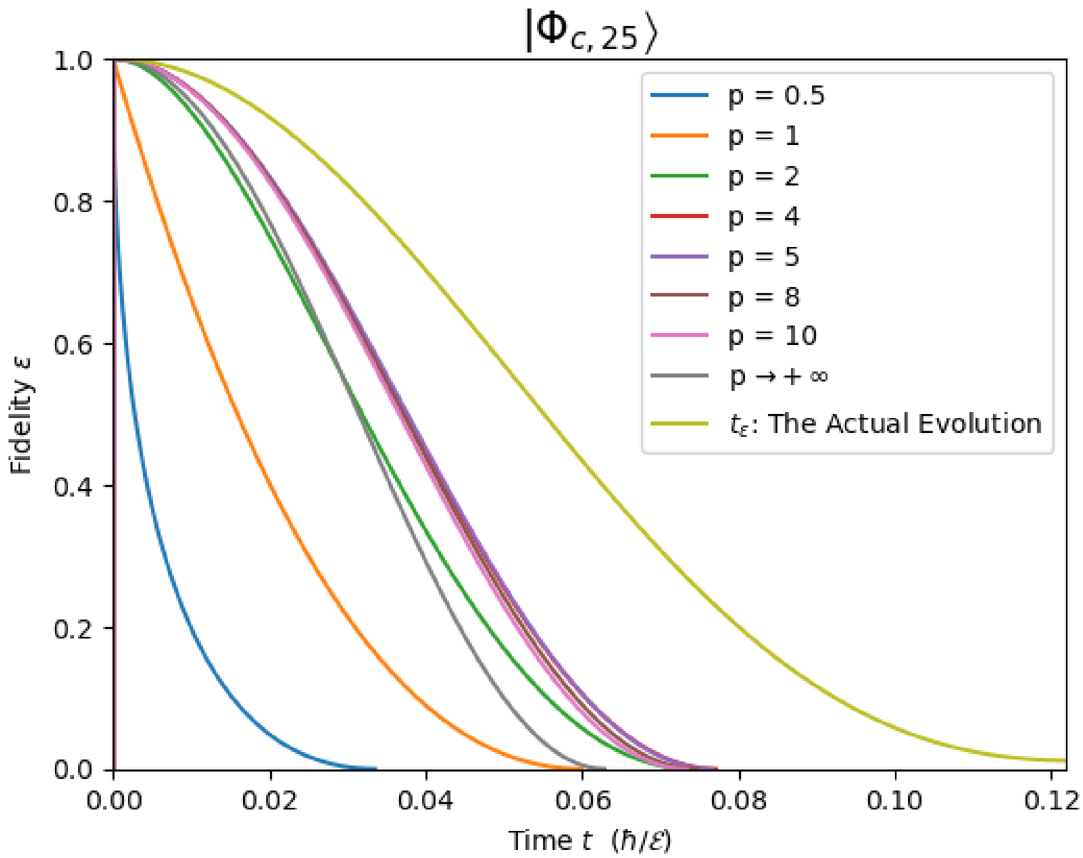
[32]:
```
mybound(25, [0.5, 1, 2, 4, 5, 8, 10])
```



$|\Phi_{a,25}\rangle$
Fidelity ε
Time t $(\hbar/\mathcal{E})$
p = 0.5
p = 1
p = 2
p = 4
p = 5
p = 8
p = 10
p $\to +\infty$
t_ε: The Actual Evolution
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.02
0.04
0.06
0.08
0.10
0.12



$|\Phi_{b,25}\rangle$
1.0
0.8
0.6
0.4
0.2
0.0
Fidelity ε
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
Time t $(\hbar/\mathcal{E})$
p = 0.5
p = 1
p = 2
p = 4
p = 5
p = 8
p = 10
p $\to +\infty$
t_ε: The Actual Evolution



two-level

[33]:
```
def two_level_compare(p):
    #2level
    system2, state2, t_list2, fidelity_list2, x_lim2 = two_level()
    E_2level = np.diag(system2)
    w_2level = state2 ** 2
    #the strengthened MT for random states
    tau_eMT_list_2level = np.linspace(0, 1, 1001)
    fidelity_eMT_2level = np.linspace(0, 1, 1001)
    for i in range(1001):
        tau_eMT_list_2level[i] = np.arccos(np.sqrt(fidelity_eMT_2level[i]))/
↪H_p(E_2level, w_2level, mu_ep(fidelity_eMT_2level[i]))
    #my bound
    fidelity_2level_infty, tau_2level_list_infty = bound_inf(E_2level)
    fidelity_ml_p_2level = [np.linspace(0, 1, 1001) for _ in range(len(p))]
    tau_ML_list_p_2level = [np.linspace(0, 1, 1001) for _ in range(len(p))]
    plt.figure()
    plt.ylabel(r"Fidelity $\epsilon$")
```

```
    plt.xlabel(r"Time $t\ \ (\hbar/\mathcal{E})$")
    plt.ylim(0, 1)
    plt.xlim(0, np.pi)
    for i in range(len(p)):
        fidelity_ml_p_2level[i], tau_ML_list_p_2level[i] = bound(E_2level,␣
↪w_2level, p[i])
        plt.plot(tau_ML_list_p_2level[i], fidelity_ml_p_2level[i], label = r"p␣
↪= "+str(p[i]))
    plt.plot(tau_2level_list_infty, fidelity_2level_infty, label = r"p␣
↪$\to+\infty$")
    plt.plot(tau_eMT_list_2level, fidelity_eMT_2level, label = r"$\tau_{sMT}$:␣
↪The Strengthened MT QSL")
    plt.plot(t_list2, fidelity_list2, label = r"$t_{\epsilon}$: The Actual␣
↪Evolution")
    plt.legend()
```

[34]:
```
two_level_compare([8, 10])
```