MAS 291

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Friday, 18^{th} October 2024

Exercise 1

The probability of finding at least one error is given by the complement of this probability:

$$P(\text{at least 1 error out of } n) = 1 - P(\text{no error out of } n)$$

$$P(\text{no error out of } n) = 1 - \frac{48}{50} \times \frac{47}{49} \times \cdots \times \frac{48-n+1}{50-n+1}$$

$$= 1 - \frac{48!}{50!} \times \frac{(50-n)!}{(48-n)!} = 1 - \frac{(50-n)(49-n)}{49 \times 50 \times (48-n)!}$$

$$= 1 - \frac{(50-n)(49-n)(48-n)!}{49 \times 50 \times (48-n)!}$$

$$= 1 - \frac{(50-n)(49-n)}{50 \times 49}$$

$$= \frac{99n - n^2}{2450}$$

We are looking for the minimum n such that

$$P(\text{at least 1 error out of } n) \geq 0.9$$

That is equivalent to:

$$99n - n^2 \geq 2450 \times 0.9 = 2205$$

Since we are looking for the minimum n, we can replace $\geq$ with $=$:

$$n^2 - 99n + 2205 = 0$$

Of course,

$$n = \frac{99 \pm \sqrt{99^2 - 4 \times 2205}}{2} \approx 33.8 \text{ or } 65.16$$

Because n must be an integer and the minimum so n=34

Conclusion

Therefore, the minimum sample size n such that the probability of finding at least one error is greater than or equal to 0.90 is:

$$\boxed{34}$$

Comment

Sampling with a minimum sample size of 34 documents provides a 90% chance of detecting at least one error. However, this suggests that sampling might not be the most effective approach when accuracy is crucial, as relying on a sample leaves room for undetected errors.

A full inspection of all 50 documents, although more labor-intensive, guarantees a 100% probability of discovering errors, which might be necessary in high-stakes situations where even minor errors can have significant consequences.

Exercise 2

Let A and B be independent events. We want to prove that A' and B', the complements of A and B, are also independent. According to the definition of independence, we need to show that:

$$P(A' \cap B') = P(A') \cdot P(B').$$

Using De Morgan's Law

We begin by using De Morgan's Law to express the complement of the intersection:

$$P(A' \cap B') = P((A \cup B)')$$

which simplifies to:

$$P(A' \cap B') = 1 - P(A \cup B)$$

Applying the Formula for Union of Events

Using the formula for the union of two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since A and B are independent, we know that $P(A \cap B) = P(A) \cdot P(B)$. Substituting this into the previous expression:

$$P(A' \cap B') = 1 - (P(A) + P(B) - P(A) \cdot P(B))$$

Simplifying the expression:

$$P(A' \cap B') = 1 - P(A) - P(B) + P(A) \cdot P(B)$$

Factoring the right-hand side:

$$P(A' \cap B') = (1 - P(A)) \cdot (1 - P(B))$$

Finally, since $1 - P(A) = P(A')$ and $1 - P(B) = P(B')$, we get:

$$P(A' \cap B') = P(A') \cdot P(B').$$

Thus, by the definition of independence, the events A' and B' are independent.

Exercise 3

Call X is the number of bolt is identified as being incorrectly torqued.

We have the probability that an incorrectly torqued bolt is identified is 0.95, so the probability that an incorrectly torqued bolt is not identified is $1 - 0.95 = 0.05$.

We need to find: $\mathbf{P}(X \geq 1)$

$$\mathbf{P}(X \geq 1) \Rightarrow 1 - \mathbf{P}(X < 1)$$

Hence, we have to fine the probability that **NO** bolt in the sample of four is identified as being incorrectly torqued and there are all in all 5 possible situations when the operator comes into play:

- A: all bolts are OK, 0 bad ones
- B: 1 bad bolt but be not identified
- C: 2 bad bolts but be not identified
- D: 3 bad bolts but be not identified
- E: 4 bad bolts but be not identified.

$$\mathbf{P}(A) = \frac{\mathbf{C}_5^0 . \mathbf{C}_{15}^4}{\mathbf{C}_{20}^4} * 0.05^0 = \frac{91}{323} \approx 0.2817$$

With,
$\mathbf{C}_5^0$ is possible situations when the operator comes into play.
$\mathbf{C}_{15}^4$ is choosing random 4 bolts in 15 bolts which are torqued to the proper limit.
$\mathbf{C}_{20}^4$ is choosing random 4 bolts in 20 bolts.
Similar to P(B),

$$\mathbf{P}(B) = \frac{\mathbf{C}_5^1 . \mathbf{C}_{15}^3}{\mathbf{C}_{20}^4} * 0.05^1 = \frac{91}{3876} \approx 0.0235$$

So, we have a formula:

$$\mathbf{P}(X < 1) = \sum_{x=0}^{4} \frac{\mathbf{C}_5^x.\mathbf{C}_{15}^{4-x}}{\mathbf{C}_{20}^4} * 0.05^x$$

$\Rightarrow \mathbf{P}(C) = \frac{7}{12920}$
$\Rightarrow \mathbf{P}(D) = \frac{1}{258400}$
$\Rightarrow \mathbf{P}(E) \approx 6.449.10^{-9}$
$\Rightarrow \mathbf{P}(X < 1) = P(A) + P(B) + P(C) + P(D) + P(E) \approx 0.3058$
$\Rightarrow 1 - \mathbf{P}(X < 1) \approx 0.6942.$

Exercise 4

Two events **A**,**B** are independent if:

$$\mathbf{P}(A \cap B) = \mathbf{P}(A).\mathbf{P}(B)$$

According to the table below, we have:

- **A**: Be the event that a part from Supplier 1.
- **B**: Be the event that a part Conforms.
- $\mathbf{P}(A) = \frac{\textit{Total number of parts from Supplier 1}}{\textit{Total number of parts}} = \frac{ka+kb}{ka+kb+a+b}$
- $P(B) = \frac{\textit{Total number of parts that Conforms}}{\textit{Total number of parts}} = \frac{ka+a}{ka+kb+a+b}$
- $\mathbf{P}(A \cap B) = \frac{ka}{ka+kb+a+b}(1)$

Now, we calculate $\mathbf{P}(A).\mathbf{P}(B)$:

$$\mathbf{P}(A).\mathbf{P}(B) = \frac{ka + kb}{ka + kb + a + b} * \frac{ka + a}{ka + kb + a + b} = \frac{(ka + kb).(ka + a)}{(ka + kb + a + b)^2}(2)$$

Compare (1) and (2) we have:

$$(1) * 1 = (1) * \frac{ka + kb + a + b}{ka + kb + a + b} = (2)$$

Hence, we proved that **A** and **B** are independent events.

Exercise 5

A geometric random variable X represents the number of trials until the first success occurs, with a success probability of p. The probability mass function (PMF) is given by:

$$P(X = k) = (1 - p)^{k-1}p \quad \text{for } k = 1, 2, 3, \ldots$$

Mean of a Geometric Random Variable

The expected value $E(X)$ of a geometric random variable is:

$$E(X) = \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = \frac{1}{p}$$

To understand the formula above, let's go through some steps:

- **Step 1:** Let S be the sum of the infinite geometric series:

  $$S = x^0 + x^1 + x^2 + x^3 + \dots$$

  Which can be written as:

  $$S = 1 + x + x^2 + x^3 + \dots$$

  Multiply both sides of the equation by x:

  $$xS = x + x^2 + x^3 + \dots$$

  Subtract the two equations:

  $$S - xS = 1$$

  This leads to:

  $$S(1 - x) = 1$$

  Finally, we solve for S:

  $$S = \frac{1}{1-x} \quad \text{for } |x| < 1$$

- **Step 2:** Based on Step 1, we have that:

  $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1$$

  Taking the derivative of both sides with respect to x:

  $$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

  Substituting $x = 1 - p$:

  $$\sum_{k=0}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}$$

  Thus, we have:

  $$E(X) = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

Variance of a Geometric Random Variable

The variance of a random variable X is defined as:

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

Calculating $E(X^2)$:

$$E(X^2) = \sum_{k=1}^{\infty} k^2 P(X = k) = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p = \frac{2-p}{p^2}$$

To understand the equation above, we have:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1$$

Taking the derivative of both sides with respect to x:

$$\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

Taking the derivative one more time:

$$\sum_{k=0}^{\infty} k(k-1) x^{k-2} = \frac{2}{(1-x)^3}$$

Multiply both sides of the equation by x:

$$\sum_{k=0}^{\infty} k(k-1) x^{k-1} = \frac{2x}{(1-x)^3}$$

Equivalently, we can write the equation as:

$$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{2x}{(1-x)^3} + \sum_{k=0}^{\infty} k x^{k-1}$$

Which equals:

$$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$$

Substituting $x = 1 - p$ and multiplying both sides of the equation by p, we have:

$$p \sum_{k=0}^{\infty} k^2 (1-p)^{k-1} = p \left(\frac{2(1-p)}{p^3} + \frac{1}{p^2} \right)$$

The result is:

$$E(X^2) = \frac{2-p}{p^2}$$

Now, substituting back to find the variance:

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Conclusion

The variance of a geometric random variable X with parameter p is:

$$\text{Var}(X) = \frac{1-p}{p^2}$$

Exercise 6

1. **Problem A Statement:**

   An airplane can seat 120 passengers. Each passenger who has a reserved seat shows up with a probability of 0.95. We are asked to find the minimum number of seats the airline should reserve such that the probability of the flight being full is at least 0.90.

   Solution Outline

   Let X be the number of passengers who show up, which follows a binomial distribution:

   $$X \sim \text{Binomial}(n, p = 0.95)$$

   We are tasked with finding n such that:

   $$P(X \geq 120) \geq 0.90$$

   For large n, the binomial distribution $X \sim \text{Binomial}(n, 0.95)$ can be approximated by a normal distribution $N(\mu, \sigma^2)$, where $\mu = np$ and $\sigma^2 = np(1-p)$. Therefore:

   $$P(X \geq 120) = P\left(X_{normal} \geq 120 - 0.5\right) = P\left(X_{normal} \geq 119.5\right)$$

   Using the standard normal transformation, we have:

   $$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{119.5 - 0.95n}{\sqrt{0.95 \cdot 0.05 \cdot n}} = \frac{119.5 - 0.95n}{\sqrt{0.0475n}}$$

   This leads to:

$$P\left(Z \geq \frac{119.5 - 0.95n}{\sqrt{0.0475n}}\right) \geq 0.9$$

Since we are looking for the minimum n, we can replace $\geq$ with $=$:

$$P\left(Z \geq \frac{119.5 - 0.95n}{\sqrt{0.0475n}}\right) = 0.9$$

This is equivalent to:

$$P\left(Z \leq \frac{119.5 - 0.95n}{\sqrt{0.0475n}}\right) = 0.1$$

For $P(Z \leq z) = 0.1$, the z-score corresponding to this probability is approximately -1.28. Thus, we have:

$$\frac{119.5 - 0.95n}{\sqrt{0.0475n}} = -1.28$$

Solving this equation for n, we find that:

$$n \approx 129.13$$

Since we don't want the probability to drop below 0.9, we need to round up the result. Therefore, the airline should reserve at least 130 seats to meet the probability requirement.

2. **Problem B Statement:**

   We need to determine the maximum number of seats the airline should reserve such that the probability of more passengers arriving than the plane can seat is less than 0.10. In other words, we want to find n such that:

   Solution Outline

   Let X be the number of passengers who show up, which follows a binomial distribution:

   $$X \sim \text{Binomial}(n, 0.95)$$

   We need to find n such that:

   $$P(X > 120) < 0.10$$

   $$\iff P(X \geq 121) < 0.10$$

For large n, the binomial distribution $X \sim \text{Binomial}(n, 0.95)$ can be approximated by a normal distribution $N(\mu, \sigma^2)$, where $\mu = np$ and $\sigma^2 = np(1-p)$. Therefore:

$$P(X \geq 121) = P\left(X_{\text{normal}} > 121 - 0.5\right) = P\left(X_{\text{normal}} \geq 120.5\right)$$

Using the standard normal transformation:

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{120.5 - 0.95n}{\sqrt{0.95 \cdot 0.05 \cdot n}} = \frac{120.5 - 0.95n}{\sqrt{0.0475n}}$$

We want to satisfy:

$$P\left(Z > \frac{120.5 - 0.95n}{\sqrt{0.0475n}}\right) < 0.1$$

This is equivalent to:

$$P\left(Z \leq \frac{120.5 - 0.95n}{\sqrt{0.0475n}}\right) \geq 0.9$$

For $P(Z \leq z) = 0.90$, the z-score corresponding to this probability is approximately 1.28. Therefore, we have:

$$\frac{120.5 - 0.95n}{\sqrt{0.0475n}} = 1.28$$

Solving this equation for n, we find that:

$$n \approx 123.58$$

Since we are looking for the maximum number of seats and we don't want the probability to exceed 0.1, we round down the result. Thus, the airline should reserve at most 123 seats to meet the probability requirement.

3. **Problem C Statement:**:

   a. **Overbooking Policy**: Reserve more than 120 seats (e.g., 130) based on the probability of no-shows, ensuring the light is as full as possible.

   b. **Conservative Policy**: Reserve 123 seats, minimizing the risk of overbooking while filling most seats.

   c. **Dynamic Policy**: Adjust seat reservations dynamically based on historical data to balance risks and costs effectively.

Exercise 7

We are provided with the following information:

$$\mu = 1.5, \quad \sigma = 0.025, \quad n = 10$$

The goal is to determine the probability that the maximum of the sample exceeds 1.6.

Step 1: Basic Probability Rules

We use two important probability rules:

1. Multiplication Rule for Independent Events:

$$P(A \cap B) = P(A) \times P(B)$$

2. Complement Rule:

$$P(A') = 1 - P(A)$$

Step 2: Probability Calculation

We need to find the probability:

$$P(X_{\max} > 1.6)$$

Using the complement rule:

$$P(X_{\max} > 1.6) = 1 - P(X_{\max} \leq 1.6)$$

The maximum is smaller than 1.6 if all individual sample values X_i are smaller than 1.6. Thus:

$$P(X_{\max} > 1.6) = 1 - P(X_1 \leq 1.6, X_2 \leq 1.6, \ldots, X_{10} \leq 1.6)$$

Since the events are independent, we can use the multiplication rule:

$$P(X_{\max} > 1.6) = 1 - P(X_1 \leq 1.6) \cdot P(X_2 \leq 1.6) \cdot \cdots \cdot P(X_{10} \leq 1.6)$$

This simplifies to:

$$P(X_{\max} > 1.6) = 1 - (P(X \leq 1.6))^{10}$$

Step 3: Finding $P(X \leq 1.6)$

The sampling distribution of the sample mean $\bar{x}$ has mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$. The z-score is calculated as:

$$z = \frac{x - \mu}{\sigma} = \frac{1.6 - 1.5}{0.025} = 4.00$$

Using the z-score table, we find:

$$P(X \leq 1.6) = P(Z < 4.00) \approx 0.9999683$$

Step 4: Final Probability Calculation

The probability for the maximum value becomes:

$$P(X_{\max} > 1.6) = 1 - (P(X \leq 1.6))^{10}$$

$$P(X_{\max} > 1.6) = 1 - (0.9999683)^{10} \approx 0.000316667$$

Result

Thus, the probability that the maximum value exceeds 1.6 is:

$$P(X_{\max} > 1.6) \approx 0.000316667$$

Exercise 8

Let the random variable X denote a measurement from a manufactured product. Suppose that the target value for the measurement is m. The quality loss of the process producing the product is defined to be the expected value of $k(X - m)^2$, where k is a constant that relates a deviation from target to a loss measured in dollars.

Part (a): Quality Loss with $E(X) = m$ and $V(X) = \sigma^2$

Given that the expected value of X is equal to the target m (i.e., $E(X) = m$) and the variance of X is $V(X) = \sigma^2$, we can calculate the quality loss as follows:

The quality loss is defined by:

$$\text{Quality Loss} = E[k(X - m)^2] = kE[(X - m)^2]$$

Using the properties of variance, we know that:

$$E[(X - m)^2] = V(X) = \sigma^2$$

Thus, we can express the quality loss as:

$$\text{Quality Loss} = kE[(X - m)^2] = k\sigma^2$$

Part (b): Quality Loss with $E(X) = \mu$ and $V(X) = \sigma^2$

In this case, the expected value of X is not equal to the target m, but instead, it is μ (i.e., $E(X) = \mu$) while the variance remains $V(X) = \sigma^2$.

The quality loss can again be expressed as:

$$\text{Quality Loss} = E[k(X - m)^2]$$

To expand this, we can use the formula for the expected value of a squared deviation:

$$E[(X-m)^2] = E[(X-\mu+\mu-m)^2]$$

Expanding the expression, we have:

$$E[(X-m)^2] = E[(X-\mu)^2] + 2E[(X-\mu)(\mu-m)] + (\mu-m)^2$$

Using properties of expectation, we can simplify this:
1. The first term is the variance:

$$E[(X-\mu)^2] = V(X) = \sigma^2$$

2. The second term is zero because $E[(X-\mu)] = 0$.
3. The last term is simply $(\mu-m)^2$.
Thus, we have:

$$E[(X-m)^2] = \sigma^2 + (\mu-m)^2$$

Substituting this back into the quality loss formula gives:

$$\text{Quality Loss} = kE[(X-m)^2] = k(\sigma^2 + (\mu-m)^2)$$

Conclusion

In summary, the quality loss of the process is:

- For part (a):
$$\text{Quality Loss} = k\sigma^2$$
- For part (b):
$$\text{Quality Loss} = k(\sigma^2 + (\mu-m)^2)$$

Exercise 9

Problem Setup

- X_1: The lifetime of an amplifier with a mean lifetime of 20,000 hours.
- X_2: The lifetime of an amplifier with a mean lifetime of 50,000 hours.
- $P_1 = 0.1$: The proportion of amplifiers with a mean lifetime of 20,000 hours.
- $P_2 = 1 - 0.1 = 0.9$: The proportion of amplifiers with a mean lifetime of 50,000 hours.

Exponential Distribution Parameters

The mean of an exponential distribution is given by:

$$\mu = \frac{1}{\lambda}$$

, where λ is the rate parameter.

For amplifiers with a mean lifetime of 20,000 hours:

$$\mu_1 = 20000 \quad \Rightarrow \quad \lambda_1 = \frac{1}{20000}$$

For amplifiers with a mean lifetime of 50,000 hours:

$$\mu_2 = 50000 \quad \Rightarrow \quad \lambda_2 = \frac{1}{50000}$$

Failure Probability Before a Certain Time t

We'll use the exponential distribution model for both X_1 and X_2, compute the individual probabilities, and then combine them using the given proportions of the two types of amplifiers.

$$P(X \leq t) = 1 - \exp(-\lambda t)$$

For X_1 (amplifiers with a mean lifetime of 20,000 hours):

$$P(X_1 \leq 60000) = 1 - \exp\left(-\frac{60000}{20000}\right) = 1 - \exp(-3)$$

For X_2 (amplifiers with a mean lifetime of 50,000 hours):

$$P(X_2 \leq 60000) = 1 - \exp\left(-\frac{60000}{50000}\right) = 1 - \exp(-1.2)$$

Total Proportion of Failures Before 60,000 Hours

The total proportion of amplifiers that fail before 60,000 hours is calculated by combining the failure probabilities of both types of amplifiers, weighted by their respective proportions:

$$P(\text{fail before 60000 hours}) = P_1 \cdot P(X_1 \leq 60000) + P_2 \cdot P(X_2 \leq 60000)$$

Substituting the known values:

$$P(\text{fail before 60000 hours}) = 0.1 \cdot (1 - \exp(-3)) + 0.9 \cdot (1 - \exp(-1.2))$$

Evaluating this expression:

$$P(\text{fail before 60000 hours}) = 0.1 \cdot (1 - 0.0498) + 0.9 \cdot (1 - 0.3012)$$

$$P(\text{fail before 60000 hours}) = 0.1 \cdot 0.9502 + 0.9 \cdot 0.6988$$

$$P(\text{fail before 60000 hours}) = 0.09502 + 0.62892 = 0.7239$$

Conclusion

Approximately 72.39% of the amplifiers will fail before reaching 60,000 hours of operation.

Exercise 10

1. Calculate the conditional probability

We have to calculate $\mathbf{P}(X < t_1 + t_2 | X > t_1)$.

Using the definition of conditional probability, we have:

$$\mathbf{P}(X < t_1 + t_2 | X > t_1) = \frac{\mathbf{P}(X < t_1 + t_2 \cap X > t_1)}{\mathbf{P}(X > t_1)}$$

2. Calculate $\mathbf{P}(X > t_1)$

According to the exponential distribution, we have:

$$P(X > t_1) = 1 - P(X \leq t_1) = e^{-\lambda t_1}$$

3. Calculate $\mathbf{P}(X < t_1 + t_2 \cap X > t_1)$

Because $X < t_1 + t_2$ and $X > t_1$ can be expressed $t_1 < X < t_1 + t_2$, we have:

$$\mathbf{P}(t_1 < X < t_1 + t_2) = \mathbf{P}(X < t_1 + t_2) - \mathbf{P}(X \leq t_1)$$

According to the exponential distribution, we have:

$$\mathbf{P}(X < t_1 + t_2) = 1 - e^{-\lambda(t_1+t_2)}$$

Hence,

$$\mathbf{P}(X < t_1 + t_2 \cap X > t_1) = \left(1 - e^{-\lambda(t_1+t_2)}\right) - \left(1 - e^{-\lambda t_1}\right) = e^{-\lambda t_1} - e^{-\lambda(t_1+t_2)}$$

4. Combine elements

Now, we have:

$$\mathbf{P}(X < t_1 + t_2 | X > t_1) = \frac{e^{-\lambda t_1} - e^{-\lambda(t_1+t_2)}}{e^{-\lambda t_1}}$$

Simplify an equation below, we have:

$$\mathbf{P}(X < t_1 + t_2 | X > t_1) = 1 - e^{-\lambda t_2}$$

5. Compare with $\mathbf{P}(X < t_2)$

Calculate $\mathbf{P}(X < t_2)$:

$$\mathbf{P}(X < t_2) = 1 - e^{-\lambda t_2}$$

Conclusion

So, we proved that:

$$\mathbf{P}(X < t_1 + t_2 | X > t_1) = \mathbf{P}(X < t_2)$$

Exercise 11

Problem Setup

- The bakery can produce 0, 1000, 2000, or 3000 rolls per day.
- The production cost is \$0.10 per roll.
- The selling price for rolls in demand is \$0.30 per roll.
- Rolls not in demand can be sold in a secondary market for \$0.05 per roll.
- The demand for rolls follows the distribution:

Demand (rolls)	Probability
0	0.3
1000	0.2
2000	0.3
3000	0.2

Expected Profit Calculation

Let x represent the number of rolls produced daily, and D represent the demand. We need to compute the expected profit for each production scenario:

Case $x = 0$ (Producing 0 rolls)

$$\text{Profit}(x = 0) = 0$$

Case $x = 1000$ **(Producing 1000 rolls)**

For each demand scenario:

- $D = 0$: All 1000 rolls sold in the secondary market at \$0.05 per roll:

$$1000 \times 0.05 - 1000 \times 0.10 = -50$$

- $D = 1000$: All 1000 rolls sold at \$0.30 per roll:

$$1000 \times 0.30 - 1000 \times 0.10 = 200$$

- $D = 2000$ or $D = 3000$: Only 1000 rolls are produced and sold at \$0.30 per roll:

$$1000 \times 0.30 - 1000 \times 0.10 = 200$$

Expected profit:

$$\text{Profit}(x = 1000) = 0.3 \times (-50) + 0.2 \times 200 + 0.3 \times 200 + 0.2 \times 200$$

$$\text{Profit}(x = 1000) = -15 + 40 + 60 + 40 = 125$$

Case $x = 2000$ **(Producing 2000 rolls)**

For each demand scenario:

- $D = 0$: All 2000 rolls sold in the secondary market at \$0.05 per roll:

$$2000 \times 0.05 - 2000 \times 0.10 = -100$$

- $D = 1000$: 1000 rolls sold at \$0.30 per roll and 1000 rolls in the secondary market:

$$1000 \times 0.30 + 1000 \times 0.05 - 2000 \times 0.10 = 150$$

- $D = 2000$ or $D = 3000$: All 2000 rolls sold at \$0.30 per roll:

$$2000 \times 0.30 - 2000 \times 0.10 = 400$$

Expected profit:

$$\text{Profit}(x = 2000) = 0.3 \times (-100) + 0.2 \times 150 + 0.3 \times 400 + 0.2 \times 400$$

$$\text{Profit}(x = 2000) = -30 + 30 + 120 + 80 = 200$$

Case $x = 3000$ **(Producing 3000 rolls)**

For each demand scenario:

- $D = 0$: All 3000 rolls sold in the secondary market at \$0.05 per roll:

$$3000 \times 0.05 - 3000 \times 0.10 = -150$$

- $D = 1000$: 1000 rolls sold at \$0.30 per roll and 2000 rolls in the secondary market:

$$1000 \times 0.30 + 2000 \times 0.05 - 3000 \times 0.10 = 100$$

- $D = 2000$: 2000 rolls sold at \$0.30 per roll and 1000 rolls in the secondary market:

$$2000 \times 0.30 + 1000 \times 0.05 - 3000 \times 0.10 = 350$$

- $D = 3000$: All 3000 rolls sold at \$0.30 per roll:

$$3000 \times 0.30 - 3000 \times 0.10 = 600$$

Expected profit:

$$\text{Profit}(x = 3000) = 0.3 \times (-150) + 0.2 \times 100 + 0.3 \times 350 + 0.2 \times 600$$
$$\text{Profit}(x = 3000) = -45 + 20 + 105 + 120 = 200$$

Conclusion

The expected profits are:

x	Expected Profit
0	0
1000	125
2000	200
3000	200

The expected profits for producing 2000 and 3000 rolls are both \$200. However, producing 2000 rolls is preferable due to the following reasons:

- **Lower risk of overproduction**: Producing 3000 rolls could lead to more rolls being unsold in the primary market, requiring them to be sold in the secondary market at a lower price (\$0.05 instead of \$0.30).
- **Minimized wastage**: By producing 2000 rolls, the bakery is less likely to produce more than what is demanded, reducing the number of rolls sold at the lower price.
- **Operational efficiency**: Producing 2000 rolls reduces the resources used (e.g., ingredients, labor, energy), potentially leading to more efficient operations and lower overhead costs compared to producing 3000 rolls.

Thus, producing **2000 rolls per day** is the more efficient and risk-averse option while still maximizing the mean profit.