Theorem 1: Suppose that K is a compact set in $\mathbf{R}^n$, U is a compact set in $C(K)$, then for any $\epsilon > 0$, there exist a positive integer N, a real constant c, a N-dim state function $|t\rangle$, which is independent of $f \in C(K)$ and a N-dim state function $|b\rangle$ depending on f, such that

$$|f(x) - c\langle b(f)|t(x)\rangle| \le \epsilon \tag{1}$$

holds for all $x \in K$ and $f \in U$.

Lemma 1: Suppose that K is a compact set in $\mathbf{R}^n$, $f \in C(K)$, then there is a continuous function $E(f) \in C(\mathbf{R}^n)$, such that (1) $f(x) = E(f)(x)$ for all $x \in K$; (2) $\sup_{x\in\mathbf{R}^n} |E(f)(x)| \le \sup_{x\in K} |f(x)|$; (3) there is a constant c such that

$$\sup_{|x'-x''|<\delta} |E(f)(x') - E(f)(x'')| \le c \sup_{|x'-x''|<\delta, x',x''\in K} |f(x') - f(x'')| \tag{2}$$

Proof: The proof of Lemma 1 can be found in [1] (p. 175)

Lemma 2: [2].

V is a compact set in $C(K)$, if and only if

1. V is a closed set in $C(K)$.
2. There is a constant M, such that $\|f(x)\|_{C(K)}$ for all $f \in V$.
3. V is equicontinuous, i.e. for any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x') - f(x'')| < \epsilon$ for all $f \in V$, provided that $x', x'' \in K$ and $\|x' - x''\|_K < \delta$.

Lemma 3: Suppose that K is a compact set in $\mathbf{I^n} = [0,1]^n$, V is a compact set in $C(K)$, then V can be extended to a compact set in $C_p[-1,1]^n$.

Proof. By Lemmas 1 and 2, V can be extended to be a compact set V_1 in $C[0,1]^n$.

Now, for every $f \in V_1$, define an even extension of f as follows

$$f^*(x_1, \cdots, x_k, \cdots, x_n) = f(x_1, \cdots, -x_k, \cdots, x_n) \tag{3}$$

then $U = \{f^* : f \in V_1\}$ is the required compact set in $C_p[-1,1]^n$.

Lemma 4: Suppose that U is a compact set in $C_p[-1,1]$,

$$B_R(f;x) = \sum_{|m|\le R} (1 - \frac{|m|^2}{R^2})^\alpha c_m(f) e^{i\pi m\cdot x} \tag{4}$$

is the Bochner-Riesz means of Fourier series of f, where $m = (m_1, \cdots, m_n)$, $|m|^2 = \sum_{i=1}^n |m_i|^2$, $c_m(f)$ are Fourier coecients of f , then for any $\epsilon > 0$, there is $R > 0$ such that

$$|B_R(f;x) - f(x)| \le \epsilon \tag{5}$$

For every $f \in U$ and $x \in [-1,1]^n$, provided that $\alpha > (n-1)/2$

Proof. The proof of Lemma 4 can be found in [3].

Lemma 5: Suppose that $|\psi(x)\rangle$ and $|\phi(y)\rangle$ are two bounded m-dim state functions, there exist a real constant c, $(m+2)$-dim normalized state functions $|\Psi(x)\rangle$ and $|\Phi(y)\rangle$, such that

$$\langle\psi(x)|\phi(y)\rangle = c\langle\Psi(x)|\Phi(y)\rangle \tag{6}$$

Proof: Without loss of generality, we can write $|\psi(x)\rangle$ and $|\phi(y)\rangle$ as

$$\begin{aligned} |\psi(x)\rangle &= \sum_{i=1}^{m} f_i(x)|i\rangle \\ |\phi(y)\rangle &= \sum_{i=1}^{m} g_i(y)|i\rangle \end{aligned} \tag{7}$$

Since their amplitudes are bounded, we define the 1-norm of the state function $\||\psi(x)\rangle\|_1 = \max_i \|f_i(x)\|_1$. Because the case of $\||\psi(x)\rangle\|_1 = 0$ or $\||\phi(x)\rangle\|_1 = 0$ is trival, we can write $|\Psi(x)\rangle$ and $|\Phi(y)\rangle$ as

$$\begin{aligned} |\Psi(x)\rangle &= \frac{1}{\||\psi(x)\rangle\|_1}(\sum_{i=1}^{m} f_i(x)|i\rangle + f_{sub}(x)|i+1\rangle) \\ |\Phi(y)\rangle &= \frac{1}{\||\phi(y)\rangle\|_1}(\sum_{i=1}^{m} g_i(y)|i\rangle + g_{sub}(x)|i+2\rangle) \end{aligned} \tag{8}$$

Since the norms of the first m dimensions i less than or equal to one, there exist f_{sub} and g_{sub} to normalize $|\Psi(x)\rangle$ and $|\Phi(x)\rangle$. Thus exist a real constant c, such that

$$\langle\psi(x)|\phi(y)\rangle = c\langle\Psi(x)|\Phi(y)\rangle \tag{9}$$

Proof of Theorem 1. Without loss of generality, we can assume that $K \subseteq [0,1]^n$. By Lemma 3, we can assume that $K = [-1,1]^n$ and $U \subseteq [-1,1]^n$. By Lemma 4, for any $\epsilon > 0$, there exists $R > 0$, such that for any $x = (x_1, \cdots, x_n) \in [-1,1]^n$ and $f \in U$, there holds

$$|\sum_{|m|\leq R}(1-\frac{|m|^2}{R^2})^\alpha c_{m_1\cdots m_n}(f^**)\exp(i\pi(m_1x_1+\cdots+m_nx_n)) - f^*(x_1,\cdots,x_n)| < \epsilon \tag{10}$$

By the denition of the Fourier coecients and evenness of $f(x)$, we can rewrite it as

$$|\sum_{|m|\leq R} d_{m_1\cdots m_n}(f^*)\cos(\pi(m_1x_1+\cdots+m_nx_n)) - f^*(x_1,\cdots,x_n)| < \epsilon \tag{11}$$

Where $d_{m_1\cdots m_n}$ are real numbers. It is obvious that for every $x \in [-1,1]^n$, the first item is the inner product of two states:

$$\begin{gathered}\sum_{|m|\leq R} d_{m_1\cdots m_n}(f^*)|m\rangle \\ \sum_{|m|\leq R} \cos(\pi(m_1x_1+\cdots+m_nx_n))|m\rangle\end{gathered} \tag{12}$$

By Lemma 5, exist a positive intege N, a real constant c and N-dim state functions $|b(f^*)\rangle$, $|t(x)\rangle$, such that

$$|f^*(x) - c\langle b(f^*)|t(x)\rangle| < \epsilon \tag{13}$$

is true for all $x \in [-1, 1]^n$ and $f^* \in V$. Thus

$$|f(x) - c\langle b(f)|t(x)\rangle| < \epsilon \tag{14}$$

is true for all $x \in [0, 1]^n$ and $f \in U$.

Theorem 2: Suppose that X is a Banach Space, $K \subseteq$ is a compact set, V is a compact set in $C(K)$, f is a continuous functional defined on V ,then for any $\epsilon > 0$, there exist a positive integer N, a real constant c, a N-dim state function $|t\rangle$, which is independent of $f \in C(K)$ and a state function $|b\rangle$ depending on f, such that

$$|f(x) - c\langle b(f)|t(x)\rangle| \leq \epsilon \tag{15}$$

holds for all $u \in V$.

Lemma 6: [4]

1. For each fixed k, V_{η_k} is a compact set in a subspace of dimension $n(\eta_k) \in C(K)$.
2. For every $u \in V$, there holds
$$\|u - u_{\eta_k}\|_{C(K)} < \delta_k \tag{16}$$
3. V^* is a compact set in $C(K)$

Proof of Theorem 2*: By Tietze Extension Theorem, we can define a continuous functional on V^* such that

$$f^*(x) = f(x) \text{ if} x \in V \tag{17}$$

Because f^* is a continuous functional defined on the compact set V^*, therefore for any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f^*(u) - f^*(v)| < \epsilon/2$ provided that $u, v \in V^*$ and $\|u - v\|_{C(K)} < \delta$/

Let k be fixed such that $\delta_k < \delta$, then by proposition 2. of Lemma 6 for every $u \in V$,

$$\|u - u_{\eta_k}\|_X < \delta_k \tag{18}$$

which implies

$$|f^*(u) - f^*(u_{\eta_k}) < \epsilon/2 \tag{19}$$

for all $u \in V$.

By proposition 1. of Lemma 6, we see that $f^*(u_{\eta_k})$ is a continuous functional defined on the compact set V_{η_k} in $\mathbf{R}^{n(\eta_k)}$.By theorem 1, we can find N, c, $|b\rangle$, $|t\rangle$, such that

$$|f^*(u_{\eta_k}) - c\langle b(f^*)|t(x)\rangle| < \epsilon/2 \tag{20}$$

we conclude that

$$|f(u) - c\langle b(f^*)|t(x)\rangle| < \epsilon/2 \tag{21}$$

Thus, Theorem 2 is proved.

Reference:

[1] E. M. Stein, Singular Integrals and Dierentiability Properties of Functions, Princeton University Press, (1970).
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