CHAPTER 15

LOG-PEARSON TYPE III DISTRIBUTION

The log-Pearson type 3 (LP3) distribution has been one of the most frequently used distributions for hydrologic frequency analyses since the recommendation of the Water Resources Council (1967, 1982) of the United States as to its use as the base method. The Water Resources Council also recommended that this distribution be fitted to sample data by using mean, standard deviation and coefficient of skewness of the logarithms of flow data [i.e., the method of moments (MOM)]. A large volume of literature on the LP3 distribution has since been published with regard to its accuracy and methods of fitting or parameter estimation. McMahon and Srikanthan (1981) and Srikanthan and McMahon (1981) examined the applicability of LP3 distribution to Australian rivers and questioned the assumption of setting to zero the coefficient of skewness of logarithms of peak discharges that were not statistically different from zero. They evaluated the effect of sample size, distribution parameters and dependence on peak annual flood estimates. Gupta and Deshpande (1974) applied LP3 distribution to evaluate design earthquake magnitudes. Phien and Jivajirajah (1984) applied LP3 distribution to annual maximum rainfall, annual streamflow and annual rainfall. Wallis and Wood (1985) found, based on Monte Carlo experiments, that the flood quantile estimates obtained by using an index flood type approach with either a generalized extreme value distribution or a Wakeby distribution fitted by PWM were superior to those obtained by LP3 distribution with MOM -based parameters. This finding was challenged later by several investigators (Beard, 1986; Landwehr et al., 1986).

The Water Resources Council recommended the use of a generalized skew coefficient. Bobee and Robitaille (1975) proposed a correction for bias in estimation of the coefficient of skewness. Tung and Mays (1981) investigated various methods of determining generalized skew coefficients. They introduced a method for determining generalized skew coefficients using a weighting procedure based on the varaince of regional map skew coefficients and variance of sample skew coefficients. Oberg and Mades (1987) evaluated several techniques of estimating generalized skew of LP3 distribution for Illinois rivers: (1) a generalized skew map of U.S., (2) an isoline map, (3) a prediction equation, (4) a regional mean skew. They found no appreciable difference between flood estimates computed using the variations of the regional mean technique and flood estimates using the skew map prepared by the Water Resources Council. Bobee (1975) showed that the method of moments recommended by the Water Resources Council (1967) would introduce bias in fitting LP3 distribution because the method of moments (MOM) used logarithms of observed data and not the moments of the observed values. He used a method which retained the moments of the original data. Ashkar and Bobee (1987) and Bobee and Ashkar (1988) used four different versions of MOM and obtained a generalized MOM (GMOM).

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They concluded that one version of GMOM might be best for estimating high flows (flows above the range covered by the data) and another for estimating low flows.

Using simulation, Ouarda and Ashkar (1998) evaluated the effect of trimming on LP3 flood quantile estimates. They examined the effect of various proportions of symmetric trimming on the estimation of moments, distribution parameters, and quantiles. The influence of sample size and parent distribution parameters on the estimation performance was also investigated for several return periods and for three fitting methods. Cohen et al. (1997) developed the expected moments algorithm (EMA) for computing moments-based flood quantile estimates when historical flood information was available. EMA can utilize three types of at-site flood information: systematic stream gage record, information about the magnitude of historical flood, and knowledge of the number of years in the historical period when no large flood occurred. Based on Monte Carlo simulations, they showed that EMA was more efficient than MOM and nearly as efficient as MLE. Rao (1980a, b, 1981, 1983a,b 1988) evaluated the properties and results of LP3 distribution in a general fashion and proposed a method of mixed moments (MIX) to estimate LP3 parameters. He found that MIX using the mean and variance of real data and the mean of log data possessed superior statistical properties to MOM. Song and Ding (1988) applied the probability weighted moments (PWM) to estimate the parameters of LP3 distribution. Their results of Monte Carlo experiments showed that PWM compared favorably with MOM and was equally efficient when compared with MLE and other curve fitting methods. Condie (1977) derived LP3 parameters using the method of maximum likelihood estimation (MLE). By fitting 37 long-term unregulated flood data sets in Canada, he concluded that MLE was superior to MOM. However, Nozdryn-Plotnicki and Watts (1979) found in their simulation study of the standard error of the T-year flood that MLE and MOM were almost comparable, and hence they suggested the use of MOM because of its computational ease.

Phien and Hsu (1985) compared a number of techniques for LP3 parameter estimation. These were MOM and modified versions of MOM. Singh and Singh (1988) estimated LP3 parameters using the principle of maximum entropy (POME) and found it comparable to MOM and MLE for historical data used. Phien and Hira (1983) estimated LP3 parameters using four methods: MLE, direct and indirect MOM, MIX, method of Bobee (1975), and other versions of MOM. They found the MIX method, consisting of the first two moments of the original data and the variance of the log-transformed values, to be providing the best estimates. Arora and Singh (1987a, b, 1989) made a comparative evaluation of different estimators of LP3 distribution: Direct and indirect MOM, MIX, MLE, and POME. Using Monte Carlo experiments, they found MIX to be markedly superior to other methods in terms of both resistance and efficiency of estimation.

Benson (1967) reported on uniform flood-frequency estimating methods for federal agencies. Among 2-parameter gamma, Gumbel, log-Gumbel, log-normal, Hazen and LP3 distributions, the LP3 distrribution was selected as the base method, with provisions for departures where justified. Reich (1970) analyzed flood peaks from Pennsylavanian streams using Gumble, log-Gumbel, and LP3 distributions. He found the Gumbel distribution to be generally applicable. Shen et al. (1980) investigated the tail behavior in extreme events using LP3 and Gumbel distributions and found LP3 distribution to be a better description of field data. Rao (1981) compared 3-parameter distributions, including LP3, Pearson type 3 (P3), lognormal (LN3), and Weibull (W); and presented bounds, negative areas of distribution and selected quantiles.The choice of the best distribution was not clear and depended on the sample statistics and the choice of the T-year flood. Loganathan et al. (1986) analyzed frequencies of low flows using a mixed LP3, a double bounded probability density function, partial duration series, and a physically based approach. The results of LP3 model were consistent with other methods.

Tasker (1987) estimated 7-day, 10-year and 7-day 20-year low flows using bootstrap using the hypothetical LP3 and W distributions. The use of these distributions led to lower mean square error than did the Box-Cox transformation and the log-Boughton method. In statistical modeling of annual maximum flows of Turkish rivers, Haktanir (1991) compared LN3, P3, LP3, EV1, log-Boughton, log-logistic (LL), and smemax distributions at 112 sites representing 23 major basins and and did not find a single distribution performing consistently better. LP3 and LL performed better more times than others. Vogel et al. (1992) discussed flood-flow frequency model selection in southwestern United States. Using flood flow data at 383 sites, they found LP3, generalized extreme value (GEV) and two-parameter and three-parameter LN distributions to provide a good approximation to flood flow data in this region. Bobee et al. (1993) reported on a systematic approach to comapring distributions in flood frequency analysis.

Hoshi and Burges (1981a) investigated sampling covariance structures of estimated parameters for LP3 dsitribution from sample estimates of mean, coefficient of variation and skew coefficient in the natural domain. They showed that there was no justification for use of logarithmic skew coefficients or the regional skew estimates in log space. Hoshi and Burges (1981b) developed an approximate method for computing the derivative of a standard gamma quantile with respect to the distribution shape parameter necessary for estimating the sampling variance of a specified quantile. Ashkar and Bobee (1988) derived confidence intervals for flood events under LP3 distribution. Condie (1977) used MLE to derive the T-year event and its asymptotic standard error. Philon and Admowski (1993) derived the asymptotic standard error of estimate of the T-year flood.

Let Y = ln X where X is a positive random variable. If Y has a Pearson type (P) III distribution then X will have a log-Pearson type (LP) III distribution with probability density function (pdf) given by

$$f(x) = \frac{1}{a\,x\,\Gamma(b)} \left(\frac{\ln x - c}{a}\right)^{b-1} \exp\left[-\left(\frac{\ln x - c}{a}\right)\right] \tag{15.1}$$

where a > 0, b > 0 and 0 < c < ln x are the scale, shape and location parameters, respectively. The LP IIII distribution is a three-parameter distribution. Its cumulative distribution function (cdf) can be expressed as

$$F(x) = \frac{1}{a\,\Gamma(b)} \int_0^\infty \frac{1}{x} \left(\frac{\ln x - c}{a}\right)^{b-1} \exp\left(-\left(\frac{\ln x - c}{a}\right)\right] dx \tag{15.2}$$

One can verify if f(x) given by equation (15.1) is a pdf as follows:

$$\int_{e^c}^\infty f(x)\, dx = 1 \tag{15.3a}$$

Substituting equation (15.1) in equation (15.3), one gets

$$\frac{1}{a\,\Gamma(b)} \int_{e^c}^\infty \frac{1}{x} \frac{\ln x - c}{a}\Big)^{b-1} \exp\left[-\left(\frac{\ln x - c}{a}\right)\right] dx = 1 \tag{15.3b}$$

Let $(\ln x - c)/a = $ y. Then $(dy/dx) = (a/x)$, and dx = xady.

If $x = e^{c}$ then y = $[(\ln e^{c} - c)/a] = [(c \ln e - c)/a] = [(c-c)/a]$. Substituting these quantities in equation (15.3b), one gets

$$f(x) = \frac{1}{a\,\Gamma(b)} \int_0^{\infty} \frac{1}{x}\, y^{b-1}\, e^{-y}\, x\, a\, dy = \frac{1}{\Gamma(b)} \int_0^{\infty} e^{-y}\, y^{b-1}\, dy = \frac{\Gamma(b)}{\Gamma}(b) = 1$$

(15.3c)

If y = [ln (x) - c]/ a is substituted in equation (15.2) the following is the result:

$$F(y) = \frac{1}{\Gamma(b)} \int_0^{y} y^{b-1} \exp(-y)\, dy \qquad (15.\ 4)$$

Equation (15.4) can be approximated by noting that F (y) can be expressed as a Chi-square distribution with degrees of freedom as 2b and chi-square as 2y. This approximation is given in Chapter 13.

It may be useful to briefly discuss some of the characteristics of the LP III distribution. To that end, the mean, variance and skewness coefficients of both X and Y are given. For Y these, respectively, are:

Mean: $\mu_y = c + a\,b$ (15.5a)

Variance: $\sigma_y^{2} = b\,a^2$ (15.5b)

Skew: $\gamma_y = \frac{|a|}{a} \frac{2}{b^{1/2}}$ (15.5c)

The moments of X about the origin can be written as

$$\mu'_r = \frac{\exp(r\,c)}{(1 - r\,a)^{b}}, \quad 1 - r\,a \succ 0, r = 0, 1, 2, \ldots\ldots.. \qquad (15.6a)$$

From equation (15.6a), the mean, coefficient of variation (CV), coefficient of skewness (skew), and kurtosis of X are given as

Mean: $\mu = \frac{\exp(c)}{(1-a)^{b}}$ (15.6b)

$$\text{Variance: } \sigma_x^2 = \exp(2c) \bullet A \tag{15.6c}$$

$$\text{Coefficient of Variation (CV): } \beta = (1-a)^b \bullet A^{1/2} \tag{15.6d}$$

$$\text{Skew: } \gamma = \left[\frac{1}{(1-3a)^b} - \frac{3}{(1-a)^b(1-2a)^b} + \frac{2}{(1-a)^{3b}}\right] / A^{3/2} \tag{15.6e}$$

Kurtosis:

$$\lambda = \left[\frac{1}{(1-4a)^b} - \frac{4}{(1-a)^b(1-3a)^b} + \frac{6}{(1-a)^{2b}(1-2a)^b} - \frac{3}{(1-a)^{4b}}\right] \bullet A^{-2} \tag{15.6f}$$

where

$$A = \left[\frac{1}{(1-2a)^b} - \frac{1}{(1-a)^{2b}}\right] \tag{15.6g}$$

It is to be noted that the coefficient of variation, skewness, and kurtosis in equations (15.6e) - (15. 6f) are independent of the location parameter c. It should also be noted that higher order moments of order r do not exist if the value of a is greater than 1/r (Bobee and Ashkar, 1991).

Consider equation (15.5a). If $a > 0$, then skew is greater than zero, implying that Y is positively skewed and $c < Y < \infty$. In this case, X is also positively skewed (Rao, 1980a), and $\exp(c) < X < \infty$. If $a < 0$, then skew is less than zero, implying that Y is negatively skewed and $-\infty < Y < c$. In this case , X is either positively skewed or negatively skewed depending on the values of parameters a and b, and $-\infty < X < \exp(c)$. For this case the density function f (x) = 0, and may be arbitrarily defined as zero.

The overall geometric shape of the LP III distribution is governed by parameters a and b (Rao, 1980a; Bobee, 1975). The pdf is capable of assuming diverse shapes, such as reverse J, U, J, and, of course, unimodal (skewed) bell shape. Hoshi and Burges (1981a) point out that if $\gamma < \beta^3 + 3\beta$, then $a < 0$, $0 < x < \exp(c)$, kurtosis of the LP III distribution is less than the kurtosis of the three-parameter lognrmal distribution and vice versa. The LP III distribution degenerates to the lognormal distribution when parameters a and b become zero and infintely, respectively (or equivalently, $\gamma = \beta^3 + 3\beta$, and $\gamma = 0$). For flood frequency analysis, only values of b greater than one and 1/a greater than zero are of interest. Negative coefficients of skew correspond to negative a values and the distribution would then become upper bounded. Under these conditions, this might be considered for low flow analysis but would be unsuitable for flood analysis.

15.1 Ordinary Entropy Method

15.1.1 SPECIFICATION OF CONSTRAINTS

Taking logarithm of equation (15.1.) to the base 'e' , one obtains

$$\ln f(x) = -\ln a\Gamma(b) - \ln x + (b-1)\ln[\frac{\ln x - c}{a}] - (\frac{\ln x - c}{a})$$
$$= -\ln a\,\Gamma(b) - \ln x + (b-1)\ln[\ln x - c] - (b-1)\ln a - \frac{\ln x}{a} + \frac{c}{a} \tag{15.7a}$$

Multiplying equation (15.7a) by -1, we get

$$-\ln f(x) = \ln a\,\Gamma(b) - \frac{c}{a} + (b-1)\ln a + (1 + \frac{1}{a})\ln x - (b-1)\ln[\ln x - c] \tag{15.7b}$$

Multiplying equation (15.7b) by f(x) and integrating between e^c and ∞, the result is the entropy function:

$$-\int_{e^c}^{\infty} f(x)\ln f(x)\,dx = [\ln a\,\Gamma(b) - \frac{c}{a} + (b-1)\ln a]\int_{e^c}^{\infty} f(x)\,dx$$
$$+ (\frac{a+1}{a})\int_{e^c}^{\infty} \ln x f(x)\,dx - (b-1)\int_{e^c}^{\infty} \ln(\ln x - c) f(x)\,dx \tag{15.7c}$$

From equation (15.7c) the constraints appropriate for equation (15.1) can be written as:

$$\int_{e^c}^{\infty} f(x)\,dx = 1 \tag{15.8}$$

$$\int_{e^c}^{\infty} \ln x f(x)\,dx = E[\ln x] = \bar{y} \tag{15.9}$$

$$\int_{e^c}^{\infty} \ln(\ln x - c) f(x)\,dx = E[\ln(\ln x - c)] \tag{15.10}$$

15.1.2 CONSTRUCTION OF ZEROTH LAGRANGE MULTIPLIER

The least-biased pdf f(x), consistent with equations (15.8) to (15.10) and based on the principle of maximum entropy (POME), takes the form:

$$f(x) = \exp[-\lambda_0 - \lambda_1 \ln x - \lambda_2 \ln(\ln x - c)] \tag{15.11}$$

where λ_0, λ_1, and λ_2 are Lagrange multipliers. Substitution of equation (15.11) in equation (15.8) yields

$$\int_{e^c}^{\infty} f(x)\, dx = \int_{e^c}^{\infty} \exp[-\lambda_0 - \lambda_1 \ln x - \lambda_2 \ln(\ln x - c)]\, dx = 1 \tag{15.12}$$

Equation (15.12) gives the partition function function as

$$\exp(\lambda_0) = \int_{e^c}^{\infty} \exp[-\lambda_1 \ln x - \lambda_2 \ln(\ln x - c)] dx \tag{15.13}$$

Equation (15.13) is simplified as

$$\exp(\lambda_0) = \int_{e^c}^{\infty} \exp[\ln y^{-\lambda_1}] \exp[\ln(\ln x - c^{-\lambda_2})]\, dx = \int_{e^c}^{\infty} x^{-\lambda_1} (\ln x - c)^{-\lambda_2}\, dx \tag{15.14}$$

Let ln x–c=y. Then, ln x = y + c; x = exp(y+c); (dy/dx) = (1/x); dx = xdy; and dx = exp(y+c) dy. Substituting these quantities in equation (15.14), we get

$$\begin{aligned} \exp(\lambda_0) &= \int_0^{\infty} [e^{y+c}]^{-\lambda_1} y^{-\lambda_2} e^{y+c}\, dy = \int_0^{\infty} (e^y e^c)^{-\lambda_1} y^{-\lambda_2} e^y e^c\, dy \\ &= \exp[c - c\lambda_1] \int_0^{\infty} \exp[-\lambda_1 y + y]\, y^{-\lambda_2}\, dy \\ &= \exp[-c(\lambda_1 - 1)] \int_0^{\infty} \exp[-y(\lambda_1 - 1)]\, y^{-\lambda_2}\, dy \end{aligned} \tag{15.15}$$

Let $y(\lambda_1 - 1) = z$. Then $y = [z/(\lambda_1 - 1)]$, and $(dz/dy) = (\lambda_1 - 1)$. Therefore, equation (15.15) becomes

$$\begin{aligned} \exp(\lambda_0) &= \exp[-c(\lambda_1 - 1)] \int_0^{\infty} e^{-z} \left(\frac{z}{\lambda_1 - 1}\right)^{-\lambda_2} \frac{dz}{(\lambda_1 - 1)} \\ &= \frac{\exp[-c(\lambda_1 - 1)]}{(\lambda_1 - 1)^{1-\lambda_2}} \int_0^{\infty} z^{-\lambda_2} e^{-z}\, dz \end{aligned} \tag{15.16}$$

Since

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha - 1} e^{-x}\, dx \tag{15.17}$$

equation (15.16) reduces to the partition function:

$$\exp(\lambda_0) = \frac{\exp[-c(\lambda_1 - 1)]}{(\lambda_1 - 1)^{1-\lambda_2}} \Gamma(1 - \lambda_2) \tag{15.18}$$

Therefore, the zeroth Lagrange multiplier is obtained from equation (15.18) as

$$\lambda_0 = -c(\lambda_1 - 1) + (\lambda_2 - 1)\ln(\lambda_1 - 1) + \ln\Gamma(1 - \lambda_2) \tag{15.19}$$

The zeroth Lagrange multiplier is also obtained from equation (15.13) as

$$\lambda_0 = \ln \int_{e^c}^{\infty} \exp[-\lambda_1 \ln x - \lambda_2 \ln(\ln x - c)]\, dx \tag{15.20}$$

14.1.3 RELATION BETWEEN LAGRANGE MULTIPLIERS AND CONSTRAINTS

Differentiating equation (15.11) with respect to λ_1 and λ_2, one gets

$$\frac{\partial \lambda_0}{\partial \lambda_1} = -\frac{\int_{e^c}^{\infty} \ln x \exp[-\lambda_1 \ln x - \lambda_2 \ln(\ln x - c)]\, dx}{\int_{e^c}^{\infty} \exp[-\lambda_1 \ln x - \lambda_2 \ln(\ln x - c)]\, dx}$$

$$= -\int_{e^c}^{\infty} \ln x \exp[-\lambda_0 - \lambda_1 \ln x - \lambda_2 \ln(\ln x - c)]\, dx$$

$$= -\int_{e^c}^{\infty} \ln x f(x)\, dx = -E[\ln x] \tag{15.21}$$

$$\frac{\partial \lambda_0}{\partial \lambda_2} = -\frac{\int_{e^c}^{\infty} \ln(\ln x - c) \exp[-\lambda_1 \ln x - \lambda_2 \ln(\ln x - c)\, dx}{\int_{e^c}^{\infty} \exp[-\lambda_1 \ln x - \lambda_2 \ln(\ln x - c)]\, dx}$$

$$= -\int_{e^c}^{\infty} \ln(\ln x - c) \exp[-\lambda_0 - \lambda_1 \ln x - \lambda_2 \ln(\ln x - c)]\, dx$$

$$= -\int_{e^c}^{\infty} \ln(\ln x - c) f(x)\, dx = -E[\ln(\ln x - c)] \tag{15.22}$$

Also differentiating equation (15.19) with respect to λ_1 and λ_2, respectively, one gets

$$\frac{\partial \lambda_0}{\partial \lambda_1} = -c + \frac{\lambda_2 - 1}{\lambda_1 - 1} \tag{15.23}$$

$$\frac{\partial \lambda_0}{\partial \lambda_2} = \ln(\lambda_1 - 1) + \frac{\partial}{\partial \lambda_2} \ln \Gamma(1 - \lambda_2) \tag{15.24}$$

Equation (15.24) can be simplified as

$$\frac{\partial\lambda_0}{\partial\lambda_2} = \ln(\lambda_1 - 1) + \frac{\partial}{\partial p}\ln\Gamma(p)\frac{\partial p}{\partial\lambda_2} = \ln(\lambda_1 - 1) - \psi(p) \tag{15.25}$$

where $p = 1 - \lambda_2$. Since the LP III distribution has three parameters, equations (15.23) and (15.25) are not sufficient and another equation is neded. This is obtained by recalling that

$$\frac{\partial^2\lambda_0}{\partial\lambda_1^2} = \frac{(\lambda_2 - 1)}{(\lambda_1 - 1)^2}(-1) = \frac{1-\lambda_2}{(\lambda_1 - 1)^2} = \sigma_y^2 \tag{15.26}$$

Equating equations (15.21) and (15.23), as well as equations (15.22) and (15.25), one obtains

$$\frac{p}{\lambda_1 - 1} = E[\ln x] - c \tag{15.27}$$

$$\psi(p) - \ln(\lambda_1 - 1) = E[\ln(\ln x - c)] \tag{15.28}$$

15.1.4 RELATION BETWEEN LAGRANGE MULTIPLIERS AND PARAMETERS

Inserting equation (15.19) into equation (15.11), one gets

$$\begin{aligned}
f(x) = \exp\,[&c(\lambda_1 - 1) - (\lambda_2 - 1)\ln(\lambda_1 - 1) - \ln\Gamma(1 - \lambda_2) - \lambda_1 - \ln x \\
&- \lambda_2 \ln(\ln x - c) - \ln x + \ln x] \\
= \exp[&-(\lambda_1 - 1)(\ln x - c) - \ln x + \ln(\lambda_1 - 1)^{-(\lambda_2 - 1)} \\
&+ \ln[\Gamma(1 - \lambda_2))^{-1} + \ln(\ln x - c)^{-\lambda_2}] \\
= \exp[&-(\lambda_1 - 1)(\ln x - c)\frac{1}{x}(\lambda_1 - 1)^{-(\lambda_2 - 1)}\frac{(\ln x - c)^{-\lambda_2}}{\Gamma(1 - \lambda_2)}]
\end{aligned} \tag{15.29}$$

Comparing equation (15.29) with equation (15.1), one gets

$$1 - \lambda_2 = b \tag{15.30}$$

$$\lambda_1 - 1 = \frac{1}{a} \tag{15.31}$$

Then

$$\lambda_1 = 1 + \frac{1}{a} \tag{15.32}$$

$$\lambda_2 = 1 - b \tag{15.33}$$

15.1.5 RELATION BETWEEN PARAMETERS AND CONSTRAINTS

The LP III distribution has 3 parameters a, b, and c. The known constraints are related to the Lagrange multipliers by equations (15.26), (15.27) and (15.28) which, in turn, are related to the parameters by equations (15.30) and (15.31). Eliminating the Lagrange multipliers between these two sets of equations, we obtain parameters directly in terms of the constraints as

$$ab + c = E[\ln x] \tag{15.34}$$

$$\psi(b) - \ln a = E[\ln\{\ln(x-c)\}] \tag{15.35}$$

$$ba^2 = \sigma_y^2 \tag{15.36}$$

15.1.6 DISTRIBUTION ENTROPY

Equation (15.7) gives the distribution entropy. Rewriting it,

$$I(x) = -\int_{e^c}^{\infty} f(x) \ln f(x)\, dx$$

$$= [\ln a\Gamma(b) - \frac{c}{a} + \ln a^{b-1}] \int_{e^c}^{\infty} f(x)\,dx + (\frac{a+1}{a}) \int_{e^c}^{\infty} \ln x\, f(x)\, dx$$

$$-(b-1) \int_{e^c}^{\infty} \ln(\ln x - c) f(x)\, dx$$

$$= \ln a^b\,\Gamma(b) - \frac{c}{a} + (\frac{a+1}{a})\,\bar{y} - (b-1) E[\ln(\ln x - c)]$$

$$= \ln a^b \Gamma(b) - \frac{c}{a} + (\frac{a+1}{a})\,\bar{y} - (b-1)\, E[\ln(y-c)] \tag{15.37}$$

Alternatively, since the transformation x = e^y is monotonic with the Jacobian J(y/x) = 1/x, we can write

$$I(x) = I(y) - E[\ln|J(\frac{y}{x})|] = I(y) + \bar{y}$$

$$= \ln(a^b\,\Gamma(b)) + \frac{\bar{y}}{a} - \frac{c}{a} - (b-1) E[\ln(y-c)] + \bar{y}$$

$$= \ln(a^b\,\Gamma(b)) + \frac{a+1}{a})\bar{y} - \frac{c}{a} - (b-1)\, E[\ln(y-c)] \tag{15.38}$$

which is identical to equation (15.37).

15.2 Parameter-Space Expansion Method

15.2.1 SPECIFICATION OF CONSTRAINTS

Following Singh and Rajagopal (1986), the constraints for this method are given by equation (15.8) and

$$\int_{e^c}^{\infty} [\ln x + \frac{\ln x - c}{a}]^{b-1} f(x)\, dx = E[\ln(\frac{\ln x - c}{a})^{b-1}] \tag{15.39}$$

$$\int_{e^c}^{\infty} \ln [\frac{\ln x - c}{a}]^{b-1} f(x)\, dx = E[\ln(\frac{\ln x - c}{a})^{b-1}] \tag{15.40}$$

15.2.2 DERIVATION OF ENTROPY FUNCTION

The pdf corresponding to the principle of maximum entropy (POME)and consistent with equations (15.8), (15.39), and (15.40) takes form

$$f(x) = \exp[-\lambda_0 - \lambda_1 \ln x - \lambda_1 (\frac{\ln x - c}{a}) - \lambda_2 \ln (\frac{\ln x - c}{a})^{b-1}] \tag{15.44}$$

where, λ_0, λ_1 and λ_2 are Lagrange multipliers. Insertion of equation (15.44) into equation (15.8) yields the partition function:

$$\begin{aligned} \exp(\lambda_0) &= \int_{e^c}^{\infty} \exp[-\lambda_1 \ln x - \lambda_1 (\frac{\ln x - c}{a}) - \lambda_2 \ln (\frac{\ln x - c}{a})^{b-1}]\, dx \\ &= a e^{c(1-\lambda_1)} (\frac{1}{\lambda_1 (1+a) - a})^{1-\lambda_2 (b-1)} \Gamma[1 - \lambda_2 (b-1))] \end{aligned} \tag{15.42}$$

The zeroth Lagrange multiplier is given by equation (15.42) as

$$\lambda_0 = \ln a + c(1-\lambda_1) - K \ln \alpha + \Gamma(K), K = 1 - \lambda_2 (b-1), \alpha = \lambda_1 (1+a) - a \tag{15.43}$$

Also, one gets the zeroth Lagrange multiplier from equation (15.28) as

$$\lambda_0 = \ln \int_{e^c}^{\infty} \exp\left[-\lambda_1 \ln x - \lambda_1 \left(\frac{\ln x - c}{a} \right) - \lambda_2 \ln \left(\frac{\ln x - c}{a} \right)^{b-1} \right] dx \tag{15.44}$$

Introduction of equation (15.43) in equation (15.41) yields

$$f(x) = \frac{e^{c(1-\lambda_1)}(\alpha)^K}{a\Gamma(K)} \exp\left[-\lambda_1 \ln x - \lambda_1 \frac{\ln x - c}{a} - \lambda_2 \ln \left(\frac{\ln x - c}{a} \right)^{b-1} \right] \tag{15.45}$$

A comparison of equation (15.45) with equation (15.1) shows $\lambda_1 = 1$, and $\lambda_2 = -1$.

Taking - logarithm of equation (15.45) leads to

$$\begin{aligned} -\ln f(x) = & -c(\lambda_1 - 1) + \ln a + \ln\Gamma(K) - K \ln\alpha + \lambda_1 \ln x + \lambda_1 \left(\frac{\ln x - c}{a} \right) \\ & + \lambda_2 \ln \left(\frac{\ln x - c}{a} \right)^{b-1} \end{aligned} \tag{15.46}$$

Therefore, the entropy function of the LP III distribution becomes

$$\begin{aligned} I(f) = & -c(\lambda_1 - 1) + \ln a + \ln\Gamma(K) - K \ln\alpha + \lambda_1 E\ \ln x] + \lambda_1 E\ \left(\frac{\ln x - c}{a} \right) \\ & + \lambda_2 E\left[\ln \left(\frac{\ln x - c}{a} \right)^{b-1} \right] \end{aligned} \tag{15.47}$$

15.2.3 RELATION BETWEEN PARAMETERS AND CONSTRAINTS

Taking partial derivatives of equation (15.47) with respect to λ_1, λ_2, a, and b separately, and equating each derivative to zero, one gets

$$\frac{\partial I}{\partial \lambda_1} = 0 = -c - K(1 + a)\psi(\alpha) + E[\ln x] + E\left(\frac{\ln x - c}{a} \right) \tag{15.48}$$

$$\frac{\partial I}{\partial \lambda_2} = 0 = -(b-1)\psi(K) + (b-1)\ln\alpha + E\left[\ln \left(\frac{\ln x - c}{a} \right)^{b-1} \right] \tag{15.49}$$

$$\frac{\partial I}{\partial \lambda_2} = 0 = -(b-1)\psi(K) + (b-1)\ln\alpha + E\,[\ln(\frac{\ln x - c}{a})^{b-1}] \qquad (15.49)$$

$$\frac{\partial I}{\partial a} = 0 = \frac{1}{a} - K(\lambda_1 - 1)\psi(\alpha)1 - \frac{\lambda_1}{a}E(\frac{\ln x - c}{a}) - \lambda_2\frac{b-1}{a} \qquad (15.50)$$

Simplification of equations (15.48) to (15.51) gives

$$E\,[\ln x] = c + a\,b\,\psi(1) \qquad (15.52)$$

$$E\,[\ln(\ln x - c)] = \ln a + \psi(b) \qquad (15.53)$$

$$E\,[\ln x] = c + ab \qquad (15.54)$$

$$E\,[\ln(\ln x - c)] = \ln a + \psi(b) \qquad (15.55)$$

Equations (15.53) and (15.55) are identical. The parameter estimation equations are equations (15.52) to (15.54).

15.3 Other Methods of Parameter Estimation

15.3.1 METHOD OF MOMENTS (DIRECT)

The direct method of moments (MOMD) (Bobee, 1975) uses sample estimates of moments of untransformed (real) data. Using equation (15.6a) we can write

$$\ln \mu'_1 = c - b\ln(1-a) \qquad (15.56)$$

$$\ln \mu'_2 = 2c - b\ln(1-2a) \qquad (15.57)$$

$$\ln \mu'_3 = 3c - b\ln(1-3a) \qquad (15.58)$$

Equations (15.56)-(15.58) can be rearranged to yield

$$\frac{\ln \mu'_3 - 3\ln \mu'_1}{\ln \mu'_2 - 2\ln \mu'_1} = \frac{3\ln(1-a) - \ln(1-3a)}{2\ln(1-a) - \ln(1-2a)} \; (= B,\ say) \qquad (15.59)$$

For a sample under consideration, $B = [\ln \mu_3' - 3\ln \mu_1'] / [\ln \mu_2' - 2\ln \mu_1']$ can be estimated from the sample estimates of the first three moments about the origin. The right side of equation(15.59), which is a function of parameter a only (say, B (a)), reveals that a is less than 1/3. In the limit, B(a) approaches infinity, 3, and 2, as a approaches 1/3, 0, and minus infinity, respectively. It should be possible to approximate the B(a) versus a relation by a series of polynomials, as for example discussed by Kite (1978). Then a good approximation of the sample estimate of a could directly be found from the sample estimate of B and should be good enough for most fitting problems. However, for purposes of simulation, a large number of (a-B(a)) points can be generated in the region a less than 1/3 (Bobee, 1975). Subsequently, a sample estimate of a can be interpolated corresponding to the sample estimate of the B value from the generated a-B(a) points, and refined using a method such as the Newton-Raphson method applied to equation (15.59). With the interpolated value of a being a good starting solution, the iterative scheme quickly converges to the true solution to a desired degree of significant accuracy. Parameters b and c can then be estimated using equations (15.56) and (15.57).

15.3.2 METHOD OF MOMENTS (INDIRECT)

The indirect method of moments (MOMI) is basically the method advocated by the U.S. Water Resource Council (1967). This method is applied to the log-transformed data. The method uses equations (15.6b) - (15.6d) and is described in Bulletin No. 15, 17A and 17 as well as by Rao (1980b) among others. Two variations of MOMI, designated as MOMI 1 and MOMI 2, were tested by Arora and Singh (1989a, b), which essentially differ in the sample skewness estimator used for the log-transformed data:

$$g_y = \frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} (y_i - \bar{y})^3 / s_y^3 \qquad (15.60)$$

$$g_y' = (1 + \frac{8.5}{n})\, g_y \qquad (15.61)$$

where n is the sample size, and $\bar{y}$ and S_y are the sample mean and standard deviation, respectively, of log-transformed data.

15.3.3 METHOD OF MIXED MOMENTS

Rao (1980b, 1983) proposed the method of mixed moments (MIX) for the LP III distribution, with the objective of obviating the use of the sample skewness coefficient in parameter estimation. After use of various combinations mixing the first two moments of the untransformed and log-transformed samples he found one particular combination to be preferable on the basis of sampling properties. This method conserves the sample mean and variance of the untransformed data and the sample mean of the log-transformed data. Thus, equations (15.6b), (15.6c), and (15.5a) are solved to estimate parameters a, b, and c. An improved method, as compared with the method described by Rao (1983), was developed by Arora and Singh (1989b), and will not be repeated here.

15.3.4 METHOD OF MAXIMUM LIKELIHOOD ESTIMATION

For the method of maximum likelihood estimation (MLE), the likelihood function of a sample of n observations drawn from a log-Pearson type III distribution can be expressed as

$$L = \frac{1}{\prod_{i=1}^{n}} [a\Gamma(b)]^{-n} \prod_{i=1}^{n} [\frac{\ln x_i - c}{a}]^{(b-1)} \exp[-\frac{1}{a}\sum_{i=1}^{n} (\ln(x_i - c)] \quad (15.62)$$

The log-transformed L becomes

$$\ln L = -n \ln a - n \ln \Gamma(b) - \sum_{i=1}^{n} x_i + (b-1) \sum_{i=1}^{n} ln\{[\ln x - c]/a\} - \frac{1}{a}\sum_{i=1}^{n} (\ln x_i - c)$$

(15.63)

Differentiating equation (15.63) with each respect to a, b, and c, and equating each derivative to zero produces the following:

$$\frac{\partial(\ln L)}{\partial a} = -nab + \sum_{i=1}^{n} (\ln x_i - c) = 0 \quad (15.64)$$

$$\frac{\partial(\ln L)}{\partial b} = -n\psi(b) + \sum_{i=1}^{n} [\frac{(\ln x_i - c)}{c}] = 0 \quad (15.65)$$

$$\frac{\partial(\ln L)}{\partial c} = \frac{n}{a} - (b-1)\sum_{i=1}^{n} \frac{1}{(\ln x_i - c)} = 0 \quad (15.66)$$

Equations (15.64) and (15.66) can be rearranged to give

$$a = \frac{s_1}{nb} \quad (15.67)$$

$$b = \frac{s_1 s_2}{(s_1 s_2 - n^2)}, \; s_1 = \sum_{i=1}^{n} (\ln x_i - c), \; s_2 = \sum_{i=1}^{n} \frac{1}{(\ln x_i - c)} \quad (15.68)$$

$$\ln L = -n \ln a - n \ln \Gamma(b) - \sum_{i=1}^{n} x_i + (b-1) \sum_{i=1}^{n} ln\{[\ln x - c]/a\}$$
$$-\frac{1}{a}\sum_{i=1}^{n} (\ln x_i - c) \quad (15.69)$$

For a specified value of c, parameters a and b can be explicitly found from equations (15.67) and (15.68), respectively. Substitution of these values of a, b, and c in equation (15.65) yields $\partial(\ln L)/\partial b = R\ (some\ residual\ value)$. The objective is to minimize R and this involves an iterative procedure. An efficient algorithm was developed by Arora and Singh (1988).

15.4 Comparative Evaluation of Estimation Methods

15.4.1 MONTE CARLO SIMULATION

Arora and Singh (1987a,b, 1989a, b) compared various methods of parameter esimation using Monte Carlo experiments. Noting that annual flood data generally lie in the area of the β-γ diagram delineated by $0.3 \prec \beta \prec 0.8$ and upward of 1 (Rossi et al., 1986; Wallis and Wood, 1985; Landwehr et al., 1978), they generated five cases of LP III population, representative of the real flood data, for Monte Carlo experiments. These cases are listed in Table 15.1. It is noted that $\lambda_1 \prec \lambda_{LN3} \prec \lambda_5 \prec \lambda_2 \prec \lambda_4 \prec \lambda_3$, where the subscripts of λ refer to the the LP III populations and subscript LN3 refers to the three-parameter lognormal population. For each of the population cases, 1000 random samples of size 10, 20, 30, 50, and 75 were generated, and parameters and quantiles were estimated using different parameter estimation methods. The 1000 estimated values of parameters and quantiles for each sample size and population cases were used to approximate the values of the standardized bias (BIAS), standard error (SE), and root mean square error (RMSE). Due to the limited number of random samples used, the results are not expected to reproduce the true values of BIAS, SE, RMSE, and robustness (Kuczera, 1982a, 1982b), but they do provide a means of comparing the performances of various estimation methods.

15.4.1.2 BIAS in Parameter Estimates: In general, unusually high BIAS was observed in estimates of parameters a, b, and c produced by all methods. MIX yielded considerably less bias than MOMD and was clearly superior to MOMD in terms of both mini-max BIAS and minimum average BIAS criteria. This was observed for most sample sizes and return periods.

Table 15.1 LP III population cases considered in sampling experiments (μ = 1) (after Arora and Singh, 1989a)

LPT III Population	Population Statistics			Parameters		
Cases	CV (β)	Skew (γ)	γ_y	a	b	c
Case 1	0.5	1	-0.45	-0.11832	19.82269	2.216713
Case 2	0.5	3	0.62	0.127683	10.30311	-1.407434
Case 3	0.5	5	1.12	0.205678	3.215257	-0.740366
Case 4	0.3	3	1.22	0.150978	2.681889	-0.438946
Case 5	0.7	3	0.20	0.059798	98.38009	-6.066213

15.4.1.2 RMSE in Parameter Estimates: There were wide differences in the RMSE performance of estimators, with the percent difference between the best and worst being as much as 425% for sample size of 10. Either MIX or MOMD provided the most favorable RMSE values. The MIX estimator was superior on the basis of the minimum-average RMSE criteria, and comparable to MOMD on the basis of mini-max RMSE criteria. Although MIX was expected to be the most resistant estimator, MOMD performed comparably. MIX and MOMD performed markedly superior to other methods. MOMI 1 performed poorly, as did MLE and POME.

15.4.1.3 Bias in Quantiles: In general, unusually high BIAS, SE and RMSE were observed for parameter estimates of a, b, and c of all methods. However, the intercorrelation among parameter estimates was such that reasonable quantile estimates were obtained. MOMD and MIX mostly underestimated the quantiles (negative bias), especially for T greater than 25. MIX consistently produced smaller BIAS than MOMD, and the difference became more pronounced at higher return periods. MOMI 1 and MOMI 2 mostly overestimated the quantiles (positive bias). Such trends were not discernible for MLE and POME. MOMI 1 mostly produced smaller absolute bias estimates than did MIX.

15.4.1.4 SE in Quantiles: In terms of standard error, MOMI 1 and MOMI 2 consistently produced higher standard error than other methods, especially MOMD and MIX. MOMI 2 fared worse than MOMI 1. MLE and POME seemed susceptible to smaller sample sizes, and in general produced higher standard error than other methods for such sample sizes. MIX and MOMD depicted remarkable stability even for smaller sample sizes when some of the other methods showed a deterioration in standard error. In general, MIX and MOMD outperformed other estimators in terms of SE for all population cases.

15.4.1.5 RMSE in Quantiles: As compared with other estimators, MOMI 1 and MOMI 2 performed poorly in terms of RMSE. While MLE and POME did perform well for some population cases and sample sizes, they depicted large deterioration in RMSE statistics for smaller sample sizes. MIX and MOMD consistently produced least or comparable RMSE estimates. MIX seemed to hold an edge over MOMD. Both of these estimators were remarkably stable for smaller sample sizes.

15.4.2 APPLICATION TO FIELD DATA

Singh and Singh (1988) compared POME, MOM, and MLE using annual maximum discharge data for six selected rivers. These data were selected on the basis of length, completeness, homogeneity, and independence of record. Each gaging station had a record length of more than 30 years. The methods were compared using relative mean error (RME) and relative absolute error (RAE).The parameter estimates obtained by POME and MLE were closer to each other than those for MOM. For two gaging stations observed and computed frequency curves are shown in Figures 15.1 and 15.2. The observed frequency curve was computed using the Gringorton plotting position formula. POME does not rquire the use of skewness whereas MOM does. In this way, bias is reduced when POME is used to estimate the LPT III parameters. For five of the six selected data sets, both RME and RAE yielded by POME were less than or equal to those of MLE. For only one data set, values of these measures were lower for MOM than those for POME, but the differences were marginal. For all six data sets, POME and MLE were found comparable.

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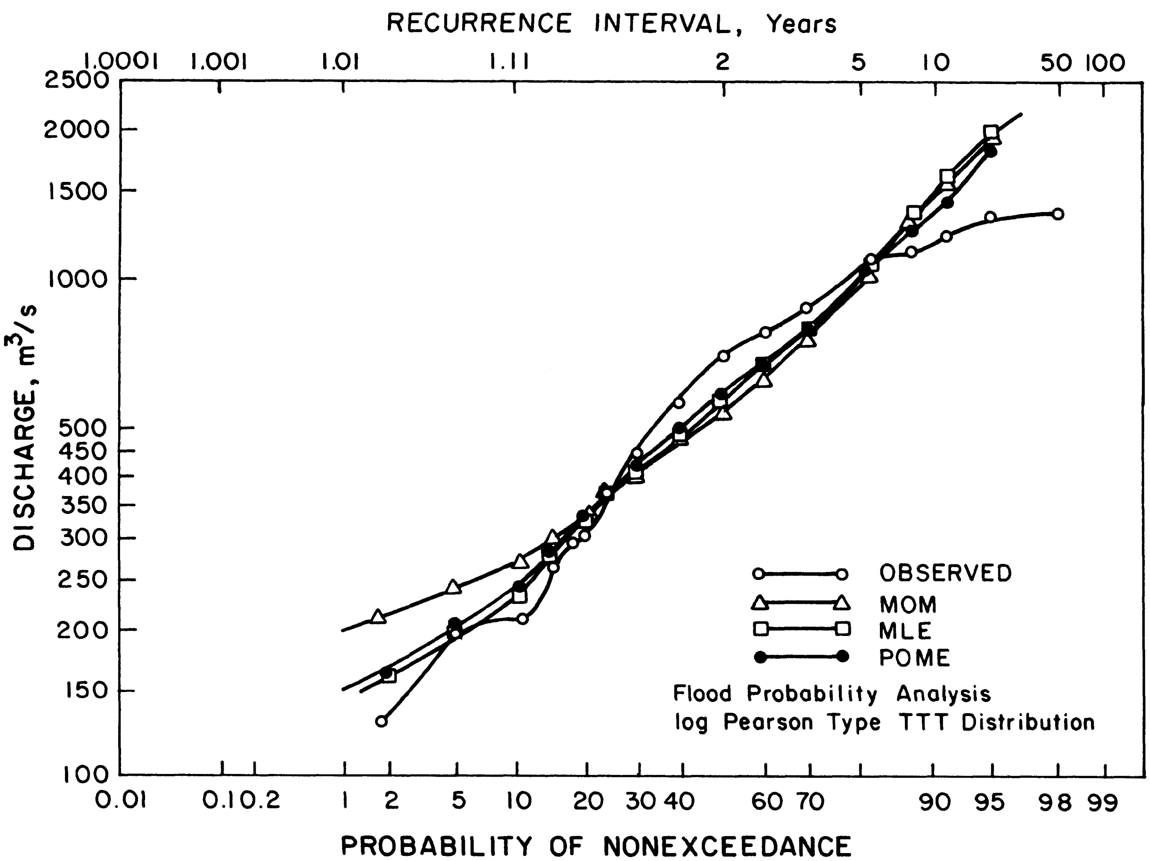

Figure 15.1 Comparison of observed and computed frequency curves using the POME, MLE and MOM methods for annual maximum discharge series for the Amite River basin at Magnoloia, Louisiana.

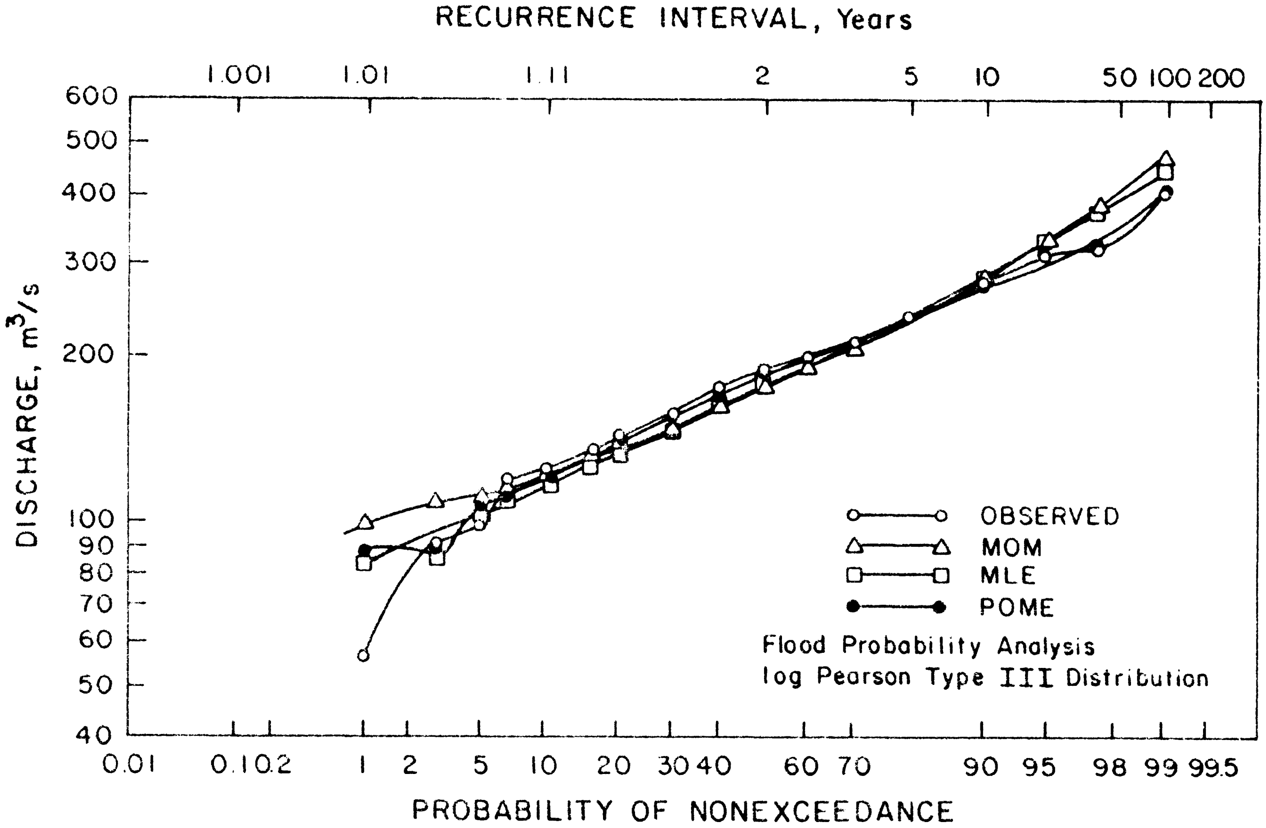

Figure 15.2 Comparison of observed and computed frequency curves using the POME, MLE and MOM methods for annual maximum discharge series for the Sebasticook River at Pittsfield, Maine.

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