北京師範大學

Theories and Observations of Dark Energy

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Contents

1 FRLW Universes

1.1 FRLW Metric

For a 3-D hyper-sphere,the distance between two points of space is:

$$ds^2 = f(r)dr^2 + r^2 d\theta + r^2 \sin^2\theta \tag{1.1.1}$$

Gaussian Curvature:

$$k = \frac{1}{2f^2(r)r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} \tag{1.1.2}$$

Solution:

$$f(r) = \frac{1}{C - kr^2} \tag{1.1.3}$$

Let $C = 1$,and introduce cosmological scale :

$$K = -\frac{k}{a^2} \tag{1.1.4}$$

$$-c^2 d\tau^2 = -c^2 dt^2 + a^2[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{1.1.5}$$

$$\mathrm{g}_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{bmatrix} \tag{1.1.6}$$

$$\mathrm{g}_{00} = -c^2, \quad \mathrm{g}_{11} = \frac{a^2}{1-Kr^2}, \quad \mathrm{g}_{22} = a^2r^2, \quad \mathrm{g}_{33} = a^2r^2\sin^2\theta \tag{1.1.7}$$

$$\mathrm{g}^{00} = -\frac{1}{c^2}, \quad \mathrm{g}^{11} = \frac{1-Kr^2}{a^2}, \quad \mathrm{g}^{22} = \frac{1}{a^2r^2}, \quad \mathrm{g}^{33} = \frac{1}{a^2r^2\sin^2\theta} \tag{1.1.8}$$

$$\mathrm{g}_{00,\gamma} = 0 \quad (\gamma = 0,1,2,3) \tag{1.1.9}$$

$$\mathrm{g}_{11,0} = \frac{1}{c^2}\frac{2a\dot{a}}{1-Kr^2}, \quad \mathrm{g}_{11,1} = \frac{2a^2Kr}{(1-Kr^2)^2}, \quad \mathrm{g}_{11,2} = \mathrm{g}_{11,3} = 0 \tag{1.1.10}$$

$$\mathrm{g}_{22,0} = 2a\dot{a}r^2, \quad \mathrm{g}_{22,1} = 2a^2r, \quad \mathrm{g}_{22,2} = \mathrm{g}_{22,3} = 0 \tag{1.1.11}$$

$$\mathrm{g}_{33,0} = 2a\dot{a}r^2\sin^2\theta, \quad \mathrm{g}_{33,1} = 2a^2r\sin^2\theta, \quad \mathrm{g}_{33,2} = 2a^2r\sin^2, \quad \mathrm{g}_{33,3} = 0 \tag{1.1.12}$$

1.2 Christoffel Symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\mathrm{g}^{\sigma\gamma}(\mathrm{g}_{\mu\gamma,\nu} + \mathrm{g}_{\nu\gamma,\mu} - \mathrm{g}_{\mu\nu,\gamma}) \tag{1.2.1}$$

Non-zero components of Christoffel Symbols

$$\Gamma^0_{11} = \frac{1}{2}g^{0\gamma}(g_{1\gamma,1} + g_{1\gamma,1} - g_{11,\gamma}) = -\frac{1}{2}g^{00}g_{11,0} = \frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2} \tag{1.2.2}$$

$$\Gamma^0_{22} = \frac{1}{2}g^{0\gamma}(g_{2\gamma,2} + g_{2\gamma,2} - g_{22,\gamma}) = -\frac{1}{2}g^{00}g_{22,0} = \frac{1}{c^2}a\dot{a}r^2 \tag{1.2.3}$$

$$\Gamma^0_{33} = \frac{1}{2}g^{0\gamma}(g_{3\gamma,3} + g_{3\gamma,3} - g_{33,\gamma}) = -\frac{1}{2}g^{00}g_{33,0} = \frac{1}{c^2}a\dot{a}r^2\sin^2\theta \tag{1.2.4}$$

$$\Gamma^i_{j0} = \frac{1}{2}g^{i\gamma}(g_{j\gamma,0} + g_{0\gamma,j} - g_{j0,\gamma}) = \frac{1}{2}g^{ii}g_{ij,0} = \frac{\dot{a}}{a}\delta^i_j \tag{1.2.5}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{1\gamma}(g_{1\gamma,1} + g_{1\gamma,1} - g_{11,\gamma}) = \frac{1}{2}g^{11}g_{11,1} = \frac{Kr}{1-Kr^2} \tag{1.2.6}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{1\gamma}(g_{2\gamma,2} + g_{2\gamma,2} - g_{22,\gamma}) = -\frac{1}{2}g^{11}g_{22,1} = -r(1-Kr^2) \tag{1.2.7}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{1\gamma}(g_{3\gamma,3} + g_{3\gamma,3} - g_{33,\gamma}) = -\frac{1}{2}g^{11}g_{33,1} = -r\sin^2\theta(1-Kr^2) \tag{1.2.8}$$

$$\Gamma^2_{12} = \frac{1}{2}g^{2\gamma}(g_{1\gamma,2} + g_{2\gamma,1} - g_{12,\gamma}) = \frac{1}{2}g^{22}g_{22,1} = \frac{1}{r} \tag{1.2.9}$$

$$\Gamma^3_{13} = \frac{1}{2}g^{3\gamma}(g_{1\gamma,3} + g_{3\gamma,1} - g_{13,\gamma}) = \frac{1}{2}g^{33}g_{33,1} = \frac{1}{r} \tag{1.2.10}$$

$$\Gamma^3_{23} = \frac{1}{2}g^{3\gamma}(g_{2\gamma,3} + g_{3\gamma,2} - g_{23,\gamma}) = \frac{1}{2}g^{33}g_{33,2} = -\sin\theta\cos\theta \tag{1.2.11}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{2\gamma}(g_{3\gamma,3} + g_{3\gamma,3} - g_{33,\gamma}) = \frac{1}{2}g^{22}g_{33,2} = \cot\theta \tag{1.2.12}$$

for$i, j = 1, 2, 3$

1.3 Ricci Tensor

$$R_{\mu\nu} = \Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma} \tag{1.3.1}$$

Non-zero components of Ricci Tensor:

$$\begin{aligned}
R_{00} &= \Gamma^\sigma_{00,\sigma} - \Gamma^\sigma_{0\sigma,0} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{00} - \Gamma^\sigma_{\rho0}\Gamma^\rho_{0\sigma} \\
&= \Gamma^0_{00,0} - \Gamma^\sigma_{0\sigma,0} + \Gamma^\sigma_{0\sigma}\Gamma^0_{00} - \Gamma^\sigma_{\sigma0}\Gamma^\sigma_{0\sigma} \\
&= 0 - 3\frac{\ddot{a}a - \dot{a}^2}{a^2} + 0 - 3(\frac{\dot{a}}{a})^2 \\
&= -3\frac{\ddot{a}}{a}
\end{aligned} \tag{1.3.2}$$

$$\begin{aligned}
R_{11} &= \Gamma^\sigma_{11,\sigma} - \Gamma^\sigma_{1\sigma,1} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{11} - \Gamma^\sigma_{\rho1}\Gamma^\rho_{1\sigma} \\
&= (\Gamma^1_{11,1} + \Gamma^2_{11,2} + \Gamma^3_{11,3}) - (\Gamma^0_{11,0} + \Gamma^1_{11,1}) \\
&\quad + (\Gamma^1_{10}\Gamma^0_{11} + \Gamma^2_{11}\Gamma^1_{10} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13}) \\
&\quad - (\Gamma^0_{11}\Gamma^1_{01} + \Gamma^0_{11}\Gamma^2_{02} + \Gamma^0_{11}\Gamma^3_{03} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^1_{11}\Gamma^3_{13}) \\
&= \frac{1}{c^2}[\frac{2}{r^2} + \frac{\dot{a}^2 + a\ddot{a}}{1-Kr^2} - \frac{2}{r^2} + \frac{\dot{a}^2}{1-Kr^2} + \frac{2K}{1-Kr^2}] \\
&= \frac{1}{c^2}\frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1-Kr^2} \\
&= \frac{1}{c^2}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2})g_{11}
\end{aligned} \tag{1.3.3}$$

We can get the following equations in using the same principle:

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{1.3.4}$$

$$R_{ij} = \frac{1}{c^2}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2})\mathrm{g}_{ij} \tag{1.3.5}$$

$$R = \frac{6}{c^2}(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}) \tag{1.3.6}$$

1.4 Friedmann Equation

Einstein Field Equation:

$$R_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}R + \Lambda\mathrm{g}_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.4.1}$$

Energy-Momentum Tensor:

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu + p\mathrm{g}_{\mu\nu} \tag{1.4.2}$$

Cosmological Principle:

$$T_{\mu\nu} = diag\{\rho c^2, p, p, p\} \tag{1.4.3}$$

According to the above equations:

$$(\frac{\dot{a}}{a})^2 + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho \tag{1.4.4}$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{Kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G}{c^2}p \tag{1.4.5}$$

Define Hubble parameter:

$$H = \frac{\dot{a}}{a} \tag{1.4.6}$$

Equation of State:

$$p = w\rho c^2 \tag{1.4.7}$$

Using 1.4.4,1.4.6 and 1.4.7,1.4.5 can be re-expressed as:

$$\dot{\rho} + 3(1+w)H\rho = 0 \tag{1.4.8}$$

Which is Continuous Equation.
Assume the expanding of universe is a Adiabatic Process, First Law of Thermodynamics:

$$dE + pdV = 0 \tag{1.4.9}$$

Where:

$$E = (\rho_m + \rho_r)V = \rho V, V \propto a^3 \tag{1.4.10}$$

1.4.9 can be re-expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) + p\frac{\mathrm{d}}{\mathrm{d}t}(a^3) = 0 \tag{1.4.11}$$

Particles of matter are non-relative,$w_m = 0$, while pressure p is mainly provided by radiation which means $w_r = 1/3$,so 1.4.8 can be re-expressed as:[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) + \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0 \tag{1.4.12}$$

Assume matter is strictly conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) = 0, \quad \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0 \tag{1.4.13}$$

Which means:

$$\rho_m \propto a^{-3}, \quad \rho_r \propto a^{-4} \tag{1.4.14}$$

Define Cosmological Critical Density:$\rho_c = \frac{3H_0^2}{8\pi G}$ **And 1.4.4 can be re-expressed as:**

$$\frac{H^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{K,0}a^{-2} + \Omega_{\Lambda,0} \tag{1.4.15}$$

Where:

$$\Omega_{r,0} = \frac{8\pi G\rho_{r0}}{3H_0^2}, \quad \Omega_{m,0} = \frac{8\pi G\rho_{m0}}{3H_0^2}, \quad \Omega_{K,0} = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2} \tag{1.4.16}$$

1.5 Cosmological Compositions Dominant Era:

Current Cosmological Parameter:

$$\begin{aligned} &\Omega_{m,0} = 0.3, \quad \Omega_{\Lambda,0} = 0.7, \quad \Omega_{r,0} = 10^{-5} \\ &H_0^{-1} = 14Gyr, \quad a_0 = 1 \end{aligned} \tag{1.5.1}$$

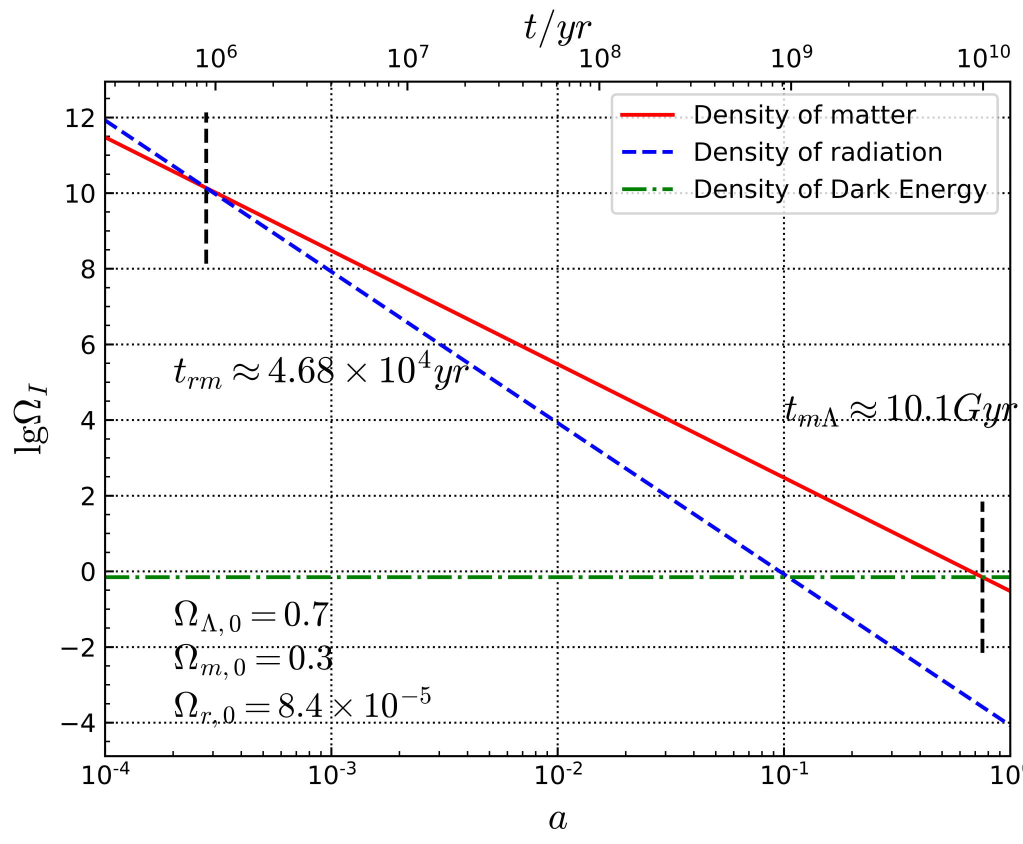

Figure 1: **The evolution of matter density and radiation density over time.,available at** `https://github.com/Wang-ZhengYi/DE_HW`

The evolution from Radiation Era to Matter Era:

$$\Omega_{r,0}\frac{a_{eq}^{-4}}{a_0^{-4}} = \Omega_{m,0}\frac{a_{eq}^{-3}}{a_0^{-3}} \tag{1.5.2}$$

$$a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.80 \times 10^{-4} \tag{1.5.3}$$

$$z_{eq} = \frac{a_0}{a_{eq}} - 1 \approx 3.57 \times 10^4 \tag{1.5.4}$$

$$\begin{aligned} t_{eq} &= \frac{1}{H_0}\int_{z_{eq}}^{\infty}\frac{dz}{(1+z)E(z)^{1/2}} \\ &= \frac{1}{H_0}\int_{z_{eq}}^{\infty}\frac{dz}{(1+z)\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}}} \\ &\approx 4.68 \times 10^4 yr \end{aligned} \tag{1.5.5}$$

The accelerating moment:

$$q_c = \frac{\Omega_m}{2} - \Omega_\Lambda = \frac{\Omega_{m,0}}{2}\frac{a_c^{-3}}{a_0^{-3}} - \Omega_{\Lambda,0} = 0 \tag{1.5.6}$$

$$a_c = (\frac{\Omega_{m,0}}{2\Omega_\Lambda})^{1/3} \approx 0.598 \tag{1.5.7}$$

$$z_c = \frac{a_0}{a_c} - 1 \approx 0.671 \tag{1.5.8}$$

$$\begin{aligned} t_c &= \frac{1}{H_0}\int_{z_c}^{\infty}\frac{dz}{(1+z)E(z)^{1/2}} \\ &\approx 7.54\mathrm{G}yr \end{aligned} \tag{1.5.9}$$

1.6 Big Bang Criterion

Critical condition of Big Bang Theory:[2]

$$H^2 = H_0^2[\Omega_{\Lambda,0} + (1-\Omega_{m,0}-\Omega_{\Lambda,0})a_c^{-2} + \Omega_{m,0}a_c^{-3}] = 0 \tag{1.6.1}$$

Which can be re-express as:

$$4(1-\Omega_{m,0}-\Omega_{\Lambda,0}) + 27\Omega_{m,0}^2\Omega_{\Lambda,0} = 0 \tag{1.6.2}$$

Introduce the following variable:

$$x = (\frac{\Omega_{\Lambda,0}}{4\Omega_{m,0}}) \tag{1.6.3}$$

1.6.2 can quickly reduce to:

$$x^3 - \frac{3}{4}x - \frac{1-\Omega_{m,0}}{\Omega_{m,0}} = 0 \tag{1.6.4}$$

Assume $\Omega_{\Lambda,0} > 0$,we can get 3 cases of 1.6.3:

- $0 < \Omega_{m,0} \leq \frac{1}{2}$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cosh[\frac{1}{3}\cosh^{-1}(\frac{1-\Omega_{m,0}}{\Omega_{m,0}})] \tag{1.6.5}$$

- $\frac{1}{2} < \Omega_{m,0} \leq 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cos[\frac{1}{3}\cos^{-1}(\frac{1-\Omega_{m,0}}{\Omega_{m,0}})] \tag{1.6.6}$$

- $\Omega_{m,0} > 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cos[\frac{1}{3}\cos^{-1}(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}) + \frac{4\pi}{3}] \tag{1.6.7}$$

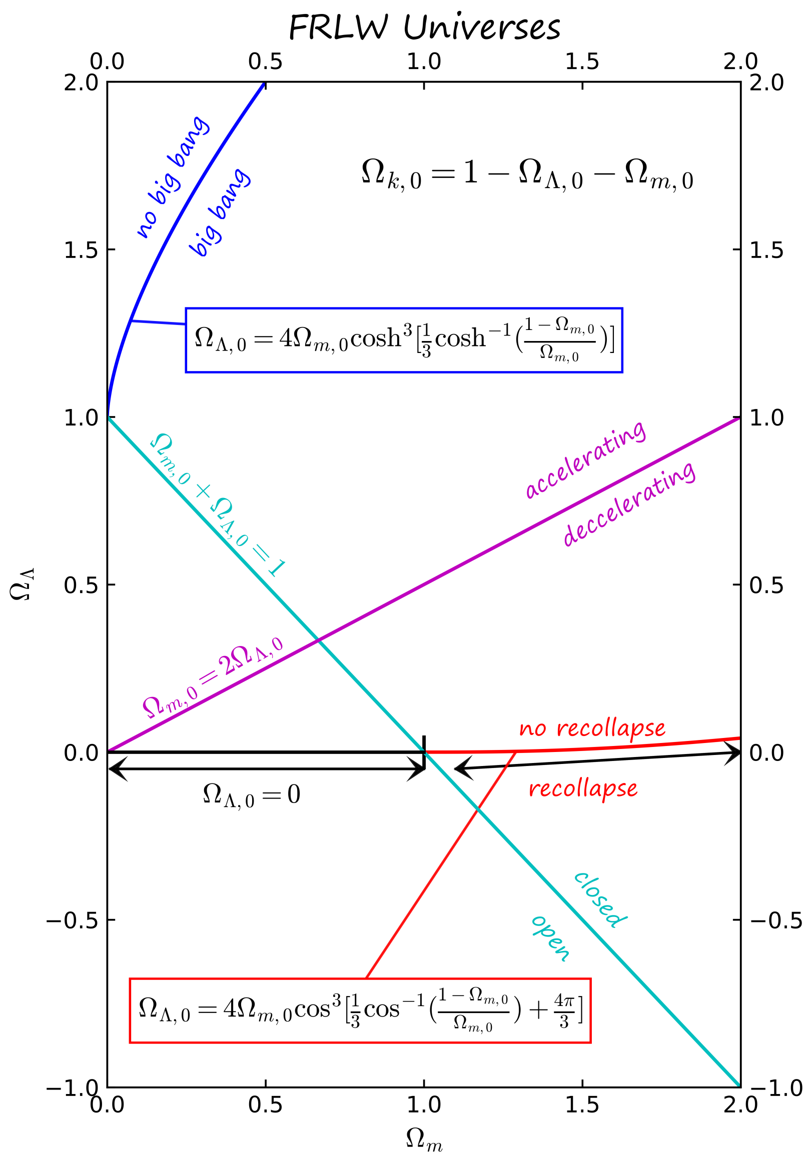

Figure 2: **FRLW universes, differences of the current density of matter and dark energy make different universes, available at** https://github.com/Wang-ZhengYi/DE_HW

The scale factor a must be negative in the past if point $(\Omega_{m,0}, \Omega_{\Lambda,0})$ is above the blue soild line in Figure 2.

2 Perturbation Theory

2.1 The Conform Newton Gauge:

$$-c^2 d\tau^2 = a^2(\eta)[-(1+2\psi)d\eta^2 + (1-2\phi)\delta_{\alpha\beta}dx^\alpha dx^\beta] \tag{2.1.1}$$

$$g_{\mu\nu} = a^2(\eta)\begin{bmatrix} -(1+2\psi) & 0 & 0 & 0 \\ 0 & 1-2\phi & 0 & 0 \\ 0 & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{bmatrix} \tag{2.1.2}$$

2.2 Christoffel Symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\gamma}(g_{\mu\gamma,\nu} + g_{\nu\gamma,\mu} - g_{\mu\nu,\gamma}) \tag{1.2.1}$$

Non-zero components of Christoffel Symbols

$$\Gamma^0_{00} = H + \dot{\psi} \tag{2.1.3}$$
$$\Gamma^0_{0i} = \psi_{,i} \tag{2.1.4}$$
$$\Gamma^0_{ij} = [H - \dot{\phi} - 2H(\psi+\phi)]\delta_{ij} \tag{2.1.5}$$
$$\Gamma^i_{00} = \frac{1}{a^2}\psi_{,i} \tag{2.1.6}$$
$$\Gamma^k_{i0} = (H - \dot{\phi})\delta_{ik} \tag{2.1.7}$$
$$\Gamma^k_{ij} = -\phi_{,j}\delta_{ik} - \phi_{,i}\delta_{jk} + \phi_{,k}\delta_{ij} \tag{2.1.8}$$

2.3 Ricci Tensor

$$R_{\mu\nu} = \Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma} \tag{1.3.1}$$

Non-zero components of Ricci Tensor:

$$\begin{aligned} R_{0i} &= \Gamma^\sigma_{0i,\sigma} - \Gamma^\sigma_{0\sigma,i} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{0i} - \Gamma^\sigma_{\rho i}\Gamma^\rho_{0\sigma} \\ &= \Gamma^0_{0i,0} + \Gamma^j_{0i,j} - (\Gamma^0_{00,i} + \Gamma^j_{0j,i}) + (\Gamma^0_{00}\Gamma^0_{0i} + \Gamma^j_{0j}\Gamma^0_{0i} + \Gamma^0_{j0}\Gamma^j_{0i} \\ &+ \Gamma^k_{jk}\Gamma^j_{0i}) - (\Gamma^0_{0i}\Gamma^0_{00} + \Gamma^j_{0i}\Gamma^0_{0j} + \Gamma^0_{ji}\Gamma^j_{00} + \Gamma^k_{ji}\Gamma^j_{0k}) \\ &= \dot{\psi}_{,i} - \dot{\phi}_{,i} - (\dot{\psi}_{,i} - 3\dot{\phi}_{,i}) + 3H\dot{\psi}_{,i} + 3H\dot{\phi}_{,i} - (H\dot{\psi}_{,i} + 3H\dot{\phi}_{,i}) \\ &= 2\dot{\phi}_{,i} + 2H\dot{\psi}_{,i} \end{aligned} \tag{2.1.9}$$

We can get the following equations in using the same principle:

$$R_{00} = -3\frac{\ddot{a}}{a} + \frac{1}{a}\psi_{,ii} + 3\ddot{\phi} + 3H(\dot{\psi} + 2\dot{\phi}) \tag{2.1.10}$$

$$\begin{aligned} R_{ij} &= \delta_{ij}[(2\dot{a}^2 + a\dot{a})(1 - 2\phi + 2\psi) - a^2 H(6\dot{\phi} + \dot{\psi}) \\ &\quad - a^2\ddot{\phi} + \phi_{,ii}] - (\psi - \phi)_{,ij} \end{aligned} \tag{2.1.11}$$

$$\begin{aligned} R &= 6(H^2 + \frac{\dot{a}}{a})(1 - 2\psi) - \frac{2}{a^2}\psi_{,ii} \\ &\quad - 6\ddot{\phi} - 6H(\dot{\psi} + 4\dot{\phi}) + \frac{4}{a^2}\phi_{,ii} \end{aligned} \tag{2.1.12}$$

3 Dark Energy models(Dark Energy Field)

3.1 Quintessence

The action of calibration ϕ:

$$S = -\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right] \tag{3.1.1}$$

Lagrangian:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) \tag{3.1.2}$$

Energy-Momentum Tensor:

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-\mathrm{g}}}\frac{\delta S}{\delta \mathrm{g}^{\mu\nu}} \\ &= \partial_\mu\phi\partial_\nu\phi - \mathrm{g}_{\mu\nu}\left[\frac{1}{2}\mathrm{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] \end{aligned} \tag{3.1.3}$$

Cosmological principle:

$$\rho = -\frac{1}{c^2}T^0_0 = \frac{1}{c^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{3.1.4}$$

$$p = T^i_i = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.1.5}$$

Continuous Equation:

$$\begin{aligned} &\dot{\rho} + 3H(\rho + \frac{p}{c^2}) \\ =&\frac{1}{c^2}\left[\dot{\phi}\ddot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\dot{\phi} + 3H\dot{\phi}^2\right] \\ =&\frac{\dot{\phi}}{c^2}(\ddot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} + 3H\dot{\phi}) \\ =&0 \end{aligned} \tag{3.1.6}$$

That is:

$$\ddot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} + 3H\dot{\phi} = 0 \tag{3.1.7}$$

3.2 Quintessence Calibration Field

Lagrangian:

$$\mathscr{L} = \frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi) \tag{3.2.1}$$

Hubble parameter and its time derivation:

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi) + \rho_m\right] \tag{3.2.2}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left[\varepsilon\dot{\phi}^2 + (1+w_m)\rho_m\right] \tag{3.2.3}$$

Where:$\kappa^2 = 8\pi G$,$\varepsilon = \pm 1$
Continuous Equation:

$$\varepsilon(\ddot{\phi} + 3H\dot{\phi}) + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \tag{3.2.4}$$

Introduce the following variables:

$$N = \ln a, \lambda = -\frac{1}{\kappa V}\frac{\mathrm{d}V}{\mathrm{d}\phi}, x = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, y = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \tag{3.2.5}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}N} &= \frac{1}{H}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\kappa}{\sqrt{3}H}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sqrt{V}}{H}\right) \\
&= \frac{\kappa}{\sqrt{3}H}\frac{\frac{1}{2\sqrt{V}}\frac{\mathrm{d}V}{\mathrm{d}\phi}\dot{\phi}H - \dot{H}\sqrt{V}}{H^2} \\
&= -\frac{\sqrt{6}}{2}\left(-\frac{1}{\kappa V}\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)\left(\frac{\kappa\dot{\phi}}{\sqrt{6}H}\right)\left(\frac{\kappa\sqrt{V}}{\sqrt{3}H}\right) - \frac{\kappa\sqrt{V}}{\sqrt{3}H}\frac{\dot{H}}{H^2} \\
&= -\frac{\sqrt{6}}{2}\lambda xy + \frac{\kappa^2}{2}y\frac{\varepsilon\dot{\phi}^2 + (1+w_m)\rho_m}{H^2} \\
&= -\frac{\sqrt{6}}{2}\lambda xy + \frac{1}{2}y\left[\frac{\varepsilon\kappa^2\dot{\phi}^2}{H^2} + \frac{\kappa^2(1+w_m)\rho_m}{H^2}\right] \\
&= -\frac{\sqrt{6}}{2}\lambda xy + \frac{1}{2}y\left[\frac{\varepsilon\kappa^2\dot{\phi}^2}{H^2} + \frac{\kappa^2(1+w_m)\left(\frac{3H^2}{\kappa^2} - \frac{1}{2}\varepsilon\dot{\phi}^2 - V\right)}{H^2}\right] \\
&= -\frac{\sqrt{6}}{2}\lambda xy + \frac{1}{2}y\left[6\frac{\varepsilon\kappa^2\dot{\phi}^2}{6H^2} + 3(1+w_m) - 3(1+w_m)\left(\frac{\varepsilon\kappa^2\dot{\phi}^2}{6H^2} + \frac{\kappa^2 V}{3H}\right)\right] \\
&= -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)\varepsilon x^2 + (1+w_m)(1-y^2)]
\end{aligned} \tag{3.2.6}$$

4 Density Evolution

4.1 Energy density perturbation

Perturbation scale factor

$$a_p = a\left(1 - \frac{\delta}{3}\right) \tag{4.1.1}$$

Friedmann Equation without Dark Energy

$$
\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{H^2}{2}\Omega_{m,0}a^{-3} && (4.1.2)\\
\frac{\ddot{a}_p}{a_p} &= \frac{\ddot{a}(1-\frac{\delta}{3}) - \frac{1}{3}(2\dot{a}\dot{\delta} + a\ddot{\delta})}{a(1-\frac{\delta}{3})}\\
&= \frac{\ddot{a}}{a} - \frac{\ddot{\delta} + 2H\dot{\delta}}{3-\delta}\\
&= -\frac{H^2}{2}\Omega_{m,0}a^{-3} - \frac{\ddot{\delta} + 2H\dot{\delta}}{3-\delta}\\
&= -\frac{H^2}{2}\Omega_m(a)\frac{1}{(1-\frac{\delta}{3})^3}\\
&\approx -\frac{H^2}{2}\Omega_m(a)(1+\delta) && (4.1.3)
\end{aligned}
$$

Ignore second order small quantity

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\Omega_m(a)\delta = 0 \tag{4.1.4}$$

Which is Perturbation Evolution Equation

4.2 Linear Growth Factor

Friedmann Equation with Dark Energy

$$
\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{H^2}{2}[\Omega_m(a) + (1+3w(a))(1-\Omega_m(a))] && (4.2.1)\\
\frac{\ddot{a}_p}{a_p} &= \frac{\ddot{a}(1-\frac{\delta}{3}) - \frac{1}{3}(2\dot{a}\dot{\delta} + a\ddot{\delta})}{a(1-\frac{\delta}{3})}\\
&= \frac{\ddot{a}}{a} - \frac{\ddot{\delta} + 2H\dot{\delta}}{3-\delta}\\
&= -\frac{H^2}{2}[\Omega_m(a) + (1+3w(a))(1-\Omega_m(a))] - \frac{\ddot{\delta} + 2H\dot{\delta}}{3-\delta}\\
&= -\frac{H^2}{2}[\Omega_m(a)\frac{1}{(1-\frac{\delta}{3})^3} + (1+3w(a))(1-\Omega_m(a)\frac{1}{(1-\frac{\delta}{3})^3})]\\
&\approx -\frac{H^2}{2}[\Omega_m(a)(1+\delta) + (1+3w(a))(1-\Omega_m(a)(1+\delta))] && (4.2.2)
\end{aligned}
$$

Define $X(a)$:

$$X(a) = \frac{\Omega_m}{1-\Omega_m} = \frac{\Omega_{m,0}}{1-\Omega_{m,0}}\exp[-3\int_a^{a_0} \mathrm{d}\ln a' w(a')] = \Omega_m a^{-3}\left(\frac{\delta H^2}{H_0^2}\right)^{-1} \tag{4.2.3}$$

Growth Factor

$$D = \frac{\delta(a)}{\delta(a_0)} \tag{4.2.4}$$

And its first order derivative and second order derivative:

$$D^{'} = \frac{1}{aH}\frac{\dot{\delta}(a)}{\delta(a_0)} \tag{4.2.5}$$

$$D^{''} = \frac{1}{a^2H^2}\frac{\ddot{\delta}(a) - \dot{\delta}(a)\frac{\ddot{a}}{a}}{\delta(a_0)} \tag{4.2.6}$$

We can get the following equation from 4.2.1-4.2.6:

$$D^{''} + \frac{3}{2}\left[1 - \frac{w(a)}{1+X(a)}\right]\frac{D^{'}}{a} - \frac{3}{2}\frac{X(a)}{1+X(a)}\frac{D}{a^2} = 0 \tag{4.2.7}$$

Linear Growth Factor

$$G = \frac{D}{a} \tag{4.2.8}$$

And its first order derivative and second order derivative:

$$\begin{aligned} G^{'} &= \frac{aD^{'} - D}{a^2} \\ &= \frac{1}{a^2}\left(\frac{1}{H}\frac{\dot{\delta}(a)}{\delta(a_0)} - \frac{\delta(a)}{\delta(a_0)}\right) \end{aligned} \tag{4.2.9}$$

$$\begin{aligned} G^{''} &= \frac{1}{a^2}(a^2D^{''} - 2aD^{'} + 2D) \\ &= \frac{1}{a^3}\left(\frac{1}{H^2}\frac{\ddot{\delta}(a) - \dot{\delta}(a)\frac{\ddot{a}}{a}}{\delta(a_0)} - \frac{2}{H}\frac{\dot{\delta}(a)}{\delta(a_0)} + \frac{2\delta(a)}{\delta(a_0)}\right) \end{aligned} \tag{4.2.10}$$

We can get the following equation from 4.2.7-4.2.10:

$$G^{''} + \left[\frac{7}{2} - \frac{3}{2}\frac{w(a)}{1+X(a)}\right]\frac{G^{'}}{a} + \frac{3}{2}\frac{1-w(a)}{1+X(a)}\frac{G}{a^2} = 0 \tag{4.2.11}$$

4.3 Numerical Simulation

4.2.7 in ΛCDM model:

$$G(a) \simeq \frac{5\Omega_m(a)}{2}\left[\Omega_m(a)^{4/7} - \Omega_\Lambda(a) + \left(1 + \frac{\Omega_m(a)}{2}\right)\left(1 + \frac{\Omega_\Lambda(a)}{70}\right)\right]^{-1} \tag{4.3.1}$$

w models:[1]

$$w(a) = w_0 + aw^{'} \tag{4.3.2}$$

$$\begin{aligned} X(a) &= \frac{\Omega_{m,0}}{1-\Omega_{m,0}}\exp[-3\int_a^1 \mathrm{d}\ln a^{'} w(a^{'})] \\ &= \frac{\Omega_{m,0}}{1-\Omega_{m,0}} a^{3w_0}\exp[-3w^{'}(1-a)] \end{aligned} \tag{4.3.3}$$

And:

$$\begin{aligned} G^{''} &= -\left[\frac{7}{2} - \frac{3}{2}\frac{w(a)}{1+X(a)}\right]\frac{G^{'}}{a} - \frac{3}{2}\frac{1-w(a)}{1+X(a)}\frac{G}{a^2} \\ &= A(a)G^{'} + B(a)G \end{aligned} \tag{4.3.4}$$

Euler Method

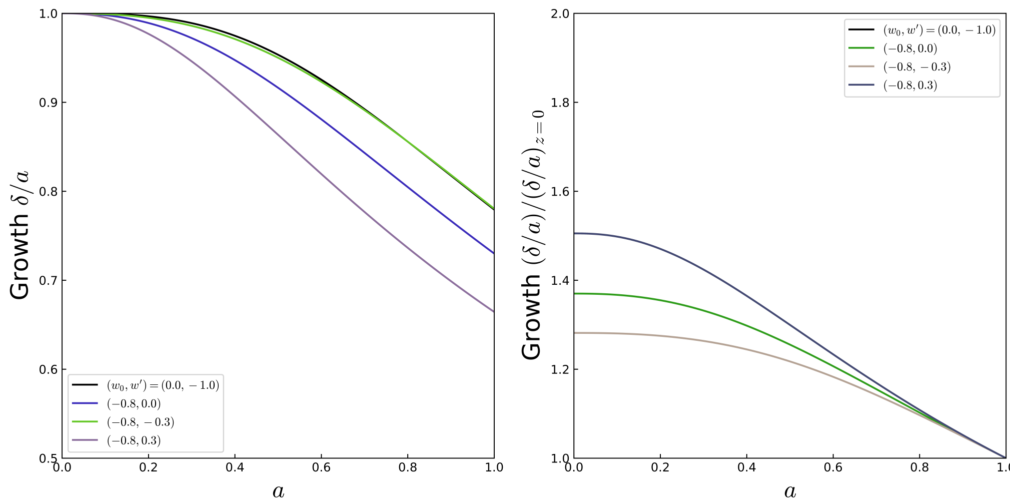

Figure 3: **Linear growth factor and its normalized form, available at** https://github.com/Wang-ZhengYi/DE_HW

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