北京師範大學

Dark Energy Theorem and Observation

Wang Zhengyi 201711160128

1 Dark Energy Theorem

1.1 FRLW Metric

For a 3-D hyper-sphere,the distance between two points of space is:

$$ds^2 = f(r)dr^2 + r^2 d\theta + r^2 \sin^2\theta \tag{1.1.1}$$

Gaussian Curvature:

$$k = \frac{1}{2f^2(r)r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} \tag{1.1.2}$$

Solution:

$$f(r) = \frac{1}{C - kr^2} \tag{1.1.3}$$

Let C=1,and introduce cosmological scale :

$$K = -\frac{k}{a^2} \tag{1.1.4}$$

$$-c^2 d\tau^2 = -c^2 dt^2 + a^2[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{1.1.5}$$

$$\mathrm{g}_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{bmatrix} \tag{1.1.6}$$

$$\mathrm{g}_{00} = -c^2, \quad \mathrm{g}_{11} = \frac{a^2}{1-Kr^2}, \quad \mathrm{g}_{22} = a^2r^2, \quad \mathrm{g}_{33} = a^2r^2\sin^2\theta \tag{1.1.7}$$

$$\mathrm{g}^{00} = -\frac{1}{c^2}, \quad \mathrm{g}^{11} = \frac{1-Kr^2}{a^2}, \quad \mathrm{g}^{22} = \frac{1}{a^2r^2}, \quad \mathrm{g}^{33} = \frac{1}{a^2r^2\sin^2\theta} \tag{1.1.8}$$

$$\mathrm{g}_{00,\gamma} = 0 \quad (\gamma = 0,1,2,3) \tag{1.1.9}$$

$$\mathrm{g}_{11,0} = \frac{1}{c^2}\frac{2a\dot{a}}{1-Kr^2}, \quad \mathrm{g}_{11,1} = \frac{2a^2Kr}{(1-Kr^2)^2}, \quad \mathrm{g}_{11,2} = \mathrm{g}_{11,3} = 0 \tag{1.1.10}$$

$$\mathrm{g}_{22,0} = 2a\dot{a}r^2, \quad \mathrm{g}_{22,1} = 2a^2r, \quad \mathrm{g}_{22,2} = \mathrm{g}_{22,3} = 0 \tag{1.1.11}$$

$$\mathrm{g}_{33,0} = 2a\dot{a}r^2\sin^2\theta, \quad \mathrm{g}_{33,1} = 2a^2r\sin^2\theta, \quad \mathrm{g}_{33,2} = 2a^2r\sin^2, \quad \mathrm{g}_{33,3} = 0 \tag{1.1.12}$$

1.2 Christoffel Symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\mathrm{g}^{\sigma\gamma}(\mathrm{g}_{\mu\gamma,\nu} + \mathrm{g}_{\nu\gamma,\mu} - \mathrm{g}_{\mu\nu,\gamma}) \tag{1.2.1}$$

Non-Zero Components of Christoffel Symbols:

$$\begin{aligned}\Gamma^0_{11} &= \frac{1}{2}\mathrm{g}^{0\gamma}(\mathrm{g}_{1\gamma,1}+\mathrm{g}_{1\gamma,1}-\mathrm{g}_{11,\gamma}) \\ &= \quad -\frac{1}{2}\mathrm{g}^{00}\mathrm{g}_{11,0} = \frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2}\end{aligned} \tag{1.2.2}$$

$$\begin{aligned}\Gamma^0_{22} &= \frac{1}{2}\mathrm{g}^{0\gamma}(\mathrm{g}_{2\gamma,2}+\mathrm{g}_{2\gamma,2}-\mathrm{g}_{22,\gamma}) \\ &= -\frac{1}{2}\mathrm{g}^{00}\mathrm{g}_{22,0} = \frac{1}{c^2}a\dot{a}r^2\end{aligned} \tag{1.2.3}$$

$$\begin{aligned}\Gamma^0_{33} &= \frac{1}{2}\mathrm{g}^{0\gamma}(\mathrm{g}_{3\gamma,3}+\mathrm{g}_{3\gamma,3}-\mathrm{g}_{33,\gamma}) \\ &= -\frac{1}{2}\mathrm{g}^{00}\mathrm{g}_{33,0} = \frac{1}{c^2}a\dot{a}r^2\sin^2\theta\end{aligned} \tag{1.2.4}$$

$$\begin{aligned}\Gamma^i_{j0} &= \frac{1}{2}\mathrm{g}^{i\gamma}(\mathrm{g}_{j\gamma,0}+\mathrm{g}_{0\gamma,j}-\mathrm{g}_{j0,\gamma}) \\ &= \frac{1}{2}\mathrm{g}^{ii}\mathrm{g}_{ij,0} = \frac{\dot{a}}{a}\delta^i_j\end{aligned} \tag{1.2.5}$$

$$\begin{aligned}\Gamma^1_{11} &= \frac{1}{2}\mathrm{g}^{1\gamma}(\mathrm{g}_{1\gamma,1}+\mathrm{g}_{1\gamma,1}-\mathrm{g}_{11,\gamma}) \\ &= \frac{1}{2}\mathrm{g}^{11}\mathrm{g}_{11,1} = \frac{Kr}{1-Kr^2}\end{aligned} \tag{1.2.6}$$

$$\begin{aligned}\Gamma^1_{22} &= \frac{1}{2}\mathrm{g}^{1\gamma}(\mathrm{g}_{2\gamma,2}+\mathrm{g}_{2\gamma,2}-\mathrm{g}_{22,\gamma}) \\ &= -\frac{1}{2}\mathrm{g}^{11}\mathrm{g}_{22,1} = -r(1-Kr^2)\end{aligned} \tag{1.2.7}$$

$$\begin{aligned}\Gamma^1_{33} &= \frac{1}{2}\mathrm{g}^{1\gamma}(\mathrm{g}_{3\gamma,3}+\mathrm{g}_{3\gamma,3}-\mathrm{g}_{33,\gamma}) \\ &= -\frac{1}{2}\mathrm{g}^{11}\mathrm{g}_{33,1} = -r\sin^2\theta(1-Kr^2)\end{aligned} \tag{1.2.8}$$

$$\begin{aligned}\Gamma^2_{12} &= \frac{1}{2}\mathrm{g}^{2\gamma}(\mathrm{g}_{1\gamma,2}+\mathrm{g}_{2\gamma,1}-\mathrm{g}_{12,\gamma}) \\ &= \frac{1}{2}\mathrm{g}^{22}\mathrm{g}_{22,1} = \frac{1}{r}\end{aligned} \tag{1.2.9}$$

$$\begin{aligned}\Gamma^3_{13} &= \frac{1}{2}\mathrm{g}^{3\gamma}(\mathrm{g}_{1\gamma,3}+\mathrm{g}_{3\gamma,1}-\mathrm{g}_{13,\gamma}) \\ &= \frac{1}{2}\mathrm{g}^{33}\mathrm{g}_{33,1} = \frac{1}{r}\end{aligned} \tag{1.2.10}$$

$$\begin{aligned}\Gamma^3_{23} &= \frac{1}{2}\mathrm{g}^{3\gamma}(\mathrm{g}_{2\gamma,3}+\mathrm{g}_{3\gamma,2}-\mathrm{g}_{23,\gamma}) \\ &= \frac{1}{2}\mathrm{g}^{33}\mathrm{g}_{33,2} = -\sin\theta\cos\theta\end{aligned} \tag{1.2.11}$$

$$\begin{aligned}\Gamma^2_{33} &= \frac{1}{2}\mathrm{g}^{2\gamma}(\mathrm{g}_{3\gamma,3}+\mathrm{g}_{3\gamma,3}-\mathrm{g}_{33,\gamma}) \\ &= \frac{1}{2}\mathrm{g}^{22}\mathrm{g}_{33,2} = \cot\theta\end{aligned} \tag{1.2.12}$$

for$i, j = 1, 2, 3$

1.3 Ricci Tensor

$$R_{\mu\nu} = \Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma} \tag{1.3.1}$$

Non-Zero Components of Ricci Tensor:

$$
\begin{aligned}
R_{00} &= \Gamma^{\sigma}_{00,\sigma} - \Gamma^{\sigma}_{0\sigma,0} + \Gamma^{\sigma}_{\rho\sigma}\Gamma^{\rho}_{00} - \Gamma^{\sigma}_{\rho 0}\Gamma^{\rho}_{0\sigma} \\
&= \Gamma^{0}_{00,0} - \Gamma^{\sigma}_{0\sigma,0} + \Gamma^{\sigma}_{0\sigma}\Gamma^{0}_{00} - \Gamma^{\sigma}_{\sigma 0}\Gamma^{\sigma}_{0\sigma} \\
&= 0 - 3\frac{\ddot{a}a - \dot{a}^2}{a^2} + 0 - 3(\frac{\dot{a}}{a})^2 \\
&= -3\frac{\ddot{a}}{a} \qquad (1.3.2)
\end{aligned}
$$

$$
\begin{aligned}
R_{11} &= \Gamma^{\sigma}_{11,\sigma} - \Gamma^{\sigma}_{1\sigma,1} + \Gamma^{\sigma}_{\rho\sigma}\Gamma^{\rho}_{11} - \Gamma^{\sigma}_{\rho 1}\Gamma^{\rho}_{1\sigma} \\
&= (\Gamma^{1}_{11,1} + \Gamma^{2}_{11,2} + \Gamma^{3}_{11,3}) - (\Gamma^{0}_{11,0} + \Gamma^{1}_{11,1}) \\
&\quad + (\Gamma^{1}_{10}\Gamma^{0}_{11} + \Gamma^{2}_{11}\Gamma^{1}_{10} + \Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{2}_{12}\Gamma^{2}_{12} + \Gamma^{3}_{13}\Gamma^{3}_{13}) \\
&\quad - (\Gamma^{0}_{11}\Gamma^{1}_{01} + \Gamma^{0}_{11}\Gamma^{2}_{02} + \Gamma^{0}_{11}\Gamma^{3}_{03} + \Gamma^{1}_{11}\Gamma^{1}_{11} + \Gamma^{1}_{11}\Gamma^{2}_{12} + \Gamma^{1}_{11}\Gamma^{3}_{13}) \\
&= \frac{1}{c^2}[\frac{2}{r^2} + \frac{\dot{a}^2 + a\ddot{a}}{1 - Kr^2} - \frac{2}{r^2} + \frac{\dot{a}^2}{1 - Kr^2} + \frac{2K}{1 - Kr^2}] \\
&= \frac{1}{c^2}\frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1 - Kr^2} \\
&= \frac{1}{c^2}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2})\mathrm{g}_{11} \qquad (1.3.3)
\end{aligned}
$$

We can get the following equations by using the same principle:

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{1.3.4}$$

$$R_{ij} = \frac{1}{c^2}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2})\mathrm{g}_{ij} \tag{1.3.5}$$

$$R = \frac{6}{c^2}(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}) \tag{1.3.6}$$

1.4 Friedmann Equation

Einstein Field Equation:

$$R_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}R + \Lambda\mathrm{g}_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.4.1}$$

Energy-Momentum Tensor:

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu + p\mathrm{g}_{\mu\nu} \tag{1.4.2}$$

Cosmological Principle:

$$T_{\mu\nu} = diag\{\rho c^2, p, p, p\} \tag{1.4.3}$$

According to the above equations:

$$(\frac{\dot{a}}{a})^2 + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho \tag{1.4.4}$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{Kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G}{c^2}p \tag{1.4.5}$$

Define Hubble parameter:

$$H = \frac{\dot{a}}{a} \tag{1.4.6}$$

Equation of State:

$$p = w\rho c^2 \tag{1.4.7}$$

Using 1.4.4 ,1.4.6 and 1.4.7 ,1.4.5 can be re-expressed as:

$$\dot{\rho} + 3(1+w)H\rho = 0 \tag{1.4.8}$$

Which is Continuous Equation.
Assume the expanding of universe is a Adiabatic Process, First Law of Thermodynamics:

$$dE + pdV = 0 \tag{1.4.9}$$

Where:

$$E = (\rho_m + \rho_r)V = \rho V, V \propto a^3 \tag{1.4.10}$$

1.4.9 can be re-expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) + p\frac{\mathrm{d}}{\mathrm{d}t}(a^3) = 0 \tag{1.4.11}$$

Particles of matter are non-relative,$w_m = 0$, while pressure p is mainly provided by radiation which means $w_r = 1/3$,so 1.4.8 can be re-expressed as:[2]

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) + \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0 \tag{1.4.12}$$

Assume matter is strictly conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) = 0, \quad \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0 \tag{1.4.13}$$

Which means:

$$\rho_m \propto a^{-3}, \quad \rho_r \propto a^{-4} \tag{1.4.14}$$

Define Cosmological Critical Density: And 1.4.4 can be re-expressed as:

$$\frac{H^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{K,0}a^{-2} + \Omega_{\Lambda,0} \tag{1.4.15}$$

Where:

$$\Omega_{r,0} = \frac{8\pi G\rho_{r0}}{3H_0^2}, \quad \Omega_{m,0} = \frac{8\pi G\rho_{m0}}{3H_0^2}, \quad \Omega_{K,0} = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2} \tag{1.4.16}$$

1.5 Cosmological Compositions Dominant Era:

Current Cosmological Parameters:

$$\begin{aligned} &\Omega_{m,0} = 0.3, \quad \Omega_{\Lambda,0} = 0.7, \quad \Omega_{r,0} = 10^{-5} \\ &H_0^{-1} = 14Gyr, \quad a_0 = 1 \end{aligned} \tag{1.5.1}$$

The evolution from Radiation Era to Matter Era:

$$\Omega_{r,0}\frac{a_{eq}^{-4}}{a_0^{-4}} = \Omega_{m,0}\frac{a_{eq}^{-3}}{a_0^{-3}} \tag{1.5.2}$$

$$a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.80 \times 10^{-4} \tag{1.5.3}$$

$$z_{eq} = \frac{a_0}{a_{eq}} - 1 \approx 3.57 \times 10^4 \tag{1.5.4}$$

$$
\begin{aligned}
t_{eq} &= \frac{1}{H_0}\int_{z_{eq}}^{\infty}\frac{dz}{(1+z)E(z)^{1/2}} \\
&= \frac{1}{H_0}\int_{z_{eq}}^{\infty}\frac{dz}{(1+z)\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}}} \\
&\approx 4.68\times 10^4 yr
\end{aligned} \tag{1.5.5}
$$

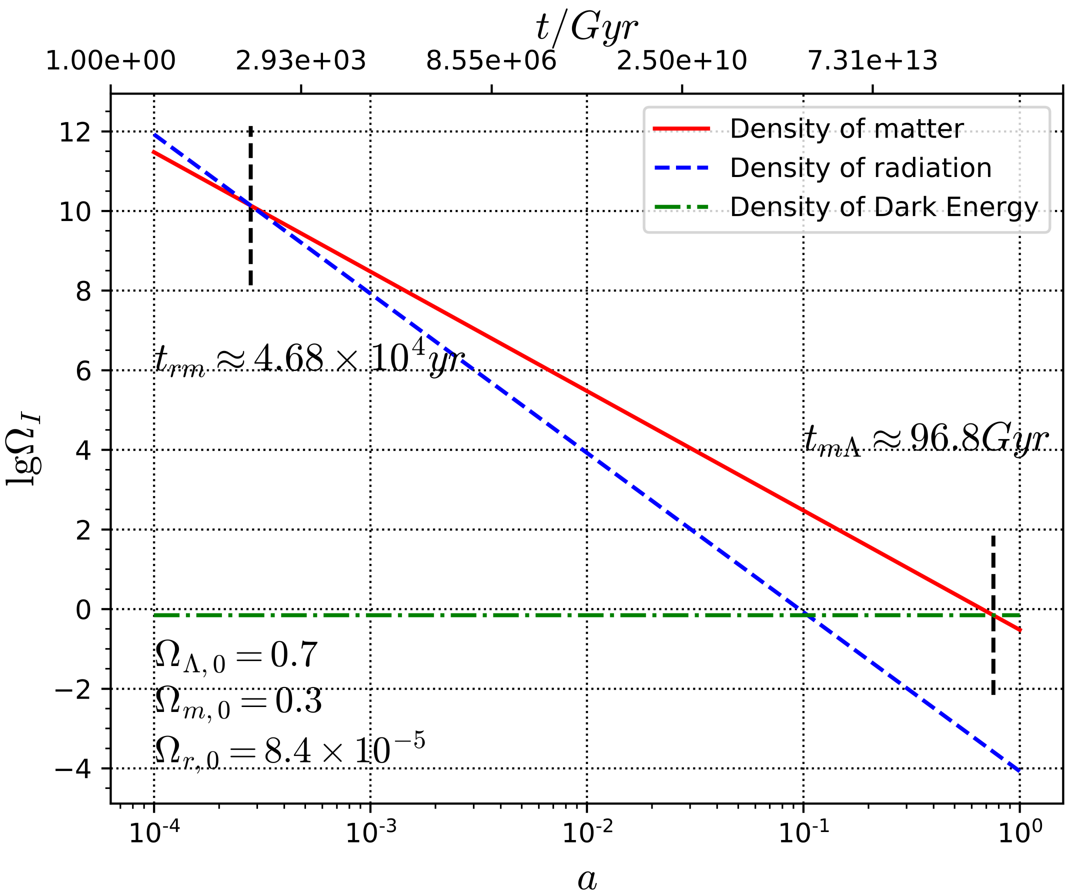

图 1: **The evolution of matter density and radiation density over time**

The moment that universe began to expand in acceleration:

$$
\begin{aligned}
q_c &= \frac{\Omega_m}{2} - \Omega_\Lambda \\
&= \frac{\Omega_{m,0}}{2}\frac{a_c^{-3}}{a_0^{-3}} - \Omega_{\Lambda,0} = 0
\end{aligned} \tag{1.5.6}
$$

$$
a_c = (\frac{\Omega_{m,0}}{2\Omega_\Lambda})^{1/3} \approx 0.598 \tag{1.5.7}
$$

$$
z_c = \frac{a_0}{a_c} - 1 \approx 0.671 \tag{1.5.8}
$$

$$
\begin{aligned}
t_c &= \frac{1}{H_0}\int_{z_c}^{\infty}\frac{dz}{(1+z)E(z)^{1/2}} \\
&\approx 96.8\text{G}yr
\end{aligned} \tag{1.5.9}
$$

1.6 Big Bang Existence Criterion[1]

Big Bang critical condition:

$$H^2 = H_0^2[\Omega_{\Lambda,0} + (1 - \Omega_{m,0} - \Omega_{\Lambda,0})a_c^{-2} + \Omega_{m,0}a_c^{-3}] = 0 \tag{1.6.1}$$

Which can be re-express as:

$$4(1 - \Omega_{m,0} - \Omega_{\Lambda,0}) + 27\Omega_{m,0}^2\Omega_{\Lambda,0} = 0 \tag{1.6.2}$$

By introducing the following variable:

$$x = (\frac{\Omega_{\Lambda,0}}{4\Omega_{m,0}}) \tag{1.6.3}$$

1.6.2 equation quickly reduces to:

$$x^3 - \frac{3}{4}x - \frac{1 - \Omega_{m,0}}{\Omega_{m,0}} = 0 \tag{1.6.4}$$

Assume that $\Omega_{\Lambda,0} > 0$,we can get the following three cases:

- $0 < \Omega_{m,0} \leq \frac{1}{2}$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cosh[\frac{1}{3}\cosh^{-1}(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}})] \tag{1.6.5}$$

- $\frac{1}{2} < \Omega_{m,0} \leq 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cos[\frac{1}{3}\cos^{-1}(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}})] \tag{1.6.6}$$

- $\Omega_{m,0} > 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cos[\frac{1}{3}\cos^{-1}(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}}) + \frac{4\pi}{3}] \tag{1.6.7}$$

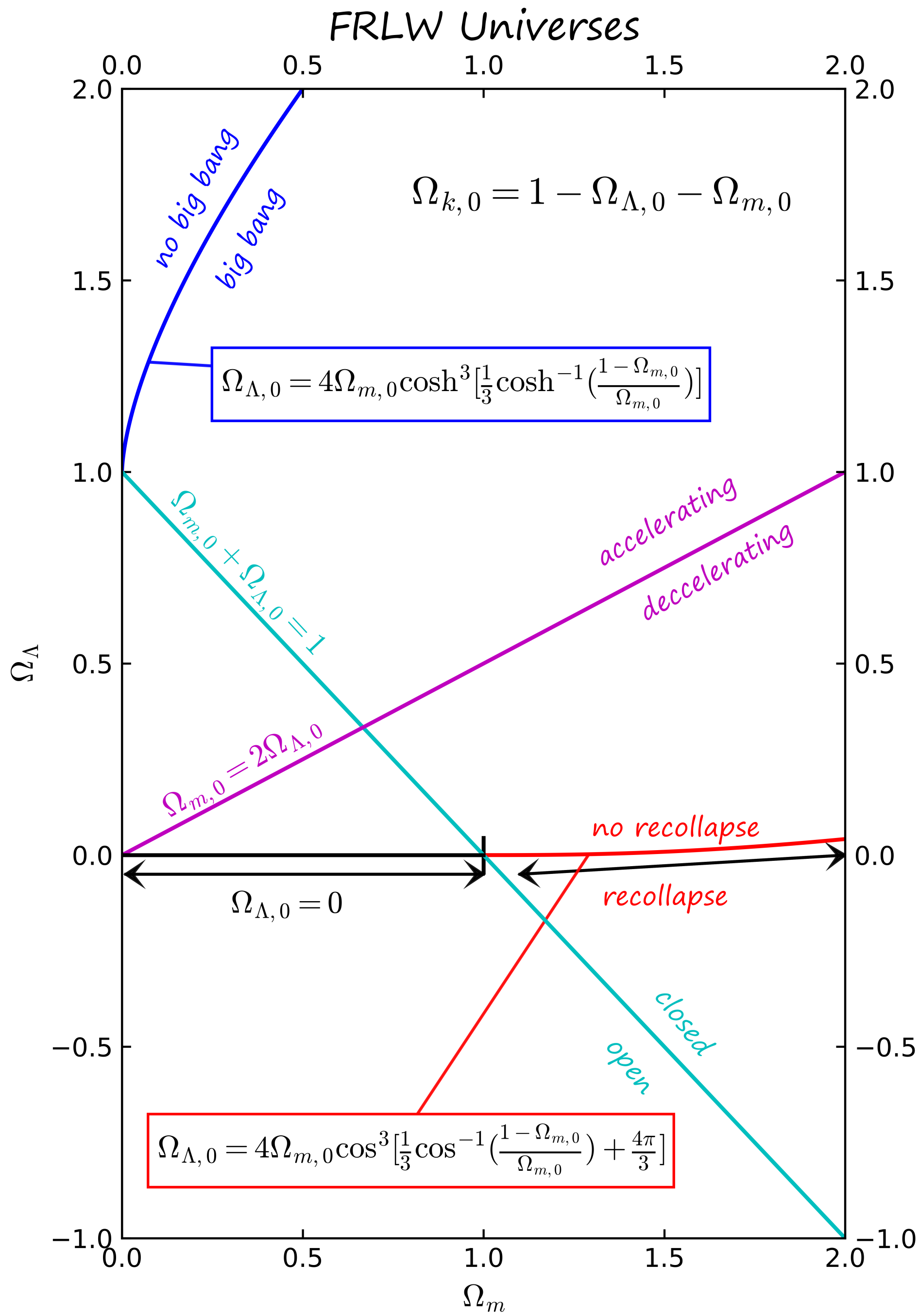

图 2: **FRLW Universe by different cosmological parameters**

参考文献

[1] G. P. Efstathiou M. P. Hobson and A. N. Lasenby. General relativity: An introduction for physicists. page 396, 2006.

[2] 冯珑珑向守平. 宇宙大尺度结构的形成. page 56, 2011.