北京師範大學

暗能量理论与观测

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1 暗能量理论

1.1 FRLW 度规

在一个三维的超球面上，空间中两点距离为:

$$ds^2 = f(r)dr^2 + r^2 d\theta + r^2 \sin^2\theta \tag{1.1.1}$$

Gauss 曲率:

$$k = \frac{1}{2f^2(r)r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} \tag{1.1.2}$$

得:

$$f(r) = \frac{1}{C - kr^2} \tag{1.1.3}$$

令 $C = 1$, 引入宇宙尺度因子

$$K = -\frac{k}{a^2} \tag{1.1.4}$$

$$-c^2 d\tau^2 = -c^2 dt^2 + a^2[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{1.1.5}$$

$$\mathrm{g}_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{bmatrix} \tag{1.1.6}$$

$$\mathrm{g}_{00} = -c^2, \quad \mathrm{g}_{11} = \frac{a^2}{1-Kr^2}, \quad \mathrm{g}_{22} = a^2r^2, \quad \mathrm{g}_{33} = a^2r^2\sin^2\theta \tag{1.1.7}$$

$$\mathrm{g}^{00} = -\frac{1}{c^2}, \quad \mathrm{g}^{11} = \frac{1-Kr^2}{a^2}, \quad \mathrm{g}^{22} = \frac{1}{a^2r^2}, \quad \mathrm{g}^{33} = \frac{1}{a^2r^2\sin^2\theta} \tag{1.1.8}$$

$$\mathrm{g}_{00,\gamma} = 0 \quad (\gamma = 0,1,2,3) \tag{1.1.9}$$

$$\mathrm{g}_{11,0} = \frac{1}{c^2}\frac{2a\dot{a}}{1-Kr^2}, \quad \mathrm{g}_{11,1} = \frac{2a^2Kr}{(1-Kr^2)^2}, \quad \mathrm{g}_{11,2} = \mathrm{g}_{11,3} = 0 \tag{1.1.10}$$

$$\mathrm{g}_{22,0} = 2a\dot{a}r^2, \quad \mathrm{g}_{22,1} = 2a^2r, \quad \mathrm{g}_{22,2} = \mathrm{g}_{22,3} = 0 \tag{1.1.11}$$

$$\mathrm{g}_{33,0} = 2a\dot{a}r^2\sin^2\theta, \quad \mathrm{g}_{33,1} = 2a^2r\sin^2\theta, \quad \mathrm{g}_{33,2} = 2a^2r\sin^2, \quad \mathrm{g}_{33,3} = 0 \tag{1.1.12}$$

1.2 Christoffel 符号

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\mathrm{g}^{\sigma\gamma}(\mathrm{g}_{\mu\gamma,\nu} + \mathrm{g}_{\nu\gamma,\mu} - \mathrm{g}_{\mu\nu,\gamma}) \tag{1.2.1}$$

Christoffel 符号非零分量:

$$\Gamma^0_{11}=\frac{1}{2}\mathrm{g}^{0\gamma}(\mathrm{g}_{1\gamma,1}+\mathrm{g}_{1\gamma,1}-\mathrm{g}_{11,\gamma}) \qquad =-\frac{1}{2}\mathrm{g}^{00}\mathrm{g}_{11,0}= \qquad \frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2} \tag{1.2.2}$$

$$\Gamma^0_{22}=\frac{1}{2}\mathrm{g}^{0\gamma}(\mathrm{g}_{2\gamma,2}+\mathrm{g}_{2\gamma,2}-\mathrm{g}_{22,\gamma}) \qquad =-\frac{1}{2}\mathrm{g}^{00}\mathrm{g}_{22,0}= \qquad \frac{1}{c^2}a\dot{a}r^2 \tag{1.2.3}$$

$$\Gamma^0_{33}=\frac{1}{2}\mathrm{g}^{0\gamma}(\mathrm{g}_{3\gamma,3}+\mathrm{g}_{3\gamma,3}-\mathrm{g}_{33,\gamma}) \qquad =-\frac{1}{2}\mathrm{g}^{00}\mathrm{g}_{33,0}= \qquad \frac{1}{c^2}a\dot{a}r^2\sin^2\theta \tag{1.2.4}$$

$$\Gamma^i_{j0}=\frac{1}{2}\mathrm{g}^{i\gamma}(\mathrm{g}_{j\gamma,0}+\mathrm{g}_{0\gamma,j}-\mathrm{g}_{j0,\gamma}) \qquad =\frac{1}{2}\mathrm{g}^{ii}\mathrm{g}_{ij,0}= \qquad \frac{\dot{a}}{a}\delta^i_j \tag{1.2.5}$$

$$\Gamma^1_{11}=\frac{1}{2}\mathrm{g}^{1\gamma}(\mathrm{g}_{1\gamma,1}+\mathrm{g}_{1\gamma,1}-\mathrm{g}_{11,\gamma}) \qquad =\frac{1}{2}\mathrm{g}^{11}\mathrm{g}_{11,1}= \qquad \frac{Kr}{1-Kr^2} \tag{1.2.6}$$

$$\Gamma^1_{22}=\frac{1}{2}\mathrm{g}^{1\gamma}(\mathrm{g}_{2\gamma,2}+\mathrm{g}_{2\gamma,2}-\mathrm{g}_{22,\gamma}) \qquad =-\frac{1}{2}\mathrm{g}^{11}\mathrm{g}_{22,1}= \qquad -r(1-Kr^2) \tag{1.2.7}$$

$$\Gamma^1_{33}=\frac{1}{2}\mathrm{g}^{1\gamma}(\mathrm{g}_{3\gamma,3}+\mathrm{g}_{3\gamma,3}-\mathrm{g}_{33,\gamma}) \qquad =-\frac{1}{2}\mathrm{g}^{11}\mathrm{g}_{33,1}= \qquad -r\sin^2\theta(1-Kr^2) \tag{1.2.8}$$

$$\Gamma^2_{12}=\frac{1}{2}\mathrm{g}^{2\gamma}(\mathrm{g}_{1\gamma,2}+\mathrm{g}_{2\gamma,1}-\mathrm{g}_{12,\gamma}) \qquad =\frac{1}{2}\mathrm{g}^{22}\mathrm{g}_{22,1}= \qquad \frac{1}{r} \tag{1.2.9}$$

$$\Gamma^3_{13}=\frac{1}{2}\mathrm{g}^{3\gamma}(\mathrm{g}_{1\gamma,3}+\mathrm{g}_{3\gamma,1}-\mathrm{g}_{13,\gamma}) \qquad =\frac{1}{2}\mathrm{g}^{33}\mathrm{g}_{33,1}= \qquad \frac{1}{r} \tag{1.2.10}$$

$$\Gamma^3_{23}=\frac{1}{2}\mathrm{g}^{3\gamma}(\mathrm{g}_{2\gamma,3}+\mathrm{g}_{3\gamma,2}-\mathrm{g}_{23,\gamma}) \qquad =\frac{1}{2}\mathrm{g}^{33}\mathrm{g}_{33,2}= \qquad -\sin\theta\cos\theta \tag{1.2.11}$$

$$\Gamma^2_{33}=\frac{1}{2}\mathrm{g}^{2\gamma}(\mathrm{g}_{3\gamma,3}+\mathrm{g}_{3\gamma,3}-\mathrm{g}_{33,\gamma}) \qquad =\frac{1}{2}\mathrm{g}^{22}\mathrm{g}_{33,2}= \qquad \cot\theta \tag{1.2.12}$$

对于 $i,j=1,2,3$

1.3 Ricci 张量

$$R_{\mu\nu}=\Gamma^\sigma_{\mu\nu,\sigma}-\Gamma^\sigma_{\mu\sigma,\nu}+\Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu}-\Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma} \tag{1.3.1}$$

Ricci 张量非零分量:

$$\begin{aligned}
R_{00}&=\Gamma^\sigma_{00,\sigma}-\Gamma^\sigma_{0\sigma,0}+\Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{00}-\Gamma^\sigma_{\rho0}\Gamma^\rho_{0\sigma}\\
&=\Gamma^0_{00,0}-\Gamma^\sigma_{0\sigma,0}+\Gamma^\sigma_{0\sigma}\Gamma^0_{00}-\Gamma^\sigma_{\sigma0}\Gamma^\sigma_{0\sigma}\\
&=0-3\frac{\ddot{a}a-\dot{a}^2}{a^2}+0-3(\frac{\dot{a}}{a})^2\\
&=-3\frac{\ddot{a}}{a}
\end{aligned} \tag{1.3.2}$$

$$\begin{aligned}
R_{11}&=\Gamma^\sigma_{11,\sigma}-\Gamma^\sigma_{1\sigma,1}+\Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{11}-\Gamma^\sigma_{\rho1}\Gamma^\rho_{1\sigma}\\
&=(\Gamma^1_{11,1}+\Gamma^2_{11,2}+\Gamma^3_{11,3})-(\Gamma^0_{11,0}+\Gamma^1_{11,1})\\
&\quad+(\Gamma^1_{10}\Gamma^0_{11}+\Gamma^2_{11}\Gamma^1_{10}+\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{12}\Gamma^2_{12}+\Gamma^3_{13}\Gamma^3_{13})\\
&\quad-(\Gamma^0_{11}\Gamma^1_{01}+\Gamma^0_{11}\Gamma^2_{02}+\Gamma^0_{11}\Gamma^3_{03}+\Gamma^1_{11}\Gamma^1_{11}+\Gamma^1_{11}\Gamma^2_{12}+\Gamma^1_{11}\Gamma^3_{13})\\
&=\frac{1}{c^2}[\frac{2}{r^2}+\frac{\dot{a}^2+a\ddot{a}}{1-Kr^2}-\frac{2}{r^2}+\frac{\dot{a}^2}{1-Kr^2}+\frac{2K}{1-Kr^2}]\\
&=\frac{1}{c^2}\frac{a\ddot{a}+2\dot{a}^2+2K}{1-Kr^2}\\
&=\frac{1}{c^2}(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+2\frac{K}{a^2})\mathrm{g}_{11}
\end{aligned} \tag{1.3.3}$$

同理:

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{1.3.4}$$

$$R_{ij} = \frac{1}{c^2}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2})\mathrm{g}_{ij} \tag{1.3.5}$$

$$R = \frac{6}{c^2}(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}) \tag{1.3.6}$$

1.4 Friedmann 方程

Einstein 场方程:

$$R_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}R + \Lambda\mathrm{g}_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.4.1}$$

能动张量:

$$T_{\mu\nu} = (\rho + \frac{p}{c^2})u_\mu u_\nu + p\mathrm{g}_{\mu\nu} \tag{1.4.2}$$

根据宇宙学原理:

$$T_{\mu\nu} = diag\{\rho c^2, p, p, p\} \tag{1.4.3}$$

综上可得:

$$(\frac{\dot{a}}{a})^2 + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho \tag{1.4.4}$$

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{Kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G}{c^2}p \tag{1.4.5}$$

定义 Hubble 参数:

$$H = \frac{\dot{a}}{a} \tag{1.4.6}$$

状态方程:

$$p = w\rho c^2 \tag{1.4.7}$$

根据 1.4.4 ,1.4.6 and 1.4.7,1.4.5 可以写为:

$$\dot{\rho} + 3(1+w)H\rho = 0 \tag{1.4.8}$$

即为连续性方程.
假设宇宙膨胀是一个绝热方程，根据热力学第一定律:

$$dE + pdV = 0 \tag{1.4.9}$$

这里:

$$E = (\rho_m + \rho_r)V = \rho V, V \propto a^3 \tag{1.4.10}$$

1.4.9 可被表示为:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) + p\frac{\mathrm{d}}{\mathrm{d}t}(a^3) = 0 \tag{1.4.11}$$

物质粒子是非相对论性的,$w_m = 0$**，压强** p **主要由辐射提供** $w_r = 1/3$**, 因此 1.4.8 可以被表示为:**[2]

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) + \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0 \tag{1.4.12}$$

假如物质是严格守恒的:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) = 0, \quad \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0 \tag{1.4.13}$$

即:

$$\rho_m \propto a^{-3}, \quad \rho_r \propto a^{-4} \tag{1.4.14}$$

则 1.4.4 可以被表示为:

$$\frac{H^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{K,0}a^{-2} + \Omega_{\Lambda,0} \tag{1.4.15}$$

其中:

$$\Omega_{r,0} = \frac{8\pi G\rho_{r0}}{3H_0^2}, \quad \Omega_{m,0} = \frac{8\pi G\rho_{m0}}{3H_0^2}, \quad \Omega_{K,0} = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2} \tag{1.4.16}$$

1.5 宇宙组分的主导时期:

当前宇宙参数:

$$\begin{aligned} &\Omega_{m,0} = 0.3, \quad \Omega_{\Lambda,0} = 0.7, \quad \Omega_{r,0} = 10^{-5} \\ &H_0^{-1} = 14Gyr, \quad a_0 = 1 \end{aligned} \tag{1.5.1}$$

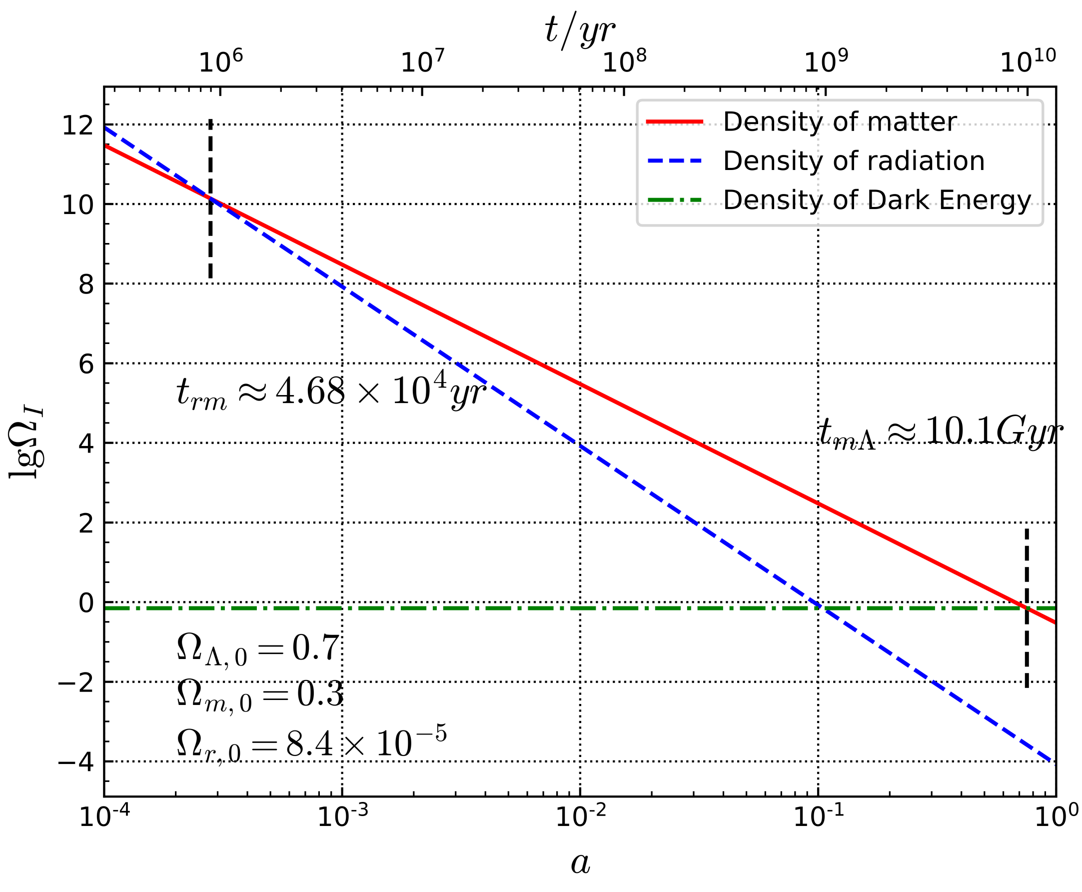

图 1: **物质、辐射和暗能量密度（宇宙学参数）随时间的演化**

辐射主导时期到物质主导时期:

$$\Omega_{r,0}\frac{a_{eq}^{-4}}{a_0^{-4}} = \Omega_{m,0}\frac{a_{eq}^{-3}}{a_0^{-3}} \tag{1.5.2}$$

$$a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.80\times 10^{-4} \tag{1.5.3}$$

$$z_{eq} = \frac{a_0}{a_{eq}} - 1 \approx 3.57\times 10^{4} \tag{1.5.4}$$

$$\begin{aligned} t_{eq} &= \frac{1}{H_0}\int_{z_{eq}}^{\infty}\frac{dz}{(1+z)E(z)^{1/2}} \\ &= \frac{1}{H_0}\int_{z_{eq}}^{\infty}\frac{dz}{(1+z)\sqrt{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}}} \\ &\approx 4.68\times 10^4 yr \end{aligned} \tag{1.5.5}$$

宇宙开始加速膨胀的时刻:

$$q_c = \frac{\Omega_m}{2} - \Omega_\Lambda = \frac{\Omega_{m,0}}{2}\frac{a_c^{-3}}{a_0^{-3}} - \Omega_{\Lambda,0} = 0 \tag{1.5.6}$$

$$a_c = (\frac{\Omega_{m,0}}{2\Omega_\Lambda})^{1/3} \approx 0.598 \tag{1.5.7}$$

$$z_c = \frac{a_0}{a_c} - 1 \approx 0.671 \tag{1.5.8}$$

$$\begin{aligned} t_c &= \frac{1}{H_0}\int_{z_c}^{\infty}\frac{dz}{(1+z)E(z)^{1/2}} \\ &\approx 7.54\mathrm{G}yr \end{aligned} \tag{1.5.9}$$

1.6 大爆炸存在判据

大爆炸理论合理性的边界条件:[1]

$$H^2 = H_0^2[\Omega_{\Lambda,0} + (1-\Omega_{m,0}-\Omega_{\Lambda,0})a_c^{-2} + \Omega_{m,0}a_c^{-3}] = 0 \tag{1.6.1}$$

即为:

$$4(1-\Omega_{m,0}-\Omega_{\Lambda,0}) + 27\Omega_{m,0}^2\Omega_{\Lambda,0} = 0 \tag{1.6.2}$$

引入以下变量代换:

$$x = (\frac{\Omega_{\Lambda,0}}{4\Omega_{m,0}}) \tag{1.6.3}$$

1.6.2 可以化简为:

$$x^3 - \frac{3}{4}x - \frac{1-\Omega_{m,0}}{\Omega_{m,0}} = 0 \tag{1.6.4}$$

假设 $\Omega_{\Lambda,0} > 0$, 我们可以得到方程 1.6.3 的以下三个解:

- $0 < \Omega_{m,0} \leq \frac{1}{2}$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cosh[\frac{1}{3}\cosh^{-1}(\frac{1-\Omega_{m,0}}{\Omega_{m,0}})] \tag{1.6.5}$$

- $\frac{1}{2} < \Omega_{m,0} \leq 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cos[\frac{1}{3}\cos^{-1}(\frac{1-\Omega_{m,0}}{\Omega_{m,0}})] \tag{1.6.6}$$

- $\Omega_{m,0} > 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0}\cos[\frac{1}{3}\cos^{-1}(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}) + \frac{4\pi}{3}] \tag{1.6.7}$$

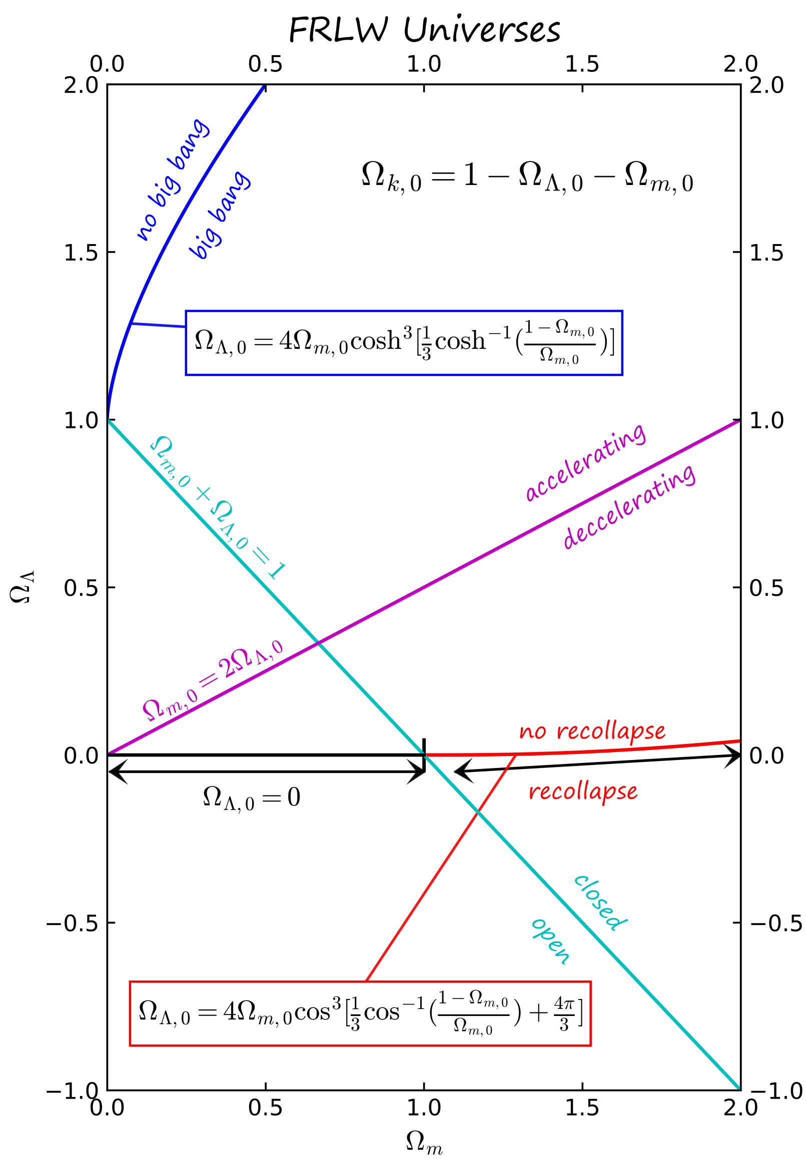

图 2: **FRLW 宇宙，当前物质和暗能量密度关系决定了宇宙的特征和演化方向**

参考文献

[1] G. P. Efstathiou M. P. Hobson and A. N. Lasenby. General relativity: An introduction for physicists. page 396, 2006.

[2] 冯珑珑向守平. 宇宙大尺度结构的形成. page 56, 2011.