北京師範大學

热力学与统计物理作业: 前 2 章

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Chapter 1

Question 1.1

试求理想气体的体胀系数 α , 压强系数 β, 等温压缩系数 κ_T
理想气体的物态方程:

$$pV = nRT$$

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_T = \frac{nR}{pV} = \frac{1}{T}$$

$$\beta = \frac{1}{p}(\frac{\partial p}{\partial T})_V = \frac{nR}{pV} = \frac{1}{T}$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = -\frac{1}{V}\left(\frac{nRT}{p^2}\right) = \frac{1}{p}$$

Question 1.2

证明任何一种具有两个独立参量 T、p 的物质，其物理状态方程可有实验测量的体胀系数 α、等温压缩系数 κ_T, 根据下述积分求得：$lnV = \int(\alpha dT - \kappa_T dp)$, 若 $\alpha = \frac{1}{T}$,$\kappa_T = \frac{1}{p}$, 试求物态方程。

$$\begin{aligned}
V &= V(T,p) \\
dV &= \left(\frac{\partial V}{\partial T}\right)_p dT + \left(\frac{\partial V}{\partial p}\right)_V dp \\
\frac{dV}{V} &= \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p dT + \frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_V dp \\
&= \alpha dT - \kappa_T dp
\end{aligned}$$

积分得 $lnV = \int(\alpha dT - \kappa_T dp)$
若 $\alpha = \frac{1}{T}$,$\kappa_T = \frac{1}{p}$

$$lnV = \int\left(\frac{1}{T}dT - \frac{1}{p}dp\right) = lnT - lnp + ln\mathrm{C}$$

即:

$$pV = \mathrm{C}T$$

□

Question 1.5

$$f(\mathscr{T}, L, T) = 0$$

得:

$$\left(\frac{\partial L}{\partial T}\right)_{\mathscr{T}}\left(\frac{\partial T}{\partial \mathscr{T}}\right)_L\left(\frac{\partial \mathscr{T}}{\partial L}\right)_T = -1$$

$$\begin{aligned}
\left(\frac{\partial \mathscr{T}}{\partial T}\right)_L &= -\left(\frac{\partial L}{\partial T}\right)_{\mathscr{T}}\left(\frac{\partial \mathscr{T}}{\partial L}\right)_T \\
&= -L\alpha\frac{A}{L}E = -\alpha AE
\end{aligned}$$

积分得:

$$\Delta\mathscr{T} = -\alpha AE(T_2 - T_1)$$

Chapter 1 continued on next page...

Question 1.11

$$p\left(z\right)=p\left(z+dz\right)+\rho\left(z\right)gdz=p\left(z\right)+\frac{\mathrm{d}}{\mathrm{d}z}p\left(z\right)dz+\rho\left(z\right)gdz$$

得

$$\frac{\mathrm{d}}{\mathrm{d}z}p\left(z\right)dz=-\rho\left(z\right)gdz$$
$$n=\frac{V}{V_0}=\frac{m}{M_r}$$
$$V_0=\frac{M_r}{\rho\left(z\right)}$$
$$p(z)\frac{M_r}{\rho(z)}=RT(z)$$
$$\frac{\mathrm{d}}{\mathrm{d}z}p(z)=-\frac{M_rg}{RT(z)}p(z)$$

绝热过程:

$$\frac{p^{\gamma-1}}{T^{\gamma}}=\mathrm{C}$$
$$(\frac{\partial T}{\partial p})_S=\frac{\gamma-1}{\gamma}\frac{T}{p}$$
$$\frac{\mathrm{d}}{\mathrm{d}z}T(z)=(\frac{\partial T}{\partial p})_S\frac{\mathrm{d}}{\mathrm{d}z}p(z)$$
$$\frac{\mathrm{d}}{\mathrm{d}z}T(z)=-\frac{\gamma-1}{\gamma}\frac{M_rg}{R}$$

得

$$\frac{\mathrm{d}T(z)}{\mathrm{d}z}=-10K/km$$

Question 1.12

准静态绝热过程:

$$c_VdT+pdV=0$$

物态方程:

$$pV=nRT$$
$$\frac{c_V}{nR}\frac{dT}{T}+\frac{dV}{V}=0$$

即:

$$\frac{1}{\gamma-1}\frac{dT}{T}+\frac{dV}{V}=0$$
$$lnF(T)=\int\frac{dT}{(\gamma-1)T}=lnV+ln\mathrm{C}$$

即:

$$VF(T)=\mathrm{C}$$

C 是常数

Question 1.17

$$\Delta S_{H_2O}=\int_{T_1}^{T_2}\frac{mc_pdT}{T}\approx 1304.6J/K$$

$$Q = mc_p\Delta T = 4.18 \times 10^5 J$$

$$\Delta S_{source} = -\frac{Q}{T} \approx -1120.6J/K$$

$$\Delta S = \Delta S_{H_2O} + \Delta S_{source} = 184J/K$$

Question 1.19

设杆长为 , 杆与　轴重合

$$dS_0 = c_p dx \int_{T_l}^{\frac{1}{2}(T_1+T_2)} \frac{dT}{T} = c_p dx ln\frac{\frac{T_1+T_2}{2}}{T_2 + \frac{T_1-T_2}{L}x}$$

c_p 是定压比热容

$$\begin{aligned}\Delta S = \int_0^L S_0 &= c_p \int_0^L [ln\frac{T_1+T_2}{2} - ln(T_2 + \frac{T_1-T_2}{2}x)]dx \\ &= c_p L ln\frac{T_1+T_2}{2} - \frac{c_p L}{T_1-T_2}(T_1 lnT_1 - T_2 lnT_2 - T_1 + T_2) \\ &= c_p(ln\frac{T_1+T_2}{2} - \frac{T_1 lnT_1 - T_2 lnT_2}{T_1-T_2} + 1)\end{aligned}$$

Question 1.21

$$\Delta S = \Delta S\ + \Delta S\ + \Delta S\ \geq 0$$

$$\Delta S\ \geq 0$$

$$Q = Q' + W$$

$$\Delta S\ = \frac{Q'}{T_2} = \frac{Q-W}{T_2}$$

$$\Delta S \geq S_1 - S_2 + \frac{Q-W}{T_2} \geq 0$$

取等号时:

$$W_{max} = Q - T_2(S_1 - S_2)$$

Chapter 2

Question extra1

$$c_V = (\frac{\partial V}{\partial T})_V = T(\frac{\partial S}{\partial T})_V$$
$$c_p = (\frac{\partial H}{\partial T})_p = T(\frac{\partial S}{\partial T})_p$$
$$c_p - c_V = T(\frac{\partial S}{\partial T})_V - T(\frac{\partial S}{\partial T})_p$$

将函数 $S(T,V)$ 转换为 $S(T,p)$:

$$\begin{aligned} dS &= (\frac{\partial S}{\partial T})_V dT + (\frac{\partial S}{\partial V})_T dV \\ &= (\frac{\partial S}{\partial T})_p dT + (\frac{\partial S}{\partial p})_T dp \end{aligned}$$

$dp = 0$:

$$\begin{aligned} (\frac{\partial S}{\partial T})_V &= (\frac{\partial S}{\partial T})_p - (\frac{\partial S}{\partial V})_T(\frac{\partial V}{\partial T})_p \\ c_p - c_V &= T(\frac{\partial S}{\partial T})_p - T[(\frac{\partial S}{\partial T})_p - (\frac{\partial S}{\partial V})_T(\frac{\partial V}{\partial T})_p] \\ &= T(\frac{\partial S}{\partial V})_T(\frac{\partial V}{\partial T})_p \\ &= T(\frac{\partial p}{\partial T})_V(\frac{\partial V}{\partial T})_p \end{aligned}$$

得 $pV = nRT$ 或 $c_p - c_V = nR$

Question 2.2

$$\begin{aligned} p &= f(V)T \\ (\frac{\partial p}{\partial T})_V &= (\frac{\partial p}{\partial T})_V \because (\frac{\partial S}{\partial V})_T \quad = (\frac{\partial p}{\partial T})_V \\ (\frac{\partial U}{\partial V})_T &= T(\frac{\partial S}{\partial T})_V - p \\ \because (\frac{\partial U}{\partial V})_T &= T(\frac{\partial p}{\partial T})_V - p \\ \therefore (\frac{\partial U}{\partial V})_T &= Tf(V) - p = 0 \end{aligned}$$

$\therefore U$ 与 V 无关

□

Question 2.3

(a)

$$dH = TdS + Vdp$$

令 $dH = 0$:

$$(\frac{\partial S}{\partial p})_H = -\frac{V}{T} < 0$$

(b)

$$dU = TdS - pdV$$

令 $dU = 0$:

$$(\frac{\partial S}{\partial p})_U = \frac{p}{T} > 0$$

□

Question 2.4

$$U(T,p) = U[T, V(T,p)]$$
$$(\frac{\partial U}{\partial p})_T = (\frac{\partial U}{\partial V})_T(\frac{\partial V}{\partial p})_T$$
$$\therefore (\frac{\partial U}{\partial V})_T = (\frac{\partial V}{\partial p})_T = 0$$

□

Question 2.7

$$pV = f(T), U = U(T)$$
$$\therefore (\frac{\partial U}{\partial V})_T = T(\frac{\partial p}{\partial T})_V - p = 0$$
$$(\frac{\partial p}{\partial T})_V = \frac{1}{V}\frac{\mathrm{d}f}{\mathrm{d}T}$$
$$\therefore T\frac{\mathrm{d}f}{\mathrm{d}T} = f$$

积分得：$lnf = lnT + lnC$ 或 $pV = CT$

Question 2.14

$$\frac{\partial E}{\partial t} = \sigma T^4 R_s^2 d\Omega$$

得:

$$T = (\frac{1.35 \times 10^3 R_{se}^2}{\sigma R_s^2})^{\frac{1}{4}} \approx 5670K$$

Question extra2

估算宇宙第一缕光产生时的宇宙年龄。

我们把宇宙的膨胀看做一个绝热过程，根据热力学第一定律:

$$dU + pdV = 0$$

其中 $U = (\rho_m + \rho_r)V = \rho V$ ρ_m 和 ρ_r 分别是物质密度和辐射密度, 宇宙尺度因子是 $a(t)$

$$\therefore V \propto a^3(t)$$

得:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) + p\frac{\mathrm{d}}{\mathrm{d}t}(a^3) = 0$$

物质粒子是非相对论性的:

$$p = p_r = \frac{\rho_r}{3}$$

得:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) + \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0$$

假设物质是严格守恒的:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_m a^3) = 0, \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_r a^4) = 0$$

得:

$$\rho_m = \rho_{m_0}(\frac{a_0}{a})^3$$
$$\rho_r = \rho_{r_0}(\frac{a_0}{a})^4$$

其中 ρ_{m_0} 和 ρ_{r_0} 分别是现在宇宙的物质密度和辐射密度

$$\rho_r \propto T^4$$

得:

$$T = T_0\frac{a_0}{a}$$

形成中性氢需要降到的温度为 T_α:

$$\epsilon_\alpha = k_B T_\alpha$$
$$H^2 = \frac{8\pi G}{3}(\rho_r + \rho_m) + \frac{\Lambda}{3} - \frac{k}{a^2}$$

H 是 $Hubble$ 常数, 对于 $\Lambda = 0, k = 0$ 的 Einstein-de Sitter 宇宙, 假设 $a(t_0) = 1$,$\rho_0 = \rho_{m_0} + \rho_{r_0} \approx \rho_{m_0}$:

$$\begin{aligned} H_0^2 &= \frac{8\pi G}{3}\rho_{m_0} \\ t_\alpha &= H_0^{-1}\int_0^{a_\alpha}\frac{ada}{(a_\alpha + a)^{\frac{1}{2}}} \\ &= \frac{2}{3}(2\sqrt{2} - 1)H_0^{-1}a_\alpha^{\frac{3}{2}} \\ &\approx 3.7 \times 10^5 yr \end{aligned}$$