流体力学导论作业: 第四次

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Assignment 1

Question A

给定平面标量场 ϕ . 设在 M 点上已知两个方向 $\vec{s_1}$ 和 $\vec{s_2}$, 的方向导数分别为 $\frac{\partial\phi}{\partial\vec{s_1}}$ 和 $\frac{\partial\phi}{\partial\vec{s_2}}$

$$\bar{M}P = \frac{\partial\phi}{\partial s_2}, \bar{M}K = \frac{\partial\phi}{\partial s_1}$$

$$|\vec{s_1}| = |\vec{s_2}| = 1$$

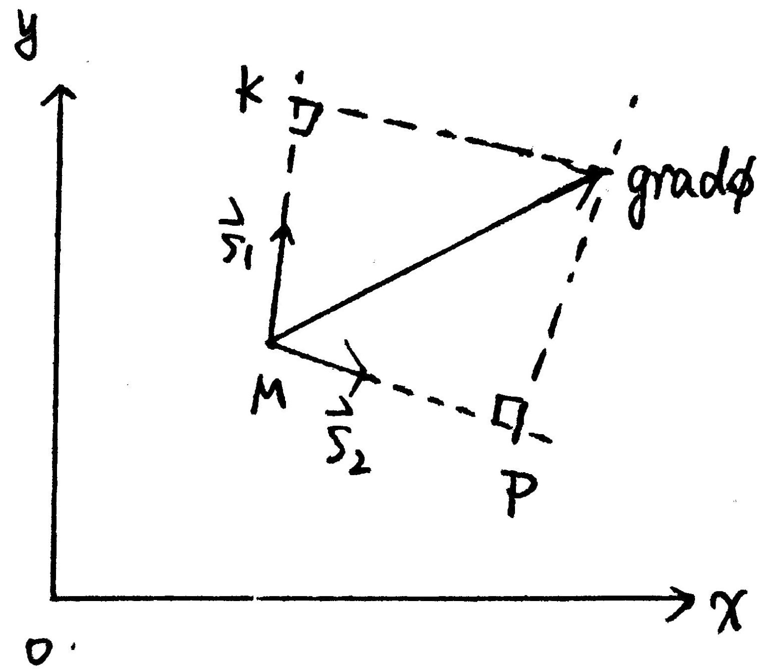


图 1: 第一题图（手绘）

Question B

利用散度 div $\vec{a}$ 的定义推导它在球坐标系中的表达式.

$$\begin{aligned}
&\nabla\cdot\vec{a}\cdot r^2\sin\theta dr d\theta d\phi \\
&= \frac{\partial}{\partial r}\left(r^2\sin\theta a_r\right) dr d\theta d\phi + \frac{\partial}{\partial\theta}\left(r\sin\theta a_\theta\right) dr d\theta d\phi + \frac{\partial}{\partial\phi}\left(r a_\phi\right) dr d\theta d\phi \\
&\nabla\cdot\vec{a} \\
&= \frac{1}{r^2\sin\theta dr d\theta d\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta a_r\right) dr d\theta d\phi + \frac{\partial}{\partial\theta}\left(r\sin\theta a_\theta\right) dr d\theta d\phi + \frac{\partial}{\partial\phi}\left(r a_\phi\right) dr d\theta d\phi\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta a_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}
\end{aligned}$$

Assignment 1 continued on next page...

Question C

已知矢量$\vec{a}$在球坐标系中的三个分量分别为:$a_r = \frac{2k\cos\theta}{r^3}, a_\theta = \frac{k\sin\theta}{r^3}, a_\phi = 0$其中 k 为一常数。试验证矢量$\vec{a}$是否为位势矢
R的通量.

$$\frac{\partial\varphi}{\partial r} = \frac{2k\cos\theta}{r^3}$$
$$\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{k\sin\theta}{r^3}$$
$$\frac{\partial\varphi}{\partial\phi} = 0$$

若加以边界条件:

$$\varphi|_{r\to\infty} = 0$$
$$\varphi = -\frac{k\cos\theta}{r^2}$$

Assignment 2

Question A

一平板重为 $mg = 1000N$, 面积为 $A = 0.16m^2$ 。板下涂满油, 沿与水平线成 $\theta = 20°$ 的斜平壁下滑, 油膜厚度为 $h = 0.005mm$。若油的粘度为 $\mu = 0.007pa \cdot s$, 求板下滑的终端速度 V。

平板受力平衡时:

$$\mu\frac{\partial v}{\partial y}A = mg\sin\theta$$

$$\frac{\partial v}{\partial y} \approx \frac{V}{h}$$

$$V = \frac{mgh\sin\theta}{\mu A} \approx 1.527m/s$$

Question B

旋转圆筒粘度计的内容的直径为 $d = 30cm$, 高为 $h = 30cm$, 外筒与内筒的间隙为 $\delta = 0.2cm$, 间隙中充满被测流体。外筒做匀速旋转，角速度为 $\omega = 15rad/s$。测出作用在精致内筒的力矩为 $M = 8.5N \cdot m$, 忽略筒底部的阻力，求被测流体的粘度 μ。

$$\mu\frac{dV}{dr}S = \frac{M}{d/2}$$

$$\mu\frac{dV}{dr} \cdot \pi dh = \mu\frac{\omega d}{2\delta} = \frac{M}{d/2}$$

$$\mu = \frac{4M}{\pi\omega d^3 h} \approx 89.074Pa \cdot s$$

Question C

根据声速的定义（微小扰动在流体中的传播速度）的定义推导其定量计算的公式 $c_s = \sqrt{dp/d\rho} = \sqrt{k/\rho}$, 其中 K 为流体的体积模量。

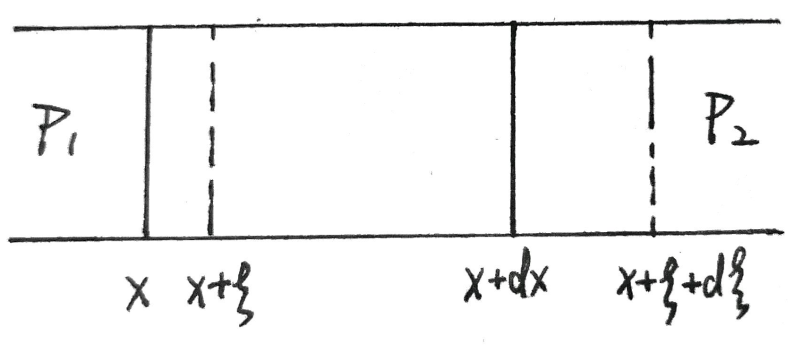


图 2: 第三题图（手绘）

$$
\begin{aligned}
&\frac{\Delta V}{V}=\frac{\Delta\xi}{\Delta x}=\frac{\partial\xi}{\partial x}\\
&dp=\frac{dp}{d\rho}d\rho=-\frac{dp}{d\rho}\rho\frac{dv}{v}=-\frac{dp}{d\rho}\rho\frac{\partial\xi}{\partial x}\\
&p_1=p_0+(dp)_x=p_0-\frac{dp}{d\rho}\rho\frac{\partial\xi}{\partial x}|_x\\
&p_2=p_0+(dp)_{x+dx}=p_0-\frac{dp}{d\rho}\rho\frac{\partial\xi}{\partial x}|_{x+dx}
\end{aligned}
$$

牛顿第二定律:

$$
\begin{aligned}
&(p_1-p_2)\,S=\rho S\frac{\partial^2\xi}{\partial t^2}dx\\
&p_1-p_2=\ \frac{dp}{d\rho}\rho\left(\frac{\partial\xi}{\partial x}|_{x+dx}-\frac{\partial\xi}{\partial x}|_x\right)=\frac{dp}{d\rho}\rho\frac{\partial^2\xi}{\partial x^2}dx=\rho\frac{\partial^2\xi}{\partial t^2}dx
\end{aligned}
$$

即:

$$
\begin{aligned}
&\frac{\partial^2\xi}{\partial t^2}-\frac{dp}{d\rho}\frac{\partial^2\xi}{\partial x^2}=0\\
&K=-\frac{dp}{dv/v}=\frac{dp}{\frac{d(1/\rho)}{1/\rho}}=\rho\frac{dp}{d\rho}
\end{aligned}
$$

所以:

$$
c_s=\sqrt{\frac{dp}{d\rho}}=\sqrt{\frac{K}{\rho}}
$$

Assignment 3

Question A

设 $u = x + t, v = -y + t, \omega = 0$, 求通过 $x = -1, y = -1$ 的流线及 $t = 0$ 时通过 $x = -1, y = -1$ 的迹线。
流线方程:

$$\frac{dx}{x+t} = \frac{dy}{-y+t}$$

积分后得:

$$(x+t)(-y+t) = C$$

而 $t = 0$, $x = -1, y = -1$, 因此$C = 1$，所以流线方程是:

$$xy = 1$$

迹线方程:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -y + t$$

解得:

$$x = C_1 e^t - t - 1,\ y = C_2 e^{-t} + t - 1$$

而 $t = 0$, $x = -1, y = -1$, 因此$C_1 = C_2 = 0$，所以迹线方程是:

$$x + y = -2$$

Question B

设 $u = \frac{cx}{x^2+y^2}, v = cyx^2 + y^2, \omega = 0$, 求流线方程并画图。
流线方程:

$$\frac{dx}{u} = \frac{dy}{v}$$

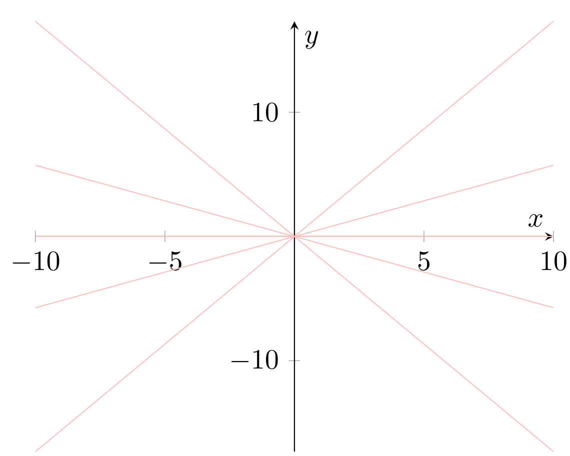
即为:

$$\frac{dx}{x} = \frac{dy}{y}$$

解得:

$$y = kx\,(k\text{是任意常数})$$

流线图:



Question C

设平面不定常流动的速度分布为 $u = x(1+2t), v = y$, 求通过点 $(1,2)$ 流线方程并画 $t_1 = 0, t_2 = 2$ 时

刻过点 $(1,2)$ 的流线图。
流线方程:

$$\frac{dx}{x\,(1+2t)}=\frac{dy}{y}$$

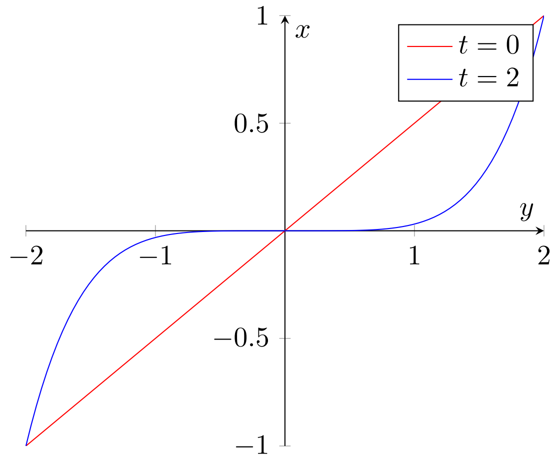
积分得:

$$\frac{1}{1+2t}\ln x=\ln Cy$$

而 $x=1,y=2,$因此$C=\frac{1}{2}$, 所以:

$$x=\left(\frac{y}{2}\right)^{1+2t}$$

流线图:



Assignment 4

Question A

已知流场的速度分布为 $u=4x^3, v=-10x^2y, \omega=2t$, 试问 (求):

1. 该流场属于几维流动?

流场在三个方向都有速度分量，所以是三维流动、

2.$t=1$ 时点 $(2,1,3)$ 处的加速度:

$$
\begin{aligned}
\vec{a} &= \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} \\
a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z} = 48x^5 \\
a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial z} = 20x^4y \\
a_z &= \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + \omega\frac{\partial \omega}{\partial z} = 2
\end{aligned}
$$

所以加速度是:

$$\vec{a} = \left(48x^5, 20x^4y, 2\right)$$

Question B

$$
\begin{aligned}
\delta\vec{r} &= \vec{r} - \vec{r_0} \\
\frac{\mathrm{d}}{\mathrm{d}t}(\delta\vec{r}) &= \delta\vec{v} = \delta\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r})
\end{aligned}
$$

即 δ 和 $\frac{\mathrm{d}}{\mathrm{d}t}()$ 可以互换。

因此:

$$\delta\vec{r} = \delta\vec{v} + \delta\vec{r}\cdot\nabla\vec{v}$$

Question C

1. 求 $\oint \vec{V}\cdot d\vec{l}$ 斯托克斯公式:

$$\oint \vec{V}\cdot d\vec{l} = \iint_S \nabla\times\vec{V}\cdot d\vec{S} = 0$$

2. 求 $\nabla\times\vec{V}$

$$
\begin{aligned}
\nabla\times\vec{V} &= \left(\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\boldsymbol{e_r} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\boldsymbol{e_\phi} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\phi) - \frac{1}{r}\frac{\partial v_r}{\partial \phi}\right]\boldsymbol{e_z} \\
&= \frac{1}{r}\frac{\partial r v_\phi}{\partial r}\boldsymbol{e_z} \\
&= \boldsymbol{0}
\end{aligned}
$$