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Theories and Observations of Dark Energy

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1 FRLW Universes

1.1 FRLW Metric

For a 3-D hyper-sphere, the distance between two points of space is:

$$ds^2 = f(r)dr^2 + r^2d\theta + r^2 \sin^2 \theta \quad (1.1.1)$$

Gaussian Curvature:

$$k = \frac{1}{2f^2(r)r} \frac{df(r)}{dr} \quad (1.1.2)$$

Solution:

$$f(r) = \frac{1}{C - kr^2} \quad (1.1.3)$$

Let $C = 1$, and introduce cosmological scale :

$$K = -\frac{k}{a^2} \quad (1.1.4)$$

$$-c^2d\tau^2 = -c^2dt^2 + a^2[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] \quad (1.1.5)$$

$$g_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2 \sin^2 \theta \end{bmatrix} \quad (1.1.6)$$

$$g_{00} = -c^2, \quad g_{11} = \frac{a^2}{1-Kr^2}, \quad g_{22} = a^2r^2, \quad g_{33} = a^2r^2 \sin^2 \theta \quad (1.1.7)$$

$$g^{00} = -\frac{1}{c^2}, \quad g^{11} = \frac{1-Kr^2}{a^2}, \quad g^{22} = \frac{1}{a^2r^2}, \quad g^{33} = \frac{1}{a^2r^2 \sin^2 \theta} \quad (1.1.8)$$

$$g_{00,\gamma} = 0 \quad (\gamma = 0, 1, 2, 3) \quad (1.1.9)$$

$$g_{11,0} = \frac{1}{c^2} \frac{2a\dot{a}}{1-Kr^2}, \quad g_{11,1} = \frac{2a^2Kr}{(1-Kr^2)^2}, \quad g_{11,2} = g_{11,3} = 0 \quad (1.1.10)$$

$$g_{22,0} = 2a\dot{a}r^2, \quad g_{22,1} = 2a^2r, \quad g_{22,2} = g_{22,3} = 0 \quad (1.1.11)$$

$$g_{33,0} = 2a\dot{a}r^2 \sin^2 \theta, \quad g_{33,1} = 2a^2r \sin^2 \theta, \quad g_{33,2} = 2a^2r \sin^2 \theta, \quad g_{33,3} = 0 \quad (1.1.12)$$

1.2 Christoffel Symbols

$$\Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\gamma} (g_{\mu\gamma,\nu} + g_{\nu\gamma,\mu} - g_{\mu\nu,\gamma}) \quad (1.2.1)$$

Non-zero components of Christoffel Symbols

$$\Gamma_{11}^0 = \frac{1}{2}g^{0\gamma}(g_{1\gamma,1} + g_{1\gamma,1} - g_{11,\gamma}) = -\frac{1}{2}g^{00}g_{11,0} = \frac{1}{c^2}\frac{a\dot{a}}{1-Kr^2} \quad (1.2.2)$$

$$\Gamma_{22}^0 = \frac{1}{2}g^{0\gamma}(g_{2\gamma,2} + g_{2\gamma,2} - g_{22,\gamma}) = -\frac{1}{2}g^{00}g_{22,0} = \frac{1}{c^2}a\dot{a}r^2 \quad (1.2.3)$$

$$\Gamma_{33}^0 = \frac{1}{2}g^{0\gamma}(g_{3\gamma,3} + g_{3\gamma,3} - g_{33,\gamma}) = -\frac{1}{2}g^{00}g_{33,0} = \frac{1}{c^2}a\dot{a}r^2 \sin^2\theta \quad (1.2.4)$$

$$\Gamma_{j0}^i = \frac{1}{2}g^{i\gamma}(g_{j\gamma,0} + g_{0\gamma,j} - g_{j0,\gamma}) = \frac{1}{2}g^{ii}g_{ij,0} = \frac{\dot{a}}{a}\delta_j^i \quad (1.2.5)$$

$$\Gamma_{11}^1 = \frac{1}{2}g^{1\gamma}(g_{1\gamma,1} + g_{1\gamma,1} - g_{11,\gamma}) = \frac{1}{2}g^{11}g_{11,1} = \frac{Kr}{1-Kr^2} \quad (1.2.6)$$

$$\Gamma_{22}^1 = \frac{1}{2}g^{1\gamma}(g_{2\gamma,2} + g_{2\gamma,2} - g_{22,\gamma}) = -\frac{1}{2}g^{11}g_{22,1} = -r(1-Kr^2) \quad (1.2.7)$$

$$\Gamma_{33}^1 = \frac{1}{2}g^{1\gamma}(g_{3\gamma,3} + g_{3\gamma,3} - g_{33,\gamma}) = -\frac{1}{2}g^{11}g_{33,1} = -r \sin^2\theta(1-Kr^2) \quad (1.2.8)$$

$$\Gamma_{12}^2 = \frac{1}{2}g^{2\gamma}(g_{1\gamma,2} + g_{2\gamma,1} - g_{12,\gamma}) = \frac{1}{2}g^{22}g_{22,1} = \frac{1}{r} \quad (1.2.9)$$

$$\Gamma_{13}^3 = \frac{1}{2}g^{3\gamma}(g_{1\gamma,3} + g_{3\gamma,1} - g_{13,\gamma}) = \frac{1}{2}g^{33}g_{33,1} = \frac{1}{r} \quad (1.2.10)$$

$$\Gamma_{23}^3 = \frac{1}{2}g^{3\gamma}(g_{2\gamma,3} + g_{3\gamma,2} - g_{23,\gamma}) = \frac{1}{2}g^{33}g_{33,2} = -\sin\theta \cos\theta \quad (1.2.11)$$

$$\Gamma_{33}^2 = \frac{1}{2}g^{2\gamma}(g_{3\gamma,3} + g_{3\gamma,3} - g_{33,\gamma}) = \frac{1}{2}g^{22}g_{33,2} = \cot\theta \quad (1.2.12)$$

for $i, j = 1, 2, 3$

1.3 Ricci Tensor

$$R_{\mu\nu} = \Gamma_{\mu\nu,\sigma}^\sigma - \Gamma_{\mu\sigma,\nu}^\sigma + \Gamma_{\rho\sigma}^\sigma \Gamma_{\mu\nu}^\rho - \Gamma_{\rho\nu}^\sigma \Gamma_{\mu\sigma}^\rho \quad (1.3.1)$$

Non-zero components of Ricci Tensor:

$$\begin{aligned} R_{00} &= \Gamma_{00,\sigma}^\sigma - \Gamma_{0\sigma,0}^\sigma + \Gamma_{\rho\sigma}^\sigma \Gamma_{00}^\rho - \Gamma_{\rho 0}^\sigma \Gamma_{0\sigma}^\rho \\ &= \Gamma_{00,0}^0 - \Gamma_{0\sigma,0}^\sigma + \Gamma_{0\sigma}^\sigma \Gamma_{00}^0 - \Gamma_{\sigma 0}^\sigma \Gamma_{0\sigma}^0 \\ &= 0 - 3\frac{\ddot{a}a - \dot{a}^2}{a^2} + 0 - 3\left(\frac{\dot{a}}{a}\right)^2 \\ &= -3\frac{\ddot{a}}{a} \end{aligned} \quad (1.3.2)$$

$$\begin{aligned} R_{11} &= \Gamma_{11,\sigma}^\sigma - \Gamma_{1\sigma,1}^\sigma + \Gamma_{\rho\sigma}^\sigma \Gamma_{11}^\rho - \Gamma_{\rho 1}^\sigma \Gamma_{1\sigma}^\rho \\ &= (\Gamma_{11,1}^1 + \Gamma_{11,2}^2 + \Gamma_{11,3}^3) - (\Gamma_{11,0}^0 + \Gamma_{11,1}^1) \\ &\quad + (\Gamma_{10}^1 \Gamma_{11}^0 + \Gamma_{11}^2 \Gamma_{10}^1 + \Gamma_{11}^1 \Gamma_{11}^1 + \Gamma_{12}^2 \Gamma_{12}^2 + \Gamma_{13}^3 \Gamma_{13}^3) \\ &\quad - (\Gamma_{11}^0 \Gamma_{01}^1 + \Gamma_{11}^0 \Gamma_{02}^2 + \Gamma_{11}^0 \Gamma_{03}^3 + \Gamma_{11}^1 \Gamma_{11}^1 + \Gamma_{11}^1 \Gamma_{12}^2 + \Gamma_{11}^1 \Gamma_{13}^3) \\ &= \frac{1}{c^2}\left[\frac{2}{r^2} + \frac{\dot{a}^2 + a\ddot{a}}{1-Kr^2} - \frac{2}{r^2} + \frac{\dot{a}^2}{1-Kr^2} + \frac{2K}{1-Kr^2}\right] \\ &= \frac{1}{c^2}\frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1-Kr^2} \\ &= \frac{1}{c^2}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2}\right)g_{11} \end{aligned} \quad (1.3.3)$$

$$\begin{aligned}
R_{22} &= \Gamma_{22,\sigma}^\sigma - \Gamma_{2\sigma,2}^\sigma + \Gamma_{\rho\sigma}^\sigma \Gamma_{22}^\rho - \Gamma_{\rho 2}^\sigma \Gamma_{2\sigma}^\rho \\
&= (\Gamma_{22,0}^0 + \Gamma_{22,1}^1) - (\Gamma_{23,2}^3) + (\Gamma_{0\sigma}^\sigma \Gamma_{22}^0 + \Gamma_{0\sigma}^\sigma \Gamma_{22}^1) \\
&\quad - (\Gamma_{22}^0 \Gamma_{20}^2 + \Gamma_{22}^1 \Gamma_{22}^2 + \Gamma_{02}^2 \Gamma_{22}^0 + \Gamma_{12}^2 \Gamma_{22}^1 + \Gamma_{32}^3 \Gamma_{23}^3) \\
&= \frac{1}{c^2} \left(\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{K}{a^2} \right) g_{22}
\end{aligned} \tag{1.3.4}$$

$$\begin{aligned}
R_{33} &= \Gamma_{33,\sigma}^\sigma - \Gamma_{3\sigma,3}^\sigma + \Gamma_{\rho\sigma}^\sigma \Gamma_{33}^\rho - \Gamma_{\rho 3}^\sigma \Gamma_{3\sigma}^\rho \\
&= (\Gamma_{33,0}^0 + \Gamma_{33,1}^1 + \Gamma_{33,2}^2) - 0 \\
&\quad + (\Gamma_{0\sigma}^\sigma \Gamma_{33}^0 + \Gamma_{1\sigma}^\sigma \Gamma_{33}^1 + \Gamma_{2\sigma}^\sigma \Gamma_{33}^2) \\
&\quad - (\Gamma_{\rho 3}^0 \Gamma_{30}^\rho + \Gamma_{\rho 3}^1 \Gamma_{31}^\rho + \Gamma_{\rho 3}^2 \Gamma_{32}^\rho + \Gamma_{\rho 3}^3 \Gamma_{33}^\rho) \\
&= \frac{1}{c^2} \left(\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{K}{a^2} \right) g_{33}
\end{aligned} \tag{1.3.5}$$

We can get the following equations in using the same principle:

$$R_{00} = -3 \frac{\ddot{a}}{a} \tag{1.3.6}$$

$$R_{ij} = \frac{1}{c^2} \left(\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{K}{a^2} \right) g_{ij} \tag{1.3.7}$$

$$R = \frac{6}{c^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \right) \tag{1.3.8}$$

1.4 Friedmann Equation

Einstein Field Equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.4.1}$$

Energy-Momentum Tensor:

$$T_{\mu\nu} = (\rho + \frac{p}{c^2}) u_\mu u_\nu + p g_{\mu\nu} \tag{1.4.2}$$

Cosmological Principle:

$$T_{\mu\nu} = \text{diag}\{\rho c^2, p, p, p\} \tag{1.4.3}$$

According to the above equations:

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{K c^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3} \rho \tag{1.4.4}$$

$$2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{K c^2}{a^2} - \Lambda c^2 = - \frac{8\pi G}{c^2} p \tag{1.4.5}$$

Define Hubble parameter:

$$H = \frac{\dot{a}}{a} \tag{1.4.6}$$

Equation of State:

$$p = w \rho c^2 \tag{1.4.7}$$

Using 1.4.4, 1.4.6 and 1.4.7, 1.4.5 can be re-expressed as:

$$\dot{\rho} + 3(1+w)H\rho = 0 \tag{1.4.8}$$

Which is Continuous Equation.

Assume the expanding of universe is a Adiabatic Process, First Law of Thermodynamics:

$$dE + pdV = 0 \quad (1.4.9)$$

Where:

$$E = (\rho_m + \rho_r)V = \rho V, V \propto a^3 \quad (1.4.10)$$

1.4.9 can be re-expressed as:

$$\frac{d}{dt}(\rho a^3) + p \frac{d}{dt}(a^3) = 0 \quad (1.4.11)$$

Particles of matter are non-relative, $w_m = 0$, while pressure p is mainly provided by radiation which means $w_r = 1/3$, so 1.4.8 can be re-expressed as:[3]

$$\frac{d}{dt}(\rho_m a^3) + \frac{1}{a} \frac{d}{dt}(\rho_r a^4) = 0 \quad (1.4.12)$$

Assume matter is strictly conserved:

$$\frac{d}{dt}(\rho_m a^3) = 0, \quad \frac{1}{a} \frac{d}{dt}(\rho_r a^4) = 0 \quad (1.4.13)$$

Which means:

$$\rho_m \propto a^{-3}, \quad \rho_r \propto a^{-4} \quad (1.4.14)$$

Define Cosmological Critical Density: $\rho_c = \frac{3H_0^2}{8\pi G}$ And 1.4.4 can be re-expressed as:

$$\frac{H^2}{H_0^2} = \Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{K,0} a^{-2} + \Omega_{\Lambda,0} \quad (1.4.15)$$

Where:

$$\Omega_{r,0} = \frac{8\pi G \rho_{r,0}}{3H_0^2}, \quad \Omega_{m,0} = \frac{8\pi G \rho_{m,0}}{3H_0^2}, \quad \Omega_{K,0} = -\frac{K}{H_0^2 a_0^2}, \quad \Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2} \quad (1.4.16)$$

1.5 Cosmological Compositions Dominant Era:

Current Cosmological Parameter:

$$\begin{aligned} \Omega_{m,0} &= 0.3, & \Omega_{\Lambda,0} &= 0.7, & \Omega_{r,0} &= 10^{-5} \\ H_0^{-1} &= 14Gyr, & a_0 &= 1 \end{aligned} \quad (1.5.1)$$

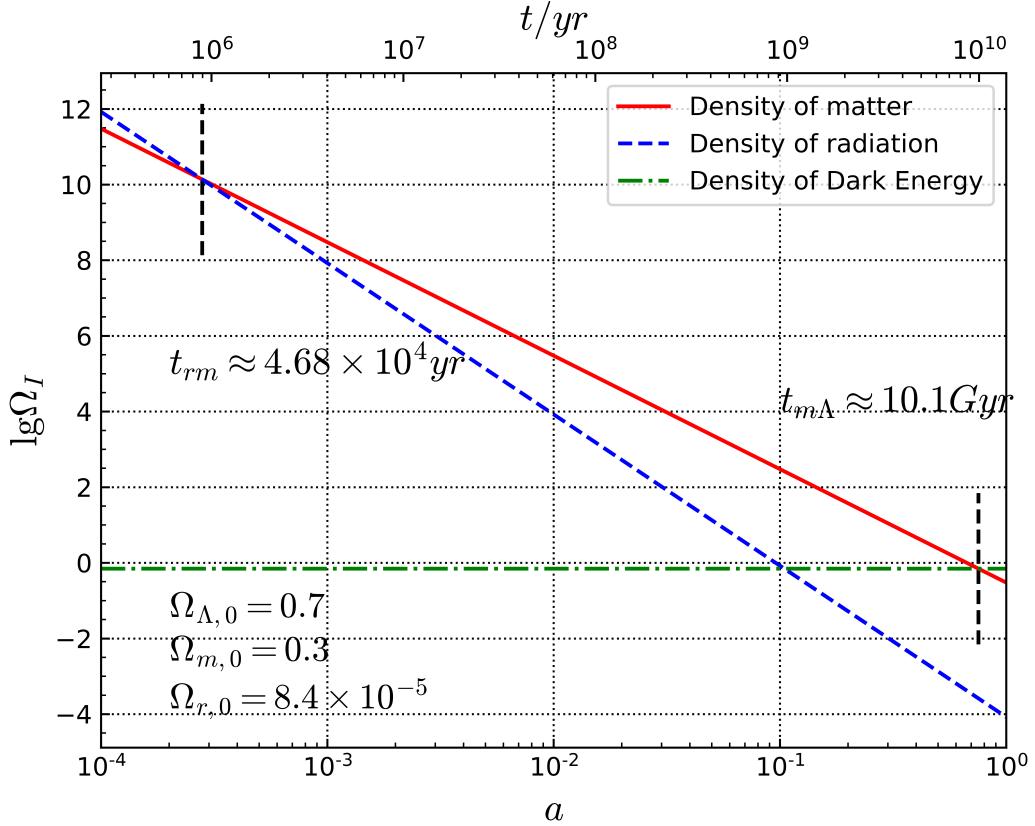


Figure 1: The evolution of matter density and radiation density over time.,available at https://github.com/Wang-ZhengYi/DE_HW

The evolution from Radiation Era to Matter Era:

$$\Omega_{r,0} \frac{a_{eq}^{-4}}{a_0^{-4}} = \Omega_{m,0} \frac{a_{eq}^{-3}}{a_0^{-3}} \quad (1.5.2)$$

$$a_{eq} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.80 \times 10^{-4} \quad (1.5.3)$$

$$z_{eq} = \frac{a_0}{a_{eq}} - 1 \approx 3.57 \times 10^4 \quad (1.5.4)$$

$$\begin{aligned} t_{eq} &= \frac{1}{H_0} \int_{z_{eq}}^{\infty} \frac{dz}{(1+z)E(z)^{1/2}} \\ &= \frac{1}{H_0} \int_{z_{eq}}^{\infty} \frac{dz}{(1+z)\sqrt{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}} \\ &\approx 4.68 \times 10^4 \text{ yr} \end{aligned} \quad (1.5.5)$$

The accelerating moment:

$$q_c = \frac{\Omega_m}{2} - \Omega_\Lambda = \frac{\Omega_{m,0}}{2} \frac{a_c^{-3}}{a_0^{-3}} - \Omega_{\Lambda,0} = 0 \quad (1.5.6)$$

$$a_c = \left(\frac{\Omega_{m,0}}{2\Omega_\Lambda}\right)^{1/3} \approx 0.598 \quad (1.5.7)$$

$$z_c = \frac{a_0}{a_c} - 1 \approx 0.671 \quad (1.5.8)$$

$$t_c = \frac{1}{H_0} \int_{z_c}^{\infty} \frac{dz}{(1+z)E(z)^{1/2}} \approx 7.54 \text{Gyr} \quad (1.5.9)$$

1.6 Big Bang Criterion

Critical condition of Big Bang Theory:[2]

$$H^2 = H_0^2 [\Omega_{\Lambda,0} + (1 - \Omega_{m,0} - \Omega_{\Lambda,0})a_c^{-2} + \Omega_{m,0}a_c^{-3}] = 0 \quad (1.6.1)$$

Which can be re-express as:

$$4(1 - \Omega_{m,0} - \Omega_{\Lambda,0}) + 27\Omega_{m,0}^2\Omega_{\Lambda,0} = 0 \quad (1.6.2)$$

Introduce the following variable:

$$x = \left(\frac{\Omega_{\Lambda,0}}{4\Omega_{m,0}}\right) \quad (1.6.3)$$

1.6.2 can quickly reduce to:

$$x^3 - \frac{3}{4}x - \frac{1 - \Omega_{m,0}}{\Omega_{m,0}} = 0 \quad (1.6.4)$$

Assume $\Omega_{\Lambda,0} > 0$, we can get 3 cases of 1.6.3:

- $0 < \Omega_{m,0} \leq \frac{1}{2}$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0} \cosh\left[\frac{1}{3} \cosh^{-1}\left(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}}\right)\right] \quad (1.6.5)$$

- $\frac{1}{2} < \Omega_{m,0} \leq 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0} \cos\left[\frac{1}{3} \cos^{-1}\left(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}}\right)\right] \quad (1.6.6)$$

- $\Omega_{m,0} > 1$

$$\Omega_{\Lambda,0} = 4\Omega_{m,0} \cos\left[\frac{1}{3} \cos^{-1}\left(\frac{1 - \Omega_{m,0}}{\Omega_{m,0}}\right) + \frac{4\pi}{3}\right] \quad (1.6.7)$$

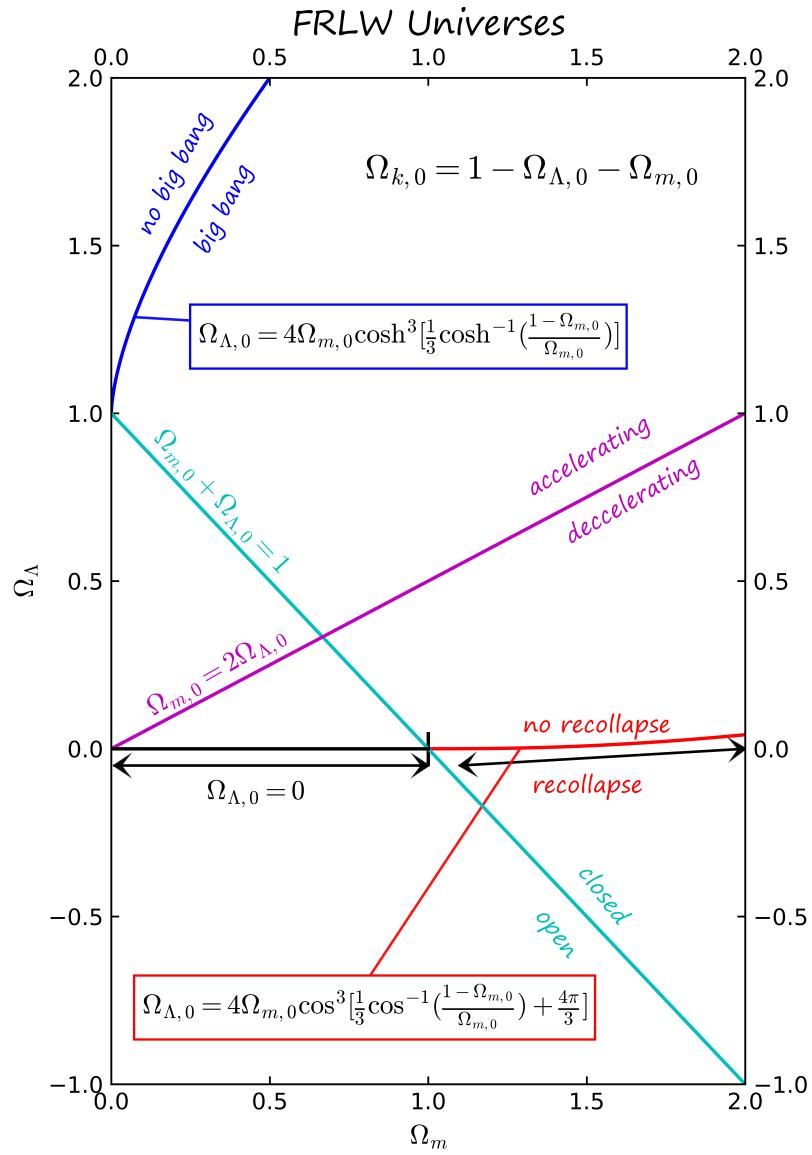


Figure 2: FRLW universes, differences of the current density of matter and dark energy make different universes, available at https://github.com/Wang-ZhengYi/DE_HW

The scale factor a must be negative in the past if point $(\Omega_{m,0}, \Omega_{\Lambda,0})$ is above the blue solid line in Figure 2.

2 Perturbation Theory

2.1 The Conform Newton Gauge:

$$-c^2 d\tau^2 = a^2(\eta) [-(1+2\psi)d\eta^2 + (1-2\phi)\delta_{\alpha\beta}dx^\alpha dx^\beta] \quad (2.1.1)$$

$$g_{\mu\nu} = a^2(\eta) \begin{bmatrix} -(1+2\psi) & 0 & 0 & 0 \\ 0 & 1-2\phi & 0 & 0 \\ 0 & 0 & 1-2\phi & 0 \\ 0 & 0 & 0 & 1-2\phi \end{bmatrix} \quad (2.1.2)$$

2.2 Christoffel Symbols

$$\Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\gamma} (g_{\mu\gamma,\nu} + g_{\nu\gamma,\mu} - g_{\mu\nu,\gamma}) \quad (1.2.1)$$

Non-zero components of Christoffel Symbols

$$\Gamma_{00}^0 = H + \dot{\psi} \quad (2.1.3)$$

$$\Gamma_{0i}^0 = \psi_{,i} \quad (2.1.4)$$

$$\Gamma_{ij}^0 = [H - \dot{\phi} - 2H(\psi + \phi)]\delta_{ij} \quad (2.1.5)$$

$$\Gamma_{00}^i = \frac{1}{a^2}\psi_{,i} \quad (2.1.6)$$

$$\Gamma_{i0}^k = (H - \dot{\phi})\delta_{ik} \quad (2.1.7)$$

$$\Gamma_{ij}^k = -\phi_{,j}\delta_{ik} - \phi_{,i}\delta_{jk} + \phi_{,k}\delta_{ij} \quad (2.1.8)$$

2.3 Ricci Tensor

$$R_{\mu\nu} = \Gamma_{\mu\nu,\sigma}^\sigma - \Gamma_{\mu\sigma,\nu}^\sigma + \Gamma_{\rho\sigma}^\sigma \Gamma_{\mu\nu}^\rho - \Gamma_{\rho\nu}^\sigma \Gamma_{\mu\sigma}^\rho \quad (1.3.1)$$

Non-zero components of Ricci Tensor:

$$\begin{aligned} R_{0i} &= \Gamma_{0i,\sigma}^\sigma - \Gamma_{0\sigma,i}^\sigma + \Gamma_{\rho\sigma}^\sigma \Gamma_{0i}^\rho - \Gamma_{\rho i}^\sigma \Gamma_{0\sigma}^\rho \\ &= \Gamma_{0i,0}^0 + \Gamma_{0i,j}^j - (\Gamma_{00,i}^0 + \Gamma_{0j,i}^j) + (\Gamma_{00}^0 \Gamma_{0i}^0 + \Gamma_{0j}^j \Gamma_{0i}^0 + \Gamma_{j0}^0 \Gamma_{0i}^j \\ &\quad + \Gamma_{jk}^k \Gamma_{0i}^j) - (\Gamma_{0i}^0 \Gamma_{00}^0 + \Gamma_{0i}^j \Gamma_{0j}^0 + \Gamma_{ji}^0 \Gamma_{00}^j + \Gamma_{ji}^k \Gamma_{0k}^j) \\ &= \dot{\psi}_{,i} - \dot{\phi}_{,i} - (\dot{\psi}_{,i} - 3\dot{\phi}_{,i}) + 3H\dot{\psi}_{,i} + 3H\dot{\phi}_{,i} - (H\dot{\psi}_{,i} + 3H\dot{\phi}_{,i}) \\ &= 2\dot{\phi}_{,i} + 2H\dot{\psi}_{,i} \end{aligned} \quad (2.1.9)$$

We can get the following equations in using the same principle:

$$R_{00} = -3\frac{\ddot{a}}{a} + \frac{1}{a}\psi_{,ii} + 3\ddot{\phi} + 3H(\dot{\psi} + 2\dot{\phi}) \quad (2.1.10)$$

$$R_{ij} = \delta_{ij}[(2\ddot{a}^2 + a\dot{a})(1 - 2\phi + 2\psi) - a^2 H(6\dot{\phi} + \dot{\psi}) - a^2 \ddot{\phi} + \phi_{,ii}] - (\psi - \phi)_{,ij} \quad (2.1.11)$$

$$R = 6(H^2 + \frac{\dot{a}}{a})(1 - 2\psi) - \frac{2}{a^2}\psi_{,ii} - 6\ddot{\phi} - 6H(\dot{\psi} + 4\dot{\phi}) + \frac{4}{a^2}\phi_{,ii} \quad (2.1.12)$$

3 Dark Energy models(Dark Energy Field)

3.1 Quintessence

The action of calibration ϕ :

$$S = - \int d^4x \sqrt{-g} \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] \quad (3.1.1)$$

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) \quad (3.1.2)$$

Energy-Momentum Tensor:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu} \left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi) \right] \quad (3.1.3)$$

Cosmological principle:

$$\rho = -\frac{1}{c^2}T_0^0 = \frac{1}{c^2} \left[\frac{1}{2}\dot{\phi}^2 + V(\phi) \right] \quad (3.1.4)$$

$$p = T_i^i = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad (3.1.5)$$

Continuous Equation:

$$\begin{aligned} & \dot{\rho} + 3H(\rho + \frac{p}{c^2}) \\ &= \frac{1}{c^2} \left[\dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi} + 3H\dot{\phi}^2 \right] \\ &= \frac{\dot{\phi}}{c^2}(\ddot{\phi} + \frac{dV}{d\phi} + 3H\dot{\phi}) \\ &= 0 \end{aligned} \quad (3.1.6)$$

That is:

$$\ddot{\phi} + \frac{dV}{d\phi} + 3H\dot{\phi} = 0 \quad (3.1.7)$$

3.2 Quintessence Calibration Field

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi) \quad (3.2.1)$$

Hubble parameter and its time derivation:

$$H^2 = \frac{\kappa^2}{3} \left[\frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi) + \rho_m \right] \quad (3.2.2)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[\varepsilon\dot{\phi}^2 + (1+w_m)\rho_m \right] \quad (3.2.3)$$

Where: $\kappa^2 = 8\pi G$, $\varepsilon = \pm 1$

Continuous Equation:

$$\varepsilon(\ddot{\phi} + 3H\dot{\phi}) + \frac{dV}{d\phi} = 0 \quad (3.2.4)$$

Introduce the following variables:

$$N = \ln a, \lambda = -\frac{1}{\kappa V} \frac{dV}{d\phi}, x = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, y = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \quad (3.2.5)$$

$$\begin{aligned} \frac{dy}{dN} &= \frac{1}{H} \frac{dy}{dt} = \frac{\kappa}{\sqrt{3}H} \frac{d}{dt} \left(\frac{\sqrt{V}}{H} \right) \\ &= \frac{\kappa}{\sqrt{3}H} \frac{\frac{1}{2\sqrt{V}} \frac{dV}{d\phi} \dot{\phi} H - \dot{H} \sqrt{V}}{H^2} \\ &= -\frac{\sqrt{6}}{2} \left(-\frac{1}{\kappa V} \frac{dV}{d\phi} \right) \left(\frac{\kappa\dot{\phi}}{\sqrt{6}H} \right) \left(\frac{\kappa\sqrt{V}}{\sqrt{3}H} \right) - \frac{\kappa\sqrt{V}}{\sqrt{3}H} \frac{\dot{H}}{H^2} \\ &= -\frac{\sqrt{6}}{2} \lambda xy + \frac{\kappa^2}{2} y \frac{\varepsilon\dot{\phi}^2 + (1+w_m)\rho_m}{H^2} \\ &= -\frac{\sqrt{6}}{2} \lambda xy + \frac{1}{2} y \left[\frac{\varepsilon\kappa^2\dot{\phi}^2}{H^2} + \frac{\kappa^2(1+w_m)\rho_m}{H^2} \right] \\ &= -\frac{\sqrt{6}}{2} \lambda xy + \frac{1}{2} y \left[\frac{\varepsilon\kappa^2\dot{\phi}^2}{H^2} + \frac{\kappa^2(1+w_m)(\frac{3H^2}{\kappa^2} - \frac{1}{2}\varepsilon\dot{\phi}^2 - V)}{H^2} \right] \\ &= -\frac{\sqrt{6}}{2} \lambda xy + \frac{1}{2} y \left[6\frac{\varepsilon\kappa^2\dot{\phi}^2}{6H^2} + 3(1+w_m) - 3(1+w_m) \left(\frac{\varepsilon\kappa^2\dot{\phi}^2}{6H^2} + \frac{\kappa^2V}{3H} \right) \right] \\ &= -\frac{\sqrt{6}}{2} \lambda xy + \frac{3}{2} y[(1-w_m)\varepsilon x^2 + (1+w_m)(1-y^2)] \end{aligned} \quad (3.2.6)$$

4 Density Evolution

4.1 Energy density perturbation

Perturbation scale factor

$$a_p = a(1 - \frac{\delta}{3}) \quad (4.1.1)$$

Friedmann Equation without Dark Energy

$$\frac{\ddot{a}}{a} = -\frac{H^2}{2}\Omega_{m,0}a^{-3} \quad (4.1.2)$$

$$\begin{aligned} \frac{\ddot{a}_p}{a_p} &= \frac{\ddot{a}(1 - \frac{\delta}{3}) - \frac{1}{3}(2\dot{a}\dot{\delta} + a\ddot{\delta})}{a(1 - \frac{\delta}{3})} \\ &= \frac{\ddot{a}}{a} - \frac{\ddot{\delta} + 2H\dot{\delta}}{3 - \delta} \\ &= -\frac{H^2}{2}\Omega_{m,0}a^{-3} - \frac{\ddot{\delta} + 2H\dot{\delta}}{3 - \delta} \\ &= -\frac{H^2}{2}\Omega_m(a) \frac{1}{(1 - \frac{\delta}{3})^3} \\ &\approx -\frac{H^2}{2}\Omega_m(a)(1 + \delta) \end{aligned} \quad (4.1.3)$$

Ignore second order small quantity

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\Omega_m(a)\delta = 0 \quad (4.1.4)$$

Which is Perturbation Evolution Equation

4.2 Linear Growth Factor

Friedmann Equation with Dark Energy

$$\frac{\ddot{a}}{a} = -\frac{H^2}{2}[\Omega_m(a) + (1 + 3w(a))(1 - \Omega_m(a))] \quad (4.2.1)$$

$$\begin{aligned} \frac{\ddot{a}_p}{a_p} &= \frac{\ddot{a}(1 - \frac{\delta}{3}) - \frac{1}{3}(2\dot{a}\dot{\delta} + a\ddot{\delta})}{a(1 - \frac{\delta}{3})} \\ &= \frac{\ddot{a}}{a} - \frac{\ddot{\delta} + 2H\dot{\delta}}{3 - \delta} \\ &= -\frac{H^2}{2}[\Omega_m(a) + (1 + 3w(a))(1 - \Omega_m(a))] - \frac{\ddot{\delta} + 2H\dot{\delta}}{3 - \delta} \\ &= -\frac{H^2}{2}[\Omega_m(a) \frac{1}{(1 - \frac{\delta}{3})^3} + (1 + 3w(a))(1 - \Omega_m(a)) \frac{1}{(1 - \frac{\delta}{3})^3}] \\ &\approx -\frac{H^2}{2}[\Omega_m(a)(1 + \delta) + (1 + 3w(a))(1 - \Omega_m(a)(1 + \delta))] \end{aligned} \quad (4.2.2)$$

Define $X(a)$:

$$X(a) = \frac{\Omega_m}{1 - \Omega_m} = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} \exp[-3 \int_a^{a_0} d \ln a' w(a')] = \Omega_m a^{-3} \left(\frac{\delta H^2}{H_0^2} \right)^{-1} \quad (4.2.3)$$

Growth Factor

$$D = \frac{\delta(a)}{\delta(a_0)} \quad (4.2.4)$$

And its first order derivative and second order derivative:

$$D' = \frac{1}{aH} \frac{\dot{\delta}(a)}{\delta(a_0)} \quad (4.2.5)$$

$$D'' = \frac{1}{a^2 H^2} \frac{\ddot{\delta}(a) - \dot{\delta}(a) \frac{\ddot{a}}{a}}{\delta(a_0)} \quad (4.2.6)$$

We can get the following equation from 4.2.1-4.2.6:

$$D'' + \frac{3}{2} \left[1 - \frac{w(a)}{1+X(a)} \right] \frac{D'}{a} - \frac{3}{2} \frac{X(a)}{1+X(a)} \frac{D}{a^2} = 0 \quad (4.2.7)$$

Linear Growth Factor

$$G = \frac{D}{a} \quad (4.2.8)$$

And its first order derivative and second order derivative:

$$\begin{aligned} G' &= \frac{aD' - D}{a^2} \\ &= \frac{1}{a^2} \left(\frac{1}{H} \frac{\dot{\delta}(a)}{\delta(a_0)} - \frac{\delta(a)}{\delta(a_0)} \right) \end{aligned} \quad (4.2.9)$$

$$\begin{aligned} G'' &= \frac{1}{a^2} (a^2 D'' - 2aD' + 2D) \\ &= \frac{1}{a^3} \left(\frac{1}{H^2} \frac{\ddot{\delta}(a) - \dot{\delta}(a) \frac{\ddot{a}}{a}}{\delta(a_0)} - \frac{2}{H} \frac{\dot{\delta}(a)}{\delta(a_0)} + \frac{2\delta(a)}{\delta(a_0)} \right) \end{aligned} \quad (4.2.10)$$

We can get the following equation from 4.2.7-4.2.10:

$$G'' + \left[\frac{7}{2} - \frac{3}{2} \frac{w(a)}{1+X(a)} \right] \frac{G'}{a} + \frac{3}{2} \frac{1-w(a)}{1+X(a)} \frac{G}{a^2} = 0 \quad (4.2.11)$$

4.3 Numerical Simulation

4.2.7 in ΛCDM model:

$$G(a) \simeq \frac{5\Omega_m(a)}{2} \left[\Omega_m(a)^{4/7} - \Omega_\Lambda(a) + \left(1 + \frac{\Omega_m(a)}{2} \right) \left(1 + \frac{\Omega_\Lambda(a)}{70} \right) \right]^{-1} \quad (4.3.1)$$

w models:[1]

$$w(a) = w_0 + aw' \quad (4.3.2)$$

$$\begin{aligned} X(a) &= \frac{\Omega_{m,0}}{1-\Omega_{m,0}} \exp \left[-3 \int_a^1 d \ln a' w(a') \right] \\ &= \frac{\Omega_{m,0}}{1-\Omega_{m,0}} a^{3w_0} \exp \left[-3w'(1-a) \right] \end{aligned} \quad (4.3.3)$$

And:

$$\begin{aligned} G'' &= - \left[\frac{7}{2} - \frac{3}{2} \frac{w(a)}{1+X(a)} \right] \frac{G'}{a} - \frac{3}{2} \frac{1-w(a)}{1+X(a)} \frac{G}{a^2} \\ &= A(a)G' + B(a)G \end{aligned} \quad (4.3.4)$$

Euler Method

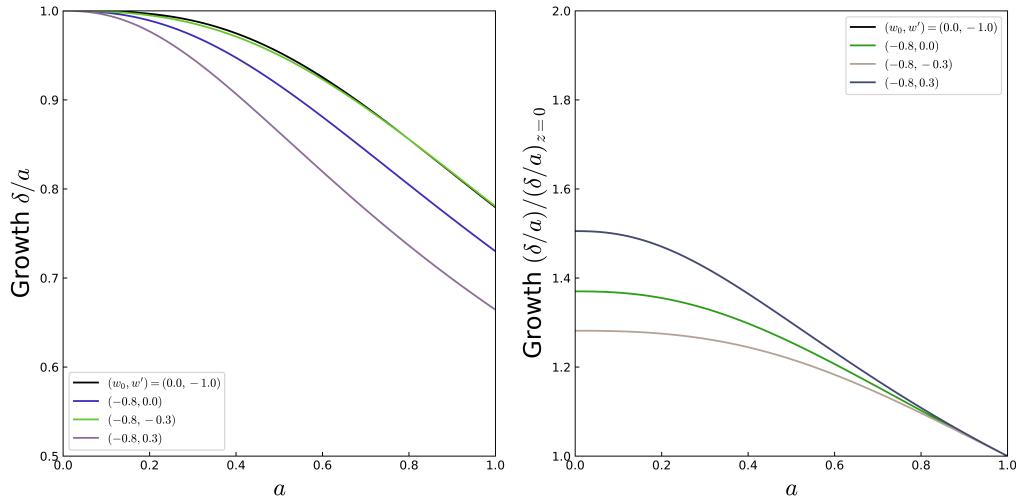


Figure 3: Linear growth factor and its normalized form, available at https://github.com/Wang-ZhengYi/DE_HW

References

- [1] E. V. Linder and A. Jenkins. Cosmic structure growth and dark energy. *Monthly Notices of the Royal Astronomical Society*, 346(2):573–583, 12 2003.
- [2] G. P. Efstathiou M. P. Hobson and A. N. Lasenby. *General Relativity: An Introduction for Physicists*. Cambridge University Press, 2 2006.
- [3] 冯珑珑向守平. 《宇宙大尺度结构的形成》. 中国科技出版社, 2011.