Math 6602 – Project 3

Assigned on: 04/10/2023 Due: 04/24/2023

You could use MATLAB, Python, C/C++, FORTRAN or any other programming language to write your code. Submit your project report as a single pdf file (named as *your last name_your first name_Project3.pdf*) on Carmen. Also, submit a copy of your code (in a zipped file, named as *your last name_your first name_Project3.zip*) through Carmen.

Write a program which solves the two-dimensional heat equation (let the computational domain be $\Omega = \{0 \le x, y \le 1\}$)

$$\begin{aligned} u_t - \Delta u &= 2t(1+\pi^2 t)\sin(\pi x)\sin(\pi y), \qquad \text{in } \Omega \times [0,1], \\ u &= 0, \qquad \text{on } \partial\Omega \times [0,1], \\ u(x,y,0) &= 0, \qquad \text{in } \Omega. \end{aligned}$$

using the finite difference method. Assuming the uniform grid points are used, i.e. $(x_i, y_j) = (ih, jh)$ where $h = \Delta x = \Delta y = 1/(J+1)$. The time step is chosen as $k = \Delta t = 1/N$. Easy to verify that $u(x,y,t) = t^2 \sin(\pi x)\sin(\pi y)$ is the unique solution to this initial-boundary value problem.

1. (5 points) Write down the explicit Euler (in time) five-point (in space) finite difference method for this problem. Convert it into a linear system problem using matrix notation.

2. (15 points) Write a program to implement this explicit Euler five-point method. Compute the numerical solution using the following choices of mesh sizes: (1): $(0.1, 0.05)$; (2): $(0.1, 0.002)$. Plot the graphs of the exact solution, the numerical solution and the error between these two solutions at the stopping time $t = 1$. Also plot the L2 norm of the error vs time (at each time step) for both runs. Explain and discuss your results. (L2 norm of the error is defined as: $\|e\|_{L^2} = (\sum_{i,j} h^2 e_{i,j}^2)^{\frac{1}{2}}$.)

3. (5 points) Write down the trapezoidal rule (in time) five-point (in space) finite difference method for this problem. Convert it into a linear system problem using matrix notation.

4. (15 points) Write a program to implement this trapezoidal rule five-point method. You can use any linear solver (such as the built-in MATLAB solver) to solve the linear system, but do mention which one you use. Compute the numerical solution using mesh sizes: $(0.1, 0.05)$. Plot the graphs of the exact solution, the numerical solution and the error between these two solutions at the stopping time $t = 1$. Also plot the L2 norm of the error vs time (at each time step) for both runs. Explain and discuss your results.