3. Evaluate the Frobenius matrix norm, 1-norm, 2-norm and ∞-norm for each matrix below. $A=\begin{pmatrix}1 & -2\\ -1 & 2\end{pmatrix}$ $B\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$ $C=\begin{pmatrix}4 & -2 & 4\\ -2 & 1 & -2\\ 4 & -2 & 4\end{pmatrix}$

A: $A^TA=\begin{pmatrix}1 & -1\\ -2 & 2\end{pmatrix}\begin{pmatrix}1 & -2\\ -1 & 2\end{pmatrix}=\begin{pmatrix}2 & -4\\ -4 & 8\end{pmatrix}$

$|\lambda E-A^TA|=\begin{vmatrix}\lambda-2 & 4\\ 4 & \lambda-8\end{vmatrix}=(\lambda-2)(\lambda-8)-16$ $\lambda_{max}=10$

$\therefore \|A\|_F=\sqrt{10}$ $\|A\|_1=4$ $\|A\|_2=\sqrt{\lambda_{max}}=\sqrt{10}$ $\|A\|_\infty=3$

B: $B^TB=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ B^TB的特征向量 $\lambda_{max}=1$

$\therefore \|B\|_F=\sqrt{3}$ $\|B\|_1=1$ $\|B\|_2=1$ $\|B\|_\infty=1$

C: $C^TC=\begin{pmatrix}4 & -2 & 4\\ -2 & 1 & -2\\ 4 & -2 & 4\end{pmatrix}\begin{pmatrix}4 & -2 & 4\\ -2 & 1 & -2\\ 4 & -2 & 4\end{pmatrix}=\begin{pmatrix}36 & -18 & 36\\ -18 & 9 & -18\\ 36 & -18 & 36\end{pmatrix}$

$|\lambda E-C^TC|=\begin{vmatrix}\lambda-36 & 18 & -36\\ 18 & \lambda-9 & 18\\ -36 & 18 & \lambda-36\end{vmatrix}=(\lambda-36)\begin{vmatrix}\lambda-9 & 18\\ 18 & \lambda-36\end{vmatrix}-18\begin{vmatrix}18 & 18\\ -36 & \lambda-36\end{vmatrix}-36\begin{vmatrix}18 & \lambda-9\\ -36 & 18\end{vmatrix}$

$=0$

$\lambda_{max}=81$

$\therefore \|C\|_F=\sqrt{81}=9$ $\|C\|_1=10$ $\|C\|_2=\sqrt{81}=9$ $\|C\|_\infty=10$

12. (a) $A=\begin{pmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 1 & 1 & -3\\ 0 & 1 & 1\end{pmatrix}$ $b=(1\ \ 1\ \ 1\ \ 1)$

(a) 求矩阵A的QR分解

$k=1$: $r_{11}=\|a_1\|=\sqrt{3}$ $q_1=\frac{a_1}{\|a_1\|}=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},0)^T$

$k=2$: $r_{12}=\langle a_2|q_1\rangle=a_2^T\cdot q_1=\sqrt{3}$ $q_2=\frac{a_2-r_{12}q_1}{\|a_2-r_{12}q_1\|}=(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},0,\frac{1}{\sqrt{3}})^T$ $r_{22}=\sqrt{3}$

$k=3$ $r_{13}=\langle a_3|q_1\rangle=a_3^Tq_1=-\sqrt{3}$ $r_{23}=\langle a_3|q_2\rangle=a_3^Tq_2=\sqrt{3}$

$r_{33}=\|a_3-r_{13}q_1-r_{23}q_2\|=\sqrt{6}$

$q_3=\frac{a_3-r_{13}q_1-r_{23}q_2}{\|a_3-r_{13}q_1-r_{23}q_2\|}=(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},0)^T$

所以 $Q=\begin{pmatrix}\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3}\\ 0 & \frac{\sqrt{3}}{3} & 0\end{pmatrix}$ $R=\begin{pmatrix}\sqrt{3} & \sqrt{3} & -\sqrt{3}\\ 0 & \sqrt{3} & \sqrt{3}\\ 0 & 0 & \sqrt{6}\end{pmatrix}$

(b) 求 $Ax=b$ 的最小二乘解

$Ax=b \Rightarrow QRx=b$ 因为Q是正交矩阵，则有 $Q^T=Q^{-1}$

$Rx=Q^Tb$，得 $\begin{pmatrix}\sqrt{3} & \sqrt{3} & -\sqrt{3}\\ 0 & \sqrt{3} & \sqrt{3}\\ 0 & 0 & \sqrt{6}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\frac{\sqrt{3}}{3}\begin{pmatrix}1 & -1 & \frac{\sqrt{2}}{2}\\ 1 & 1 & \frac{\sqrt{2}}{2}\\ 1 & 0 & -\sqrt{2}\\ 0 & 1 & 0\end{pmatrix}^T(1,1,1,1)^T$

$=(\sqrt{3}\ \ \frac{\sqrt{3}}{3}\ \ 0)^T$ 解得 $(x_1,x_2,x_3)^T=(\frac{2}{3},\frac{1}{3},0)$

16. u方向的投影矩阵为：

$$P_u = \frac{u \cdot u^T}{u^T \cdot u} = \frac{1}{15} \cdot \begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix}$$

v方向的投影矩阵为

$$P_v = \frac{v \cdot v^T}{v^T \cdot v} = \frac{1}{18} \cdot \begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix}$$

~~(a)~~ $\cancel{P_u \cdot v = \frac{1}{15}(-6, 3, 9, -3)}$

(a) $P_v \cdot u = \frac{1}{18}(3, 12, 0, -3)^T = (\frac{1}{6}, \frac{2}{3}, 0, -\frac{1}{6})^T$

(b) $P_u \cdot v = \frac{1}{15}(-6, 3, 9, -3)^T = (-\frac{2}{5}, \frac{1}{5}, \frac{3}{5}, -\frac{1}{5})^T$

(c) $P_{v\perp} \cdot u = (I - P_v) \cdot u = u - P_v \cdot u = (-\frac{13}{6}, \frac{1}{3}, -3, \frac{5}{6})^T$

(d) $P_{u\perp} \cdot v = (I - P_u) \cdot v = v - P_u \cdot v = (\frac{7}{5}, \frac{19}{5}, -\frac{3}{5}, -\frac{4}{5})^T$