4. $A(x,y,z) = (x+2y-z, -y, x+7z)$ is a linear oprator on R^3.

(a) determine $[A]_s$, where S is the standard basis.

S是标准基，则 $S=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = (x, y, z)$ 对其进行线性变换

$x' = (1,0,1)^T \quad y' = (2,-1,0)^T \quad z' = (-1,0,7)$

所以 $[A]_s = \begin{pmatrix}1&2&-1\\0&-1&0\\1&0&7\end{pmatrix}$

(b) Determine $[A]_{s'}$ as well as the nonsingular matrix Q such that

$[A]_{s'} = Q^{-1}[A]_s Q$ for $s' = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right]$

设 $S' = (y_1, y_2, y_3)$ $S = (x_1, x_2, x_3)$

$[I]_{s's} = Q_{s's} = (v_1, v_2, v_3)$ $v_1 = [y_1]_s = (1\ 0\ 0)^T$

$v_2 = [y_2]_s = (1\ 1\ 0)^T$

$v_3 = [y_3]_s = (1\ 1\ 1)^T$

$\therefore Q = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$

设 $[I]_{ss'} = (v_1', v_2', v_3')$ $v_1' = [x_1]_{s'} = (1\ 0\ 0)^T$

$v_2' = [x_2]_{s'} = (-1\ 1\ 0)^T$

$\therefore [I]_{ss'} = Q^{-1} = \begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}$ $v_3' = [x_3]_{s'} = (0\ -1\ 1)^T$

$$[A]_{S'} = Q^{-1}[A_S]\cdot Q = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & -1 \\ -1 & -1 & -7 \\ 1 & 0 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{pmatrix}$$

所以 $[A]_{S'} = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{pmatrix}$

7. Let T be the linear operator on R^4 defined by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + 2x_3 - x_4,\ x_2 + x_4,\ 2x_3 - x_4,\ x_3 + x_4)$, and let $X = \text{span}\{e_1, e_2\}$ be the subspace that is spanned by the first two unit vectors in R^4.

(a) Explain why X is invariant under T.

X的基为：$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = (e_1, e_2)$ ~~$Te_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $Te_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = e_1 + e_2$~~

$$[T]_S = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

设$\vec{v}$为在X空间内的任意一个向量，$\vec{v} = \alpha e_1 + \beta e_2 = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}$

$$[T\vec{v}]_S = \begin{pmatrix} \alpha+\beta \\ \beta \\ 0 \\ 0 \end{pmatrix} = (\alpha+\beta)e_1 + \beta e_2 \in X$$

所以X是T下的不变子空间。

(b) Determine $[T/_X]_{\{e_1,e_2\}}$

$T/_X(e_1) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = e_1$ $\quad T/_X(e_2) = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = e_1+e_2$

$\therefore [T/_X]_{\{e_1,e_2\}} = \left([T/_X(e_1)]_{\{e_1,e_2\}} \mid [T/_X(e_2)]_{\{e_1,e_2\}} \right) = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$

(c) Describe the structure of $[T]_B$ where B is any basis obtained from an extantion of $\{e_1, e_2\}$。

设 $B = \{e_1, e_2, \beta_1, \beta_2\}$

$[T(e_1)]_B = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\quad [T(e_2)]_B = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ $\quad [T(\beta_1)]_B = \begin{pmatrix}a_{11}\\a_{12}\\b_{11}\\b_{12}\end{pmatrix}$

$[T(\beta_2)]_B = \begin{pmatrix}a_{21}\\a_{22}\\b_{21}\\b_{22}\end{pmatrix}$

则$[T]_B$的结构为 $[T]_B = \left([T(e_1)]_B \mid [T(e_2)]_B \mid [T(\beta_1)]_B \mid [T(\beta_2)]_B \right)$

$= \begin{pmatrix}1 & 1 & a_{11} & a_{21}\\0 & 1 & a_{12} & a_{22}\\0 & 0 & b_{11} & b_{21}\\0 & 0 & b_{12} & b_{22}\end{pmatrix} = \begin{pmatrix}[T/_X]_{\{e_1,e_2\}} & A_{2\times 2}\\ 0 & B_{2\times 2}\end{pmatrix}$