[redacted]

1.(1) Householder reduction

$$\vec{w}_1 = \frac{A_{*1} - \|A_{*1}\| \vec{e}_1}{\| A_{*1} - \|A_{*1}\| \vec{e}_1 \|} = \frac{1}{\sqrt{3}}(-1,-1,1)^T$$

$$R_1 = I - 2\vec{w}_1 \cdot \vec{w}_1^T = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \qquad R_1 A = \left(\begin{array}{c|cc} 3 & 15 & 0 \\ \hline 0 & -9 & 54 \\ 0 & 12 & 3 \end{array}\right)$$

$$A_2 = \begin{pmatrix} -9 & 54 \\ 12 & 3 \end{pmatrix} \qquad \vec{w}_2 = \frac{[A_2]_{*1} - \|[A_2]_{*1}\| \vec{e}_1}{\| [A_2]_{*1} - \|[A_2]_{*1}\| \vec{e}_1 \|} = \frac{1}{\sqrt{5}}(-2,1)^T$$

$$\hat{R}_2 = I - 2\vec{w}_2 \cdot \vec{w}_2^T = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\hat{R}_2 A_2 = \begin{pmatrix} 15 & -30 \\ 0 & 45 \end{pmatrix} \qquad R_2 R_1 A = \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{pmatrix} = R$$

$$Q = (R_2 R_1)^{-1} = (R_2 R_1)^T = R_1^T R_2^T = \begin{pmatrix} \frac{1}{3} & \frac{14}{15} & -\frac{2}{15} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{15} & \frac{11}{15} \end{pmatrix}$$

(2) Givens reduction

$$c_1 = \frac{1}{\sqrt{5}} \quad s_1 = -\frac{2}{\sqrt{5}} \qquad P_{12} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_{12} \cdot A = \begin{pmatrix} \sqrt{5} & \frac{29}{\sqrt{5}} & -34 \\ 0 & \frac{33}{\sqrt{5}} & 20 \\ 2 & \frac{4}{\sqrt{5}} & 37 \end{pmatrix}$$

$$c_2 = \frac{\sqrt{5}}{3} \quad s_2 = \frac{2}{3}$$

$$P_{13}=\begin{pmatrix}\frac{\sqrt5}{3} & 0 & \frac{2}{3}\\ 0 & 1 & 0\\ -\frac{2}{3} & & \frac{\sqrt5}{3}\end{pmatrix}\qquad P_{13}P_{12}A=\left(\begin{array}{c|cc}3 & 15 & 0\\ \hline 0 & \frac{33}{\sqrt5} & -\frac{48}{\sqrt5}\\ 0 & -\frac{6}{\sqrt5} & \frac{111}{\sqrt5}\end{array}\right)$$

$$c_3=\frac{11}{5\sqrt5}\qquad s_3=\frac{-2}{5\sqrt5}\qquad A_2=\begin{pmatrix}\frac{33}{\sqrt5} & -\frac{48}{\sqrt5}\\ -\frac{6}{\sqrt5} & \frac{111}{\sqrt5}\end{pmatrix}$$

$$\hat P_{23}'=\begin{pmatrix}\frac{11}{5\sqrt5} & -\frac{2}{5\sqrt5}\\ \frac{2}{5\sqrt5} & \frac{11}{5\sqrt5}\end{pmatrix}\qquad \hat P_{23}'\cdot A_2=\begin{pmatrix}15 & -30\\ 0 & 45\end{pmatrix}\qquad P_{23}=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{11}{5\sqrt5} & -\frac{2}{5\sqrt5}\\ 0 & \frac{2}{5\sqrt5} & \frac{11}{5\sqrt5}\end{pmatrix}$$

$$P_{23}P_{13}P_{12}A=\begin{pmatrix}3 & 15 & 0\\ 0 & 15 & -30\\ 0 & 0 & 45\end{pmatrix}=R$$

$$Q=(P_{23}P_{13}P_{12})^{-1}=(P_{23}P_{13}P_{12})^T=P_{12}^TP_{13}^TP_{23}^T=\begin{pmatrix}\frac13 & \frac{14}{15} & -\frac{2}{15}\\ -\frac23 & \frac13 & \frac23\\ \frac23 & -\frac{2}{15} & \frac{11}{15}\end{pmatrix}$$

2. 先对A进行 Householder 约减

$$\vec{w}_1=\frac{A_{*1}-\|A_{*1}\|\vec{e}_1}{\|A_{*1}-\|A_{*1}\|\vec{e}_1\|}=\frac{1}{\sqrt{10}}(-1\ 2\ -2\ 1)^T$$

$$\cancel{R_1=I-2\vec{w}_1\cdot\vec{w}_1^T=\frac{2}{\sqrt{10}}\begin{pmatrix}1 & -2 & 2 & -1\\ -2 & 4 & -4 & 2\\ 2 & -4 & 4 & -2\\ -1 & 2 & -2 & 1\end{pmatrix}}\qquad \cancel{RA=\sqrt{10}\left(\begin{array}{c|cc}-1 & 12 & -1\\ \hline 2 & -24 & 2\\ -2 & 24 & -2\\ 1 & -12 & 1\end{array}\right)}$$

$$\cancel{A_2=\sqrt{10}\begin{pmatrix}-24 & 2\\ 24 & -2\\ -12 & 1\end{pmatrix}}\qquad \cancel{\vec{w}_2=\frac{[A_2]_{*1}-\|[A_2]_{*1}\|\vec{e}_1}{\|[A_2]_{*1}-\|[A_2]_{*1}\|\vec{e}_1\|}=\frac{1}{\sqrt{30}}(-5\ 2\ -1)^T}$$

$$\cancel{\hat R_2=I-2\vec{w}_2\cdot\vec{w}_2^T=}$$

$$R_1=I-2\vec{w}_1\cdot\vec{w}_1^T=\frac15\begin{pmatrix}4 & 2 & -2 & 1\\ 2 & 1 & 4 & -2\\ -2 & 4 & 1 & 2\\ 1 & -2 & 2 & 4\end{pmatrix}\qquad R_1A=\left(\begin{array}{c|cc}5 & -15 & 5\\ \hline 0 & 10 & -5\\ 0 & -10 & 2\\ 0 & 5 & 14\end{array}\right)$$

$$A_2=\begin{pmatrix}10 & -5\\ -10 & 2\\ 5 & 14\end{pmatrix} \qquad \vec{w}_2=\frac{[A_2]_{*1}-\|[A_2]_{*1}\|\vec{e}_1}{\|[A_2]_{*1}-\|[A_2]_{*1}\|\vec{e}_1\|}=\frac{1}{\sqrt{6}}(-1,-2,1)^T$$

$$\hat{R}_2=I-2\vec{w}_2\cdot\vec{w}_2^{\,T}=\frac{1}{3}\begin{pmatrix}2 & -2 & 1\\ -2 & -1 & 2\\ 1 & 2 & 2\end{pmatrix} \qquad R_2=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ 0 & -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}$$

$$\hat{R}_2A_2=\begin{pmatrix}15 & 0\\ 0 & 12\\ 0 & 9\end{pmatrix} \qquad R_2R_1A=\begin{pmatrix}5 & -15 & 5\\ 0 & 15 & 0\\ 0 & 0 & 12\\ 0 & 0 & 9\end{pmatrix} \qquad A_3=\begin{pmatrix}12\\ 9\end{pmatrix} \quad \hat{R}_3=($$

$$(R_2R_1)^{-1}=(R_2R_1)^T=R_1^TR_2^T=\frac{1}{5}\begin{pmatrix}4 & 3 & 0 & 0\\ 2 & -\frac{8}{3} & -\frac{10}{3} & \frac{5}{3}\\ -2 & \frac{8}{3} & -\frac{5}{3} & \frac{10}{3}\\ 1 & -\frac{4}{3} & \frac{10}{3} & \frac{10}{3}\end{pmatrix}=Q$$

Q中的~~每~~前三列即构成了R(A)的一组基.

7. (1) 证明 $\mathcal{X}$ 和 $\mathcal{Y}$ 互为 R^3的 互补子空间，需要证明

① $\mathcal{X}\cup\mathcal{Y}=R^3$ $\quad [\mathcal{X}|\mathcal{Y}]=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{pmatrix}\xrightarrow[R_3-R_1]{R_2-R_1}\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 1 & 2\end{pmatrix}\xrightarrow{R_3-R_2}\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$

$\mathrm{Rank}([\mathcal{X}|\mathcal{Y}])=3$ 证毕

② 证明 $\mathcal{X}\cap\mathcal{Y}=\vec{0}$ 假设存在 $\vec{v}\in\mathcal{X}\cap\mathcal{Y}\neq\vec{0}$ 则有 ~~[illegible]~~

$$\vec{v}=x\begin{pmatrix}1\\1\\1\end{pmatrix}+y\begin{pmatrix}1\\2\\2\end{pmatrix}=z\begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \begin{cases}x+y=z\\ x+2y=2z\\ x+2y=3z\end{cases} \qquad \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \vec{v}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

得出矛盾，所以假设前不成立，证毕.

(2) 求沿 $\mathcal{Y}$ 向 $\mathcal{X}$ 的投影

$P = [X|0]\,[X|Y]^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_2]{\cancel{R_2-R_1},\ \cancel{R_3-R_2}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R_2-R_3]{R_1-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix}$$

互补投影：(沿X往Y方向投影)

$$Q = I - P = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix}$$

(c) $$Q\vec{v} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

(d) $$P \cdot P = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} = P$$

$$Q \cdot Q = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} = Q$$

(e)① $R(P) = \{b \mid Px = b\} =$

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

$N(Q)=\{x \mid Qx=\vec{0}\} = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

$\mathcal{X} = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\} \xRightarrow{\text{列变换}} \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

$\therefore R(P)=N(Q)=\mathcal{X}$

② $N(P)=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ $\quad R(Q)=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}=\mathcal{Y}$

$\therefore N(P)=R(Q)=\mathcal{Y}$