6. verify that $rank(A^TA) = rank(A) = rank(AA^T)$ for $A = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix} \xrightarrow{r_2+r_1} \begin{pmatrix} 1 & 3 & 1 & -4 \\ 0 & 0 & 2 & -4 \\ 2 & 6 & 2 & -8 \end{pmatrix} \xrightarrow[r_2-2r_1]{\cancel{r_3 \leftrightarrow r_2}} \begin{pmatrix} 1 & 3 & 1 & -4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rank(A) = 2$

$$A^T \cdot A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 1 & 1 & 2 \\ -4 & 0 & -8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix} = \begin{pmatrix} 6 & 18 & 4 & -20 \\ 18 & 54 & 12 & -60 \\ 4 & 12 & 6 & -20 \\ 12 & 36 & 12 & -48 \end{pmatrix}$$

$$A^T \cdot A \xrightarrow[\substack{r_4-2r_1 \\ r_3-\frac{2}{3}r_1}]{r_2-3r_1} \begin{pmatrix} 6 & 18 & 4 & -20 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{10}{3} & -\frac{20}{3} \\ 0 & 0 & 4 & -8 \end{pmatrix} \xrightarrow[r_2 \leftrightarrow r_3]{r_4-\frac{6}{5}r_3} \begin{pmatrix} 6 & 18 & 4 & -20 \\ 0 & 0 & \frac{10}{3} & -\frac{20}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rank(A^T \cdot A) = 2$

$$A \cdot A^T = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 1 & 1 & 2 \\ -4 & 0 & -8 \end{pmatrix} = \begin{pmatrix} 27 & -9 & -10 \\ -9 & 11 & -18 \\ 54 & -18 & -20 \end{pmatrix}$$

$$A \cdot A^T \xrightarrow{r_2-(-\frac{1}{3})r_1} \begin{pmatrix} 27 & -9 & -10 \\ 0 & 8 & -\frac{64}{3} \\ 54 & -18 & -20 \end{pmatrix} \xrightarrow{r_3-2r_1} \begin{pmatrix} 27 & -9 & -10 \\ 0 & 8 & -\frac{64}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$rank(A \cdot A^T) = 2$

$\therefore \quad rank(A^TA) = rank(A) = rank(A \cdot A^T)$

9. Using least squares techniques, fit the following data.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	2	7	9	12	13	14	14	13	10	8	4

(1) $y = a_0 + a_1 x$

$b^T = [2, 7, 9, 12, 13, 14, 14, 13, 10, 8, 4]$

$x = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

相当于求解 $Ax = b$ 的解，求标准化方程组 $A^TAx = A^Tb$ 的解

$A^TA = \begin{pmatrix} 11 & 0 \\ 0 & 110 \end{pmatrix}$ $A^T \cdot b = \begin{pmatrix} 106 \\ 20 \end{pmatrix}$ 即

$$\begin{pmatrix} 11 & 0 \\ 0 & 110 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 106 \\ 20 \end{pmatrix}$$ 解得 $$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \frac{106}{11} \\ \frac{2}{11} \end{pmatrix}$$

预测值 $\hat{y}$ = 8.74, 8.92, 9.1, 9.28, 9.46, 9.64, 9.82, 10.00, 10.18, 10.36, 10.54

方差 $\sigma_1 = 162.91$

(2) $y = a_0 + a_1 x + a_2 x^2$

$x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ 25 & 16 & 9 & 4 & 1 & 0 & 1 & 4 & 9 & 16 & 25 \end{bmatrix}$

相当于求解 $Ax = b$ 的解，求标准化方程组 $A^TAx = A^Tb$ 的解

$\cancel{A^TA}$

$$A^TA=\begin{pmatrix}11 & 0 & 110\\ 0 & 110 & 0\\ 110 & 0 & 1958\end{pmatrix}\qquad A^Tb=\begin{pmatrix}106\\20\\688\end{pmatrix}$$

即

$$\text{即}\quad\begin{pmatrix}11 & 0 & 110\\ 0 & 110 & 0\\ 110 & 0 & 1958\end{pmatrix}\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}=\begin{pmatrix}106\\20\\688\end{pmatrix}$$

$$\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}=\begin{pmatrix}\frac{86}{11}\\ \frac{2}{11}\\ -\frac{86}{979}\end{pmatrix}\approx\begin{pmatrix}7.82\\0.182\\ \cancel{-0.088}\end{pmatrix}-0.0878$$

$$\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}=\begin{pmatrix}\\ \frac{2}{11}\\ \cancel{\frac{299}{644}}\end{pmatrix}\cancel{\frac{97}{308}}\qquad\begin{pmatrix}13.976\\0.182\\-0.434\end{pmatrix}$$

预测值 $\hat{y}$ = 2.216, 6.304, 9.524, 11.876, 13.36, 13.976, 13.724, 12.604, 10.616, 7.76, 4.036

方差 $\sigma_2=1.623$

$\sigma_2<\sigma_1$

所以用 $y=a_1\cancel{x+}\ a_0+a_1x+a_2x^2$ 拟合效果更好。

[redacted]