Signal Processing Tutorial

Birds: Quick Phase
Slow Phase

Just like losing hair cells, vestibular gets older

Plant Dynamics: Input-Output Response

m = mass (negligible)
c = damping coefficient
k = spring constant
u(t) = eye position
x(t) = applied force (input)

Output

$$\underbrace{m\ddot{u}}_{\text{Acceleration}} + \underbrace{c\dot{u}}_{\text{Velocity}} + \underbrace{ku}_{\text{Position}} = \underbrace{x(u,t)}_{\text{Input}}$$

How to solve:

- Characteristic polynomial for homogeneous component
- Modify general solution to satisfy full inhomogeneous ODE via:
  - Method of undetermined coefficients
  - Method of variation of parameters

can be hard!

- Use of numerical integration algorithms (ode23...)

computationally intensive!

The Laplace Transformation and Frequency Domain

Laplace VS Fourier : $s = \sigma + i\omega$

Laplace Transform: $X(s) = \mathcal{L}\{x(t)\}$, $x(t) = \mathcal{L}^{-1}\{X(s)\}$

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad , \quad x(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} X(s) e^{st} ds}$$

*Let computers do this! (or use tables for analytic results)

Think of S as frequency

$$\mathcal{L}\{\frac{d}{dt} x(t)\} = \frac{d}{dt} \mathcal{L}\{x(t)\} = \frac{d}{dt} X(s)$$ (Linearity)

Properties

Laplace of derivative

$$\rightarrow \mathcal{L}\{\dot{x}\} = sX(s) + \ldots$$

$$\rightarrow \mathcal{L}\{\ddot{x}\} = s^2 X(s) + \ldots$$

Unnecessary Details

Time Domain

$\mathcal{L}\,,\,\mathcal{L}^{-1}$

Frequency Domain

Time Domain		Frequency Domain
t		s
x(t)	lower case / Upper Cases	X(s)
u(t)		U(s)

No more derivatives

$$m\ddot{u} + c\dot{u} + ku = x(u,t) \qquad ms^2U(s) + csU(s) + kU(s) = X(s)$$

differential equation:(algebraic equation!

$$\rightarrow (ms^2 + cs + k)U(s) = X(s)$$

$\frac{\text{Output}}{\text{Input}}$

$$H(s) \equiv \frac{U(s)}{X(s)} = \frac{1}{ms^2 + cs + k}$$

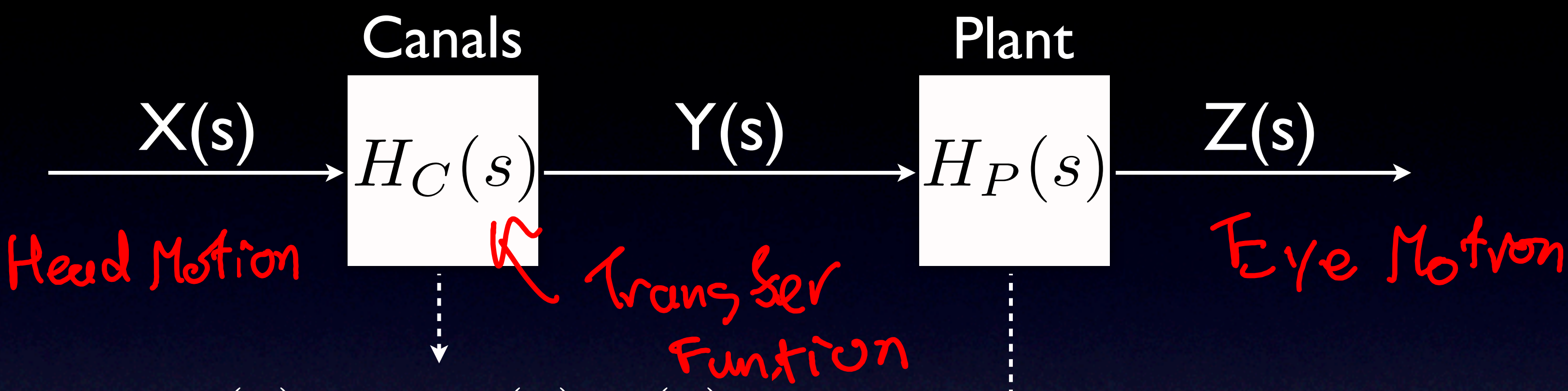
(transfer function)

Input → output

Magic Box

factoring... $\longrightarrow$ $H(s) = \dfrac{1}{(1+T_1 s)(1+T_2 s)}$

Transfer Functions in The Laplace Domain



$$Y(s) = H_C(s)X(s)$$

$$Z(s) = H_P(s)Y(s)$$

$$\boxed{Z(s) = H_P(s)H_C(s)X(s)}$$

easy!

Transfer Function Representations

Standard polynomial representation:

$$H_C(s) = \frac{Y(s)}{X(s)} = \frac{G_Y(s)}{G_X(s)} = \frac{a_0 + a_1 s + \ldots}{b_0 + b_1 s + b_2 s^2 + \ldots}$$

Factor it

$$G_i = i_0 + i_1 s + i_2 s^2 + \ldots = (s - z_1)(s - z_2)\ldots$$

$$\{z_j\} = \text{set of roots of polynomial G(s)}$$

ZPK representation:

- if G(s) in *numerator*, then roots of G are *zeros*(Z)of H.
- if G(s) in *denominator*, then roots of G are *poles(P)* of H.
- K represents the *gain* of the transfer function

Quadratic

$$H(s) = K\frac{(s - z_1)(s - z_2)\ldots}{(s - p_1)(s - p_2)\ldots}$$

Can cancel and simplify terms

Torsion Pendulum Transform

Acceleration → Velocity

Goldberg: $H_{\mathrm{aff}} = H_{\mathrm{TP}} H_L$

$$H_{\mathrm{TP}} = \frac{1}{(1+\tau_1 s)(1+\tau_2 s)} = \frac{\xi(s)}{A(s)}$$ (wrt acc.)

Acceleration

$$H_{\mathrm{L}} = (1+\tau_{\mathrm{L}} s)$$

Firing rate

Cupula Pos & Vel: $R(t) = \xi + \tau_{\mathrm{L}} \dot{\xi}$ ⇒ $\dfrac{\tilde{R}(s)}{\tilde{\xi}(s)} = (1+\tau_{\mathrm{L}} s)$

Goldberg: $H_{\mathrm{aff}} = H_{\mathrm{TP}} H_L$

Derivative

$$H_{\mathrm{TP}} = \frac{s}{(1+\tau_1 s)(1+\tau_2 s)} = \frac{\xi(s)}{V(s)}$$ (wrt vel.)

$\frac{0}{0}$ can happen

0, can pull to infinity

$$H_{\mathrm{L}} = (1+\tau_{\mathrm{L}} s)$$

Cupula Pos & Vel: $R(t) = \xi + \tau_{\mathrm{L}} \dot{\xi}$ ⇨ $\frac{\tilde{R}(s)}{\tilde{\xi}(s)} = (1+\tau_{\mathrm{L}} s)$

Position Velocity

Transfer Function Characterization

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

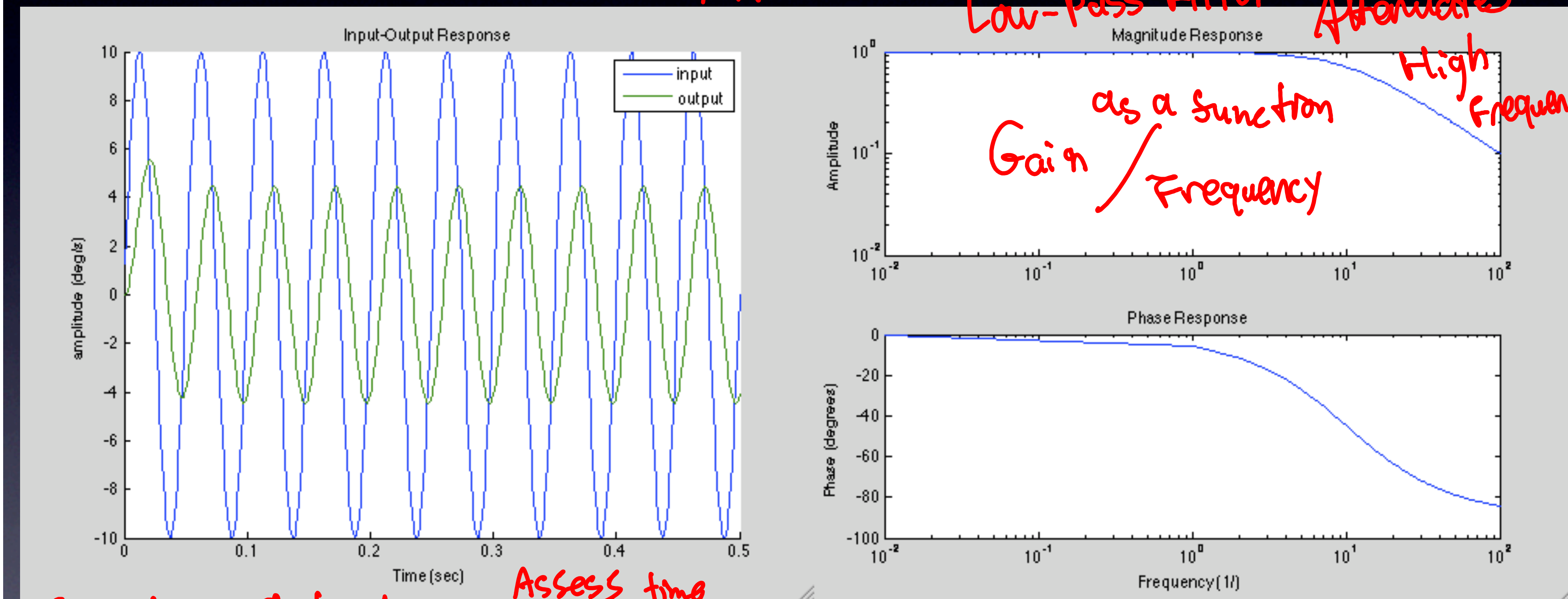
← Try H(s)

angle

$$H(s) = \underbrace{|H(s)|}_{\text{gain}}\underbrace{e^{i\phi(s)}}_{\text{phase}}$$

Low-Pass Filter

Attenuates High frequency



Gain / Frequency as a function

Input vs Output
Different Amplitude
Phase

Assess time difference
Time → Degree

$$H_{\mathrm{LP}}(s) = g\frac{1}{1+\tau s} = \frac{g\tau}{(s+1/\tau)}$$

FIN