Communications

Simple Approximate Formulas for Input Resistance, Bandwidth, and Efficiency of a Resonant Rectangular Patch

David R. Jackson and Nicolaos G. Alexopoulos

Abstract—**Simple approximate formulas for the input resistance, bandwidth, and radiation efficiency of a resonant rectangular microstrip patch are derived. These formulas become increasingly accurate as the substrate thickness decreases. Because the formulas are derived from approximations of a rigorous Sommerfeld solution, they provide insight into the effect of the substrate parameters on the patch properties, in addition to providing approximate design equations.**

I. Introduction

It is the aim of this communication to derive explicit closed-form expressions for the radiation resistance, bandwidth (surface-wave ratio (SWR) ≤ 2.0 definition), and radiation efficiency of a resonant rectangular patch antenna, using an approach similar to that in [1]. The starting point for the derivation of these results is asymptotic formulas giving the radiated and surface-wave power for a horizontal Hertzian electric dipole on a grounded layer. These formulas are based on a rigorous Sommerfeld solution, and become increasingly accurate as the substrate thickness decreases, regardless of ϵ_r and μ_r. A cavity model analysis is then used in conjunction with these formulas to obtain results for the radiation efficiency, radiation resistance, and bandwidth of the resonant patch.

Because the resulting patch formulas are based on approximations of a rigorous solution, and not an empirical derivation, they yield insight into the effect of the substrate parameters such as ϵ_r, μ_r, and substrate thickness, on the patch behavior. These results should also be useful for providing rough design information.

II. Asymptotic Formulas for Hertzian Dipole

A unit strength Hertzian electric dipole on a substrate layer is shown in Fig. 1. The exact radiation pattern is given by [2].

$$E_\phi = \sin\phi\left(\frac{j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}F(\theta) \tag{1}$$

$$E_\theta = -\cos\phi\left(\frac{j\omega\mu_0}{4\pi r}\right)e^{-jk_0 r}G(\theta) \tag{2}$$

where

$$F(\theta) = \frac{2\tan(\beta_1 b)}{\tan(\beta_1 b) - j\dfrac{N(\theta)}{\mu_r}\sec\theta} \tag{3}$$

$$G(\theta) = \frac{2\tan(\beta_1 b)\cos\theta}{\tan(\beta_1 b) - j\dfrac{\epsilon_r}{N(\theta)}\cos\theta} \tag{4}$$

$$\beta_1 = k_0 N(\theta)$$

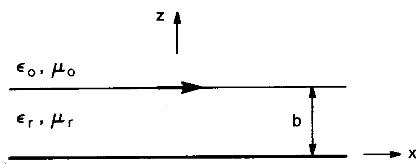

Fig. 1. Horizontal Hertzian electric dipole on top of a grounded substrate layer.

with

$$N(\theta) = \sqrt{n^2 - \sin^2\theta}$$
$$n = \sqrt{\epsilon_r \mu_r}.$$

The radiated (space-wave) power is

$$P_r^h = \int_0^{2\pi}\int_0^{\pi/2} S_r^h(\theta,\phi) r^2 \sin\theta \, d\theta \, d\phi \tag{5}$$

with

$$S_r^h(\theta,\phi) = \frac{1}{2\eta_0}(|E_\theta|^2 + |E_\phi|^2). \tag{6}$$

The ϕ-integration in (5) is trivial, involving only $\sin^2\phi$ and $\cos^2\phi$. The θ-integration on the exact expression cannot apparently be done in closed form. However, for a thin substrate ($b/\lambda_0 \ll 1$), (3) and (4) may be approximated as

$$F(\theta) \simeq 2j\mu_r(k_0 b)\cos\theta \tag{7}$$

$$G(\theta) \simeq 2j(k_0 b)\frac{N^2(\theta)}{\epsilon_r}. \tag{8}$$

The integrations may now be done in closed form, with the result

$$P_r^h \simeq \frac{1}{\lambda_0^2}(k_0 b)^2\left[80\pi^2\mu_r^2\left(1 - \frac{1}{n^2} + \frac{2/5}{n^4}\right)\right]. \tag{9}$$

An asymptotic formula for the surface-wave power may also be obtained. The surface-wave power is given by

$$P_{sw}^h = -\frac{1}{2}\,\mathrm{Re}\int_0^\infty\int_0^{2\pi} E_z H_\phi^* \rho \, d\phi \, dz \tag{10}$$

where E_z, H_ϕ are the fields of the dominant TM_0 surface wave. To calculate P_{sw}^h, the Sommerfeld solution [3] for the magnetic vector potential Π_z is considered, which for $z > b$ is

$$\Pi_z = \int_0^\infty J_1(\lambda\rho)g(\lambda)e^{-u_0(z-b)}\cos\phi \, d\lambda \tag{11}$$

where

$$g(\lambda) = \frac{2}{D_e(\lambda)D_m(\lambda)}\frac{(1-n^2)}{\mu_r}\lambda^2 \tag{12}$$

$$D_e(\lambda) = u_0 + \frac{1}{\mu_r}u_1\coth(u_1 b) \tag{13}$$

$$D_m(\lambda) = \frac{1}{\mu_r}u_0 n^2 + u_1\tanh(u_1 b) \tag{14}$$

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with

$$u_0 = \left(\lambda^2 - k_0^2\right)^{1/2}$$
$$u_1 = \left(\lambda^2 - k_1^2\right)^{1/2}$$
$$k_1 = k_0 n.$$

A factor of $-j\omega\mu_0/4\pi k_0^2$ is suppressed for convenience in (12). A Π_x component is also present in the Sommerfeld solution, but is not needed to find the fields of the TM_0 surface wave.

Extending the integration in (11) to $(-\infty, +\infty)$, and taking the residue at the TM_0 surface-wave pole after closing the contour, yields the surface wave field for $z > b$ in the far field ($k_0\rho \gg 1$) as

$$\Pi_z = \cos\phi \frac{e^{-j\lambda\rho}}{\sqrt{\lambda\rho}} e^{-u_0(z-b)} S_1 \tag{15}$$

where

$$S_1 = \frac{1}{D_e(\lambda) D_m'(\lambda)} \left[\frac{60\sqrt{\pi}}{k_0} (-1+j) \frac{(n^2-1)\lambda^2}{\mu_r} \right] \tag{16}$$

with $\lambda = \lambda_{TM_0}$. Because the surface-wave power in the air region dominates the power in the dielectric region as $b \to 0$, only the air region need be considered. Equation (10) allows for a closed form expression for the surface-wave power, since the resulting ϕ and z integrations may both be performed analytically. The result involves $D_e(\lambda_{TM_0})$ and $D_m'(\lambda_{TM_0})$, which may also be approximated for $b/\lambda_0 \ll 1$. To do this, (14) is approximated and set to zero to obtain

$$\frac{1}{\mu_r} u_0 n^2 + u_1^2 b = 0. \tag{17}$$

This yields the asymptotic solution

$$\lambda_{TM_0} \sim k_0 \left[1 + \frac{(k_0 b)^2}{\epsilon_r^2} (n^2-1)^2 \right]^{1/2}. \tag{18}$$

Substituting this result into (13) and the derivative of (14) then yields the final result, which after simplification is

$$P_{SW}^h \sim \frac{1}{\lambda_0^2} (k_0 b)^3 \left[60\pi^3 \left(1 - \frac{1}{n^2}\right)^3 \mu_r^3 \right]. \tag{19}$$

The only approximation in (9) and (19) is that $b/\lambda_0 \ll 1$, while ϵ_r and μ_r may both be arbitrary. The radiation efficiency is then defined as

$$e_r = \frac{P_r^h}{P_r^h + P_{SW}^h}. \tag{20}$$

II. Formulas For Patch

A rectangular patch is shown in Fig. 2. As an approximation, the radiation efficiency of the patch may be taken as that of the infinitesimal dipole, so (9), (19), and (20) may be used directly. Bandwidth and radiation resistance information cannot be obtained quit as directly, however. A cavity model analysis is introduced first, so that the Q of the patch may be obtained.

Within the cavity, it is assumed that

$$E_z = A_{10} \sin\left(\frac{\pi x}{L}\right) \tag{21}$$

with

$$k_1 L \doteq \pi. \tag{22}$$

From cavity model theory a I-A probe current will excite a dominant-mode field amplitude [4], [5]

$$A_{10} = \frac{j\omega\mu_0\mu_r}{k_{1e}^2 - k_{10}^2} \left(\frac{2}{WL}\right) \sin\left(\frac{\pi x_f}{L}\right) \tag{23}$$

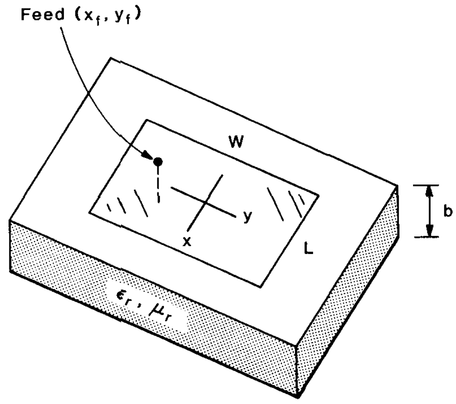

Fig. 2. Rectangular patch antenna with coaxial feed at (x_f, y_f).

where

$$k_{10}^2 = k_1^2 \tag{24}$$
$$k_{1e}^2 = k_0^2 \epsilon_r \mu_r (1 - jl) \tag{25}$$

where l is the effective loss tangent which will dissipate the same total power as the actual power (radiated + surface wave) produced by the patch current. The power dissipated by the fictitious loss tangent would be

$$P_d = \frac{1}{4} \omega \epsilon_0 \epsilon_r l b W L A_{10}^2. \tag{26}$$

The total power produced by the corresponding patch surface current

$$J_{sx}(x, y) = \frac{A_{10}}{j\omega\mu_0\mu_r} \left(\frac{\pi}{L}\right) \cos\left(\frac{\pi x}{L}\right) \tag{27}$$

can be approximated as

$$P_T = \frac{P_r}{e_r} = \frac{P_r^h m_{eq}^2}{e_r} \cdot \frac{P_r}{P_r^h m_{eq}^2} \tag{28}$$

where P_r is the space-wave power radiated by the patch current, and e_r is the radiation efficiency of the Hertzian dipole. The expression for P_T in (28) is shown factored into two parts, where m_{eq} is the current moment of the patch, given as

$$m_{eq} = \int_s J_{sx}(x, y)\, dx\, dy. \tag{29}$$

The term

$$p = \frac{P_r}{P_r^h m_{eq}^2} \tag{30}$$

appearing as the second term in (28) is the ratio of the power radiated by the patch with that radiated by an equivalent Hertzian dipole of the same moment. This term will be approximately 1.0 for a physically small patch.

After substituting (9) into (28) and performing the elementary integration in (29), the result is

$$P_T = p(320) \left(\frac{b}{\lambda_0}\right)^2 \frac{W^2}{\eta_0^2} \pi^2 \frac{c_1}{e_r} A_{10}^2 \tag{31}$$

where

$$c_1 = 1 - \frac{1}{n^2} + \frac{2/5}{n^4}. \tag{32}$$

To calculate the p factor in (31), (5) and (30) are used to express

p as

$$p = \frac{\int_0^{2\pi}\int_0^{\pi/2} S_r^h(\theta,\phi)\,|A(\theta,\phi)|^2 r^2 \sin\theta\, d\theta\, d\phi}{\int_0^{2\pi}\int_0^{\pi/2} S_r^h(\theta,\phi) r^2 \sin\theta\, d\theta\, d\phi} \tag{33}$$

where $A(\theta, \phi)$ is the element factor of the patch with a unit-strength current moment in *free space*, and is given by

$$A(\theta,\phi) = \left(\frac{\pi}{2}\right)^2 T_1(\theta,\phi) T_2(\theta,\phi) \tag{34}$$

with

$$T_1(\theta,\phi) = \frac{\sin\left(\frac{k_0 w}{2}\sin\theta\sin\phi\right)}{\frac{k_0 w}{2}\sin\theta\sin\phi} \tag{35}$$

$$T_2(\theta,\phi) = \frac{\cos\left(\frac{k_0 L}{2}\sin\theta\cos\phi\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{k_0 L}{2}\sin\theta\cos\phi\right)^2}. \tag{36}$$

The exact expression for p, using the exact $S_r^h(\theta, \phi)$, depends on W, L, b, ϵ_r and μ_r. However, in (8) for $G(\theta)$, assume that $n^2 \gg 1$ so that

$$G(\theta) \simeq 2j\mu_r(k_0 b). \tag{37}$$

Because of the common μ_r and $k_0 b$ factors in (7) and (37), (33) will then be a function only of the patch dimensions W and L. To obtain an approximate closed-form expression for p, under the assumption of (37), series expansions for the sin and cos terms in the numerators of (35) and (36) may be used [6], as well as a $(1 - x)^{-1}$ type of expansion for the denominator of (36). Keeping only terms of order $(k_0 w)^4$ and $(k_0 L)^4$ and lower, the following result is obtained:

$$p = 1 + \frac{a_2}{20}(k_0 w)^2 + a_4\left(\frac{3}{560}\right)(k_0 w)^4 + b_2\left(\frac{1}{10}\right)(k_0 L)^2 \tag{38}$$

where

$$a_2 = -0.16605$$

$$a_4 = 0.00761$$

$$b_2 = -0.09142.$$

For $L/\lambda_0 \le 0.5$ and $W/L \le 2$, the error in (38) is less then 2%. Equation (31) together with (38), then gives the power P_T which is equated to the dissipated power P_d of (26) in order to solve for l. This is then substituted into (23)–(25) to solve for A_{10}. Neglecting the effect of higher order modes, the input resistance of the patch at resonance (as seen by the coaxial feed probe) is then

$$R_{in} \simeq -A_{10} b \sin\left(\frac{\pi x_f}{L}\right). \tag{39}$$

The final result is

$$R_{in} = 90\frac{e_r}{pc_1}\epsilon_r\mu_r\left(\frac{L}{W}\right)^2 \sin^2\left(\frac{\pi x_f}{L}\right) \tag{40}$$

where R_{in} is expressed in ohms.

The SWR ≤ 2 bandwidth of the patch may be computed by first calculating the Q of the patch, defined as

$$Q = \omega_0 \frac{U_s}{P_T} \tag{41}$$

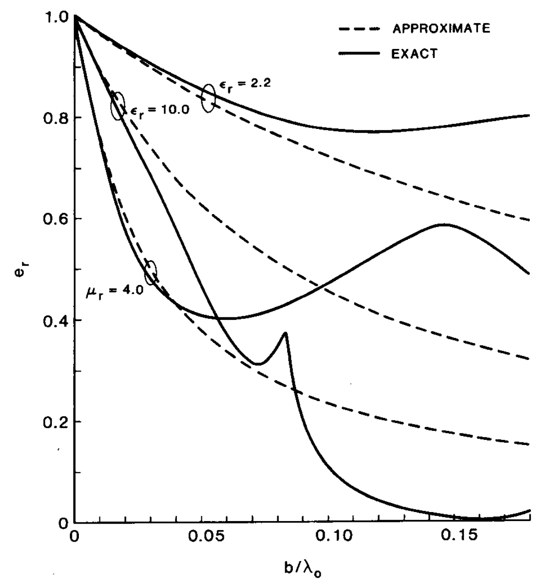

Fig. 3. Radiation efficiency versus substrate thickness for three different substrates. $W/L = 2.0$.

where U_S = energy stored within the patch cavity. Using $H_y = J_{sx}$ and the fact that the energy stored in the electric and magnetic fields are equal at resonance, results in

$$U_s = \frac{1}{4} b\mu_0\mu_r WLH_y^2 = \frac{1}{4} b\pi^2 \frac{W}{L}\frac{A_{10}^2}{\omega_0^2\mu_0\mu_r}. \tag{42}$$

The SWR ≤ 2 bandwidth is calculated from [7, p. 62] as

$$\mathrm{BW} = \frac{2\,\Delta f}{f_0} = \frac{1}{\sqrt{2}\,Q}. \tag{43}$$

Using (31), (41), and (42) in (43) gives the result

$$\mathrm{BW} = \frac{16}{3\sqrt{2}}\frac{c_1 p}{e_r}\left(\frac{1}{\epsilon_r}\right)\left(\frac{b}{\lambda_0}\right)\left(\frac{W}{L}\right). \tag{44}$$

An interesting point is that ϵ_r appears in this final expression for BW, while μ_r does not. This is a consequence of the μ_r^2 term in (9). See [8] for a further discussion of magnetic substrates.

III. Sample Results

To show the accuracy obtainable with the approximate formulas, results are given for different substrate parameters and thicknesses. Results are compared with an "exact" solution, which is based on a dominant-mode spectral domain theory with a filamentary feed current [9]–[11]. Results are shown for a low permittivity substrate ($\epsilon_r = 2.2$, $\mu_r = 1.0$), a high permittivity substrate ($\epsilon_r = 10.0$, $\mu_r = 1.0$), and a magnetic substrate ($\epsilon_r = 1.0$, $\mu_r = 4.0$). Results are shown versus normalized substrate thickness b/λ_0 for a resonant length patch. In the "exact" method, the resonant length L/λ_0 is found by trial and error, while the approximate formulas use Hammerstad's formula [12]. For a further discussion of resonant length, see [13].

The radiation efficiency, resonant input resistance, and bandwidth are shown in Figs. 3, 4, and 5 respectively. For the input resistance and bandwidth, the probe feed is taken at $x_f = L/8$, $y_f = 0$ (although x_f and y_f do not affect the approximate bandwidth formula). The exact bandwidth was found from a trial and error search of the SWR = 2.0 points, assuming a matched feed at f_0.

The input resistance formula is seen to be the least accurate of the three formulas. This is due to the fact that the amplitude of the patch

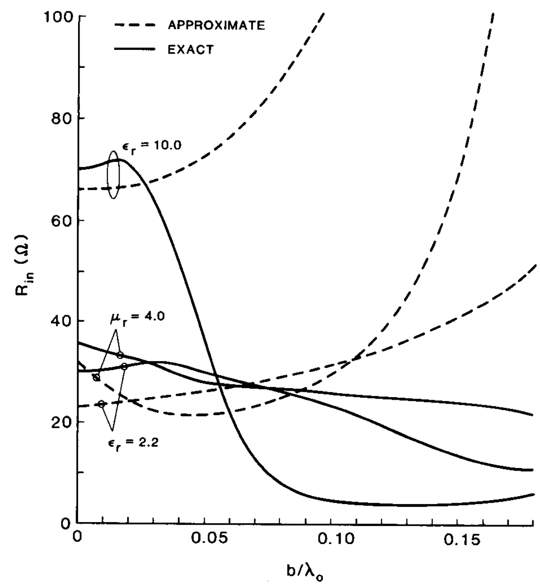

Fig. 4. Resonant input resistance versus substrate thickness for three different substrates. $W/L = 1.5, 2x_f/L = 0.25, y_f = 0$.

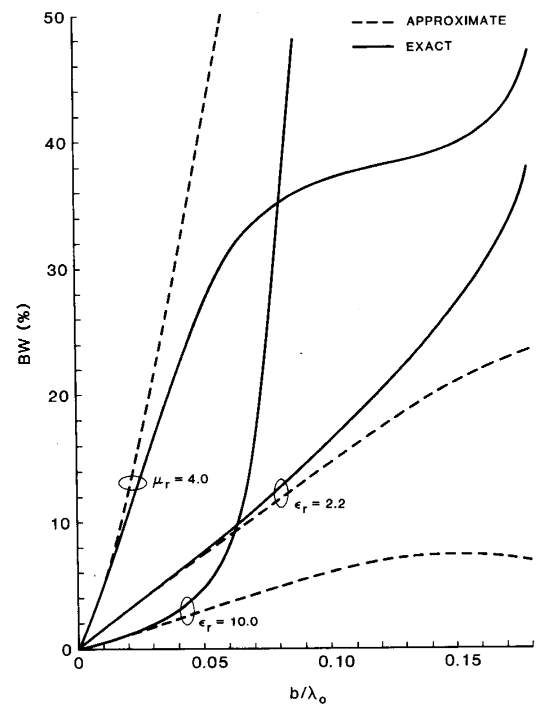

Fig. 5. Bandwidth (SWR ≤ 2.0) in percent versus substrate thickness for three different substrates. $W/L = 1.5, 2x_f/L = 0.25, y_f = 0$.

current determines the input resistance (see (39)), but does not affect the efficiency or bandwidth calculations.

V. Conclusion

Simple approximate formulas for the resonant input resistance, bandwidth, and radiation efficiency of a rectangular microstrip patch on a thin substrate have been derived. Although the formulas are approximate, they are derived by approximating rigorous results, and are not empirical. One use of such formulas would be to obtain rough design data. The bandwidth and efficiency formulas are the most accurate, and may even be sufficiently accurate for final design purposes. In addition to design applications, these formulas also provide insight into the fundamental influence of the substrate parameters on the patch behavior. One important conclusion is that the patch bandwidth decreases inversely with increasing ϵ_r for a fixed substrate thickness, while μ_r has little effect. However, the radiation efficiency decreases much more rapidly with increasing substrate thickness when using a magnetic substrate.

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Time-Domain Extrapolation to the Far Field Based on FDTD Calculations

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***Abstract*—A scheme to extrapolate finite-difference time-domain (FDTD) calculated scattered fields to the far zone is developed here. It could replace the usual extrapolation scheme for a single frequency, in**

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