Computer Vision

Spring 2006 15-385,-685

Instructor: S. Narasimhan

Wean 5403

T-R 3:00pm – 4:20pm

Announcements

- Homework 1 is due today in class.
- Homework 2 will be out later this evening (due in 2 weeks).
- Start homeworks early.
- Post questions on bboard.

Image Processing and Filtering

Lecture #5

Image as a Function

- We can think of an **image** as a function, *f*,
- f: $R^2 \rightarrow R$
  - $f(x, y)$ gives the **intensity** at position (x, y)
  - Realistically, we expect the image only to be defined over a rectangle, with a finite range:
    - f: $[a,b]\times[c,d] \rightarrow [0,1]$
- A color image is just three functions pasted together. We can write this as a "vector-valued" function:

$$f(x,y) = \begin{bmatrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{bmatrix}$$

Image as a Function

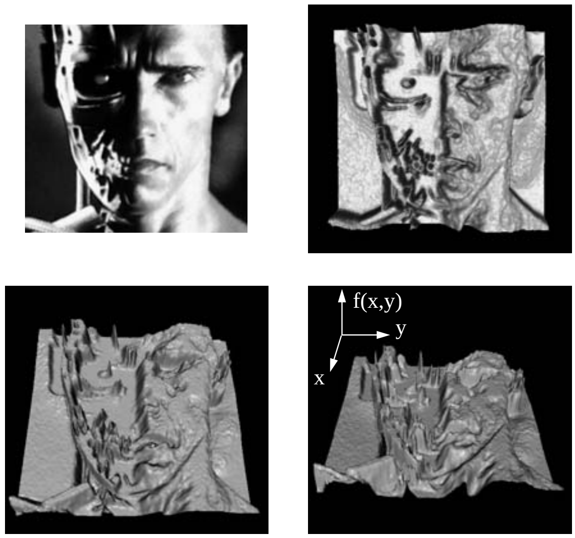


Image Processing

- Define a new image *g* in terms of an existing image *f*
  - We can transform either the domain or the range of *f*
- Range transformation:

$$g(x, y) = t(f(x, y))$$

What kinds of operations can this perform?

Image Processing

- Some operations preserve the range but change the domain of *f* :

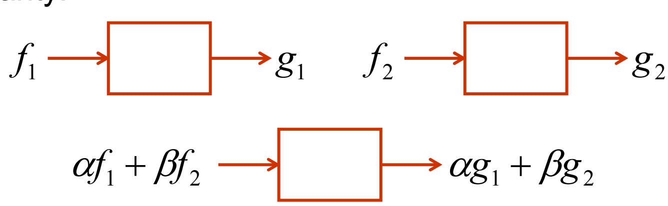
$$g(x,y) = f(t_x(x,y), t_y(x,y))$$

What kinds of operations can this perform?

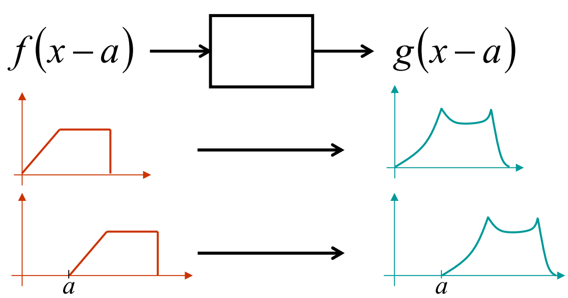
- Still other operations operate on both the domain and the range of f .

Linear Shift Invariant Systems (LSIS)

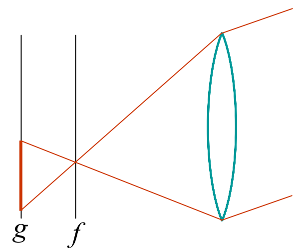
Linearity:



Shift invariance:



Example of LSIS



Defocused image (*g*) is a processed version of the focused image (*f*)

<u>Ideal lens is a LSIS</u> $f(x) \longrightarrow \boxed{\text{LSIS}} \longrightarrow g(x)$

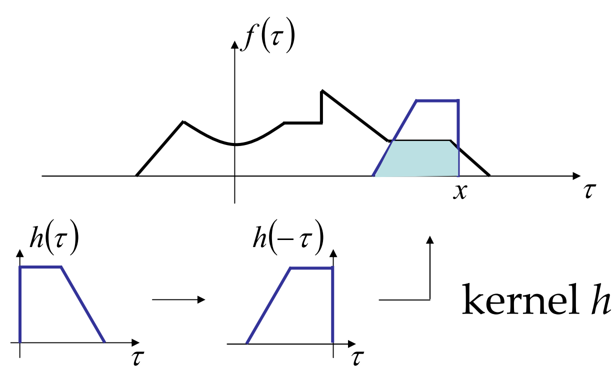
Linearity: Brightness variation
Shift invariance: Scene movement

(not valid for lenses with non-linear distortions)

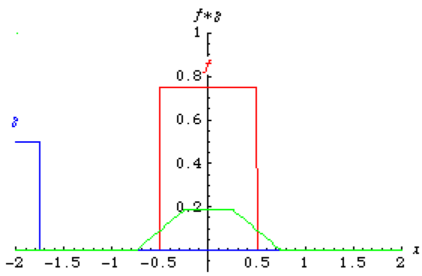
Convolution

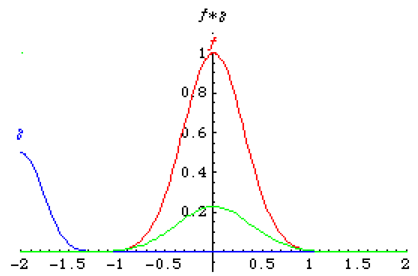
LSIS is doing convolution; convolution is linear and shift invariant

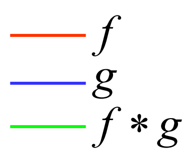
$$\boxed{g(x) = \int_{-\infty}^{\infty} f(\tau) h(x-\tau) d\tau \qquad g = f * h}$$



Convolution - Example

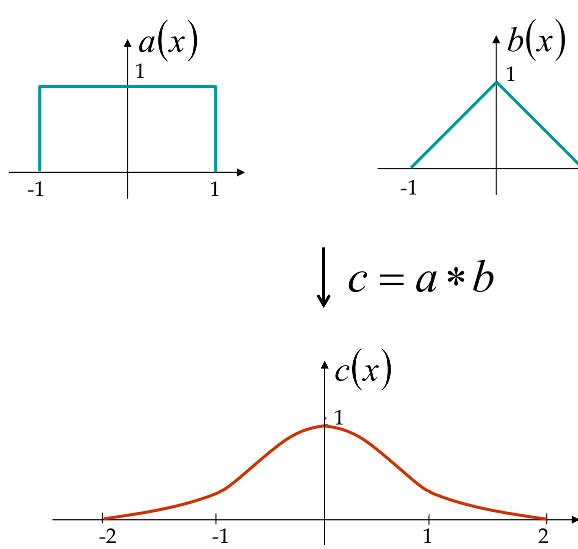






Eric Weinstein's Math World

Convolution - Example

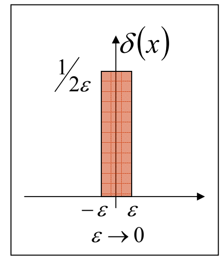


Convolution Kernel – Impulse Response

$$f \longrightarrow \boxed{h} \longrightarrow g \qquad\qquad g = f * h$$

- What h will give us $g = f$?

Dirac Delta Function (Unit Impulse)



Sifting property:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(0)\delta(x)dx$$

$$= f(0)\int_{-\infty}^{\infty} \delta(x)dx = f(0)$$

$$g(x) = \int_{-\infty}^{\infty} f(\tau)\delta(x-\tau)d\tau = f(x)$$

$$= \int_{-\infty}^{\infty} \delta(\tau)h(x-\tau)d\tau = h(x)$$

Point Spread Function

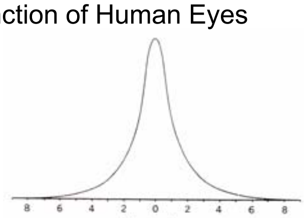
scene $\longrightarrow$ **Optical System** $\longrightarrow$ image

- Ideally, the optical system should be a Dirac delta function.
- However, optical systems are never ideal.

$\delta(x)$ $\longrightarrow$ **Optical System** $\longrightarrow$ $PSF(x)$

point source — point spread function

- Point spread function of Human Eyes



Point Spread Function


normal vision


myopia


hyperopia


astigmatism

Images by Richmond Eye Associates

Properties of Convolution

- Commutative

$$a * b = b * a$$

- Associative

$$(a * b) * c = a * (b * c)$$

- Cascade system

$$f \longrightarrow \boxed{h_1} \longrightarrow \boxed{h_2} \longrightarrow g$$

$$= f \longrightarrow \boxed{h_1 * h_2} \longrightarrow g$$

$$= f \longrightarrow \boxed{h_2 * h_1} \longrightarrow g$$

How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

- Polynomials are not the best - unstable and not very physically meaningful.
- Easier to talk about "signals" in terms of its "frequencies" (how fast/often signals change, etc).

Jean Baptiste Joseph Fourier (1768-1830)

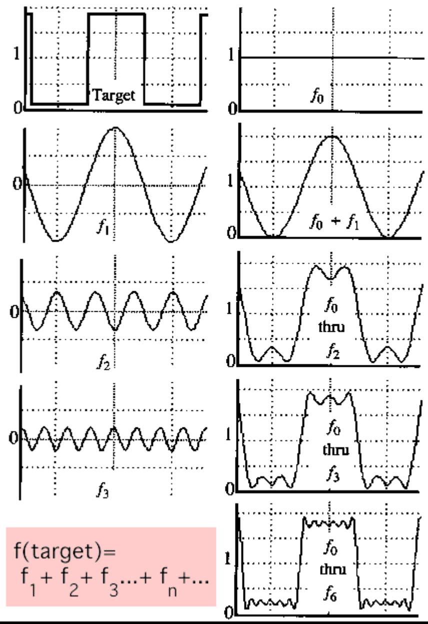
- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

A Sum of Sinusoids

- Our building block:

$$A \sin(\omega x + \phi)$$

- Add enough of them to get any signal *f(x)* you want!
- How many degrees of freedom?
- What does each control?
- Which one encodes the coarse vs. fine structure of the signal?



f(target)= $f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of *x*:

f(x) → Fourier Transform → ***F(ω)***

- For every ω from 0 to inf, ***F(ω)*** holds the amplitude *A* and phase ϕ of the corresponding sine $A \sin(\omega x + \phi)$
  - How can *F* hold both? Complex number trick!

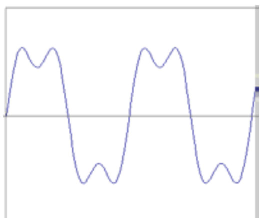
$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

F(ω) → Inverse Fourier Transform → ***f(x)***

Time and Frequency

- example : $g(t) = \sin(2pf\,t) + (1/3)\sin(2p(3f)\,t)$

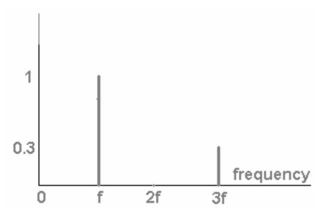


Time and Frequency

- example : $g(t) = \sin(2pf\,t) + (1/3)\sin(2p(3f)\,t)$

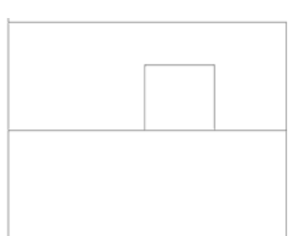
Frequency Spectra

- example : $g(t) = \sin(2pf\,t) + (1/3)\sin(2p(3f)\,t)$

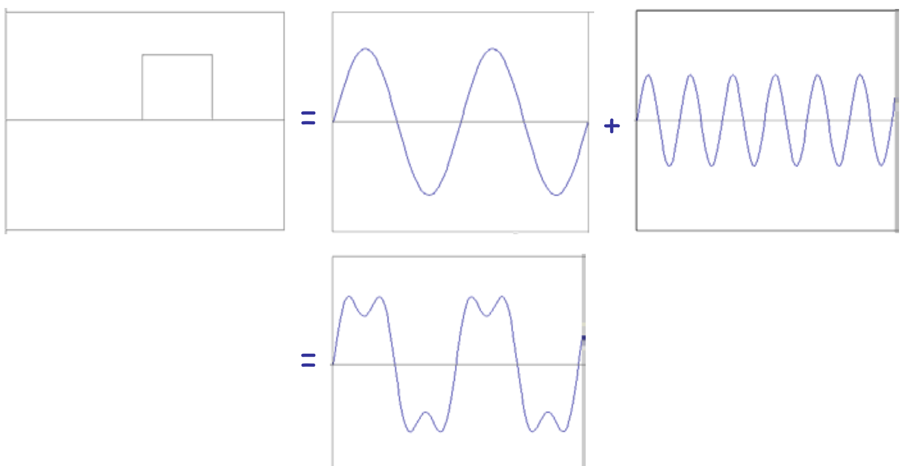


Frequency Spectra

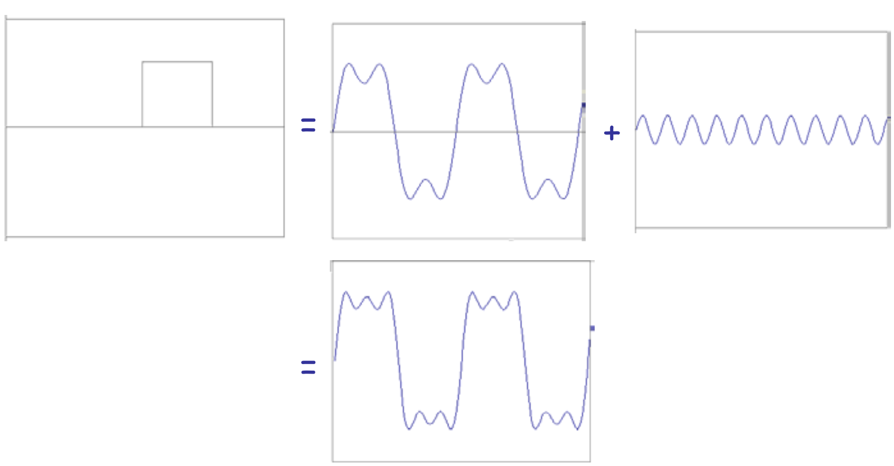
- Usually, frequency is more interesting than the phase



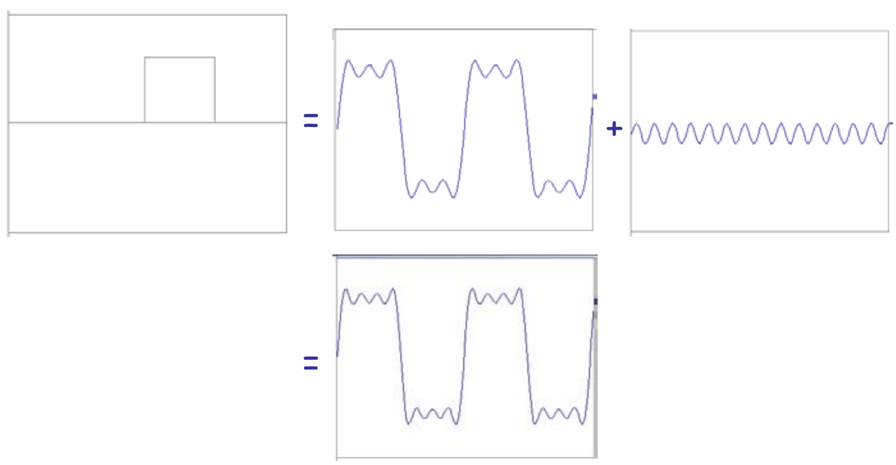
Frequency Spectra



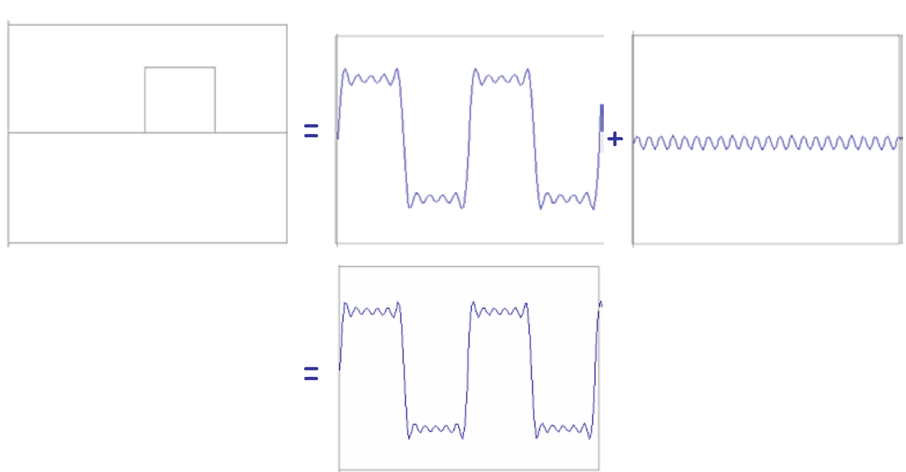
Frequency Spectra



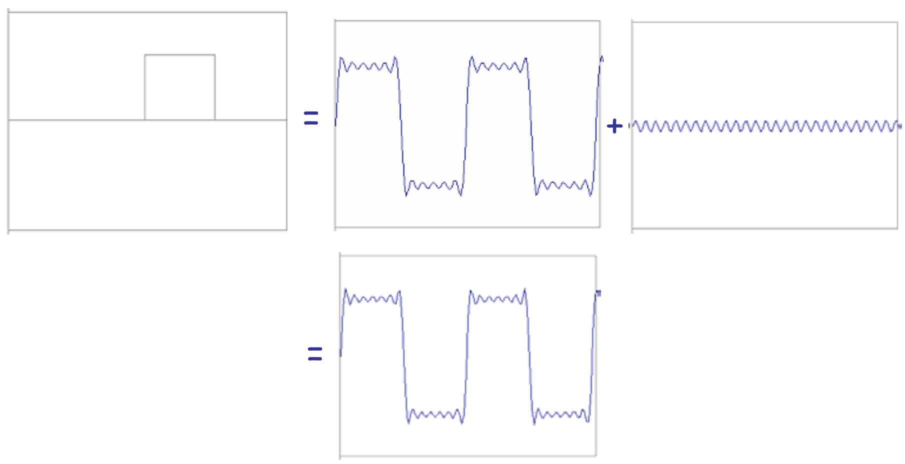
Frequency Spectra



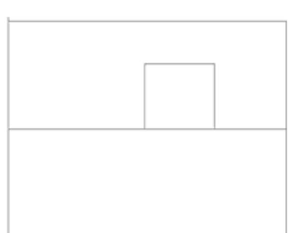
Frequency Spectra



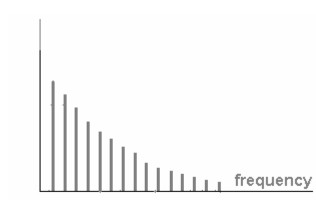
Frequency Spectra



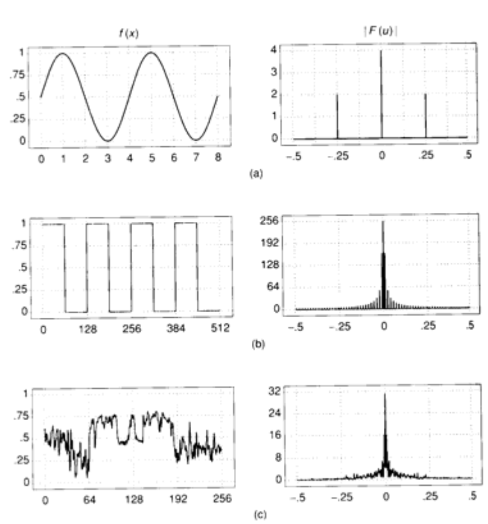
Frequency Spectra



$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$

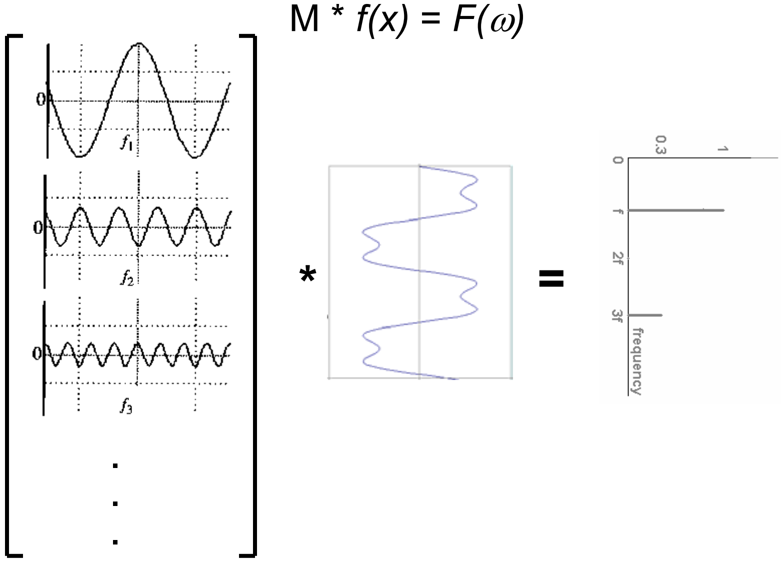


Frequency Spectra



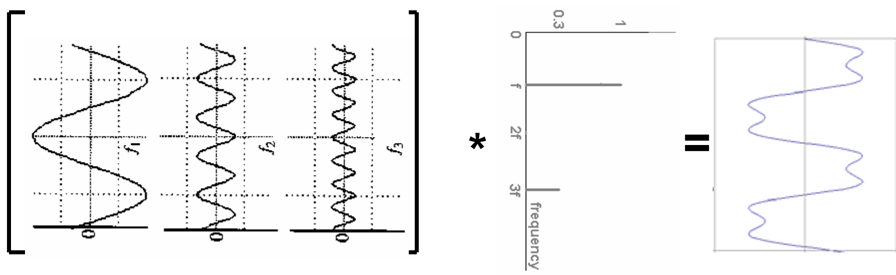
FT: Just a change of basis

M * $f(x) = F(\omega)$



IFT: Just a change of basis

$M^{-1} * F(\omega) = f(x)$



.
.
.

Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Arbitrary function $\longrightarrow$ Single Analytic Expression

Spatial Domain (x) $\longrightarrow$ Frequency Domain (u)

(Frequency Spectrum $F(u)$)

Inverse Fourier Transform (IFT)

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} dx$$

Fourier Transform

- Also, defined as:

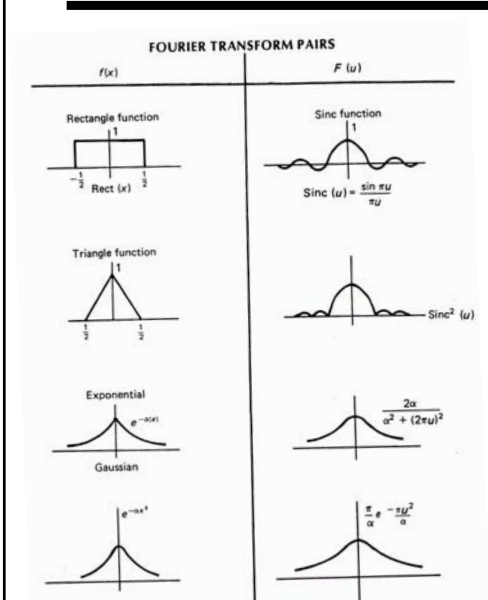
$$F(u)=\int_{-\infty}^{\infty} f(x)e^{-iux}dx$$

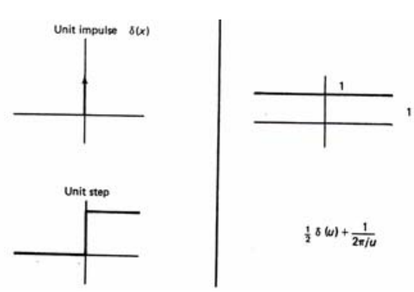
Note: $e^{ik}=\cos k+i\sin k \qquad i=\sqrt{-1}$

- Inverse Fourier Transform (IFT)

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{iux}dx$$

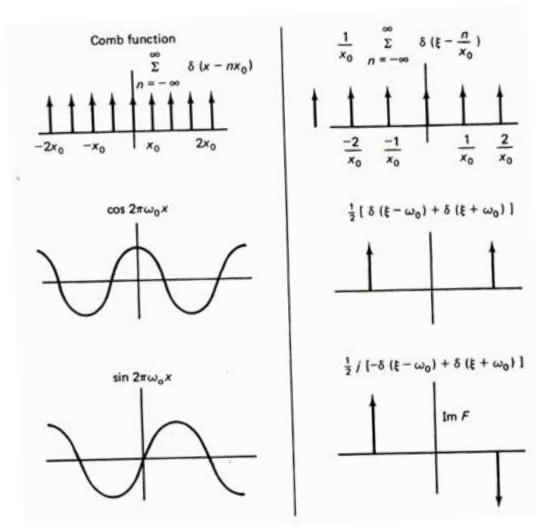
Fourier Transform Pairs (I)





Note that these are derived using angular frequency (e^{-iux})

Fourier Transform Pairs (I)



Note that these are derived using angular frequency (e^{-iux})

Fourier Transform and Convolution

Let $g = f * h$

Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

$$= F(u) H(u)$$

Convolution in spatial domain

$\Leftrightarrow$ Multiplication in frequency domain

Fourier Transform and Convolution

Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

So, we can find *g(x)* by Fourier transform

$$\begin{tikzcd} g & = & f \arrow[d, "\mathrm{FT}"] & * & h \arrow[d, "\mathrm{FT}"] \\ G \arrow[u, "\mathrm{IFT}"] & = & F & \times & H \end{tikzcd}$$

Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

Properties of Fourier Transform

Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$$

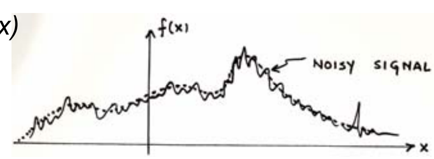
$$\int_{-\infty}^{\infty} f(x) g^*(x)\, dx = \int_{-\infty}^{\infty} F(\xi) G^*(\xi)\, d\xi$$

$f(x)$	$F(\xi)$
Real(R)	Real part even (RE) Imaginary part odd (IO)
Imaginary (I)	RO,IE
RE,IO	R
RE,IE	I
RE	RE
RO	IO
IE	IE
IO	RO
Complex even (CE)	CE
CO	CO

Example use: Smoothing/Blurring

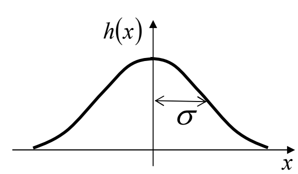
- We want a smoothed function of *f(x)*

$$g(x) = f(x) * h(x)$$



- Let us use a Gaussian kernel

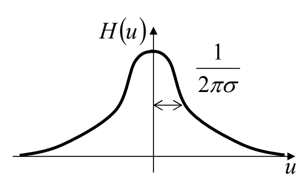
$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2} \right]$$



- Then

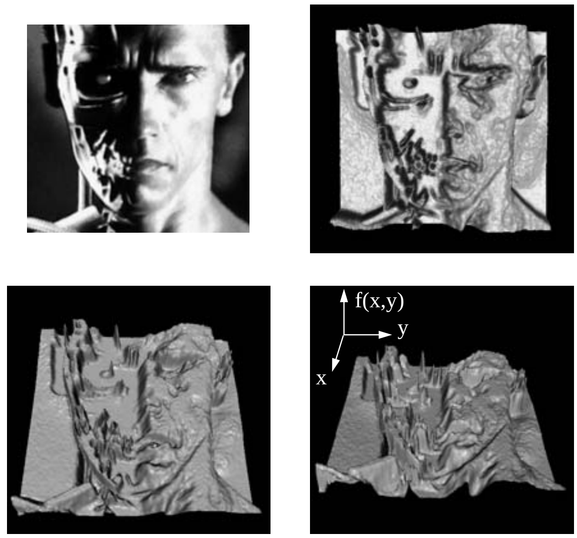
$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2 \right]$$

$$G(u) = F(u) H(u)$$



H(u) attenuates high frequencies in *F(u)* (Low-pass Filter)!

Image as a Discrete Function



Digital Images

The scene is

- **projected** on a 2D plane,
- **sampled** on a regular grid, and each sample is
- **quantized** (rounded to the nearest integer)

$$f(i,j) = \text{Quantize}\{f(i\Delta, j\Delta)\}$$

Image as a matrix

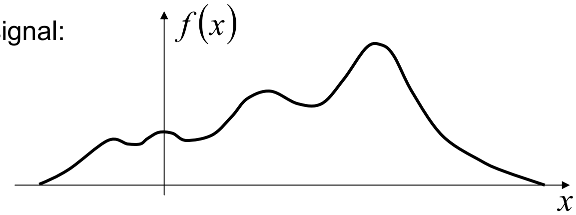
$j \rightarrow$

$i \downarrow$

62	79	23	119	120	105	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

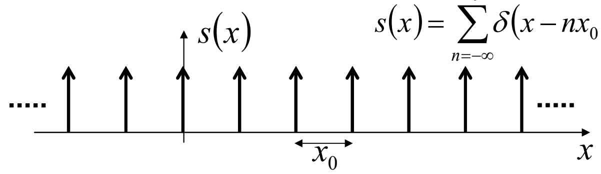
Sampling Theorem

Continuous signal:



Shah function (Impulse train):

$$s(x) = \sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$

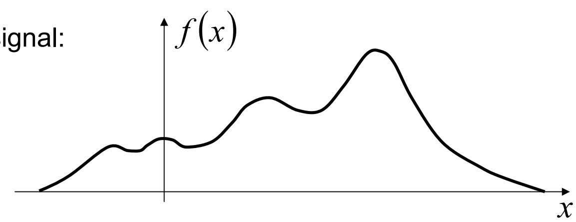


Sampled function:

$$f_s(x) = f(x)s(x) = f(x)\sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$

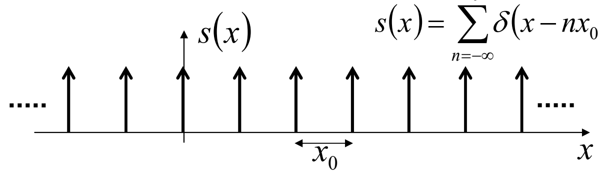
Sampling Theorem

Continuous signal:



Shah function (Impulse train):

$$s(x) = \sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$



Sampled function:

$$f_s(x) = f(x)s(x) = f(x)\sum_{n=-\infty}^{\infty} \delta(x - nx_0)$$

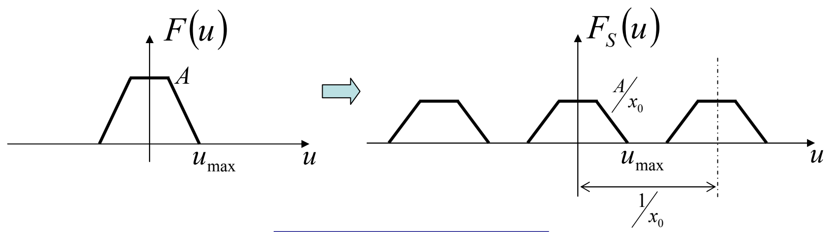
Sampling Theorem

Sampled function:

$$f_s(x) = f(x)s(x) = f(x)\sum_{n=-\infty}^{\infty}\delta(x - nx_0)$$

Sampling frequency $\frac{1}{x_0}$

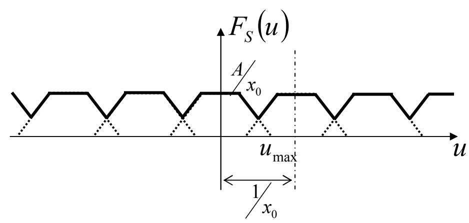
$$F_S(u) = F(u) * S(u) = F(u) * \frac{1}{x_0}\sum_{n=-\infty}^{\infty}\delta\left(u - \frac{n}{x_0}\right)$$



Only if $u_{\max} \le \frac{1}{2x_0}$

Nyquist Theorem

If $u_{\max} > \frac{1}{2x_0}$



Aliasing

When can we recover $F(u)$ from $F_S(u)$?

Only if $u_{\max} \le \frac{1}{2x_0}$ (Nyquist Frequency)

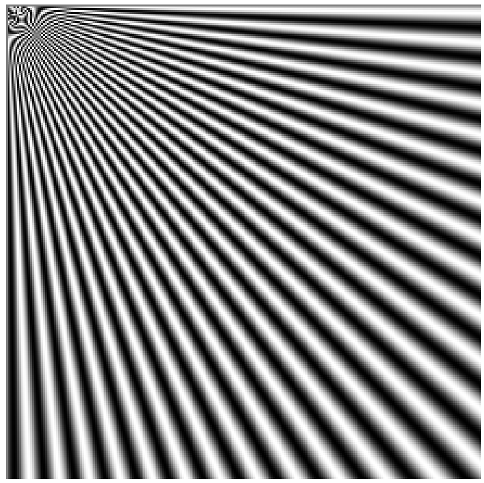
We can use

$$C(u) = \begin{cases} x_0 & |u| < 1/2x_0 \\ 0 & \text{otherwise} \end{cases}$$

Then $F(u) = F_S(u)C(u)$ and $f(x) = \mathrm{IFT}[F(u)]$

Sampling frequency must be greater than $2u_{\max}$

Aliasing



Announcements

- Homework 1 is due today in class.
- Homework 2 will be out later this evening.
- Start homeworks early.
- Post questions on bboard.

Next Class

- Image Processing and Filtering (continued)
- Horn, Chapter 6