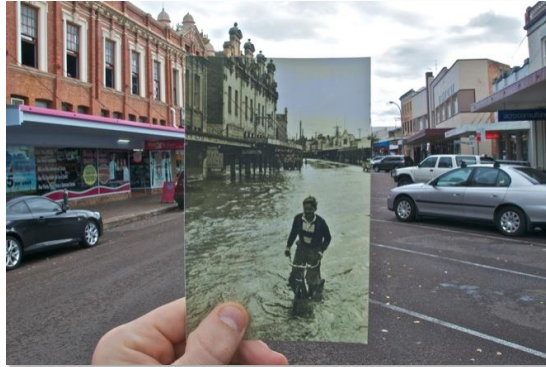# CS4670/5760: Computer Vision
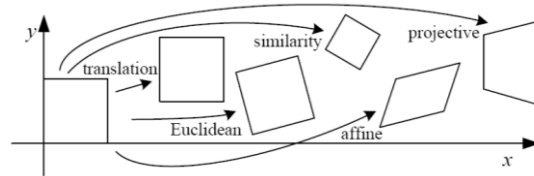Noah Snavely

Lecture 10: Image alignment

http://www.wired.com/gadgetlab/2010/07/camera-software-lets-you-see-into-the-past/

Reading

- Szeliski: Chapter 6.1

2D image transformations

Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\left[\begin{array}{c\|c} I & t \end{array}\right]_{2\times 3}$	2	orientation + $\cdots$	
rigid (Euclidean)	$\left[\begin{array}{c\|c} R & t \end{array}\right]_{2\times 3}$	3	lengths + $\cdots$	
similarity	$\left[\begin{array}{c\|c} sR & t \end{array}\right]_{2\times 3}$	4	angles + $\cdots$	
affine	$\left[\begin{array}{c} A \end{array}\right]_{2\times 3}$	6	parallelism + $\cdots$	
projective	$\left[\begin{array}{c} \tilde{H} \end{array}\right]_{3\times 3}$	8	straight lines	

These transformations are a nested set of groups
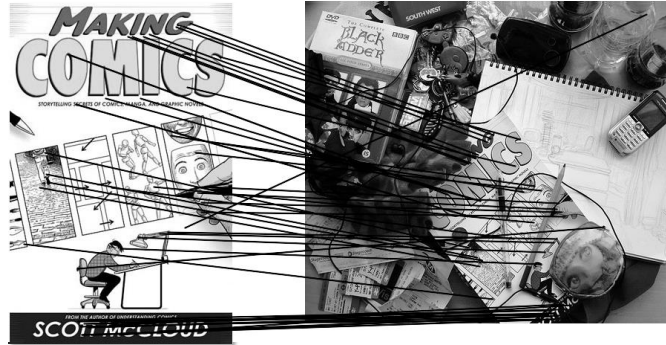 • Closed under composition and inverse is a member

Last time: Image Warping

• Given a coordinate xform $(x',y') = T(x,y)$ and a source image $f(x,y)$, how do we compute an xformed image $g(x',y') = f(T(x,y))$?

$T(x,y)$

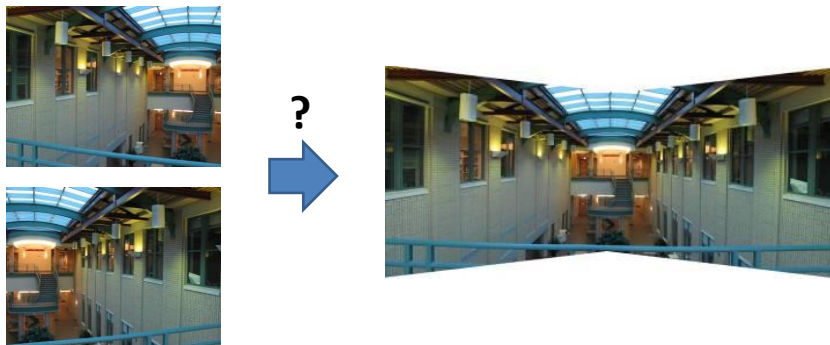$f(x,y)$ $g(x',y')$
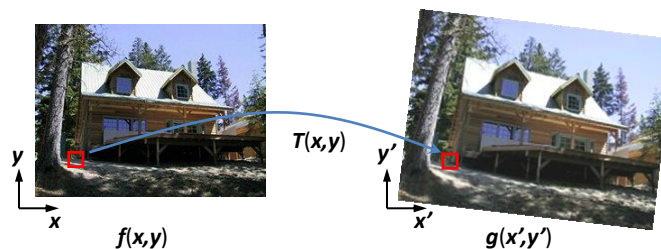
Computing transformations

- Given a set of matches between images A and B
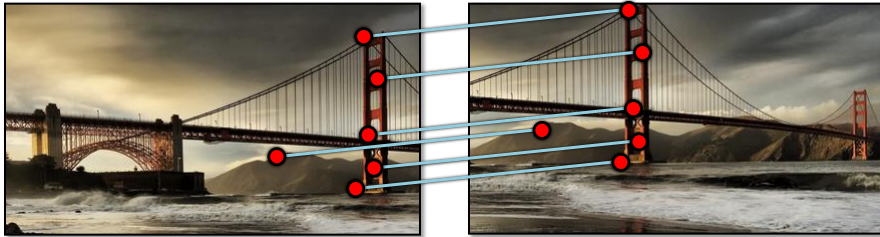 - How can we compute the transform T from A to B?

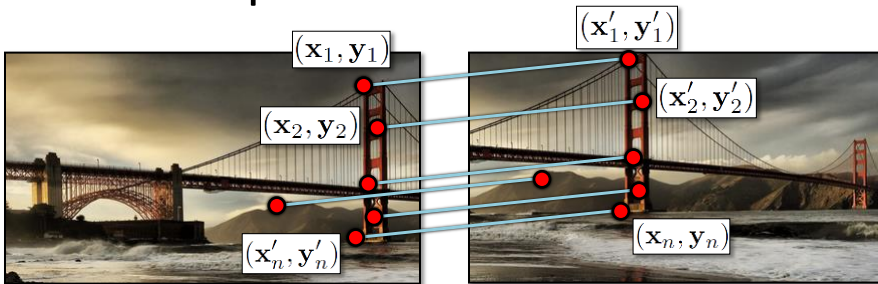 - Find transform T that best "agrees" with the matches

Computing transformations

?

Simple case: translations

How do we solve for $(\mathbf{x}_t, \mathbf{y}_t)$**?**

$(\mathbf{x}_t, \mathbf{y}_t)$

Simple case: translations

$(\mathbf{x}_1, \mathbf{y}_1)$

$(\mathbf{x}_1', \mathbf{y}_1')$

$(\mathbf{x}_2, \mathbf{y}_2)$

$(\mathbf{x}_2', \mathbf{y}_2')$

$(\mathbf{x}_n', \mathbf{y}_n')$

$(\mathbf{x}_n, \mathbf{y}_n)$

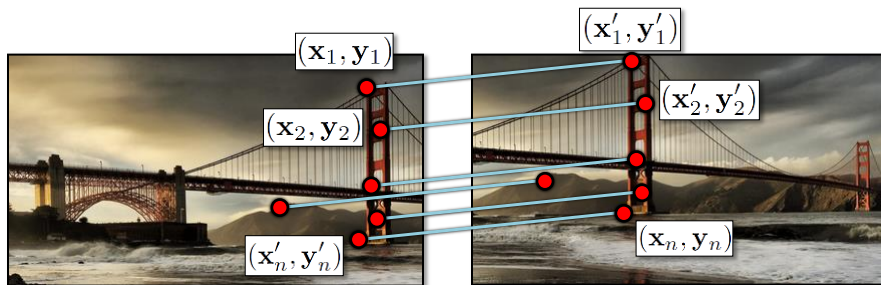Displacement of match i = $\left(\mathbf{x}_i' - \mathbf{x}_i, \mathbf{y}_i' - \mathbf{y}_i \right)$

$$(\mathbf{x}_t, \mathbf{y}_t) = \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i' - \mathbf{x}_i, \frac{1}{n} \sum_{i=1}^{n} \mathbf{y}_i' - \mathbf{y}_i \right)$$

Another view

$$\mathbf{x}_i + \mathbf{x_t} = \mathbf{x}'_i$$
$$\mathbf{y}_i + \mathbf{y_t} = \mathbf{y}'_i$$
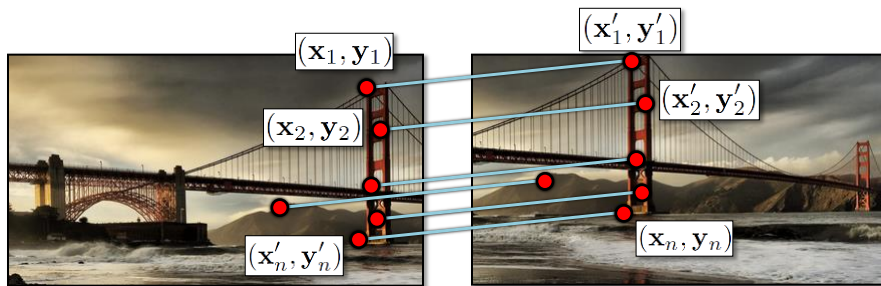
- System of linear equations
  - What are the knowns? Unknowns?
  - How many unknowns? How many equations (per match)?

Another view

$$\mathbf{x}_i + \mathbf{x_t} = \mathbf{x}'_i$$
$$\mathbf{y}_i + \mathbf{y_t} = \mathbf{y}'_i$$

- Problem: more equations than unknowns
  - "Overdetermined" system of equations
  - We will find the *least squares* solution

Least squares formulation

- For each point $(\mathbf{x}_i, \mathbf{y}_i)$

$$
\begin{aligned}
\mathbf{x}_i + \mathbf{x_t} &= \mathbf{x}_i' \\
\mathbf{y}_i + \mathbf{y_t} &= \mathbf{y}_i'
\end{aligned}
$$

- we define the *residuals* as

$$
\begin{aligned}
r_{\mathbf{x}_i}(\mathbf{x}_t) &= (\mathbf{x}_i + \mathbf{x}_t) - \mathbf{x}_i' \\
r_{\mathbf{y}_i}(\mathbf{y}_t) &= (\mathbf{y}_i + \mathbf{y}_t) - \mathbf{y}_i'
\end{aligned}
$$

Least squares formulation

- Goal: minimize sum of squared residuals

$$
C(\mathbf{x}_t, \mathbf{y}_t) = \sum_{i=1}^{n} \left(r_{\mathbf{x}_i}(\mathbf{x}_t)^2 + r_{\mathbf{y}_i}(\mathbf{y}_t)^2 \right)
$$

- "Least squares" solution
- For translations, is equal to mean displacement

Least squares formulation

- Can also write as a matrix equation

$$
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1 \\
& \vdots \\
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix} x_t \\ y_t \end{bmatrix}
=
\begin{bmatrix}
x_1' - x_1 \\
y_1' - y_1 \\
x_2' - x_2 \\
y_2' - y_2 \\
\vdots \\
x_n' - x_n \\
y_n' - y_n
\end{bmatrix}
$$

$$
\underset{2n \,x\, 2}{\mathbf{A}} \quad \underset{2 \,x\, 1}{\mathbf{t}} = \underset{2n \,x\, 1}{\mathbf{b}}
$$

Least squares

$$\mathbf{At} = \mathbf{b}$$

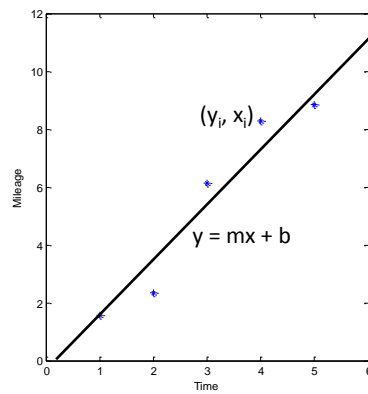- Find **t** that minimizes

$$||\mathbf{At} - \mathbf{b}||^2$$

- To solve, form the *normal equations*

$$\mathbf{A}^{\mathrm{T}}\mathbf{At} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

$$\mathbf{t} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Questions?

Least squares: linear regression

The chart shows points labeled (y_i, x_i) fitted by a line $y = mx + b$, with "Mileage" on the y-axis and "Time" on the x-axis.
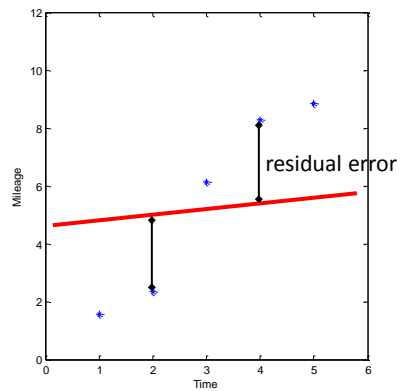
Linear regression

$$\text{Cost}(m, b) = \sum_{i=1}^{n} |y_i - (mx_i + b)|^2$$

Linear regression

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Affine transformations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- How many unknowns?
- How many equations per match?
- How many matches do we need?

Affine transformations

- Residuals:

$$r_{x_i}(a, b, c, d, e, f) = (ax_i + by_i + c) - x'_i$$
$$r_{y_i}(a, b, c, d, e, f) = (dx_i + ey_i + f) - y'_i$$

- Cost function:

$$C(a, b, c, d, e, f) =$$
$$\sum_{i=1}^{n} \left(r_{x_i}(a, b, c, d, e, f)^2 + r_{y_i}(a, b, c, d, e, f)^2 \right)$$
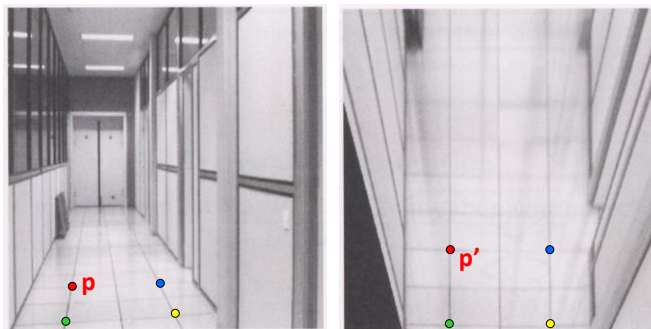
Affine transformations

- Matrix form

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ & & \vdots & & & \\ x_n & y_n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ \vdots \\ x'_n \\ y'_n \end{bmatrix}$$

$$\mathbf{A} \qquad \mathbf{t} = \mathbf{b}$$

$2n \text{ x } 6$ $6 \text{ x } 1$ $2n \text{ x } 1$

Homographies

To unwarp (rectify) an image
- solve for homography **H** given **p** and **p'**
- solve equations of the form: w**p'** = **Hp**
  - linear in unknowns: w and coefficients of **H**
  - H is defined up to an arbitrary scale factor
  - how many points are necessary to solve for **H**?

Solving for homographies

$$\begin{bmatrix} x_i' \\ y_i' \\ 1 \end{bmatrix} \cong \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

$$x_i' = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

Not linear!

$$y_i' = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

$$x_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02}$$
$$y_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12}$$

Solving for homographies

$$x_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02}$$
$$y_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12}$$

$$\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & -x_i'x_i & -x_i'y_i & -x_i' \\ 0 & 0 & 0 & x_i & y_i & 1 & -y_i'x_i & -y_i'y_i & -y_i' \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for homographies

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 & -x_1' \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\ & & & & \vdots & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & -x_n'x_n & -x_n'y_n & -x_n' \\ 0 & 0 & 0 & x_n & y_n & 1 & -y_n'x_n & -y_n'y_n & -y_n' \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} \qquad \qquad \mathbf{h} \qquad \mathbf{0}$$

2n × 9   **9**   **2n**

Defines a least squares problem: minimize $\|\mathbf{Ah} - \mathbf{0}\|^2$

- Since $\mathbf{h}$ is only defined up to scale, solve for unit vector $\hat{\mathbf{h}}$
- Solution: $\hat{\mathbf{h}}$ = eigenvector of $\mathbf{A}^T\mathbf{A}$ with smallest eigenvalue
- Works with 4 or more points

Questions?

Image Alignment Algorithm

Given images A and B

1. Compute image features for A and B
2. Match features between A and B
3. Compute homography between A and B using least squares on set of matches

What could go wrong?