# Fast Solver for 2D Stokes Flow

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## 1 Introduction

### 1.1 Problem Description

### 1.2 Why does it matter

### 1.3 Related works

**hemholtz eq, the normal mode analysis**

**stokes eq, similar complex networks**

### 1.4 What I have done

### 1.5 This paper is organized as following

## 2 Math Preliminaries

It is well-known that plane Stokes equation is closely related to the biharmonic equation. In this section, we transform the Stokes boundary value problem (1-4) into a biharmonic boundary value problem (6,7). Then, Goursat's formulae and Sherman-Lauricella representation will give a boundary integral equation, which will be solved numerically.

The previous paragraph is no more than what's done in (cite professor greengard's paper). The new idea in this paper is to exploit the **return to Poiseuille** phenomenon: Stokes flow in a straight pipe will converge to the Poiseuille flow in the axial direction at an exponential rate. This is proved analytically in (cite some paper proving this, using a figure as an example). As in (some figure), return to Poiseuille allow us to assume the flow is almost Poiseuille away from irregular (not straight pipe) area.

Therefore we can break the domain, at where the flow is almost Poiseuille, into subdomains. And we can assume that the subdomains have the boundary value of a Poiseuille velocity profile. Then, we can solve the Stokes equation for each sub-domain and then glue the solutions of each subdomain to get a very accurate solution for the global domain.

Other way around, we can build the solvers for some standard shapes of pipes. Connecting these pipes can give us a very complicated network of pipes. And we can solve the Stokes equation on this pipe network by simply matching the local solvers at the boundary. This saves much time compared to solving the Stokes equation directly.

### 2.1 Stokes Boundary Value Problem

The plane linear Stokes equation is

$$\nu\Delta u = \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \nu\Delta v = \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{1}$$

$$u_x + v_y = 0 \tag{2}$$

where $u, v$ are components of velocity, $\rho$ is the density, $\nu$ is the viscosity, and $p$ is the pressure. Additionally, vorticity is defined as $\zeta = u_y - v_x$. It is easy to see that vorticity is harmonic because of (1):

$$\Delta\zeta = \Delta u_y - \Delta v_x = \frac{1}{\nu\rho}p_{xy} - \frac{1}{\nu\rho}p_{xy} = 0 \tag{3}$$

We are interested in the boundary value problems on a finite domain $D \subset \mathbb{R}^2$, with the the $(M+1)$-ply connected boundary $\partial D = \Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_M$ is , where $\Gamma_0$ is the outer boundary of $D$, and $\Gamma_1, \cdots, \Gamma_M$ are the interior contours of $D$. The boundary value of our interest is velocities on the boundary:

$$u = h_2(t), \quad v = -h_1(t), \quad t \in \Gamma \tag{4}$$

(insert a picture here of an example domain.)

### 2.2 The Biharmonic Boundary Value Problem

(2) is equivalent to the existence of the stream function $W(x, y)$ such that:

$$W_x = -v, \quad W_y = u \tag{5}$$

(3) indicates that stream function satisfies the biharmonic equation:

$$\Delta^2 W(x, y) = \Delta(W_{xx} + W_{yy}) = \Delta(-v_x + u_y) = \Delta\zeta = 0 \tag{6}$$

And the velocity boundary condition (4) can be understood as the boundary conditions :

$$W_x(t) = h_1(t), \quad W_y(t) = h_2(t), \quad t \in \Gamma \tag{7}$$

## 2.3 Goursat's Formulae

Any plane biharmonic function $W(x, y)$ can be expressed by Goursat's formulae as

$$W(x, y) = \mathrm{Re}(\bar{z}\phi(z) + \chi(z)) \tag{8}$$

where $\phi, \chi$ are analytic functions of complex variable $z = x + yi$.

Simple calculations lead to the following useful formulae:

$$u(x, y) + iv(x, y) = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{9}$$

$$\zeta(x, y) + \frac{i}{\nu}p(x, y) = 4\phi'(z) \tag{10}$$

where $\psi = \chi'$.

(9) transforms the boundary condition (7) to

$$\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)} = h(t), \quad t \in \Gamma \tag{11}$$

where $h(t) = h_1(t) + ih_2(t)$. $t$ need to be understood as a complex number in the LHS and a plane coordinate in the RHS. This is a slight **abuse of notation** that will not be explained later.

## 2.4 Sherman-Lauricella Representation

(citing professor Greengard's paper) suggested the following Sherman-Lauricella integral representation:

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi - z}d\xi + \sum_{k=1}^{M} C_k \log(z - z_k) \tag{12}$$

$$\begin{aligned}\psi(z) = {} & \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\omega(\xi)}d\xi + \omega(\xi)\overline{d\xi}}{\xi - z} - \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\xi}\omega(\xi)}{(\xi - z)^2}d\xi \\ & + \sum_{k=1}^{M}\frac{b_k}{z - z_k} + \sum_{k=1}^{M}\overline{C}_k \log(z - z_k) - \sum_{k=1}^{M} C_k \frac{\overline{z}_k}{z - z_k}\end{aligned} \tag{13}$$

where $\omega$ is an unknown complex density on $\Gamma$, $z_k$ is an arbitrarily prescribed point inside the component curves $\Gamma_k$, and

$$C_k = \int_{\Gamma_k} \omega(\xi)|d\xi|$$

It is important to obeserve that $\phi, \psi$ might be multiply-valued functions, but velocity, pressure, and vorticity should be single-valued functions.

Letting a point $z$ in the interior of $D$ approach to a point on the boundary $\Gamma$, the classical formulae for the limiting values of Cauchy-type integral gives us the an integral equation for $\omega$:

$$\begin{aligned}
&\omega(t) + \frac{1}{2\pi i}\int_\Gamma \omega(\xi) d\ln\frac{\xi - t}{\overline{\xi - t}} - \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\xi)} d\ln\frac{\xi - t}{\overline{\xi - t}} && (14)\\
&\quad + \frac{\bar{b}_0}{\bar{t} - \bar{z}^*} + \sum_{k=1}^{M}\frac{\bar{b}_k}{\bar{t} - \bar{z}_k} + \sum_{k=1}^{M} 2C_k \log|t - z_k| + \sum_{i=1}^{M}\bar{C}_k\frac{t - z_k}{\bar{t} - \bar{z}_k} && (15)\\
&=h(t) && (16)
\end{aligned}$$

where $b_0$ vanishes on the natural compatibility condition $\mathrm{Re}\int_\Gamma \bar{h}(t)dt = 0$, which means there is zero net flux across $\Gamma$. The invertibility of the integral equation should follow from (citing some paper).[1]

### 2.5 Return to Poiseuille

Return to Poiseuille is formulated on the domain of a semi-infinite pipe $D_L = \{(x, y) \mid x \geq 0, |y| \leq L\}$, with the boundaries

$$\begin{aligned}
\Gamma_L = &\quad \Gamma_L^1 && \cup\Gamma_L^2 && \cup\Gamma_L^3 && (17)\\
= &\quad \{(0, y)||y| \leq L\} && \cup\{(x, L)|x \geq 0\} && \cup\{(x, -L) \mid x \geq 0\}
\end{aligned}$$

where $\Gamma_L^1$ is the inflow, $\Gamma_L^2, \Gamma_L^3$ are walls, which means they have the non-slippery boundary conditions. So, the boundary conditions on $\Gamma_L$ is of the following form:

$$\begin{aligned}
W_x(0, y) + iW_y(0, y) &= -v(0, y) + iu(0, y), && |y| \leq L && (18)\\
W_x(x, \pm L) + iW_y(x, \pm L) &= 0, && x \geq 0 && (19)
\end{aligned}$$

where $u, v$ are velocities given at the boundary. Therefore the flux at the inflow $x = 0$ is $F = \int -u(0, y)dy$.

The Poiseuille velocity profile on this domain with flux $F$ is

$$\begin{aligned}
u_{poi}(x, y) &= \frac{4F}{3L^3}L^2 - y^2, && (20)\\
v_{poi}(x, y) &= 0 && (21)
\end{aligned}$$

where the subscript *poi* stands for Poiseuille.

We want to prove that the difference

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} = \begin{pmatrix}u\\ v\end{pmatrix} - \begin{pmatrix}u_{poi}\\ v_{poi}\end{pmatrix} \tag{22}$$

[1] or should I try to prove it by myself? That would be a good practice on understanding the invertibility of integral equations. And I don't really understand them...

decays exponentially as $x \to \infty$. Since all PDE in this paper is homogeneous and linear, $\bar{u}, \bar{v}$ can be solved through the biharmonic equation for $\tilde{W} = W - W_{poi}$:

$$
\begin{aligned}
\Delta\tilde{W}(x,y) &= 0, && (x,y) \in D_L && (23)\\
\tilde{W}_x(0,y) &= -v(0,y), && |y| \le L && (24)\\
\tilde{W}_y(0,y) &= u(0,y) - u_{poi}(0,y), && |y| \le L && (25)\\
\tilde{W}_x(x,\pm L) = 0, &\quad \tilde{W}_y(x,\pm L) = 0, && x \ge 0 && (26)
\end{aligned}
$$

Without lost of generality, assuming that $\tilde{W}(0,-1) = 0$. Then boundary conditions (24),(25), and (26) can be rewritten as:

$$
\begin{aligned}
\tilde{W}(x,\pm L) &= 0, && \frac{\partial\tilde{W}}{\partial\nu}(x,\pm L) = 0, && x \ge 0 && (27)\\
\tilde{W}(0,y) &= \int_{-L}^{y} \tilde{W}_y(0,\eta)d\eta, && \frac{\partial\tilde{W}}{\partial\nu}(0,y) = v(0,y), && |y| \le L && (28)
\end{aligned}
$$

where $\frac{\partial}{\partial\nu}$ denotes the out normal derivative.

For return to Poiseuille phenomenon, one should expect that

$$
A(x) = \sqrt{\int_{-L}^{L} \tilde{u}^2(x,y) + \tilde{v}(x,y)^2 dy} \le A\exp(-kx) \tag{29}
$$

where $A$ depends on $L, u(0,y), v(0,y)$, and $k$ depends only on $L$. [Horgan 1989] has summarized the result on $k$:

- $k = \frac{4.2}{2L}$. [2] This is the predicted value given by the analysis of Papkovich-Fadle eigenfunctions, which is complete basis for the problem (23-28) given that $u(0,y), v(0,y)$ and their first few derivative has bounded-variation. See Gregory 1980 for this result. However, this is not proven yet to the best of author's knowledge.

- $k = \frac{2.7}{2L}$. Although this value is smaller than the previous one. It has been proven in Knowles 1983. This is the best value of $k$ that I know of.

**Remark** Pretending that $k = \frac{4.2}{2L}$ is a true. Then, if the pipe has the length of $10L$, the flow should be Poiseuille at the outlet, up to an error of order of machine precision, regardless of the flow at the inlet. [3] In real situations, the difference to Poiseuille converges to $10^{-13}$, the numerical error of my Nyestorm discretization, at $7.5L$. So this return to poiseuille approximation really makes sense.

[2] this is the smallest real part of the roots of $\sin z + z = 0, \operatorname{Re} z > 0$

[3] I should spell out the regularity conditions along with the equations

### 2.5.1 plotting the estimate against the true error

todo.

## 2.6 perhaps also some potential theory

hmmm, I don't know that to write in this section.

## 2.7 Summary

In this section, we have articulated the integral equation for Stokes flow in terms of the stream function $W$. And we have supplied the reasons why **return to Poiseuille** is a reasonable assumption to make.

Given the above, we can do the following

- For each pipe, with the inlets and outlets being sufficiently long, $\geq 7.5L$ as suggested by numerical experiment, straight pipe, we can build a solver of this pipe with Poiseuille boundary conditions.
- We can connect these pipes and their solvers on their inlets and outlets, based on the restriction of flux and single-valuedness of the pressure function.
- Connecting the solvers of thes pipes will give us a solver for a network of pipes, which could be rather complicated and takes days for a direct solution.

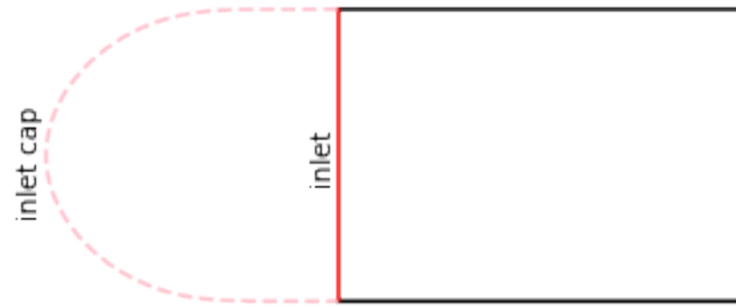
# 3 Numerical Algorithms

## 3.1 discretizing the geometry

## 3.2 Integral equation

## 3.3 Smooth

Nyestorm discretization will give spectral convergence for smooth geometry, which is much better compare to handling geometry with the corners. For this reason it is wise to smooth the geometry.




**smoothing the corner at inlet or outlet**

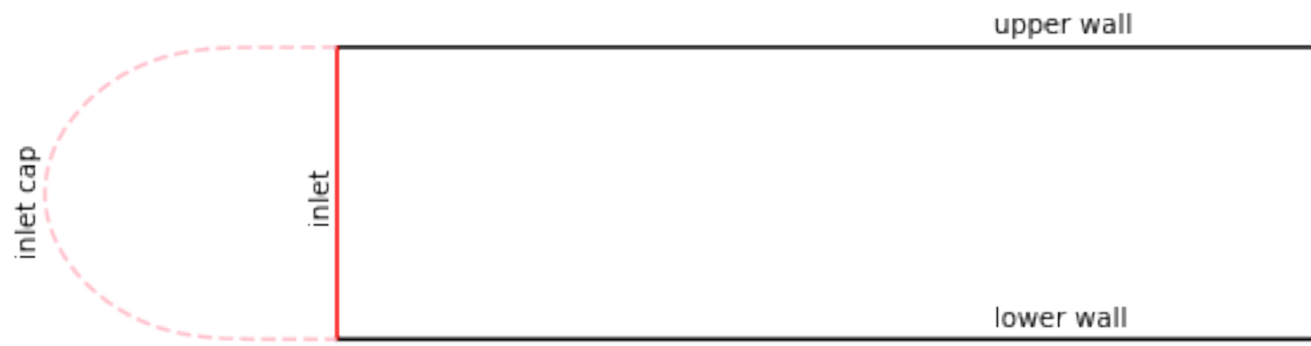



Figure 1: Smoothed corner at the inlet/outlet

### 3.4 how to combine local solvers

## 4 numerical results

### 4.1 plot the analytic error bound in assuming return to poiseuille

### 4.2 plot the numerical error for such assumptions

### 4.3 plotting the error of combining local to global

### 4.4 solve for a complicated network of pipes to show the power of this method

## 5 Conclusions

### 5.1 summarize what I've done

### 5.2 outlook. What other work might be followed?