Final Exam

Due June 17, 2024

June 6, 2024

- One defining characteristic of topological phases is the appearance of robust topologically protected modes at the system’s boundary.

- Via this problem, we are going to explore the topological phases in the prototypical **antisymmetric Lotka-Volterra equation** (ALVE).

- As a variant of the previously assigned homework problem, the ALVE is a nonlinear dynamical system and captures the evolutionary dynamics of a rock-paper-scissors cycle.

- On a one-dimensional chain of rock-paper-scissor cycles, topological phases become manifest as robust **polarization states**.

- At the transition point between left and right polarization, **solitary waves** are observed.

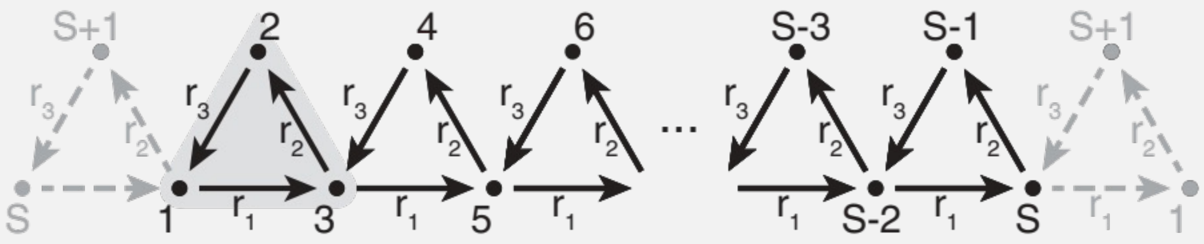
- Intriguingly, this topological phase transition lies in the symmetry class D within the famous “tenfold way” classification of topological matter, as can be also realized by the 1D topological superconductors.

Problem 1.1

The antisymmetric Lotka-Volterra equation is a nonlinear, mass-conserving dynamical system defined on S sites. The mass at each site α is denoted as x_α evolving according to the coupled ordinary differential equations,

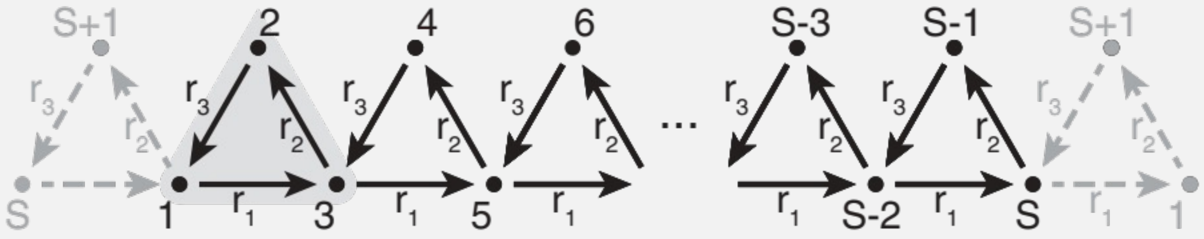
$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = x_\alpha \sum_{\beta=1}^{S} A_{\alpha\beta}\, x_\beta, \qquad\qquad \alpha = 1,2,\ldots,S$$

where the real-valued $S\times S$ matrix $\mathbf{A}$ is antisymmetric, meaning $A_{\alpha\beta} = -A_{\beta\alpha}$, which defines how mass is transported between two sites controlled by a nonlinear mutual interaction that is proportional to $x_\alpha x_\beta$. See the following diagram for an illustration.



The chain of the rock-paper-scissors cycles is shown above. An arrow from one site to another indicates that mass is transported in this direction at a rate of $r_1, r_2, r_3 > 0$. We choose $r_1 = 1$ as the unit and define the skewness $r = \frac{r_2}{r_3}$ as the control parameter.

Problem 1.1



Now choose the total length of the chain to be 13, namely $S = 13$, and fix the two end boundary conditions to be open. The initial mass distribution is the following,

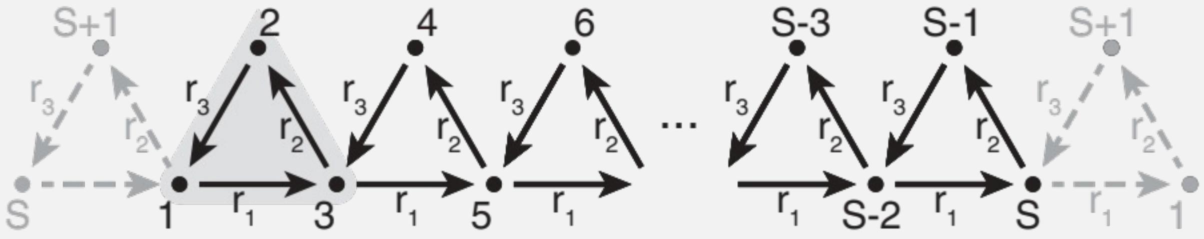
$$x_\alpha(t=0) = \begin{cases} \dfrac{2 - 0.01(S+1)}{S-1}, & \text{if } \alpha \text{ is even} \\ 0.01, & \text{if } \alpha \text{ is odd} \end{cases}$$

See also the above picture for the labelling.

Then try to solve the antisymmetric Lotka-Volterra equations for the following **first** circumstance using the fourth-order Runge-Kutta method from $t = 0$ up to $t = 5000$ using a step size $h = 0.1$. Make a plot for the mass evolutions as a function of time at the two end points and the middle central point.

Case I: $\boldsymbol{r = 0.5}$ meaning $\boldsymbol{r_1 = 1, r_2 = 0.5, r_3 = 1}$

Problem 1.2



Now choose the total length of the chain to be 13, namely $S = 13$, and fix the two end boundary conditions to be open. The initial mass distribution is the following,

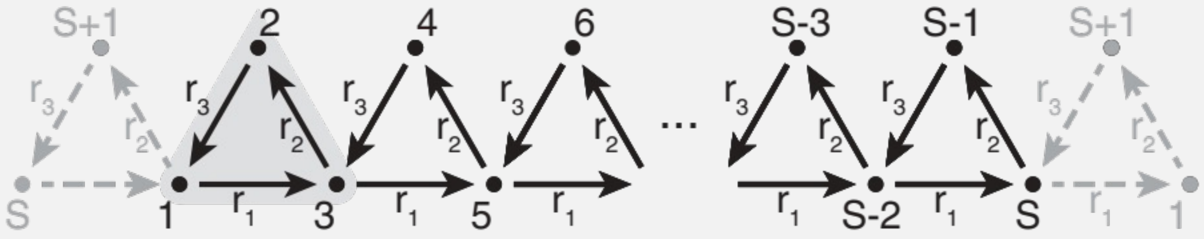
$$x_\alpha(t=0) = \begin{cases} \dfrac{2 - 0.01(S+1)}{S-1}, & \text{if } \alpha \text{ is even} \\ 0.01, & \text{if } \alpha \text{ is odd} \end{cases}$$

See also the above picture for the labelling.

Then try to solve the antisymmetric Lotka-Volterra equations for the following **second** circumstance using the fourth-order Runge-Kutta method from $t = 0$ up to $t = 5000$ using a step size $h = 0.1$. Make a plot for the mass evolutions as a function of time at the two end points and the middle central point.

Case II: $r = 1$ meaning $r_1 = 1, r_2 = 1, r_3 = 1$

Problem 1.3



Now choose the total length of the chain to be 13, namely $S = 13$, and fix the two end boundary conditions to be open. The initial mass distribution is the following,

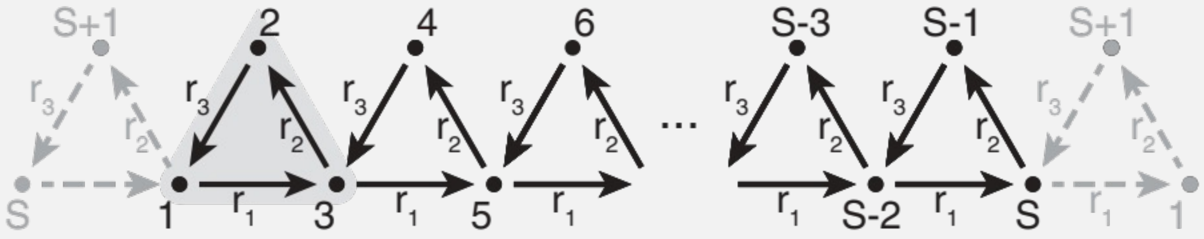
$$x_\alpha(t=0) = \begin{cases} \dfrac{2 - 0.01(S+1)}{S-1}, & \text{if } \alpha \text{ is even} \\ 0.01, & \text{if } \alpha \text{ is odd} \end{cases}$$

See also the above picture for the labelling.

Then try to solve the antisymmetric Lotka-Volterra equations for the following **third** circumstance using the fourth-order Runge-Kutta method from $t = 0$ up to $t = 5000$ using a step size $h = 0.1$. Make a plot for the mass evolutions as a function of time at the two end points and the middle central point.

Case III: $r = 2$ meaning $r_1 = 1, r_2 = 2, r_3 = 1$

Problem 1.4



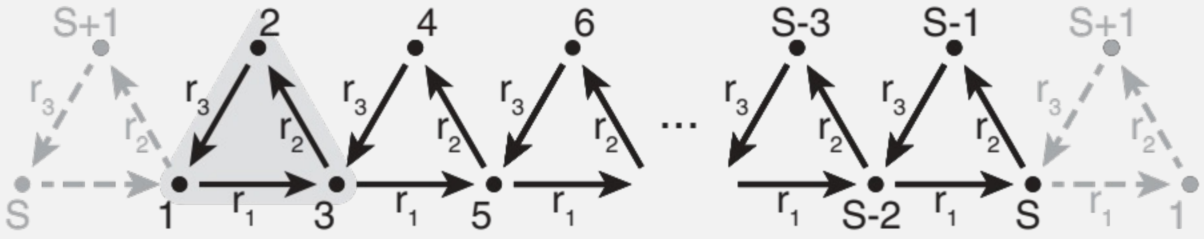
To further clarify the physical meaning of your findings, we introduce a quantity called **mass polarization** by averaging the mass at every site over a time window T as follows,

$$\langle x_\alpha \rangle_T = \frac{1}{T} \int_0^T x_\alpha(t) \mathrm{d}t$$

For **Case I:** $\boldsymbol{r = 0.5}$ **meaning** $\boldsymbol{r_1 = 1, r_2 = 0.5, r_3 = 1}$, using the results of your time evolution data and the Simpson's method to numerically evaluate this mass polarization for every site of the $S = 13$ chain over the time window $T = 5000$.

Make a plot of the mass polarization as a function of site α. What is the possible functional form between $\langle x_\alpha \rangle_T$ and α for case I? Can you fit the data $\langle x_\alpha \rangle_T$ vs α?

Problem 1.5



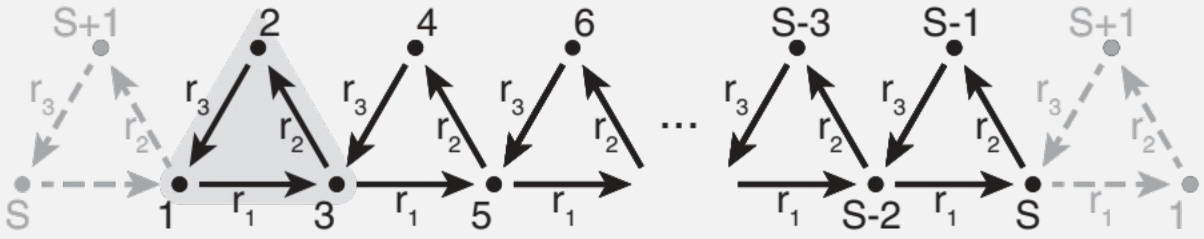
To further clarify the physical meaning of your findings, we introduce a quantity called **mass polarization** by averaging the mass at every site over a time window T as follows,

$$\langle x_\alpha \rangle_T = \frac{1}{T} \int_0^T x_\alpha(t) \mathrm{d}t$$

For **Case II:** $\boldsymbol{r = 1}$ **meaning** $\boldsymbol{r_1 = 1, r_2 = 1, r_3 = 1}$, using the results of your time evolution data and the Simpson's method to numerically evaluate this mass polarization for every site of the $S = 13$ chain over the time window $T = 5000$.

Make a plot of the mass polarization as a function of site α. What is the possible functional form between $\langle x_\alpha \rangle_T$ and α for case II? Can you fit the data $\langle x_\alpha \rangle_T$ vs α?

Problem 1.6



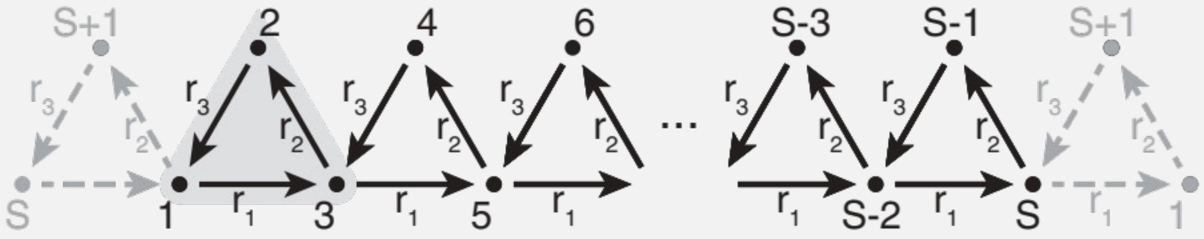
To further clarify the physical meaning of your findings, we introduce a quantity called **mass polarization** by averaging the mass at every site over a time window T as follows,

$$\langle x_\alpha \rangle_T = \frac{1}{T}\int_0^T x_\alpha(t)\mathrm{d}t$$

For **Case III:** $\boldsymbol{r = 2}$ **meaning** $\boldsymbol{r_1 = 1, r_2 = 2, r_3 = 1}$, using the results of your time evolution data and the Simpson's method to numerically evaluate this mass polarization for every site of the $S = 13$ chain over the time window $T = 5000$.

Make a plot of the mass polarization as a function of site α. What is the possible functional form between $\langle x_\alpha \rangle_T$ and α for case III? Can you fit the data $\langle x_\alpha \rangle_T$ vs α?

Problem 1.7



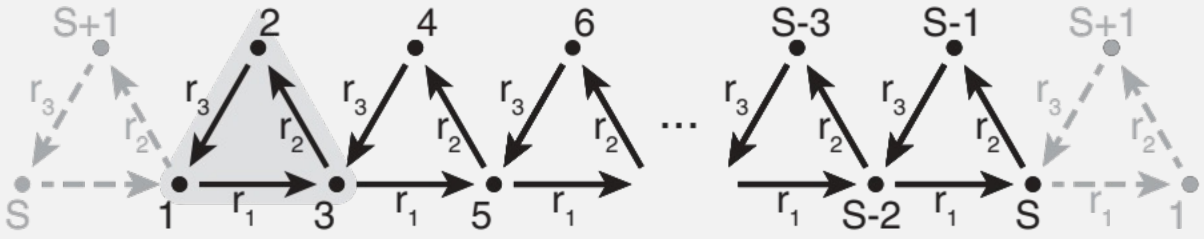
From the above 6 subproblems, what have you learnt about this **antisymmetric Lotka-Volterra system**, in particular its topological characters?

Is there a topological phase transition?

If so, where is the transition point?

Make a derivation and Explain to me, why the total mass of the chain is **conserved**?

Problem 1.8



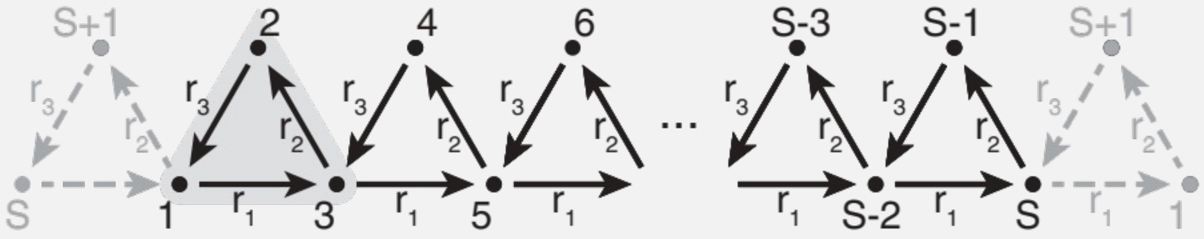
Now try to refine your code, and we are going to move on to a longer chain with 69 sites, i.e., $S = 69$. Still using open boundary conditions, but change the initial mass distribution to the following:

$$x_6(t=0) = x_{12}(t=0) = 0.01 \times 0.45$$
$$x_7(t=0) = x_{11}(t=0) = 0.15 \times 0.45$$
$$x_8(t=0) = x_{10}(t=0) = 0.05 \times 0.45$$
$$x_9(t=0) = 0.3 \times 0.45$$

$$x_{22}(t=0) = x_{28}(t=0) = 0.005 \times 0.45$$
$$x_{23}(t=0) = x_{27}(t=0) = 0.07 \times 0.45$$
$$x_{24}(t=0) = x_{26}(t=0) = 0.015 \times 0.45$$
$$x_{25}(t=0) = 0.1 \times 0.45$$

Masses on all other sites at the initial time is 0.01.

Problem 1.8

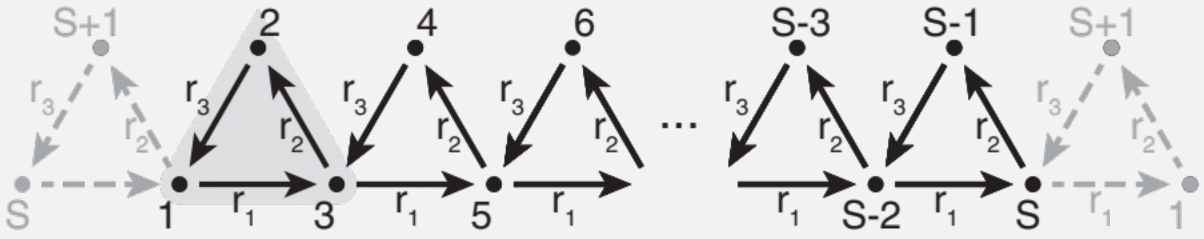


Specify first to the **Case II:** $\boldsymbol{r = 1}$ **meaning** $\boldsymbol{r_1 = 1, r_2 = 1, r_3 = 1}$, using fourth-order Runge-Kutta to solve the mass evolution of the $S = 69$ system at a time step $h = 0.01$ from $t = 0$ up to $t = 240$.

Make a plot of $x_\alpha(t)$ as a function of site α at $t = 0$, $t = 120$, $t = 240$, respectively.

Explain the physical meaning or the physical picture behind your results.

Problem 1.9



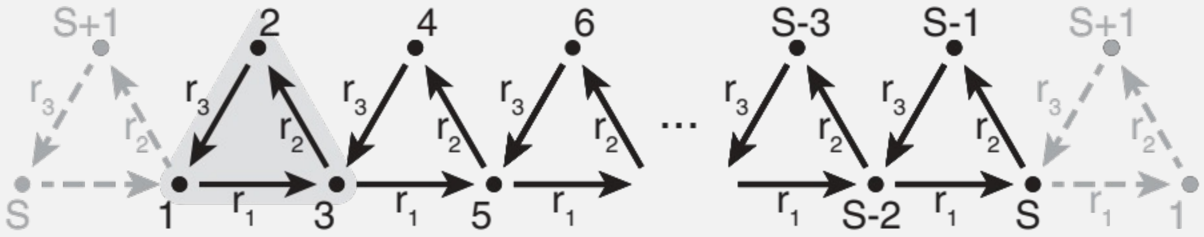
Specify next to the **Case I:** $\boldsymbol{r = 0.5}$ **meaning** $\boldsymbol{r_1 = 1, r_2 = 0.5, r_3 = 1}$, using fourth-order Runge-Kutta to solve the mass evolution of the $S = 69$ system at a time step $h = 0.01$ from $t = 0$ up to $t = 240$.

Make a plot of $x_\alpha(t)$ as a function of site α at $t = 0$, $t = 120, t = 240$, respectively.

Explain the physical meaning or the physical picture behind your results.

Particularly, what is the key difference between the results of Case I and Case II? Why?

Problem 1.10



Specify then to the **Case III:** $\boldsymbol{r = 2}$ **meaning** $\boldsymbol{r_1 = 1, r_2 = 2, r_3 = 1}$, using fourth-order Runge-Kutta to solve the mass evolution of the $S = 69$ system at a time step $h = 0.01$ from $t = 0$ up to $t = 240$.

Make a plot of $x_\alpha(t)$ as a function of site α at $t = 0,\ t = 120, t = 240$, respectively.

Explain the physical meaning or the physical picture behind your results.

Particularly, what is the key difference between the results of Case III and Case II? Why?