A Unifying Framework of Galois Connections: From Adjoint Functors to Galois Theory and Topological Covering Spaces

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2025.12.1

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1 Abstract

This paper presents a unified categorical perspective on two classical correspondence theorems: the fundamental theorem of Galois theory and the calssification theorm of covering spaces. We begin by very basic category theory, then introduce the definition of Galois connection between partially ordered sets, and then show that this concept naturally categorifies to the theory of adjoint functors.

In second part we will introduce Galois theory and demonstrates how both Galois theory and covering space theory in algebraic topology can be viewed as concrete instances of the same abstract adjunction. The core idea of this paper is that **Galois theory and covering space theory are actually the same things as they are the applications of Galois connection in algebra and topology**

Reader should be at least a undergraduate student who has basic knowledge of abstract algebra and algebraic topology, and have some basic understanding of category theory, although we will introduce in this paper.

Ketwords: Galois connection, Adjoint functors, Galois theory, Covering space theory, mathematical unification.

2 Category theory

Any readers who are familiar with category theory can skip this section directly to next part. In this section we will introduce some basic concepts in category theory that are necessary for understanding the rest of this paper.

2.1 Category

Definition 2.1 *(Category) A* ***category*** $\mathscr{C}$ *consists of a collection* $ob(\mathscr{C})$ *of* ***objects****, and* $\forall A, B \in ob(\mathscr{C})$*, there is a collection* $\mathscr{C}(A, B)$ *of* ***morphisms*** *from* A *to* B*. With following three axioms satisfied:*
1. ***Compostion****:* $\forall A, B, C \in ob(\mathscr{C})$*, if* $f \in \mathscr{C}(A, B)$ *and* $g \in \mathscr{C}(B, C)$*, then there is a function* $g \circ f \in \mathscr{C}(A, C)$ *called compostion of* f *and* g*.*
2. ***Associativity****:* $(h \circ g) \circ f = h \circ (g \circ f),\ \forall f \in \mathscr{C}(A, B),\ g \in \mathscr{C}(B, C),\ h \in \mathscr{C}(C, D)$*.*
3. ***Identity laws****:* $\forall A \in ob(\mathscr{C}),\ \exists 1_A \in \mathscr{C}(A, A)$*, called the identity of* A*. And* $\forall f \in \mathscr{C}(A, B)$*, we have* $f \circ 1_A = f = 1_B \circ f$*.*

Example 2.1 *(Category of Sets:* ***Set****) For a category* $\mathscr{C}$*, where* $ob(\mathscr{C})$ *is a collection of sets, here we consider each set as one object, no matter the cardinality of it. Given two set* A *and* $B \in ob(\mathscr{C})$*, the mapping or morphism between two sets is exactly a function from* A *to* B*. Together with all the sets we have and the functions between each two sets we call them* ***Set****, the category of set.*

Example 2.2 *(Category of Groups and Rings:* ***Grp*** *and* ***Ring****) We have a collection of groups, and a morphism between every two given group* G *and* H *which is so-called group homomorphism. Then all these groups together with the group homomorphisms are called category* ***Grp*** *of groups. Similarly, there is a category* ***Ring*** *of rings and ring homomorphisms.*

Example 2.3 *(Category of Vector Spaces over field k:* $\textbf{Vect}_k$*) For a field k,* $\textbf{Vect}_k$ *consists the vector fields over k and the mapping between two vector spaces* H *to* W *which will be the k linear transformations from* H *to* W*, i.e.* $\mathcal{L}(H, W)$*.*

Example 2.4 *(Category of Topological Spaces:* ***Top****) There is a collection of topological spaces and the mapping between topological spaces are continous maps. Together the topological spaces and maps are called* ***Top****.*

Example 2.5 *(Category of nothing:* $\emptyset$*) There is a collection of nothing and no morphisms between nothing, these called empty category* $\emptyset$*.*

Example 2.6 *(Category of one object: **1**) There is a category **1** with only one object in the collection and only Identity map.*

Example 2.7 *(Discrete Category) A category $\mathscr{C}$ is discrete if $\forall A, B \in ob(\mathscr{C})$ and $A \neq B$, $\mathrm{Hom}_{\mathscr{C}}(A, B) = \emptyset$. This does not mean there is no mapping in $\mathscr{C}$, notice that $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A),\ \forall A$.*

Example 2.8 *(One object category constructed by a group) A group actually is a one object category. Differ this from category **1** that **1** has only identity but for group one element in it is a morphism. Let us put in a clear way. We have a group $G = \{e, g_1, g_2, ...\}$, consider a category $\mathscr{C}$ that $\mathrm{Ob}_{\mathscr{C}} = G$, the identity morphism in $\mathscr{C}(G, G)$ is actually $e \in G$:*

$$e(G) = e \cdot G = \{e \cdot e, e \cdot g_1, e \cdot g_2 ...\} = \{e, g_1, g_2, ...\} = G$$

and $\forall g \in G$, $g(G) \subseteq G$ from the closure of group structure, if $|G| < \infty$, then $g(G) = G,\ \forall g \in G$. The corresponding table below helps you to understand the isomorphism between mathematical structure.

Category $\mathscr{C}$ with single object A	**Group G**
Maps in $\mathscr{C}$	Elements in G
$\circ$ in $\mathscr{C}$	$\cdot$ in G
$1_A \in \mathscr{C}(A, A)$	$e_G \in G$

We remark that the category of one mathematical object is a collection of some structural objects not necessarily all the objects, and we provide a example say one object category of group.

Now we put our focus into the morphisms in category, given A and B as object of category $\mathscr{C}$, the mapping in $\mathscr{C}(A, B)$ should not necessarily be so-called functions or transformations, we name the morphisms as transformations it is because for $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, we have $f : A \longrightarrow B$, gives us feeling that the morphism f trans A into B.

We should consider it more abstract, the type of $f(A) = B$ is actually a one directional relation of A and B. If f is not some machine but a not comprehensive statement, for example: $f :=$ 'is bigger than', then $f(A) = B$ is a full statement:

$$f(A) = B \Longleftrightarrow A \longrightarrow B \Longleftrightarrow A \textit{ is bigger than } B$$

Consider mapping as relation between different objects is one of core idea in category theory, it is a great abstraction and acrroding to this we can find many isomorphism between different mathematical structures.

Definition 2.2 *(Opposite Category) a category noted $\mathscr{C}^{op}$ is said to be the opposite or dual category of given category $\mathscr{C}$, it has exactly the same object with all the arrows in $\mathscr{C}$ reversed, thas is:*

$$ob(\mathscr{C}^{op}) = ob(\mathscr{C}) \quad \textit{and} \quad \mathscr{C}^{op}(B, A) = \mathscr{C}(A, B)$$

The 'Category' is not a unique object but a structure of mathematical objects. For instance, Group is something in a set satisfy associativity and there is an identity and inverce under specific operation, the set in definition of group can be anything, they could be functions, person, computer program, formula for Rubik's cube... So similarly, here the category of some mathematical object is not represent all these object in the collection. You can construct any category you want with few objects as long as you give the morphisms and they satisfy the axioms.

Take **Set** as example, the category of set not necessarily contain all sets, if you construct a set contains limited number of sets and follow the axioms to be a category, it still called **Set**.

You may have an intuition that if we not restrict the size of $\mathrm{Hom}_{\mathscr{C}}$ for specific $\mathscr{C}$, we have a trouble with Russell's Paradox because we are talking about "set of sets". To avoid such hindrance, we introduce the concept of small category.

Definition 2.3 *(Locally Small Category) We say a category $\mathscr{C}$ is locally small if $\forall A, B \in \mathrm{Ob}_{\mathscr{C}}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a set.*

Example 2.9 *The category **Set** is locally small because for any two sets $A, B \in ob(\textbf{Set})$, the morphism set $\mathrm{Hom}_{\textbf{Set}}(A, B)$ is the set of all functions from A to B, which is a set.*

Example 2.10 *The category **Grp** is locally small because for any two groups $G, H \in ob(\textbf{Grp})$, the morphism set $\mathrm{Hom}_{\textbf{Grp}}(G, H)$ is the set of all group homomorphisms from G to H, which is a set.*

2.2 Functor

Definition 2.4 *(Functor) We say a map of categories $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor, if it sends every A in $ob(\mathscr{C})$ to a $\mathcal{F}(A)$ in $ob(\mathscr{D})$ and a morphism $f : A \longrightarrow B$ of $\mathscr{C}$ to a morphism $\mathcal{F}(f) : \mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ of $\mathscr{D}$, while satisfies two axioms that*

$$\mathcal{F}(id_A) = id_{\mathcal{F}(A)} \quad \text{and} \quad \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

where $A \in ob(\mathscr{C})$ and $f, g \in mor(\mathscr{C})$

The fundamental group map $\pi_1(*, *)$ can be considered as a functor from the category of topological space with the basepoint **Top*** to **Grp**. For any topological space X with a basepoint x specified, the functor send the pair $(X, x) \in ob(\textbf{Top*})$ to its fundamental group $\pi_1(X, x) \in \textbf{Grp}$, with give us the algebric structure of the loops start at basepoint x in space X. In this case any $f \in \textbf{Top*}(X, Y)$ not only a continous map from X to Y but also a basepoint-preserving $f : (X, x) \longrightarrow (Y, y)$, with its image under the functor $\pi_1(f) : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$.

For examples apart from algebric topology, we have forgetful functors from **Grp** to **Set** which just by its name, the group forgets its structure under the operation but keep its members as a set. And free functors can be considered as dual functor of forgetful, send a set to a group with an operation and add more elements in the set to make it a group.

One type a functor is widely used in categorical language, for a locally small category $\mathscr{C}$ and $A \in ob(\mathscr{C})$, we have $H^A = \mathrm{Hom}(A, -) : \mathscr{C} \longrightarrow \textbf{Set}$. The morphism functor send every element X in the category to the morphism set $\mathrm{Hom}(A, X)$, which is the set of all morphisms from A to X. And the morphism map under functor is $H^A(g) = \mathrm{Hom}(A, g) : \mathrm{Hom}(A, X) \longrightarrow \mathrm{Hom}(A, Y)$, simply by $f \longmapsto g \circ f$ for all $f : X \longrightarrow Y$.

But if the postion of given $A \in ob(\mathscr{C})$ switch then everything changed. For similar morphism functor $H_A = \mathrm{Hom}(-, A)$, the narrow preserving diagram make no sense, the image $H_A(g)$ cannot be defined for certain $g \in \mathrm{Hom}(X, A)$, so the functor is actually defined on the opposite category of $\mathscr{C}$, that is, $H_A = \mathrm{Hom}(-, A) : \mathscr{C} \longrightarrow \textbf{Set}$ (See diagram below). This gives us the motivation to define the special functor on the opposite category.

$$
\begin{tikzcd}
X \arrow[r, "H^A"] \arrow[d, "f"'] & \mathrm{Hom}(X,A) \arrow[d] \\
Y \arrow[r, "H^A"] & \mathrm{Hom}(Y,A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & A \\
Y \arrow[ur, "cannot\ define"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "H^A"] & \mathrm{Hom}(X,A) \arrow[d] \\
Y \arrow[u, "f'"] \arrow[r, "H^A"] & \mathrm{Hom}(Y,A)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] & A \\
Y \arrow[u, "f'"] \arrow[ur, "g \circ f"'] &
\end{tikzcd}
$$

Definition 2.5 *(contravariant functor) Let $\mathscr{C}$ and $\mathscr{D}$ be categories, a functor $\mathscr{C}^{op} \longrightarrow \mathscr{D}$ is said to be the contravariant functor from $\mathscr{C}$ to $\mathscr{D}$.*

Having introduced functors and contravariant functors, a natural question arises: when do two functors form a "symmetric pair" that encode a reversible translation between categories? This leads to the central concept of **adjoint functors**, which is the categorical formalisation of Galois connections.

Definition 2.6 *(Adjoint functors) Let $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}$ be two functors. We say $\mathcal{F}$ is **left adjoint** to $\mathcal{G}$ (and $\mathcal{G}$ is **right adjoint** to $\mathcal{F}$), written $\mathcal{F} \dashv \mathcal{G}$, if there exists a natural bijection*

$$\Phi_{c,d} : \mathrm{Hom}_{\mathscr{D}}\big(\mathcal{F}(c), d\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{C}}\big(c, \mathcal{G}(d)\big)$$

for every $c \in \mathscr{C}$ and $d \in \mathscr{D}$, such that for any morphisms $f : c' \to c$ in $\mathscr{C}$ and $g : d \to d'$ in $\mathscr{D}$ the following diagram commutes:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}\big(\mathcal{F}(c), d\big) \arrow[r, "\Phi_{c,d}"] \arrow[d, "\mathcal{F}(f)^* \circ g_*"'] & \mathrm{Hom}_{\mathscr{C}}\big(c, \mathcal{G}(d)\big) \arrow[d, "f^* \circ \mathcal{G}(g)_*"] \\
\mathrm{Hom}_{\mathscr{D}}\big(\mathcal{F}(c'), d'\big) \arrow[r, "\Phi_{c',d'}"] & \mathrm{Hom}_{\mathscr{C}}\big(c', \mathcal{G}(d')\big)
\end{tikzcd}
$$

where f^* denotes pre-composition with f and g_* denotes post-composition with g.

The definition precisely captures the idea that morphisms starting from $\mathcal{F}(c)$ in $\mathscr{D}$ correspond one-to-one to morphisms ending at $\mathcal{G}(d)$ in $\mathscr{C}$, and this correspondence respects composition in both categories. In the special case where $\mathscr{C}$ and $\mathscr{D}$ are preordered sets viewed as categories, the adjunction condition reduces exactly to the monotone Galois connection $F(a) \leq b \iff a \leq G(b)$.

Example 2.11 (Free–forgetful adjunction) *The most ubiquitous example of an adjunction is between the **free functor** $\mathcal{F} : \mathbf{Set} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ and the **forgetful functor** $\mathcal{G} : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Set}$. Here $\mathcal{F}$ sends a set X to the free real vector space $\mathbb{R}^{(X)}$ with basis X, and $\mathcal{G}$ simply discards the vector-space structure, returning the underlying set.*

The adjunction $\mathcal{F} \dashv \mathcal{G}$ is witnessed by the natural bijection

$$\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{R}}}\big(\mathbb{R}^{(X)}, V\big) \;\simeq\; \mathrm{Hom}_{\mathbf{Set}}\big(X, \mathcal{G}(V)\big),$$

which states that a linear map from the free vector space on X to a vector space V is uniquely determined by its values on the basis X, i.e., by an ordinary set map $X \to \mathcal{G}(V)$. This is exactly the universal property of a free object.

Example 2.12 (Hom–tensor adjunction) *In the category* $\mathbf{Vect}_{\mathbb{R}}$*, fix a vector space* W*. Define two functors:*

$$\mathcal{F}(V) = V \otimes W, \qquad \mathcal{G}(U) = \mathrm{Hom}_{\mathbb{R}}(W, U).$$

Then $\mathcal{F} \dashv \mathcal{G}$ *because there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{R}}(V \otimes W, U) \;\simeq\; \mathrm{Hom}_{\mathbb{R}}\big(V, \mathrm{Hom}_{\mathbb{R}}(W, U)\big),$$

which is the familiar "currying" operation for linear maps. This adjunction underlies many dualities in algebra and geometry.

Example 2.13 (Galois connection as adjunction) *Let* $(P, \leq)$ *and* $(Q, \leq)$ *be two partially ordered sets, regarded as categories. A monotone Galois connection* $(F : P \to Q, G : Q \to P)$ *is precisely an adjunction* $F \dashv G$ *in this categorical setting. Indeed, the condition*

$$F(p) \leq q \iff p \leq G(q)$$

is exactly the statement that the Hom-sets (which are either empty or singletons) are in bijection. Thus the theory of adjoint functors is a genuine generalisation of the order-theoretic Galois connection. We will come back to this later.

The power of the adjunction language becomes apparent when one notices that many fundamental constructions in mathematics—free objects, limits, exponentials, sheafification—are most cleanly described as left or right adjoints. In the following sections we will see how both Galois theory and the classification of covering spaces fit perfectly into this pattern, revealing a deep structural unity between algebra and topology.

Beyond covariant and contravariant functors and adjoint functors, several important variants appear naturally in mathematical practice.

Definition 2.7 *(Faithful, full, and fully faithful functors) A functor* $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ *is said to be faithful if for every pair of objects* $X, Y \in \mathscr{C}$*, the map* $\mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ *is injective; full if the same map is surjective; and fully faithful if it is both full and faithful.*

Example 2.14 *Consider the forgetful functor* $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ *which sends a group to its underlying set and a group homomorphism to the corresponding set function. This functor is faithful: if two group homomorphisms* $f, g : G \to H$ *induce the same function on the underlying sets, then* $f = g$ *as homomorphisms. However,* U *is not full because not every set function between the underlying sets of two groups is a group homomorphism. For instance, any permutation of* $\mathbb{Z}$ *as a set that does not preserve addition is a morphism in* **Set** *but not in* **Grp***.*

Example 2.15 *Let* **FinSet** *denote the category whose objects are all finite sets and whose morphisms are all functions between them. The inclusion functor* $\iota : \mathbf{FinSet} \longrightarrow \mathbf{Set}$ *sends each finite set to itself (as an object of* **Set***) and each function between finite sets to the same function (as a morphism in* **Set***). This functor is faithful because the inclusion of Hom-sets* $\mathrm{Hom}_{\mathbf{FinSet}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ *is injective—indeed, it is literally the identity map on the set of functions. It is full because for any two finite sets* X, Y*, every set map* $f : X \to Y$ *is automatically a morphism in* **FinSet***, so the inclusion map on Hom-sets is surjective as well. Hence* ι *is fully faithful: it induces a bijection* $\mathrm{Hom}_{\mathbf{FinSet}}(X, Y) \cong \mathrm{Hom}_{\mathbf{Set}}(X, Y)$ *for every pair of finite sets* X, Y*.*

Example 2.16 *Another example is the fundamental group functor* $\pi_1 : \mathbf{Top}_* \longrightarrow \mathbf{Grp}$*. This functor is not faithful: two different pointed maps may induce the same group homomorphism on fundamental groups. For instance, any two null-homotopic maps from* $(S^1, *)$ *to itself both induce the trivial homomorphism* $\mathbb{Z} \to \mathbb{Z}$*, yet they need not be equal as continuous maps. It is also not full: not every group homomorphism* *between fundamental groups is realized by a continuous map. The full and faithful properties are therefore independent and capture different aspects of how a functor relates the structures of two categories. For example, the forgetful functor* **Grp** $\longrightarrow$ **Set** *is faithful but not full, as not every set map between groups is a homomorphism. The inclusion functor from the category of finite sets into all sets is fully faithful.*

Another fundamental construction is the product of categories and the corresponding notion of a bifunctor.

Definition 2.8 *(Bifunctor) Given three categories $\mathscr{C}, \mathscr{D}, \mathscr{E}$, a bifunctor is a functor*

$$\mathcal{F} : \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{E},$$

where $\mathscr{C} \times \mathscr{D}$ is the product category whose objects are pairs (c, d) and morphisms are pairs (f, g) with componentwise composition.

Example 2.17 *The most important example is the Hom-bifunctor. For a locally small category $\mathscr{C}$, we have*

$$\mathrm{Hom}_{\mathscr{C}}(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathbf{Set},$$

which sends a pair (X, Y) to the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, and a pair of morphisms $(f : X' \to X, g : Y \to Y')$ to the map

$$\mathrm{Hom}_{\mathscr{C}}(f, g) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X', Y'), \quad h \longmapsto g \circ h \circ f.$$

This bifunctor plays a crucial role in the theory of adjunctions, as the adjunction isomorphism is precisely a natural isomorphism between two such Hom-bifunctors.

2.3 Natural transformations

We have defined the categories and functors as the mapping of two categories so far, the definition of functors is actually equivalent to say the diagram below commutes.

$$\begin{tikzcd}
\mathscr{C} \arrow[d, shift right=2] \arrow[d] \arrow[d, shift left=2, "\mathcal{F}\ \mathcal{G}\ \mathcal{H}"'] \\
\mathscr{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, "\mathcal{F}"'] \arrow[dd, "f"] \arrow[dr, "\mathcal{G}"] \arrow[drr, "\mathcal{H}"] & & \\
\mathcal{F}(X) \arrow[rr, dotted, no head] \arrow[dd, "\mathcal{F}(f)"] & & \mathcal{G}(X) \arrow[r, dotted, no head] \arrow[dd, "\mathcal{G}(f)"] & \mathcal{H}(X) \arrow[dd] \\
& Y \arrow[dl, "\mathcal{F}"] \arrow[dr, "\mathcal{G}"] \arrow[drr, "\mathcal{H}"] & & \\
\mathcal{F}(Y) \arrow[rr, dotted, no head] & & \mathcal{G}(Y) \arrow[r, dotted, no head] & \mathcal{H}(Y)
\end{tikzcd}$$

So it is so natural to define a new mapping to fill in the gaps in the dashed line at the base of the triangle. To be precise, we have to make the bottom rectangular commutes. Such mapping between two functors comes up in a natural way thus we call it natural transformations.

Definition 2.9 *Given two categories $\mathscr{C}$ and $\mathscr{D}$, and two functors $\mathcal{F}, \mathcal{G} : \mathscr{C} \longrightarrow \mathscr{D}$, the natural transformation is a map of functors $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$, which consists a morphism $\alpha_X : \mathcal{F}(X) \longrightarrow \mathcal{G}(X)$ for all $X \in ob(\mathscr{C})$ such that for all $f : X \longrightarrow Y$, the following diagram commutes*

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\alpha_X"] & \mathcal{F}(Y) \arrow[d, "\alpha_Y"] \\
\mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(Y)
\end{tikzcd}$$

Example 2.18 *Consider the category* ***Top*** *of topological spaces, and two functors* $\mathcal{F}, \mathcal{G} : \textbf{\textit{Top}} \longrightarrow \textbf{\textit{Top}}$*, where* $\mathcal{F}(X) = X$ *and* $\mathcal{G}(X) = X \times [0,1]$ *for any* $X \in ob(\textbf{\textit{Top}})$*, and for any morphism* $f : X \longrightarrow Y$*,* $\mathcal{F}(f) = f$ *and* $\mathcal{G}(f) = f \times id_{[0,1]}$*. Then there is a natural transformation* $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$*, where for each topological space* X*, the morphism* $\alpha_X : X \longrightarrow X \times [0,1]$ *is defined by* $\alpha_X(x) = (x, 0)$ *for all* $x \in X$*. This natural transformation satisfies the commutativity condition for all morphisms in* ***Top****.*

Example 2.19 *Consider the category* $\mathbf{Vect}_{\mathbb{R}}$ *of real vector spaces and linear maps. Define two functors* $\mathcal{F}, \mathcal{G} : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ *by:*

$$\mathcal{F}(V) = V \oplus V, \quad \mathcal{G}(V) = V \otimes \mathbb{R}^2,$$

where $\oplus$ *denotes direct sum and* $\otimes$ *denotes tensor product. On a linear map* $f : V \to W$*, we set* $\mathcal{F}(f) = f \oplus f$ *and* $\mathcal{G}(f) = f \otimes \mathrm{id}_{\mathbb{R}^2}$*.*

There is a natural transformation $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ *whose component at a vector space* V *is the canonical isomorphism*

$$\alpha_V : V \oplus V \longrightarrow V \otimes \mathbb{R}^2, \quad (v_1, v_2) \longmapsto v_1 \otimes e_1 + v_2 \otimes e_2,$$

where $\{e_1, e_2\}$ *is the standard basis of* $\mathbb{R}^2$*. For any linear map* $f : V \to W$*, the diagram*

$$\begin{tikzcd} V \oplus V \arrow[r, "f \oplus f"] \arrow[d, "\alpha_V"] & W \oplus W \arrow[d, "\alpha_W"] \\ V \otimes \mathbb{R}^2 \arrow[r, "f \otimes \mathrm{id}_{\mathbb{R}^2}"] & W \otimes \mathbb{R}^2 \end{tikzcd}$$

commutes because both paths send (v_1, v_2) *to* $f(v_1) \otimes e_1 + f(v_2) \otimes e_2$*. Hence* α *is indeed a natural transformation.*

Example 2.20 *Let* $\mathbf{Grp}$ *be the category of groups and homomorphisms. Consider the abelianization functor* $\mathrm{Ab} : \mathbf{Grp} \longrightarrow \mathbf{Ab}$*, where* $\mathbf{Ab}$ *is the category of abelian groups, defined by*

$$\mathrm{Ab}(G) = G/[G, G],$$

the quotient of G *by its commutator subgroup* $[G, G]$*, and* $\mathrm{Ab}(f)$ *is the induced homomorphism on quotients.*

There is a natural transformation $\eta : \mathrm{id}_{\mathbf{Grp}} \longrightarrow U \circ \mathrm{Ab}$*, where* $U : \mathbf{Ab} \longrightarrow \mathbf{Grp}$ *is the forgetful functor (treating an abelian group as a group). The component at a group* G *is the canonical projection*

$$\eta_G : G \longrightarrow G/[G, G], \quad g \longmapsto g[G, G].$$

For any group homomorphism $f : G \to H$*, the diagram*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\eta_G"] & H \arrow[d, "\eta_H"] \\ G/[G, G] \arrow[r, "\mathrm{Ab}(f)"] & H/[H, H] \end{tikzcd}$$

commutes because $\mathrm{Ab}(f)(g[G, G]) = f(g)[H, H] = \eta_H(f(g))$*. This natural transformation encodes the universal property of abelianization: every homomorphism from* G *to an abelian group factors uniquely through* η_G*.*

A natural transformation $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is called a *natural isomorphism* if each component $\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is an isomorphism in $\mathscr{D}$. In this case we write $\mathcal{F} \cong \mathcal{G}$ and say the two functors are naturally isomorphic.

Example 2.21 *In the category* $\mathbf{FinVect}_k$ *of finite-dimensional vector spaces over a field* k*, the double dual functor* $(-)^{**}$ *is naturally isomorphic to the identity functor. The natural isomorphism* $\eta : \mathrm{id} \longrightarrow (-)^{**}$ *has components*

$$\eta_V : V \longrightarrow V^{**}, \quad \eta_V(v)(\varphi) = \varphi(v),$$

which are isomorphisms precisely when V *is finite-dimensional.*

Given two categories $\mathscr{C}$ and $\mathscr{D}$, we can form the *functor category* $\mathscr{D}^{\mathscr{C}}$, whose objects are functors from $\mathscr{C}$ to $\mathscr{D}$ and whose morphisms are natural transformations between them. Composition in this category is given by vertical composition of natural transformations.

Definition 2.10 *(Vertical composition) Given three functors $\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathscr{C} \longrightarrow \mathscr{D}$ and natural transformations $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ and $\beta : \mathcal{G} \longrightarrow \mathcal{H}$, the vertical composition $\beta \circ \alpha : \mathcal{F} \longrightarrow \mathcal{H}$ is defined componentwise by $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ for each $X \in \mathscr{C}$.*

There is also a horizontal composition of natural transformations. Suppose we have functors $\mathcal{F}, \mathcal{G} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{H}, \mathcal{K} : \mathscr{D} \longrightarrow \mathscr{E}$, together with natural transformations $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ and $\beta : \mathcal{H} \longrightarrow \mathcal{K}$. Then the horizontal composite $\beta * \alpha : \mathcal{H} \circ \mathcal{F} \longrightarrow \mathcal{K} \circ \mathcal{G}$ is defined by $(\beta * \alpha)_X = \beta_{\mathcal{G}(X)} \circ \mathcal{H}(\alpha_X)$.

Proposition 2.1 *(Naturality as a family of commutative diagrams) A collection of morphisms $\{\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)\}_{X \in \mathscr{C}}$ constitutes a natural transformation $\alpha : \mathcal{F} \to \mathcal{G}$ if and only if for every morphism $f : X \to Y$ in $\mathscr{C}$, the diagram*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \Big\downarrow{\scriptstyle \alpha_X} & & \Big\downarrow{\scriptstyle \alpha_Y} \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y) \end{array}$$

commutes. In particular, a natural transformation is completely determined by its components on objects.

An important application of natural transformations is the alternative description of adjunctions. An adjunction $\mathcal{F} \dashv \mathcal{G}$ can be equivalently described by two natural transformations: the *unit* $\eta : \mathrm{id}_{\mathscr{C}} \longrightarrow \mathcal{G}\mathcal{F}$ and the *counit* $\varepsilon : \mathcal{F}\mathcal{G} \longrightarrow \mathrm{id}_{\mathscr{D}}$, satisfying the triangle identities $(\varepsilon\mathcal{F}) \circ (\mathcal{F}\eta) = \mathrm{id}_{\mathcal{F}}$ and $(\mathcal{G}\varepsilon) \circ (\eta\mathcal{G}) = \mathrm{id}_{\mathcal{G}}$. This formulation highlights that adjunctions are not merely bijections of Hom-sets, but coherent families of morphisms connecting the functors.

Example 2.22 *For the free-forgetful adjunction $\mathcal{F} \dashv \mathcal{G}$ between* **Set** *and* $\mathbf{Vect}_{\mathbb{R}}$*, the unit $\eta_X : X \longrightarrow \mathcal{G}\mathcal{F}(X)$ sends an element $x \in X$ to the basis vector e_x in the free vector space $\mathbb{R}^{(X)}$, while the counit $\varepsilon_V : \mathcal{F}\mathcal{G}(V) \longrightarrow V$ is the linear extension map that evaluates a formal linear combination of vectors in V.*

2.4 Limits and colimits

Now we introduce two of the most important and powerful concepts in category theory. Limits and colimits give us a way to discover the mathematical structure universally and uniquely. Both of them are defined after a special functor.

Definition 2.11 *(Category-shaped diagram) Let $\mathscr{C}$ and $\mathscr{D}$ be category and small category. The functor $\mathcal{F} : \mathscr{D} \longrightarrow \mathscr{C}$ is a $\mathscr{D}$-shaped diagram in $\mathscr{C}$.*

We have the category $\mathscr{D}[\mathscr{C}]$ called $\mathscr{D}$-shaped diagram category in $\mathscr{C}$ where $\mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ are the natural transformations. And our limits will be defined in the image of one specific $\mathscr{D}$-shaped diagram in the category $\mathscr{C}$. We already know for any two $D, D' \in \mathscr{D}$, the image of $\mathcal{F}$ make the triangle commutes. Moreover, for fixed $A \in \mathscr{C}$, it should be commutative with every image of elements in $\mathscr{D}$ under the functor.

Definition 2.12 *(Cone) A cone on the $\mathscr{D}$-shaped diagram functor $\mathcal{F}$ consists a vertex $A \in \mathrm{Ob}(\mathscr{C})$ and the family of maps in $\mathrm{Hom}_{\mathscr{C}}$, that $f_{D_i} : A \longrightarrow \mathcal{F}(D_i)$ for every $D_i \in \mathrm{Ob}(\mathscr{D})$, such that $\forall g_i : D_i \longrightarrow D_{i+1}$ in $\mathscr{D}$, the cone diagram commutes:*

$$\begin{tikzcd}
 & & & & A \arrow[dllll, "f_1"'] \arrow[dlll, "f_2"'] \arrow[dll, "f_3"'] \arrow[d, "f_4"] \arrow[drr, "f_5"] \arrow[drrrr, "f_{k-1}"] \arrow[drrrrrr, "f_k"] & & & & & & \\
\mathcal{F}(D_1) \arrow[r, "\mathcal{F}(g_1)"'] & \mathcal{F}(D_2) \arrow[r, "\mathcal{F}(g_2)"'] & \mathcal{F}(D_3) \arrow[r, "\mathcal{F}(g_3)"'] & \mathcal{F}(D_4) \arrow[r, "\mathcal{F}(g_4)"'] & (D_5) \arrow[rr, dotted] & & \mathcal{F}(D_{k-1}) \arrow[rr, "\mathcal{F}(g_{k-1})"'] & & \mathcal{F}(D_k)
\end{tikzcd}$$

The cone is unnessarily to be defined on the family of all objects D_i in $\mathscr{D}$, we denote this cone as $(f_i : A \longrightarrow \mathcal{F}(D_i))_{i\in I}$ if we pick out a specific countable set I in $\mathrm{Ob}(\mathscr{C})$.

Such cone-shaped structure is easy to find in any category and the number of such cones is large. So what we want to find is a so-called best cone which can represented as the top cone of every cone and give us an universal structure of the category.

Definition 2.13 *(Limits and colimits) The limits of $\mathcal{F}$ is a vertex of the cone $(\varphi_i\colon \lim\mathcal{F} \longrightarrow \mathcal{F}(D_i))_{i\in I}$ satisfying that for any cone of $\mathcal{F}$: $(f_i : A \longrightarrow \mathcal{F}(D_i))_{i\in I}$, there is a unique map in mor$(\mathscr{C})$ $\tilde{f} : A \longrightarrow \lim\mathcal{F}$ such that $\varphi_i \circ \tilde{f} = f_i$, $\forall i \in I$. The colimits is the dual of limits which is defined by reversing arrows. These are equivalent to say such diagram commutes for the simple case of any two $D_1, D_2 \in \mathrm{ob}(\mathscr{D})$:*

$$\begin{tikzcd}
 & \lim\mathcal{F} \arrow[ddl, "\varphi_1"'] \arrow[ddr, "\varphi_2"] & \\
 & A \arrow[u, "\tilde{f}"'] \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\mathcal{F}(D_1) \arrow[rr, "\mathcal{F}(d_1)"'] & & \mathcal{F}(D_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{colim}\mathcal{F} \arrow[d, "\tilde{g}"] & \\
 & A & \\
\mathcal{F}(D_1) \arrow[uur, "\phi_1"] \arrow[ur, "g_1"'] \arrow[rr, "\mathcal{F}(d_1)"'] & & \mathcal{F}(D_2) \arrow[uul, "\phi_2"'] \arrow[ul, "g_2"]
\end{tikzcd}$$

In categorical way, for one $\mathscr{D}$-shaped diagram functor $\mathcal{F}$, the image category $\mathscr{C}$ can considered as the category with objects are the vertex of cones defined on the functor $\mathcal{F}$, and $\lim\mathcal{F}$ is the terminal object while $\mathrm{colim}\mathcal{F}$ is the initial in such category.

The abstract definition of limits becomes concrete in familiar mathematical settings. Several special cases of limits and colimits appear ubiquitously across mathematics.

Example 2.23 *(Products and coproducts) Let $\mathscr{D}$ be a discrete category (no non-identity morphisms). A $\mathscr{D}$-shaped diagram in $\mathscr{C}$ is simply a family of objects $\{C_i\}_{i\in I}$. A limit of such a diagram is called a* product*, denoted $\prod_{i\in I} C_i$, with projection maps $\pi_j : \prod_i C_i \to C_j$. In* **Set***, the product is the Cartesian product; in* **Top***, it is the product topology; in* **Grp***, it is the direct product of groups.*

Dually, a colimit of a discrete diagram is a coproduct*, denoted $\coprod_{i\in I} C_i$, with inclusion maps $\iota_j : C_j \to \coprod_i C_i$. In* **Set***, the coproduct is the disjoint union; in* **Top***, it is the disjoint union topology; in* **Grp***, it is the free product.*

Example 2.24 *(Equalizers and coequalizers) Let $\mathscr{D}$ be the category $\bullet \rightrightarrows \bullet$ with two parallel arrows. A diagram of shape $\mathscr{D}$ in $\mathscr{C}$ is a pair of morphisms $f, g : A \to B$. The limit of this diagram, called an* equalizer*, is an object E with a morphism $e : E \to A$ such that $f \circ e = g \circ e$, universal among all such pairs. In* **Set***, the equalizer of $f, g : X \to Y$ is $\{x \in X \mid f(x) = g(x)\}$ with the inclusion map.*

The colimit of the same diagram is a coequalizer*, given by a morphism $q : B \to Q$ with $q \circ f = q \circ g$, universal among such maps. In* **Set***, the coequalizer of $f, g : X \to Y$ is the quotient of Y by the equivalence relation generated by $f(x) \sim g(x)$.*

Limits and colimits are not guaranteed to exist in an arbitrary category. A category is called *complete* if it has all (small) limits, and *cocomplete* if it has all (small) colimits. Many familiar categories such as **Set**, **Top**, **Grp**, and **Vect**$_k$ are both complete and cocomplete.

Proposition 2.2 *(Preservation of limits) A functor $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ is said to* preserve limits *if whenever $\{p_i : L \to C_i\}$ is a limit cone for a diagram in $\mathscr{C}$, then $\{\mathcal{F}(p_i) : \mathcal{F}(L) \to \mathcal{F}(C_i)\}$ is a limit cone for the image diagram in $\mathscr{D}$. Right adjoint functors preserve limits, and left adjoint functors preserve colimits.*

Example 2.25 *The forgetful functor U : **Grp** $\longrightarrow$ **Set** is a right adjoint (to the free group functor), hence it preserves limits. Indeed, the underlying set of a product of groups is the Cartesian product of their underlying sets. Similarly, the free group functor $\mathcal{F}$: **Set** $\longrightarrow$ **Grp** is a left adjoint, so it preserves colimits: the free group on a disjoint union is the free product of the free groups on the components.*

Limits and colimits provide a universal language for describing constructions that are "glued together" from simpler pieces (colimits) or that "simultaneously satisfy a family of conditions" (limits). In the next section, we will see how this universality is precisely captured by the concept of an adjoint functor, linking the abstract notion of limits to the concrete Galois connections we aim to study.

3 Galois connection

Now we will introduce a fantastic concept which is likely a univseral property in many correspondence relation, which is the Galois connection. The categorical equivalence of Galois theory and covering space theory can be hold is actually that they are the special case of Galois connection in algebra and topology.

A Galois connection is a particular correspondence between two partially ordered sets, or posets which as defined below.

Definition 3.1 *(Posets) A partial ordered set or poset is a pair $(P, \leq)$ consists a set P and a binary operation leq called partial relation and satisfying following axioms:*
(i) Reflexivity: $\forall a \in P, a \leq a$. (ii) Antisymmetry: $\forall a, b \in P$, $a \leq b$ and $\leq a$ implies $a = b$. (iii) Transitivity: $\forall a, b, c \in P$, $a \leq b$ and $b \leq c$ implies $a \leq c$.

We say two element a and b in the poset P are comparable if $a \leq b$ or $b \leq a$ otherwise they are incomparable. If all elements are comparable then the parital ordered set is a totally ordered set.

Example 3.1 *Easy to verify $(\mathbb{R}, \leq)$ is a totally ordered set.*

Example 3.2 *Take $P = \mathcal{P}(\{1, 2, 3\})$ as example, $(P, \subseteq)$ is a poset not totally ordered because $\{1\}$ and $\{2\}$ are incomparable.*

Example 3.3 *$(\mathbb{N}^+, |)$ is a parital ordered set but it is not totally ordered since $2 \nmid 3$.*

Definition 3.2 *(preordered sets) A preordered set $(P, \leq)$ is a poset whithout the antisymmetry.*

Example 3.4 *$(N, |)$ is a preordered, consider 2 and -2.*

Definition 3.3 *(Monotone Galois connection) A monotone Galois connection consists two monotone functions between two posets $(A, \leq)$ and $(B, \leq)$ where $F : A \longrightarrow B$ and $G : B \longrightarrow A$ such that*

$$\forall a \in A, b \in B, we\ have\ F(a) \leq b \Longleftrightarrow a \leq G(b)$$

We call F the lower adjoint of G and G the upper adjoint of F. In posets we say there is a morphism $a \longrightarrow b$ if and only if $a \leq b$, so in diagram we have

$$\begin{tikzcd}
\bullet \arrow[r, dashed] & \bullet \arrow[r, dashed] & a \arrow[r, "\leq"] \arrow[d, "F"] & G(b) \arrow[r, dashed] & \bullet \arrow[r, dashed] & \bullet \\
\bullet \arrow[r, dashed] & \bullet \arrow[r, dashed] & F(a) \arrow[r, "\leq"] & b \arrow[u, "G"] \arrow[r, dashed] & \bullet \arrow[r, dashed] & \bullet
\end{tikzcd}$$

You should easily have the intuition that F and G are not inverse of each other otherwise the Galois connection will be trivial. Or in other words an upper or lower adjoint of a Galois connection uniquely determines the other. Thus we have another definition of Galois connection, or, a proposition followed immediately by the original definition.

Proposition 3.1 *Let $(F : A \longrightarrow B, G : B \longrightarrow A)$ be the Galois connection between posets $(A, \leq_A)$ and $(B, \leq_B)$. Then $\forall a \in A, b \in B$, we have*

$$F(a) = \min\{b \in B | a \leq_A G(b)\},$$
$$G(b) = \max\{a \in A | F(a) \leq_B b\}.$$

Proof We only need to prove one side of the Galois connection and the other side comes dually. To show $F(a) = \min\{b \in B | a \leq_A G(b)\}$, is to show $F(a)$ is the minimal element of the set $S_a = \{b \in B | a \leq_A G(b)\}$. By the definition of Galois connection we take $b = F(a)$ then

$$F(a) \leq_B F(a) \Longleftrightarrow a \leq_A G(F(a))$$

where left side always holds so does the right hand side, so $a \leq_A G(F(a))$, which is $F(a) \in S_a$. Now we want to show $F(a)$ is greatest lower bound of this set
Take any $b' \in S_a$, by the definition of S_a we have $a \leq_A G(b')$ implies $F(a) \leq_B b$, thus $F(a)$ is the lower bound. Moreover, let $b_0 \in S_a$ be any lower bound that $\forall b \in S_a, b_0 \leq_B b$, we take $b = F(a) \in S_a$ then $b_0 \leq_B F(a)$, with $F(a) \leq_B b_0$ we have $b_0 = F(a)$ by antisymmetry. And the other side is dual. $\square$

A consequence of this is that if F or G is bijective then $F = G^{-1}$ and the Galois connection is just an isomorphism of posets. So the Galois connection is a weaker version of isomorphism. And you may notice that actually the composition of two functions are the automorphism of the posets.

Definition 3.4 *(Closure operator and kernel operator) A closure operator is the composition of the Galois connection by first applying the lower adjoint as $FG : B \longrightarrow B$ and the dual is called the kernel operator $GF : A \longrightarrow A$ which first applying the upper adjoint.*

Proposition 3.2 *The closure operator GF and kernel operator FG are both monotone, idempotent and extensive (or contractive).*

Proof We say a operator C on a poset is monotone if $\forall a, b \in P, a \leq b$ implies $C(a) \leq C(b)$, and the two are monotone natrually by they are composition of monotone functions. We say a operator C is idempotent if $C^2 = Id$. It is easy to verify $GF(GF(a)) = a, \forall a \in A$ and so does the other one. We say GF is extensive if $\forall a \in A, a \leq GF(a)$ and FG is contractive if $\forall b \in B, FG(b) \leq b$, which will be proved in the next proposition. $\square$

Proposition 3.3 *$\forall a \in A, b \in B$, we have $a \leq_A GF(a)$ and $FG(b) \leq_B b$.*

Proof Take $b = F(a)$ in the Galois connection condition $F(a) \leq_B b \Longleftrightarrow a \leq_A G(b)$. Since $F(a) \leq_B F(a)$ holds trivially, we obtain $a \leq_A G(F(a))$. The other side is dual. $\square$

From the extensive and contractive property we can see we may lose information after applying the Galois connection, which is another reason why Galois connection is weaker than isomorphism. Thus we define a special Galois connection called Galois insertion or Galois embedding.

Definition 3.5 *(Galois embedding) A Galois connection* (F, G) *is a Galois embedding if the kernel operator is the identity, in mathematical*

$$FG(b) = b, \forall b \in B$$

Proposition 3.4 *The following conditions are equivalent for a Galois connection* (F, G) *between posets* $(A, \leq_A)$ *and* $(B, \leq_B)$*: (i)* (F, G) *is a Galois embedding. (ii)* G *is injective. (iii)* F *is surjective.*

Proof $(i) \Rightarrow (ii)$: Supp $G(b_1) = G(b_2)$, then $FG(b_1) = FG(b_2)$ implies $b_1 = b_2$ by Galois embedding, thus G is an injection.
$(ii) \Rightarrow (iii)$: For any $b \in B$ we have $FG(b) \leq_B b$ which $G(FG(b)) \leq_A G(b)$ by monotonicity of G. And we know $G(b) \leq GF(G(b))$ because GF is extensive, so $GF(G(B)) = G(b)$ by antisymmetry. By the injectivity of G we have $F(G(b)) = b$, thus $\forall b \in B$ take $a = G(b) \in A$ we have $F(a) = b$, F is surjective.
$(iii) \Rightarrow (i)$: By surjectivity for any $b \in B$ we pick $a \in A$ such that $F(a) = b$. Then $F(a) = b \leq_B b$ always holds, implies $a \leq_A G(b)$. Thus $b = F(a) \leq_B F(G(b))$ by monotonicity of F, and we know $FG(b) \leq_B b$ because FG is contractive, so $FG(b) = b$ by antisymmetry. □

The Galois connection can also be defined on preordered sets or classes. And the real power of Galois connection will release once you realize any preordered sets $(P, \leq)$ can be considered as a category, where we say a and b are comparable if there is a morphism in $\mathrm{Hom}_P(a, b)$, and id_a is the morphism satisfies the reflexivity.
We should generalize our Galois connection to have better view in different mathematical fields, to formalize it, we have

Definition 3.6 *(Categorical Galois connection) For categories* $\mathscr{C}$ *and* $\mathscr{D}$ *with the functors* $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ *and* $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}$*. We say* $\mathcal{F}$ *and* $\mathcal{G}$ *satisfy th Galois connection if there exists a natural transformation*

$$\Phi : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(-), -) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(-, \mathcal{G}(-))$$

which $\forall c \in \mathscr{C}, d \in \mathscr{D}$*, gives a bijection*

$$\Phi_{c,d} : \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(c, \mathcal{G}(d))$$

makes the below diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c), d) \arrow[r, "\Phi_{c,d}"] \arrow[d, "\mathcal{F}(f)^* \circ g_*"] & \mathrm{Hom}_{\mathscr{C}}(c, \mathcal{F}(d)) \arrow[d, "f^* \circ \mathcal{G}(g)_*"] \\
\mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(c'), d') \arrow[r, "\Phi_{c',d'}"] & \mathrm{Hom}_{\mathscr{C}}(c', \mathcal{F}(d'))
\end{tikzcd}$$

One should very trivial to verify this is actually the definition of adjoint functors where $\mathcal{F} \dashv \mathcal{G}$, that is to say, the Galois connection is actually the adjoint functors in preordered set category.

Example 3.5 *Consider the categories* $\mathscr{C} = \boldsymbol{Set}$ *and* $\mathscr{D} = \boldsymbol{Set}$ *itself. Fix a set* A *and define functors:*

$$\mathcal{F} : \boldsymbol{Set} \longrightarrow \boldsymbol{Set}, \quad \mathcal{F}(X) = X \times A,$$

$$\mathcal{G} : \boldsymbol{Set} \longrightarrow \boldsymbol{Set}, \quad \mathcal{G}(Y) = Y^A,$$

where Y^A *denotes the set of all functions from* A *to* Y*.*

For any sets X *and* Y*, there is a natural bijection*

$$\Phi_{X,Y} : \mathrm{Hom}_{\boldsymbol{Set}}(X \times A, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\boldsymbol{Set}}(X, Y^A),$$

*given by sending a map $f : X \times A \to Y$ to its curried form $\tilde{f} : X \to Y^A$, where $\tilde{f}(x)(a) = f(x, a)$. This is precisely the definition of the Cartesian closed structure of **Set**, and it makes the required naturality diagram commute. Hence $(\mathcal{F}, \mathcal{G})$ form a categorical Galois connection (an adjunction $\mathcal{F} \dashv \mathcal{G}$).*

Example 3.6 *As we remarked before, if we lift the definition of Galois connection to categorical language, then any adjoint functors $\mathcal{F} \dashv \mathcal{G}$ form a Galois connection. Let $\mathscr{C} = \textbf{Set}$ and $\mathscr{D} = Vect_{\mathbb{R}}$ be the category of vector space over the filed $\mathbb{R}$.*
A classesical example of adjoint functors is the free-forgetful adjunction which free functor is given by $\mathcal{F} : \textbf{Set} \longrightarrow Vect_{\mathbb{R}}$:

$$\mathcal{F}(X) = \mathbb{R}^{(X)} = \oplus_{x \in X}\mathbb{R}$$
$$\mathcal{F}(f) \sum_{x \in X} a_x e_x = \sum_{x \in X} a_x e_{f(x)}$$

Where $\mathbb{R}^{(X)} := \{f : X \longrightarrow \mathbb{R} \mid finitely\ many\ x \in X\ s.t.\ f(x) \neq 0\}$. The free functor will add sufficient structure to make the set X a vector space over $\mathbb{R}$ and $f \in \mathrm{Hom}_{\textbf{Set}}(X, Y)$. The forgetful functor is given by $\mathcal{G} : Vect_{\mathbb{R}} \longrightarrow \textbf{Set}$:

$$\mathcal{G}(V) = U(V)$$
$$\mathcal{G}(\varphi) = \varphi$$

The forgetful functor will simply forget the vector space structure and consider V as a set and any linear transformation $\varphi : V \longrightarrow W$ get forgeted to a set function. It is easy to verify these two functors form an adjoint pair $\mathcal{F} \dashv \mathcal{G}$.
For andy set X and any vector space V, there is a natrual

$$\Phi_{X,V} : \mathrm{Hom}_{Vect_{\mathbb{R}}}(\mathbb{R}^{(X)}, V) \xrightarrow{\simeq} \mathrm{Hom}_{\textbf{Set}}(X, U(V))$$

Given a linear map $L : \mathbb{R}^{(X)} \longrightarrow V$, we can define $\Phi_{X,V}(L) : X \longrightarrow U(V)$ by $\Phi_{X,V}(L)(x) = L(e_x)$, where e_x is the basis vector corresponding to $x \in X$. Conversely, given a set map $\psi : X \longrightarrow U(V)$, we difine $\Phi^{-1}_{X,V}(\psi) : \mathbb{R}^{(X)} \longrightarrow V$ by the linear extension

$$\Phi^{-1}_{X,V}(\psi)\left(\sum_{x \in X} a_x e_x\right) = \sum_{x \in X} a_x \psi(x)$$

It is clear the map $\Phi_{X,V}$ is a bijection and natural in both X and V, thus $(\mathcal{F}, \mathcal{G})$ is a categorical Galois connection.

Galois connection can be considered as an universal property in many mathematical areas, any adjoint functors can be verified to form a Galois connection, or precisely they are the same thing.

Now we introduce a pretty example of Galois connection in the area intersect category and logic, which is a adjoint bridge between existential quantification and the categorical pullback.

Example 3.7 *Let $\mathscr{C} = \textbf{Set}$ and consider the slice category $\mathscr{D} = \textbf{Set}/A$ for any fixed set A, which its object are pairs (X, f_X) where $X \in \mathrm{Ob}(\textbf{Set})$ and $f_X \in \mathrm{Hom}_{\textbf{Set}}(X, A)$. So any morphism $\phi : (X, f_X) \longrightarrow (Y, f_Y)$ is a function $\phi \in \mathrm{Hom}_{\textbf{Set}}(X, Y)$ such that $f_Y \circ \phi = f_X$.*
Now for fixed element $A, B \in \textbf{Set}$ we have a funcion $p : A \longrightarrow B$ in $\mathrm{Hom}_{\textbf{Set}}(A, B)$. We first define the pullback functor along p as

$$p^* : \textbf{Set}/B \longrightarrow \textbf{Set}/A, p^*(Y, f_Y) = (Y \times_B A, \pi_A)$$

where $Y \times_B A = \{(y, a) \in Y \times A \mid g(y) = p(a)\}$ is the pullback, and π_A is the projection to A. For another functor we have the Post-compostion with p as

$$\exists_p : \textbf{Set}/A \longrightarrow \textbf{Set}/B, \exists_p(X, f_X) = (X, p \circ f_X)$$

To verfy the Galois connection, for $(X, f) \in \textbf{Set}/A$ and $(Y, g) \in \textbf{Set}/B$, we need a natrual bijection

$$\mathrm{Hom}_{\textbf{Set}/B}(\exists_p(X, f_X), (Y, g_Y)) \xrightarrow{\simeq} \mathrm{Hom}_{\textbf{Set}/A}((X, f_X), p^*(Y, g_Y))$$

A morphism on the left is a function $\alpha : X \longrightarrow Y$ such that $g \circ \alpha = p \circ f$, where a morphism on the right is a function $\beta : X \longrightarrow Y \times_B A$ such that $\pi_A \circ \beta = f$ and $\pi_Y \circ \beta$ is a map $X \longrightarrow Y$ satisfying the same compatibility condition $g \circ \pi_Y \circ \beta = p \circ f$. Thus we can define the bijection as

$$\alpha \longmapsto (x \mapsto (\alpha(x), f(x)))$$
$$\beta \longmapsto \pi_Y \circ \beta$$

The condition $g(\alpha(x)) = p(f(x))$ guarantees that $(\alpha(x), f(x))$ indeed lies in the pullback $Y \times_B A$. And nutruality follows directly from the definition of the functors on morphisms. Thus we have the categorical Galois connection $(\exists_p, p^*)$.

To reach our primry goal, we should introduce some important properties of adjoint functors or Galois connection. The first one is the fixed-point equivalence which gives us an equivalence between two subcategories defined by the unit and counit of the adjoint functors.

Proposition 3.5 *(Fixed-point equivalence) Let $\mathcal{F} : \mathscr{C} \longrightarrow \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \longrightarrow \mathscr{C}$ be adjoint functors $\mathcal{F} \dashv \mathcal{G}$, with unit $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathcal{G}\mathcal{F}$ and counit $\varepsilon : \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}_{\mathscr{D}}$. Define the* fixed-point subcategories

$$\mathscr{C}_{\mathrm{fix}} := \{c \in \mathscr{C} \mid \eta_c \textit{ is an isomorphism}\}, \quad \mathscr{D}_{\mathrm{fix}} := \{d \in \mathscr{D} \mid \varepsilon_d \textit{ is an isomorphism}\}.$$

Then the adjoint pair $(\mathcal{F}, \mathcal{G})$ restricts to an equivalence of categories

$$\mathcal{F}|_{\mathscr{C}_{\mathrm{fix}}} : \mathscr{C}_{\mathrm{fix}} \xrightarrow{\simeq} \mathscr{D}_{\mathrm{fix}}, \qquad \mathcal{G}|_{\mathscr{D}_{\mathrm{fix}}} : \mathscr{D}_{\mathrm{fix}} \xrightarrow{\simeq} \mathscr{C}_{\mathrm{fix}}.$$

4 Galois theory

Galois theroy is one of the most powerful algebric theory. Consider the quadratic equation $ax^2 + bx + c = 0$, we say $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the formula of solution, namely the radical. For any higher n degree polynomial equation $\sum_{i=0}^{n} a_i x^i = 0$, the radical to this equation is a family of functions $x_i = f_i(a_0, a_1, ...a_n)$ which satisfies the equations, where any functions only required the operations of $+, -, \times, \div, \sqrt{\cdot}$. The biggest contribution of Galois Theory is the proof of the non-existence of the radical for any equations more than degree 5. In this chapter we will introduce some basic techique of Galois theory and see how it is connected to covering space.

The genius idea of Galois is to connect every equations to its symmetric group. More precisely, every equations's spliting field has a Galois groups and the existence of radical is related to the Galois group is solvable or not.

Definition 4.1 *(Ring and Field) A ring R is a set with binary operation $+$ and $\times$ where $(R, +)$ is a commutative group and $\times$ is associative and distributive over addition. A field F is a ring such that $(F \backslash \{0\}, \times)$ is a commutative group.*

The definition is not new to us, fields play very provital role in Galois theory, where Galois theory build a bridge between field extension and group theory.

Definition 4.2 *(Field Extension) Let F be any field, a field extension of F is a field L with a injective inclusion homomorphism $i : F \hookrightarrow L$. We denote the field extension as L/F. We can consider the extension field L as a vector space over F, and the dimension of L over F is called the degree of the extension, denoted as $[L : F]$.*

Example 4.1 *The complex field $\mathbb{C}$ is a field extension of real field $\mathbb{R}$ with the inclusion $i : \mathbb{R} \hookrightarrow \mathbb{C}$, and the degree of this extension is 2 because $\{1, i\}$ form a basis of $\mathbb{C}$ over $\mathbb{R}$.*

Example 4.2 *The field $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ is a field extension of rational field $\mathbb{Q}$ with inclusion $i : \mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$, and the degree of this extension is also 2 because $\{1, \sqrt{2}\}$ form a basis of $\mathbb{Q}(\sqrt{2})$.*

Definition 4.3 *(spliting field) Let F be any field and $f(x) \in F[x]$ be any polynomial over F, a splitting field of $f(x)$ over F is a field extension L/F such that*

$$f(x) = a(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$$

in $L[x]$ where $\alpha_i \in L$ are all the roots of $f(x)$ in L, and $L = F(\alpha_1, \alpha_2, ..., \alpha_n)$ is generated by adjoining all the roots to F. Morevore, such L is the minimal field with this property, meaning if there is another field extension K/F such that $f(x)$ splits in $K[x]$, then $L \subseteq K$.

Consider the polynomial $f(x) = x^2 - 2$ over $\mathbb{Q}$, the roots of $f(x)$ are $\pm\sqrt{2} \notin \mathbb{Q}$ which is not what we want. Our primry goal is to find a minimal field that contains $f(x)$'s parameters also with roots, so that we can study $f(x)$ by studying the structure of this minimal field.

Example 4.3 *Consider the polynomial $f(x) = x^2 + 1$ over $\mathbb{R}$, then the splitting field of $f(x)$ is $\mathbb{R}(i)$ because $f(x) = (x - i)(x + i)$ in $\mathbb{R}(i)[x]$ and $\mathbb{R}(i, -i) = \mathbb{R}(i) = \{a + bi|\ a, b \in \mathbb{R}\}$ is generated by adjoining the roots to $\mathbb{R}$, where i and $-i$ contribute the same extension.*

Example 4.4 *Consider the polynomial $f(x) = x^3 - 2$ over $\mathbb{Q}$, then the splitting field of $f(x)$ is $\mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{2\pi i/3}$ because $f(x) = (x - \sqrt[3]{2})(x - \omega\sqrt[3]{2})(x - \omega^2\sqrt[3]{2})$ in $\mathbb{Q}(\sqrt[3]{2}, \omega)[x]$ and $\mathbb{Q}(\sqrt[3]{2}, \omega) = \mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2})$.*

If you have good mathematical intuition, you may find the permutation of the roots play very important role in the structure of the splitting field, and this is exactly what Galois theory is trying to do, build a bridge between field extension and group theory by studying the permutation of the roots.

Before we introduce Galois group and Galois extension, we first need to introduce normal extension and separable extension.

Definition 4.4 *(Normal Extension) Let L/F be any field extension, we say L/F is a normal extension if L is the splitting field of some polynomial over F.*

Example 4.5 *Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has splitting field $\mathbb{R}(i)$, then the extension $\mathbb{R}(i)/\mathbb{R}$ is a normal extension.*

Example 4.6 *Consider the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ has splitting field $\mathbb{Q}(\sqrt[3]{2}, \omega)$, then the extension $\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}$ is a normal extension.*

Example 4.7 *The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is* ***not normal****. Although $\sqrt[3]{2}$ is a root of $f(x) = x^3 - 2 \in \mathbb{Q}[x]$, the polynomial does not split completely in $\mathbb{Q}(\sqrt[3]{2})$ since the other two roots $\omega\sqrt[3]{2}$ and $\omega^2\sqrt[3]{2}$ are not contained in this field ($\mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{R}$ but $\omega \notin \mathbb{R}$). Thus, $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field of any polynomial over $\mathbb{Q}$.*

Definition 4.5 *(Separable Extension) Let L/F be any field extension, we say L/F is a separable extension if every element $\alpha \in L$ is the root of some separable polynomial over F, that is a polynomial whose all roots are distinct in its splitting field.*

Example 4.8 *Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has splitting field $\mathbb{R}(i)$, then the extension $\mathbb{R}(i)/\mathbb{R}$ is a separable extension because $f(x)$ has two distinct roots $\pm i$ in $\mathbb{R}(i)$.*

Example 4.9 *Let k be a field of characteristic $p > 0$, and let $F = k(t)$ be the field of rational functions over k. Consider the polynomial $f(x) = x^p - t \in F[x]$. This polynomial is irreducible over F but is **not separable**. In any extension field containing a root α of $f(x)$, we have $\alpha^p = t$, and thus*

$$f(x) = x^p - t = x^p - \alpha^p = (x - \alpha)^p$$

*so α is the only root with multiplicity p. The extension $F(\alpha)/F$ is a **non-separable** (purely inseparable) extension.*

Now we can introduce Galois group and Galois extension.

Definition 4.6 *(Galois Group) Let L/F be any field extension, the Galois group of L/F is the group of all field automorphisms of L that fix every element of F, denoted as $\mathrm{Gal}(L/F)$, that is*

$$\mathrm{Gal}(L/F) := \{\sigma \in \mathrm{Aut}(L) | \sigma(a) = a, \forall a \in F\}$$

Example 4.10 *Consider $f(x) = x^2+1 \in R[x]$ has roots in the extension field $\mathbb{R}(i)$, then to find the Galois group $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) = \{\sigma \in \mathrm{Aut}(\mathbb{R}(i)) | \sigma(a) = a, \forall a \in \mathbb{R}\}$, it is easy to see that $\sigma(i)^2 = \sigma(i^2) = \sigma(-1) = -1$ because the filed automorphism is first a ring homomorphism, thus $\sigma(i) = \pm i$. So*

$$\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) = \{id, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$$

where $\mathrm{id}(a + bi) = a + bi$ and $\sigma(a + bi) = a - bi$ for all $a, b \in \mathbb{R}$.

Example 4.11 *Consider $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ has roots in the extension field $\mathbb{Q}(\sqrt[3]{2}, \omega)$, then to find the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) = \{\sigma \in \mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \omega)) | \sigma(a) = a, \forall a \in \mathbb{Q}\}$, we first see that $\sigma(\sqrt[3]{2})$ is also a root of $f(x)$ so $\sigma(\sqrt[3]{2}) = \omega^k \sqrt[3]{2}$ for some $k = 0, 1, 2$. Similarly, $\sigma(\omega)$ is also a root of $x^2 + x + 1$ so $\sigma(\omega) = \omega$ or ω^2. Thus we have six different automorphisms given by the choice of $\sigma(\sqrt[3]{2})$ and $\sigma(\omega)$, and it is easy to verify these six automorphisms form a group isomorphic to S_3.*

Next we will introduce Galois extension and see how Galois group act on the extension field.

Definition 4.7 *(Galois Extension) Let L/F be any field extension, we say L/F is a Galois extension if $|\mathrm{Gal}(L/F)| = [L : F]$.*

Proposition 4.1 *A Galois extension is both normal and separable.*

Proof Let L/F be a Galois extension. By definition, L/F is finite and $|\mathrm{Gal}(L/F)| = [L : F]$. We first show that L/F is separable. Let $\alpha \in L$ and let $m(x) \in F[x]$ be its minimal polynomial. Consider the orbit of α under the action of $G = \mathrm{Gal}(L/F)$:

$$\{\sigma(\alpha) : \sigma \in G\}$$

Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r$ be the distinct elements in this orbit. Every $\sigma \in G$ permutes the α_i, so the polynomial

$$f(x) = \prod_{i=1}^{r} (x - \alpha_i)$$

has coefficients fixed by all $\sigma \in G$. Since L/F is Galois, the fixed field of G is exactly F, so $f(x) \in F[x]$. Since $m(x)$ is the minimal polynomial of α over F, we have $m(\alpha) = 0$, and for any $\sigma \in G$, $m(\sigma(\alpha)) = \sigma(m(\alpha)) = 0$. Thus all α_i are roots of $m(x)$, so $f(x)$ divides $m(x)$ in $L[x]$, hence also in $F[x]$ by Gauss's Lemma. But $m(x)$ is irreducible over F, so $m(x) = f(x)$ up to a constant factor. Therefore $m(x)$ has distinct roots $\alpha_1, \ldots, \alpha_r$, so α is separable over F. Since α was arbitrary, L/F is separable.
Then we show L/F is a normal extension. Let $f(x) \in F[x]$ be an irreducible polynomial that has a root $\alpha \in L$. We need to show that $f(x)$ splits completely in L. Consider the minimal polynomial $m(x)$ of α

over F. Since $f(x)$ is irreducible and has α as a root, $f(x) = m(x)$ up to a constant factor. From Step 1, we know $m(x)$ splits into distinct linear factors in some extension.
Let β be any root of $m(x)$ in an algebraic closure. Then there exists an F-isomorphism $\sigma : F(\alpha) \to F(\beta)$ with $\sigma(\alpha) = \beta$. Since L/F is separable and L is the splitting field of some polynomial over $F(\alpha)$, the isomorphism σ extends to an automorphism $\tilde{\sigma} \in \mathrm{Gal}(L/F)$.
Therefore $\beta = \tilde{\sigma}(\alpha) \in L$, so all roots of $m(x)$ lie in L. Hence $m(x)$ splits completely in $L[x]$, and thus $f(x)$ splits completely in $L[x]$. This shows that L/F is normal. □

Example 4.12 *Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has splitting field $\mathbb{R}(i)$, then the Galois group $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ has order 2 and the degree of the extension $[\mathbb{R}(i) : \mathbb{R}] = 2$, so $\mathbb{R}(i)/\mathbb{R}$ is a Galois extension.*

Theorem 4.1 *(Characterization of Galois Extension) Let L/F be any field extension, then the following statements are equivalent:*

1. *L/F is a Galois extension.*
2. *F is the fixed field of $\mathrm{Gal}(L/F)$, that is $F = \{a \in L | \sigma(a) = a, \forall \sigma \in \mathrm{Gal}(L/F)\}$.*
3. *L is a splitting field of a separable polynomial over F.*

Proof We prove the equivalences cyclically: (1) ⇒ (2) ⇒ (3) ⇒ (1).
(1) ⇒ (2): Assume L/F is Galois. Let $G = \mathrm{Gal}(L/F)$ and define the fixed field

$$L^G = \{a \in L \mid \sigma(a) = a \text{ for all } \sigma \in G\}$$

Clearly $F \subseteq L^G$. Suppose for contradiction that $F \subsetneq L^G$.
Then there exists $\alpha \in L^G \backslash F$ with minimal polynomial $m(x) \in F[x]$ of degree > 1. Since L/F is separable, $m(x)$ has distinct roots. Since L/F is normal, all roots of $m(x)$ lie in L.
Let $\beta \neq \alpha$ be another root of $m(x)$. There exists an F-embedding $\sigma : F(\alpha) \to L$ with $\sigma(\alpha) = \beta$. This extends to an F-automorphism $\tilde{\sigma} \in G$ because L/F is normal.
But $\alpha \in L^G$ implies $\tilde{\sigma}(\alpha) = \alpha$, contradicting $\tilde{\sigma}(\alpha) = \beta \neq \alpha$. Therefore $L^G = F$.
(2) ⇒ (3): Assume $F = L^{\mathrm{Gal}(L/F)}$. Since L/F is finite, take an F-basis $\{\alpha_1, \ldots, \alpha_n\}$ of L. Let $G = \mathrm{Gal}(L/F) = \{\sigma_1, \ldots, \sigma_n\}$. Consider the polynomial

$$f(x) = \prod_{i=1}^{n} \prod_{j=1}^{n} (x - \sigma_i(\alpha_j))$$

The coefficients of $f(x)$ are symmetric polynomials in the $\sigma_i(\alpha_j)$, and are fixed by all $\sigma \in G$ since applying σ just permutes the factors. By assumption (2), these coefficients lie in F. Thus $f(x) \in F[x]$.
Since each α_j appears among the roots $\sigma_i(\alpha_j)$ (take $\sigma_1 = \mathrm{id}$), L is generated over F by roots of $f(x)$. Also, $f(x)$ is separable because its roots are distinct (they are images of basis elements under automorphisms). Therefore L is the splitting field of the separable polynomial $f(x)$ over F.
(3) ⇒ (1): Assume L is the splitting field of a separable polynomial $f(x) \in F[x]$. Then L/F is normal by definition of splitting field, and separable because $f(x)$ is separable. Since L/F is finite, normal, and separable, it is Galois. □

The Galois extension build a strong connection between field extension and its Galois group, and the next step is to see how Galois group act on the extension field.

Theorem 4.2 *(Fundamental Theorem of Galois Theory) Let L/F be a Galois extension with Galois group $G = \mathrm{Gal}(L/F)$, then there is a one-to-one correspondence between the set of intermediate fields K with $F \subseteq K \subseteq L$ and the set of subgroups H of G, given by the maps*

$$K \longmapsto \mathrm{Gal}(L/K) = \{\sigma \in G | \sigma(a) = a, \forall a \in K\}$$
$$H \longmapsto L^H = \{a \in L | \sigma(a) = a, \forall \sigma \in H\}$$

Furthermore, the following properties hold:

1. *K is a Galois extension of F if and only if H is a normal subgroup of G, in which case* $\mathrm{Gal}(K/F) \cong G/H$*.*
2. *$[L : K] = |H|$ and $[K : F] = [G : H]$.*

Proof We divide the proof into several parts. First we show the maps are well-defined and inverses. Let K be an intermediate field with $F \subseteq K \subseteq L$. Since L/F is Galois, L/K is also Galois (as it is the splitting field of the same separable polynomial over K). Thus $\mathrm{Gal}(L/K)$ is a subgroup of G. Let H be a subgroup of G. The fixed field L^H is clearly an intermediate field with $F \subseteq L^H \subseteq L$. We now show these maps are inverses. For any intermediate field K:

$$L^{\mathrm{Gal}(L/K)} = K$$

This holds because L/K is Galois, so the fixed field of $\mathrm{Gal}(L/K)$ is exactly K.
For any subgroup $H \leq G$:

$$\mathrm{Gal}(L/L^H) = H$$

To see this, let $H' = \mathrm{Gal}(L/L^H)$. Clearly $H \subseteq H'$ since every $\sigma \in H$ fixes L^H by definition.
Now consider the degrees:

$$[L : L^H] = |H'| \quad \text{(since } L/L^H \text{ is Galois)}$$
$$[L : L^H] \leq |H| \quad \text{(by Artin's lemma)}$$

Thus $|H'| \leq |H|$, but since $H \subseteq H'$, we must have $H = H'$. Morevore, for any corresponding pair $K = L^H$ and $H = \mathrm{Gal}(L/K)$:

$$[L : K] = |\mathrm{Gal}(L/K)| = |H|$$
$$[K : F] = \frac{[L : F]}{[L : K]} = \frac{|G|}{|H|} = [G : H]$$

which proves property (2).
Suppose $H \triangleleft G$ is normal and $K = L^H$. We show K/F is Galois. For any $\sigma \in G$ and $a \in K$, and any $\tau \in H$, we have: $\tau(\sigma(a)) = \sigma(\sigma^{-1}\tau\sigma(a)) = \sigma(a)$, since $\sigma^{-1}\tau\sigma \in H$ by normality, and thus fixes $a \in K = L^H$. Therefore $\sigma(a) \in L^H = K$, so $\sigma(K) = K$. This allows us to define a restriction map:

$$\rho : G \to \mathrm{Gal}(K/F), \quad \sigma \mapsto \sigma|_K$$

The kernel of ρ is exactly $\mathrm{Gal}(L/K) = H$, so we get an injection: $G/H \hookrightarrow \mathrm{Gal}(K/F)$. Comparing orders:

$$[G : H] = [K : F] \leq |\mathrm{Gal}(K/F)| \leq [K : F]$$

Thus equality holds throughout, so K/F is Galois with $\mathrm{Gal}(K/F) \cong G/H$. Conversely, if K/F is Galois, then for any $\sigma \in G$ and $\tau \in H = \mathrm{Gal}(L/K)$, and any $a \in K$ we have $\sigma^{-1}\tau\sigma(a) = \sigma^{-1}(\sigma(a)) = a$, since τ fixes $\sigma(a) \in K$ (as K/F is normal). Thus $\sigma^{-1}\tau\sigma \in H$, so $H \triangleleft G$. □

Example 4.13 *Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has splitting field $\mathbb{R}(i)$, then the Galois group* $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ *has only two subgroups: the trivial group $\{id\}$ and the whole group itself. By the fundamental theorem of Galois theory, there are only two intermediate fields: $\mathbb{R}$ corresponding to* $\mathrm{Gal}(\mathbb{R}(i)/\mathbb{R})$ *and $\mathbb{R}(i)$ corresponding to the trivial group.*

Example 4.14 *Consider the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ has splitting field $\mathbb{Q}(\sqrt[3]{2}, \omega)$, then the Galois group* $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q}) \cong S_3$ *has several subgroups, for example, the subgroup $H = \{id, (123), (132)\} \cong \mathbb{Z}/3\mathbb{Z}$ is a normal subgroup of S_3. By the fundamental theorem of Galois theory, there is an intermediate field K corresponding to H, and $K/\mathbb{Q}$ is a Galois extension with Galois group* $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3/H \cong \mathbb{Z}/2\mathbb{Z}$*. The diagram below showclearly how the one-to-one correspondence goes*

$$
\begin{tikzcd}
 & \{(1)\} \arrow[ddl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
 & \langle(12)\rangle \arrow[dd, "3"] & \langle(13)\rangle \arrow[ddl, "3"] & \langle(23)\rangle \arrow[ddll, "3"] \\
A_3 \arrow[dr, "2"] & & & \\
 & S_3 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathbb{Q}(\sqrt[3]{2},\omega) & & \\
 & \mathbb{Q}(\omega^2\sqrt[3]{2}) \arrow[u] & \mathbb{Q}(\omega\sqrt[3]{2}) \arrow[ul] & \mathbb{Q}(\sqrt[3]{2}) \arrow[ull] \\
\mathbb{Q}(\omega) \arrow[uur] & & & \\
 & \mathbb{Q} \arrow[ul, "2"] \arrow[uu, "3"] \arrow[uur, "3"] \arrow[uurr, "3"] & &
\end{tikzcd}
$$

5 Topological covering Space

We have already talked about the fundamental group and fundamental groupoid for topological space, then comes up with van Kampen theorem to calculate the fundamental structure for some nontrivial space. What we more focus on is the equivalent description to van Kampen theorem, show us the power of categorical language also give us some intuition to homotopy theory.

Now we come to covering space, recall how we prove $\pi_1(S^1) = \mathbb{Z}$ by covering the real line $\mathbb{R}$ circle by circle to S^1. But covering space is more than that to compute fundamental group. Unlike any other textbook, we will see covering space in categorical language and in homotopic way.

Definition 5.1 *(Evenly Covering Space) An evenly covering space for space X consists a pair $(\hat{X}, p)$ such that $p : \hat{X} \longrightarrow X$ is surjective and $\forall x \in X$, there is an open neighborhood U_x where*

$$p^{-1}(U_x) = \bigsqcup_{i \in I} V_i$$

for some disjoint open set V_i in $\hat{X}$, and $p|_{V_i} : V_i \longrightarrow U_x$ is homeomorphism for every $i \in I$. We call $\{V_i\}$ is the sheets of $\hat{X}$ over X and $F_x = p^{-1}(x) \in \bigsqcup_{i \in I} V_i$ is a fiber of the covering p.

It should be mentioned that the difference between evenly covering space and covering space is that the preimage of any neighborhood can be written as disjoint union or not. But in Algebric Topological the covering we describe are almost all evenly, so in later words we consider cover is an evenly cover. One should verify any homomorphism is a cover. Most classical example is the covering we used to show $\pi_1(S^1) = \mathbb{Z}$ by $p = \mathbb{R} \longrightarrow S^1$. We will see the strong connection between covering space and the fundamental groupoid or group, the following theorem describes how p acts on the space.

Theorem 5.1 *(Unique path lifting theorem) For a cover $p : \hat{X} \longrightarrow X$ with $e_0 \in F_{x_0}$ where $x_0 \in X$, we* *have*

$$\forall f \in \{f \text{ a path} : I \longrightarrow X | f(0) = x_0\},$$
$$\exists! g \in \{g \text{ a path} : I \longrightarrow \hat{X} | g(0) = e_0\}$$

such that $p \circ g = f$

Proof For a evenly covered base space X, if f lies entirely on the sheet $S_{x_0} = \{V_i\}$ with respect to U_{x_0}, then $g = q \circ f$ where q is the homomorphic inverse of p such that $q|_{V_i} \circ p|_{V_i} = id|V_i$, $\forall V_i \in S_{x_0}$, and the uniqueness comes from the unique inverse for homeomorphism. For a general case, by Lebesgue's Covering Lemma I is compact that we can partition I by $0 = t_0 < t_1 < ... < t_n = 1$ according to the covering, making f maps every closed $[t_i, t_{t+1}]$ to an evenly covered neighborhood of $f(t_i)$, then by the evenly cover case and induction we can obatin the unique lifting. ☐

What the unique path lifting theorem means is that for the cover $p : \hat{X} \longrightarrow X$*, any fundamental points* x_0 *in based space* X *lifted into the fiber* F_{x_0}*, and* $\forall e_i \in F_{x_0}$*. This theorem give us the intuition that the lifting also act on the homotopic calss of each path.*

Corollary 5.1 *For a covering* $p : \hat{X} \longrightarrow X$ *and a homotopy in* X *that* $h : f \simeq f'$ *start from* x_0 *lifts uniquely to a homotopy* $H : g \simeq g'$ *in* $\hat{X}$ *start from* e_0*.*

Proof For the homotopy $h : I \times I \longrightarrow X$, by Lebesgue's Covering Lemma we can subdivide the compact $I \times I$ into many subsquares each of which maps to a fundamental neighborhood of f, by **Theorem 5.1** we see any paths in the homotopic class lifts to a unique path that h lifts uniquely to a homotopy $H : I \times I \longrightarrow \hat{X}$ where f and f' lift to g and g' with $H : g \simeq g'$ that $H(0,0) = e_0$, and one should be easily verified that $h = p \circ H$. ☐

Proposition 5.1 *The induced group homomorphism* $p_* : \pi_1(\hat{X}, e_i) \longrightarrow \pi_1(X, x_0)$ *is a monomorphism, for all* $e_i \in F_{x_0}$*.*

Before discussing the algebric structure of covering space, we first define two important covering space according to special group structures.

Definition 5.2 *Consider the covering* $p : \hat{X} \longrightarrow X$*. We say* p *is regular if* $\pi_1(X, x_0)/p_*(\pi_1(\hat{X}, e_i))$ *is still a group,* p *is universal if* $\pi_1(X, x_0)/p_*(\pi_1(\hat{X}, e_i)) = \pi_1(X, x_0)$ *and it is the normal subgroup of* $\mathbb{Z}$ *so it's regular.*

Example 5.1 *Any integer multiple winding of* S^1 *is a normal cover, say if a covering wrap the* S^1 *three times it is easy to calculate* $p_*(\hat{X}, e_i) = 3\mathbb{Z}$*. Moreover, the covering* $p : \mathbb{R} \longrightarrow S^1$ *is universal because* $\mathbb{R}$ *is simply connected that* $\pi_1(\mathbb{R}, e_i) = 0$*, we also call* $\mathbb{R}$ *is the universal cover of* S^1*.*

The relationship between covering space $\hat{X}$ *and based space* X *is defined on the structure of the underlying fundamental groups. But we want to study the structure more specifically, says we should generalize the idea of covering from topological space to any algebric objects. One way to do this is to define the covering space on categories. Once we have the categorical covering, the induced homomorphism* p_* *will become a covering contain more information rather than a mapping. The way to generalize the covering space is working on the category called morphism category.*

Definition 5.3 *(Morphisms Category) Let* $\mathscr{C}$ *be any category and a* $x \in \mathrm{ob}(\mathscr{C})$*, the morphism category of* $\mathscr{C}$ *under* x *is a category* $x\backslash\mathscr{C}$ *such that*

$$\mathrm{Ob}(x\backslash\mathscr{C}) := \bigcup_{y_i \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(x, y_i)$$

which is all the morphisms in $\mathscr{C}$ that from x to others. For any $f, g \in \mathrm{Ob}(x\backslash\mathscr{C})$, $f : x \longrightarrow y_1,\ g : x \longrightarrow .y_2$ for some $y_1, y_2 \in \mathrm{Ob}(\mathscr{C})$, then the morphism in $x\backslash\mathscr{C}$ is defined as the compostion

$$\mathrm{Hom}_{x\backslash\mathscr{C}}(f,g) := \mathrm{Hom}_{\mathscr{C}}(y_1,y_2)$$

which means the morphisms $\gamma : f \longrightarrow g$ are the morphisms $\gamma : y_1 \longrightarrow y_2$ such that $\gamma(f) := \gamma \circ f = g$ makes the diagram below commutes

$$\begin{tikzcd}
 & x \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\xrightarrow[\gamma\in\mathrm{Hom}_{x\backslash\mathscr{C}}(f,g)]{}"] & \\
y_1 \arrow[rr, "\gamma\in\mathrm{Hom}_{\mathscr{C}}(y_1,y_2)"'] & & y_2
\end{tikzcd}$$

The motivation of defining this is that we want to generalize our covering idea to fundamental group or groupoid, as you can see both of them can be considered as categorical objects, so the covering should be a functor.

Definition 5.4 *(Covering of Groupoids) Let $\mathscr{B}$ and $\mathscr{C}$ be two small connected category, meaning the objects form a set not a proper class, and any two object have at least one invertalbe morphism between them. Then the covering between $\mathscr{B}$ and $\mathscr{C}$ is a surjective functor $p : \mathscr{B} \longrightarrow \mathscr{C}$ such that*

$$p : \mathrm{Ob}(b\backslash\mathscr{B}) \longrightarrow \mathrm{Ob}(p(b)\backslash\mathscr{C})$$

is a bijection $\forall b \in \mathscr{B}$. Similar to the covering of topological space, for a $c \in \mathscr{C}$, the fiber of c is the set $F_c = \{b \in \mathscr{B} | p(b) = c\}$ and $\forall c \in \mathscr{C}$ we have

$$p^{-1}(\mathrm{Ob}(c\backslash\mathscr{C})) = \bigsqcup_{b_i \in F_c} \mathrm{Ob}(b_i\backslash\mathscr{B})$$

The objects in $x\backslash\mathscr{C}$ is a set of morphisms of $\mathscr{C}$ with source x we denote is as $\mathrm{St}_{\mathscr{C}}(x)$, so the covering restricts to the bijection $p : \mathrm{St}_{\mathscr{B}}(b) \longrightarrow \mathrm{St}_{\mathscr{C}}(p(b))$ and $p^{-1}(\mathrm{St}_{\mathscr{C}}(c)) = \bigsqcup_{b_i\in F_c} \mathrm{St}_{\mathscr{B}}(b_i)$.

The definition of covering of groupoids is strictly followed by the covering of topological space, with the based point x_0 changed to the morphism with source x_0. We have the statement below naturally.

Proposition 5.2 *The induced functor $\prod(p) : \prod(\hat{X}) \longrightarrow \prod(X)$ is a covering of groupoid if $p : \hat{X} \longrightarrow X$ is a covering of topological space.*

Proof This is actually the same statement of **Theorem 5.1** and **Corollary 5.1** as long as consider any path is a invertible morphism in the groupoid as a category. ☐

The key of algebric topology is to use algebric structure to study and verfy difference topological space, and our main work is also trying to discover any potential example to build a bridge between difference mathematical field. We will see how covering space and Galois theory connected to each other by the underlying group of topological space through covering space. Now we will study more on the algebric side of covering space.

Definition 5.5 *(Automorphism group of groupoid) Let $\mathscr{C}$ be any groupoid, the automorphism of $x \in \mathscr{C}$ are all the objects in $x\backslash\mathscr{C}$ which from x sends to x itself and denoted as $\pi(\mathscr{C}, x)$, in mathematical words*

$$\pi(\mathscr{C}, x) := \mathrm{Hom}_{\mathscr{C}}(x, x)$$

It is obvious that if the $\mathscr{C}$ is any fundamental groupoid of a topological space then the automorphism group is actually the fundamental group with respect to the chosen x_0, that is

$$\pi(\prod(X), x_0) = \mathrm{Hom}_{\prod(X)}(x_0, x_0) = \pi_1(X, x_0)$$

So for any covering space $p : \hat{X} \longrightarrow X$ we have the covering of underlying groupoid with the connected of groups, the covering of groupoid indicate some information on groups.

Proposition 5.3 *Let $p : \hat{X} \longrightarrow X$ be a covering of topological space, then the induced morphism $p_* : \pi(\prod(\hat{X}), e_i) \longrightarrow \pi(\prod(X), x_0)$ is a monomorphism, where $x_0 \in X$ is the based point and $e_i \in F_{x_0} = \{e_i\}_{i \in I}$. Moreover, $\forall e_j, e_k \in F_{x_0}$, $p_*(\pi(\prod(\hat{X}), e_j))$ is conjugate to $p_*(\pi(\prod(\hat{X}), e_k))$ in $\pi(\prod(X), x_0)$.*

Proof The injectivity is trivial by the bijection of $\prod(p)$ on $\mathrm{St}(e_i)$. $\prod(\hat{X})$ is a groupoid so there is a path $g : e_j \longrightarrow e_k$, the conjugation is given by the $p(g) \in \pi(\prod(X), x_0)$, that is

$$p_*(\pi(\prod(\hat{X}), e_k)) = [p_*(g)] \circ [p_*(\pi(\prod(\hat{X}), e_j))] \circ [p_*(g)]^{-1} \qquad \square$$

The proposition is saying the underlying group of based space at x_0 is hiden in any copy of group at $e_i \in F_{x_0}$, when the covering space covers to X by p, all the group structure on F_{x_0} will onto the one group at x_0.

Proposition 5.4 *The group $p_*(\pi(\prod(\hat{X}), e_j))$ runs through all conjugates of $p_*(\pi(\prod(\hat{X}), e_i)) \in \pi(\prod(X), x_0)$ as e_j runs through F_{x_0}.*

Proof This is trivial that the surjectivity of the groupoid covering functor $\prod(p)$ on $\mathrm{St}(e_i)$ and apply the same method in **Proof 5.3** $\square$

Now the connection between covering space of topological space and groupoid, together with the underlying fundamental group is very clear as the commutative diagram shows below, we consider the fundamental group of a space as the automorphism group under the groupoid category at specific object, all our work is trying to build a categorical eyesight of algebric topology.

$$\begin{tikzcd}
\hat{X} \arrow[r, "\prod"] \arrow[d, "p"] & \prod(\hat{X}) \arrow[r, "\pi(\mathscr{C},x)"] \arrow[d, "\prod(p)"] & \pi(\prod(\hat{X}), e_i) \arrow[r, equal] \arrow[d, "p_*"] & \pi_1(\hat{X}, e_i) \arrow[d, "p^*"] \\
X \arrow[r, "underlying\ groupoid"'] & \prod(X) \arrow[r, "automorphism\ group"'] & \pi(\prod(X), x_0) \arrow[r, equal] & \pi_1(X, x_0)
\end{tikzcd}$$

It is clear that the fundamental group of covering space should have a same corresponding relation as groupoid, if we consider the group as the automorphism of the groupoid category under the covering, then there should be some algebric relation between fundamental groups.