CHARACTERIZE ON THE HEISENBERG GROUP WITH LEFT INVARIANT LORENTZIAN METRIC

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Abstract. In this paper, we consider the biharmonicity conditions for maps between Riemannian manifolds and we characterize non-geodesic biharmonic curve in Heisenberg group H_3 which is endowed with left invariant Lorentzian metric.

1. Introduction

In this paper, we study some geometric proporties of the three-dimensional Heisenberg group H_3 endowed with a left invariant Lorentzian metric.

A map φ from a compact Riemannian manifold (M, g) to another Riemannian manifold (N, h) is harmonic if it is a critical point of the energy;

$$E^1(\varphi) = \frac{1}{2}\int_M \mid d\varphi \mid^2 v_g. \tag{1.1}$$

A map $\varphi : (M, g) \to (N, h)$ is biharmonic if it is a critical point of the bienergy:

$$E^2(\varphi) = \frac{1}{2}\int_M \mid \tau_2(\varphi) \mid^2 v_g, \tag{1.2}$$

where $\tau_2(\varphi) = trace\nabla d\varphi$ is the tension field of φ. Using the first variational formula one see that φ is biharmonic map if and only if its bitension field vanishes identically,

$$\tau^2(\varphi) := -\Delta(\tau_2(\varphi)) - traceR^N(d\varphi, \tau_2(\varphi))d\varphi = 0, \tag{1.3}$$

where

$$\Delta = -trace_g(\nabla)^2 = -trace(\nabla\nabla - \nabla_{\nabla^M}) \tag{1.4}$$

is the Laplacian on sections of the pull-back bundle $\varphi^{-1}TN$ and R^N is the curvature operator of (N, h) defined by

$$R^N(P_1, P_2)P_3 = [\nabla^N_{P_1}, \nabla^N_{P_2}]P_3 - \nabla^N_{[P_1,P_2]}P_3. \tag{1.5}$$

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Note that

$$\tau^2(\varphi) = -J(\tau_2(\varphi)) \\ = -\Delta(\tau_2(\varphi)) - traceR^N(d\varphi, \tau_2(\varphi))d\varphi, \tag{1.6}$$

where J is the Jacobi operator which plays an important role in the study of harmonic maps.

2. Riemannian structure of H_3

The Heisenberg group H_3 can be seen as the Euclidean space $\mathbb{R}^3$ endowed with the multiplication

$$(x', y', z')(x, y, z) = \left(x' + x, y' + y, z' + z + \frac{1}{2}x'y - \frac{1}{2}y'x\right) \tag{2.1}$$

and the Riemannian metric g given by

$$g = ds^2 = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy\right)^2. \tag{2.2}$$

Also

$$ds^2 = \sum_{i=1}^{3} w^i \otimes w^i,$$

$$w^1 = dx \ , \ w^2 = dy \ , \ w^3 = dz + \frac{y}{2}dx - \frac{x}{2}dy.$$

The metric g is invariant with respect to the translations corresponding to that multiplication. This metric is isometric to the others also quite standard, which is left invariant with respect to the composition arising form the multiplication of the 3×3 Heisenberg matrices.

First of all we shall determine the Levi-Civita connection ∇ of the metric g with respect to the left invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \ , \ e_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \ , \ e_3 = \frac{\partial}{\partial z},$$

which is dual to the coframe

$$w^1 = dx \ , \ w^2 = dy \ , \ w^3 = dz + \frac{y}{2}dx - \frac{x}{2}dy.$$

We obtain

$$\begin{aligned} &\nabla_{e_1}e_1 = 0, \nabla_{e_1}e_2 = \frac{1}{2}e_3, \nabla_{e_1}e_3 = -\frac{1}{2}e_2, \\ &\nabla_{e_2}e_1 = -\frac{1}{2}e_3, \nabla_{e_2}e_2 = 0, \nabla_{e_2}e_3 = \frac{1}{2}e_1, \\ &\nabla_{e_3}e_1 = -\frac{1}{2}e_2, \nabla_{e_3}e_2 = \frac{1}{2}e_1, \nabla_{e_3}e_3 = 0, \end{aligned} \tag{2.3}$$

also, we have the well-known Heisenberg bracket relations.

$$[e_1, e_2] = e_3\ ,\ [e_3, e_1] = [e_2, e_3] = 0.$$

We shall adopt the following notation and sign conventation: the curvature operator is

$$R(P_1, P_2)P_3 = -\nabla_{P_1}\nabla_{P_2}P_3 + \nabla_{P_2}\nabla_{P_1}P_3 + \nabla_{[P_1,P_1]}P_3. \tag{2.4}$$

Biharmonic curves in (H_3, g)**.** Let $\gamma : I \to (H_3, g)$ be a curve on the Heisenberg group H_3 parametrized by arclength. Let $\{P_1, P_2, P_3\}$ be a frame fields tangent to H_3 along γ defined as follows; P_1 is the unit vector field γ' tangent to γ, P_2 is the unit vector field in direction of $\nabla_{P_1}P_1$ (normal to γ) and P_3 is choosen so that $\{P_1, P_2, P_3\}$ is a positively oriented orthonormal basis. Then we have the following Frenet formulas:

$$\begin{aligned} \nabla_{P_1}P_1 &= \kappa P_2, \\ \nabla_{P_1}P_2 &= -\kappa P_1 - \tau P_3, \\ \nabla_{P_1}P_3 &= \tau P_2, \end{aligned} \tag{2.5}$$

where $\kappa = \mid \nabla_{P_1}P_1 \mid$ is the curvature and τ is the torsion of γ. With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} P_1 &= \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3, \\ P_2 &= \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3, \\ P_3 &= \zeta_1 e_1 + \zeta_2 e_2 + \zeta_3 e_3 \end{aligned} \tag{2.6}$$

and we have the biharmonic equation for γ

$$\begin{aligned} \tau^2(\gamma) = -J(\tau_2(\gamma)) &= \nabla^3_{P_1}P_1 + R(P_1, \kappa P_2)P_1 \\ &= (-3\kappa'\kappa)P_1 + (\kappa'' - \kappa^3 - \kappa\tau^2 + \frac{\kappa}{4} - \kappa\zeta_3^2)P_2 \\ &\quad + (-2\kappa'\tau - \kappa\tau' + \kappa\eta_3\zeta_3)P_3 = 0. \end{aligned}$$

THEOREM 2.1. [4] *Let $\gamma : I \to (H_3, g)$ be a differentiable curve on the Heisenberg group H_3 parametrized by arclength. Then γ is a non-geodesic biharmonic curve if and only if*

$$\begin{aligned} \kappa &= const. \neq 0, \\ \kappa^2 + \tau^2 &= \frac{1}{4} - \zeta_3^2, \\ \tau' &= \eta_3\zeta_3. \end{aligned} \tag{2.7}$$

COROLLARY 2.2. [4] *Let $\gamma : I \to (H_3, g)$ be a differentiable curve parametrized by arclength. If $\zeta_3 = 0$, then γ is not biharmonic.*

COROLLARY 2.3. *Let $\gamma : I \to (H_3, g)$ be a differentiable curve parametrized by arclength. If $\zeta_3 = const.$ and $\eta_3\zeta_3 \neq 0$, then γ is not biharmonic.*

Similar to the terminology used for curves in $\mathbb{R}^3$, we keep the name helix for curve in a Riemannian 3-manifold having constant both geodesic curvature and geodesic torsion.With this terminology, we can use equation (2.7) to deduce the following

COROLLARY 2.4. *Let $\gamma : I \to (H_3, g)$ be a non-geodesic biharmonic helix parametrized by arclength, then*

$$\begin{aligned} \zeta_3 &= const. \neq 0, \\ \eta_3 &= 0, \\ \kappa^2 + \tau^2 &= 2\zeta_3^2 - 1. \end{aligned} \tag{2.8}$$

3. Left invariant Lorentzian metric on the Heisenberg group (H_3, g_L)

The Lorentzian Heisenberg group H_3 can be seen as the space $\mathbb{R}^3$ endowed with multiplication

$$(x', y', z')(x, y, z) = (x' + x, y' + y, z' + z + x'y - y'x). \tag{3.1}$$

H_3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

THEOREM 3.1. [1] *Left invariant Lorentz metric on the Heisenberg group H_3 is isometric to the following metric,*

$$g_L = dx^2 + (xdy + dz)^2 - ((1 - x)dy - dz)^2. \tag{3.2}$$

THEOREM 3.2. *The Lorentz metric* (3.2) *on the Heisenberg group H_3 is flat.*

Proof. The Lie algebra of H_3 has a basis

$$e_1 = \frac{\partial}{\partial x}\ , \ e_2 = \frac{\partial}{\partial y} + (1 - x)\frac{\partial}{\partial z}\ , \ e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}. \tag{3.3}$$

For this basis the Lie bracket are:

$$[e_2, e_1] = e_2 - e_3\ , \ [e_3, e_1] = e_2 - e_3, \ [e_2, e_3] = 0 \tag{3.4}$$

and we have

$$g_L(e_1, e_1) = 1, \ g_L(e_2, e_2) = 1, \ g_L(e_3, e_3) = -1. \tag{3.5}$$

If ∇ is the Levi-Civita connection and R is the curvature tensor of ∇, we have

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = 0, \\ \nabla_{e_1} e_1 &= \nabla_{e_3} e_1 = e_2 - e_3, \\ \nabla_{e_2} e_2 &= \nabla_{e_2} e_3 = \nabla_{e_3} e_3 = -e_1. \end{aligned} \tag{3.6}$$

So we obtain that

$$R(e_1, e_3) = R(e_1, e_2) = R(e_2, e_3) = 0. \tag{3.7}$$

Then the Lorentz metric g_L is flat.

4. Biharmonic curves in (H_3, g_L)

The biharmonic curves in H_3 we shall use their Frenet vector fields and equations. Let $\gamma : I \to (H_3, g)$ be a differentiable curve parametrized by arclength. Let $\{P_1, P_2, P_3\}$ be a frame fields tangent to H_3 along γ defined as follows; P_1 is the unit vector field γ' tangent to γ, P_2 is the unit vector field in direction of $\nabla_{P_1} P_1$ (normal to γ) and P_3 is choosen so that $\{P_1, P_2, P_3\}$ is a positively oriented orthonormal basis. Then we have the following Frenet equations.

$$\begin{aligned} \nabla_{P_1} P_1 &= \kappa P_2 \\ \nabla_{P_1} P_2 &= \kappa P_1 + \tau P_3 \\ \nabla_{P_1} P_3 &= -\tau P_2 \end{aligned} \tag{4.1}$$

where $\kappa = \mid \tau_2(\gamma) \mid$ is the curvature and τ its torsion. The Lie algebra of H_3 has a basis

$$e_1 = \frac{\partial}{\partial x}\ ,\ e_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z}\ ,\ e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}.$$

By (2.6) and by the biharmonic map equation (1.3) it reduces to,

$$\nabla^3_{P_1} P_1 - R(P_1, \nabla_{P_1} P_1)P_1 = 0 \tag{4.2}$$

THEOREM 4.1. *Let $\gamma : I \to (H_3, g_L)$ be a differentiable curve on the Heisenberg group H_3. γ is non-geodesic biharmonic curve if and only if γ is a helix. In this case $\kappa = \pm\tau$.*

Proof. Suppose that γ is biharmonic, from (3.7) we obtain that

$$R(P_1, \nabla_{P_1} P_1)P_1 = 0.$$

So by (4.2),

$$\nabla^3_{P_1} P_1 = 0.$$

Direct computation shows that

$$\nabla^3_{P_1} P_1 = 3\kappa'\kappa P_1 + (\kappa'' + \kappa^3 - \kappa\tau^2)P_2 + (2\kappa'\tau - \kappa\tau')P_3 = 0. \tag{4.3}$$

Also we obtain

$$\begin{aligned} \kappa'\kappa &= 0, \\ \kappa' + \kappa^3 - \kappa\tau^2 &= 0, \\ 2\kappa'\tau - \kappa\tau' &= 0. \end{aligned} \tag{4.4}$$

From (4.4) we obtain

$$\kappa = const. \qquad (4.5)$$
$$\kappa^2 - \tau^2 = 0.$$

Since (4.5) we have γ is a helix.

Conversely suppose that γ is a helix. Then from (4.2) γ is biharmonic curve.

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