ON THE SECOND FUNDAMENTAL FORM OF FINSLERIAN ISOMETRIC EMBEDDINGS

1. Introduction

.

Classical theorem of Gauss.

intrinsic geometry, extrinsic geometry of embedding.

definition of second fundamental form.

Definition 1.1. Banach-Minkowski space
norm (possibly non-symmetric) on V, smooth C^∞ on $V\backslash\{0\}$
strongly-convex

Here we use V^n to denote n-dimensional Minkowski-Finsler space.

Definition 1.2. Finsler metric
function $\varphi : TM \to \mathbb{R}$ such that $\varphi \in C^\infty(TM\backslash 0)$.
For all $x \in M$, $\varphi|_{T_xM}$ is Banach-Minkowski.

Use definition of second-order flat from the paper ... .

Definition 1.3. We denote $M = (M, \varphi)$ be a Finsler manifold. We say that the metric φ is second-order flat at a point $p \in M$ if there exists a flat Finsler metric φ_0 in a neighbourhood of p such that for $x \in M$ near p and $v \in T_xM\backslash\{0\}$, we have $\frac{\varphi(x,v)}{\varphi_0(x,v)} = 1 + o(|x-p|^2)$
Here $|x-p|$ denotes the distance from x to p in an arbitrary local coordinate system.

Theorem 1. *Let $M = (M^3, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^4 be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. Then second fundamental form of f at p is degenerate and its nullspace has dimension 2.*

Theorem 2. *Let $M = (M^n, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^{n+1} be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. If we assume second fundamental form $S(\text{-}, \text{-})$ at p has mixed signature, then second fundamental form at p is also degenerate and its nullspace has dimension $n-1$.*

Riemannian cases.
1. Riemannian metric is second-order flat at point p is equivalent to Riemannian curvature tensor is identically zero at point p.
2. Riemannian case of Theorem 1 has been proved for every dimension n. Actually in Riemannian case there exists a stronger result:
Theorem: If we assume that Riemannian curvature tensor is identically zero at point p of Riemannian manifold M^n and there exists isometric embedding f of Riemannian manifold M^n into R^{n+m}, then second fundamental form at point $f(p)$ is degenerate and has nullspace of dimension $n-m$.
There exists several different versions of proofs, see ... .
3. There exists isometric embedding $f : T^n \to \mathbb{R}^{2n}$, s.t. second fundamental form is non-degenerate at every point of its image. Indeed, considering the coordinate chart on torus $T^n \cong S^1 \times S^1 \times ... \times S^1$ constructed by $(\alpha_1, \alpha_2, ..., \alpha_n)$, here α_i is the angle of each circle. An example of such a map is constructed by $f : (\alpha_1, \alpha_2, ..., \alpha_n) \to \frac{\sqrt{n}}{n}(cos\alpha_1, sin\alpha_1, cos\alpha_2, sin\alpha_2, ..., cos\alpha_n, sin\alpha_n)$.
So result in 2 is sharp.

Notations.
smooth manifold, metric spaces
$M^n, \mathbb{R}^n, \mathbb{V}^n, H, U$
tangent vector in the tangent space
u, v, w, v_1, v_2
image of tangent vector induced by isometric embedding
$\tilde{u}, \tilde{v}, \tilde{w}, \tilde{v_1}, \tilde{v_2}$
Some vectors in $\mathbb{V}^n$, not necessarily in tangent plane
τ, λ_0
Metric
φ, Φ, Ψ, G_u
Vector-valued symmetric tensor
$L(\text{-}), B(\text{-},\text{-}), T(\text{-},\text{-},\text{-})$
Vector-valued symmetric tensor by projections into tangent plane
$\tilde{L}(\text{-}), \tilde{B}(\text{-},\text{-}), \tilde{T}(\text{-},\text{-},\text{-})$
Scalar symmetric tensor by projections into 1-dim linear subspace
$\hat{L}(\text{-}), \hat{B}(\text{-},\text{-}), \tilde{B}(\text{-},\text{-}), \hat{T}(\text{-},\text{-},\text{-})$

2. Preliminary and assumptions

First we assume Finsler metric in this section is second-order flat.
Let $V = (V, \Phi)$ be a 4-dimensional Banach-Minkowski space and $f : (U, \varphi) \to V$ a smooth isometric embedding. Then

$\Phi(d_x f(v)) = \varphi(x, v)$
for all $x \in U$ and $v \in \mathbb{R}^3$

Now we can assume that we choose local coordinates such that φ_0 is the restriction of a Banach-Minkowski norm. Thus $\varphi_0(x, v) = \varphi_0(v)$ does not depend on x. Then ... implies that for every $v \in \mathbb{R}^3$, $\varphi(p, v) = \varphi_0(v)$,
$d_p(\varphi(\cdot, v)) = 0$,
$d_p d_v(\varphi(\cdot, v)) = 0$,
and
$d_p^2(\varphi(\cdot, v)) = 0$.
$d_p^2 d_v(\varphi(\cdot, v)) = 0$.
$d_p^2 d_v^2(\varphi(\cdot, v)) = 0$.

Now we use some notations and properties in Finsler geometry.
If we assume $\Psi = \frac{1}{2}\Phi^2$, we define *osculating Riemannian metric* at direction u as

$$G_u(v, w) = \frac{\partial^2(\Psi(u + sv + tw))}{\partial s \partial t} = d_p^2(\Psi(u))(v, w) \tag{2.0.1}$$

By *Euler's homogeneous theorem*, we have a property of Legendre transform

(2.0.2)
$$L_u(v) = d_p(\Psi(u))(v) = \frac{\partial^2(\Psi(u + su + tv))}{\partial s \partial t}|_{(s,t)=(0,0)} = d_p^2(\Psi(u))(u, v) = G_u(u, v)$$

Cartan tensor
$A_u(v_1, v_2, v_3) = \frac{1}{2}\frac{\partial^3 \Psi(u + r \cdot v_1 + s \cdot v_2 + t \cdot v_3)}{\partial r \partial s \partial t}|_{(r,s,t)=(0,0,0)}$

3. Proposition

Proposition 3.1. $(\mathbb{R}^n, \varphi)$ *normed space,* $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *symmetric bilinear form,* $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *vector-valued symmetric bilinear tensor. If for all* $v, w \in \mathbb{R}^n$ *such that* $S(v, w) = 0$*, we have that* $Q(v, w)$ *is tangent to the level surface of* φ *at* v *at* w*, then there exists* $\lambda \in \mathbb{R}^n$*, s.t.*

$$Q(u, v) = S(u, v) \cdot \lambda \tag{3.0.1}$$

for all $v, w \in \mathbb{R}^n$ *and a fixed vector* $\lambda \in \mathbb{R}^n$*.*

We can prove that this proposition holds for dim $= 3$ *and it holds for arbitrary dimension when* S*(-, -) is indefinite.*

Case 1. If S has mixed signature, then exist nonzero v s.t. $S(v, v) = 0$.

$v = u + \epsilon \cdot u'$

$$\begin{cases} S(v,v) = 0 \\ S(u',u') = 0 \end{cases} \tag{3.0.2}$$

$$\begin{cases} v = u + \epsilon \cdot u' \\ w = u - \epsilon \cdot u' \end{cases} \tag{3.0.3}$$

For a map $\epsilon \to L_{v(\epsilon)}(Q(v(\epsilon), w(\epsilon))) = Q(u,u) - \epsilon^2 Q(u',u')$
we take its differentiation of ϵ at $\epsilon = 0$, we get

$$g_u(u', Q(u,u)) = 0 \tag{3.0.4}$$

which means $B(u,u) \perp_{g_u} u'$.

Then use lemma 3.2 below, we get $B(u,u) = S(u,u) \cdot \lambda$. So $B(u,v) = S(u,v) \cdot \lambda$.

Case 2. If S is positive semi-definite or negative semi-definite.

$$\begin{cases} x = cos\alpha \\ y = sin\alpha \end{cases} \tag{3.0.5}$$

$Q(x \cdot e_1 + y \cdot e_2, y \cdot e_1 - x \cdot e_2) = xy \cdot (Q(e_1,e_1) - Q(e_2,e_2)) + (y^2 - x^2) \cdot Q(e_1,e_2)$

Since when (e_1, e_2) map to $(e_2, -e_1)$, Q map to $-Q$.
There exists α_0, s.t. Q(-,-) lies in the hyperplane spanned by $\{e_1, e_2\}$.
So $Q(e_1,e_1) - Q(e_2,e_2)$ is the canonical normal vector of this 2-dimensional hyperplane.
Because

$$\begin{cases} Q(x,y) \perp x \\ Q(x,y) \perp y \end{cases} \tag{3.0.6}$$

for all $x, y \in \mathbb{R}^n$
So $Q(x,y)$ is the canonical normal vector of hyperplane spanned by x, y.
For every 2-dimensional hyperplane, there exists a linear projector of norm 1, s.t. the convex body can be projected into the hyperplane.
Using Blaschke-Kakutani theorem, then this convex body is Euclidean.

Lemma 3.2. *In $dim = 2,3$ case, if for a symmetric bilinear form S_1, $S(u,u) = 0$ implies $S_1(u,u) = 0$, then there exist a constant c such that $S_1(u,u) = c \cdot S(u,u)$.*

Proof.
We prove first for dim $= 2$ and condition that $S(u,u) = 0$ implies $S_1(u,u) = 0$.
By choosing a basis e_1, e_2, we get
$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e. $S(e_1,e_1) = 1, S(e_1,e_2) = 0, S(e_2,e_2) = -1$.
So $S(e_1 - e_2, e_1 - e_2) = 0, S(e_1 + e_2, e_1 + e_2) = 0$.
By condition, we get that $S_1(e_1 - e_2, e_1 - e_2) = 0, S_1(e_1 + e_2, e_1 + e_2) = 0$.

We choose constant $c = \frac{S_1(e_1-e_2,e_1+e_2)}{S(e_1-e_2,e_1+e_2)}$.
By changing the basis into e_1-e_2, e_1+e_2 and condition that $S(e_1-e_2,e_1+e_2) = S(e_1,e_1) - S(e_2,e_2) = 2$, we get
$S = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $S_1 = \begin{pmatrix} 0 & 2c \\ 2c & 0 \end{pmatrix}$, so $S_1(u,u) = c \cdot S(u,u)$ for all u.

Notice that now we know $S_1(u,u) = c \cdot S(u,u)$, we could choose $c = \frac{S_1(e_2,e_2)}{S(e_2,e_2)}$

Consider when dim $= 3$, $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$,

by restricting on submatrices of dim $= 2$ and apply result above, we could get that
$S_1 = \begin{pmatrix} c & \lambda & 0 \\ \lambda & c & 0 \\ 0 & 0 & -c \end{pmatrix}$
Here $c = \frac{S_1(e_3,e_3)}{S(e_3,e_3)}$

Since $S(1,1,\sqrt{2}) = 1+1-2 = 0$, so $S_1(1,1,\sqrt{2}) = 1+1-2+2\lambda = 0$ which means $\lambda = 0$. So $S_1 = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{pmatrix}$.

We prove that $S_1(u,u) = c \cdot S(u,u)$.

□

Lemma 3.3. *Let S be an indefinite nonsingular symmetric bilinear form in $\mathbb{R}^3$ and let B be a vector-valued symmetric bilinear map defined as $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$. If $S(u,u) = 0$ can imply $B(u,u) = 0$ for all $\{u \mid S(u,u) = 0\}$, then*

$$B(u,u) = S(u,u) \cdot \tau \tag{3.0.7}$$

for a fixed $\tau \in \mathbb{R}^4$

Proof. Now we know that if $S(u,u) = 0$, then vector-valued symmetric bilinear form $B(u,u) = 0$.
We denote $B(v_1,v_2) = (B_1(v_1,v_2), B_2(v_1,v_2), B_3(v_1,v_2), B_4(v_1,v_2))$ and each $B_i(v_1,v_2)$ is a symmetric bilinear form, so we get if $S(u,u) = 0$, then $B_i(u,u) = 0$.
From above, we know that $B_i(u,u) = c_i \cdot B(u,u)$ for some constant c_i. So $B(u,u) = S(u,u) \cdot (c_1, c_2, ..., c_n)$. Here denote $\tau = (c_1, c_2, ..., c_n)$, we get $B(u,u) = S(u,u) \cdot \tau$. □

We notice that from above, we can get
$B(u,v) = \frac{1}{2}(B(u+v,u+v) - B(u-v,u-v))$, which means $B(u,v) = S(u,v)\tau$

4. Proof of Theorem

Definition 4.1. We say that there exists *conormal vector*, if for an isometric embedding of M^n into V^{n+1}, there exists a fixed vector in V^{n+1} for all osculating Riemannian metric of Finsler metric restricted on the tangent plane $d_p f(T_p M)$.

Definition 4.2. We say that a symmetric k-linear form $T(v_1, v_2, \ldots, v_k)$ is *decomposable* if there exists 1-form L, s.t. $T(v_1, .v_2, \ldots, v_k) = L(v_1) \cdot L(v_2) \cdot \cdots \cdot L(v_k)$. Also, we say that a vector-valued symmetric k-linear form $\tilde{T}$ is *decomposable*, if there exists a fixed vector λ and 1-form L, s.t. $\tilde{T}(v_1, .v_2, \ldots, v_k) = L(v_1) \cdot L(v_2) \cdot \cdots \cdot L(v_k) \cdot \lambda$.

Here we assume all maps f, Finsler manifold M^n and Finsler metric Φ satisfy the condition in Preliminary.

Using conjecture above for $Q(\text{-},\text{-}) = P \circ d_p^2 f(\text{-},\text{-})$, there exists $\tau \in \mathbb{R}^n$, s.t. $Q(u, v) = S(u, v) \cdot \lambda$
So $P \circ d_p^2 f(u, v) = \hat{B}(u, v) = S(u, v) \cdot \lambda$, with λ in the tangent plane $d_p f(T_p M)$. Since $d_{\tilde{u}} \Phi(\tilde{v}, B(u, u)) = 0$ with $\tilde{v} = d_p f(v)$ for all v.
So there exists conormal vector τ for all osculating Riemannian metric.

Using conjecture above for $Q(\text{-},\text{-}) = P \circ d_p^3 f(\text{-},\text{-}, w)$ with arbitrary $w \in T_p M$
So we have $T(u, v, w) = S(u, v) \cdot \lambda(w)$.
Using lemma 4.1 below, we get that

$$S(u, v) = C_0 \cdot L(u) \cdot L(v) \tag{4.0.1}$$

for some constant C_0.

$$T(u, v, w) = C_1 \cdot L(u) \cdot L(v) \cdot L(w) \cdot \lambda_0 \tag{4.0.2}$$

for some constant C_1.

Lemma 4.3. *If a 3-linear vector-valued symmetric form $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $T(u, v, w) = S(u, v) \cdot \lambda(w)$, here $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form, $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued 1-form, then there exists 1-form $L : \mathbb{R}^n \to \mathbb{R}$, s.t. $S(u, v) = L(u) \cdot L(v)$ and $T(u, v, w) = L(u) \cdot L(v) \cdot L(w) \cdot \lambda_0$ for a fixed vector λ_0.*

Proof. If

$$\begin{cases} S(u, v) \neq 0 \\ S(u, w) \neq 0 \end{cases} \tag{4.0.3}$$

Then from $T(u, v, w) = S(u, v) \cdot \lambda(w) = S(u, w) \cdot \lambda(v)$, we know $\lambda(v) // \lambda(w)$.

Since set

$$\begin{cases} S(u, v) = 0 \\ S(u, w) = 0 \end{cases} \tag{4.0.4}$$

is nowhere dense and closed. It does not cover $\mathbb{R}^n$.
Denote $\overline{\mathcal{U}} = \{$non-null vectors of $S\}$ is open and dense, unless S is identically zero.
So for all $v, w \in \overline{\mathcal{U}}$, $\lambda(v)//\lambda(w)$.
So there exists 1-dimensional subspace l, s.t. for all $v \in \overline{\mathcal{U}}$, $\lambda(v) \in l$.
The same holds for all $v \in$ Closure$(\overline{\mathcal{U}}) = \mathbb{R}^n$.
We denote λ for a nonzero vector from l, so $\lambda(v) = L(v) \cdot \lambda$. Here $L : \mathbb{R}^n \to \mathbb{R}$ is a function.
Then from $T(u, v, w) = S(u, v) \cdot L(w) \cdot \lambda = S(u, w) \cdot L(v) \cdot \lambda$
So $S(u, v) = \frac{S(u,w)}{L_0(w)} \cdot L(v) = L(u) \cdot \frac{S(v,w)}{L(w)}$
Since S(-,-) is symmetric, so $S(u, v) = L(u) \cdot L(v)$.
So $T(u, v, w) = L(u) \cdot L(v) \cdot L(w) \cdot \lambda$

□

Proposition 4.4. *Let $M = (M^n, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^{n+1} be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. Then $B(u, v) = L(u) \cdot L(v) \cdot \tau$ for $k = 2$ and $T(u, v, w) = L(u) \cdot L(v) \cdot L(w) \cdot \lambda$ for $k = 3$.*

Proposition 4.5. *Let $M = (M^n, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^{n+1} be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. Then there exists a conormal vector for every osculating Riemannian metric restricted on the hyperplane $d_pf(T_pM)$.*

Alternative proof.

Lemma 4.6. *Assume Finsler metric satisfies condition above, then all osculating Riemannian metric G_u satisfy the condition below*

$$G_{\tilde{u}}(\tilde{v}, B(u, u)) = 0 \tag{4.0.5}$$

for any $u, v \in T_pM$.

Proof. Let (M, φ) and p defined in the same way as Theorem ... and φ_0 defined as in Definition ... . Because the statement of the theorem is local, it suffices to consider a small coordinate neighbourhood U of p. So we now assume that $M = U \subset \mathbb{R}^3$ and use $TU = U \times \mathbb{R}^3$. In this way, φ is a function of $x \in U$ and $v \in \mathbb{R}^3$.

Define $H =$Imd_pf, H is the tangent plane to $f(U)$ at $f(p)$, which can be considered as a linear subspace of V. The map d_pf is a linear isometry between $(\mathbb{R}^3, \varphi_0)$ and (H, Φ_H). Fix an isomorphism between $V\backslash H$ and $\mathbb{R}$ and denote S the second fundamental form of f at p with this isomorphism. So S is a symmetric real-valued bilinear form on $\mathbb{R}^3$ given by $S(v, w) = \pi(d_p^2 f(v, w))$ for all $v, w \in \mathbb{R}^3$ where $\pi : V \to V\backslash H \cong \mathbb{R}$ is the quotient map. We say that vectors $v, w \in \mathbb{R}^3$ are $S-$orthogonal if $S(v, w) = 0$.

Let $u \in \mathbb{R}^3$ and $\tilde{u} = d_p f(u)$. Differentiating ... with respect to x at $x = p$ in the direction $v \in \mathbb{R}^3$ and taking into account ... yields

$$d_{\tilde{u}}\Phi(d_p^2 f(u, v)) = 0 \tag{4.0.6}$$

Here $d_{\tilde{u}}\Phi$ denotes the differential of Φ at $\tilde{u}$, this differential is a linear map from V to $\mathbb{R}$, and $d_p^2 f(u, v) \in V$ is an argument of this linear map.

Differentiating ... twice with respect to x at $x = p$ in directions $v_1, v_2 \in \mathbb{R}^3$ and taking into account ... yields

$d_{\tilde{u}}\Phi(d_p^3 f(u, v_1, v_2)) + d_{\tilde{u}}^2\Phi(d_P^2 f(u, v_1), d_p^2 f(u, v_2)) = 0,$
We fix the notation $\tilde{u} = d_p f(u), \tilde{v} = d_p f(v)$.

By $d_p d_u(\varphi(u))(v_1)(v_2) = 0$, we get

$$d_{\tilde{u}}\Phi(d_p^2 f(v_1, v_2)) + d_{\tilde{u}}^2\Phi(d_p f(v_1), d_p^2 f(u, v_2)) = 0 \tag{4.0.7}$$

By substitute $v_1 = v, v_2 = u$ into (4.0.7), we get that

$$d_{\tilde{u}}\Phi(d_p^2 f(v, u)) + d_{\tilde{u}}^2\Phi(d_p f(v), d_p^2 f(u, u)) = 0 \tag{4.0.8}$$

Notice that $\frac{1}{2}\Phi^2$ also satisfies above formulas. So let us denote $\Psi = \frac{1}{2}\Phi^2$. We have

$$d_{\tilde{u}}(\Psi)(d_p^2 f(u, v)) = 0 \tag{4.0.9}$$

$$d_{\tilde{u}}(\Psi)(d_p^2 f(u, v)) + d_{\tilde{u}}^2(\Psi)(d_p f(u), d_p^2 f(u, v)) = 0 \tag{4.0.10}$$

Now we fix notation $B(u, v) = d_p^2 f(u, v)$.
Substitute formulas (2.0.1) into (4.0.9); Substitute (2.0.1) and (2.0.2) into (4.0.10) , we get

$$G_{\tilde{u}}(\tilde{u}, B(u, v)) = 0 \tag{4.0.11}$$
$$G_{\tilde{u}}(\tilde{u}, B(u, v)) + G_{\tilde{u}}(\tilde{v}, B(u, u)) = 0 \tag{4.0.12}$$

We subtract (4.0.12) by (4.0.11), we get

$$G_{\tilde{u}}(\tilde{v}, B(u,u)) = 0 \tag{4.0.13}$$

for all $v, u \in T_pM$

□

When second fundamental form is indefinite, using lemma, there exists canonical normal vector τ.

Using canonical normal vector τ, and $B(v_1, v_2) = S(v_1, v_2) \cdot \tau$, we have

$$G_u(v_1, B(v_2, v_3)) = 0. \tag{4.0.14}$$

After taking derivative, and using symmetry and property of Cartan tensor, we can get

$$G_u(B(u,u), B(v_1, v_2)) - G_u(B(u, v_1), B(u, v_2)) = 0 \tag{4.0.15}$$

So we have

$$S(u,u) \cdot S(v_1, v_2) = S(u, v_1) \cdot S(u, v_2) \tag{4.0.16}$$

$$S(v_1, v_2) = S(u, v_1) \cdot \frac{S(u, v_2)}{S(u,u)} \tag{4.0.17}$$

$$S(v_1, v_2) = C_0 \cdot L(v_1) \cdot L(v_2) \tag{4.0.18}$$

, here C_0 is constant.

Remark 4.7.
1. Local version of Banach space says that:
Theorem: Let (V^3, Φ) be a Banach-Minkowski space and Σ its unit sphere. Suppose that $\mathcal{U} \subset Gr_2(V)$ is an open set s.t. for every $H \in \mathcal{U}$ there exists a vector $\tau = \tau_H \in V \backslash \{0\}$ which is tangent to Σ at every point of $H \cap \Sigma$. Then $\Phi|_H$ is a Euclidean norm for every $H \in \mathcal{U}$.

2. Global flat Finsler metric Φ is monochromatic. Restrictions of monochromatic Finsler metric Φ to all planes from a neighbourhood of point p are isometric.

3. If second fundamental form of f is non-degenerate at $p \in M$, considering image of map $G : M \to Gr_2(V)$ induced by $G(x) = \operatorname{Im} d_x f$ for $x \in M$, second fundamental form of f is non-degenerate and image of G contains a neighbourhood of $G(p)$ in $Gr_2(V)$. So restrictions of Φ to all planes from this neighbourhood are isometric. This satisfies the condition for theorem in 1.

4. So combine 2, 3, we can get the corollary of theorem in 1 that if there exists an isometric embedding of globally flat Finsler metric M^n into $\mathbb{V}^{n+1}$, second fundamental form is degenerate at every point.

Indeed, if second fundamental form is non-degenerate, then theorem gives that this monochromatic(flat) Finsler metric has to be Euclidean norm. Because of flat metric, using result 3 in Riemannian case, second fundamental form of isometric embedding of Euclidean norm is degenerate.