ON THE SECOND FUNDAMENTAL FORM OF FINSLERIAN ISOMETRIC EMBEDDINGS

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Abstract. For a 3-dimensional Finsler manifold which is second-order flat at a point, if it can be isometrically embedded into 4-dimensional Banach-Minkowski space, we prove that its second fundamental form at this point is degenerate and has 2-dimensional nullspace. We also prove that for all cases of isometric embedding of n-dimensional Finsler manifold into $(n+1)$-dimensional Banach-Minkowski space, its second fundamental form at this point has to be semi-definite.

1. Introduction

In Riemannian geometry, Gauss' Theorema Egregium says that Riemannian curvature is an intrinsic invariant of Riemannian manifold, which is independent of ambient Euclidean space where Riemannian manifold is isometrically embedded and of how it is embedded. On the other hand, second fundamental form, which can be computed via second derivatives of the embedding map, descibes how curved the submanifold is embedded.

The beauty of differential geometry lies in the delicate connection between intrinsic geometry and extrinsic geometry. Using Gauss-Codazzi equation, we can observe an explicit connection between Riemannian curvature tensor and second fundamental form. Considering its special case, when Riemannian curvature tensor is zero, we can indicate that its second fundamental form is degenerate.

We try to find an analog in Finsler geometry. Our main results are the following:

Theorem 1. *Let $M = (M^3, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^4 be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. Then second fundamental form of f at p is degenerate and its nullspace has dimension 2.*

Theorem 2. *Let $M = (M^n, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^{n+1} be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. Then second fundamental form of f at p is semi-definite.*

In Finsler geometry, each tangent space is endowed with a norm(possibly non-symmetric) rather than an inner product. In general, there is no normal vector or normal subspace. Besides, there is no simple form of curvature tensor. The relation between curvature tensor and second fundamental form is obscure.

The first result of such an isometric embedding of 2-dimension Finsler manifold into 3-dimensional Banach-Minkowski space, was proved by Sergei Ivanov in his paper[1]. We generalize it to a higher dimension. First, we introduces several definitions which are later used in our theorem.

Definition 1.1. A *Banach-Minkowski space* (V^n, Φ) is finite n-dimensional vector space V with norm (possibly non-symmetric) $\Phi : V \to \mathbb{R}_+$, which is smooth outside 0 and Φ^2 is strictly convex.

Banach-Minkowski space is a Finslerian analog of Euclidean space. An equivalent definition is first to take a smooth convex body, then we could obtain a Banach-Minkowski norm by taking the Minkowski functional of this convex body. When this convex body is an ellipsoid, the Banach-Minkowski norm is an inner product.

Recall that Finsler manifold is a smooth manifold of which tangent space at every point is a Banach-Minkowski space. Here is the formal definition.

Definition 1.2. A *Finsler manifold* M is a smooth manifold equipped with a function $\varphi : TM \to \mathbb{R}_+$ which is smooth on $TM\backslash 0$, and at each point x, $\varphi|_{T_xM}$ is Banach-Minkowski norm.

In this paper, the main condition is that we have a Finsler metric which is second-order flat at a point p. In Riemannian geometry, it means existence of local coordinates such that second derivatives of Riemannian metric tensor g vanishes at p. It also implies that Riemannian curvature tensor at p is zero.

Definition 1.3. Let $M = (M, \varphi)$ be a Finsler manifold. A Finsler metric φ_0 is called *flat* if it is locally isometric to a Banach-Minkowski space. We say that the metric φ is *second-order flat* at a point $p \in M$ if there exists a flat Finsler metric φ_0 in a neighbourhood of p such that for $x \in M$ near p and $v \in T_xM\backslash\{0\}$, we have

$$\frac{\varphi(x,v)}{\varphi_0(x,v)} = 1 + o(|x-p|^2) \tag{1.0.1}$$

Here $|x-p|$ denotes the distance from x to p in an arbitrary local coordinate system.

As we said, in general, there is no normal vector in Finsler case. Therefore we adopt an affine invariant version of second fundamental form. First we choose an arbitrary vector in the complement of the tangent plane and then take the projection of the second derivatives of our map. We claim that our theorems is independent of the vector or projection we choose.

Definition 1.4. Let M^n be a n-dimensional Finsler manifold, V^{n+1} a $(n+1)$-dimensional Banach-Minkowski space, $f : M^n \to V^{n+1}$ an isometric embedding. We use H to define tangent plane at $f(p)$, that is $H = Imd_pf$. We choose an arbitrary vector w in the complement $V\backslash H$ and this gives a projection map $\pi : V \to V/H$. Then the *second fundamental form* S of f at a point $p \in M$ is a symmetric bilinear form defined by

$$S(v,w) = \pi(d_p^2 f(v,w)) \tag{1.0.2}$$

for all $v, w \in T_pM$.

Before we start to prove our theorem. We recall some of the known results in Riemannian geometry. They are contained in paper[4] and textbook[5].

Riemannian cases.

1. Recall that derivatives of Riemannian metric are expressed by Christoffel symbols. When there exists a local coordinate such that Christoffel symbols and their first derivatives are all zeros at point p, it implies that Riemannian curvature tensor is zero at point p.
2. A Riemannian analog of Theorem 1 has been proved for every dimension n. We reformulate it as below:

If we assume that Riemannian curvature tensor is zero at point p of Riemannian manifold M^n and there exists isometric embedding f of Riemannian manifold M^n into R^{n+m}, then second fundamental form at point p is degenerate and has nullspace of dimension $n-m$.
3. There exists isometric embedding $f : T^n \to \mathbb{R}^{2n}$, s.t. second fundamental form is non-degenerate at every point of its image. Indeed, considering the coordinate chart on torus $T^n \cong S^1 \times S^1 \times ... \times S^1$ constructed by $(\alpha_1, \alpha_2, ..., \alpha_n)$, here α_i is the angle of each circle. An example of such a map is constructed by $f : (\alpha_1, \alpha_2, ..., \alpha_n) \to \frac{\sqrt{n}}{n}(cos\alpha_1, sin\alpha_1, cos\alpha_2, sin\alpha_2, ..., cos\alpha_n, sin\alpha_n)$.
So result in 2 is sharp.

2. Notations and Preliminaries

Notations

- $M^n, \mathbb{R}^n, \mathbb{V}^n, H, U$: smooth manifold, hyperplane, open subset
- u, v, w, v_1, v_2: tangent vector in the tangent space
- $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{v_1}, \tilde{v_2}$: image of tangent vector induced by isometric embedding
- τ, λ_0: Some vectors in $\mathbb{V}^n$, not necessarily in tangent plane
- φ, Φ, Ψ, G_u: Finsler Metric, Osculating Riemannian Metric
- L(-), B(-,-), T(-,-,-) Vector-valued symmetric tensor
- $\tilde{L}$(-), $\tilde{B}$(-,-), $\tilde{T}$(-,-,-) Vector-valued symmetric tensor by projections into tangent plane
- $\hat{L}$(-), $\hat{B}$(-,-), $\hat{\tilde{B}}$(-,-), $\hat{T}$(-,-,-) Scalar symmetric tensor by projections into 1-dim linear subspace

Preliminaries

First we assume Finsler metric φ_0 is flat and Finsler metric φ is second-order flat at a point p.

Let $V = (V, \Phi)$ be a 4-dimensional Banach-Minkowski space and $f : (U, \varphi) \to V$ a smooth isometric embedding. Then by definition of isometric embedding:

$$\Phi(d_x f(v)) = \varphi(x, v) \tag{2.0.1}$$

for all $x \in U$ and $v \in \mathbb{R}^3$

By definition of flat Finsler metric, it means that there exists local coordinates such that $\varphi_0(x, v) = \varphi_0(v)$ does not depend on x. By definition 1.3, the Finsler metric φ at point p satisfies $\varphi(p, v) = \varphi_0(v)$. Then (1.0.1) implies that for every $v \in \mathbb{R}^3$, we have the following:

$$d_p(\varphi(\cdot, v)) = 0, \tag{2.0.2}$$

$$d_p^2(\varphi(\cdot, v)) = 0. \tag{2.0.3}$$

Consider that when Finsler metric φ_0 is flat, i.e. locally isometric to Banach-Minkowski space. It also implies that there exists local coordinate such that $d_v\varphi_0$ and $d_v^2\varphi_0$ on the manifold M are constant forms. This implies the following:

$$d_p d_v(\varphi(\cdot, w)) = 0, \tag{2.0.4}$$

$$d_p^2 d_v(\varphi(\cdot, w)) = 0, \tag{2.0.5}$$

$$d_p^2 d_v^2(\varphi(\cdot, w)) = 0. \tag{2.0.6}$$

Now we introduce some notations and properties in Finsler geometry.

By definition, at a point of Finsler manifold, its tangent space is equipped with a Banach-Minkowski norm. Its unit sphere is a convex body. Geometrically, for every point u of this unit sphere, there exists a unique ellipsoid tangent to the convex body at this point u. This is *osculating Riemannian metric* G_u at a direction u.

Analytically, if we assume $\Psi = \frac{1}{2}\Phi^2$, we define *osculating Riemannian metric* at direction u as

$$G_u(v, w) = \frac{\partial^2(\Psi(u + sv + tw))}{\partial s \partial t} = d_p^2(\Psi(u))(v, w) \tag{2.0.7}$$

Since Banach-Minkowski norm is 1-homogeneous, by *Euler's homogeneous theorem*, we have a property of Legendre transform

$$\begin{aligned} L_u(v) = d_p(\Psi(u))(v) &= \frac{\partial^2(\Psi(u + su + tv))}{\partial s \partial t}|_{(s,t)=(0,0)} \\ &= d_p^2(\Psi(u))(u, v) = G_u(u, v) \end{aligned} \tag{2.0.8}$$

Another important notation is *Cartan tensor*. Cartan tensor is third derivative of Finsler metric. When Cartan tensor is identically zero, Finsler metric is an inner product.

$$A_u(v_1, v_2, v_3) = \frac{1}{2}\frac{\partial^3 \Psi(u + r \cdot v_1 + s \cdot v_2 + t \cdot v_3)}{\partial r \partial s \partial t}|_{(r,s,t)=(0,0,0)} \tag{2.0.9}$$

Also, by *Euler's homogeneous theorem* , we have a property of Cartan tensor

$$A_u(u, v_1, v_2) = 0 \tag{2.0.10}$$

for any v_1, v_2.

3. Proposition

In small dimension, we could use symmetry of second and third derivative of second-order flat Finsler metric to deduce strong result. However, when we try to generalize this result to higher dimensions, the same technique does not work. In order to prove our theorem, we need to prove a proposition first.

Proposition 3.1. *$(\mathbb{R}^n, \varphi)$ normed space, $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ symmetric bilinear form, $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ vector-valued symmetric bilinear map. If for all $v, w \in \mathbb{R}^n$ such that $S(v, w) = 0$, we have that $Q(v, w)$ is tangent to the level surface of φ at v and tangent at w, then there exists $\lambda \in \mathbb{R}^n$, s.t.*

$$Q(u, v) = S(u, v) \cdot \lambda \tag{3.0.1}$$

for all $v, w \in \mathbb{R}^n$ and a fixed vector $\lambda \in \mathbb{R}^n$.

We prove that this proposition holds for dim $= 3$ and it holds for arbitrary dimension when S(-, -) is indefinite.

Proof. Analytically, the condition that $Q(v,w)$ is tangent to the level surface of φ at v and tangent at w means that

$$d_v\varphi(Q(v,w)) = d_w\varphi(Q(v,w)) = 0 \tag{3.0.2}$$

We classify our *proposition* into two cases:

(1) $S(\text{-},\text{-})$ is indefinite,
(2) $S(\text{-},\text{-})$ is semi-definite.

(1) indefinite case.

$S(\text{-},\text{-})$ is indefinite, i.e. there exists nonzero vector u, s.t. $S(u,u) = 0$.

If $S(\text{-},\text{-})$ is not identically zero, then there exists $u_1 \in \mathbb{R}^n$ such that $S(u_1,u_1) > 0$ and $S(-u_1,u_1) < 0$. Thus the set of zeros, i.e. $\{u \mid S(u,u) = 0\}$ is a codimension 1 hypersurface separating the set that has positive second fundamental form and the set that has negative second fundamental form. So there exists a linear independent vector u' such that $S(u',u') = 0$. We combine them below:

$$\begin{cases} S(u,u) = 0 \\ S(u',u') = 0 \end{cases} \tag{3.0.3}$$

Now we denote

$$\begin{cases} v(\epsilon) = u + \epsilon \cdot u' \\ w(\epsilon) = u - \epsilon \cdot u' \end{cases} \tag{3.0.4}$$

We construct a map using Legendre transform and denote it as

$$F(\epsilon) = L_{v(\epsilon)}(Q(v(\epsilon),w(\epsilon))) = L_{v(\epsilon)}(Q(u,u) - \epsilon^2 Q(u',u')) \tag{3.0.5}$$

Since we choose this special linear combinations $v(\epsilon)$ and $w(\epsilon)$ of u and u', we have

$$S(v(\epsilon),w(\epsilon)) = S(u,u) - S(u',u') = 0 \tag{3.0.6}$$

Thus using our condition, we can get that for any $\epsilon \in \mathbb{R}$

$$L_{v(\epsilon)}(Q(v(\epsilon),w(\epsilon))) = 0 \tag{3.0.7}$$

Notice that differentiation of osculating Riemannian metric gives us Cartan tensor. We take its differentiation of ϵ at $\epsilon = 0$. By (2.0.8), (3.0.29),

$$\begin{aligned} G_{v(\epsilon)}(v'(\epsilon), Q(v(\epsilon),w(\epsilon))) + G_{v(\epsilon)}(v(\epsilon), \frac{d}{d\epsilon}Q(v(\epsilon),w(\epsilon))) \\ + A_{v(\epsilon)}(v(\epsilon), Q(v(\epsilon),w(\epsilon)), v'(\epsilon)) = 0 \end{aligned} \tag{3.0.8}$$

Notice that $\frac{d}{d\epsilon}Q(v(\epsilon),w(\epsilon)) = -2\epsilon \cdot Q(u',u')$. When $\epsilon = 0$, we have

$$G_u(u', Q(u,u)) + A_u(u, Q(u,u), u') = 0 \tag{3.0.9}$$

By condition (2.0.10), Cartan tensor will vanish. So we get a property of osculating Riemannian metric at u,

$$G_u(u', Q(u,u)) = 0 \tag{3.0.10}$$

Geometrically, since the set of zeros of indefinite second fundamental form is a hypersurface rather than a linear subspace. They span the whole space $\mathbb{R}^n$. Notice that above formula is linear w.r.t. u'. So it holds for all vectors in $\mathbb{R}^n$.

Rigorously, we formulate above in lemma 3.2. Using this lemma, we get that

$$G_u(u', Q(u,u)) = 0 \tag{3.0.11}$$

for all $u' \in \mathbb{R}^n$, which also means $Q(u,u) \perp_{G_u} u'$.

Then use lemma 3.3, we get

$$Q(u,u) = S(u,u) \cdot \lambda \tag{3.0.12}$$

for all $u \in \mathbb{R}^n$.

Since $Q(\text{-},\text{-})$ is symmetric bilinear form, we have $Q(u,v) = S(u,v) \cdot \lambda$ for all $u, v \in \mathbb{R}^n$.

(2) semi-definite case.
We first denote

$$\begin{cases} x = cos\alpha \\ y = sin\alpha \end{cases} \tag{3.0.13}$$

We define rotation matrix using $O(\alpha) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$

Consider a rotation acts on the symmetric bilinear form $Q(e_1, e_2)$, that is $Q(O(\alpha)(e_1, e_2)^T)$. More explicitly,

$$Q(x \cdot e_1 - y \cdot e_2, y \cdot e_1 + x \cdot e_2) = xy \cdot (Q(e_1, e_1) - Q(e_2, e_2)) + (x^2 - y^2) \cdot Q(e_1, e_2) \tag{3.0.14}$$

Since when (e_1, e_2) map to $(e_2, -e_1)$, $Q(e_1, e_2)$ map to $Q(e_2, -e_1) = -Q(e_1, e_2)$. So by continuity of $Q(\text{-},\text{-})$ and action $O(\alpha)$, there exists α_0 and we denote image of its action as $(u_1, u_2)^T = O(\alpha_0)(e_1, e_2)^T$, such that $Q(u_1, u_2)$ lies in the hyperplane spanned by $\{e_1, e_2\}$. Using our condition, we have

$$d_{u_1}\varphi(Q(u_1, u_2)) = d_{u_2}\varphi(Q(u_1, u_2)) = 0 \tag{3.0.15}$$

This means that $Q(u_1, u_2) = 0$.

Now we change the basis of our coordinate from $\{e_1, e_2\}$ to $\{u_1, u_2\}$. We have

$$\begin{aligned} Q(x \cdot u_1 - y \cdot u_2, y \cdot u_1 + x \cdot u_2) &= xy \cdot (Q(u_1, u_1) - Q(u_2, u_2)) + (x^2 - y^2) \cdot Q(u_1, u_2) \\ &= xy \cdot (Q(u_1, u_1) - Q(u_2, u_2)) \end{aligned} \tag{3.0.16}$$

By our condition, we have $Q(u_1, u_2)$ always tangent the the level surface at u_1 and at u_2. So $Q(u_1, u_1) - Q(u_2, u_2)$ is tangent at any point of this 2-dimensional hyperplane, i.e.

$$d_w(Q(u_1, u_1) - Q(u_2, u_2)) = 0 \tag{3.0.17}$$

for all w in the 2-dimensional subspace spanned by $\{e_1, e_2\}$.

Therefore, for every 2-dimensional hyperplane, there exists a linear projector of norm 1, s.t. the convex body can be projected into the hyperplane. In the hyperplane spanned by $\{e_1, e_2\}$, its linear projector is $Q(u_1, u_1) - Q(u_2, u_2)$.

Using Blaschke-Kakutani ellipsoid characterization, then this convex body is Euclidean.

Because Euclidean sphere has unique normal vector tangent to the level surface of v and w. We denote this normal vector as λ. So we have

$$Q(u,v) = S_1(u,v) \cdot \lambda \tag{3.0.18}$$

for some symmetric bilinear form S_1.

So if $S(v,w) = 0$, then $Q(v,w) = S_1(v,w) \cdot \lambda = 0$, so $S_1(v,w) = 0$. Use the **Fact** below, we have that $S_1(v,w) = C_0 \cdot S(v,w)$. So $Q(v,w) = S(v,w) \cdot (C_0 \lambda)$

Fact: Let S and S_1 be symmetric bilinear forms on $\mathbb{R}^3$. For all linearly independent vectors $v, w \in \mathbb{R}^3$ such that $S(v,w) = 0$, we have $S_1(v,w) = 0$. Then $S_1 = \lambda \cdot S$ for some $\lambda \in \mathbb{R}$.

□

Lemma 3.2. *Let S be an nonsingular indefinite symmetric bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, g be a linear map $\mathbb{R}^3 \to \mathbb{R}$. If for all $u' \in \mathbb{R}^n$ such that $S(u', u') = 0$, we have $g(u') = 0$. Then we have*

$$g(u') = 0 \tag{3.0.19}$$

for all $u' \in \mathbb{R}^3$.

Proof. By choosing basis e_1, e_2, e_3, it suffices to assume $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

So we have $u_1 = (1,0,1), u_2 = (0,1,1), u_3 = (1,1,\sqrt{2})$ and

$$S(u_1,u_1) = 0, S(u_2,u_2) = 0, S(u_3,u_3) = 0 \tag{3.0.20}$$

So by our condition, we have

$$g(u_1) = g(u_2) = g(u_3) = 0 \tag{3.0.21}$$

We notice that $(u_1,u_2,u_3)^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & \sqrt{2} \end{pmatrix}$ is a rank 3 matrix.

So they form a basis, that is, for any vector $u' \in \mathbb{R}^3$, there exists $a_1, a_2, a_3 \in \mathbb{R}$, such that $u' = a_1 u_1 + a_2 u_2 + a_3 u_3$. Then we have

$$g(u') = g(a_1 u_1 + a_2 u_2 + a_3 u_3) = 0 \tag{3.0.22}$$

□

Lemma 3.3. *In $dim = 2, 3$ case, if for a symmetric bilinear form S_1, $S(u,u) = 0$ implies $S_1(u,u) = 0$, then there exist a constant c such that $S_1(u,u) = c \cdot S(u,u)$.*

Proof. We first prove for $dim = 2$. Then we apply this result to $dim = 3$ to simply the proof.

(1) case 1: $dim = 2$

Notice that these two symmetric bilinear forms corresponds to two symmetric matrices. Our condition says that one symmetric bilinear form S is zero can imply another form S_1 is zero.

Our idea is that we need to choose an basis such that all diagonal elements are zero and there is only one free variable in our matrix. This implies that two second fundamental forms differ by multiplying a constant.

Here we denote S, S_1 using their matrix forms. By choosing a basis e_1, e_2, we get

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.0.23}$$

i.e. $S(e_1, e_1) = 1, S(e_1, e_2) = 0, S(e_2, e_2) = -1$.
Notice that $S(e_1 - e_2, e_1 - e_2) = 0, S(e_1 + e_2, e_1 + e_2) = 0$, after changing the basis into $\{e_1 - e_2, e_1 + e_2\}$, we have $S(e_1 - e_2, e_1 + e_2) = S(e_1, e_1) - S(e_2, e_2) = 2$. Thus we get

$$S = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

By our condition, we get that $S_1(e_1 - e_2, e_1 - e_2) = 0, S_1(e_1 + e_2, e_1 + e_2) = 0$. So there is only one free variable in our matrix S_1, that means under the same basis, we can express S_1 as

$$S_1 = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_0 & 0 \end{pmatrix} \tag{3.0.25}$$

for some constant λ_0

Now we can choose constant $c = \dfrac{\lambda_0}{2}$. So we have

$$S_1(u, u) = c \cdot S(u, u) \tag{3.0.26}$$

for all u.

Notice that now we know $S_1(u, u) = c \cdot S(u, u)$, we could choose $c = \frac{S_1(e_2, e_2)}{S(e_2, e_2)}$

(2)case 2: $dim = 3$

In this case, we don't need to change basis. We apply above result and use one special vector which vanishes the second fundamental form.

It suffices to choose a basis $\{e_1, e_2, e_3\}$, such that $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Notice that there are two indefinite submatrices of rank 2. That is

$$\begin{pmatrix} 1 & * & 0 \\ * & * & * \\ 0 & * & -1 \end{pmatrix} \text{ and } \begin{pmatrix} * & * & * \\ * & 1 & 0 \\ * & 0 & -1 \end{pmatrix}$$

Also notice that we could choose constant $c = \frac{S_1(e_3, e_3)}{S(e_3, e_3)}$ such that these two pairs of indefinite submatrices from S and from S_1 share the same constant c.

By restricting on those submatrices and apply result in $dim = 2$, we get that

$$S_1 = \begin{pmatrix} c & \lambda & 0 \\ \lambda & c & 0 \\ 0 & 0 & -c \end{pmatrix} \tag{3.0.27}$$

Since $S(1, 1, \sqrt{2}) = 1+1-2 = 0$, so by our condition $S_1(1, 1, \sqrt{2}) = 1+1-2+2\lambda = 0$. We get $\lambda = 0$.
So under the same basis $\{e_1, e_2, e_3\}$, we have

$$S_1 = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{pmatrix} \tag{3.0.28}$$

It means that $S_1(u,u) = c \cdot S(u,u)$.

$\square$

Lemma 3.4. *Let S be an indefinite nonsingular symmetric bilinear form in $\mathbb{R}^3$ and let B be a vector-valued symmetric bilinear map defined as $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$. If $S(u,u) = 0$ can imply $B(u,u) = 0$ for all $\{u \mid S(u,u) = 0\}$, then*

$$B(u,u) = S(u,u) \cdot \tau \tag{3.0.29}$$

for a fixed $\tau \in \mathbb{R}^4$

Proof. We denote $B(v_1,v_2) = (B_1(v_1,v_2), B_2(v_1,v_2), B_3(v_1,v_2), B_4(v_1,v_2))$ and each $B_i(v_1,v_2)$ is a symmetric bilinear form, so if $S(u,u) = 0$, then $B_i(u,u) = 0$. Now we can use our previous lemma 3.3 for these four elements.

From lemma 3.3, we know that $B_i(u,u) = c_i \cdot B(u,u)$ for some constant c_i. So $B(u,u) = S(u,u) \cdot (c_1, c_2, ..., c_n)$. Here denote $\tau = (c_1, c_2, ..., c_n)$, we get $B(u,u) = S(u,u) \cdot \tau$.

$\square$

Noticing that using $B(u,v) = \frac{1}{2}(B(u+v,u+v) - B(u-v,u-v))$, if for all u we have $B(u,u) = S(u,u) \cdot \tau$, we could get $B(u,v) = S(u,v) \cdot \tau$ for all u, v.

4. Proof of Theorem

Suppose there exists a vector tangent at every point of cross-section of codimension 1 hyperplane with the convex body. Such a vector also suffices the condition of linear projector for Blaschke-Kakutani theorem. To define it, we introduce a term called *conormal vector*.

Definition 4.1. We say that there exists *conormal vector*, if for an isometric embedding of M^n into V^{n+1}, there exists a fixed vector in V^{n+1} for all osculating Riemannian metric of Finsler metric restricted on the tangent plane $d_p f(T_p M)$.

Here we assume all maps f, Finsler manifold M^n and Finsler metric Φ satisfy the condition in our theorem. Now we can prove our theorem.

Proof. The idea is first to use condition of second-order flatness by taking differentiation of the norm. Then in order to solve the technical problems, we use projections of second derivative and third derivative as the condition for our proposition 3.1.

From Section 2 of *Preliminary*, we have

$$\Phi(d_x f(v)) = \varphi(x,v) \tag{4.0.1}$$

$$d_p(\varphi(\cdot,v)) = 0, \tag{4.0.2}$$

$$d_p^2(\varphi(\cdot,v)) = 0. \tag{4.0.3}$$

By applying condition of isometric embedding, which is the first formula (4.0.1) into condition of first-order flat, which is the second formula (4.0.2). It gives that

$$d_v\Phi(d_p^2 f(v,w)) = 0 \tag{4.0.4}$$

for all v, w.

We separate our proof into two steps. The first step is to prove that using condition of first-order flat, there exists a conormal vector. The second step is to prove that when Finsler metric is second-order flat, second fundamental form is degenerate.

(1). Step 1:

Our second formula (4.0.2) means symmetric bilinear form $d_p^2 f(v,w)$ is tangent to the level surface of Φ at v and at w, of which projection into tangent plane satisfies the condition of Proposition 3.1.

In order to use our proposition, we need to define projection map and affine invariant version of second fundamental form first.

We choose a vector τ_0 in the complement of tangent plane, and we can define projection map $P : V \to Imd_p f$ and quotient map $\pi : V \to V/Imd_p f$ by its decomposition along τ_0, that is, for all $u' \in V$

$$u' = \pi(u')\cdot\tau_0 + P(u') \tag{4.0.5}$$

We define second fundamental form as $S(u,v) = \pi(d_p^2 f(u,v))$.

Notice that $Q = P\circ d_p^2 f$ is a symmetric bilinear form which satisfies our condition of Proposition 3.1. Because second fundamental form is indefinite, from result of proposition 3.1, we have

$$Q(u,v) = S(u,v)\cdot\lambda \tag{4.0.6}$$

with λ in the tangent plane $d_p f(T_pM)$.

Combine formulas in (4.0.5), (4.0.6), we can explicitly express second derivative.

$$\begin{aligned} d_p^2 f(u,v) &= \pi(d_p^2 f(u,v))\cdot\tau_0 + Q(u,v) \\ &= S(u,v)\cdot\tau_0 + S(u,v)\cdot\lambda \\ &= S(u,v)\cdot(\tau_0+\lambda) \end{aligned} \tag{4.0.7}$$

Here in formula (4.0.5) $u' = d_p^2 f(u,v)$.

Let $\tau = \tau_0 + \lambda$, we have $d_p^2 f(u,v) = S(u,v)\cdot\tau$. From condition that second derivative is tangent to the level surface of Φ, we get that τ is tangent to the unit convex body at every point of tangent plane, i.e. τ is conormal vector. Explicitly,

$$d_v\Phi(d_p^2 f(u,v)) = S(u,v)\cdot d_v\Phi(\tau) = 0 \tag{4.0.8}$$

When S is not identically zero, set of vectors which makes second fundmental form nonvanishing is dense. So by continuity, we have

$$d_v\Phi(\tau) = 0 \tag{4.0.9}$$

for all v.

(2) Step 2:

Using the third formula from second-order flat condition, we have

$$d_v\Phi(d_p^3 f(v,w,w_1)) + d_v^2\Phi(d_p^2 f(v,w), d_p^2 f(v,w_1)) = 0 \tag{4.0.10}$$

Now we can second derivative is parallel to conormal vector. So

$$d_v\Phi(d_p^3 f(v,w,w_1)) + S(v,w)S(v,w_1)\cdot d_v^2\Phi(\tau,\tau) = 0 \tag{4.0.11}$$

When $S(v,w)=0$, we have $d_v\Phi(d_p^3 f(v,w,w_1))=0$.

We define a new projection P_τ as projection along conormal vector τ into tangent plane. Fix a vector $w_1 \in T_pM$, let $Q' = P_\tau \circ d_p^3 f(\text{-},\text{-},w_1)$. By property (4.0.9) of conormal vector, this also implies that $d_v\Phi(Q'(v,w))=0$.

Thus now it satisfies the assumption of Proposition 3.1. From result of Proposition 3.1, we have

$$Q'(v,w) = S(v,w)\cdot\lambda_{w_1} \tag{4.0.12}$$

λ_{w_1} is dependent on vector w_1. So we rewrite it as $\lambda(w_1)=\lambda_{w_1}$

Now we have $T(u,v,w) = S(u,v)\cdot\lambda(w)$.
Using lemma 4.2 below, we get that

$$S(u,v) = C_0\cdot L(u)\cdot L(v) \tag{4.0.13}$$

for some constant C_0.
Thus we can express the third derivative as

$$T(u,v,w) = C_1\cdot L(u)\cdot L(v)\cdot L(w)\cdot\lambda_1 \tag{4.0.14}$$

for some constant C_1 and some constant vector λ_1.

Notice that 1-form $L:\mathbb{R}^3\to\mathbb{R}$ has nullspace of dimension 2 and second fundamental form S has the same nullspace as L. So second fundamental form S is degenrate and has 2-dimensional nullspace.

□

Lemma 4.2. *If a 3-linear vector-valued symmetric form $T:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies $T(u,v,w)=S(u,v)\cdot\lambda(w)$, here $S:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is a symmetric bilinear form, $\lambda:\mathbb{R}^n\to\mathbb{R}^n$ is a vector-valued 1-form, then there exists 1-form $L:\mathbb{R}^n\to\mathbb{R}$, s.t. $S(u,v)=L(u)\cdot L(v)$ and $T(u,v,w)=L(u)\cdot L(v)\cdot L(w)\cdot\lambda_0$ for a fixed vector λ_0.*

Proof. First we claim a fact.

Fact: If

$$\begin{cases} S(u,v)\neq 0 \\ S(u,w)\neq 0 \end{cases} \tag{4.0.15}$$

Then from symmetry $T(u,v,w)=S(u,v)\cdot\lambda(w)=S(u,w)\cdot\lambda(v)$, we know $\lambda(v)//\lambda(w)$.

For a vector u not in the nullspace, the set $\{v \mid S(u,v)=0\}$ is closed and is nowhere dense when S is not identically zero. So the set $\{v \mid S(u,v)\neq 0\}$ is dense. It means that the assumption of **Fact** is satisfied.

So for almost all v,w, $\lambda(v)//\lambda(w)$. By continuity, it holds for all v,w. So there exists 1-dimensional subspace l, s.t. for all v, $\lambda(v)\in l$.

We denote λ_0 for a nonzero vector from l, so $\lambda(v)=L(v)\cdot\lambda_0$. Here $L:\mathbb{R}^n\to\mathbb{R}$ is a function.

Then we have $T(u,v,w)=S(u,v)\cdot L(w)\cdot\lambda_0 = S(u,w)\cdot L(v)\cdot\lambda_0$

So $S(u,v)=\frac{S(u,w)}{L_0(w)}\cdot L(v) = L(u)\cdot\frac{S(v,w)}{L(w)}$

Since S is symmetric bilinear form, then $S(u,v) = L(u) \cdot L(v)$.
So $T(u,v,w) = L(u) \cdot L(v) \cdot L(w) \cdot \lambda_0$

□

From above, we have proved some extra properties when the Finsler metric is second-order flat at a point.

Proposition 4.3. *Let $M = (M^n, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^{n+1} be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. Then $B(u,v) = L(u) \cdot L(v) \cdot \tau$ for $k = 2$ and $T(u,v,w) = L(u) \cdot L(v) \cdot L(w) \cdot \lambda$ for $k = 3$.*

Proposition 4.4. *Let $M = (M^n, \varphi)$ be a Finsler manifold whose metric is second-order flat at a point $p \in M$. Let V^{n+1} be a Banach-Minkowski space and $f : M \to V$ a smooth isometric embedding. Then there exists a conormal vector for every osculating Riemannian metric restricted on the hyperplane $d_p f(T_p M)$.*

Alternative proof of indefinite case.

Lemma 4.5. *Assume Finsler metric satisfies condition above, then all osculating Riemannian metric G_u satisfy the condition below*

$$G_{\tilde{u}}(\tilde{v}, B(u,u)) = 0 \tag{4.0.16}$$

for any $u, v \in T_p M$.

Proof. Recall that in Section 2 of Preliminary, from our condition, we have several formulas:

$$d_{\tilde{u}}\Phi(d_p^2 f(u,v)) = 0 \tag{4.0.17}$$

$$d_{\tilde{u}}\Phi(d_p^3 f(u,v_1,v_2)) + d_{\tilde{u}}^2\Phi(d_P^2 f(u,v_1), d_p^2 f(u,v_2)) = 0 \tag{4.0.18}$$

We fix the notation $\tilde{u} = d_p f(u), \tilde{v} = d_p f(v)$.

By $d_p d_u(\varphi(u))(v_1)(v_2) = 0$, we get

$$d_{\tilde{u}}\Phi(d_p^2 f(v_1,v_2)) + d_{\tilde{u}}^2\Phi(d_p f(v_1), d_p^2 f(u,v_2)) = 0 \tag{4.0.19}$$

By substitute $v_1 = v, v_2 = u$ into (4.0.19), we get that

$$d_{\tilde{u}}\Phi(d_p^2 f(v,u)) + d_{\tilde{u}}^2\Phi(d_p f(v), d_p^2 f(u,u)) = 0 \tag{4.0.20}$$

Notice that $\frac{1}{2}\Phi^2$ also satisfies above formulas. So let us denote $\Psi = \frac{1}{2}\Phi^2$. We have

$$d_{\tilde{u}}(\Psi)(d_p^2 f(u,v)) = 0 \tag{4.0.21}$$

$$d_{\tilde{u}}(\Psi)(d_p^2 f(u,v)) + d_{\tilde{u}}^2(\Psi)(d_p f(u), d_p^2 f(u,v)) = 0 \tag{4.0.22}$$

Now we fix notation $B(u,v) = d_p^2 f(u,v)$.
Substitute formulas (2.0.7) into (4.0.21); Substitute (2.0.7) and (2.0.8) into (4.0.22) , we get

$$G_{\tilde{u}}(\tilde{u}, B(u,v)) = 0 \tag{4.0.23}$$

$$G_{\tilde{u}}(\tilde{u}, B(u,v)) + G_{\tilde{u}}(\tilde{v}, B(u,u)) = 0 \tag{4.0.24}$$

We subtract (4.0.24) by (4.0.23), we get

$$G_{\tilde{u}}(\tilde{v}, B(u,u)) = 0 \tag{4.0.25}$$

for all $v, u \in T_pM$

□

Proof of indefinite case:
When second fundamental form is indefinite, using lemma, there exists canonical normal vector τ.

Using canonical normal vector τ, and $B(v_1, v_2) = S(v_1, v_2) \cdot \tau$, we have

$$G_u(v_1, B(v_2, v_3)) = 0. \tag{4.0.26}$$

After taking derivative, and using symmetry and property of Cartan tensor, we can get

$$G_u(B(u,u), B(v_1, v_2)) - G_u(B(u, v_1), B(u, v_2)) = 0 \tag{4.0.27}$$

So we have

$$S(u,u) \cdot S(v_1, v_2) = S(u, v_1) \cdot S(u, v_2) \tag{4.0.28}$$

$$S(v_1, v_2) = S(u, v_1) \cdot \frac{S(u, v_2)}{S(u,u)} \tag{4.0.29}$$

$$S(v_1, v_2) = C_0 \cdot L(v_1) \cdot L(v_2) \tag{4.0.30}$$

, here C_0 is constant.

Remark 4.6.
1. Local version of Banach space says that:
Theorem: Let (V^3, Φ) be a Banach-Minkowski space and Σ its unit sphere. Suppose that $\mathcal{U} \subset Gr_2(V)$ is an open set s.t. for every $H \in \mathcal{U}$ there exists a vector $\tau = \tau_H \in V \backslash \{0\}$ which is tangent to Σ at every point of $H \cap \Sigma$. Then $\Phi|_H$ is a Euclidean norm for every $H \in \mathcal{U}$.

2. Global flat Finsler metric Φ is monochromatic. Restrictions of monochromatic Finsler metric Φ to all planes from a neighbourhood of point p are isometric.

3. If second fundamental form of f is non-degenerate at $p \in M$, considering image

of map $G : M \to Gr_2(V)$ induced by $G(x) = \text{ Im } d_x f$ for $x \in M$, second fundamental form of f is non-degenerate and image of G contains a neighbourhood of $G(p)$ in $Gr_2(V)$. So restrictions of Φ to all planes from this neighbourhood are isometric. This satisfies the condition for theorem in 1.

4. So combine 2, 3, we can get the corollary of theorem in 1 that if there exists an isometric embedding of globally flat Finsler metric M^n into $\mathbb{V}^{n+1}$, second fundamental form is degenerate at every point.
Indeed, if second fundamental form is non-degenerate, then theorem gives that this monochromatic(flat) Finsler metric has to be Euclidean norm. Because of flat metric, using result 3 in Riemannian case, second fundamental form of isometric embedding of Euclidean norm is degenerate.

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