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Adaptive Control for Passivity and Synchronization of Coupled Reaction-Diffusion Neural Networks with Multiple State Couplings

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Abstract: In this paper, a coupled reaction-diffusion neural networks with multiple state couplings is presented. By selecting appropriate adaptive control scheme and employing inequality techniques, several passivity conditions for the proposed network model are given. In addition, a sufficient condition for ensuring the synchronization of the proposed network model is also established by exploiting the output-strictly passivity. Finally, we give a numerical example to verify the effectiveness of the derived criteria.

Key Words: Passivity, Synchronization, Coupled Reaction-Diffusion Neural Networks, Multiple State Couplings

1 Introduction

Recently, the dynamical behaviors of neural networks (NNs) have been widely studied in various fields due to their diversity of applications in optimization, pattern classification, image processing, signal processing, etc.. As is well known, the phenomenon of reaction diffusion is caused by the movement of electrons in a nonuniform electromagnetic field, which is unavoidable [1, 2]. Therefore, it is very important to consider the reaction-diffusion phenomenon in the neural networks. Until now, the dynamical behaviors of reaction-diffusion neural networks (RDNNs) have been studied extensively and deeply [3–5]. With the help of some inequality techniques and Lyapunov functional method, Wang et al. [3] not only analyzed the passivity of the RDNNs, but also considered the robust passivity under the circumstance that the parameter uncertainties appear in RDNNs. In [4], the robust passivity and passivity criteria were derived for stochastic RDNNs with time-varying delays by employing stochastic analysis and inequality techniques. In [5], the global exponential stability of RDNNs with time-varying delays was researched by exploiting the Wirtinger's inequality and a diffusion-dependent Lyapunov functional.

More recently, coupled RDNNs (CRDNNs) consisting of several identical or nonidentical RDNNs has attracted much attention in different research areas, such as harmonic oscillation generation, pattern recognition, and chaotic generator design. Therefore, the dynamical behaviors of CRDNNs have been extensively studied, in particular the synchronization [6, 7] and passivity [8, 9]. Wang et al. [6] respectively discussed the pinning synchronization for state coupled and spatial diffusion coupled CRDNNs, and several synchronization criteria for these networks were derived by exploiting several inequality techniques and Lyapunov functional method. In [7], the synchronization problem for CRDNNs with time delays was studied by exploiting the adaptive feedback control method, and the authors presented several sufficient conditions to ensure the synchronization based on the LaSalle invariant principle. In [8], the authors considered the passivity of the CRDNNs with switching topology, several sufficient conditions were given to guarantee the output-strictly passivity and input-strictly passivity by using the Lyapunov functional method and inequality techniques. In [9], Huang et al. proposed a nonlinear CRDNNs, and the pinning passivity and passivity problems of such network model were studied. Unfortunately, in these existing results on the passivity of the CRDNNs, the input and output were required to have the same dimension [8, 9]. As far as we know, passivity of CRDNNs with different input and output dimensions was seldom studied [10, 11]. Wang et al. [10] discussed the passivity for two types of CRDNNs with different dimensions of input and output, and analyzed the internal stability of the CRDNNs. In [11], some passivity criteria for the CRDNNs with undirected and directed topologies were gained based on the designed adaptive laws, and the authors also studied the relationship between the output strict passivity and synchronization.

However, in the above-mentioned works about the dynamical behaviors of CRDNNs are all based upon the single weighted network models [6–11]. Actually, many networks in the real life should be modeled by multi-weights models of complex network, examples are social networks, public traffic roads networks, etc. [12]. The dynamical behaviors for complex networks with multiple weights have been studied by some authors in recent years [13, 14]. In [13], the authors gave some passivity criteria for complex networks with multi-weights by employing Lyapunov functional method and inequality techniques, and designed the pinning control strategy based on nodes and edges. Qin et al. [14] discussed the H_∞ synchronization and synchronization of the complex dynamical networks with multi-weights for switching and fixed topologies, and presented several H_∞ synchronization and synchronization criteria by employing Lyapunov functional method and inequality techniques. Regretfully, very few authors have considered the dynamical behaviors about the CRDNNs with multi-weights [15]. In [15], Wang et al. discussed the finite-time synchronization and finite-time passivity problems for CRDNNs with multiple state couplings and multiple delayed state couplings, and several sufficient conditions for ensuring the finite-time passivity and synchronization were given by selecting appropriate controllers. Therefore, it is very meaningful to further study the passivity and synchronization for CRDNNs with multiple state couplings.

In this paper, we respectively study the passivity and synchronization of CRDNNs with multiple state couplings. First, several passivity criteria for multiple state coupled CRDNNs are obtained by employing some inequality techniques and the adaptive state feedback controller. Second, we investigate the relationship between passivity and synchronization for CRDNNs with multiple state couplings, and a synchronization criterion is established by exploiting the obtained output strictly passivity result.

Notations: $\mathbb{R}^{\epsilon} \ni \Omega = \{a = (a_1, a_2, \cdots, a_\epsilon)^T | \ |a_h| < c_h, h = 1, 2, \cdots, \epsilon\}$. $\gamma_m(\cdot)$ and $\gamma_M(\cdot)$ denote the minimum and the maximum eigenvalue of the corresponding matrix. $\mathcal{D} = \{1, 2, \cdots, Z\}$ represents node set, $\mathcal{B} \subseteq \mathcal{D} \times \mathcal{D}$ represents undirected edge set. $\mathcal{Z}_p = \{q \in \mathcal{D} : (p, q) \in \mathcal{B}\}$.

2 Preliminaries

2.1 Definitions

Definition 2.1. (see [10]) If there exists a non-negative function W and a constant matrix $H \in \mathbb{R}^{\beta \times \zeta}$ such that

$$\int_{t_0}^{t_r} \int_{\Omega} y^T(a,t) H u(a,t) da dt \geqslant W(t_r) - W(t_0)$$

for any $t_r, t_0 \in \mathbb{R}^+$ and $t_r \geqslant t_0$, the system with input $u(a,t) \in \mathbb{R}^{\zeta}$ and output $y(a,t) \in \mathbb{R}^{\beta}$ is passive.

Definition 2.2. (see [10]) If there exists a non-negative function W and two constant matrices $\mathbb{R}^{\zeta \times \zeta} \ni Q_1 > 0$ and $H \in \mathbb{R}^{\beta \times \zeta}$ such that

$$\begin{aligned} & \int_{t_0}^{t_r} \int_{\Omega} y^T(a,t) H u(a,t) da dt \\ \geqslant \ & W(t_r) - W(t_0) + \int_{t_0}^{t_r} \int_{\Omega} u^T(a,t) Q_1 u(a,t) da dt \end{aligned}$$

for any $t_r, t_0 \in \mathbb{R}^+$ and $t_r \geqslant t_0$, the system is input-strictly passive.

Definition 2.3. (see [10]) If there exists a non-negative function W and two constant matrices $\mathbb{R}^{\beta \times \beta} \ni Q_2 > 0$ and $H \in \mathbb{R}^{\beta \times \zeta}$ such that

$$\begin{aligned} & \int_{t_0}^{t_r} \int_{\Omega} y^T(a,t) H u(a,t) da dt \\ \geqslant \ & W(t_r) - W(t_0) + \int_{t_0}^{t_r} \int_{\Omega} y^T(a,t) Q_2 y(a,t) da dt \end{aligned}$$

for any $t_r, t_0 \in \mathbb{R}^+$ and $t_r \geqslant t_0$, the system is output-strictly passive.

2.2 Lemma

Lemma 2.1. (see [16]) The following inequality obviously holds:

$$\int_{\Omega} g^2(a) da \leqslant c_h^2 \int_{\Omega} \left(\frac{\partial g(a)}{\partial a_h} \right)^2 da, \quad h = 1, 2, \cdots, \epsilon,$$

where $C^1(\Omega) \ \ni \ g(a)$ is a real-valued function and $g(a)|_{\partial\Omega = 0}$.

3 Adaptive passivity and synchronization of CRDNNs with multiple state couplings

3.1 Network model

The network model considered in this section is described by:

$$\begin{aligned} \frac{\partial \chi_p(a,t)}{\partial t} \ = \ & M \triangle \chi_p(a,t) - A \chi_p(a,t) + E f(\chi_p(a,t)) \\ & + K u_p(a,t) + \sum_{s=1}^{d} \sum_{q=1}^{Z} b_s B_{pq}^s \Gamma^s \chi_q(a,t) \\ & + J + v_p(a,t), p = 1, 2, \cdots, Z, \end{aligned} \quad (1)$$

where $\chi_p(a,t) = (\chi_{p1}(a,t), \chi_{p2}(a,t), \cdots, \chi_{pz}(a,t))^T \in \mathbb{R}^z$ is the state vector of node p; $u_p(a,t) \in \mathbb{R}^{\eta}$ is the external input of node p; $v_p(a,t) \in \mathbb{R}^z$ is the control input of node p; $0 < M = \operatorname{diag}(m_1, m_2, \cdots, m_z) \in \mathbb{R}^{z \times z}$; $0 < A = \operatorname{diag}(\Lambda_1, \Lambda_2, \cdots, \Lambda_z) \in \mathbb{R}^{z \times z}$; $\triangle = \sum_{h=1}^{\epsilon} (\partial^2 / \partial a_h^2)$; $f(\chi_p(a,t)) = (f_1(\chi_{p1}(a,t)), f_2(\chi_{p2}(a,t)), \cdots, f_z(\chi_{pz}(a,t)))^T \in \mathbb{R}^z$; $J = (J_1, J_2, \cdots, J_z)^T \in \mathbb{R}^z$; $E \in \mathbb{R}^{z \times z}$ and $K \in \mathbb{R}^{z \times \eta}$ are two known matrices; $0 < b_s \in \mathbb{R}$ $(s = 1, 2, \cdots, d)$ represents the coupling strength; $0 < \Gamma^s \in \mathbb{R}^{z \times z} (s = 1, 2, \cdots, d)$ denotes the inner coupling relationship; $B^s = (B_{pq}^s)_{Z \times Z} \in \mathbb{R}^{Z \times Z} (s = 1, 2, \cdots, d)$ is the outer coupling matrix, where B_{pq}^s is defined as follows: if there is an edge between node p and node $q (p \neq q)$, then $B_{pq}^s = B_{qp}^s > 0$; otherwise, $B_{pq}^s = B_{qp}^s = 0$; and $B_{pp}^s = -\sum_{\substack{q=1 \\ q \neq p}}^{Z} B_{pq}^s$.

The boundary value and initial value for network (1) are given as follows:

$$\begin{aligned} \chi_p(a,t) \ &= \ 0, \qquad (a,t) \in \partial\Omega \times [0, +\infty), \\ \chi_p(a,0) \ &= \ \varphi_p(a), \quad a \in \Omega, \end{aligned}$$

in which $\varphi_p(a) (p = 1, 2, \cdots, z)$ is continuous on Ω.

In this paper, the network (1) is connected and the function $f_p(\cdot) (p = 1, 2, \cdots, z)$ satisfies the following condition:

$$|f_p(\theta_1) - f_p(\theta_2)| \leqslant \mathcal{Q}_p |\theta_1 - \theta_2|$$

for any $\theta_1, \theta_2 \in \mathbb{R}$, where $\mathcal{Q}_p$ is a positive constant.

Defining $\bar{\chi}(a,t) = \frac{1}{Z} \sum_{p=1}^{Z} \chi_p(a,t)$, one gets

$$\begin{aligned} \frac{\partial \bar{\chi}(a,t)}{\partial t} \ = \ & M \triangle \bar{\chi}(a,t) - A \bar{\chi}(a,t) + \frac{1}{Z} \sum_{p=1}^{Z} K u_p(a,t) \\ & + \frac{1}{Z} \sum_{p=1}^{Z} E f(\chi_p(a,t)) + J + \frac{1}{Z} \sum_{p=1}^{Z} v_p(a,t). \end{aligned}$$

Selecting $e_p(a,t) = \chi_p(a,t) - \bar{\chi}(a,t)$, one obtains

$$\begin{aligned} \frac{\partial e_p(a,t)}{\partial t} \ = \ & M \triangle e_p(a,t) - A e_p(a,t) + E f(\chi_p(a,t)) \\ & - \frac{1}{Z} \sum_{l=1}^{Z} E f(\chi_l(a,t)) - \frac{1}{Z} \sum_{l=1}^{Z} K u_l(a,t) \\ & + K u_p(a,t) + \sum_{s=1}^{d} \sum_{q=1}^{Z} b_s B_{pq}^s \Gamma^s e_q(a,t) \end{aligned}$$

$$+v_p(a,t)-\frac{1}{Z}\sum_{l=1}^{Z}v_l(a,t). \qquad (2)$$

Choose the following output vector $y_p(a,t)\in\mathbb{R}^{\zeta}$ for the network (2):

$$y_p(a,t)=P_1e_p(a,t)+P_2u_p(a,t), \qquad (3)$$

where $P_1\in\mathbb{R}^{\zeta\times z}$ and $P_2\in\mathbb{R}^{\zeta\times\eta}$.

Take

$$\begin{aligned}u(a,t)&=(u_1^T(a,t),u_2^T(a,t),\cdots,u_Z^T(a,t))^T,\\y(a,t)&=(y_1^T(a,t),y_2^T(a,t),\cdots,y_Z^T(a,t))^T,\\e(a,t)&=(e_1^T(a,t),e_2^T(a,t),\cdots,e_Z^T(a,t))^T.\end{aligned}$$

Design the following adaptive state feedback controller for network (1):

$$\begin{aligned}v_p(a,t)&=\sum_{s=1}^{d}\sum_{q=1}^{Z}b_sG_{pq}^s(t)\Gamma^s\chi_q(a,t),\\\dot{G}_{pq}^s(t)&=\alpha_{pq}^s\int_\Omega(\chi_p(a,t)-\chi_q(a,t))^T\Gamma^s(\chi_p(a,t)\\&\quad-\chi_q(a,t))da,(p,q)\in\mathcal{B}, \qquad (4)\end{aligned}$$

where $p=1,2,\cdots,Z$; if $q\in\mathcal{Z}_p$, then $\mathbb{R}\ni\alpha_{pq}^s=\alpha_{qp}^s>0$ and $\mathbb{R}\ni G_{pq}^s(t)=G_{qp}^s(t)>0$; otherwise, $\mathbb{R}\ni\alpha_{pq}^s=\alpha_{qp}^s=0$ and $\mathbb{R}\ni G_{pq}^s(t)=G_{qp}^s(t)=0$; and $G_{pp}^s(t)=-\sum_{\substack{q=1\\q\neq p}}^{Z}G_{pq}^s(t)$.

3.2 Passivity criteria

Theorem 3.1. If there is a matrix $H\in\mathbb{R}^{\zeta Z\times\eta Z}$ satisfying

$$(I_Z\otimes P_2)^TH+H^T(I_Z\otimes P_2)>0,$$

the network (2) is passive.

Proof. Define the following Lyapunov functional for the network (2):

$$\begin{aligned}V_1(t)&=\sum_{p=1}^{Z}\int_\Omega e_p^T(a,t)e_p(a,t)da\\&\quad+\sum_{s=1}^{d}\sum_{p=1}^{Z}\sum_{q\in\mathcal{Z}_p}\frac{b_s}{2\alpha_{pq}^s}(G_{pq}^s(t)-\delta_{pq}^s)^2, \qquad (5)\end{aligned}$$

where $\delta_{pq}^s=\delta_{qp}^s(p\neq q)$ are non-negative constants, and $\delta_{pq}^s=0(p\neq q)$ if and only if $G_{pq}^s(t)=0$.

Then, we acquire

$$\begin{aligned}\dot{V}_1(t)&=2\sum_{p=1}^{Z}\int_\Omega e_p^T(a,t)\Big[M\triangle e_p(a,t)-Ae_p(a,t)\\&\quad+\sum_{s=1}^{d}\sum_{q=1}^{Z}b_sB_{pq}^s\Gamma^se_q(a,t)+Ku_p(a,t)\\&\quad-\frac{1}{Z}\sum_{l=1}^{Z}Ku_l(a,t)+Ef(\chi_p(a,t))\\&\quad-Ef(\bar{\chi}(a,t))+Ef(\bar{\chi}(a,t))+v_p(a,t)\\&\quad-\frac{1}{Z}\sum_{l=1}^{Z}Ef(\chi_l(a,t))-\frac{1}{Z}\sum_{l=1}^{Z}v_l(a,t)\Big]da\\&\quad+\sum_{s=1}^{d}\sum_{p=1}^{Z}\sum_{q\in\mathcal{Z}_p}b_s(G_{pq}^s(t)-\delta_{pq}^s)\int_\Omega(\chi_p(a,t)\\&\quad-\chi_q(a,t))^T\Gamma^s(\chi_p(a,t)-\chi_q(a,t))da. \qquad (6)\end{aligned}$$

Obviously,

$$\begin{aligned}\sum_{p=1}^{Z}e_p(a,t)&=\sum_{p=1}^{Z}\Big(\chi_p(a,t)-\bar{\chi}(a,t)\Big)\\&=\sum_{p=1}^{Z}\chi_p(a,t)-\sum_{q=1}^{Z}\chi_q(a,t)\\&=0.\end{aligned}$$

Then, we can get

$$\begin{aligned}\sum_{p=1}^{Z}e_p^T(a,t)\Big(Ef(\bar{\chi}(a,t))-\frac{1}{Z}\sum_{l=1}^{Z}Ef(\chi_l(a,t))\Big)&=0,\\\sum_{p=1}^{Z}e_p^T(a,t)\Big(\sum_{l=1}^{Z}Ku_l(a,t)\Big)&=0,\\\sum_{p=1}^{Z}e_p^T(a,t)\Big(\sum_{l=1}^{Z}v_l(a,t)\Big)&=0. \qquad (7)\end{aligned}$$

From the Lemma 2.1, one obtains

$$\begin{aligned}&2\int_\Omega e_p^T(a,t)M\Delta e_p(a,t)da\\=&-\sum_{h=1}^{\epsilon}\frac{2}{c_h^2}\int_\Omega e_p^T(a,t)Me_p(a,t)da. \qquad (8)\end{aligned}$$

Let $\mathcal{Q}=\mathrm{diag}\,(\mathcal{Q}_1^2,\mathcal{Q}_2^2,\cdots,\mathcal{Q}_z^2)$. Obviously

$$\begin{aligned}&2e_p^T(a,t)E(f(\chi_p(a,t))-f(\bar{\chi}(a,t)))\\\leqslant\ &e_p^T(a,t)(EE^T+\mathcal{Q})e_p(a,t). \qquad (9)\end{aligned}$$

Define the matrix $\delta^s=(\delta_{pq}^s)_{Z\times Z}$, where $\delta_{pp}^s=-\sum_{\substack{q=1\\q\neq p}}^{Z}\delta_{pq}^s$. Then, we can easily derive

$$\begin{aligned}&\sum_{s=1}^{d}\sum_{p=1}^{Z}\sum_{q\in\mathcal{Z}_p}(G_{pq}^s(t)-\delta_{pq}^s)(e_p(a,t)\\&-e_q(a,t))^T\Gamma^s(e_p(a,t)-e_q(a,t))\\&=-2\sum_{s=1}^{d}\sum_{p=1}^{Z}\sum_{q=1}^{Z}(G_{pq}^s(t)-\delta_{pq}^s)e_p^T(a,t)\Gamma^se_q(a,t). \qquad (10)\end{aligned}$$

By (4), (7)-(10), one gets

$$\begin{aligned}\dot{V}_1(t)&\leqslant\int_\Omega e^T(a,t)\Big[I_Z\otimes\Big(-\sum_{h=1}^{\epsilon}\frac{2}{c_h^2}M-2A+EE^T\\&\quad+\mathcal{Q}\Big)+2\sum_{s=1}^{d}b_s(B^s\otimes\Gamma^s)\Big]e(a,t)da\\&\quad+2\sum_{s=1}^{d}b_s\int_\Omega e^T(a,t)(\delta^s\otimes\Gamma^s)e(a,t)da\\&\quad+2\int_\Omega e^T(a,t)(I_Z\otimes K)u(a,t)da.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\dot{V}_1(t) - 2\int_\Omega y^T(a,t)Hu(a,t)da \\
\leqslant\ & \int_\Omega e^T(a,t)\Big\{I_Z\otimes\Big(-\sum_{h=1}^{\epsilon}\frac{2}{c_h^2}M - 2A + EE^T + \mathcal{Q}\Big) \\
&+2\sum_{s=1}^{d} b_s(B^s\otimes\Gamma^s) + [(I_Z\otimes K) - (I_Z\otimes P_1^T)H] \\
&\times[(I_Z\otimes P_2^T)H + H^T(I_Z\otimes P_2)]^{-1}[(I_Z\otimes K^T) \\
&-H^T(I_Z\otimes P_1)]\Big\}e(a,t)da \\
&+2b_1\int_\Omega e^T(a,t)(\delta^1\otimes\Gamma^1)e(a,t)da \\
=\ & \int_\Omega e^T(a,t)(\Xi + 2b_1\delta^1\otimes\Gamma^1)e(a,t)da, \qquad (11)
\end{aligned}
$$

where $\Xi = I_Z\otimes[-2\sum_{h=1}^{\epsilon} M/c_h^2 - 2A + EE^T + \mathcal{Q}] + 2\sum_{s=1}^{d} b_s(B^s\otimes\Gamma^s) + [(I_Z\otimes K) - (I_Z\otimes P_1^T)H][(I_Z\otimes P_2^T)H + H^T(I_Z\otimes P_2)]^{-1}[(I_Z\otimes K^T) - H^T(I_Z\otimes P_1)]$.

On the other hand, there exists an orthogonal matrix $\mathcal{K} = (\kappa_1, \kappa_2, \cdots, \kappa_Z) \in \mathbb{R}^{Z\times Z}$ such that $\mathcal{K}^T\delta^1\mathcal{K} = \Upsilon = \mathrm{diag}(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_Z)$, in which $0 = \Upsilon_1 > \Upsilon_2 \geqslant \Upsilon_3 \geqslant \cdots \geqslant \Upsilon_Z$. Let $\varrho(a,t) = (\varrho_1^T(a,t), \varrho_2^T(a,t), \cdots, \varrho_Z^T(a,t))^T = (\mathcal{K}^T\otimes I_z)e(a,t)$. Since $\kappa_1 = \frac{1}{\sqrt{Z}}(1,1,\cdots,1)^T$, one has $\varrho_1(a,t) = (\kappa_1^T\otimes I_z)e(a,t) = 0$. Then, we can derive from (11) that

$$
\begin{aligned}
&\dot{V}_1(t) - 2\int_\Omega y^T(a,t)Hu(a,t)da \\
\leqslant\ & \int_\Omega e^T(a,t)\Xi e(a,t)da \\
&+2b_1\Upsilon_2\int_\Omega \varrho^T(a,t)(I_Z\otimes\Gamma^1)\varrho(a,t)da \\
=\ & \int_\Omega e^T(a,t)(\Xi + 2b_1\Upsilon_2 I_Z\otimes\Gamma^1)e(a,t)da.
\end{aligned}
$$

By choosing δ_{pq} large enough such that

$$\gamma_M(\Xi) + 2b_1\Upsilon_2\gamma_m(\Gamma^1) \leqslant 0,$$

one has

$$\Xi + 2b_1\Upsilon_2 I_Z\otimes\Gamma^1 \leqslant 0.$$

Therefore,

$$\frac{\dot{V}_1(t)}{2} \leqslant \int_\Omega y^T(a,t)Hu(a,t)da.$$

Letting $V(t) = V_1(t)/2$, one has

$$\int_{t_0}^{t_r}\int_\Omega y^T(a,t)Hu(a,t)dadt \geqslant V(t_r) - V(t_0)$$

for any $0 \leqslant t_0 \leqslant t_r \in \mathbb{R}$. □

Theorem 3.2. If there are two matrices $\mathbb{R}^{\eta Z\times\eta Z} \ni Q_1 > 0$ and $H \in \mathbb{R}^{\zeta Z\times\eta Z}$ satisfying

$$(I_Z\otimes P_2)^T H + H^T(I_Z\otimes P_2) - Q_1 > 0,$$

the network (2) is input-strictly passive.

Theorem 3.3. If there are two matrices $\mathbb{R}^{\zeta Z\times\zeta Z} \ni Q_2 > 0$ and $H \in \mathbb{R}^{\zeta Z\times\eta Z}$ satisfying

$$
\begin{aligned}
&(I_Z\otimes P_2)^T H + H^T(I_Z\otimes P_2) \\
-\ &(I_Z\otimes P_2)^T Q_2(I_Z\otimes P_2) > 0,
\end{aligned}
$$

the network (2) is output-strictly passive.

3.3 Synchronization criteria

Definition 3.1. The network (1) with the input $u_p(a,t) = 0$ $(p = 1,2,\cdots,Z)$ is synchronized if

$$\lim_{t\to+\infty}\left\|\chi_p(\cdot,t) - \frac{1}{Z}\sum_{q=1}^{Z}\chi_q(\cdot,t)\right\| = 0, \quad p = 1,2,\cdots,Z.$$

Theorem 3.4. Under the adaptive state feedback controller (4), the network (1) can achieves synchronization if the network (2) is output-strictly passive with regard to the storage function $W(t) = V_1(t)/2$ and $P_1^T P_1 > 0$.

Proof. If the network (2) under the adaptive state feedback controller (4) is output-strictly passive with regard to $W(t)$, we have

$$
\begin{aligned}
\frac{W(t+\phi) - W(t)}{\phi} \leqslant\ & \frac{\int_t^{t+\phi}\int_\Omega y^T(a,t)Hu(a,s)dads}{\phi} \\
&-\frac{\int_t^{t+\phi}\int_\Omega y^T(a,t)Q_2 y(a,s)dads}{\phi}
\end{aligned}
$$

for any $t \in \mathbb{R}^+$ and $\phi > 0$, where $\mathbb{R}^{\zeta Z\times\zeta Z} \ni Q_2 > 0$ and $H \in \mathbb{R}^{\zeta Z\times\eta Z}$.

Then, we can easily derive

$$\dot{W}(t) \leqslant \int_\Omega y^T(a,t)Hu(a,t)da - \int_\Omega y^T(a,t)Q_2 y(a,t)da.$$

Letting $u(a,t) = 0$, one gets

$$\dot{W}(t) \leqslant -\gamma_m(P_1^T P_1)\gamma_m(Q_2)\left\|e(\cdot,t)\right\|^2. \qquad (12)$$

By (12), we get that $\lim_{t\to+\infty} W(t)$ exists and $G_{pq}^s(t)((p,q)\in\mathcal{B})$ is bounded. Therefore, $\lim_{t\to+\infty} G_{pq}^s(t)$ exists based on (4). Then, it follows from (5) that

$$\lim_{t\to\infty}\left\|e(\cdot,t)\right\|^2 = \wp \geqslant 0.$$

Suppose that

$$\lim_{t\to\infty}\left\|e(\cdot,t)\right\|^2 = \wp > 0.$$

Then, there obviously exists $0 < S \in \mathbb{R}$ such that

$$\left\|e(\cdot,t)\right\|^2 > \wp/2, \quad t \geqslant S. \qquad (13)$$

By (12) and (13), one has

$$\dot{W}(t) < -\frac{\gamma_m(P_1^T P_1)\gamma_m(Q_2)\wp}{2}, \quad t \geqslant S. \qquad (14)$$

In light of (14), one derives

$$-W(S) \leqslant W(+\infty) - W(S)$$

$$
\begin{aligned}
&= \int_S^{+\infty} \dot{W}(t)dt \\
&< -\int_S^{+\infty} \frac{\gamma_m(P_1^T P_1)\gamma_m(Q_2)\wp}{2} dt \\
&= -\infty,
\end{aligned}
$$

which results in a contradiction.

Therefore, $\lim_{t\to\infty} \|e(\cdot,t)\| = 0$. Namely, the network (1) can achieves synchronization under the adaptive state feedback controller (4).

Based on the Theorems 3.3 and 3.4, the following conclusion can be obtained.

Corollary 3.1. The network (1) under the adaptive controller (4) realizes synchronization if there exist $\mathbb{R}^{\zeta Z\times\zeta Z} \ni Q_2 > 0$ and $H \in \mathbb{R}^{\zeta Z\times\eta Z}$ satisfying

$$
\begin{pmatrix} Y & 0 \\ 0 & P_1^T P_1 \end{pmatrix} > 0,
$$

where $Y = (I_Z \otimes P_2)^T H + H^T(I_Z \otimes P_2) - (I_Z \otimes P_2)^T Q_2 (I_Z \otimes P_2)$.

4 Numerical example

Example. The CRDNNs with multiple state couplings is given as follows:

$$
\begin{aligned}
\frac{\partial \chi_p(a,t)}{\partial t} &= M\triangle\chi_p(a,t) - A\chi_p(a,t) + Ef(\chi_p(a,t)) \\
&\quad +2\sum_{q=1}^{5} B_{pq}^1 \Gamma^1 \chi_q(a,t) + J + Ku_p(a,t) \\
&\quad +3\sum_{q=1}^{5} B_{pq}^2 \Gamma^2 \chi_q(a,t) + v_p(a,t) \\
&\quad +4\sum_{q=1}^{5} B_{pq}^3 \Gamma^3 \chi_q(a,t), \qquad (15)
\end{aligned}
$$

where $p = 1,2,\cdots,5$, $f_k(\xi) = \frac{|\xi+1|-|\xi-1|}{4}, k = 1,2,3$, $\Gamma^1 = \mathrm{diag}(0.5,0.6,0.5), \Gamma^2 = \mathrm{diag}(0.7,0.4,0.8), \Gamma^3 = \mathrm{diag}(0.5,0.7,0.8), \Omega = \{a \mid -0.5 < a < 0.5\}, J = (0.6,0.4,0.8)^T, M = \mathrm{diag}(0.3,0.5,0.2), A = \mathrm{diag}(0.7,0.9,0.8)$.

$$
E = \begin{pmatrix} 0.4 & 0.7 & 0.5 \\ 0.5 & 0.5 & 0.6 \\ 0.2 & 0.3 & 0.7 \end{pmatrix}, \quad K = \begin{pmatrix} 0.4 & 0.3 \\ 0.4 & 0.4 \\ 0.5 & 0.3 \end{pmatrix},
$$

$$
B^1 = \begin{pmatrix} -0.4 & 0.1 & 0 & 0.2 & 0.1 \\ 0.1 & -0.4 & 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & -0.6 & 0 & 0.5 \\ 0.2 & 0.1 & 0 & -0.6 & 0.3 \\ 0.1 & 0.1 & 0.5 & 0.3 & -1 \end{pmatrix},
$$

$$
B^2 = \begin{pmatrix} -0.5 & 0.2 & 0 & 0.2 & 0.1 \\ 0.2 & -0.7 & 0.3 & 0.1 & 0.1 \\ 0 & 0.3 & -0.8 & 0 & 0.5 \\ 0.2 & 0.1 & 0 & -0.6 & 0.3 \\ 0.1 & 0.1 & 0.5 & 0.3 & -1 \end{pmatrix},
$$

$$
B^3 = \begin{pmatrix} -0.8 & 0.3 & 0 & 0.3 & 0.2 \\ 0.3 & -0.7 & 0.1 & 0.2 & 0.1 \\ 0 & 0.1 & -0.4 & 0 & 0.3 \\ 0.3 & 0.2 & 0 & -0.6 & 0.1 \\ 0.2 & 0.1 & 0.3 & 0.1 & -0.7 \end{pmatrix}.
$$

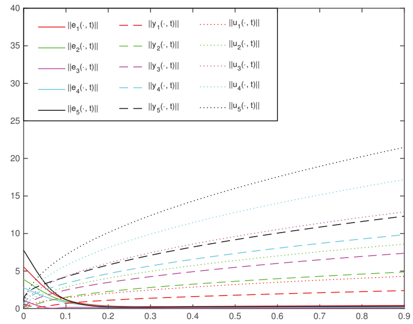

Fig. 1: $\|e_p(t)\|, \|y_p(t)\|, \|u_p(t)\|, p = 1,2,\cdots,5.$

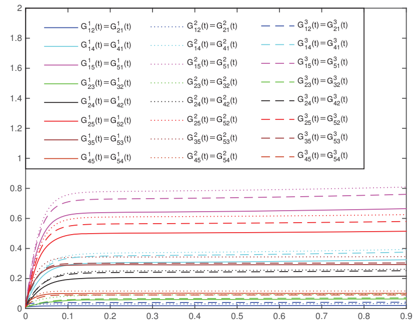

Fig. 2: Adaptive feedback gains

Apparently, function $f_k(\cdot)$ satisfies Lipschitz condition with $\mathcal{Q}_k = 0.5$.

The output vector $y_p(a,t) \in \mathbb{R}^{3\times 2}$ is selected as follows:

$$
y_p(a,t) = P_1 e_p(a,t) + P_2 u_p(a,t), \quad p = 1,2,\cdots,5,
$$

where

$$
P_1 = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.2 & 0.2 \\ 0.2 & 0.3 \\ 0.3 & 0.2 \end{pmatrix}.
$$

Case 1: By taking advantage of the MATLAB YALMIP Toolbox, the following matrix H can be obtained:

$$
H = 10^8 I_5 \otimes \begin{pmatrix} 0.5800 & 0.5800 \\ -4.0599 & 5.5099 \\ 5.5099 & -4.0599 \end{pmatrix}.
$$

Therefore, under the adaptive state feedback controller (4), the network (15) achieves passivity. The simulation results are displayed in Figs. 1 and 2.

Case 2: By taking advantage of the MATLAB YALMIP Toolbox, the following matrices H and Q_2 that satisfy the condition of Corollary 3.1 can be obtained:

$$
\begin{aligned}
H &= 10^8 I_5 \otimes \begin{pmatrix} 0.7185 & 0.7185 \\ -3.6033 & 5.3995 \\ 5.3995 & -3.6032 \end{pmatrix}, \\
Q_2 &= 10^8 I_5 \otimes \begin{pmatrix} 1.7827 & -0.0000 & -0.0000 \\ -0.0000 & 1.7827 & 0.0000 \\ -0.0000 & 0.0000 & 1.7827 \end{pmatrix}.
\end{aligned}
$$

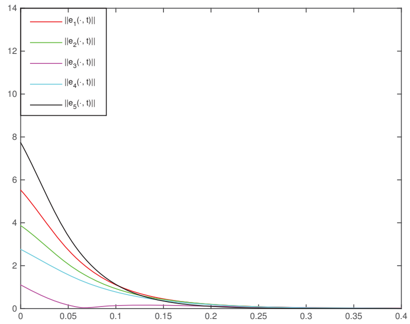

Fig. 3: $\|e_p(t)\|, p = 1, 2, \cdots, 5$.

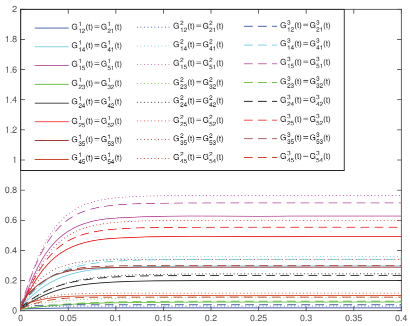

Fig. 4: Adaptive feedback gains

Therefore, under the adaptive state feedback controller (4), the network (15) achieves synchronization. The simulation results are displayed in Figs. 3 and 4.

5 Conclusion

In this paper, the CRDNNs with multiple state couplings has been studied. By choosing appropriate adaptive state feedback controller and making use of some inequality techniques, we have given several passivity conditions for these network model. Furthermore, the synchronization criterion for the network also has been established by exploiting the obtained output strictly passivity result. Finally, a numeral example has been provided to verify the effectiveness of the derived criteria.

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