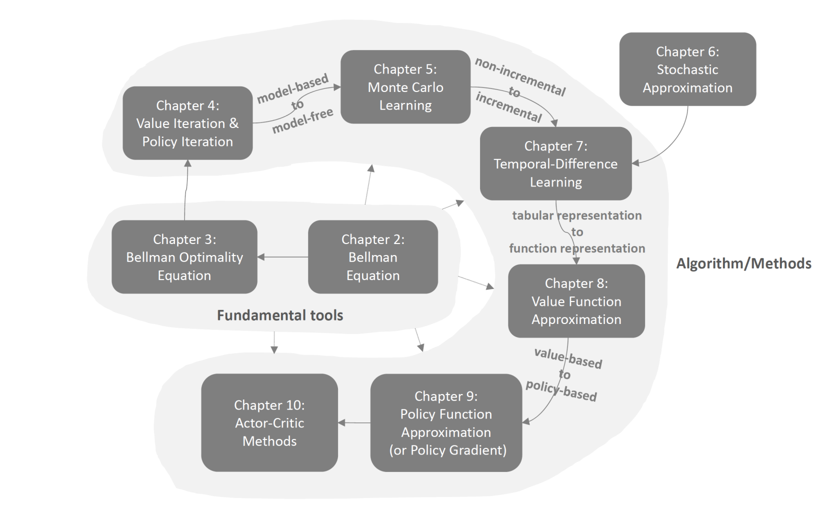
Lecture 7: Temporal-Difference Learning

Shiyu Zhao

Outline



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Introduction

- This lecture introduces temporal-difference (TD) learning, which is one of the most well-known methods in reinforcement learning (RL).
- Monte Carlo (MC) learning is the first model-free method. TD learning is the second model-free method. TD has some advantages compared to MC.
- We will see how the stochastic approximation methods studied in the last lecture are useful.

Outline

Outline

Motivating example: stochastic algorithms

We next consider some stochastic problems and show how to use the RM algorithm to solve them.

First, consider the simple mean estimation problem: calculate

$$w = \mathbb{E}[X],$$

based on some iid samples $\{x\}$ of X.

- By writing $g(w) = w - \mathbb{E}[X]$, we can reformulate the problem to a root-finding problem

$$g(w) = 0.$$

- Since we can only obtain samples $\{x\}$ of X, the noisy observation is

$$\tilde{g}(w, \eta) = w - x = (w - \mathbb{E}[X]) + (\mathbb{E}[X] - x) \doteq g(w) + \eta.$$

- Then, according to the last lecture, we know the RM algorithm for solving $g(w) = 0$ is

$$w_{k+1} = w_k - \alpha_k \tilde{g}(w_k, \eta_k) = w_k - \alpha_k (w_k - x_k)$$

Motivating example: stochastic algorithms

Second, consider a little more complex problem. That is to estimate the mean of a function $v(X)$,

$$w = \mathbb{E}[v(X)],$$

based on some iid random samples $\{x\}$ of X.

- To solve this problem, we define

$$\begin{aligned} g(w) &= w - \mathbb{E}[v(X)] \\ \tilde{g}(w, \eta) &= w - v(x) = (w - \mathbb{E}[v(X)]) + (\mathbb{E}[v(X)] - v(x)) \doteq g(w) + \eta. \end{aligned}$$

- Then, the problem becomes a root-finding problem: $g(w) = 0$. The corresponding RM algorithm is

$$w_{k+1} = w_k - \alpha_k \tilde{g}(w_k, \eta_k) = w_k - \alpha_k [w_k - v(x_k)]$$

Motivating example: stochastic algorithms

Third, consider an even more complex problem: calculate

$$w = \mathbb{E}[R + \gamma v(X)],$$

where R, X are random variables, γ is a constant, and $v(\cdot)$ is a function.

- Suppose we can obtain samples $\{x\}$ and $\{r\}$ of X and R. we define

$$\begin{aligned} g(w) &= w - \mathbb{E}[R + \gamma v(X)], \\ \tilde{g}(w, \eta) &= w - [r + \gamma v(x)] \\ &= (w - \mathbb{E}[R + \gamma v(X)]) + (\mathbb{E}[R + \gamma v(X)] - [r + \gamma v(x)]) \\ &\doteq g(w) + \eta. \end{aligned}$$

- Then, the problem becomes a root-finding problem: $g(w) = 0$. The corresponding RM algorithm is

$$w_{k+1} = w_k - \alpha_k \tilde{g}(w_k, \eta_k) = w_k - \alpha_k [w_k - (r_k + \gamma v(x_k))]$$

Motivating example: stochastic algorithms

Quick summary:

- The above three examples are more and more complex.
- They can all be solved by the RM algorithm.
- We will see that the TD algorithms have similar expressions.

Outline

TD learning of state values

Note that

- TD learning often refers to a broad class of RL algorithms.
- TD learning also refers to a specific algorithm for estimating state values as introduced below.

TD learning of state values – Algorithm description

The data/experience required by the algorithm:

- $(s_0, r_1, s_1, \ldots, s_t, r_{t+1}, s_{t+1}, \ldots)$ or $\{(s_t, r_{t+1}, s_{t+1})\}_t$ generated following the given policy π.

The TD learning algorithm is

$$v_{t+1}(s_t) = v_t(s_t) - \alpha_t(s_t)\Big[v_t(s_t) - [r_{t+1} + \gamma v_t(s_{t+1})]\Big], \tag{1}$$

$$v_{t+1}(s) = v_t(s), \quad \forall s \neq s_t, \tag{2}$$

where $t = 0, 1, 2, \ldots$. Here, $v_t(s_t)$ is the estimated state value of $v_\pi(s_t)$; $\alpha_t(s_t)$ is the learning rate of s_t at time t.

- At time t, only the value of the visited state s_t is updated whereas the values of the unvisited states $s \neq s_t$ remain unchanged.
- The update in (2) will be omitted when the context is clear.

TD learning of state values – Algorithm properties

The TD algorithm can be annotated as

$$\underbrace{v_{t+1}(s_t)}_{\text{new estimate}} = \underbrace{v_t(s_t)}_{\text{current estimate}} - \alpha_t(s_t)\Big[\overbrace{v_t(s_t) - [\underbrace{r_{t+1} + \gamma v_t(s_{t+1})}_{\text{TD target } \bar{v}_t}]}^{\text{TD error } \delta_t}\Big], \qquad (3)$$

Here,

$$\bar{v}_t \doteq r_{t+1} + \gamma v(s_{t+1})$$

is called the TD target.

$$\delta_t \doteq v(s_t) - [r_{t+1} + \gamma v(s_{t+1})] = v(s_t) - \bar{v}_t$$

is called the TD error.

It is clear that the new estimate $v_{t+1}(s_t)$ is a combination of the current estimate $v_t(s_t)$ and the TD error.

TD learning of state values – Algorithm properties

First, why is $\bar{v}_t$ called the TD target?

That is because the algorithm drives $v(s_t)$ towards $\bar{v}_t$.

To see that,

$$\begin{aligned} v_{t+1}(s_t) &= v_t(s_t) - \alpha_t(s_t)\big[v_t(s_t) - \bar{v}_t\big] \\ \Longrightarrow \quad v_{t+1}(s_t) - \bar{v}_t &= v_t(s_t) - \bar{v}_t - \alpha_t(s_t)\big[v_t(s_t) - \bar{v}_t\big] \\ \Longrightarrow \quad v_{t+1}(s_t) - \bar{v}_t &= [1 - \alpha_t(s_t)][v_t(s_t) - \bar{v}_t] \\ \Longrightarrow \quad |v_{t+1}(s_t) - \bar{v}_t| &= |1 - \alpha_t(s_t)||v_t(s_t) - \bar{v}_t| \end{aligned}$$

Since $\alpha_t(s_t)$ is a small positive number, we have

$$0 < 1 - \alpha_t(s_t) < 1$$

Therefore,

$$|v_{t+1}(s_t) - \bar{v}_t| \leq |v_t(s_t) - \bar{v}_t|$$

which means $v(s_t)$ is driven towards $\bar{v}_t$!

TD learning of state values – Algorithm properties

Second, what is the interpretation of the TD error?

$$\delta_t = v(s_t) - [r_{t+1} + \gamma v(s_{t+1})]$$

- It is a difference between two consequent time steps.
- It reflects the deficiency between v_t and v_π. To see that, denote

$$\delta_{\pi,t} \doteq v_\pi(s_t) - [r_{t+1} + \gamma v_\pi(s_{t+1})]$$

  Note that

$$\mathbb{E}[\delta_{\pi,t}|S_t = s_t] = v_\pi(s_t) - \mathbb{E}\big[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s_t\big] = 0.$$

  - If $v_t = v_\pi$, then δ_t should be zero (in the expectation sense).
  - Hence, if δ_t is not zero, then v_t is not equal to v_π.
- The TD error can be interpreted as innovation, which means new information obtained from the experience (s_t, r_{t+1}, s_{t+1}).

TD learning of state values – Algorithm properties

Other properties:

- The TD algorithm in (3) only estimates the state value of a given policy.
  - *It does not estimate the action values.*
  - *It does not search for optimal policies.*
- Later, we will see how to estimate action values and then search for optimal policies.
- Nonetheless, the TD algorithm in (3) is fundamental for understanding the core idea.

TD learning of state values – The idea of the algorithm

Q: What does this TD algorithm do mathematically?

A: It solves the Bellman equation of a given policy π.

TD learning of state values – The idea of the algorithm

First, a new expression of the Bellman equation.

The definition of state value of π is

$$v_\pi(s) = \mathbb{E}\big[R + \gamma G | S = s\big], \quad s \in \mathcal{S} \tag{4}$$

where G is discounted return. Since

$$\mathbb{E}[G|S = s] = \sum_a \pi(a|s) \sum_{s'} p(s'|s,a) v_\pi(s') = \mathbb{E}[v_\pi(S')|S = s],$$

where S' is the next state, we can rewrite (4) as

$$v_\pi(s) = \mathbb{E}\big[R + \gamma v_\pi(S') | S = s\big], \quad s \in \mathcal{S}. \tag{5}$$

Equation (5) is another expression of the Bellman equation. It is sometimes called the Bellman expectation equation, an important tool to design and analyze TD algorithms.

TD learning of state values – The idea of the algorithm

Second, solve the Bellman equation in (5) using the RM algorithm.

In particular, by defining

$$g(v(s)) = v(s) - \mathbb{E}\big[R + \gamma v_\pi(S')|s\big],$$

we can rewrite (5) as

$$g(v(s)) = 0.$$

Since we can only obtain the samples r and s' of R and S', the noisy observation we have is

$$\begin{aligned}\tilde{g}(v(s)) &= v(s) - \big[r + \gamma v_\pi(s')\big] \\ &= \underbrace{\Big(v(s) - \mathbb{E}\big[R + \gamma v_\pi(S')|s\big]\Big)}_{g(v(s))} + \underbrace{\Big(\mathbb{E}\big[R + \gamma v_\pi(S')|s\big] - \big[r + \gamma v_\pi(s')\big]\Big)}_{\eta}.\end{aligned}$$

TD learning of state values – The idea of the algorithm

Therefore, the RM algorithm for solving $g(v(s)) = 0$ is

$$\begin{aligned} v_{k+1}(s) &= v_k(s) - \alpha_k \tilde{g}(v_k(s)) \\ &= v_k(s) - \alpha_k \Big(v_k(s) - \big[r_k + \gamma v_\pi(s'_k) \big] \Big), \quad k = 1, 2, 3, \ldots \end{aligned} \tag{6}$$

where $v_k(s)$ is the estimate of $v_\pi(s)$ at the kth step; r_k, s'_k are the samples of R, S' obtained at the kth step.

The RM algorithm in (6) has two assumptions that deserve special attention.

- We must have the experience set $\{(s, r, s')\}$ for $k = 1, 2, 3, \ldots$.
- We assume that $v_\pi(s')$ is already known for any s'.

TD learning of state values – The idea of the algorithm

Therefore, the RM algorithm for solving $g(v(s)) = 0$ is

$$\begin{aligned} v_{k+1}(s) &= v_k(s) - \alpha_k \tilde{g}(v_k(s)) \\ &= v_k(s) - \alpha_k \Big(v_k(s) - \big[r_k + \gamma v_\pi(s'_k) \big] \Big), \quad k = 1, 2, 3, \ldots \end{aligned}$$

where $v_k(s)$ is the estimate of $v_\pi(s)$ at the kth step; r_k, s'_k are the samples of R, S' obtained at the kth step.

To remove the two assumptions in the RM algorithm, we can modify it.

- One modification is that $\{(s, r, s')\}$ is changed to $\{(s_t, r_{t+1}, s_{t+1})\}$ so that the algorithm can utilize the sequential samples in an episode.
- Another modification is that $v_\pi(s')$ is replaced by an estimate of it because we don't know it in advance.

TD learning of state values – Algorithm convergence

Theorem (Convergence of TD Learning)

By the TD algorithm (1), $v_t(s)$ *converges with probability 1 to* $v_\pi(s)$ *for all* $s \in \mathcal{S}$ *as* $t \to \infty$ *if* $\sum_t \alpha_t(s) = \infty$ *and* $\sum_t \alpha_t^2(s) < \infty$ *for all* $s \in \mathcal{S}$.

Remarks:

- This theorem says the state value can be found by the TD algorithm for a given a policy π.
- $\sum_t \alpha_t(s) = \infty$ and $\sum_t \alpha_t^2(s) < \infty$ must be valid for all $s \in \mathcal{S}$. At time step t, if $s = s_t$ which means that s is visited at time t, then $\alpha_t(s) > 0$; otherwise, $\alpha_t(s) = 0$ for all the other $s \neq s_t$. That requires every state must be visited an infinite (or sufficiently many) number of times.
- The learning rate α is often selected as a small constant. In this case, the condition that $\sum_t \alpha_t^2(s) < \infty$ is invalid anymore. When α is constant, it can still be shown that the algorithm converges in the sense of expectation sense.

For the proof of the theorem, see my book.

TD learning of state values – Algorithm properties

While TD learning and MC learning are both model-free, what are the **advantages and disadvantages** of TD learning compared to MC learning?

TD/Sarsa learning	MC learning
Online: TD learning is online. It can update the state/action values immediately after receiving a reward.	**Offline:** MC learning is offline. It has to wait until an episode has been completely collected.
Continuing tasks: Since TD learning is online, it can handle both episodic and continuing tasks.	**Episodic tasks:** Since MC learning is offline, it can only handle episodic tasks that has terminate states.

Table: Comparison between TD learning and MC learning.

TD learning of state values – Algorithm properties

While TD learning and MC learning are both model-free, what are the **advantages and disadvantages** of TD learning compared to MC learning?

TD/Sarsa learning	MC learning
Bootstrapping: TD bootstraps because the update of a value relies on the previous estimate of this value. Hence, it requires initial guesses.	**Non-bootstrapping:** MC is not bootstrapping, because it can directly estimate state/action values without any initial guess.
Low estimation variance: TD has lower than MC because there are fewer random variables. For instance, Sarsa requires $R_{t+1}, S_{t+1}, A_{t+1}$.	**High estimation variance:** To estimate $q_\pi(s_t, a_t)$, we need samples of $R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots$. Suppose the length of each episode is L. There are $\|\mathcal{A}\|^L$ possible episodes.

Table: Comparison between TD learning and MC learning (continued).

Outline

TD learning of action values – Sarsa

- The TD algorithm introduced in the last section can only estimate state values.
- Next, we introduce, Sarsa, an algorithm that can directly estimate action values.
- We will also see how to use Sarsa to find optimal policies.

Sarsa – Algorithm

First, our aim is to estimate the action values of a given policy π.

Suppose we have some experience $\{(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})\}_t$.

We can use the following *Sarsa* algorithm to estimate the action values:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q_t(s_t, a_t) - [r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})]\Big],$$

$$q_{t+1}(s, a) = q_t(s, a), \quad \forall (s, a) \neq (s_t, a_t),$$

where $t = 0, 1, 2, \ldots$.

- $q_t(s_t, a_t)$ is an estimate of $q_\pi(s_t, a_t)$;
- $\alpha_t(s_t, a_t)$ is the learning rate depending on s_t, a_t.

Sarsa – Algorithm

- **Why is this algorithm called Sarsa?** That is because each step of the algorithm involves $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})$. Sarsa is the abbreviation of state-action-reward-state-action.
- **What is the relationship between Sarsa and the previous TD learning algorithm?** We can obtain Sarsa by replacing the state value estimate $v(s)$ in the TD algorithm with the action value estimate $q(s,a)$. As a result, Sarsa is an action-value version of the TD algorithm.
- **What does the Sarsa algorithm do mathematically?** The expression of Sarsa suggests that it is a stochastic approximation algorithm solving the following equation:

$$q_\pi(s,a) = \mathbb{E}\left[R + \gamma q_\pi(S', A')|s,a\right], \quad \forall s,a.$$

This is another expression of the Bellman equation expressed in terms of action values. The proof is given in my book.

Sarsa – Algorithm

Theorem (Convergence of Sarsa learning)

By the Sarsa algorithm, $q_t(s, a)$ converges with probability 1 to the action value $q_\pi(s, a)$ as $t \to \infty$ for all (s, a) if $\sum_t \alpha_t(s, a) = \infty$ and $\sum_t \alpha_t^2(s, a) < \infty$ for all (s, a).

Remarks:

- This theorem says the action value can be found by Sarsa for a given a policy π.

Sarsa – Implementation

The ultimate goal of RL is to find optimal policies.

To do that, we can combine Sarsa with a policy improvement step.

The combined algorithm is also called Sarsa.

Pseudocode: Policy searching by Sarsa

For each episode, do

If the current s_t is not the target state, do

Collect the experience $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})$: In particular, take action a_t following $\pi_t(s_t)$, generate r_{t+1}, s_{t+1}, and then take action a_{t+1} following $\pi_t(s_{t+1})$.

Update q-value:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q_t(s_t, a_t) - [r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})]\Big]$$

Update policy:

$\pi_{t+1}(a|s_t) = 1 - \frac{\epsilon}{|\mathcal{A}|}(|\mathcal{A}| - 1)$ if $a = \arg\max_a q_{t+1}(s_t, a)$

$\pi_{t+1}(a|s_t) = \frac{\epsilon}{|\mathcal{A}|}$ otherwise

Remarks about this algorithm:

- The policy of s_t is updated immediately after $q(s_t, a_t)$ is updated. This is based on the idea of generalized policy iteration.
- The policy is ϵ-greedy instead of greedy to well balance exploitation and exploration.

Be clear about the core idea and complication:

- The core idea is simple: that is to use an algorithm to solve the Bellman equation of a given policy.
- The complication emerges when we try to find optimal policies and work efficiently.

Sarsa – Examples

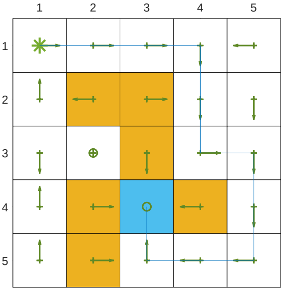
Task description:

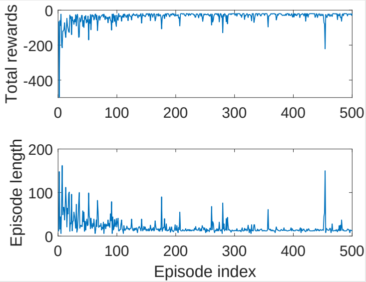
- The task is to find a good path from a specific starting state to the target state.
  - This task is different from all the previous tasks where we need to find out the optimal policy for every state!
  - Each episode starts from the top-left state and end in the target state.
  - In the future, pay attention to what the task is.
- $r_{\text{target}} = 0$, $r_{\text{forbidden}} = r_{\text{boundary}} = -10$, and $r_{\text{other}} = -1$. The learning rate is $\alpha = 0.1$ and the value of ϵ is 0.1.

Sarsa – Examples

Results:

- The left figures above show the final policy obtained by Sarsa.
    - Not all states have the optimal policy.
- The right figures show the total reward and length of every episode.
    - The metric of total reward per episode will be frequently used.





Outline

TD learning of action values: Expected Sarsa

A variant of Sarsa is the **Expected Sarsa** algorithm:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q_t(s_t, a_t) - (r_{t+1} + \gamma\mathbb{E}[q_t(s_{t+1}, A)])\Big],$$

$$q_{t+1}(s, a) = q_t(s, a), \quad \forall(s, a) \neq (s_t, a_t),$$

where

$$\mathbb{E}[q_t(s_{t+1}, A)]) = \sum_a \pi_t(a|s_{t+1})q_t(s_{t+1}, a) \doteq v_t(s_{t+1})$$

is the expected value of $q_t(s_{t+1}, a)$ under policy π_t.

Compared to Sarsa:

- The TD target is changed from $r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})$ as in Sarsa to $r_{t+1} + \gamma\mathbb{E}[q_t(s_{t+1}, A)]$ as in Expected Sarsa.
- Need more computation. But it is beneficial in the sense that it reduces the estimation variances because it reduces random variables in Sarsa from $\{s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1}\}$ to $\{s_t, a_t, r_{t+1}, s_{t+1}\}$.

TD learning of action values: Expected Sarsa

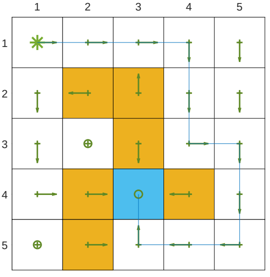
What does the algorithm do mathematically? Expected Sarsa is a stochastic approximation algorithm for solving the following equation:

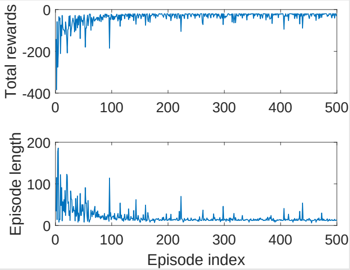
$$q_\pi(s,a) = \mathbb{E}\Big[R_{t+1} + \gamma \mathbb{E}_{A_{t+1}\sim\pi(S_{t+1})}[q_\pi(S_{t+1},A_{t+1})]\Big|S_t = s, A_t = a\Big], \quad \forall s,a.$$

The above equation is another expression of the Bellman equation:

$$q_\pi(s,a) = \mathbb{E}\Big[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s, A_t = a\Big],$$

Illustrative example:





Outline

TD learning of action values: n-step Sarsa

n-step Sarsa: can unify Sarsa and Monte Carlo learning

The definition of action value is

$$q_\pi(s,a) = \mathbb{E}[G_t|S_t = s, A_t = a].$$

The discounted return G_t can be written in different forms as

$$\begin{aligned}
\text{Sarsa} \longleftarrow \quad G_t^{(1)} &= R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}), \\
G_t^{(2)} &= R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, A_{t+2}), \\
&\vdots \\
n\text{-step Sarsa} \longleftarrow \quad G_t^{(n)} &= R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^n q_\pi(S_{t+n}, A_{t+n}), \\
&\vdots \\
\text{MC} \longleftarrow \quad G_t^{(\infty)} &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots
\end{aligned}$$

It should be noted that $G_t = G_t^{(1)} = G_t^{(2)} = G_t^{(n)} = G_t^{(\infty)}$, where the superscripts merely indicate the different decomposition structures of G_t.

TD learning of action values: n-step Sarsa

- Sarsa aims to solve

$$q_\pi(s,a) = \mathbb{E}[G_t^{(1)}|s,a] = \mathbb{E}[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1})|s,a].$$

- MC learning aims to solve

$$q_\pi(s,a) = \mathbb{E}[G_t^{(\infty)}|s,a] = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots |s,a].$$

- An intermediate algorithm called *n-step Sarsa* aims to solve

$$q_\pi(s,a) = \mathbb{E}[G_t^{(n)}|s,a] = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \dots + \gamma^n q_\pi(S_{t+n}, A_{t+n})|s,a].$$

- The algorithm of n-step Sarsa is

$$\begin{aligned} q_{t+1}(s_t,a_t) &= q_t(s_t,a_t) \\ &\quad - \alpha_t(s_t,a_t)\Big[q_t(s_t,a_t) - [r_{t+1} + \gamma r_{t+2} + \dots + \gamma^n q_t(s_{t+n}, a_{t+n})]\Big]. \end{aligned}$$

n-step Sarsa is more general because it becomes the (one-step) Sarsa algorithm when $n = 1$ and the MC learning algorithm when $n = \infty$.

TD learning of action values: n-step Sarsa

- n-step Sarsa needs $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1}, \ldots, r_{t+n}, s_{t+n}, a_{t+n})$.
- Since $(r_{t+n}, s_{t+n}, a_{t+n})$ has not been collected at time t, we are not able to implement n-step Sarsa at step t. However, we can wait until time $t+n$ to update the q-value of (s_t, a_t):

$$\begin{aligned}&q_{t+n}(s_t, a_t) = q_{t+n-1}(s_t, a_t)\\ &- \alpha_{t+n-1}(s_t, a_t)\Big[q_{t+n-1}(s_t, a_t) - [r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^n q_{t+n-1}(s_{t+n}, a_{t+n})]\Big]\end{aligned}$$

- Since n-step Sarsa includes Sarsa and MC learning as two extreme cases, its performance is a blend of Sarsa and MC learning:
  - If n is large, its performance is close to MC learning and hence has a large variance but a small bias.
  - If n is small, its performance is close to Sarsa and hence has a relatively large bias due to the initial guess and relatively low variance.
- Finally, n-step Sarsa is also for policy evaluation. It can be combined with the policy improvement step to search for optimal policies.

Outline

TD learning of optimal action values: Q-learning

- Next, we introduce Q-learning, one of the most widely used RL algorithms.
- Sarsa can estimate the action values of a given policy. It must be combined with a policy improvement step to find optimal policies.
- Q-learning can directly estimate optimal action values and hence optimal policies.

Q-learning – Algorithm

The Q-learning algorithm is

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\left[q_t(s_t, a_t) - [r_{t+1} + \gamma \max_{a \in \mathcal{A}} q_t(s_{t+1}, a)]\right],$$

$$q_{t+1}(s, a) = q_t(s, a), \quad \forall (s, a) \neq (s_t, a_t),$$

Q-learning is very similar to Sarsa. They are different only in terms of the TD target:

- The TD target in Q-learning is $r_{t+1} + \gamma \max_{a \in \mathcal{A}} q_t(s_{t+1}, a)$
- The TD target in Sarsa is $r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})$.

What does Q-learning do mathematically?

It aims to solve

$$q(s,a)=\mathbb{E}\left[R_{t+1}+\gamma\max_a q(S_{t+1},a)\middle|S_t=s,A_t=a\right],\quad \forall s,a.$$

This is the Bellman optimality equation expressed in terms of action values. See the proof in my book.

Off-policy vs on-policy

Before further studying Q-learning, we first introduce two important concepts: on-policy learning and off-policy learning.

There exist two policies in a TD learning task:

- The behavior policy is used to generate experience samples.
- The target policy is constantly updated toward an optimal policy.

On-policy vs off-policy:

- When the behavior policy is the same as the target policy, such kind of learning is called on-policy.
- When they are different, the learning is called off-policy.

Off-policy vs on-policy

Advantages of off-policy learning:

- It can search for optimal policies based on the experience samples generated by any other policies.
  - As an important special case, the behavior policy can be selected to be exploratory. For example, if we would like to estimate the action values of all state-action pairs, we can use a exploratory policy to generate episodes visiting every state-action pair sufficiently many times.

Off-policy vs on-policy

How to judge if a TD algorithm is on-policy or off-policy?

- First, check what the algorithm does mathematically.
- Second, check what things are required to implement the algorithm.

It deserves special attention because it is one of the most confusing problems to beginners.

Off-policy vs on-policy

Sarsa is on-policy.

- First, Sarsa aims to solve the Bellman equation of a given policy π:

$$q_\pi(s,a) = \mathbb{E}\left[R + \gamma q_\pi(S',A')|s,a\right], \quad \forall s,a.$$

  where $R \sim p(R|s,a)$, $S' \sim p(S'|s,a)$, $A' \sim \pi(A'|S')$.

- Second, the algorithm is

$$q_{t+1}(s_t,a_t) = q_t(s_t,a_t) - \alpha_t(s_t,a_t)\Big[q_t(s_t,a_t) - [r_{t+1} + \gamma q_t(s_{t+1},a_{t+1})]\Big],$$

  which requires $(s_t,a_t,r_{t+1},s_{t+1},a_{t+1})$:

  - If (s_t,a_t) is given, then r_{t+1} and s_{t+1} do not depend on any policy!
  - a_{t+1} is generated following $\pi_t(s_{t+1})$!

- π_t is both the target and behavior policy.

Off-policy vs on-policy

Monte Carlo learning is on-policy.

- First, the MC method aims to solve

$$q_\pi(s,a) = \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \ldots | S_t = s, A_t = a\right], \quad \forall s, a.$$

  where the sample is generated following a given policy π.

- Second, the implementation of the MC method is

$$q(s,a) \approx r_{t+1} + \gamma r_{t+2} + \ldots$$

- A policy is used to generate samples, which is further used to estimate the action values of the policy. Based on the action values, we can improve the policy.

Off-policy vs on-policy

Q-learning is off-policy.

- First, Q-learning aims to solve the Bellman optimality equation

$$q(s,a) = \mathbb{E}\left[R_{t+1} + \gamma \max_a q(S_{t+1},a)\middle| S_t = s, A_t = a\right], \quad \forall s, a.$$

- Second, the algorithm is

$$q_{t+1}(s_t,a_t) = q_t(s_t,a_t) - \alpha_t(s_t,a_t)\left[q_t(s_t,a_t) - [r_{t+1} + \gamma \max_{a\in\mathcal{A}} q_t(s_{t+1},a)]\right]$$

which requires $(s_t, a_t, r_{t+1}, s_{t+1})$.

  - If (s_t, a_t) is given, then r_{t+1} and s_{t+1} do not depend on any policy!

- The behavior policy to generate a_t from s_t can be anything. The target policy will converge to the optimal policy.

Q-learning – Implementation

Since Q-learning is off-policy, it can be implemented in an either off-policy or on-policy fashion.

Pseudocode: Policy searching by Q-learning (on-policy version)

For each episode, do

If the current s_t is not the target state, do

Collect the experience $(s_t, a_t, r_{t+1}, s_{t+1})$: In particular, take action a_t following $\pi_t(s_t)$, generate r_{t+1}, s_{t+1}.

Update q-value:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q_t(s_t, a_t) - [r_{t+1} + \gamma \max_a q_t(s_{t+1}, a)]\Big]$$

Update policy:

$$\pi_{t+1}(a|s_t) = 1 - \frac{\epsilon}{|\mathcal{A}|}(|\mathcal{A}| - 1) \text{ if } a = \arg\max_a q_{t+1}(s_t, a)$$

$$\pi_{t+1}(a|s_t) = \frac{\epsilon}{|\mathcal{A}|} \text{ otherwise}$$

See the book for more detailed pseudocode.

Q-learning – Algorithm

Pseudocode: Optimal policy search by Q-learning (off-policy version)

For each episode $\{s_0, a_0, r_1, s_1, a_1, r_2, \ldots\}$ generated by π_b, do

For each step $t = 0, 1, 2, \ldots$ of the episode, do

Update q-value:

$$q_{t+1}(s_t, a_t) \;=\; q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q(s_t, a_t) - [r_{t+1} + \gamma \max_a q_t(s_{t+1}, a)]\Big]$$

Update target policy:

$\pi_{T,t+1}(a|s_t) = 1$ if $a = \arg\max_a q_{t+1}(s_t, a)$

$\pi_{T,t+1}(a|s_t) = 0$ otherwise

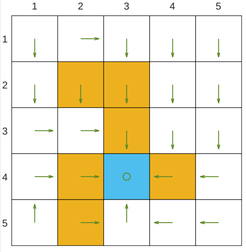
See the book for more detailed pseudocode.

Q-learning – Examples

Task description:

- The task in these examples is to find an optimal policy for all the states.
- The reward setting is $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, and $r_{\text{target}} = 1$. The discount rate is $\gamma = 0.9$. The learning rate is $\alpha = 0.1$.

Ground truth: an optimal policy and the corresponding optimal state values.



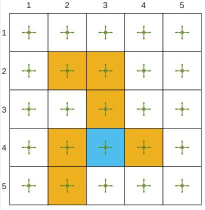
(a) Optimal policy

	1	2	3	4	5
1	5.8	5.6	6.2	6.5	5.8
2	6.5	7.2	8.0	7.2	6.5
3	7.2	8.0	10.0	8.0	7.2
4	8.0	10.0	10.0	10.0	8.0
5	7.2	9.0	10.0	9.0	8.1

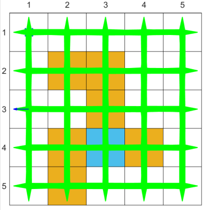
(b) Optimal state value

Q-learning – Examples

The behavior policy and the generated experience (10^5 steps):

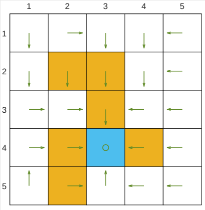


(a) Behavior policy

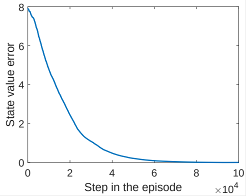


(b) Generated episode

The policy found by off-policy Q-learning:



(a) Estimated policy

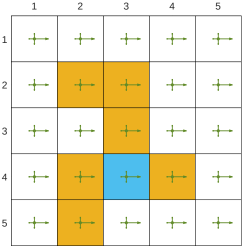


(b) State value error

Q-learning – Examples

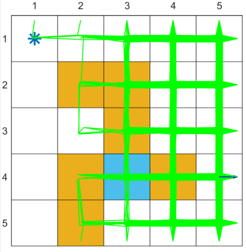
The importance of exploration: episodes of 10^5 steps

If the policy is not sufficiently exploratory, the samples are not good.

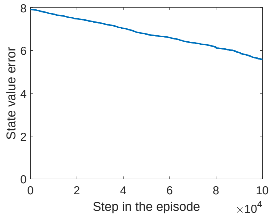


(a) Behavior policy

$\epsilon = 0.5$

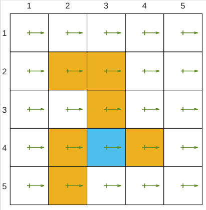


(b) Generated episode



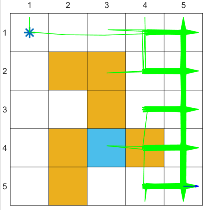
(c) Q-learning result

Q-learning – Examples

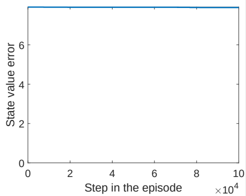


(a) Behavior policy

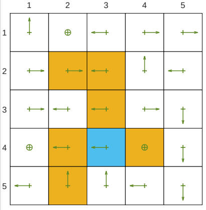
$\epsilon = 0.1$



(b) Generated episode

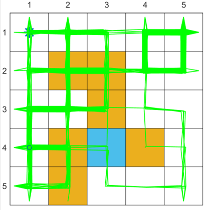


(c) Q-learning result

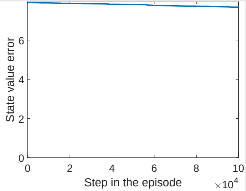


(a) Behavior policy

$\epsilon = 0.1$



(b) Generated episode



(c) Q-learning result

Outline

A unified point of view

All the algorithms we introduced in this lecture can be expressed in a unified expression:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)[q_t(s_t, a_t) - \bar{q}_t],$$

where $\bar{q}_t$ is the *TD target*.

Different TD algorithms have different $\bar{q}_t$.

Algorithm	**Expression of $\bar{q}_t$**
Sarsa	$\bar{q}_t = r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})$
n-step Sarsa	$\bar{q}_t = r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^n q_t(s_{t+n}, a_{t+n})$
Expected Sarsa	$\bar{q}_t = r_{t+1} + \gamma \sum_a \pi_t(a \vert s_{t+1}) q_t(s_{t+1}, a)$
Q-learning	$\bar{q}_t = r_{t+1} + \gamma \max_a q_t(s_{t+1}, a)$
Monte Carlo	$\bar{q}_t = r_{t+1} + \gamma r_{t+2} + \ldots$

The MC method can also be expressed in this unified expression by setting $\alpha_t(s_t, a_t) = 1$ and hence $q_{t+1}(s_t, a_t) = \bar{q}_t$.

A unified point of view

All the algorithms can be viewed as stochastic approximation algorithms solving the Bellman equation or Bellman optimality equation:

Algorithm	**Equation aimed to solve**
Sarsa	BE: $q_\pi(s,a) = \mathbb{E}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \vert S_t = s, A_t = a\right]$
n-step Sarsa	BE: $q_\pi(s,a) = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^n q_\pi(s_{t+n}, a_{t+n}) \vert S_t = s, A_t = a]$
Expected Sarsa	BE: $q_\pi(s,a) = \mathbb{E}\big[R_{t+1} + \gamma \mathbb{E}_{A_{t+1}}[q_\pi(S_{t+1}, A_{t+1})]\big\vert S_t = s, A_t = a\big]$
Q-learning	BOE: $q(s,a) = \mathbb{E}\left[R_{t+1} + \max_a q(S_{t+1}, a)\middle\vert S_t = s, A_t = a\right]$
Monte Carlo	BE: $q_\pi(s,a) = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \ldots \vert S_t = s, A_t = a]$

Outline

Summary

- Introduced various TD learning algorithms
- Their expressions, math interpretations, implementation, relationship, examples
- Unified point of view