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Chen Shaoxiang, M. Lambrecht,

TECHNICAL NOTE

X-Y BAND AND MODIFIED (s, S) POLICY

CHEN SHAOXIANG
M. LAMBRECHT

Department of Applied Economic Sciences, K. U. Leuven, Belgium

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This paper considers the stochastic, single-item, periodic review inventory problem. Most importantly we assume a finite production capacity per period and a production cost function containing a fixed (as well as a variable) component. With stationary data, a convex expected holding and shortage cost function, we show that generally the modified (s, S) policy is not optimal to the finite horizon problems. The optimal policy does, however, show a systematic pattern which we call the **X-Y band** structure. This **X-Y** band policy is interpreted as follows: whenever the inventory level drops below **X**, order up to capacity; when the inventory level is above **Y**, do nothing; if the inventory level is between **X** and **Y**, however, the ordering pattern is different from problem to problem. Although the **X** and **Y** bounds may change from period to period, we prove the existence of a pair of finite **X** and **Y** values that can apply for all the periods (i.e., bounds on individual bounds). One calculation for such **X** and **Y** bounds that are tight in some cases is also provided. By exploring the **X-Y** band structure, we can drastically reduce the computation effort for finding the optimal policies.

This paper deals with one of the most fundamental production and inventory models, i.e., the one-product, periodic review inventory replenishment system. The general description of the system is that at the beginning of each time period, upon a review of the inventory position, the decision on how much to order has to be made; an order, when placed, is delivered after a fixed number of time periods; demands for the item are random variables, and independently and identically distributed (i.i.d.) from period to period; the one-period expected holding and shortage cost function is assumed convex; and any unfilled demand in a period is entirely backlogged to be eventually satisfied by future deliveries. The problem is: what is the optimal ordering policy, and most importantly, what special pattern, if any, characterizes the optimal policy?

Numerous papers tackled this problem, and the research findings mainly depend on the treatments of the problem w.r.t. the production (ordering) cost function, the production capacity, the planning horizon, and the cost criterion of the objective function.

If the production (ordering) cost is linear (i.e., setup cost $K = 0$) and the production capacity (order quantity) per period is unlimited ($CP = \infty$), then the optimal ordering policy can be described by a single critical number noted as S: when initial stock is below S, enough should be produced (ordered) to bring total stock up to S; otherwise, nothing should be ordered. This policy is called the base-stock policy (see Scarf 1960, Scarf et al. 1963, and Veinott 1965).

If $K = 0$ and $CP < \infty$ (finite capacity), then the modified base-stock policy, proved by Federgruen and Zipkin (1986), is optimal: follow a base-stock policy when possible; when the prescribed production quantity would exceed the capacity, produce to capacity.

If $K > 0$ and $CP = \infty$, then the optimal policy is of the (s, S) type: when the inventory level falls below a critical number s, produce enough to bring total stock up to S; otherwise nothing should be produced. See Scarf and many others (e.g., Wagner 1972).

Although the critical number(s) of the above optimal policies (base-stock, modified base-stock, or (s, S)-policy) may vary from period to period, there exist stationary policies of corresponding types that are optimal to the infinite-horizon problems. Moreover, the optimal policy type is the same under either the expected-discounted-cost criterion or the average-cost criterion. See, e.g., Veinott and Wagner (1965).

The last case is the one in which $K > 0$ and $CP < \infty$. This problem has not yet been fully solved. Wijngaard (1972) gives an example of a 2-period, deterministic-demand problem having a complex optimal policy. Extension of the (s, S) policy in the uncapacitated case would result in the modified (s, S) policy for the capacitated problem, as stated in Federgruen and Zipkin:

> If the production costs have a fixed (as well as a variable) component, it might be reasonable to expect that the modified (s, S) policy would be optimal: when the inventory level falls below a critical number s, produce enough to bring total stock up to S, or as close as possible, given the production capacity; otherwise do not produce."

Figure 1 is a graphical summary of the four cases.

Subject classifications: Inventory/production· policies, review/lead times. Production/scheduling: planning. Dynamic programming/optimal control: Markov infinite slate.
Area of review: MANUFACTURING, OPERATIONS AND SCHEDULING.

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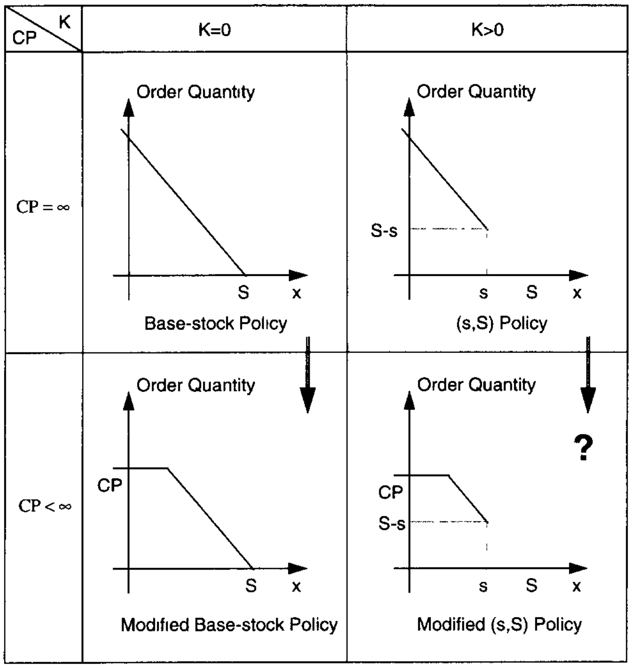

Figure 1. A literature review.

In this paper, we show that generally the modified (s, S)-policy is not optimal to the capacity constrained problems of finite horizon under the expected-discounted-cost criterion. Interestingly, the optimal policy does exhibit a special structure: the **X-Y band**. This structure implies that when the inventory level drops below **X**, produce at full capacity; when the inventory level is above **Y**, produce nothing. Unfortunately there is no clear pattern for inventory levels between **X** and **Y**. Although the **X** and **Y** bounds may change from period to period, we prove the existence of and provide one calculation for a pair of finite **X** and **Y** bounds that can apply for all the periods, and these bounds are tight in some cases. It is clear that since we need only to find the optimal decisions for those inventory levels that lie between the **X** and **Y** bounds, the computational effort for finding the optimal policies can be greatly reduced.

Section 1 sets the notation, the basic assumptions, and the dynamic program formulation. A 20-period counter example is given in Section 2, followed by the proofs of the existence of the **X** and **Y** bounds in Section 3 and 4.

1. NOTATION, ASSUMPTIONS AND THE DYNAMIC PROGRAMMING FORMULATION

We assume that the demand D in a period is a nonnegative, discrete random variable and is independently and identically distributed from period to period with probability distribution $p(j)$, i.e., $p(j) = \Pr(D = j), j = 0, 1, \ldots$ We also assume that D is upper bounded, i.e., $D \leqslant MD$ for some positive integer MD. The rationale for this will be given later. Practically, this is of no restriction at all. Although discrete demand is assumed for notational convenience, most of the results obtained in this paper can be applied to the continuous demand case as well. Let

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x = inventory level (on hand plus on order) prior to placing any order in a period.

y = inventory on hand plus on order subsequent to an order decision but before the demand occurs in a period.

α = single period discount factor, $0 \leqslant \alpha < 1$. 必须小于 1

CP = production capacity, or limit on order size, a positive integer.

$K\delta(q) + cq$ = production cost (ordering cost) of producing (ordering) q units, where K ($\geqslant 0$) is the setup cost, c ($\geqslant 0$) is the unit cost, and $\delta(0) = 0$, $\delta(q) = 1$ if $q > 0$.

$L(y)$ = one period expected holding and shortage penalty cost function. See, e.g. Wagner (1972), for the calculations of $L(y)$ under different postulations.

Finally, let

$f_n(x)$ = expected discounted cost for an n-period horizon problem, if the beginning inventory level is x, and if optimal policies are followed over the n periods.

Assumption. **(I)** $L(y)$ is a convex function.
(II) $\lim_{|y|\to\infty}[cy + L(y)] = \infty$.

The dynamic program formulation of the problem is given by:

$$f_n(x) = \underset{y\in[x,x+CP]}{\text{minimum}} \Big\{K\delta(y-x) + c(y-x) + L(y) + \alpha \sum_{j=0}^{\infty} f_{n-1}(y-j)p(j)\Big\}, \; f_0(x) \equiv 0 . \tag{1}$$

In order to guarantee that $\sum_{j=0}^{\infty} f_{n-1}(y-j)p(j) < \infty$ (convergence), one has to make an assumption about the probability distribution $p(j)$, or the cost function $L(y)$, or both. To avoid this trouble, we simply assume that the demand cannot be infinite. Hence there is an upper bound MD such that $D \leqslant MD$, or $p(j) = 0$ for $j > MD$. The latter expression allows Equation (1) to be stated as it is. If appropriate, we will use $\sum_{j\geqslant 0}$, or simply $\sum$, instead of $\sum_{j=0}^{\infty}$ for simplicity. 需求有限

Define $g_n(y)$ as

$$g_n(y) = L(y) + cy + \alpha \sum_{j=0}^{\infty} f_{n-1}(y-j)p(j) . \tag{2}$$

Then,

$$f_n(x) = \underset{y\in[x,x+CP]}{\text{minimum}} \Big\{K\delta(y-x) + c(y-x) + L(y) + \alpha \sum_{j=0}^{\infty} f_{n-1}(y-j)p(j)\Big\} = \min\begin{cases} g_n(x) - cx, \\ \underset{y\in[x,x+CP]}{\text{minimum}}\, g_n(y) - cx + K. \end{cases} \tag{3}$$

Suppose $\text{minimum}_{y\in[x,x+CP]}\, g_n(y) = g_n(y^*)$, where $y^* \in [x, x + CP]$. (This is justified by the continuity of $g_n(y)$.

闭区间连续函数必有最值

See Shaoxiang and Lambrecht for the proof.) Then the optimal decision is

$$y_n^*(x) = \begin{cases} y^* & \text{if } g_n(x) \geqslant g_n(y^*) + K, \\ x & \text{if } g_n(x) < g_n(y^*) + K. \end{cases} \tag{4}$$

In (4), if $g_n(x) = g_n(y^*) + K$, both $y_n^*(x) = y^*$ and $y_n^*(x) = x$ are optimal. Thus in the following sections we may select either of them as the optimal decision.

2. A COUNTEREXAMPLE

In the example below, we assume a linear holding and shortage cost function with h denoting the holding cost per unit-period and π the shortage cost per unit-period.

Example 1. Assume that $h = 1.0$, $\pi = 10.0$, $K = 22.0$, $c = 1.0$, $\lambda = 0$, $\alpha = 0.90$, $CP = 9$, and the demand distribution is $p(6) = 0.95$, $p(7) = 0.05$.

The optimal order quantities, $O_n^*(x)$, obtained through the dynamic program (1) up to 20 periods are given below:

x \ n	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
−5	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
−4	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
−3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
−2	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
−1	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	9	7	7
0	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	6	6
1	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	5	5
2	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	9	4
3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	3
4	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	0
5	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	0	7	7	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

As an illustration of how to read the above results, suppose that the beginning inventory level (x) is 0, and there are 19 periods (n) to go, then the optimal order quantity is 9 units.

We should pause here to remind the reader that under a modified (s, S) policy, the order quantity should be a **non-increasing** function of the inventory level at the beginning of a period (Figure 1).

Now, two important facts can be observed from Example 1. First, it clearly shows that the optimal order policies for periods from 2 up to 20 are not the modified (s, S) type. Second, the optimal order quantity is always 9 ($=CP$) units if $x \leqslant -3$, and is always 0 if $x \geqslant 6$, and it is true for **all** the periods. This suggests there exist two inventory level bounds, **X** and **Y**, the lower and upper bound respectively, such that if the beginning inventory level is lower then or equal to **X**, the optimal decision is to order at capacity; if the inventory level is above or equal to **Y**, then nothing should be ordered. The question is: does this special structure prevail in general?

3. THE X BOUND

Lemma 1. *There exists* **X**, *such that it is optimal to order up to full capacity for all* $x \leqslant \mathbf{X}$, *and all* $\mathbf{n} \geqslant 1$, *i.e.*, $y_n^*(x) = x + CP$, *for* $\forall x \leqslant \mathbf{X}$, *and* $\forall n > 1$.

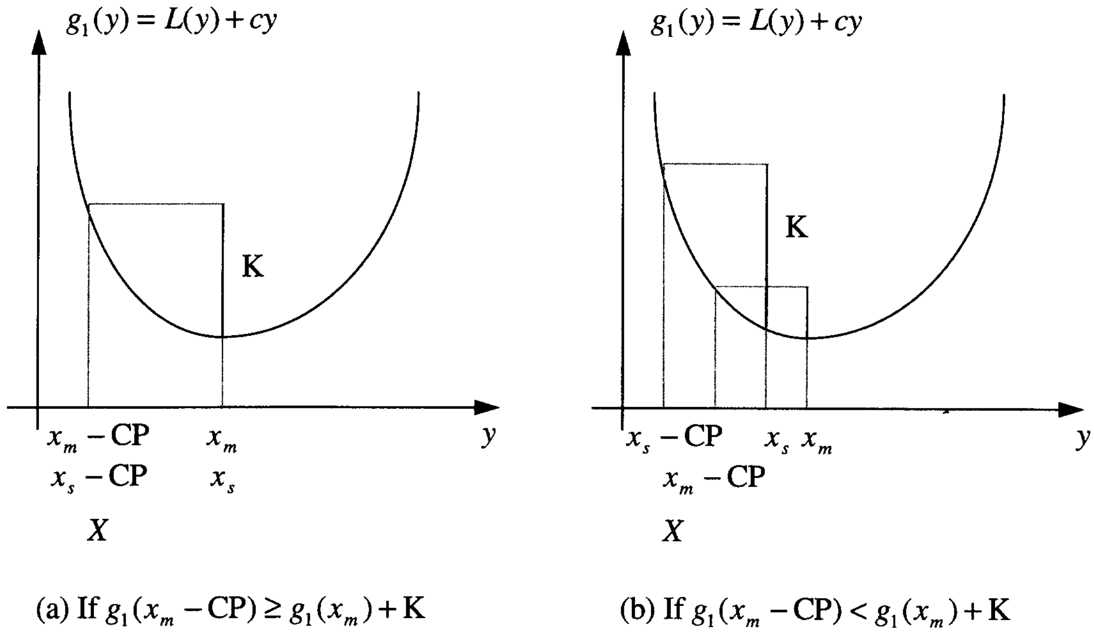
Proof. Let x_m be the point at which the convex function $cy + L(y)$ reaches its minimum. Then, $cy + L(y)$ is nonincreasing for $y \leqslant x_m$, and nondecreasing for $y \geqslant x_m$. For simplicity, we will use decreasing to mean nonincreasing, and increasing to mean nondecreasing.

Note that $g_1(y) = cy + L(y)$ by the definition of $g_n(y)(2)$ and of $f_0(x)(\equiv 0)$. Let x_s ($\leqslant x_m$) be the maximum point at which

x_s-CP 是最大能力订货的最大库存水平

$$c \cdot (x_s - CP) + L(x_s - CP) = g_1(x_s - CP) \geqslant cx_s + L(x_s) + K = g_1(x_s) + K. \tag{5}$$

That is, $(x_s - CP)$ is the highest inventory level at which full capacity ordering is required by the optimal policy for a single-period model ($n = 1$). Note that if $g_1(x_m - CP) \geqslant g_1(x_m) + K$, then $x_s = x_m$; if $g_1(x_m - CP) < g_1(x_m) + K$, then x_s should be the highest point at which $g_1(x_s - CP) \geqslant g_1(x_s) + K$. Let $\mathbf{X} = x_s - CP$. Graphically, x_s, **X** can be determined as follows:



x_m 是 L(y)+cy 的最小值点

Figure 2. Determining x_s and X bound.

Now we will prove the following for all $n \geqslant 1$:

(a) $f_{n-1}(x)$ is a decreasing function for $x \leqslant x_m$.
(b) $g_n(y)$ is a decreasing function for $y \leqslant x_m$.
(c) $g_n(y) \geqslant g_n(y + CP) + K$ for $y \leqslant x_s - CP \equiv \mathbf{X}$.
(d) $y_n^*(x) = x + CP$ for $x \leqslant x_s - CP = \mathbf{X}$.

x_s 的定义 X bound x_s <= x_m (定义)

For $n = 1$, (a) is true by the definition of $f_0(x)$.

(b): $g_1(y) = cy + L(y)$, which is a decreasing function for $y \leqslant x_m$.
(c) is derived from the convexity of $g_1(y)$ and the definition of x_s.

根据定义直接得出

(d) is derived from (c), (b), and (4).

Now, suppose the results for n. We prove the results for $n + 1$.

For (a), define $F_n(x)$ as

$$F_n(x) = \underset{y \in [x, x+CP]}{\text{minimum}}\, g_n(y).$$

By (b) (in case of n), $g_n(y)$ is a decreasing function for $y \leqslant x_m$, thus $F_n(x) = g_n(x + CP)$ for $x \leqslant x_m - CP$, and hence $F_n(x)$ is a decreasing function for $x \leqslant x_m - CP$. Now, we show that $F_n(x)$ is also a decreasing function for $x \in [x_m - CP, x_m]$. By (b), for $x \in [x_m - CP, x_m]$:

$$F_n(x) = \underset{y \in [x, x+CP]}{\text{minimum}}\, g_n(y) = \min\begin{cases} g_n(x_m) \\ \underset{y \in [x_m, x+CP]}{\text{minimum}}\, g_n(y), \end{cases}$$

根据最小推出

clearly, $F_n(x)$ is a decreasing function of x. And finally since

$$f_n(x) = \min\begin{cases} g_n(x) - cx \\ F_n(x) - cx + K, \end{cases}$$

and both $(g_n(x) - cx)$ and $(F_n(x) - cx + K)$ are decreasing functions for $x \leqslant x_m$, so is $f_n(x)$.

For part (b), since

$$g_{n+1}(y) = L(y) + cy + \alpha \sum_{j \geqslant 0} f_n(y - j)p(j),$$

and $y - j \leqslant x_m$ if $y \leqslant x_m$. Hence, (b) follows from (a) and the definition of x_m.

只有需求是 stationary 时才能这样

For part (c). If $y \leqslant x_s - CP$, then $y + CP - j \leqslant x_s \leqslant x_m$. Thus, by (a)

$$\alpha \sum_{j \geqslant 0} f_n(y - j)p(j) \geqslant \alpha \sum_{j \geqslant 0} f_n(y + CP - j)p(j), \tag{6}$$

and by the definition of x_s and the convexity of $L(y) + cy$, for $y \leqslant x_s - CP$,

$$L(y) + cy \geqslant L(y + CP) + c \cdot (y + CP) + K, \tag{7}$$

so, (c) follows from (6), (7), and the definition of $g_{n+1}(y)$.

Part (d) follows immediately from (b), (c), and (4). Lemma 1, which is (d), is thus proved by induction. □

Two points should be stressed here. First, the **X** bound in Lemma 1 is a **global** one, i.e., as long as the inventory drops below **X**, the production should be set at full capacity, no matter how many periods to go. Second, Lemma 1 also provides a calculation for such a **global X** bound.

引理1也提供了一种计算方法

4. THE Y BOUND

Proposition 1. *For any $x \in \Re$, and $0 \leqslant a \leqslant CP$, there exists a', $0 \leqslant a' \leqslant CP$, such that for any $n \geqslant 1$,*

$$f_n(x) - f_n(x + a) \leqslant \max\begin{cases} K + ca \\ g_n(y') - g_n(y' + a') + ca, \quad \text{where } y' = x + CP. \end{cases}$$

Proof. For any $x \in \Re$, and $a \in [0, CP]$, consider $g_n(y)$ and $f_n(x)$:

If at $x + a$, it is optimal not to order, then by (3), $f_n(x + a) = g_n(x + a) - c, (x + a)$. And again by (3), $f_n(x) \leqslant g_n(x + a) - cx + K$. Hence

$$\begin{aligned} &f_n(x) - f_n(x+a) \\ &\quad \leqslant g_n(x+a) - cx + K - (g_n(x+a) - c\cdot(x+a)) \\ &\quad = K + ca\,; \end{aligned}$$

Otherwise, it is optimal at $x + a$ to order, say, b units, $0 < b \leqslant CP$, then

$$f_n(x+a) = g_n(x+a+b) - c\cdot(x+a) + K\,.$$

If $x + a + b \leqslant x + CP$, then by (3), $f_n(x) \leqslant g_n(x + a + b) - cx + K$, and hence

$$f_n(x) - f_n(x+a) \leqslant ca\,;$$

Otherwise, $x + CP < x + a + b \leqslant x + CP + CP$. Let $y' = x + CP$, $a' = a + b - CP$. Then, $0 < a' \leqslant CP$, $x + a + b = y' + a'$, and by (3),

$$\begin{aligned} &f_n(x) - f_n(x+a) \\ &\quad \leqslant g_n(x+CP) - cx + K - (g_n(x+a+b) \\ &\quad - c\cdot(x+a) + K) \\ &\quad = g_n(y') - g_n(y'+a') + ca\,. \end{aligned}$$

Hence, Proposition 1, as a summary, is proven. □

4.1. The Y Bound, If $MD \leqslant CP$

As denoted earlier, MD stands for the maximum demand in a period. We first consider the case $MD \leqslant CP$.

Let x_L be the point at which $L(y)$ is minimized. Recall that x_m is the point at which $g_1(y)$ $(= L(y) + cy)$ is minimized. It can be shown very easily that

$$x_L \geqslant x_m \geqslant x_s\,. \tag{8}$$

Now, we show that x_L is a **global** **Y** bound in case $MD \leqslant CP$.

Lemma 2. *If $MD \leqslant CP$, then for all $x \geqslant x_L \equiv \mathbf{Y}$, all $n \geqslant 1$, $y_n^*(x) = x$, i.e., it is optimal not to order if the starting inventory level $x \geqslant x_L = \mathbf{Y}$. Specifically, the following are true:*

(e) $f_{n-1}(x) \leqslant f_{n-1}(x + a) + ca + K$ for all $x \geqslant x_L - CP$, and all $0 \leqslant a \leqslant CP$;

(f) $g_n(y) < g_n(y + a) + K$ for all $y \geqslant x_L = \mathbf{Y}$, and for all $0 \leqslant a \leqslant CP$;

(g) $y_n^*(x) = x$ for all $x \geqslant x_L = \mathbf{Y}$.

Proof. For $n = 1, f_{n-1}(x) = f_0(x) \equiv 0$, so (e) is true. Since $x_L \geqslant x_m$ (see (8)), so (f) follows by the convexity of $g_1(y)$. (g) follows immediately from (f) and (4).

Now, suppose the results for n, we prove for $n + 1$.

For part (e), first note that for any a, $0 \leqslant a \leqslant CP$, and for any $x \geqslant x_L$, by (3), (4) and (f) (in case of n),

$f_n(x) = g_n(x) - cx$, and hence

$$f_n(x) - f_n(x+a) = g_n(x) - g_n(x+a) + ca \leqslant K + ca\,.$$

Second, for any given $x \in [x_L - CP, x_L]$, $x + CP \geqslant x_L$. Hence, by Proposition 1 and (f),

$$f_n(x) - f_n(x+a) \leqslant K + ca\,.$$

Thus (e) follows. Indeed, (e) can be directly derived from Proposition 1 and (f).

需要 j(MD)<=CP 的假设

For (f), if $y \geqslant x_L$, then, $y - j \geqslant x_L - j \geqslant x_L - CP$. Thus by the definition of x_L and from (e)

适用于 stationary 需求的情况

$$\begin{aligned} g_{n+1}(y) &= L(y) + cy + \alpha \sum f_n(y-j)p(j) \\ &\leqslant L(y+a) + cy + \alpha \sum (f_n(y+a-j) \\ &\qquad + ca + K)p(j) \\ &< g_{n+1}(y+a) + K\,. \end{aligned}$$

Part (g) follows immediately from (f) and (4).

Thus, Lemma 2 is proven by induction. □

Return to Example 1 for an illustration of how to calculate the **X** and **Y** bounds. The one-period expected holding and shortage cost function of Example 1 can be derived easily:

$$L(y) = \begin{cases} 60.5 - 10y & \text{for } y \leqslant 6, \\ -2.2 + 0.45y & \text{for } 6 \leqslant y \leqslant 7, \\ -6.05 + y & \text{for } y \geqslant 7 \end{cases} \qquad (\text{note: } L(y) \in C).$$

It is obvious that for our example, $x_s = x_m = x_L = 6$. Hence, by Lemma 2 and Lemma 3,

$$\mathbf{X} = x_s - CP = 6 - 9 = -3, \quad \mathbf{Y} = x_L = 6\,,$$

and the optimal policy (optimal order quantity) can then be described as

$$O_n^*(x) = \begin{cases} 9 & \text{for } x \leqslant -3, \\ \text{undetermined} & \text{for } -2 \leqslant x \leqslant 5, \\ 0 & \text{for } x \geqslant 6. \end{cases}$$

This solution is exactly the same as shown in Table 1. Interestingly, the **X** and **Y** bounds calculated above are exactly reached in this example. Thus, the **X** and **Y** bounds given by Lemma 1 and Lemma 2 are sometimes tight. It should be noted here that, of many examples we have tried, there is no additional "pattern" within the **X-Y** band, though the example given above does.

4.2. The Y Bound, If $MD > CP$

The crucial assumption needed for Lemma 2 is that the demand never exceeds capacity. The following example is designed to show what might happen if the condition is violated.

Example 2. Let $h = 0.2$, $\pi = 10.0$, $K = 15$, $c = 1.0$, $\lambda = 0$, $\alpha = 0.95$, $CP = 8$, and $p(0) = 0.3$, $p(10) = 0.7$.

一个MD > CP时的反例

After writing down the one-period expected holding and shortage cost function, one can clearly see that $x_s = x_m = x_L = 10$. Hence, by Lemma 1, $\mathbf{X} = x_s - CP = 10 - 8 = 2$. If Lemma 2 were still true, $\mathbf{Y}$ would be: $\mathbf{Y} =$

$x_L = 10$. However, the optimal order quantities, $O_n^*(x)$, from the dynamic program (1) up to 20 periods are as follows:

		n																			
		20	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	2	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
	3	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	8	7
	4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	8	6
	5	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	8	7	5
	6	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	8	8	6	4
	7	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	8	8	7	5	3
	8	·	·	·	·	·	·	·	·	·	·	·	·	·	·	8	8	8	6	4	0
	9	·	·	·	·	·	·	·	·	·	·	·	·	·	8	8	8	7	5	3	0
	10	·	·	·	·	·	·	·	·	·	·	·	·	8	8	8	8	6	4	0	·
	11	·	·	·	·	·	·	·	·	·	·	·	8	8	8	7	7	5	0	·	·
x	**12**	·	·	·	·	·	·	·	·	·	·	8	8	8	8	6	6	8	0	·	·
	13	·	·	·	·	·	·	·	·	·	8	8	8	8	7	5	8	0	·	·	·
	14	·	·	·	·	·	·	·	·	8	8	8	8	8	6	4	0	·	·	·	·
	15	·	·	·	·	·	·	·	8	8	8	8	8	7	5	0	·	·	·	·	·
	16	·	·	·	·	·	·	8	8	8	8	8	8	6	0	·	·	·	·	·	·
	17	·	·	·	·	·	8	8	8	8	8	7	7	8	0	·	·	·	·	·	·
	18	·	·	·	·	8	8	8	8	8	8	6	0	0	·	·	·	·	·	·	·
	19	·	·	·	8	8	8	8	8	7	7	0	·	·	·	·	·	·	·	·	·
	20	·	·	8	8	8	8	8	0	0	0	·	·	·	·	·	·	·	·	·	·
	21	·	8	8	8	8	0	0	·	·	·	·	·	·	·	·	·	·	·	·	·
	22	8	0	0	0	0	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·

It seems that the **Y** bound is increasing as n increases. Note that in this example, the expected demand in a period (ED) is 7, which is less than the capacity CP. Thus, even if the assumption $MD \leqslant CP$ in Lemma 2 is relaxed to $ED \leqslant CP$, it is still hard to see the existence of **a global Y** bound. Yet, we show in the next section that such a **Y** bound still exists, and can be calculated.

In this section, we make an additional assumption about $L(y)$, which is similar to the one made in Federgruen and Zipkin.

Assumption (III). $L(y) - L(y + MD) \leqslant A + B|y - x_L|^{\tau}$ for some nonnegative integer τ, and for some positive constants A and B.

Assumption **(III)** says that the one-period expected holding and shortage cost function $L(y)$ is polynomially bounded. If it is not the case, Chen and Lambrecht show that $f_n(x)$ may diverge as n increases.

Let M be set as

$$M = \sum_{i=1}^{\infty} \alpha^i (A + B(i\ MD)^{\tau}) .$$

Since $\alpha^{i+1}(i+1)^{\tau}/\alpha^i i^{\tau} \to \alpha < 1$, so, M is a finite value.

Let N be the smallest integer such that

$$\alpha^N M \leqslant K , \qquad (9)$$

N 会很大

then, we prove that

$$\mathbf{Y} \equiv x_L + N\ MD , \qquad (10)$$

is one of the global **Y** bounds, independent of n.

Thus, suppose that the beginning inventory x is greater than or equal to **Y** (set in (10)), and there are n periods to go. The question is: should we order something now?

First of all, if $n \leqslant N$, then nothing should be ordered now (and in the future), since the beginning inventory is already high enough to satisfy all possible demands over the rest n periods.

Now, we assume $n > N$, and we study two scenarios:

1. order nothing;
2. order, say, b units.

In order to compare and be able to say that Scenario 1 is better than 2, we assume:

a. an optimal ordering strategy is to be followed by the system under Scenario 2 over the rest $(n - 1)$ periods (the decision for the first period has been made);
b. in each future period, the system under Scenario 1 will order exactly the same order quantity as that implied in a.

Denote $f_n(x)$ and $f_n(x+b)$ as the expected discounted costs under Scenario 1 and 2, respectively, and let y_i (y_i') be the inventory level subsequent to an ordering decision but before the demand D_i occurs in period i under Scenario 1 (2). Then, it is clear by a and b above that for whatever sample path of demands D_i,

$$y_i' = y_i + b\,. \tag{11}$$

Since (or can be easily derived that)

$$f_n(x) = \sum_{i=1}^{n} \alpha^i E(L(y_i))$$

之后的决策（是否订货，订货多少）全一样

$$+ \text{ Expected discounted ordering cost, and}$$

$$f_n(x+b) = \sum_{i=1}^{n} \alpha^i E(L(y_i')) + K + cb$$

$$+ \text{ Expected discounted ordering cost}\,.$$

(We leave implicit in the notation above (and below) the dependence of y_i (y_i') on $D_1, D_2, \ldots, D_{i-1}$.) Note that, because of b, the ordering costs under Scenario 1 are the same as that under Scenario 2 (except of course in the first period), so is the expected discounted ordering cost. Thus, by (11),

$$f_n(x) - f_n(x+b) = \sum_{i=1}^{n} \alpha^i E[L(y_i) - L(y_i + b)] - K - cb\,.$$

Intuitively, the inventory holding cost will be lower under Scenario 1 than that of 2, but the shortage penalty cost could be higher under Scenario 1. However, since the beginning inventory is "very high" ($x \geqslant \mathbf{Y}$), the first shortage may only occur at least N periods later. Mathematically, $L(y_i) - L(y_i + b) \leqslant 0$ in the first N periods (at least), but because of the convexity of $L(y)$, it increases as y decreases. Yet, y can at most decrease MD units a period. Thus, it is clear that

$$\begin{aligned} f_n(x) - f_n(x+b) &\leqslant \alpha^N E \sum_{i=1}^{n-N} \alpha^i [L(x_L - i\,MD) - L(x_L - (i-1)MD)] \\ &\quad - K - cb \\ &\leqslant \alpha^N E \sum_{i=1}^{n-N} \alpha^i (A + B(i\,MD)^{\tau}) - K - cb \leqslant \alpha^N M - K \\ &\quad - cb < 0\,, \end{aligned}$$

where the last three inequalities are derived by Assumption (III), the definition of M, and by (9). The above inequality shows that the system under Scenario 1 has a lower expected discounted cost than that under 2, even if it followed a suboptimal strategy. This result proves that it is optimal not to order if $x \geqslant \mathbf{Y}$.

Note that one calculation for a **global** $\mathbf{Y}$ bound is also provided in the above proof. However, unlike the case of $MD \leqslant CP$, where the $\mathbf{Y}$ bound (given in Lemma 2) is tight in some cases, this time the $\mathbf{Y}$ bound is quite conservative. Interested readers are referred to Chen and Lambrecht for a tighter calculation of the $\mathbf{Y}$ bound for the case $MD > CP$.

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REFERENCES

FEDERGRUEN, A., AND P. ZIPKIN. 1986. An Inventory Model with Limited Production Capacity and Uncertain Demands I. The Average-cost Criterion. II. The Discounted-cost Criterion. *Math. Opns. Res.* **11,** 2, 193–215.

DE GHELLINCK, G. T., AND G. D. EPPEN. 1967. Linear Programming Solution for Separable Markovian Decision Problems. *Mgmt. Sci.* **13,** 5, 371–394.

IGLEHART, D. L. 1963. Optimality of (s, S) Policies in the Infinite Horizon Dynamic Inventory Problem. *Mgmt. Sci.* **9,** 259–267.

SCARF, H. 1960. The Optimality of (S, s) Policies in the Dynamic Inventory Problems. *Mathematical Methods in the Social Science*, K. J. Arrow, S. Karlin, and P. Suppes (Eds.), Stanford University Press.

SCARF, H., D. GILFORD, AND M. SHELLY (EDS.). 1963. *Multistage Inventory Models and Techniques*, Stanford University Press, Stanford.

SHAOXIANG, C., AND M. LAMBRECHT. 1990. **X-Y** Band and (s, S) Policies. Research Paper, No. 9011, ETEW, K. U. Leuven.

VEINOTT, A. 1966. The Status of Mathematical Inventory Theory. *Mgmt. Sci.* **12,** 745–766.

VEINOTT, A., AND H. M. WAGNER. 1965. Computing Optimal (s, S) Inventory Policies. *Mgmt. Sci.* **11,** 5, 525–552.

WAGNER, H. M. 1972. Principles of Operations Research. Appendix II. Prentice-Hall International Editions.

WIJNGAARD, J. 1972. An Inventory Problem with Constrained Order Capacity. TH-Report 72-WSK-63, Eindhoven University of Technology.