Available online at www.sciencedirect.com

SCIENCE DIRECT®

European Journal of Operational Research 174 (2006) 581–599

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/ejor

Production, Manufacturing and Logistics

# Modelling and computing $(R^n, S^n)$ policies for inventory systems with non-stationary stochastic demand

S. Armagan Tarim [a,*], Brian G. Kingsman [b]

[a] Department of Computer Science, Cork Constraint Computation Centre, University College Cork, Cork, Ireland
[b] Department of Management Science, Management School, Lancaster University, Lancaster, LA1 4YX, UK

Received 17 October 2003; accepted 21 January 2005
Available online 8 April 2005

**Abstract**

This paper addresses the single-item, non-stationary stochastic demand inventory control problem under the non-stationary $(R, S)$ policy. In non-stationary $(R, S)$ policies two sets of control parameters—the review intervals, which are not necessarily equal, and the order-up-to-levels for replenishment periods—are fixed at the beginning of the planning horizon to minimize the expected total cost. It is assumed that the total cost is comprised of fixed ordering costs and proportional direct item, inventory holding and shortage costs. With the common assumption that the actual demand per period is a normally distributed random variable about some forecast value, a certainty equivalent mixed integer linear programming model is developed for computing policy parameters. The model is obtained by means of a piecewise linear approximation to the non-linear terms in the cost function. Numerical examples are provided.


*Keywords:* Inventory; Modelling; Non-stationary $(R, S)$ policy; Stochastic demand

## 1. Introduction

An important problem faced in coordinating suppliers and buyers in supply chain partnerships is the system nervousness. Nervousness arises when a formerly fixed order decision for a certain period is replanned later. The deviations causing nervousness may be in the form of quantity adjustments and/or changes in order setups. It is noted in [5,6] that nervousness due to deviations in order setups is considered as the most serious in practice and is referred to as *setup instability*.

* Corresponding author.
*E-mail address:* at@4c.ucc.ie (S.A. Tarim).

0377-2217/$ - see front matter 
doi:10.1016/j.ejor.2005.01.053

The $(s, S)$ type of control policy, which can be shown to be optimal for a wide class of problems under pure cost consideration, has been questioned with respect to nervousness criterion by Inderfurth [6] and De Kok and Inderfurth [8]. Their research reveals that, in terms of nervousness, the $(s, S)$ policy exhibits the worst performance among a number of policies considered. Blackburn et al. [1] examine the effectiveness of different strategies for dealing with the problem of nervousness and suggest a strategy which is based on the work of Carlson et al. [4]. The essence of this strategy is to encourage setups in periods where they are scheduled previously. In this regard, the $(R, S)$ policy—in which the schedules are set, particularly for the timing of future orders with the associated order-up-to-levels—provides a means of dampening the setup instability and coping with the demand uncertainty in production and inventory systems.

If demand were stationary, the optimal inventory replenishment policy determined using $(R, S)$ would be cyclical, that is, a sufficient amount of order would be placed every fixed number of $R$ periods to raise the inventory position to the level $S$. However, under non-stationary stochastic demand assumption this presents a difficult problem where the two control parameters, $R^n$ and $S^n$, are non-stationary (see [9]).

The ‘‘static-dynamic uncertainty’’ strategy proposed by Bookbinder and Tan [2] is one approach to work out the parameters of non-stationary $(R, S)$ policies. Bookbinder–Tan's solution heuristic is a two-stage process of firstly fixing the replenishment periods and then secondly determining what adjustments should be made to the planned orders as demand is realized. The total cost, composed of ordering and inventory holding costs, is minimized under a minimal service level constraint. A formulation of the same problem that determines both the replenishment periods and the associated order-up-to-levels simultaneously, hence gives the optimal solution, is presented by Tarim and Kingsman [11]. In addition to the inventory holding and ordering costs of [2], Tarim and Kingsman take into account the direct item costs. The expected total cost during the planning horizon is minimized also under a minimal service level constraint.

The model developed in this paper addresses the non-stationary $(R, S)$ policy under the same assumptions, except the adoption of shortage costs for backorders rather than the imposition of minimal service level constraints. This version of the problem, due to the non-linearity of the cost function, is significantly more complicated than that addressed in [2,11]. In order to deal with this complexity, a piecewise linear approximation to the cost function is proposed and, hence, the non-linear model is reduced to a mixed integer linear one. The resultant model gives the replenishment periods, from which one can determine the non-stationary review intervals $R^n$, and the order-up-to-levels, $S^n$.

The issue of system nervousness is an active current research area. If there is to be more co-operation and co-ordination in supply chains, then a model that attempts to determine a frozen schedule for the timing of orders in advance taking account of the stochastic demand is a contribution of practical interest. This need to fix the deliveries in advance, whilst allowing reasonable flexibility in the order size, has been at the heart of the problem of buying raw materials on fluctuating price markets [7]. In this regard, an investigation of models and solution procedures for non-stationary $(R, S)$ policies is potentially important from the practical application perspective.

The rest of the paper is organized as follows. Section 2 defines the problem and sets the notation. Section 3 explores certain properties of the stochastic component of the expected total cost function. Section 4 presents a piecewise linear approximation to the non-linear cost function of the stochastic lot-sizing problem. A certainty equivalent mixed ILP model for the stochastic inventory problem is given in Section 5. Section 6 is devoted to illustrative numerical examples. In the final section some concluding remarks are stated.

## 2. Multi-period stochastic lot-sizing problem

The demand, $d_t$ in period $t$, is considered as a random variable with known probability density function, $g_t(d_t)$, and is assumed to occur instantaneously at the beginning of each period. The mean rate of demand may vary from period to period. Demands in different time periods are assumed independent. A fixed holding cost $h$ is incurred on any unit carried in inventory over from one period to the next. Demands occurring when the system is out of stock are assumed to be backordered, and satisfied immediately the next replenishment order arrives. A fixed shortage cost $s$ is incurred for each unit of demand backordered. A fixed procurement (ordering or set-up) cost $a$ is incurred each time a replenishment order is placed, whatever the size of the order. In addition to the fixed ordering cost, a proportional direct item cost $v$ is incurred. For convenience, without loss of generality, the initial inventory level is set to zero and the delivery lead-time is not incorporated. It is assumed that negative orders are not allowed, so that if the actual stock exceeds the order-up-to-level for that review, this excess stock is carried forward and not returned to the supply source. However, such occurrences are regarded as rare events and accordingly the cost of carrying the excess stock is ignored. The above assumptions are valid for the rest of the paper.

The general multi-period production/inventory problem with stochastic demands can be formulated as finding the timing of the stock reviews and the size of the non-negative replenishment orders, $X_t$ in period $t$, then placed that minimize the expected total cost over a finite planning horizon of $N$ periods, as given below:

$$E\{TC\} = \int_{d_1} \cdots \int_{d_N} \sum_{t=1}^{N} \left(a\delta_t + vX_t + hI_t^+ + sI_t^-\right) g_1(d_1) \cdots g_N(d_N) \mathrm{d}(d_1) \cdots \mathrm{d}(d_N), \tag{1}$$

subject to

$$X_t - M\delta_t \leqslant 0, \quad t = 1, \ldots, N, \tag{2}$$

$$I_t = \sum_{i=1}^{t} (X_i - d_i), \quad t = 1, \ldots, N, \tag{3}$$

$$I_t^+ = \max(0, I_t), \quad t = 1, \ldots, N, \tag{4}$$

$$I_t^- = -\min(0, I_t), \quad t = 1, \ldots, N, \tag{5}$$

$$X_t, I_t^+, I_t^- \geqslant 0, \quad I_t \in \mathbb{R}, \quad \delta_t \in \{0, 1\}, \quad t = 1, \ldots, N, \tag{6}$$

where

- $d_t$ the demand in period $t$, a random variable with probability density function, $g_t(d_t)$,
- $a$ the fixed ordering cost,
- $v$ the proportional direct item cost,
- $h$ the proportional stock holding cost,
- $s$ the proportional shortage cost,
- $\delta_t$ a $\{0, 1\}$ variable that takes the value of 1 if a replenishment occurs in period $t$ and 0 otherwise,
- $I_t$ the inventory level at the end of period $t$, $-\infty < I_t < +\infty$, $I_0 = 0$,
- $I_t^+$ the excess inventory at the end of period $t$ carried over to the next period, $0 \leqslant I_t^+$,
- $I_t^-$ the shortages at the end of period $t$, magnitude of the negative inventory, $0 \leqslant I_t^-$,
- $X_t$ the replenishment order placed and received in period $t$, $0 \leqslant X_t$,
- $M$ some large positive number.

The non-stationary $(R, S)$ policy proposed consists of a series of review times and associated order-up-to-levels. Consider a review schedule which has $m$ reviews over the $N$ period planning horizon with orders arriving at $\{T_1, T_2, \ldots, T_m\}$, $T_j > T_{j-1}$. For convenience $T_1 = 1$ is defined as the start of the planning horizon and $T_{m+1} = N + 1$ the period immediately after the end of the horizon. The total expected cost function, Eq. (1), for this replenishment policy can be expressed in the form

$$E\{TC\} = \int_{d_1} \ldots \int_{d_N} \sum_{i=1}^{m} \sum_{t=T_i}^{T_{i+1}-1} \left(a\delta_t + vX_t + hI_t^+ + sI_t^-\right) g_1(d_1) \ldots g_N(d_N) \mathrm{d}(d_1) \ldots \mathrm{d}(d_N). \tag{7}$$

In this dynamic review and replenishment policy, clearly the orders $X_t$ are all equal to zero except at the review times $T_1, T_2, \ldots, T_m$.

The inventory level $I_t$ carried from period $t$ to period $t+1$ is any orders that have arrived up to and including period $t$ less the total demand to date. Hence it is given as

$$I_t = \sum_{j=1}^{i} X_{T_j} - \sum_{k=1}^{t} d_k, \quad T_i \leqslant t < T_{i+1}, \quad i = 1, \ldots, m. \tag{8}$$

Following the transformation,

$$S_t = X_t + I_{t-1}, \quad t = 1, \ldots, N, \tag{9}$$

the decision variable $X_{T_i}$ is expressed in terms of a new variable $S_t \in \mathbb{R}$, where $S_t$ may be interpreted as the opening stock level for period $t$, if there is no replenishment in this period (i.e., $t \neq T_i$ and $X_t = 0$) and the order-up-to-level for the $i$th review period $T_i$ if there is a replenishment (i.e., $t = T_i$ and $X_t > 0$). Hence, by means of Eq. (8), it follows $\sum_{j=1}^{i} X_{T_j} = S_{T_i} + \sum_{k=1}^{T_i-1} d_k$, $i = 1, \ldots, m$ and that the inventory level $I_t$ at the end of period $t$ can be expressed as

$$I_t = S_{T_i} - \sum_{k=T_i}^{t} d_k, \quad T_i \leqslant t < T_{i+1}, \quad i = 1, \ldots, m. \tag{10}$$

Since, from Eqs. (4), (5) and (10),

$$I_t^+ = \max(0, I_t) = \max\left(0, S_{T_i} - \sum_{k=T_i}^{t} d_k\right), \quad T_i \leqslant t < T_{i+1}, \quad i = 1, \ldots, m, \tag{11}$$

$$I_t^- = -\min(0, I_t) = -\min\left(0, S_{T_i} - \sum_{k=T_i}^{t} d_k\right), \quad T_i \leqslant t < T_{i+1}, \quad i = 1, \ldots, m, \tag{12}$$

the random components of the excess inventory $I_t^+$ and the shortage $I_t^-$ depend only on the demand since the most recent stock review in period $T_i$, $i = 1$ to $m$. Note that $S_{T_i}$ is not a random variable; rather, it is a deterministic decision variable. Thus, the expected cost function, Eq. (7), can be written as the summation of $m$ intervals, $T_i$ to $T_{i+1}$ for $i = 1$ to $m$, defining $D_{t_1,t_2} = \sum_{j=t_1}^{t_2} d_j$ (see Appendix A for details):

$$\begin{aligned} E\{TC\} = {} & \sum_{i=1}^{m} \left( a\delta_{T_i} + \sum_{t=T_i}^{T_{i+1}-1} \underbrace{\int_{d_{T_i}} \cdots \int_{d_t} (hI_t^+ + sI_t^-) g_{T_i}(d_{T_i}) \cdots g_t(d_t) \mathrm{d}(d_{T_i}) \cdots \mathrm{d}(d_t)}_{E\{C_{T_i,t}\}} \right) + vI_N \\ & + v \int_{D_{1,N}} D_{1,N} \times g(D_{1,N}) \mathrm{d}(D_{1,N}), \end{aligned} \tag{13}$$

or

$$E\{TC\} = \sum_{i=1}^{m} \left( a\delta_{T_i} + \sum_{t=T_i}^{T_{i+1}-1} E\{C_{T_i,t}\} \right) + vI_N + v \int_{D_{1,N}} D_{1,N} \times g(D_{1,N}) \mathrm{d}(D_{1,N}). \tag{14}$$

Substituting for $I_t^+$ and $I_t^-$, $E\{C_{T_i,t}\}$ of Eq. (13) becomes

$$E\{C_{T_i,t}\} = \int_{d_{T_i}} \cdots \int_{d_t} \left[ h \max\left(0, S_{T_i} - \sum_{j=T_i}^{t} d_j\right) - s \min\left(0, S_{T_i} - \sum_{j=T_i}^{t} d_j\right) \right] g_{T_i}(d_{T_i}) \cdots g_t(d_t) \mathrm{d}(d_{T_i}) \cdots \mathrm{d}(d_t). \tag{15}$$

Since the demands $d_t$ are independent, the density of their sum equals the Fourier convolution of their respective densities. So, Eq. (15) can be expressed as

$$E\{C_{T_i,t}\} = \int_{-\infty}^{S_{T_i}} h(S_{T_i} - D_{T_i,t}) \times g(D_{T_i,t}) \mathrm{d}(D_{T_i,t}) - \int_{S_{T_i}}^{\infty} s(S_{T_i} - D_{T_i,t}) \times g(D_{T_i,t}) \mathrm{d}(D_{T_i,t}). \tag{16}$$

It is obvious that $E\{C_{T_i,t}\}$ is the expected cost function of a single period inventory problem where the single period demand is $D_{T_i,t}$. Since $S_{T_i}$ may be interpreted as the order-up-to-level for the $i$th review period $T_i$ and $S_{T_i} - D_{T_i,t}$ is the end of period inventory for the "single period" with demand $D_{T_i,t}$, the expected total subcosts $E\{C_{T_i,t}\}$ are the sums of single period inventory costs where the demands are the cumulative demands over increasing periods. The next section addresses certain properties of the single period cost function.

## 3. Stochastic cost component

For simplicity, let us drop the $T_i$ and $t$ subscripts in Eq. (16), so the single period expected cost model is expressed as

$$E\{C\} = h \int_{-\infty}^{S} (S - D) g(D) \mathrm{d}(D) - s \int_{S}^{\infty} (S - D) g(D) \mathrm{d}(D), \tag{17}$$

which is the well-known single period cost expression with the two components of a holding cost on the positive end of period inventory and a shortage cost for any demands backordered.

Let the mean demand over the period be $\mu$ and assume that the stochastic component is a normally distributed random variable, $x$, with zero mean, variance $\sigma^2$ and distribution $\phi(x;\sigma)$. Note that this would generally be the situation if the demand estimates were the results of some statistical forecasting system. Thus the expected costs can be rewritten as

$$E\{C\} = h \int_{-\infty}^{S-\mu} (S - \mu - x) \phi(x;\sigma) \mathrm{d}x - s \int_{S-\mu}^{\infty} (S - \mu - x) \phi(x;\sigma) \mathrm{d}x. \tag{18}$$

Rearranging Eq. (18) leads to two alternative simplified functions as below:

$$E\{C\} = h(S - \mu) - (h + s)(S - \mu) \int_{S-\mu}^{+\infty} \phi(x;\sigma) \mathrm{d}x + (h + s) \frac{\sigma}{\sqrt{2\pi}} \exp\left(\frac{-(S - \mu)^2}{2\sigma^2}\right), \tag{19}$$

or

$$E\{C\} = -s(S - \mu) + (h + s)(S - \mu) \int_{-\infty}^{S-\mu} \phi(x;\sigma) \mathrm{d}x + (h + s) \frac{\sigma}{\sqrt{2\pi}} \exp\left(\frac{-(S - \mu)^2}{2\sigma^2}\right). \tag{20}$$

Since the first term of Eq. (19), $h(S - \mu)$, is the holding cost for the expected excess closing inventory, the remaining two terms give the expected cost related to the stochastic disturbances when the closing inventory is positive. Similarly, since the first term of Eq. (20), $-s(S - \mu)$, is the shortage cost for the expected end of period backorders, the remaining two terms give the expected cost related to the stochastic disturbances when the closing inventory is negative. Thus the expected single period inventory cost can be expressed as the sum of the effects due to the expected end of period inventory plus a stochastic component. The third terms of the both equations are the same, therefore they are excluded from the comparison of the other terms in the stochastic cost components, which is

$$\left.\begin{array}{l} -(h+s)(S-\mu)\int_{S-\mu}^{+\infty}\phi(x;\sigma)\,\mathrm{d}x \text{ for } S \geqslant \mu \\ (h+s)(S-\mu)\int_{-\infty}^{S-\mu}\phi(x;\sigma)\,\mathrm{d}x \text{ for } S \leqslant \mu \end{array}\right\} = -(h+s)|S-\mu|\int_{|S-\mu|}^{+\infty}\phi(x;\sigma)\,\mathrm{d}x. \tag{21}$$

Note that since the distribution is assumed normal and hence symmetric, Eq. (21) implies that this stochastic cost component is a function of the magnitude of the closing inventory (i.e. a symmetric function). Therefore, the complete stochastic cost component in general is

$$-(h+s)|S-\mu|\int_{|S-\mu|}^{\infty}\phi(x;\sigma)\,\mathrm{d}x + (h+s)\frac{\sigma}{\sqrt{2\pi}}\exp\left(\frac{-(S-\mu)^2}{2\sigma^2}\right). \tag{22}$$

Hence the cost incurred as a result of the stochastic component of the demand is a function only of the sum of the inventory costs $h + s$, the absolute value of the end of period inventory and the standard deviation of the stochastic disturbance.

Figs. 1 and 2 illustrate the shape of the single-period inventory costs function (for both the deterministic and stochastic cases) and the stochastic cost component (i.e., the difference between the deterministic and stochastic cases) respectively.

Letting $\Phi(k;\sigma) = \int_{-\infty}^{k}\phi(x;\sigma)\,\mathrm{d}x$, then the stochastic cost component can be written as

$$-(h+s)|S-\mu| + (h+s)|S-\mu|\Phi(|S-\mu|;\sigma) + (h+s)\frac{\sigma}{\sqrt{2\pi}}\exp\left(\frac{-(S-\mu)^2}{2\sigma^2}\right), \tag{23}$$

where we have used the fact that $\Phi(k;\sigma) = 1 - \Phi(-k;\sigma)$.

## 4. A piecewise linear cost approximation

Defining $y = |S-\mu|/\sigma$, then the stochastic component of the single period inventory costs, Eq. (23), can be written as

$$\sigma(h+s)(-y + y\Phi(y) + \phi(y)) = \sigma(h+s)f(y) \tag{24}$$

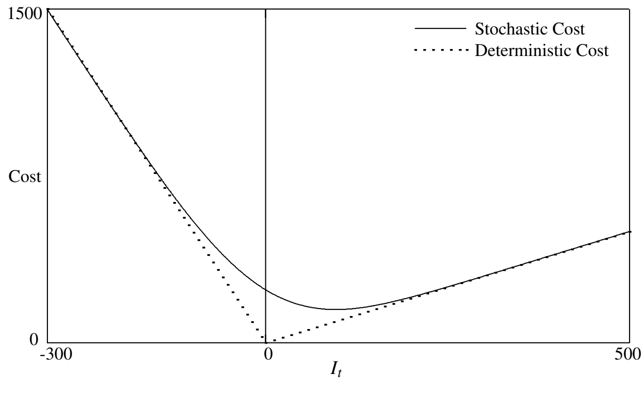


Fig. 1. Cost functions for deterministic and stochastic cases ($h = 1, s = 5, \sigma = 100$).

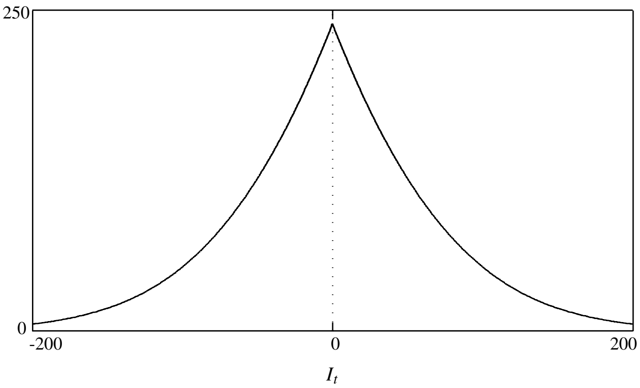


Fig. 2. Stochastic cost component vs $I_t$ ($h = 1, s = 5, \sigma = 100$).

say, which is a product of the standard deviation of the stochastic demand, the sum of the inventory costs and some function of the magnitude of the normalized expected closing inventory. The function $f(y)$ in Eq. (24) is in terms of the unit normal distribution, so Eq. (24) for the stochastic cost component is a general result for any normal distribution. The stochastic cost component, $\sigma(h+s)f(y)$, is far from being a tractable expression. Therefore a piecewise linear approximation to $f(y)$ with parameters $\alpha$ and $\beta$ is developed as

$$L(y) = \begin{cases} \beta(1 - y/\alpha) & \text{for } y \leqslant \alpha, \\ 0 & \text{for } y > \alpha. \end{cases} \tag{25}$$

The variable $y$ may take on any of a large range of values. The eventual objective of the analysis is to determine the best values for the $S_t$. A reasonable approximation for all possible values of $y$ is thus required. This indicates that it is most appropriate to use the minimax criterion rather than the minimum sum of squared errors to determine the best approximation.

A minimax criterion is used, to find the values of parameters $\alpha$ and $\beta$ which minimize the maximum error between $f(y)$ and the linear approximation. This relies on some geometric insight. Consider the graphs of the function $f(y)$ and the linear approximation $L(y)$ shown in Fig. 3. Let $\epsilon = \max|f(y) - L(y)|$. Clearly the two functions must be equal at two points in the range $[0, \alpha]$; otherwise, we could improve the approximation by moving the graph of $L(y)$ appropriately. Hence looking at the shape of the difference between the two functions, $e = L(y) - f(y)$, given in Fig. 3, it is clear that the largest values of $\epsilon = |e|$ will be at $y = 0$, $\alpha$

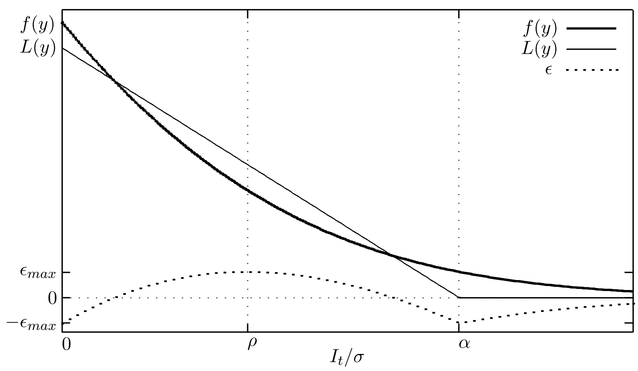


Fig. 3. Piecewise linear minimax approximation, $L(y)$.

(the breakpoint of the piecewise linear approximation) and at some intermediate point $\rho$ where the difference $e = L(y) - f(y)$ has a relative maximum value. The minimum maximum error will be achieved when the magnitude of the errors at these three points are equal. The details of the derivation and the results are presented in Appendix B.

Rounding the numerical values of the coefficients given in Eq. (B.6) to three places of decimals, the expected inventory costs for the single period decision problem, given as Eqs. (19) and (20), can be expressed as

$$E\{C\} = \begin{cases} h(S-\mu) + \max(0, (h+s)(0.362\sigma - 0.260|S-\mu|)) & \text{for } S \geqslant \mu, \\ -s(S-\mu) + \max(0, (h+s)(0.362\sigma - 0.260|S-\mu|)) & \text{for } S < \mu. \end{cases} \tag{26}$$

An illustrative example is given in Fig. 4. In this figure, the inventory cost function and the piecewise linear approximation to it are plotted for $I_t$ in the range $[-300, 500]$, $h = 1$, $s = 5$ and $\sigma = 100$.

In order to evaluate the accuracy of the approximation, the worst-case percentage error in replacing the cost function with the linear approximation is presented in Fig. 5 for a number of line segments in the piece-

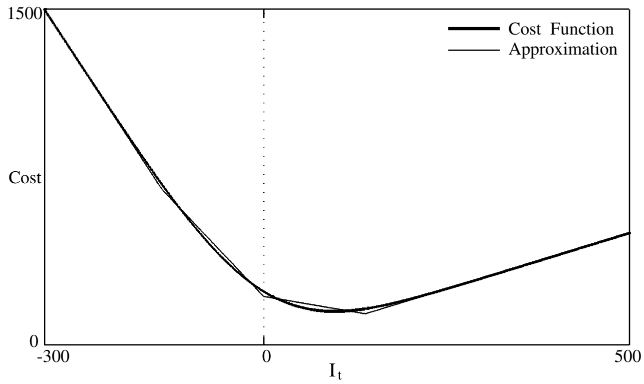


Fig. 4. Single-period cost function and single breakpoint approximation ($h = 1, s = 5, \sigma = 100$).

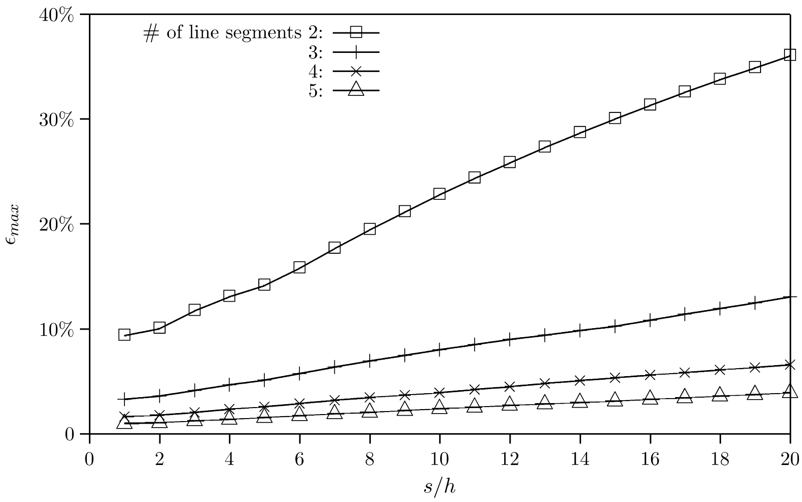


Fig. 5. Worst-case error in piecewise approximation ($0 \leqslant \sigma < +\infty, \; -\infty < |S-\mu| < +\infty$).

wise approximation changing from 2 to 5. In this graph, the worst-case error, for $0 \leqslant \sigma < +\infty$ and $-\infty < |S - \mu| < +\infty$, is plotted with respect to $s/h$ ratio over the range $[1, 20]$. In this range, the worst-case error is 3.92% when there are five line segments (i.e., four breakpoints) in the approximation. Note that the accuracy of the piecewise linear approximation can be improved further by introducing new breakpoints.

The above results clearly show that Eq. (26) is an accurate approximation to the stochastic cost expressions $E\{C_{T_i,t}\}$ of Section 2, Eq. (16).

## 5. Certainty equivalent mixed ILP model

The single period inventory model has a total inventory cost made up of a cost on the expected end of period inventory plus a stochastic disturbance cost term. The stochastic disturbance effect depends on the expected end of period inventory and the standard deviation of demand over the period. As demonstrated in Section 2, the total inventory costs model can be decomposed into a sum of the costs of a series of single period models, one for each period of the planning horizon. The demands in each "single period" are the cumulative demands since the most recent review occurred.

Using the results of the single period analysis, given in Eq. (26), it follows that a linear approximation to $E\{C_{T_i,t}\}$ in terms of the order-up-to-level $S_{T_i}$ at the $i$th review period $T_i$ is

$$E\{C_{T_i,t}\} = \begin{cases} h(S_{T_i} - \widetilde{D}_{T_i,t}) + \max(0, (h+s)(0.362\sigma_{T_i,t} - 0.260|S_{T_i} - \widetilde{D}_{T_i,t}|)) & \text{for } S_{T_i} \geqslant \widetilde{D}_{T_i,t}, \\ -s(S_{T_i} - \widetilde{D}_{T_i,t}) + \max(0, (h+s)(0.362\sigma_{T_i,t} - 0.260|S_{T_i} - \widetilde{D}_{T_i,t}|)) & \text{for } S_{T_i} < \widetilde{D}_{T_i,t}, \end{cases} \tag{27}$$

where $\widetilde{D}_{T_i,t} = \sum_{j=T_i}^{t} \tilde{d}_j$ is the expected cumulative demand over periods $T_i$ to $t$ inclusive, $\tilde{d}_j$ is the forecast mean demand in period $j$, and $\sigma_{T_i,t}$ is the standard deviation of the cumulative demand, $D_{T_i,t}$, over periods $T_i$ to $t$ about the mean forecast value.

From the definition of surplus inventory $I_t^+$, the expected surplus inventory $\widetilde{I}_t^+$ at the end of period $t$ is $\widetilde{I}_t^+ = \max(0, \widetilde{I}_t)$ or $\widetilde{I}_t^+ = S_{T_i} - \widetilde{D}_{T_i,t}$ if $S_{T_i} \geqslant \widetilde{D}_{T_i,t}$, $t = 1, \ldots, N$. Similarly, the expected shortage is $\widetilde{I}_t^- = -\min(0, \widetilde{I}_t)$ or $\widetilde{I}_t^- = \widetilde{D}_{T_i,t} - S_{T_i}$ if $S_{T_i-L} < \widetilde{D}_{T_i,t}$, $t = 1, \ldots, N$.

Both $\widetilde{I}_t^+$ and $\widetilde{I}_t^-$ are non-negative variables, which clearly must not be non-zero at the same time. It is shown in Appendix C that if they satisfy

$$\widetilde{I}_t^+ \geqslant \widetilde{I}_t \quad \text{and} \quad \widetilde{I}_t^- \geqslant -\widetilde{I}_t \tag{28}$$

plus the equality condition

$$\widetilde{I}_t - \widetilde{I}_t^+ + \widetilde{I}_t^- = 0, \quad t = 1, \ldots, N, \tag{29}$$

then there are sufficient conditions to ensure that either $\widetilde{I}_t^+$ or $\widetilde{I}_t^-$, but not both, can be non-zero at a time. Since $\widetilde{I}_t^+$ and $\widetilde{I}_t^-$ cannot both be non-zero at the same time, Eq. (27) can be rewritten as

$$E\{C_{T_i,t}\} = h\widetilde{I}_t^+ + s\widetilde{I}_t^- + \max(0, (h+s)[0.362\sigma_{T_i,t} - 0.260(\widetilde{I}_t^+ + \widetilde{I}_t^-)]). \tag{30}$$

Let the stochastic cost term in the above expression be denoted by the continuous non-negative variable $Q_{T_i,t}$ so that Eq. (30) can be expressed as a simple linear sum of three variables:

$$E\{C_{T_i,t}\} = h\widetilde{I}_t^+ + s\widetilde{I}_t^- + Q_{T_i,t}. \tag{31}$$

Substituting the above expression into Eq. (14) shows that the total expected costs can be expressed as a linear sum of variables:

$$E\{TC\} = v\widetilde{I}_N + v\widetilde{D}_{1,N} + \sum_{i=1}^{m} \sum_{t=T_i}^{T_{i+1}-1} (a\delta_t + h\widetilde{I}_t^{+} + s\widetilde{I}_t^{-} + Q_{T_i,t}). \tag{32}$$

The original stochastic objective function has now been transformed to an equivalent deterministic one based on the expected end of period stock.

The expected inventory level $\widetilde{I}_t$ at the end of period $t$ can be simply expressed as

$$\widetilde{I}_t = S_t - \widetilde{d}_t, \quad t = 1, \ldots, N. \tag{33}$$

It follows that $S_t$ must be equal to $\widetilde{I}_{t-1}$ if no order is received in period $t$. This will be achieved by the two linear inequalities below:

$$\left.\begin{array}{l} S_t \geqslant \widetilde{I}_{t-1} \\ S_t \leqslant M\delta_t + \widetilde{I}_{t-1} \end{array}\right\} \quad t = 1, \ldots, N, \tag{34}$$

where $M$ is some very large positive number. If $\delta_t = 0$ then $S_t$ must equal $\widetilde{I}_{t-1}$, for the constraints become $S_t \leqslant \widetilde{I}_{t-1}$ and $S_t \geqslant \widetilde{I}_{t-1}$ respectively. Whilst if $\delta_t = 1$ then $S_t$ is any order-up-to-level which is greater than or equal to $\widetilde{I}_{t-1}$. The values of $t$ for which $\delta_t = 1$ are the periods when stock reviews should be carried out and a replenishment order placed. The desired opening stock levels, $S_{T_i}$, as required for the solution to the problem, will then be those values of $S_t$ for which $\delta_t = 1$.

In the objective function, Eq. (32), the factor $Q_{T_i,t}$, representing the stochastic disturbance cost element, depends upon the standard deviation over the periods between the stock review at $T_i$ and the current period $t$. Clearly this can only be determined after the review times $T_i$ have been determined. But, as these are chosen to minimize the expected costs, the stock review times cannot be determined until the appropriate standard deviation values to use in the piecewise linear approximation to the stochastic disturbances are known, which depend upon the number of periods that have elapsed since the stock review for which the most recent order received was placed. There is an obvious circularity here in trying to solve the problem. Tarim and Kingsman [11] overcome this problem by formulating it as a mixed integer linear programming model. Their approach is employed here.

Let $\xi_{tj}$ be the standard deviation of the cumulative demands over the $j$ periods ending in period $t$, i.e. periods $t-j+1$ up to and including $t$. Since it is a finite planning horizon of $N$ periods, the standard deviations can be found for all relevant cases. If the 0/1 integer variable $P_{tj}$ is defined as taking a value of 1 if the order most recently received was placed at a stock review in period $t-j+1$ and zero elsewhere, then the standard deviation $\sigma_{T_i,t}$ can be expressed as

$$\sigma_{T_i,t} = \sum_{j=1}^{t} P_{tj}\xi_{tj}, \quad t = 1, \ldots, N. \tag{35}$$

The result $P_{tt} = 1$ means that the stock review was in period 1, the start of the planning horizon, so the most recent order being received in period 1, whilst $P_{t1} = 1$ means that the stock review was in period $t$, with the most recent order was received at the start of period $t$ itself.

There can at most be only one most recent order received prior to period $t$. Thus the $P_{tj}$ must satisfy

$$\sum_{j=1}^{t} P_{tj} = 1, \quad t = 1, \ldots, N. \tag{36}$$

Three other conditions as given below are necessary to identify uniquely the period in which the most recent review prior to any period $t$ took place.

- If $\delta_{t-j+1} = 1$ and $\sum_{k=t-j+2}^{t}\delta_k = 0$, so all subsequent $\delta_k$ for $k = t - j + 2,\ t - j + 3, \ldots, t$ are 0, then we must have $P_{tj} = 1$, as in these circumstances period $t - j + 1$ had the most recent stock review prior to period $t$.
- If $\delta_{t-j+1} = 0$ and $\sum_{k=t-j+2}^{t}\delta_k = 0$, then $P_{tj} = 0$ since the most recent review prior to period $t$ must have been earlier than $t - j + 1$.
- If $\delta_{t-j+1} = 1$ and $\sum_{k=t-j+2}^{t}\delta_k \geqslant 1$, then $P_{tj} = 0$, since other reviews prior to period $t$ occur after period $t + j - 1$.

All of these three conditions can be satisfied by the equality condition given in Eq. (36) and the single constraint below, which are designed to identify uniquely the periods in which the most recent order prior to $t$ took place, that

$$P_{tj} \geqslant \delta_{t-j+1} - \sum_{k=t-j+2}^{t} \delta_k, \quad t = 1, \ldots, N, \quad j = 1, \ldots, t. \tag{37}$$

Exploiting Eq. (35) for $\sigma_{T_i,t}$, from Eqs. (30) and (31) the factor $Q_{T_i,t}$ can be expressed as linear inequalities in terms of the decision variables $P_{tj}$, $\widetilde{I}_t^+$ and $\widetilde{I}_t^-$ and now be rewritten as $Q_t$ since it is a function of $t$ only:

$$Q_t \geqslant \begin{cases} (h+s)(0.362\sum_{j=1}^{t} P_{tj}\xi_{tj} - 0.260(\widetilde{I}_t^+ + \widetilde{I}_t^-)), \\ 0, \end{cases} \quad t = 1, \ldots, N, \tag{38}$$

and, since

$$\sum_{i=1}^{m} \sum_{t=T_i}^{T_{i+1}-1} Q_{T_i,t} = \sum_{t=1}^{N} Q_t \tag{39}$$

from Eq. (32),

$$E\{TC\} = v\widetilde{I}_N + v\widetilde{D}_{1,N} + \sum_{t=1}^{N}(a\delta_t + h\widetilde{I}_t^+ + s\widetilde{I}_t^- + Q_t). \tag{40}$$

This model thus determines the number of replenishments as well as the replenishment schedule, the timing of the replenishments, together with the values to use for dynamically determining the sizes of the replenishment orders as demand is realized, that give the minimum expected total costs. The problem is to determine the values of the 0/1 integer variables, $\delta_t$ for $t = 1, \ldots, N$ and $P_{tj}$ for $j = 1, \ldots, t$, $t = 1, \ldots, N$, and the non-negative continuous variables $\widetilde{I}_t^+$, $\widetilde{I}_t^-$ and $Q_t$ for $t = 1, \ldots, N$, plus the unconstrained continuous variables $\widetilde{I}_t$ and $S_t$ for $t = 1, \ldots, N$, that minimize the objective function Eq. (40) subject to the constraints as outlined above. For the sake of convenience, the complete certainty equivalent model for the non-stationary stochastic demand inventory control problem under the non-stationary $(R, S)$ policy is given below:

$$\min E\{TC\} = v\widetilde{I}_N + v\widetilde{D}_{1,N} + \sum_{t=1}^{N}(a\delta_t + h\widetilde{I}_t^+ + s\widetilde{I}_t^- + Q_t), \tag{M-1}$$

s.t.

$$\widetilde{I}_t = S_t - \widetilde{d}_t, \quad t = 1, \ldots, N, \tag{M-2}$$

$$S_t \geqslant \widetilde{I}_{t-1}, \quad t = 1, \ldots, N, \tag{M-3}$$

$$S_t \leqslant M\delta_t + \widetilde{I}_{t-1}, \quad t = 1, \ldots, N, \tag{M-4}$$

$$\widetilde{I}_t^{+} \geqslant \widetilde{I}_t, \quad \widetilde{I}_t^{-} \geqslant -\widetilde{I}_t, \quad \widetilde{I}_t - \widetilde{I}_t^{+} + \widetilde{I}_t^{-} = 0, \quad t = 1, \ldots, N, \tag{M-5}$$

$$\sum_{j=1}^{t} P_{tj} = 1, \quad t = 1, \ldots, N, \tag{M-6}$$

$$P_{tj} \geqslant \delta_{t-j+1} - \sum_{k=t-j+2}^{t} \delta_k, \quad t = 1, \ldots, N, \quad j = 1, \ldots, t, \tag{M-7}$$

$$Q_t \geqslant (h+s)\left(0.362 \sum_{j=1}^{t} P_{tj}\xi_{tj} - 0.260(\widetilde{I}_t^{+} + \widetilde{I}_t^{-})\right), \quad t = 1, \ldots, N, \tag{M-8}$$

$$Q_t, \widetilde{I}_t^{+}, \widetilde{I}_t^{-} \geqslant 0, \quad -\infty < S_t, \widetilde{I}_t < +\infty, \quad \delta_t, P_{tj} \in \{0, 1\}. \tag{M-9}$$

The times of the stock reviews are given by the values of $i$ such that $\delta_i = 1$. The associated order-up-to-levels, for $\delta_i = 1$, are given as $S_i$.

In the above model, a piecewise linear approximation to the stochastic cost component is given by considering only one breakpoint. However, the accuracy of the approximation can be improved by introducing new breakpoints. Each additional breakpoint adds a new set of constraints in the form of Eq. (M-8) but with different parameters. The number of decision variables remains the same.

It should be noted that, from Eqs. (M-6) and (M-7), it follows that $P_{tj}$ must still take a binary value even if it is declared as a continuous variable. Therefore, the total number of binary variables reduces to the total number of periods, $N$.

## 6. Numerical examples

The effect of higher noise levels—uncertainty in the demand forecasts—on the order schedule is illustrated (see Figs. 6–11) by means of a simple numerical example, in which $N = 1$, $h = 1$, $s = 10$, $a = 250$ and $v \in \{0, 7\}$. The forecasts of demand in each period are given in Table 1. It is assumed that the demand

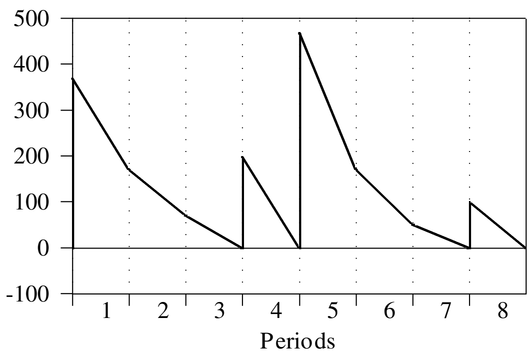


Fig. 6. $h = 1$, $s = 10$, $a = 250$, $v = 0$, $\tau = 0.0$.

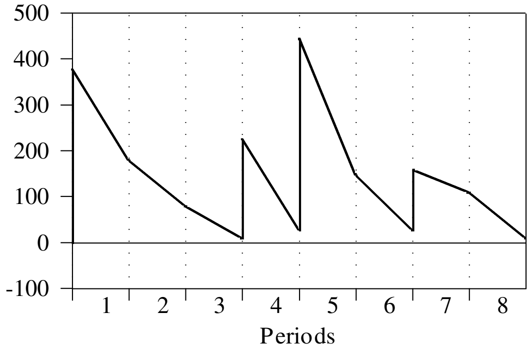


Fig. 7. $h = 1$, $s = 10$, $a = 250$, $v = 0$, $\tau = 0.1$.

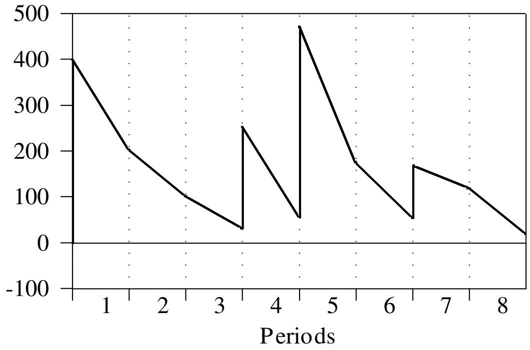


Fig. 8. $h = 1$, $s = 10$, $a = 250$, $v = 0$, $\tau = 0.2$.

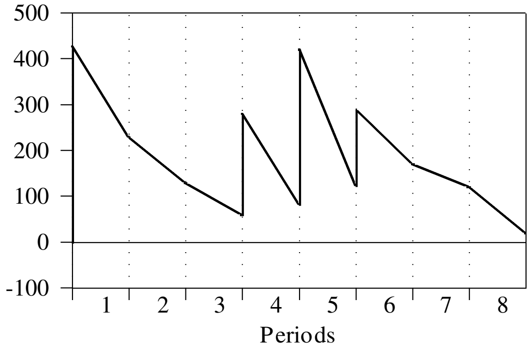


Fig. 9. $h = 1$, $s = 10$, $a = 250$, $v = 0$, $\tau = 0.3$.

in each period is normally distributed about the forecast value with the same coefficient of variation, $\tau$. Thus the standard deviation of demand in period $t$ is $\sigma_t = \tau \widetilde{d}_t$.

In all cases, initial inventory levels, delivery lead-times, and salvage values are set to zero. Figs. 6–11 depict optimal replenishment policies for six different instances, with respect to $\tau$ and $v$, of the above problem.

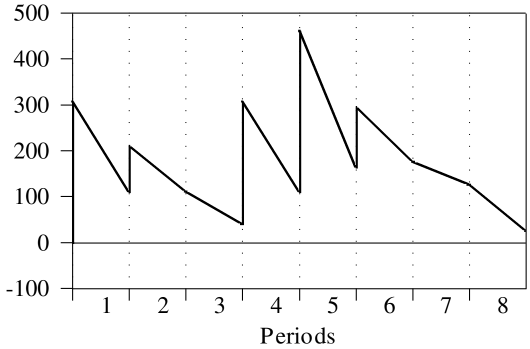


Fig. 10. $h = 1$, $s = 10$, $a = 250$, $v = 0$, $\tau = 0.4$.

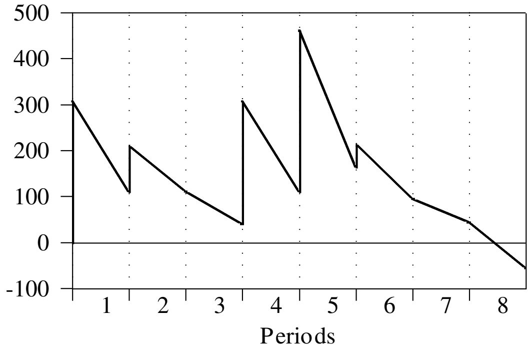


Fig. 11. $h = 1$, $s = 10$, $a = 250$, $v = 7$, $\tau = 0.4$.

Table 1
Forecasts of period demands

| Period, $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\tilde{d}_t$ | 200 | 100 | 70 | 200 | 300 | 120 | 50 | 100 |

- Fig. 6 shows the optimal replenishment policy for the deterministic case. The direct item cost is taken as zero, $v = 0$. It requires 4 replenishments, arriving at the beginning of Period 1, 4, 5 and 8. The $(R^n, S^n)$ policy parameters are $R = [3, 1, 3, 1]$ and $S = [370, 200, 470, 100]$ in this instance.
- Fig. 7, in which the coefficient of variation of the period demand is 0.1, illustrates that for low levels of forecast uncertainty the number of replenishments remains the same; however, the replenishment periods are now 1, 4, 5 and 7. As one would expect, buffer stocks are introduced. This amounts to 9 units for the first order covering 3 periods, 27 units for the second order covering 1 period, 27 for the third order covering 2 periods and 9 for the final order covering the last 2 periods. In this case, $R = [3, 1, 2, 2]$ and $S = [379, 227, 447, 159]$.
- When the coefficient of variation of demand increases to 0.2, as in Fig. 8, the number of replenishments remains still the same: 1st, 4th, 5th and 7th periods. But, the buffer stock levels of 32, 55, 54, 19 units are significantly higher. Now, $R = [3, 1, 2, 2]$ and $S = [402, 255, 474, 169]$.

Table 2
Sox's example

| Period, $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\widetilde{d}_t$ | 110 | 40 | 10 | 62 | 12 | 80 | 122 | 130 |
| $v_t$ | 5.6 | 4.2 | 3.0 | 2.0 | 1.2 | 0.6 | 0.2 | 0.0 |

- In Fig. 9, the case where the coefficient of variation is further increased to 0.3, the number of replenishments is still 4; however, now the replenishment periods are the 1st, 4th, 5th and 6th periods. The buffer stock levels are 59, 82, 123 and 20 units, respectively. It follows $R = [3, 1, 1, 3]$ and $S = [429, 282, 423, 290]$.
- At a relatively high value of coefficient of variation, 0.4, there are 5 replenishments (1st, 2nd, 4th, 5th and 6th periods) as displayed in Fig. 10. The buffer stock level now takes a value between 26 and 164 units. In this case, the policy parameters are $R = [1, 2, 1, 1, 3]$ and $S = [310, 211, 310, 464, 296]$.
- Finally, Fig. 11 shows the effect of non-zero variable unit cost, $v = 7$. The coefficient of variation is set to $\tau = 0.4$ for a comparison with the last case. It is observed that the replenishment periods do not change (i.e., $1, 2, 4, 5, 6$) but the expected total order quantity is decreased, as one would expect. For the last replenishment, even a stockout has been planned. The review intervals and the order-up-to-levels are $R = [1, 2, 1, 1, 3]$ and $S = [310, 211, 310, 464, 215]$, respectively.

The order-up-to-levels and buffer stocks do not appear to be reducible to some simple model. These results clearly have implications for Materials Requirements Planning under uncertainty. For it has been commonly suggested in the literature that the solution to uncertainty is to carry a constant level of buffer stock.

Sox [10] describes a formulation of the dynamic lot-sizing problem when demand is stochastic and the costs are non-stationary. Assuming that the distribution of the cumulative demand is known for each period and that all unsatisfied demand is backordered, an optimal solution algorithm is developed that resembles the Wagner–Whitin algorithm. Sox also provides an example to demonstrate his algorithm. In what follows, the solution to this example is calculated using both Sox's stochastic dynamic lot-sizing policy and the non-stationary $(R, S)$ policy developed in this paper, and a comparison of the results is presented.

In Sox's example, the initial inventory is $I_0 = 98$, $N = 8$, $h = 0.5$, $s = 12$, $a = 48$, $\tau = 0.2$. The rest of the problem data is given in Table 2. Note that the direct unit costs are non-stationary.

It is reported that in the optimal solution, under the dynamic lot-sizing policy, the replenishment periods are 1, 2, 4, 6, 7 and 8 with lot-sizes 34.6, 49.6, 81.4, 95.8, 139.4 and 150.0, respectively. The expected total cost is 1233. The non-stationary $(R, S)$ policy chooses the same replenishment periods with order-up-to-levels 128.5, 56.9, 84.6, 101.9, 155.4 and 165.6. The expected cost of the $(R, S)$ policy is 1031. It is clear that the expected cost difference between these two policies is very significant.

## 7. Conclusion and further remarks

This paper presents an approach to tackle the non-stationary stochastic demand inventory problem under the non-stationary $(R, S)$ policy. By means of a piecewise linear approximation to the non-linear cost function, a certainty equivalent mixed integer programming model is built. The accuracy of the approximation can be improved by introducing new breakpoints—each adding a new set of constraints without requiring additional variables—to the piecewise linear approximation. The resultant MIP model gives the approximately optimal solution in terms of the number and timing of the replenishments and the associated order-up-to-levels. Using these $(R, S)$ policy parameters and observing the realized demand, the size of the actual replenishment orders for the periods when stock reviews take place are determined. Unlike the Bookbinder–Tan [2] and Tarim–Kingsman [11] models, which incorporate service level constraints, this new MIP model takes into account shortage costs.

Although, it has been assumed that the replenishment lead time is zero, it is possible to extend the model for the non-zero replenishment lead time situation without any loss of generality.

Revisions to replenishment plans due to differences in realized demand from that forecasted are common occurrences in production and inventory systems. These revisions are disruptive and undesirable since existing orders are cancelled and new orders are placed. Freezing the schedule within the planning horizon is one of the commonly used methods to reduce nervousness. In this regard, the authors believe that the $(R^n, S^n)$ policy, which provides a means of reducing especially the setup instability in production/inventory systems, offers interesting directions for future research.

## Acknowledgements

This material is based upon works supported by Science Foundation Ireland under Grant No. 03/CE3/I405 as part of the Centre for Telecommunications Value-Chain-Driven Research (CTVR) and Grant No. 00/PI.1/C075.

## Appendix A

The total direct unit cost component of Eq. (7) is

$$\int_{d_1} \cdots \int_{d_N} \sum_{i=1}^{m} \sum_{t=T_i}^{T_{i+1}-1} (vX_t) g_1(d_1) \cdots g_N(d_N) \mathrm{d}(d_1) \cdots \mathrm{d}(d_N). \tag{A.1}$$

From the definition of $X_{T_i}$, $X_t = 0$ for $t \neq T_i$, Eq. (A.1) can be written as

$$v \int_{d_1} \cdots \int_{d_N} \sum_{i=1}^{m} (X_{T_i}) g_1(d_1) \cdots g_N(d_N) \mathrm{d}(d_1) \cdots \mathrm{d}(d_N) \tag{A.2}$$

and, exploiting $\sum_{j=1}^{i} X_{T_j} = S_{T_i} + \sum_{k=1}^{T_i-1} d_k$, $i = 1, \ldots, m$,

$$v \int_{d_1} \cdots \int_{d_N} \sum_{i=1}^{m} \left( S_{T_m} + \sum_{k=1}^{T_m-1} d_k \right) g_1(d_1) \cdots g_N(d_N) \mathrm{d}(d_1) \cdots \mathrm{d}(d_N). \tag{A.3}$$

Since there is no replenishment between $T_m + 1$ and $T_{m+1}$, it follows that

$$S_{T_m} + \sum_{k=1}^{T_m-1} d_k = S_{N+1} + \sum_{k=1}^{N} d_k = I_N + \sum_{k=1}^{N} d_k. \tag{A.4}$$

The demands $d_t$ are independent, and therefore, the density of their sum equals the Fourier convolution of their respective densities. So, defining $D_{1,N} = \sum_{k=1}^{N} d_k$, Eq. (A.3) can now be expressed as

$$vI_N + v \int_{D_{1,N}} D_{1,N} \times g_{D_{1,N}}(D_{1,N}) \times \mathrm{d}(D_{1,N}). \tag{A.5}$$

It is clear from Eq. (A.5) that the total unit cost component is a function of the single decision variable $I_N$ and the constants $v$ and $\int_{D_{1,N}} D_{1,N} \times g_{D_{1,N}}(D_{1,N}) \times \mathrm{d}(D_{1,N})$.

## Appendix B

Zelen and Severo [12] proposed an approximation to the Gaussian unit normal cumulative distribution function, which was shown to have a maximum error of less than 0.00025, as below:

$$\Phi(y) = 1 - \frac{1}{2}\left(\sum_{i=0}^{4} a_i y^i\right)^{-4}, \quad \mathbf{a} = [1.0, 0.196854, 0.115194, 0.000344, 0.019527]. \tag{B.1}$$

Using this approximation, it follows that

$$f(y) = \frac{1}{\sqrt{2\pi}}\exp(-0.5y^2) - \frac{y}{2}\left(\sum_{i=0}^{4} a_i y^i\right)^{-4}, \tag{B.2}$$

and

$$e = f(y) - L(y) = \begin{cases} -\beta(1 - y/\alpha) + (1/\sqrt{2\pi})\exp(-y^2/2) - (y/2)\left(\sum_{i=0}^{4} a_i y^i\right)^{-4} & \text{for } y \leqslant \alpha, \\ (1/\sqrt{2\pi})\exp(-y^2/2) - (y/2)\left(\sum_{i=0}^{4} a_i y^i\right)^{-4} & \text{for } y > \alpha. \end{cases} \tag{B.3}$$

Thus the maximum error is given by

$$\begin{aligned} y = 0: &\quad \epsilon = \frac{1}{\sqrt{2\pi}} - \beta, \\ y = \alpha: &\quad \epsilon = \frac{1}{\sqrt{2\pi}}\exp(-\alpha^2/2) - \frac{\alpha}{2}\left(\sum_{i=0}^{4} a_i \alpha^i\right)^{-4}, \\ y = \rho: &\quad \epsilon = \frac{1}{\sqrt{2\pi}}\exp(-\rho^2/2) - \frac{\rho}{2}\left(\sum_{i=0}^{4} a_i \rho^i\right)^{-4} - \beta(1 - \rho/\alpha). \end{aligned} \tag{B.4}$$

Also, since the error difference, $e = L(y) - f(y)$, has a relative maximum value at the point $y = \rho$, it must satisfy $\mathrm{d}(L(y) - f(y))/\,\mathrm{d}y = 0$, giving a fourth equation:

$$\frac{\rho}{\sqrt{2\pi}}\exp(-\rho^2/2) - \frac{1}{2}\left(\sum_{i=0}^{4} a_i \rho^i\right)^{-4} + 2\rho\left(\sum_{i=0}^{4} a_i \rho^i\right)^{-5}\left(\sum_{i=1}^{4} i a_i \rho^{i-1}\right) + \frac{\beta}{\alpha} = 0. \tag{B.5}$$

This gives four non-linear equations in the unknowns $\epsilon$, $\rho$, $\alpha$ and $\beta$. The equations are solved by using Brown's Newton-like method [3] based upon Gaussian elimination. The solution to these four non-linear equations is

$$\alpha = 1.391822049, \quad \beta = 0.361644769, \quad \rho = 0.645610556, \quad \epsilon = 0.037297511. \tag{B.6}$$

Hence the piecewise linear approximation to the stochastic part of the cost function is

$$\approx 0.362\sigma(h + s)\left(1 - \frac{|S - \mu|}{1.392\sigma}\right), \tag{B.7}$$

where $|S - \mu|/\sigma \leqslant \alpha$. Remembering that the linear approximation has to be non-negative, it can be expressed as

$$\approx \max(0, 0.362(h + s)(\sigma - 0.7185|S - \mu|)). \tag{B.8}$$

The slope of the piecewise linear minimax approximation to the stochastic cost function is independent of the standard deviation of the disturbances and has value $-0.2598(h+s)$. Thus the above is a general result which can be used for any normal distribution, given the mean value, $\mu$, and standard deviation, $\sigma$.

## Appendix C

If $\widetilde{I}^{+}$ and $\widetilde{I}^{-}$ are both to have positive values then from the definition it follows $\widetilde{I}^{+} = \max(0,\widetilde{I}) + \varepsilon$ and $\widetilde{I}^{-} = -\min(0,\widetilde{I}) + \eta$, where $\varepsilon$ and $\eta$ are non-negative values. From Eq. (29), it follows that

$$\widetilde{I} - \max(0,\widetilde{I}) - \varepsilon - \min(0,\widetilde{I}) + \eta = 0. \tag{C.1}$$

There are clearly three possible cases that might occur; $\widetilde{I} > 0$, $\widetilde{I} < 0$, $\widetilde{I} = 0$. From Eq. (C.1) it is easy to see that, in all cases, $\varepsilon = \eta$. So, the expected costs $E\{C\}$ of Eq. (30) become,

$$\begin{aligned} &h(\max(0,\widetilde{I}) + \varepsilon) + s(-\min(0,\widetilde{I}) + \varepsilon) \\ &\quad + \max\Big(0, (h+s)[0.362\sigma - 0.26(\max(0,\widetilde{I}) - \min(0,\widetilde{I}) + 2\varepsilon)]\Big), \end{aligned} \tag{C.2}$$

or

$$\begin{aligned} &h\max(0,\widetilde{I}) - s\min(0,\widetilde{I}) + (h+s)\varepsilon \\ &\quad + \max\Big(0, (h+s)[-0.52\varepsilon + 0.362\sigma - 0.26(\max(0,\widetilde{I}) - \min(0,\widetilde{I}))]\Big). \end{aligned} \tag{C.3}$$

In Eq. (C.3) the coefficient of $\varepsilon$, $C_\varepsilon$, is positive; because

$$\text{either} \quad (h+s)[-0.52\varepsilon + 0.362\sigma - 0.26(\max(0,\widetilde{I}) - \min(0,\widetilde{I}))] > 0 \quad \Rightarrow \quad C_\varepsilon = 0.48(h+s), \tag{C.4}$$

$$\text{or} \quad (h+s)[-0.52\varepsilon + 0.362\sigma - 0.26(\max(0,\widetilde{I}) - \min(0,\widetilde{I}))] \leqslant 0 \quad \Rightarrow \quad C_\varepsilon = 1.00(h+s). \tag{C.5}$$

Therefore, since the objective is to find a value of $\varepsilon$ minimizing the total cost subject to $\varepsilon \geqslant 0$, $\varepsilon$ must equal 0. This follows that either $\widetilde{I}^{+} = \widetilde{I}$ and $\widetilde{I}^{-} = 0$ or $\widetilde{I}^{+} = 0$ and $\widetilde{I}^{-} = -\widetilde{I}$. Thus satisfying Eq. (29) is sufficient to ensure that $\widetilde{I}_t^{+}$ and $\widetilde{I}_t^{-}$ cannot be positive simultaneously.

## References

[1] J.D. Blackburn, D.H. Kropp, R.A. Millen, A comparison of strategies to dampen nervousness in MRP systems, Management Science 32 (1986) 413–429.
[2] J.H. Bookbinder, J.Y. Tan, Strategies for the probabilistic lot-sizing problem with service-level constraints, Management Science 34 (1988) 1096–1108.
[3] K.M. Brown, A quadratically convergent method for solving simultaneous nonlinear equations. PhD thesis, Purdue University, Lafayette, Ind., 1966.
[4] R.C. Carlson, J.V. Jucker, D.H. Kropp, Less nervous MRP systems: A dynamic economic lot-sizing approach, Management Science 25 (1979) 754–761.
[5] G. Heisig, Comparison of $(s,S)$ and $(s,nQ)$ inventory control rules with respect to planning stability, International Journal of Production Economics 73 (2001) 59–82.
[6] K. Inderfurth, Nervousness in inventory control, OR Spektrum 16 (1994) 113–123.
[7] B.G. Kingsman, Raw Materials Purchasing: An Operational Research Approach, Pergamon Press, New York, 1985.
[8] T. De Kok, K. Inderfurth, Nervousness in inventory management: Comparison of basic control rules, European Journal of Operational Research 103 (1997) 55–82.
[9] E.A. Silver, D.F. Pyke, R. Peterson, Inventory Management and Production Planning and Scheduling, John Wiley and Sons, New York, 1998.

[10] C.R. Sox, Dynamic lot sizing with random demand and non-stationary costs, Operations Research Letters 20 (1997) 155–164.
[11] S.A. Tarim, B.G. Kingsman, The stochastic dynamic production/inventory lot-sizing problem with service-level constraints, International Journal of Production Economics 88 (2004) 105–119.
[12] M. Zelen, N.C. Severo, Probability functions, in: M. Abromowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, vol. 5, Applied Mathematics Series, US Department of Commerce, 1964, pp. 925–995.