

# Cash-Flow Based Dynamic Inventory Management

Michael N. Katehakis

Department of Management Science & Information Systems, Rutgers Business School—Newark and New Brunswick, 100 Rockefeller Rd., Piscataway, New Jersey 08554, USA, mnk@rutgers.edu

Benjamin Melamed

Department of Supply Chain Management, Rutgers Business School—Newark and New Brunswick, 100 Rockefeller Rd., Piscataway, New Jersey 08554, USA, melamed@business.rutgers.edu

Jim (Junmin) Shi\*

Martin Tuchman School of Management, New Jersey Institute of Technology, University Heights, Newark, New Jersey 07102, USA  
junmin.shi@njit.edu

Small-to-medium-sized enterprises (SMEs), including many startup firms, need to manage interrelated flows of cash and inventories of goods. In this study, we model a firm that can finance its inventory (ordered or manufactured) with loans in order to meet random demand which in general may not be time stationary. The firm earns interest on its cash on hand and pays interest on its debt. The objective is to maximize the expected value of the firm's capital at the end of a finite planning horizon. The firm's state at the beginning of each period is characterized by the inventory level and the capital level measured in units of the product, whose sum represents the "net worth" of the firm. Our study shows that the optimal ordering policy is characterized by a pair of threshold parameters as follows. (i) If the net worth is less than the lower threshold, then the firm employs a base stock order up to the lower threshold. (ii) If the net worth is between the two thresholds, then the firm orders exactly as many units as it can afford, without borrowing. (iii) If the net worth is above the upper threshold, then the firm employs a base stock order up to the upper threshold. Further, upper and lower bounds for the threshold values are developed using two simple-to-compute myopic ordering policies which yield lower bounds for the value function. We also derive an upper bound for the value function by considering a sell-back policy. Subsequently, it is shown that policies of similar structure are optimal when the loan and deposit interest rates are piecewise linear functions, when there is a maximal loan limit and when unsatisfied demand is backordered. Finally, further managerial insights are provided with extensive numerical studies.

*Key words:* inventory-finance decision; threshold values; myopic policy; sell-back policy

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## 1. Introduction

In the current competitive environment, small-to-medium-sized enterprises (SMEs), including many startup firms, must make joint decisions concerning interrelated flows of cash and products (cf. MaRS 2014). Cash is the lifeblood of any business and is often in short supply. Cash flows stem from operations, financing and investing activities. To maximize profits, firms need to manage operations, financing and investing activities, efficiently and comprehensively. For example, most startup firms need access to funds, typically for ordering or manufacturing products (cf. Zwilling 2013). As another example, some small businesses are seasonal in nature, particularly retail businesses. If a firm makes most of its sales during the holiday season, it may need a bank loan prior to the holiday season to finance a large amount of inventory to gear up for that season. Bank loans to purchase inventory are generally short-term in nature

and firms usually pay them off after the season is over with proceeds from their seasonal sales.

To facilitate SME borrowing, the *Small Business Administration* (SBA) has established a loan program with a successful record. For example, *Wells Fargo Capital Finance* is the nation's leading provider of loans (by dollar volume) guaranteed by the SBA, providing loans to companies across a broad range of industries throughout the United States. Building on a strong track record, Wells Fargo works with numerous companies in diverse industries, including apparel, electronics, furniture, housewares, sporting goods, toys and games, food products, hardware, and industrial goods. Accordingly, small business loans at Wells Fargo rose 18% in 2014 (cf. Finance.Yahoo.com 2015, Reuters 2015, WellsFargo.com 2015).

In this study, we model and analyze the optimal financial and operational policy of an SME firm whose inventory is subject to lost sales, zero replenishment lead times and periodic review over a finite

planning horizon. The firm's state (i.e., *inventory-cash profile*) consists of inventory level and cash level, where a positive cash level represents cash on hand while a negative cash level represents a loan position. In each period, cash flows are managed from the perspectives of operations, financing and investing, as follows. (i) From the operations perspective, the firm stocks up and sells inventory; (ii) From the financing perspective, the firm can use its cash on hand or an external short-term loan (if needed) to procure products for inventory; (iii) From the investing perspective, any cash on hand is deposited in a bank account to earn interest at a given rate, while debt incurs interest at a higher given rate. The objective of the firm is to dynamically optimize the order quantities in each period, given the current state of cash and inventory, via joint operational/financial decisions, so as to maximize the expected value of the firm's capital (i.e., total wealth level) at the end of a finite time horizon. To this end, we develop and study a discrete-time model for a single-product inventory system over multiple periods.

The inventory flow is described as follows. At the beginning of each period, the firm decides on an order quantity, and the corresponding replenishment order materializes with zero lead time. During the remainder of the period, no inventory transactions (demand fulfillment or replenishment) take place. Rather, all such transactions are settled at the end of the period. Incoming demand is aggregated over that period, and the total period's demand draws down on-hand inventory. However, if the demand exceeds the on-hand inventory, then all excess demand is lost (the backordering setting is studied in section 7.3). All the leftover inventory (if any) is carried forward to the next period subject to a holding cost, and at the last period, the remaining inventory (if any) is disposed of either at a salvage value or at a disposal cost.

Cash flows take place as follows. All transactions taking place in a period are settled at its end. More precisely, the firm updates its cash position with the previous period's revenue from sales and interest earned on a deposit, or paid on outstanding debt (if any). The firm then decides the order quantity for the next period and pays for replenishment as follows: first, with cash on hand, and if insufficient, with a withdrawal from cash on deposit (there is no withdrawal penalty), and if still insufficient, by a loan. If not all cash on hand is used for replenishment, then any unused cash is deposited in a bank where it earns interest. At the end of each period, the resulting cash on hand or debt are carried forward to the next period. Thus, the firm's state in each period  $n = 1, \dots, N$ , is a two-dimensional vector  $(x_n, y_n)$ , where  $x_n$  represents the value of inventory and  $y_n$  the cash level (cash in deposit if positive, debit if negative)

measured in units of the product, available at the start of the period; together they constitute a "net worth" of the firm,  $\xi_n = x_n + y_n$ .

The main contributions of the study are as follows. First, it establishes the structure of the optimal ordering policy in terms of the net worth  $\xi_n$  of the firm at the beginning of each period. It is shown that the optimal policy is characterized by a sequence of two threshold critical values,  $\alpha_n$  and  $\beta_n$ , where  $\alpha_n < \beta_n$ ; cf. Theorem 1 for the single-period problem and Theorem 3 for the multi-period problem, both in the sequel. This optimal policy has the following structure: (i) If  $\xi_n \leq \alpha_n$ , then the firm employs a base stock ordering up to  $\alpha_n$ ; (ii) If  $\alpha_n < \xi_n < \beta_n$ , then the firm orders exactly as many units as it can afford, without borrowing; (iii) If  $\beta_n \leq \xi_n$ , then the firm employs a base stock ordering up to  $\beta_n$ . The above cases will be referred to as the *over-utilization*, *full-utilization*, and *under-utilization* of available cash, respectively.

Second, for the single-period problem, we show that the  $(\alpha, \beta)$  optimal policy yields a positive expected value even with zero values for both initial inventory and cash. For the multi-period problem, we construct two myopic policies which respectively provide upper and lower bounds for the threshold values. The two myopic policies yield lower bounds for the value function. Since the underlying state space is two dimensional, it leads to optimization algorithms of high computational complexity, but by considering a sell-back policy, we derive a one-dimensional approximation algorithm that provides upper bounds for the value function. Subsequently, it is shown that policies of similar structure are optimal when the loan and deposit interest rates are piecewise linear functions (cf. Theorem 6), when there is a maximal loan limit (cf. Theorem 7) and when unsatisfied demand is backordered (cf. Theorem 8). Finally, further managerial insights are provided with numerical studies.

The remainder of this study is organized as follows. Section 2 reviews related literature and section 3 formulates the model. In section 4, the single-period model is developed and the optimal  $(\alpha, \beta)$  policy is derived, where the threshold values are functions of the demand distribution and the cost parameters of the problem. Section 5 extends the analysis to the dynamic multi-period problem and derives a two-threshold  $(\alpha_n, \beta_n)$  optimal policy via a dynamic programming analysis. Section 6 introduces two myopic policies that provide upper and lower bounds for each  $\alpha_n$  and  $\beta_n$ . The two myopic policies yield two lower bounds for the value function. Further, an upper bound for the value function is also introduced. In section 7, it is shown that simple modifications of the underlying model allow the extension of the results to the case of piecewise linear interest rate functions, a maximal loan limit and backordering of

unsatisfied demand. Numerical studies are presented in section 8. Finally, section 9 concludes the study. All proofs are relegated to Appendix S1.

## 2. Literature Review

In a seminal study, Modigliani and Miller (1958) show that in a perfect capital market, a firm's operational and financial decisions can be made separately. Since then, most literature in inventory management extended the classical newsvendor problem in a variety of ways, but assuming that decision makers are not subject to financial constraints. However, in view of the imperfection of real-life capital markets, a growing body of literature has begun to consider operational decision making subject to financial constraints. In those studies, inventories of goods are often treated as special financial instruments (cf. Singhal 1988). Accordingly, portfolios of physical products and financial instruments have been studied using finance/investment principles such as *Modern Portfolio Theory* and *Capital Asset Pricing Model*. For an up-to-date and comprehensive literature review on the interplay between inventories and finance, the reader is referred to Birge (2014) and references therein.

The growing literature on the interface between operations management and financial decisions can be categorized into two major strands: *single-agent* and *game related multiple-agent*. For each strand, attendant models can be further classified as single-period and multi-period models. Our study belongs to the single-agent strand and addresses both single-period and multi-period models.

In the single-agent strand, a firm's operational and financial decisions are typically made simultaneously, but without interaction with other firms. Considering an imperfect market, Xu and Birge (2004) develop models with simultaneous production and financing decisions in the presence of demand uncertainty, which illustrates how a firm's production decisions are affected by the existence of financial constraints. Recently, Birge and Xu (2011) present an extension of a model in Xu and Birge (2004) by assuming that debt and production scale decisions have fixed costs necessary to maintain operations, variable costs of production, and volatility in future demand forecasts. Building on their previous work of Xu and Birge (2004, 2008), they consider the effect of various operating conditions on capital structure, and exhibit some empirical support for their previously predicted relationship between production margin and market leverage. Cai et al. (2014) investigate the roles of bank and trade credits in a supply chain with a capital-constrained retailer facing demand uncertainty. The studies above focus on single-period problems.

Some single-agent, multi-period problems are addressed in Hu and Sobel (2005) and Li et al. (2013). Here, multi-period models posit different interest rates on cash on hand and outstanding loans. These papers also demonstrate the importance of joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence on bank financing. For example, Hu and Sobel (2005) examine the interdependence of a firm's capital structure and its short-term operating decisions concerning inventories, dividends, and liquidity. To this end, Hu and Sobel (2005) formulate a dynamic model to maximize the expected present value of dividends. Li et al. (2013) study a dynamic model of managerial decisions in a manufacturing firm in which inventory and financial decisions interact and are coordinated in the presence of demand uncertainty, financial constraints, and default risk. It is shown that the coordination between financial and operational decisions can significantly improve the expected present value of dividends. In addition, Gupta and Wang (2009) study a retailer's dynamic inventory problem (both discrete and continuous review) in the presence of random demand while the retailer is financially supported by trade credits from its supplier, contingent on the age of the inventory. It is shown that the optimal policy is a base stock policy where the base stock level is affected by the offered credit terms. Luo and Shang (2015) study a centralized supply chain consisting of two divisions (modeled as echelons), of which the headquarters division manages the financial and operational decisions with a cash pool, and the focus is on analyzing the value of cash pooling.

The work most closely related to this study is Chao et al. (2008) and Gong et al. (2014). The first reference considers a single-agent multi-period problem for a self-financing retailer without external loan availability. It shows that the optimal (cash flow dependent) policy in each period, is uniquely determined by a single critical value. Our study differs from Chao et al. (2008) in several ways. First, from a modeling perspective, Chao et al. (2008) consider a self-financing firm without access to external credit, while our model admits loans that afford the retailer the flexibility of ordering larger quantities to achieve a higher fill rate. The admissibility of loans intertwines financial decisions (e.g., how much to borrow) with operational decisions (e.g., how much to order) as well as investment decisions (how much to deposit for additional interest revenue). In this case, the complexity of the problem is raised from one dimension (operational decisions alone) to two dimensions (joint operational and financial decisions). Further, Chao et al. (2008) assume an i.i.d demand process and time-stationary costs, while we

consider a non-stationary demand process and time varying (loan and deposit) interest rates. Second, in terms of results, Chao et al. (2008) structure the optimal policy as a base stock policy with a single threshold value, while our study shows that the optimal policy is characterized by two threshold values, which divide the ordering decision space into three intervals to guide financing, operational, and investment decisions. Gong et al. (2014) study a similar model aiming to maximize the expected terminal total capital but along lines different from ours. In particular, Gong et al. (2014) consider an increasing concave loan rate while this study considers a piecewise linear loan rate. Importantly, our study differs in several aspects as follows: (i) Concerning model setting, we consider non-stationary demand processes and time-varying (loan and deposit) interest rates as well as an extension to realistic piecewise linear loan rates. (ii) Concerning results, our optimal policy is presented in terms of two thresholds per period (multiple thresholds for the case of piecewise linear loan rates), while Gong et al. (2014) analyze a single-threshold policy. (iii) Computationally, we develop simple easy-to-compute policies (based on single-period information) which provide lower and upper bounds for each of the two threshold values, and we propose an efficient algorithm for the computation of the optimal policy. Furthermore, we provide upper and lower bounds for the value function. (iv) Finally, we develop managerial insights into the interplay between operational, financial, and investment decisions. We note that our setting of the problem is in-between the models studied in Chao et al. (2008) and Gong et al. (2014) but differs in emphasis. In summary, this study contributes to such extant literature by tackling the interplay between operational and financial decisions (i.e., inventory flow and cash flow, respectively) reflected by loans or deposits, and developing bounds for the threshold values as well as the objective function.

A second strand of the literature has investigated multi-agent competition among firms and financial institutions using game theoretic approaches. This literature includes Buzacott and Zhang (2004), Dada and Hu (2008), Yasin and Gaur (2012), Raghavan and Mishra (2011), etc. Most such papers deal with a single-period problem. Buzacott and Zhang (2004) analyze a Stackelberg game between the bank and the retailer in a newsvendor inventory model. Dada and Hu (2008) assume that loan interest is charged by the bank endogenously and use a game model to capture the interaction between the bank and the inventory controller through which an equilibrium is derived and a nonlinear loan schedule is obtained to coordinate the channel. Yasin and Gaur (2012) study the implications of asset based lending to operational

investment, probability of bankruptcy, and capital structure for a borrower firm. Raghavan and Mishra (2011) study a short-term financing problem in a cash-constrained supply chain.

As part of the second literature strand, multi-agent game-theoretic approaches have also been used to model the competition between suppliers and retailers. Such recent studies include Kouvelis and Zhao (2011a, b) among many others. An important portion of this strand addresses the impact of trade credits provided by suppliers to retailers. Lee and Rhee (2010) study the impact of inventory financing costs on supply chain coordination by considering four coordination mechanisms: all-unit quantity discount, buy backs, two-part tariff, and revenue sharing. It is shown that using trade credits in addition to contracts, a supplier can fully coordinate the supply chain and achieve maximal joint profit. Further, Lee and Rhee (2011) model a firm with a supplier that grants trade credits and markdown allowances. Given the supplier's offer, it determines the order quantity and the financing option for the inventory under either trade credits or direct financing from a financial institution. The impact of trade credits from an operational perspective is also studied in Yang and Birge (2013), which investigates the role that trade credit plays in channel coordination and inventory financing. It is shown that when offering trade credits, the supplier balances its impact on operational profit and costs.

### 3. Model Formulation

Consider a time horizon  $\mathcal{N} := \{1, 2, \dots, N\}$  where the periods are indexed forward. At the beginning of period  $n \in \mathcal{N}$ , let the state of the system be represented by a vector  $(x_n, y_n)$ , where  $x_n \geq 0$  denotes the amount of on-hand inventory (number of product units) and  $y_n$  denotes the amount of product that can be purchased using all the available cash  $Y_n$ , that is,  $y_n$  is the cash position measured in product units. Let  $q_n \geq 0$  denote the order quantity (in product units) that arrives at the beginning of period  $n \in \mathcal{N}$ . The ordering cost per unit is  $c_n \geq 0$ , so that  $Y_n = c_n \times y_n$ . We assume that any order quantity can be fully satisfied within zero lead time of replenishment. Let  $D_n \in \mathbb{R}^+$  denote the random demand during period  $n$ , and assume it follows a general distribution. We assume that demands of different periods are independent but could be generally non-stationary across periods. Let  $f_n(\cdot)$  and  $F_n(\cdot)$  denote the probability density function (pdf) and the cumulative distribution function (cdf) of  $D_n$ , respectively. The selling price per unit is  $p_n \geq c_n$ .

Throughout all periods  $n = 1, \dots, N - 1$ , any unsold units are carried over to subsequent periods,

非稳态需求

提出了一个库存控制策略

并且设计了求解算法

将资金用最大可订销量表示

提前期为0



inventory on hand is subject to a constant holding cost  $h_n \geq 0$  per unit over period  $n$ , and any unmet demand is lost. The backorder setting of the problem is studied in subsection 7.3. At the end of the horizon, that is, the end of period  $N$ , all leftover inventory (if any) is salvaged (or disposed of) at a constant price (cost) of  $s$  per unit. Note that we allow a negative  $s$ , in which case  $s$  represents a disposal cost per unit, for example, the unit cost of vehicle tire disposal. For notational convenience, we denote in the sequel  $h_N = -s$ , so that the salvage value (or disposal cost)  $s$  is treated uniformly as a type of inventory cost.

To finance inventory, a loan can be obtained at interest rate  $\ell_n \geq 0$ . Interest rates are fixed and depend entirely upon current market conditions. For example, *Lending Club* (a US peer-to-peer lending company) can provide Small Business Loans at a fixed rate up to a sufficiently large amount. When unused cash remains after an ordering decision, the unused cash is deposited in a bank account to earn interest at rate  $i_n \geq 0$ . To avoid trivialities, we assume that  $i_n < \ell_n$  and it is possible to achieve a positive profit with the aid of a loan, that is,  $(1 + \ell_n)c_n < p_n$ . This assumption can be equivalently written as  $1 + \ell_n < p_n/c_n$ . Note also, that the above assumption implies  $1 + i_n < p_n/c_n$  since  $i_n < \ell_n$ ; namely, investing in inventory is more profitable than depositing available cash in the bank.

Given an initial inventory-cash state  $(x_n, y_n)$ , if an order of size  $q_n \geq 0$  is placed and the demand during the period is  $D_n$ , then we have the following operational and financial cash inflows:

- The cash flow from inventory operations (i.e., the cash received from inventory sales) at the end of the period is given by

$$\begin{aligned} R_n(D_n, q_n, x_n) & \quad \text{巧妙地将库存与资金分开了} \\ &= p_n \cdot \min\{q_n + x_n, D_n\} - h_n \cdot [q_n + x_n - D_n]^+ \\ &= p_n \cdot [q_n + x_n - (q_n + x_n - D_n)^+] \\ &\quad - h_n \cdot [q_n + x_n - D_n]^+ \\ &= p_n(q_n + x_n) - (p_n + h_n) \cdot [q_n + x_n - D_n]^+, \quad (1) \end{aligned}$$

where  $[a]^+ = \max\{a, 0\}$  and the second equality holds by  $\min\{a, b\} = a - [a - b]^+$ .

- The cash flow from financial transactions at the end of the period is computed for each of the following investing and financing scenarios:

- (i) Investment transactions: If the order quantity satisfies  $0 \leq q_n \leq y_n$ , then the leftover amount  $c_n \cdot (y_n - q_n)$  of cash is deposited in the bank and will yield a positive inflow of  $c_n \cdot (y_n - q_n)(1 + i_n)$  at the end of period  $n$ .

- (ii) Financing transactions: If the order quantity satisfies  $q_n > y_n$  (including the case  $q_n = 0 > y_n$ ), then a loan amount of  $c_n \cdot (q_n - y_n)$  will be borrowed during the period and will result in a cash outflow of  $c_n \cdot (q_n - y_n)(1 + \ell_n)$  at the end of the period.

未满足需求丢失；也在后面讨论了backorder的情况

In summary, the net cash flow from financial transaction (positive or negative) can be written in general as

$$K_n(q_n, y_n) = c_n \cdot (y_n - q_n) [(1 + i_n) \mathbf{1}_{\{q_n \leq y_n\}} + (1 + \ell_n) \mathbf{1}_{\{q_n > y_n\}}]. \quad (2)$$

Note that the cash inflow from inventory operations,  $R_n(D_n, q_n, x_n)$ , is dependent on  $D_n$  but independent of  $y_n$ . In a similar vein, cash flow from financial transactions,  $K_n(q_n, y_n)$ , is independent of the initial on-hand inventory,  $x_n$ , and the demand size,  $D_n$ , but is dependent on  $y_n$ . Note also that the ordering cost,  $c_n \cdot q_n$ , has been accounted for in Equation (2) while the remaining cash, if any, is invested in the bank and its value at the end of the period is given by  $K_n(q_n, y_n)$ .

Since the order quantity  $q_n = q_n(x_n, y_n)$  is decided at the beginning of period  $n$  as a function of  $(x_n, y_n)$ , it is readily shown that the state process  $\{(x_n, y_n)\}$  under study is a Markov decision process (MDP) with decision variable  $q_n$  (cf. Ross 1996). The state dynamics of the system, that is, inventory flow and cash flow, are given as follows, for  $n = 1, 2, \dots, N - 1$

$$x_{n+1} = [x_n + q_n - D_n]^+; \quad (3)$$

$$y_{n+1} = [R_n(D_n, q_n, x_n) + K_n(q_n, y_n)]/c_{n+1}, \quad (4)$$

where  $R_n$  and  $K_n$  are given by Equations (1) and (2), respectively.

Note that, at the beginning of period  $n$ , the interplay between finance and operations is typically one way. Namely, it is feasible to purchase products with available cash  $y_n$  (when  $y_n > 0$ ), but conversion of on-hand inventory  $x_n$  into cash is restricted or impossible. Conversely, the sales interplay between finance and operations is the opposite, namely, sales decrease the inventory and increase cash on hand. Thus, the requisite optimization can be formulated as follows:

$$\begin{aligned} V_n(x_n, y_n) &= \sup_{q_n \geq 0} \mathbb{E} \left[ V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n \right], \\ n &= 1, 2, \dots, N - 1, \end{aligned} \quad (5)$$

最优方程

where the expectation is taken with respect to  $D_n$ ,  $x_{n+1}$  and  $y_{n+1}$  are given by Equations (3) and (4), respectively. For the final period  $N$ , we have

$$V_N(x_N, y_N) = \sup_{q_N \geq 0} \mathbb{E} \left[ R_N(D_N, q_N, x_N) + K_N(q_N, y_N) \right]. \quad (6)$$

#### 4. The Single-Period Problem

The analysis of the last period,  $N$ , is a typical single-period problem and its objective function is given by Equation (6). To simplify notation, we omit the subscript  $N$  from any related variable when denoting counterparts under the single-period setting, without causing any confusion. Accordingly, for any given initial “inventory-cash” state  $(x, y)$  and order size,  $q \geq 0$ , the expected net worth at the end of the period is given by

$$G(q, x, y) = p(x + q) - (p - s) \int_0^{x+q} (x + q - t)f(t)dt + c(y - q) \left[ (1 + i)\mathbf{1}_{\{q \leq y\}} + (1 + \ell)\mathbf{1}_{\{q > y\}} \right],$$

$c$  是可变成本,  $s$  是持有成本  
利息只能二选一 (7)

where the first term above is the initial inventory value, the second is the expected value of inventory sold over the period adjusted for salvage/disposal, and the third is the cash flow from financial transactions.

The following lemma summarizes the important properties of the function  $G(q, x, y)$ .

LEMMA 1. The function  $G(q, x, y)$  is continuous in  $q, x$  and  $y$ , and satisfies the following

- (i) It is concave in  $q \in [0, \infty)$ , for all  $x, y$  and all  $s < p$ .
- (ii) It is increasing and concave in  $x$ , for  $s \geq 0$ .
- (iii) It is increasing and concave in  $y$ , for all  $s < p$ .

REMARKS. 这个不需要

1. It is important to point out that  $G(q, x, y)$  might not be increasing in  $x$  if  $s < 0$ . To see this, we first write

$$\frac{\partial}{\partial x} G(q, x, y) = p\bar{F}(q + x) + sF(q + x). \quad (8)$$

In particular, if  $s$  represents a disposal cost, that is,  $s < 0$ , then the right side of Equation (8) might be negative, which implies that  $G(q, x, y)$  is decreasing for sufficiently high values of  $x$ . Further, for the special case  $s < 0$ , it is of interest to locate the critical value,  $x'$  such that  $G(q, x, y)$  is decreasing for  $x > x'$ . To this end, we set Equation (8) to zero, which yields

$$(p - s) \cdot F(q + x) = p. \quad (9)$$

Therefore,

$$x' = F^{-1} \left( \frac{p}{p - s} \right) - q, \quad (10)$$

where  $F^{-1}(\cdot)$  is the inverse function of  $F(\cdot)$ . Equation (10) shows that a higher disposal cost implies a lower threshold value for  $x'$  above.

2. Lemma 1 implies that higher values of initial assets,  $x$  and  $y$ , or of net worth,  $\zeta = x + y$ , yield a higher expected revenue  $G(q, x, y)$ . Further, for any assets  $(x, y)$  there is a unique optimal order quantity  $q^*$  such that

$$q^*(x, y) = \arg \max_{q \geq 0} G(q, x, y). \quad \text{结论只适合最后一阶段}$$

We next define the critical values,  $\alpha$  and  $\beta$ , as follows:

$$\alpha = F^{-1}(a); \quad (11)$$

$$\beta = F^{-1}(b), \quad (12)$$

where

$$a = \frac{p - c \cdot (1 + \ell)}{p - s};$$

$$b = \frac{p - c \cdot (1 + i)}{p - s}.$$

It is readily seen that  $a \leq b$ , since  $0 \leq i \leq \ell$  by assumption. This implies that  $\alpha \leq \beta$ , since  $F^{-1}(z)$  is increasing in  $z$ . The critical value  $\beta$  can be interpreted as the optimal order quantity for the classical newsvendor problem (cf. Zipkin 2000 and many others) corresponding to the case of sufficiently large  $Y$  of our model, in which case no loan is involved, but the unit “price”  $c \cdot (1 + i)$  has been inflated to reflect the opportunity cost of cash not invested in the bank at an interest rate  $i$ . Similarly,  $\alpha$  can be interpreted as the optimal order quantity for the classical newsvendor problem corresponding to the case  $Y = 0$  of our model, that is, all the order is financed by a loan at an interest rate  $\ell$ . Note also that in contrast to the classical newsvendor model, the critical values  $\alpha$  and  $\beta$  above, are now functions of the corresponding interest rates and represent opportunity costs that take into account the value of time using the interest factors  $1 + i$  and  $1 + \ell$ .

The following theorem identifies the optimal  $(\alpha, \beta)$  policy.

**THEOREM 1.** For any given inventory-cash state  $(x, y)$ , the optimal order quantity is

$$q^*(x, y) = \begin{cases} (\beta - x)^+, & \beta \leq x + y; \\ y^+, & \alpha \leq x + y < \beta; \\ (\alpha - x)^+, & x + y < \alpha, \end{cases} \quad (13)$$

where  $\alpha$  and  $\beta$  are given by Equations (11) and (12), respectively.

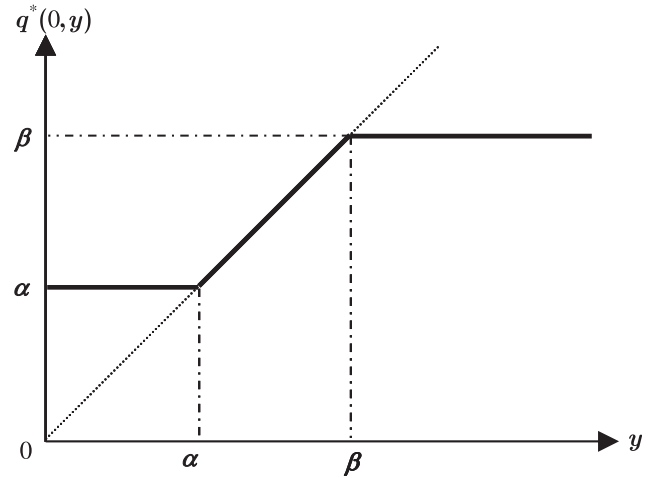
图像形状，因为2选1，

Note that the optimal ordering quantity in the classical newsvendor model, can be obtained from Theorem 1 as the solution for the extreme case with  $i = \ell = 0$ , in which case the optimal order quantity is given by:  $\alpha = \beta = F^{-1}\left(\frac{p-c}{p-s}\right)$ .

Next we elucidate the structure of the optimal  $(\alpha, \beta)$  policy below by discussing the utilization level of the initially available cash  $Y$ .

1. *Over-utilization:*  $(x + y < \alpha)$ . If  $x \leq \alpha$ , then it is optimal to order  $q^* = \alpha - x = y + (\alpha - x - y)$ . In this case,  $y = Y/c$  units are purchased using all available cash  $Y$ , if  $Y > 0$ , and the remaining  $(\alpha - x - y)$  units are purchased with the proceeds of a loan in the amount  $c \cdot (\alpha - x - y)$ . If  $x > \alpha$ , then it is optimal not to order, that is,  $q^* = 0$ .
2. *Full-utilization:*  $(\alpha \leq x + y < \beta)$ . It is optimal to order  $q^* = y = Y/c$  with all the available cash  $Y$ , if  $Y > 0$ ; otherwise, no ordering. In this case, all available cash is fully utilized and no deposit and no loan are involved.
3. *Under-utilization:*  $(x + y \geq \beta)$ . In this case, it is optimal to order  $q^* = (\beta - x)^+$ . In particular,

**Figure 1** The Optimal Order Quantity When  $x = 0$



We mention that the structure of this type of two-threshold values is essentially caused by the two distinct (loan and deposit) interest rates. Actually, the optimal threshold policy with multiple critical values caused by a piecewise linear ordering cost has been studied in the literature (cf. Bensoussan et al. 1983, Fu et al. 2012). In section 7.1, we will derive the structural result of multiple threshold values for piecewise-type loan and deposit functions.

We are now in a position to derive an explicit representation for the optimal expected net worth,  $V(x, y) = \max_{q \geq 0} G(q, x, y)$ .

**THEOREM 2.** For the single-period problem with initial state  $(x, y)$ ,

- (i)  $V(x, y)$  is given by

$$V(x, y) = \begin{cases} p \cdot x - (p - s) \cdot T(x) + c \cdot y \cdot (1 + i), & x > \beta; \\ p \cdot \beta - (p - s) \cdot T(\beta) + c \cdot (x + y - \beta)(1 + i), & x \leq \beta, \beta \leq x + y; \\ p \cdot (x + y^+) - (p - s) \cdot T(x + y^+), & \alpha \leq x + y < \beta; \\ p \cdot \alpha - (p - s) \cdot T(\alpha) + c \cdot (x + y - \alpha)(1 + i), & x + y < \alpha, \end{cases} \quad (14)$$

if  $x < \beta$ , it is optimal to order  $\beta - x$  using  $c(\beta - x)$  units of the available cash  $Y$ , and deposit the remaining cash to earn interest. However, if  $x \geq \beta$ , then  $q^* = 0$ , that is, it is optimal not to order and deposit the entire amount  $Y$ , if  $Y > 0$ , to earn interest.

The above interpretation is illustrated in Figure 1 for the case  $x = 0$ , which plots the optimal order quantity  $q^*$  as a function of  $y$ . Note that for  $y \in (0, \alpha)$  there is over-utilization of  $y$ ; for  $y \in [\alpha, \beta)$  there is full-utilization of  $y$  and for  $y \in [\beta, \infty)$  there is under-utilization of  $y$ .

where  $T(x) = \int_0^x (x - t)f(t)dt$ ;

- (ii) The function  $V(x, y)$  is increasing in  $x$  and  $y$ , and jointly concave in  $(x, y)$ , for  $x, y \geq 0$ .

From an investment perspective, it is of interest to consider the possibility of speculation (cf. Hull 2002). The following result shows that the  $(\alpha, \beta)$  policy given in Theorem 1 yields a positive value with zero investment. Specifically, when the firm has zero initial inventory and cash, that is,  $x = 0$  and  $y = 0$ , the optimal policy yields a positive expected final asset value.

COROLLARY 1. For  $x = 0$  and  $y = 0$ , one has

$$V(0,0) = (p - s) \int_0^\alpha t \cdot f(t) dt > 0.$$

Note that arbitrage usually means that it is possible to have a positive profit for any realized demand, that is, of a net risk-free profit at zero cost (cf. Hull 2002). Thus, the above speculation possibility does not in general imply an arbitrage, which only exists if the demand is deterministic.

## 5. Optimal Solution for the Multiple Period Problem

For ease of exposition, we shall denote  $\xi_n = x_n + y_n$  and take  $z_n = x_n + q_n$  as the decision variable in lieu of  $q_n$ . Here,  $z_n$  represents on-hand inventory in product units after replenishment, and is constrained by  $z_n \geq x_n \geq 0$  for each period  $n$ . The attendant MDP model defined by Equations (5)–(6) can be written as follows:

$$V_n(x_n, y_n) = \max_{z_n \geq x_n} G_n(z_n, x_n, y_n), \quad (15)$$

where  $G_n(z_n, x_n, y_n) = \mathbb{E}[V_{n+1}(x_{n+1}, y_{n+1})]$ . The inventory-cash states are periodically updated as follows:

$$x_{n+1} = [z_n - D_n]^+; \quad (16)$$

$$y_{n+1} = p'_n \cdot z_n - (p'_n + h'_n)[z_n - D_n]^+ + c'_n \cdot (\xi_n - z_n) \times \left[ (1 + i_n) \mathbf{1}_{\{z_n \leq \xi_n\}} + (1 + \ell_n) \mathbf{1}_{\{z_n > \xi_n\}} \right], \quad (17)$$

where  $p'_n = p_n/c_{n+1}$ ,  $h'_n = h_n/c_{n+1}$  and  $c'_n = c_n/c_{n+1}$ .

We first present the following result whose proof appears in Appendix S1.

LEMMA 2. For  $n \in \mathcal{N}$ ,

- (1)  $G_n(z_n, x_n, y_n)$  is increasing in  $x_n$  and  $y_n$ , and is concave in  $z_n$ ,  $x_n$  and  $y_n$ .
- (2)  $V_n(x_n, y_n)$  is increasing and concave in  $x_n$  and  $y_n$ .

We next present and prove the main result of this section.

THEOREM 3 (THE  $(\alpha_n, \beta_n)$  OPTIMAL ORDERING POLICY). For every period  $n \in \mathcal{N}$  with state  $(x_n, y_n)$  at the beginning of the period, there exist positive constants  $\alpha_n = \alpha_n(x_n, y_n)$  and  $\beta_n = \beta_n(x_n, y_n)$  such that  $\alpha_n \leq \beta_n$ , which give rise to an optimal order quantity as follows:

$$q_n^*(x_n, y_n) = \begin{cases} (\beta_n - x_n)^+, & \xi_n \geq \beta_n; \\ (y_n)^+, & \alpha_n \leq \xi_n < \beta_n; \\ (\alpha_n - x_n)^+, & \xi_n < \alpha_n. \end{cases} \quad (18)$$

Further,  $\alpha_n$  is uniquely determined by

$$\mathbb{E} \left[ \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n) \frac{\partial V_{n+1}}{\partial y_{n+1}} \right) \mathbf{1}_{\{x_n > D_n\}} \right] = [c'_n(1 + \ell_n) - p'_n] \mathbb{E} \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right], \quad (19)$$

and  $\beta_n$  is uniquely determined by

$$\mathbb{E} \left[ \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n) \frac{\partial V_{n+1}}{\partial y_{n+1}} \right) \mathbf{1}_{\{\beta_n > D_n\}} \right] = [c'_n(1 + i_n) - p'_n] \mathbb{E} \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right], \quad (20)$$

where the expectations are with respect to  $D_n$ , conditioned on the initial state  $(x_n, y_n)$ .

Theorem 3 establishes that the optimal ordering policy for period  $n$  is characterized by two threshold parameters,  $\alpha_n$  and  $\beta_n$ . More importantly, these two threshold values can be computed recursively by solving the implicit equations, Equations (19) and (20), respectively.

REMARK. The study in Chao et al. (2008) assumes that borrowing is not allowed, and thus the firm is strictly limited to order at most  $y_n$  units in period  $n$ . It was shown there that the optimal policy is determined in each period by one critical value. Our results in Theorem 3 subsume that study as a special case. To see that, set  $\ell_n$  to be sufficiently large such that a loan is financially prohibitive. In this case,  $\alpha_n$  becomes zero and  $\beta_n$  can be interpreted as the critical value developed in Chao et al. (2008).

COROLLARY 2. For any period  $n < N$  and the associated initial state  $(x_n, y_n)$ , the threshold parameters,  $\alpha_n$  and  $\beta_n$ , are determined only by the total worth  $\xi_n = x_n + y_n$ , that is, they are of the form  $\alpha_n = \alpha_n(\xi_n)$  and  $\beta_n = \beta_n(\xi_n)$ . However, for the last period  $N$ ,  $\alpha_N$  and  $\beta_N$  are independent of either  $x_N$  or  $y_N$ , or both.

Not surprisingly, Corollary 2 implies that the operational decisions are affected by the firm's current financial status. Furthermore, one can first compute the  $\alpha_n$  and  $\beta_n$  at the beginning of the period based on the total worth  $\xi_n$ . The decision on the order quantity  $q_n$  can then be made according to Equation (18).



## 6. Myopic Policies and Bounds Analysis

As shown by Theorem 3, the computation of  $\alpha_n$  and  $\beta_n$  is complex and costly. This fact motivates the study to follow of myopic ordering policies that are relatively simple to compute and implement. Such myopic policies optimize a given objective function with respect to any single period and ignore multi-period interactions and cumulative effects. To this end, we introduce two types of myopic policies. In the following two subsections, we will show that myopic policy (I) yields lower bounds,  $\underline{\alpha}_n$  and  $\underline{\beta}_n$  for the two threshold values,  $\alpha_n$  and  $\beta_n$ , while myopic policy (II) yields upper bounds,  $\bar{\alpha}_n$  and  $\bar{\beta}_n$  for the same values.

Before presenting the myopic policies, we present the following lemma, which will be used to derive the upper and lower bounds.

LEMMA 3. For real functions  $f(x)$  and  $g(x)$ ,

- (a) if  $f(x)$  and  $g(x)$  are both monotonically increasing or both decreasing, then

$$\mathbb{E}[f(X) \cdot g(X)] \geq \mathbb{E}[f(X)] \cdot \mathbb{E}[g(X)],$$

where the expectation is taken with respect to the random variable  $X$ .

- (b) If  $f(x)$  is increasing (decreasing), while  $g(x)$  is decreasing (increasing), then

$$\mathbb{E}[f(X) \cdot g(X)] \leq \mathbb{E}[f(X)] \cdot \mathbb{E}[g(X)].$$

### 6.1. Myopic Policy (I) and Lower Bounds for $\alpha_n$ and $\beta_n$

Myopic policy (I) is the one-period optimal policy obtained by assuming that only holding cost is assessed on any leftover inventory, that is, we assume the following modified “salvage value” cost structure:

$$\underline{s}_n = \begin{cases} -h_n, & n < N, \\ s, & n = N. \end{cases}$$

Let further,

$$\underline{a}_n = \frac{p_n - c_n \cdot (1 + \ell_n)}{p_n - \underline{s}_n}; \quad (21)$$

$$\underline{b}_n = \frac{p_n - c_n \cdot (1 + i_n)}{p_n - \underline{s}_n}. \quad (22)$$

and the corresponding critical values are given by

$$\underline{\alpha}_n = F_n^{-1}(\underline{a}_n); \quad (23)$$

$$\underline{\beta}_n = F_n^{-1}(\underline{b}_n). \quad (24)$$

For  $n \in \mathcal{N}$ , the order quantity below defines the *myopic policy (I)*:

$$q_n(x_n, y_n) = \begin{cases} (\underline{\beta}_n - x_n)^+, & x_n + y_n \geq \underline{\beta}_n; \\ (y_n)^+, & \underline{\alpha}_n \leq x_n + y_n < \underline{\beta}_n; \\ \underline{\alpha}_n - x_n, & x_n + y_n < \underline{\alpha}_n. \end{cases}$$

The next theorem establishes the lower bound properties of the myopic policy (I).

THEOREM 4. The following are true:

- (i) For the last period  $N$ ,  $\alpha_N = \underline{\alpha}_N$  and  $\beta_N = \underline{\beta}_N$ .  
 (ii) For any period  $n = 1, 2, \dots, N - 1$ ,

$$\begin{aligned} \alpha_n &\geq \underline{\alpha}_n, \\ \beta_n &\geq \underline{\beta}_n. \end{aligned}$$

### 6.2. Myopic Policy (II) and Upper Bounds for $\alpha_n$ and $\beta_n$

Myopic policy (II) is the one-period optimal policy obtained by assuming that not only the holding cost is assessed but also the cost in the next period on any leftover inventory, that is, we assume the following modified “salvage value” cost structure:

$$\bar{s}_n = \begin{cases} c_{n+1} - h_n, & n < N; \\ s, & n = N. \end{cases} \quad (25)$$

One can interpret the modified salvage values  $\bar{s}_n$  of Equation (25) as representing a fictitious income from *inventory liquidation* (or pre-salvage at full current cost) at the beginning of the next period,  $n + 1$ , that is, it corresponds to the situation that the firm can salvage inventory at the price  $c_{n+1}$  at the beginning of the period  $n + 1$ . Note that the condition  $c_n(1 + \ell_n) + h_n \geq c_{n+1}$  is required if inventory liquidation is admitted. Otherwise, the firm will stock up at an infinite level and sell them off at the beginning of period  $n + 1$ . Such speculation is precluded by the aforementioned condition.

Let further,

$$\bar{a}_n = \frac{p_n - c_n[1 + \ell_n]}{p_n - \bar{s}_n}; \quad (26)$$

$$\bar{b}_n = \frac{p_n - c_n[1 + i_n]}{p_n - \bar{s}_n}, \quad (27)$$

and let the corresponding critical values be given by

$$\bar{\alpha}_n = F_n^{-1}(\bar{a}_n), \quad (28)$$

$$\bar{\beta}_n = F_n^{-1}(\bar{b}_n). \quad (29)$$

For  $n \in \mathcal{N}$ , the order quantity below defines the *myopic policy* (II):

$$\bar{q}_n(x_n, y_n) = \begin{cases} (\bar{\beta}_n - x_n)^+, & x_n + y_n \geq \bar{\beta}_n; \\ (y_n)^+, & \bar{\alpha}_n \leq x_n + y_n < \bar{\beta}_n; \\ \bar{\alpha}_n - x_n, & x_n + y_n < \bar{\alpha}_n. \end{cases}$$

Let  $V_n^L(x_n, y_n)$  denote the optimal expected future value when the inventory liquidation option is available only at the beginning of period  $n + 1$  (but not in the remaining periods  $n + 2, \dots, N$ ), given the initial state  $(x_n, y_n)$  of period  $n$ . For notational simplicity, let  $\xi_{n+1} = \xi_{n+1}(x_n, y_n, z_n, D_n)$  represent the total inventory and cash value in period  $n + 1$  when the firm orders  $z_n \geq x_n$  in state  $(x_n, y_n)$  and the demand is  $D_n$ . Note that  $\xi_{n+1} = x_{n+1} + y_{n+1}$ , where  $x_{n+1}$  and  $y_{n+1}$  are given by Equations (16) and (17), respectively. Therefore,  $V_n^L$  can be written as

$$V_n^L(x_n, y_n) = \max_{z_n \geq x_n} \mathbb{E}[V_{n+1}(0, \xi_{n+1}) | x_n, y_n]. \quad (30)$$

Note that for any real functions  $g(x)$  and  $f(x)$ , with  $g(x)$  increasing and concave in  $x$  and  $f(x)$  concave in  $x$ , the function  $g(f(x))$  is concave in  $x$ . Thus, the concavity of  $\mathbb{E}[V_{n+1}(0, \xi_{n+1}) | x_n, y_n]$  in  $z_n$  follows from the observations that  $V_n(0, \xi)$  is increasing and concave in  $\xi$  (cf. part 2 of Lemma 2) and  $\xi_{n+1}$  is concave in  $z_n$ , since both  $x_{n+1}$  and  $y_{n+1}$  are concave in  $z_n$  (cf. discussion in the proof of Lemma 2 regarding the second-order derivatives of  $x_{n+1}$  and  $y_{n+1}$  with respect to  $z_n$ ). Consequently,  $V_n^L$  has an optimal policy characterized by a sequence of threshold value pairs,  $(\alpha_n^L, \beta_n^L)$ .

Before exhibiting the upper bounds of  $\alpha_n$  and  $\beta_n$ , we present the following result.

**PROPOSITION 1.** *For any period  $n$  with initial state  $(x_n, y_n)$ ,*

- (i)  $V_n(x_n - d, y_n + d)$  is increasing in  $d$  where  $0 \leq d \leq x_n$ .
- (ii) The partial derivatives satisfy,  $\partial V_n(x_n, y_n) / \partial y_n \geq \partial V_n(x_n, y_n) / \partial x_n$ .

Part (i) of Proposition 1 states that, for the same total assets, incremental cash is preferable to incremental inventory. This is true because more cash provides more flexibility in tuning the inventory level. Part (ii) implies that the marginal contribution of a unit of cash is higher than that of a unit of inventory. In view of Proposition 1, we have the following results.

**PROPOSITION 2.** *For any state  $(x_n, y_n)$ , and all  $n = 1, \dots, N$ ,*

- (i)  $V_n^L(x_n, y_n) \geq V_n(x_n, y_n)$ ,
- (ii)  $\alpha_n^L \geq \alpha_n$  and  $\beta_n^L \geq \beta_n$ .

Note that inventory liquidation at the beginning of period  $n + 1$  provides the firm with additional flexibility, since the initial inventory  $x_{n+1}$  can be liquidated into cash, thereby allowing the firm to hold only  $\xi_{n+1} = x_{n+1} + y_{n+1}$  in cash. Note further that the firm will choose to stock up to a higher level of inventory when liquidation is allowed. Indeed, the firm has no inventory risk in the sense that all leftover inventory in period  $n$  (after satisfying the period demand  $D_n$ ) can be salvaged at full cost  $c_{n+1}$  at the beginning of the next period,  $n + 1$ . In other words, the firm will take advantage of inventory liquidation to stock product at a higher level than in the counterpart case which disallows such liquidation. The advantage of doing so is twofold: (i) more demand can be satisfied leading to more revenue, and (ii) no extra cost accrues from liquidation of leftover inventory.

The next result establishes the upper bound properties of the myopic policy (II).

**PROPOSITION 3.** *For any period  $n = 1, 2, \dots, N - 1$ , if  $c_n(1 + \ell_n) + h_n \geq c_{n+1}$ , then the threshold parameters of the optimal policy given in Equations (19)–(20) and the associated myopic optimal policy given in Equations (28)–(29) satisfy*

$$\begin{aligned} \bar{\alpha}_n &\geq \alpha_n; \\ \bar{\beta}_n &\geq \beta_n. \end{aligned}$$

For the last period  $N$ ,  $\alpha_N = \bar{\alpha}_N$  and  $\beta_N = \bar{\beta}_N$ .

**REMARK.** It is not easy in general to compute solutions even for the simpler multi-period problems which involve base-stock levels. In the supply chain literature, computational methods are often based on heuristic algorithms or iterative search procedures. Determination of base-stock levels in each period or for each echelon and the minimization of the total supply chain cost are notoriously complex and computationally tedious, especially in our underlying two-dimensional state space with the attendant curse of dimensionality (cf. Chen and Zheng 1994, Daniel and Rajendran 2006, Lee and Billington 1993, Minner 1997, Shang and Song 2003). Nevertheless, with the aid of the lower and upper bounds developed above, one can devise an efficient search algorithm for the problem herein, for example, a bisection or golden section search algorithm (cf. Press 2007).

### 6.3. Sell-Back to the Supplier: Upper Bound for the Value Function $V_n(x, y)$

The myopic policies above provide lower and upper bounds for  $\alpha_n$  and  $\beta_n$ , respectively, and both of them

provide lower bounds for the net worth value function  $V_n(x, y)$ . In this subsection we obtain upper bounds for the value function by considering a related problem in which the firm is given the option to sell unneeded inventory back to the supplier at full price. Thus, for some cases, the firm has more flexibility at the beginning of a period  $n \in \mathcal{N}$  to readjust the inventory level as compared to just ordering from the supplier at cost  $c_n$ . For example, the firm is allowed to sell unneeded inventory back to the supplier or at a free-trading spot market at price  $c_n$ . We assume  $c_{n+1} \leq c_n + h_n$  to preclude situations where the firm can profit from selling products in inventory back to the supplier.

In practice, this situation can occur either when there is an agreement between the supplier and the firm such as a *vendor-managed inventory* contract, or there is a commodity spot market, for example, energy sources and natural resources (cf. Secomandi 2010). We will refer to such a setting as a *sell-back* option. The sell-back option provides the firm with the flexibility to convert any excess inventory into cash prior to the regular selling season. Consequently, the state of the system is reduced from two variables,  $x_n$  and  $y_n$ , to a single variable  $\xi_n$  that represents the sum of the value of inventory on hand plus the value of cash on hand, expressed in product units, that is,  $\xi_n = x_n + y_n$ .

Adding Equations (16) and (17) yields the dynamics equations for  $\xi_{n+1} = x_{n+1} + y_{n+1}$  with the decision variables  $z_n$ ,

$$\begin{aligned} \xi_{n+1} = & p'_n \cdot z_n - (p'_n + h'_n - 1)[z_n - D_n]^+ \\ & + c'_n \cdot (\xi_n - z_n) \left[ (1 + i_n) \mathbf{1}_{\{z_n \leq \xi_n\}} + (1 + \ell_n) \mathbf{1}_{\{z_n > \xi_n\}} \right]. \end{aligned} \quad (31)$$

The dynamic programming equations for the value function  $V_n^S(\xi_n)$  are

$$V_n^S(\xi_n) = \max_{z_n \geq 0} \mathbb{E}[V_{n+1}^S(\xi_{n+1})], \quad (32)$$

where the terminal value  $V_N^S(\xi_N) = V_N(0, \xi_N)$  is given by Equation (6).

It can be readily shown that  $V_n^S(\xi)$  is increasing and concave in  $\xi$  for any  $n \in \mathcal{N}$ . Consequently, we have the following result.

**THEOREM 5. (OPTIMAL INVENTORY-TRADING POLICY WITH SELL-BACK).** *At the beginning of period  $n \in \mathcal{N}$ , if the firm has a sell-back option, then*

- (i) *The optimal policy is the  $(\alpha_n^S, \beta_n^S)$  policy, where  $0 \leq \alpha_n^S < \beta_n^S$ . That is, the optimal target inventory is*

$$z_n^*(\xi_n) = \begin{cases} \beta_n^S, & \xi_n \geq \beta_n^S; \\ \xi_n, & \alpha_n^S \leq \xi_n < \beta_n^S; \\ \alpha_n^S, & \xi_n < \alpha_n^S. \end{cases}$$

- (ii) *For any  $x_n$  and  $y_n$ , the value functions satisfy*

$$V_n^S(x_n + y_n) \geq V_n^L(x_n, y_n) \geq V_n(x_n, y_n).$$

Theorem 5 states that, with the sell-back option, the firm can freely retune its inventory via selling down or ordering up as follows. If  $\xi_n \geq \beta_n^S$ , then it is optimal to choose the target inventory  $z_n$  equal to  $\beta_n^S$ ; if  $\xi_n \leq \alpha_n^S$ , then it is optimal to choose the target inventory  $z_n$  as  $\alpha_n^S$ ; and otherwise, if  $\alpha_n^S < \xi_n < \beta_n^S$ , then it is optimal to choose the target inventory  $z_n$  equal to  $\xi_n$ .

Note that  $V_n^S$  provides an upper bound of  $V_n$  which is easier to compute than that of  $V_n^L(x, y)$ , since the former is a function of only one variable.

## 7. Model Extensions

In this section, we study three extensions of the basic model. The first extension is a piecewise linear structure of the interest rate functions for both loans and deposits, the second is the constraint of a maximal loan limit and the third is backordering of unmet demand.

### 7.1. Piecewise-Type Loan and Deposit Functions

Hitherto, the loan function was assumed to be a linear function of the form  $L(x) = (1 + \ell) \cdot x$  with a constant loan rate  $\ell$ . However, in practice,  $L(x)$  can have a more complex form. In this subsection, we investigate the common case of a piecewise-type linear  $L(x)$ , that is,

$$L(x) = (1 + \ell^{(m)}) \cdot x, \quad x \in (x^{(m-1)}, x^{(m)}],$$

where  $x^{(m-1)} < x^{(m)}$ ,  $x^{(0)} = 0$  and  $\ell^{(m)} < \ell^{(m+1)}$  for  $m = 1, 2, 3, \dots$ . In a similar vein, we assume that the deposit interest function is piecewise linear of the form

$$M(y) = (1 + i^{(k)}) \cdot y, \quad y \in (y^{(k-1)}, y^{(k)}],$$

where  $y^{(k-1)} < y^{(k)}$ ,  $y^{(0)} = 0$  and  $i^{(k)} \leq i^{(k+1)}$  for  $k = 1, 2, 3, \dots$ . Without loss of generality, we assume that the loan interest rates are always greater than the deposit interest rates, that is  $\bar{i} < \ell^{(1)}$  where  $\bar{i} = \sup_k \{i^{(k)}\}$ .

Before characterizing the optimal ordering policy, we introduce the threshold values of  $\alpha^{(m)}$  and  $\beta^{(k)}$  for  $m, k = 1, 2, 3, \dots$  as follows

$$\alpha^{(m)} = F^{-1}(a^{(m)}); \quad (33)$$

$$\beta^{(k)} = F^{-1}(b^{(k)}), \quad (34)$$

where

$$a^{(m)} = \frac{p - c \cdot (1 + \ell^{(m)})}{p - s};$$

$$b^{(k)} = \frac{p - c \cdot (1 + i^{(k)})}{p - s}.$$

It is readily seen that

$$\beta^{(1)} \geq \dots \geq \beta^{(k)} \geq \beta^{(k+1)} \dots \geq \bar{\beta} > \alpha^{(1)} \geq \dots$$

$$\alpha^{(m)} \geq \alpha^{(m+1)} \dots \geq 0,$$

where  $\bar{\beta} = F_n^{-1}(\frac{p-c \cdot (1+i)}{p-s})$ . The structure of the optimal ordering policy is exhibited in Theorem 6 below.

**THEOREM 6.** *For any initial state  $(x, y)$ , the optimal order quantity is given by*

$$q^*(x, y) = \begin{cases} (\beta^{(k)} - x)^+, & \beta^{(k+1)} \leq x + y \leq \beta^{(k)}; \\ (y)^+, & \alpha^{(1)} \leq x + y < \bar{\beta}; \\ (\alpha^{(m)} - x)^+, & \alpha^{(m+1)} \leq x + y < \alpha^{(m)}, \end{cases}$$

where  $\{\alpha^{(m)}\}$  and  $\{\beta^{(k)}\}$  are given by Equations (33) and (34), respectively.

For the multi-period problem, the optimal order quantity for each period has a structure similar to that of its single-period counterpart, and the proof is analogous to that of Theorem 6. We note that for any period  $n < N$ , the threshold values,  $\alpha_n^{(m)}$  and  $\beta_n^{(k)}$ , are determined by  $\xi_n$ . However, for the last period,  $N$ , the threshold values,  $\alpha_N^{(m)}$  and  $\beta_N^{(k)}$ , are independent of the initial state,  $x_N$  and  $y_N$ , or  $\xi_N$ . We mention that if the lender charges a sufficiently high loan rate, such that  $p \leq c \cdot (1 + \ell^{(m)})$  for loans exceeding  $x^{(m-1)}$ , then  $\alpha^{(m)} = 0$  by virtue of Equation (33), since  $a^{(m)}$  is non-positive. Theorem 6 also implies that for a negative total asset value,  $x + y < 0$ , the firm will not finance orders via a loan.

## 7.2. Financing Subject to a Maximal Loan Constraint

In practice, the outstanding loan amount is often restricted not to exceed some prescribed maximum. Let  $L_n > 0$  denote such maximum for period  $n$ ; hence the maximal loan can finance  $L'_n := L_n/c_n$  product units. In this case, we have the following structural results for the optimal ordering policy.

**THEOREM 7 (OPTIMAL ORDERING POLICY UNDER A MAXIMAL LOAN CONSTRAINT).** *For period  $n \in \mathcal{N}$  with state  $(x_n, y_n)$  at the beginning of the period and a loan maximum  $L_n$ , there exist positive constants  $\alpha_n^L = \alpha_n^L(\xi_n)$  and  $\beta_n^L = \beta_n^L(\xi_n)$  satisfying  $\alpha_n^L \leq \beta_n^L$ , which characterize the optimal order quantity as follows:*

$$q^*(x_n, y_n) = \begin{cases} (\beta_n^L - x_n)^+, & x_n + y_n \geq \beta_n^L; \\ (y_n)^+, & \alpha_n^L \leq x_n + y_n < \beta_n^L; \\ \alpha_n^L - x_n, & \alpha_n^L - L'_n \leq x_n + y_n < \alpha_n^L; \\ L'_n, & x_n + y_n < \alpha_n^L - L'_n. \end{cases}$$

Note that for period  $n$ , the inequality  $\alpha_n^L < L'_n$  is admissible. In this case, the loan maximum is not binding, since the optimal order quantity satisfies  $q_n^* \leq \alpha_n^L < L'_n$ .

## 7.3. System with Backordering

Recall that our basic model assumes the lost-sale rule for unmet demand. This subsection considers the case in which backorders are allowed and shows that an ordering policy of a similar  $(\alpha_n, \beta_n)$  structure is optimal.

Suppose that any unmet demand is backordered subject to penalty cost  $b_n$  per unit of unmet demand (Katehakis et al. 2015, Shi et al. 2013, 2014). Let  $\tilde{V}_n(x, y)$  denote the optimal expected profit function of period  $n$ , given initial state  $(x, y)$ . Accordingly, we have the following dynamic programming formulation:

$$\tilde{V}_n(x_n, y_n) = \sup_{z_n \geq x_n} \mathbb{E} \left[ \tilde{V}_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n \right], \quad (35)$$

$$n = 1, 2, \dots, N - 1,$$

where the expectation is taken with respect to  $D_n$ , and the vector  $(x_{n+1}, y_{n+1})$  is given by

$$x_{n+1} = z_n - D_n; \quad (36)$$

$$y_{n+1} = [\tilde{R}_n(D_n, z_n) + \tilde{K}_n(z_n, \xi_n)]/c_{n+1}, \quad (37)$$

where

$$\tilde{R}_n(D_n, z_n) = (p_n + b_n) \cdot z_n - (p_n + h_n + b_n)[z_n - D_n]^+ - b_n \mathbb{E}[D_n]; \quad (38)$$

$$\tilde{K}_n(z_n, \xi_n) = c_n \cdot (\xi_n - z_n) [(1 + i_n) \mathbf{1}_{\{z_n \leq \xi_n\}} + (1 + \ell_n) \mathbf{1}_{\{z_n > \xi_n\}}]. \quad (39)$$

For the terminal period  $N$ ,  $\tilde{V}_N(x_N, y_N) = \sup_{q_N \geq 0} \mathbb{E} [\tilde{R}_N(D_N, z_N) + \tilde{K}_N(z_N, \xi_N)]$ , where  $h_N = -s$ .

**THEOREM 8.** *Under the backorder rule, for any period  $n \in \mathcal{N}$  with initial state  $(x_n, y_n)$ , there exist positive*



constants  $\alpha_n = \alpha_n(\xi_n)$  and  $\beta_n = \beta_n(\xi_n)$  satisfying  $\alpha_n \leq \beta_n$ , which characterize the optimal order quantity as follows:

$$\tilde{q}^*(x_n, y_n) = \begin{cases} (\beta_n - x_n)^+, & x_n + y_n \geq \beta_n; \\ (y_n)^+, & \alpha_n \leq x_n + y_n < \beta_n; \\ \alpha_n - x_n, & x_n + y_n < \alpha_n. \end{cases} \quad (40)$$

Further,  $\alpha_n$  is uniquely determined by

$$\begin{aligned} & \mathbb{E} \left[ \left( \frac{\partial \tilde{V}_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n) \frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}} \right) \mathbf{1}_{\{x_n > D_n\}} \right] \\ &= [c'_n(1 + \ell_n) - p'_n] \mathbb{E} \left[ \frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}} \right], \end{aligned} \quad (41)$$

and  $\beta_n$  is uniquely determined by

$$\begin{aligned} & \mathbb{E} \left[ \left( \frac{\partial \tilde{V}_{n+1}}{\partial x_{n+1}} - (p'_n + h'_n) \frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}} \right) \mathbf{1}_{\{\beta_n > D_n\}} \right] \\ &= [c'_n(1 + i_n) - p'_n] \mathbb{E} \left[ \frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}} \right], \end{aligned} \quad (42)$$

where the expectations are taken with respect to  $D_n$ , conditioned on the initial state,  $(x_n, y_n)$ .

## 8. Numerical Studies

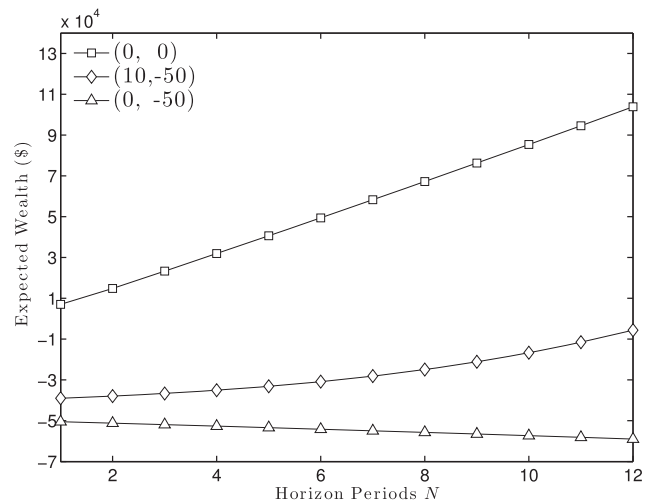
In this section, we present numerical studies designed to glean further insights into the interplay between cash flow and inventory flow. To this end, we fix the following parameters across periods: selling price  $p = 2000$ ; ordering cost  $c = 1000$ ; holding cost  $h = 500$ ; deposit interest rate  $r = 1\%$ ; loan interest rate  $\ell = 15\%$ ; and salvage value  $s = 600$ . In the sequel, subsection 8.1 considers the impacts of time horizon and initial states on the optimal solution; subsection 8.2 studies the corresponding impact of various demand distributions, and subsection 8.3 investigates the performance of the lower and upper bounds of the value function. An extensive test-bed study is provided in Appendix S2.

### 8.1. Impact of Time Horizon

To examine the impact of the time horizon,  $N$ , on the value function, we assume that demand is uniformly distributed as  $D \sim \mathcal{U}[0, 20]$ . To emphasize the dependence of  $V_1(x, y)$  on  $N$  we will write  $V_{1:N}(x, y)$  and set  $N$  varying from 1 to 12. Figure 2 depicts the resultant optimal functions  $V_{1:N}(x, y)$  for the three selected initial states  $(x, y) = (0, 0)$ ,  $(10, -50)$  and  $(0, -50)$ .

Observe that the total net worth  $\xi = x + y$  exhibits a significant impact on the value function. In particular,  $V_{1:N}(x, y)$  is increasing in  $N$  for sufficiently large net worth  $\xi$ , for example, for  $(x, y) = (0, 0)$  and even for

Figure 2 Function  $V_{1:N}(x, y)$  with Selected Initial States  $(x, y)$



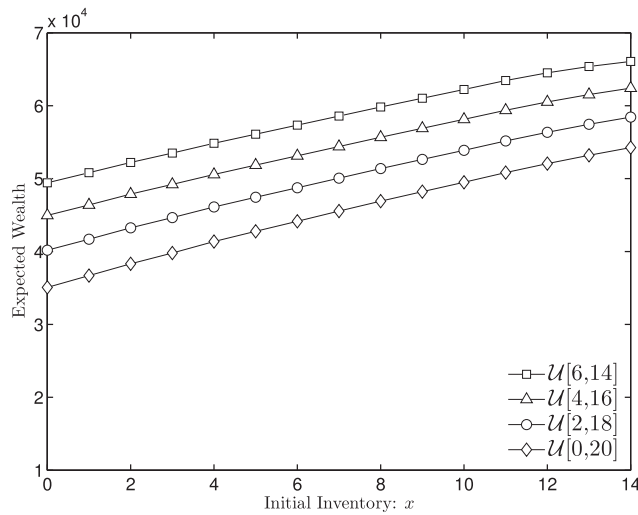
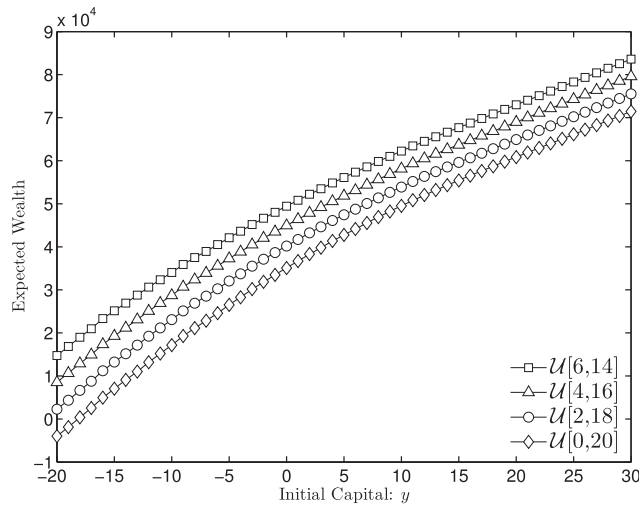
$(x, y) = (10, -50)$ ; however, it is decreasing in  $N$  for sufficiently small  $\xi$ , for example,  $(x, y) = (0, -50)$ . The former can be explained as follows: for moderately small values of  $\xi$ , the operational decisions play a more important role than the financial ones, yielding significantly larger magnitudes of  $V_{1:N}(x, y)$  over the time horizon  $N$ . Conversely, for sufficiently small  $\xi$ , the optimal value is primarily determined by financial decisions as opposed to operational ones, for example, there is a large cost stemming from loans when  $y$  is sufficiently negative, and accordingly,  $V_{1:N}(x, y)$  is decreasing over time. The dominant role of operational decisions is visually demonstrated by the slopes of  $V_{1:N}(x, y)$  in  $N$  as depicted in Figure 2.

### 8.2. Impact of Demand Distributions

In this subsection we study the impact of demand distributions on the value functions. We further provide some observations of the performances of myopic policies (I) and (II), in terms of uniform and ZIP distributed demands with respect to different values of the coefficient of variation (abbreviated as *c.v.*) of demand.

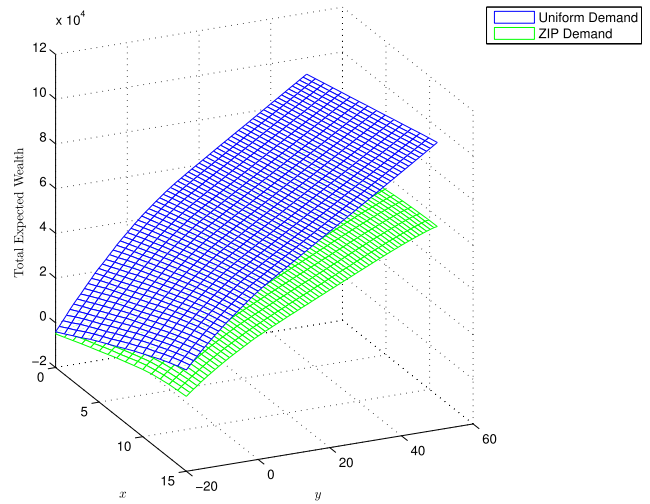
In the computational examples below, we fix the time horizon at  $N = 6$ . We first consider several uniform demand distributions with the same mean but different variances. Figures 3 and 4 depict  $V_1(x, 0)$  and  $V_1(0, y)$ , respectively, for  $D \sim \mathcal{U}[0, 20]$ ,  $D \sim \mathcal{U}[2, 18]$ ,  $D \sim \mathcal{U}[4, 16]$  and  $D \sim \mathcal{U}[6, 14]$ . Both figures show concavity for each demand type. Given the common mean, it is seen that the demand variance has a significant impact on the value function. For example, there is roughly a 5000 reduction from the case of  $D \sim \mathcal{U}[6, 14]$  to the case of  $D \sim \mathcal{U}[4, 16]$  for any  $x$  or  $y$  in each figure.

We next consider the impact of different types of demand distributions other than the uniform distribution. The Zero-Inflated Poisson (ZIP) distribution is

**Figure 3** Function  $V_1(x, 0)$  for  $N = 6$  with Selected Uniform Demand Distributions

**Figure 4** Function  $V_1(0, y)$  for  $N = 6$  with Selected Uniform Demand Distributions


a useful generalization of the Poisson distribution (cf Lambert 1992). The  $ZIP$  distribution with parameters  $\pi \in [0, 1)$  and  $\lambda \geq 0$  has the following probability mass function:  $\mathbb{P}(X = 0) = \pi + (1 - \pi)e^{-\lambda}$  and  $\mathbb{P}(X = k) = (1 - \pi)e^{-\lambda}\lambda^k/k!$  for  $k = 1, 2, 3, \dots$ . Hence, it has mean  $\mu_X = \lambda(1 - \pi)$  and standard deviation  $\sigma_X = \sqrt{\lambda(1 - \pi)(1 + \lambda\pi)}$ . Therefore, its coefficient of variation (defined as the ratio of the standard deviation to the mean) is  $c.v. = \sqrt{\frac{1 + \lambda\pi}{\lambda(1 - \pi)}}$ .

Figure 5 compares the optimal  $V_1(x, y)$  with  $D \sim ZIP(0.18, 10)$  and  $D \sim U[0, 20]$ , both having the same  $c.v. = 0.58$ . Here, it is seen that the uniform demand yields higher  $V_1(x, y)$  than the  $ZIP$  demand for any  $x$  and  $y$ . This can be explained by the mean of the uniform demand ( $\mu = 10$ ) being larger than that of

**Figure 5**  $V_1(x, y)$  for  $N = 6$ : Uniform vs.  $ZIP$  Demand Distributions


the  $ZIP$  demand ( $\mu = 8.2$ ), even though they have the same  $c.v.$

Table 1 displays the optimal objective function values and the performances of myopic policies (I) and (II), in terms of uniform and  $ZIP$  demands with respect to different  $c.v.$  values. In the table, for  $x_1 = y_1 = 0$ ,  $V_1 := V_1(0, 0)$ ,  $V_1^{(\mathcal{X})}$  is the expected net worth under myopic policy ( $\mathcal{X}$ ),  $\mathcal{X} \in \{I, II\}$ , and the performance gap percentage  $\Delta^{(\mathcal{X})}\% := (V_1 - V_1^{(\mathcal{X})})/V_1 \times 100\%$ . Observe that, as  $c.v.$  decreases, the value of  $V_1(0, 0)$  increases significantly for each type of demands. Observe further that under the uniform demand, myopic policy (II) with  $\Delta^{(II)}\% < 0.2\%$  significantly outperforms myopic policy (I) with  $\Delta^{(I)}\% > 3.5\%$ ; however, the converse is observed under the  $ZIP$  demand. For the same  $c.v.$  value, the  $ZIP$  demand has lower  $V_1$  than the Uniform demand; this can be explained by the smaller mean of the  $ZIP$  demand, and the significantly higher probability of zero demand, for example, when  $c.v. = 0.46$ ,  $\mathbb{P}(D = 0) = 9\%$  for the  $ZIP$  demand, but  $\mathbb{P}(D = 0) = 0\%$  for the Uniform demand. Consequently, other characteristics of the demand distribution, such as the shape of its pdf function (in addition to the second order characteristic of  $D$  such as its  $c.v.$ ), will influence relative myopic policy performance. Finally, note that for the last row in Table 1,  $ZIP(0, 10)$  is the standard Poisson distribution with mean  $\lambda = 10$ —the same mean as  $D \sim U(6, 14)$ . However,  $ZIP(0, 10)$  has a smaller  $V_1$ , because it has a larger standard deviation, and hence a higher  $c.v.$  value.

### 8.3. Upper and Lower Bounds of the Value Function

As discussed in section 6, each of the two myopic policies provides a lower bound for the value function, whence we can take the larger of the two as the

**Table 1** Optimal vs. Myopic Policies: Uniform and ZIP Demand with Various *c.v.* ( $x_1 = y_1 = 0$  and  $N = 6$ )

$D \sim \mathcal{U}[a, b]$							$D \sim \text{ZIP}(\pi, \lambda)$						
<i>c.v.</i>	$[a, b]$	Optimal $V_1$	Myopic (I)		Myopic (II)		<i>c.v.</i>	$(\pi, \lambda)$	Optimal $V_1$	Myopic (I)		Myopic (II)	
			$V_1^{(I)}$	$\Delta^{(I)}\%$	$V_1^{(II)}$	$\Delta^{(II)}\%$				$V_1^{(I)}$	$\Delta^{(I)}\%$	$V_1^{(II)}$	$\Delta^{(II)}\%$
0.58	[0, 20]	35,074	30,271	13.69%	35,016	0.16%	0.58	(0.18, 10)	23,130	22,800	1.43%	21,757	5.94%
0.46	[2, 18]	40,174	36,784	8.44%	40,158	0.04%	0.45	(0.09, 10)	26,920	26,910	0.04%	26,142	2.89%
0.35	[4, 16]	44,950	42,329	5.83%	44,886	0.14%	0.35	(0.02, 10)	29,612	29,484	0.43%	29,115	1.68%
0.23	[6, 14]	49,428	47,677	3.54%	49,385	0.09%	0.31	(0, 10)	30,355	30,288	0.22%	29,911	1.46%

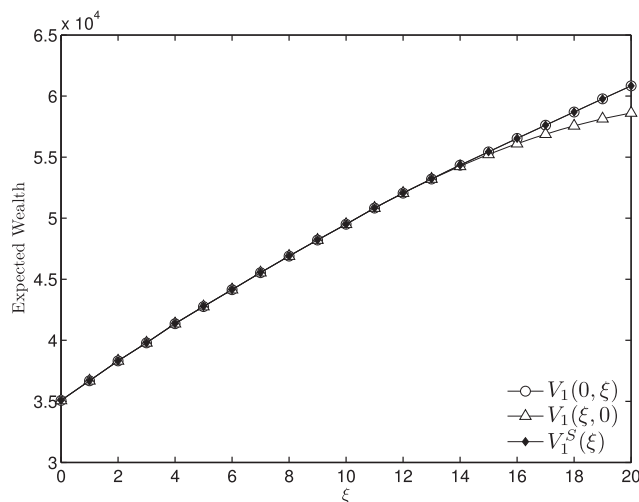
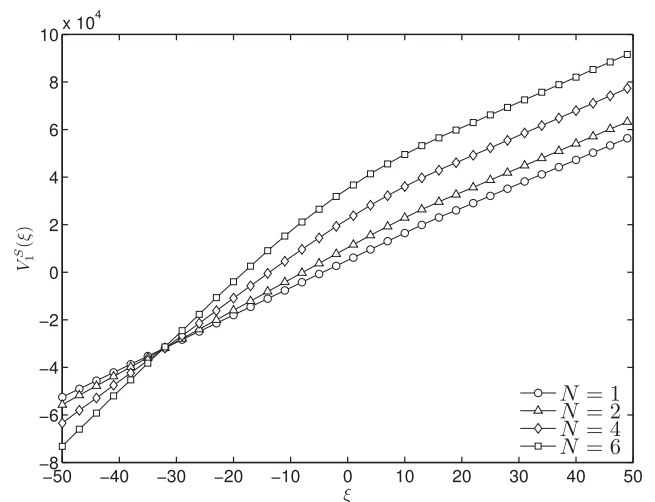
better lower bound. Furthermore,  $V_1^S$  provides an upper bound. In what follows, we numerically examine their approximating performance using the same parameters as in the previous numerical studies.

Figure 6 compares  $V_1^S(\xi)$  with  $V_1(\xi, 0)$  and  $V_1(0, \xi)$  for  $N = 6$ . Observe that for any  $\xi$ , it is shown that  $V_1^S(\xi) \geq V_1(0, \xi) \geq V_1(\xi, 0)$ . Observe further that the difference between  $V_1^S(\xi)$  and  $V_1(0, \xi)$  is almost imperceptible, which implies that the upper bound is fairly tight for this state of high cash level  $y$  (or low product level  $x$ ). Moreover, the bound gap between  $V_1^S(\xi)$  and  $V_1(\xi, 0)$  gets larger as  $\xi$  increases.

Figure 7 depicts the effect of  $N$  on  $V_{1:N}^S(\xi)$ , for  $N = 1, 2, 4, 6$ . Observe that for any  $N$ ,  $V_{1:N}^S(\xi)$  is increasing and jointly concave in  $\xi$ . Observe further that in the top-right of Figure 7,  $V_{1:N}^S(\xi)$  is increasing in  $N$  for large values of  $\xi$ , for example,  $\xi > 10$ ; however, in the far bottom-left of the figure,  $V_{1:N}^S(\xi)$  is decreasing in  $N$  for sufficiently small  $\xi$ , for example,  $\xi < -30$ . Finally, we conclude that operational decisions play a more important role than financial ones for the middle range of  $\xi$ , for example,  $\xi \in (-30, 10)$ , since the slope at each  $N$  in the interval is significantly larger than elsewhere, and gets larger as  $N$  increases.

Table 2 displays the *optimal value function* (Opt.) and its lower and upper bounds for  $D \sim \mathcal{U}[0, 20]$  and

$D \sim \mathcal{U}[6, 14]$ ,  $N = 6, 12$  and selected product level states  $x = 0, 7, 14$ . In this case, the lower bound is obtained from myopic policy (II), since it performs better than myopic policy (I). Observe that each bound provides a fairly tight approximation of the optimal value with  $\Delta^{(X)}\%$  less than 1%. It is important to point out that compared with the myopic policies, the sell-back policy provides a better approximation for the value function when the initial net worth is not too high, for example, for  $x = 0$  and 7. However, the opposite holds when the net worth gets larger, for example,  $x = 14$ . Observe further that as the *c.v.* of the demand distribution decreases from  $\mathcal{U}[0, 20]$  to  $\mathcal{U}[6, 14]$ , the optimal value function and its lower and upper bounds increase, due to the reduced volatility of the demand. Moreover, the lower bound gap  $\Delta^{(II)}$  gets smaller, whereas this change is not evident for its upper bound gap  $\Delta^S$ . Note that, for each  $N$  and each demand distribution, the bound gap  $\Delta^S$  gets larger as  $x$  increases, whereas this behavior is not evident for  $\Delta^{(II)}$ . Finally, as the time horizon  $N$  gets longer, the optimal value function and its lower and upper bounds get larger, and the bound gap  $\Delta^X$  gets larger as well for both bounds,  $X \in \{II, S\}$ . However, such behavior of the corresponding gap percentage  $\Delta^{(X)}\%$  is not evident.

**Figure 6**  $V_1^S(\xi)$  vs.  $V_1(0, y)$  and  $V_1(x, 0)$  for  $N = 6$  with  $D \sim \mathcal{U}[0, 20]$ **Figure 7** Function  $V_1^S(\xi)$  for  $N = 1, 2, 4, 6$  with  $D \sim \mathcal{U}[0, 20]$ 

**Table 2** Lower and Upper Bounds for the Value Function  $V_1(x, 0)$

$N$	$x$	$D \sim \mathcal{U}[0, 20]$							$D \sim \mathcal{U}[6, 14]$						
		Opt. $V_1$	Lower bound			Upper bound			Opt. $V_1$	Lower bound			Upper bound		
			$V_1^{(I)}$	$\Delta^{(I)}$	$\Delta^{(I)}\%$	$V_1^S$	$\Delta^S$	$\Delta^S\%$		$V_1^{(I)}$	$\Delta^{(I)}$	$\Delta^{(I)}\%$	$V_1^S$	$\Delta^S$	$\Delta^S\%$
6	0	35,074	35,016	58	0.16%	35,080	−6	−0.02%	49,428	49,386	43	0.09%	49,428	0	0.00%
	7	45,542	45,435	107	0.24%	45,550	−9	−0.02%	58,575	58,536	39	0.07%	58,575	0	0.00%
	14	54,248	54,200	48	0.09%	54,355	−107	−0.20%	66,076	66,036	40	0.06%	66,634	−558	−0.84%
12	0	75,888	75,693	194	0.26%	75,923	−36	−0.05%	103,872	103,760	112	0.11%	103,872	0	0.00%
	7	87,273	87,057	216	0.25%	87,290	−18	−0.02%	113,564	113,464	100	0.09%	113,564	0	0.00%
	14	96,528	96,410	118	0.12%	96,660	−132	−0.14%	121,521	121,419	102	0.08%	122,114	−592	−0.49%

**Table 3** Impact of Loan Maximum  $L'$

$x$	$L' = 5$					$L' = 10$					$L' = \infty$				
	$V$	$V^{(I)}$	$\Delta^{(I)}\%$	$V^{(II)}$	$\Delta^{(II)}\%$	$V$	$V^{(I)}$	$\Delta^{(I)}\%$	$V^{(II)}$	$\Delta^{(II)}\%$	$V$	$V^{(I)}$	$\Delta^{(I)}\%$	$V^{(II)}$	$\Delta^{(II)}\%$
0	17,026	7606	55.33%	9322	45.25%	17,154	10,761	37.27%	16,745	2.38%	17,163	10,784	37.17%	16,972	1.11%
2	21,008	13,472	35.87%	16,285	22.49%	21,106	14,497	31.31%	20,809	1.41%	21,113	14,511	31.27%	20,854	1.23%
4	24,511	18,738	23.55%	22,260	9.18%	24,586	19,255	21.68%	24,488	0.40%	24,591	19,261	21.67%	24,506	0.35%
6	28,206	22,063	21.78%	27,447	2.69%	28,257	22,645	19.86%	28,142	0.40%	28,260	22,649	19.86%	28,152	0.38%
8	31,841	27,481	13.69%	31,422	1.32%	31,873	27,595	13.42%	31,785	0.27%	31,875	27,597	13.42%	31,793	0.26%
10	35,052	31,644	9.72%	34,820	0.66%	35,073	31,702	9.61%	35,012	0.17%	35,074	31,704	9.61%	35,015	0.17%
12	38,156	34,619	9.27%	38,012	0.38%	38,170	34,668	9.17%	38,121	0.13%	38,171	34,670	9.17%	38,124	0.12%
14	40,620	37,155	8.53%	40,495	0.31%	40,634	37,196	8.46%	40,577	0.14%	40,634	37,198	8.46%	40,579	0.14%
16	42,607	39,160	8.09%	42,499	0.25%	42,621	39,201	8.02%	42,570	0.12%	42,622	39,203	8.02%	42,572	0.12%
18	44,155	40,750	7.71%	44,060	0.22%	44,170	40,791	7.65%	44,121	0.11%	44,171	40,794	7.65%	44,123	0.11%
20	45,214	41,911	7.30%	45,130	0.18%	45,230	41,953	7.25%	45,187	0.10%	45,232	41,955	7.24%	45,189	0.09%

#### 8.4. Impact of Maximal Loan Constraint

In this subsection, we numerically gauge the impact of the loan maximum  $L'$  which limits the firm's borrowing to purchases of at most  $L'$  product units. In this study, we assume  $D \sim \mathcal{U}[0, 20]$  and  $N = 6$ . Table 3 displays the optimal expected net worth and the performances of both myopic policies for  $y = -10$  and varying  $x$ , with  $L' = 5, 10, \infty$ .

We make the following observations. First, as  $L'$  increases, both myopic policies perform better. The impact of  $L'$  on the performance of each policy becomes significant when the total net worth  $\xi$  is relatively small and  $L'$  is low. For example, when  $x = 0$  and  $L' = 5$ ,  $\Delta^{(I)}\% = 55\%$  and  $\Delta^{(II)}\% = 45\%$ , both of which are much higher than in the other cases. Furthermore, myopic policy (II) always outperforms myopic policy (I). This can be explained by the uniform distribution of the demand (cf. the discussion in subsection 8.2). Finally, both myopic policies perform better as  $x$  increases (i.e., as the net worth  $\xi$  increases).

## 9. Concluding Remarks

In this paper, we studied the optimal operational-financial policy, for a single-item inventory system that admits both interest-bearing loans and interest-earning deposits, with the goal of maximizing the net

worth of the firm at the end of a prescribed time horizon. We showed that the optimal ordering policy, called  $(\alpha_n, \beta_n)$  policy, for each period is characterized by two threshold parameters. We presented two myopic policies which provide lower and upper bounds, respectively, for the threshold parameters. The two myopic policies provide lower bounds of the value function. We further introduced an upper bound for the value function via the assumption that the firm can sell back unneeded inventory to the supplier.

The results of this study suggest several directions for further research. These include (i) addition of a fixed ordering cost and/or a fixed financial transaction cost; (ii) addition of a fixed system operating cost proportional to the elapsed time; and (iii) non-risk neutral decision making. Finally, it would be of interest to analyze the risk-induced performance of the system, for example, credit rollover risk and bankruptcy probabilities (cf. Babich et al. 2012, Gong et al. 2014, Li et al. 2013, Shi 2016).

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## References

- Babich, V., G. Aydin, P. Y. Brunet, J. Keppo, R. Saigal. 2012. Risk, financing and the optimal number of suppliers. H. Gurnani, A. Mehrotra, S. Ray eds. *Supply Chain Disruptions: Theory and Practice of Managing Risk*, Springer, London Dordrecht Heidelberg, New York, pp. 195–240.
- Bartle, R. G. 2014. *The Elements of Integration and Lebesgue Measure*. John Wiley & Sons, New York.
- Bensoussan, A., M. Crouhy, J. M. Proth. 1983. *Mathematical Theory of Production Planning*. North-Holland, Amsterdam.
- Birge, J. R. 2014. Om forum—operations and finance interactions. *Manuf. Serv. Oper. Manage. Serv. Oper. Manag.* **17**(1): 4–15.
- Birge, J. R., X. Xu. 2011. Firm profitability, inventory volatility, and capital structure. *Inventory Volatility, and Capital Structure* (August 23, 2011).
- Buzacott, J. A., R. Q. Zhang. 2004. Inventory management with asset-based financing. *Management Sci.* **50**(9): 1274–1292.
- Cai, G. G., X. Chen, Z. Xiao. 2014. The roles of bank and trade credits: Theoretical analysis and empirical evidence. *Prod. Oper. Manag.* **23**(4): 583–598.
- Chao, X., J. Chen, S. Wang. 2008. Dynamic inventory management with cash ow constraints. *Nav. Res. Log.* **55**(8): 758–768.
- Chen, F., Y. S. Zheng. 1994. Lower bounds for multi-echelon stochastic inventory systems. *Management Sci.* **40**(11): 1426–1443.
- Dada, M., Q. Hu. 2008. Financing newsvendor inventory. *Oper. Res. Lett.* **36**(5): 569–573.
- Daniel, J. S. R., C. Rajendran. 2006. Heuristic approaches to determine base-stock levels in a serial supply chain with a single objective and with multiple objectives. *Eur. J. Oper. Res.* **175**(1): 566–592.
- Finance.Yahoo.com. 2015. Wells fargo lends \$22.6 billion to small businesses since launching five year lending goal. May 14.
- Fu, Q., S. X. Zhou, X. Chao, C. Y. Lee. 2012. Combined pricing and portfolio option procurement. *Prod. Oper. Manag.* **21**(2): 361–377.
- Gong, X., X. Chao, D. Simchi-Levi. 2014. Dynamic inventory control with limited capital and short-term financing. *Nav. Res. Log.* **61**(3): 184–201.
- Gupta, D., L. Wang. 2009. A stochastic inventory model with trade credit. *Manuf. Serv. Oper. Manag.* **11**(1): 4–18.
- Hu, Q., M. J. Sobel. 2005. Capital structure and inventory management. *Unpublished manuscript*. Case Western Reserve University.
- Hull, J. C. 2002. *Options, Futures, and Other Derivative Securities*, 5th edn. Prentice-Hall, Boston.
- Katehakis, M. N., B. Melamed, J. Shi. 2015. Optimal time-average cost for inventory systems with compound poisson demands and lost-sales. *Ann. Oper. Res.* doi: 10.1007/s10479-015-1998-y.
- Kouvelis, P., W. Zhao. 2011a. The newsvendor problem and price-only contract when bankruptcy costs exist. *Prod. Oper. Manag.* **20**(6): 921–936.
- Kouvelis, P., W. Zhao. 2011b. Financing the newsvendor: Supplier vs. bank, and the structure of optimal trade credit contracts. *Oper. Res.* **60**(3): 566–580.
- Lambert, D. 1992. Zero-in ated poisson regression with an application to defects in manufacturing. *Technometrics* **34**: 1–14.
- Lee, H. L., C. Billington. 1993. Material management in decentralized supply chains. *Oper. Res.* **41**(5): 835–847.
- Lee, C. H., B. D. Rhee. 2010. Coordination contracts in the presence of positive inventory financing costs. *Int. J. Prod. Econ.* **124**: 331–339.
- Lee, C. H., B. D. Rhee. 2011. Trade credit for supply chain coordination. *Eur. J. Oper. Res.* **214**(1): 136–146.
- Li, L., M. Shubik, M. J. Sobel. 2013. Control of dividends, capital subscriptions, and physical inventories. *Management Sci.* **59**(5): 1107–1124.
- Luo, W., K. Shang. 2015. Joint inventory and cash management for multi-divisional supply chains. *Oper. Res.* **63**(5): 1098–1116.
- MaRS. 2014. Manage your cash ow: Operations, investing and financing. Available at [www.marsdd.com/mars-library/manage-cash-flow-operations-investing-financing/](http://www.marsdd.com/mars-library/manage-cash-flow-operations-investing-financing/) (accessed date April 10, 2015).
- Minner, S. 1997. Dynamic programming algorithms for multi-stage safety stock optimization. *Oper. Res. Spektrum* **19**(4): 261–271.
- Modigliani, F., M. H. Miller. 1958. The cost of capital, corporation finance, and the theory of investment. *Am. Econ. Rev.* **48**: 261–297.
- Press, W. H. 2007. *Numerical Recipes with Source Code CD-ROM 3rd Edition: The Art of Scientific Computing*. Cambridge University Press, New York.
- Raghavan, N. R. S., V. K. Mishra. 2011. Short-term financing in a cash-constrained supply chain. *Int. J. Prod. Econ.* **134**: 407–412.
- Reuters. 2015. Wells fargo's wealth profit driven by loans, managed accounts. Available at [www.reuters.com](http://www.reuters.com) (accessed date April 14, 2016).
- Ross, S. M. 1996. *Stochastic Processes*, 2nd edn. John Wiley & Sons, New York.
- Secomandi, N. 2010. Optimal commodity trading with a capacitated storage asset. *Management Sci.* **56**(3): 449–467.
- Shang, K., J. S. Song. 2003. Analysis of serial supply chains with a service constraint. The Fuqua School of Business, Duke University.
- Shi, J. 2016. Stock-out risk of continuous production/inventory systems with compound poisson demands. Working paper, New Jersey Institute of Technology.
- Shi, J., M. N. Katehakis, B. Melamed. 2013. Martingale methods for pricing inventory penalties under continuous replenishment and compound renewal demands. *Ann. Oper. Res.* **208**(1): 593–612.
- Shi, J., M. N. Katehakis, B. Melamed, Y. Xia. 2014. Production-inventory systems with lost sales and compound poisson demands. *Operations Research* **62**(5): 1048–1063.
- Singhal, V. R. 1988. Inventories, risk, and the value of the firm. *J. Manuf. Oper. Manag.* **1**(1): 4–43.
- WellsFargo.com. 2015. Wells fargo/gallup survey: Small business optimism dips in second quarter. Available at [www.wells-fargo.com](http://www.wells-fargo.com) (accessed date May 11, 2016).
- Widder, D. V. 1989. *Advanced Calculus*. Courier Corporation, New York.
- Xu, X., J. R. Birge. 2004. Joint production and financing decisions: Modeling and analysis. Working paper, Available at SSRN 652562 (accessed date May 20, 2016).
- Xu, X., J. R. Birge. 2008. Operational decisions, capital structure, and managerial compensation: A news vendor perspective. *Eng. Econ.* **53**(3): 173–196.
- Yang, S., J. R. Birge. 2013. How inventory is (should be) financed: trade credit in supply chains with demand uncertainty and costs of financial distress. Working paper, Available at SSRN 1734682 (accessed date May 20, 2016).

- Yasin, A., V. Gaur. 2012. Operational investment and capital structure under asset based lending. Johnson School Research Paper Series.
- Zipkin, P. H. 2000. *Foundations of Inventory Management*. McGraw-Hill, Boston, MA.
- Zwilling, M. 2013. 10 more creative ways to finance your startup. Available at [www.forbes.com/sites/martinzwilling/2013/03/06/10-more-creative-ways-to-finance-your-startup](http://www.forbes.com/sites/martinzwilling/2013/03/06/10-more-creative-ways-to-finance-your-startup) (accessed date March 6, 2016).

## Supporting Information

Additional Supporting Information may be found in the online version of this article:

**Appendix S1:** Proofs.

**Appendix S2:** Test-Bed Numerical Study.