Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 0000-0000 | EISSN 0000-0000 | 00 | 0000 | 0001

INFORMS
DOI 10.1287/xxxx.0000.0000

Cash-Flow Based Dynamic Inventory Management

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Small-to-medium size enterprises (SMEs), including many startup firms, need to manage interrelated flows of cash and inventories of goods. In this paper, we model a firm that can finance its inventory (ordered or manufactured) with loans in order to meet random demand which in general may not be time stationary. The firm earns interest on its cash on hand and pays interest on its debt. The objective is to maximize the expected value of the firm's capital at the end of a finite planning horizon. The firm's state at the beginning of each period is characterized by the inventory level and the capital level measured in units of the product, whose sum represents the "net worth" of the firm. Our study shows that the optimal ordering policy is characterized by a pair of threshold parameters as follows. i) If the net worth is less than the lower threshold, then the firm employs a base stock order up to the lower threshold. ii) If the net worth is between the two thresholds, then the firm orders exactly as many units as it can afford, without borrowing. iii) If the net worth is above the upper threshold, then the firm employs a base stock order up to the upper threshold. Further, upper and lower bounds for the threshold values are developed using two simple-to-compute myopic ordering policies which yield lower bounds for the value function. We also derive an upper bound for the value function by considering a sell-back policy. Subsequently, it is shown that policies of similar structure are optimal when the loan and deposit interest rates are piecewise linear functions, when there is a maximal loan limit and when unsatisfied demand is backordered. Finally, further managerial insights are provided with extensive numerical studies.

Key words: Inventory-finance decision, threshold values, myopic policy, sell-back policy.
History: Submitted on May 5th, 2013, two round revisions with authors, accepted by Dr. John R. Birge on March 24, 2016.

1. Introduction

In the current competitive environment, small-to-medium size enterprises (SMEs), including many startup firms, must make joint decisions concerning interrelated flows of cash and products; cf. MaRS (10 Apr. 2014). Cash is the lifeblood of any business and is often in short supply. Cash flows stem from operations, financing and investing activities. To maximize profits, firms need to manage operations, financing and investing activities, efficiently and comprehensively. For example, most startup firms need access to funds, typically for ordering or manufacturing products; cf. Zwilling (6 Mar. 2013). As another example, some small businesses are seasonal in nature, particularly retail businesses. If a firm makes most of its sales during the holiday season, it may need a bank loan prior to the holiday season to finance a large amount of inventory to gear up for that season. Bank loans to purchase inventory are generally short-term in nature and firms usually pay them off after the season is over with proceeds from their seasonal sales.

To facilitate SME borrowing, the *Small Business Administration* (SBA) has established a loan program with a successful record. For example, *Wells Fargo Capital Finance* is the nation's leading provider of loans (by dollar volume) guaranteed by the SBA, providing loans to companies across a broad range of industries throughout the United States. Building on a strong track record, Wells Fargo works with numerous companies in diverse industries, including apparel, electronics, furniture, housewares, sporting goods, toys and games, food products, hardware, and industrial goods. Accordingly, small business loans at Wells Fargo rose 18% in 2014; cf. Reuters (14 April 2015), WellsFargo.com (11 May 2015) and Finance.Yahoo.com (14 May 2015).

In this paper we model and analyze the optimal financial and operational policy of an SME firm whose inventory is subject to lost sales, zero replenishment lead times and periodic review over a finite planning horizon. The firm's state (i.e., *inventory-cash profile*) consists of inventory level and cash level, where a positive cash level represents cash on hand while a negative cash level represents a loan position. In each period, cash flows are managed from the perspectives of operations, financing and investing, as follows. (i) From the operations perspective, the firm stocks up and sells inventory; (ii) From the financing perspective, the firm can use its cash on hand or an external short-term loan (if needed) to procure products for inventory; (iii) From the investing perspective, any cash on hand is deposited in a bank account to earn interest at a given rate, while debt incurs interest at a higher given rate. The objective of the firm is to dynamically optimize the order quantities in each period, given the current state of cash and inventory, via joint operational/financial decisions, so as to maximize the expected value of the firm's capital (i.e., total wealth level) at the end of a finite time horizon. To this end, we develop and study a discrete-time model for a single-product inventory system over multiple periods.

The inventory flow is described as follows. At the beginning of each period, the firm decides on an order quantity, and the corresponding replenishment order materializes with zero lead time. During the remainder of the period, no inventory transactions (demand fulfillment or replenishment) take place. Rather, all such transactions are settled at the end of the period. Incoming demand is aggregated over that period, and the total period's demand draws down on-hand inventory. However, if the demand exceeds the on-hand inventory, then all excess demand is lost (the backordering setting is studied in §7.3). All the leftover inventory (if any) is carried forward to the next period subject to a holding cost, and at the last period, the remaining inventory (if any) is disposed of either at a salvage value or at a disposal cost.

Cash flows take place as follows. All transactions taking place in a period are settled at its end. More precisely, the firm updates its cash position with the previous period's revenue from sales and interest earned on a deposit, or paid on outstanding debt (if any). The firm then decides the order quantity for the next period and pays for replenishment as follows: first, with cash on hand, and if insufficient, with a withdrawal from cash on deposit (there is no withdrawal penalty), and if still insufficient, by a loan. If not all cash on hand is used for replenishment, then any unused cash is deposited in a bank where it earns interest. At the end of each period the resulting cash on hand or debt are carried forward to the next period. Thus, the firm's state in each period $n = 1, \ldots, N$, is a two dimensional vector (x_n, y_n), where x_n represents the value of inventory and y_n the cash level (cash in deposit if positive, debit if negative) measured in units of the product, available at the start of the period; together they constitute a "net worth" of the firm, $\xi_n = x_n + y_n$.

The main contributions of the paper are as follows. First, it establishes the structure of the optimal ordering policy in terms of the net worth ξ_n of the firm at the beginning of each period. It is shown that the optimal policy is characterized by a sequence of two threshold critical values, α_n and β_n, where $\alpha_n < \beta_n$; cf. Theorem 1 for the single period problem and Theorem 3 for the multi-period problem, both in the sequel. This optimal policy has the following structure: i) If $\xi_n \leq \alpha_n$, then the firm employs a base stock ordering up to α_n; ii) If $\alpha_n < \xi_n < \beta_n$, then the firm orders exactly as many units as it can afford, without borrowing; iii) If $\beta_n \leq \xi_n$, then the firm employs a base stock ordering up to β_n. The above cases will be referred to as the *over-utilization*, *full-utilization*, and *under-utilization* of available cash, respectively.

Second, for the single period problem, we show that the (α, β) optimal policy yields a positive expected value even with zero values for both initial inventory and cash. For the multi-period problem, we construct two myopic policies which respectively provide upper and lower bounds for the threshold values. The two myopic policies yield lower bounds for the value function. Since the underlying state space is two-dimensional, it leads to optimization algorithms of high computational complexity, but by considering a sell-back policy, we derive a one-dimensional approximation algorithm that provides upper bounds for the value function. Subsequently, it is shown that policies of similar structure are optimal when the loan and deposit interest rates are piecewise linear functions (cf. Theorem 6), when there is a maximal loan limit (cf. Theorem 7) and when unsatisfied demand is backordered (cf. Theorem 8). Finally, further managerial insights are provided with numerical studies.

The remainder of this paper is organized as follows. Section 2 reviews related literature and Section 3 formulates the model. In Section 4, the single-period model is developed and the optimal (α, β) policy is derived, where the threshold values are functions of the demand distribution and the cost parameters of the problem. Section 5 extends the analysis to the dynamic multi-period problem and derives a two-threshold (α_n, β_n) optimal policy via a dynamic programming analysis. Section 6 introduces two myopic policies that provide upper and lower bounds for each α_n and β_n. The two myopic policies yield two lower bounds for the value function. Further, an upper bound for the value function is also introduced. In Section 7, it is shown that simple modifications of the underlying model allow the extension of the results to the case of piecewise linear interest rate functions, a maximal loan limit and backordering of unsatisfied demand. Numerical studies are presented in Section 8. Finally, Section 9 concludes the paper. All proofs are relegated to the appendix.

2. Literature Review

In a seminal study, Modigliani and Miller (1958) show that in a perfect capital market, a firm's operational and financial decisions can be made separately. Since then, most literature in inventory management extended the classical newsvendor problem in a variety of ways, but assuming that decision makers are not subject to financial constraints. However, in view of the imperfection of real-life capital markets, a growing body of literature has begun to consider operational decision making subject to financial constraints. In those studies, inventories of goods are often treated as special financial instruments [cf. Singhal (1988)]. Accordingly, portfolios of physical products and financial instruments have been studied using finance/investment principles such as *Modern Portfolio Theory* (MPT) and *Capital Asset Pricing Model* (CAPM). For an up-to-date and comprehensive literature review on the interplay between inventories and finance, the reader is referred to Birge (2014) and references therein.

The growing literature on the interface between operations management and financial decisions can be categorized into two major strands: *single-agent* and *game related multiple-agent*. For each strand, attendant models can be further classified as single-period and multi-period models. Our study belongs to the single-agent strand and addresses both single-period and multi-period models.

In the single-agent strand, a firm's operational and financial decisions are typically made simultaneously, but without interaction with other firms. Considering an imperfect market, Xu and

Birge (2004) develop models with simultaneous production and financing decisions in the presence of demand uncertainty, which illustrates how a firm's production decisions are affected by the existence of financial constraints. Recently, Birge and Xu (2011) present an extension of a model in Xu and Birge (2004) by assuming that debt and production scale decisions have fixed costs necessary to maintain operations, variable costs of production, and volatility in future demand forecasts. Building on their previous work of Xu and Birge (2004), Xu and Birge (2008) consider the effect of various operating conditions on capital structure, and exhibit some empirical support for their previously predicted relationship between production margin and market leverage. Cai et al. (2014) investigate the roles of bank and trade credits in a supply chain with a capital-constrained retailer facing demand uncertainty. The studies above focus on single-period problems.

Some single-agent, multi-period problems are addressed in Hu and Sobel (2005) and Li et al. (2013). Here, multi-period models posit different interest rates on cash on hand and outstanding loans. These papers also demonstrate the importance of joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence on bank financing. For example, Hu and Sobel (2005) examine the interdependence of a firm's capital structure and its short-term operating decisions concerning inventories, dividends, and liquidity. To this end, Hu and Sobel (2005) formulate a dynamic model to maximize the expected present value of dividends. Li et al. (2013) study a dynamic model of managerial decisions in a manufacturing firm in which inventory and financial decisions interact and are coordinated in the presence of demand uncertainty, financial constraints, and default risk. It is shown that the coordination between financial and operational decisions can significantly improve the expected present value of dividends. In addition, Gupta and Wang (2009) study a retailer's dynamic inventory problem (both discrete and continuous review) in the presence of random demand while the retailer is financially supported by trade credit from its supplier, contingent on the age of the inventory. It is shown that the optimal policy is a base stock policy where the base stock level is affected by the offered credit terms. Luo and Shang (2015) study a centralized supply chain consisting of two divisions (modeled as echelons), of which the headquarter division manages the financial and operational decisions with a cash pool, and the focus is on analyzing the value of cash pooling.

The work most closely related to this study is Chao et al. (2008) and Gong et al. (2014). The first reference considers a single-agent multi-period problem for a self-financing retailer without external loan availability. It shows that the optimal (cash flow dependent) policy in each period, is uniquely determined by a single critical value. Our study differs from Chao et al. (2008) in several ways. First, from a modeling perspective, Chao et al. (2008) consider a self-financing firm without access to external credit, while our model admits loans that afford the retailer with the flexibility of ordering larger quantities to achieve a higher fill rate. The admissibility of loans intertwines financial decisions (e.g., how much to borrow) with operational decisions (e.g., how much to order) as well as investment decisions (how much to deposit for additional interest revenue). In this case, the complexity of the problem is raised from one dimension (operational decisions alone) to two dimensions (joint operational and financial decisions). Further, Chao et al. (2008) assume an iid demand process and time-stationary costs, while we consider a non-stationary demand process and time varying (loan and deposit) interest rates. Second, in terms of results, Chao et al. (2008) structure the optimal policy as a base stock policy with a single threshold value, while our paper shows that the optimal policy is characterized by two threshold values, which divide the ordering decision space into three intervals to guide financing, operational and investment decisions. Gong et al. (2014) study a similar model aiming to maximize the expected terminal total capital but along lines different from ours. In particular, Gong et al. (2014) consider an increasing concave loan rate while this paper considers a piecewise linear loan rate. Importantly, our study differs in several aspects as follows: (1) Concerning model setting, we consider non-stationary demand processes and time varying (loan and deposit) interest rates as well as an extension to realistic piecewise loan rates. (2) Concerning results, our optimal policy is presented in terms of two thresholds per period (multiple thresholds for the case of piecewise linear loan rates), while Gong et al. (2014) analyze a single-threshold policy. (3) Computationally, we develop simple easy-to-compute policies (based on single period information) which provide lower and upper bounds for each of the two threshold values, and we propose an efficient algorithm for the computation of the optimal policy. Furthermore we provide upper and lower bounds for the value function. (4) Finally we develop managerial insights into the interplay between operational, financial and investment decisions. We note that our setting of the problem is in-between the models studied in Chao et al. (2008) and Gong et al. (2014) but differs in emphasis. In summary, this study contributes to such extant literature by tackling the interplay between operational and financial decisions (i.e., inventory flow and cash flow, respectively) reflected by loans or deposits, and developing bounds for the threshold values and the objective function as well.

A second strand of the literature has investigated multi-agent competition between firms and financial institutions using game theoretic approaches. This literature includes Buzacott and Zhang (2004), Dada and Hu (2008), Yasin and Gaur (2012), Raghavan and Mishra (2011), etc. Most such papers deal with a single-period problem. Buzacott and Zhang (2004) analyzes a Stackelberg game between the bank and the retailer in a newsvendor inventory model. Dada and Hu (2008) assume that loan interest is charged by the bank endogenously and use a game model to capture the interaction between the bank and the inventory controller through which an equilibrium is derived and a non-linear loan schedule is obtained to coordinate the channel. Yasin and Gaur (2012) study the implications of asset based lending to operational investment, probability of bankruptcy, and capital structure for a borrower firm. Raghavan and Mishra (2011) study a short-term financing problem in a cash-constrained supply chain.

As a part of the second literature strand, multi-agent game-theoretic approaches have also been used to model the competition between suppliers and retailers. Such recent studies include Kouvelis and Zhao (2011a) and Kouvelis and Zhao (2011b) among many others. An important portion of this strand addresses the impact of trade credits provided by suppliers to retailers. Lee and Rhee (2010) study the impact of inventory financing costs on supply chain coordination by considering four coordination mechanisms: all-unit quantity discount, buy backs, two-part tariff, and revenue-sharing. It is shown that using trade credits in addition to contracts, a supplier can fully coordinate the supply chain and achieve maximal joint profit. Further, Lee and Rhee (2011) model a firm with a supplier that grants trade credits and markdown allowances. Given the supplier's offer, it determines the order quantity and the financing option for the inventory under either trade credit or direct financing from a financial institution. The impact of trade credits from an operational perspective is also studied in Yang and Birge (2013), which investigates the role that trade credit plays in channel coordination and inventory financing. It is shown that when offering trade credits, the supplier balances its impact on operational profit and costs.

3. Model Formulation

Consider a time horizon $\mathscr{N} := \{1, 2, \ldots, N\}$ where the periods are indexed forward. At the beginning of period $n \in \mathscr{N}$, let the state of the system be represented by a vector (x_n, y_n), where $x_n \geq 0$ denotes the amount of on-hand inventory (number of product units) and y_n denotes the amount of product that can be purchased using all the available cash Y_n, i.e., y_n is the cash position measured in *product units*. Let $q_n \geq 0$ denote the order quantity (in product units) that arrives at in the beginning of period $n \in \mathscr{N}$. The ordering cost per unit is $c_n \geq 0$, so that $Y_n = c_n \cdot y_n$. We assume that any order quantity can be fully satisfied within zero lead time of replenishment. Let $D_n \in \mathbb{R}^+$ denote the random demand during period n, and assume it follows a general distribution. We assume that demands of different periods are independent but could be generally non-stationary across periods. Let $f_n(\cdot)$ and $F_n(\cdot)$ denote the *probability density function* (pdf) and the *cumulative distribution function* (cdf) of D_n, respectively. The selling price per unit is $p_n \geq c_n$.

Throughout all periods $n = 1, \ldots, N-1$, any unsold units are carried over subsequent periods, inventory on hand is subject to a constant holding cost $h_n \geq 0$ per unit over period n, and any unmet demand is lost. The backorder setting of the problem is studied in §7.3. At the end of the horizon, i.e., the end of period N, all leftover inventory (if any) is salvaged (or disposed of) at a constant price (cost) of s per unit. Note that we allow a negative s, in which case s represents a disposal cost per unit, e.g., the unit cost of vehicle tire disposal. For notational convenience, we denote in the sequel $h_N = -s$, so that the salvage value (or disposal cost) s is treated uniformly as a type of inventory cost.

To finance inventory, a loan can be obtained at interest rate $\ell_n \geq 0$. Interest rates are fixed and depend entirely upon current market conditions. For example, *Lending Club* (a US peer-to-peer lending company) can provide Small Business Loans at a fixed rate up to a sufficiently large amount. When unused cash remains after an ordering decision, the unused cash is deposited in a bank account to earn interest at rate $i_n \geq 0$. To avoid trivialities, we assume that $i_n < \ell_n$ and it is possible to achieve a positive profit with the aid of a loan, i.e., $(1+\ell_n)c_n < p_n$. This assumption can be equivalently written as $1+\ell_n < p_n/c_n$. Note also, that the above assumption implies $1+i_n < p_n/c_n$ since $i_n < \ell_n$; namely, investing in inventory is more profitable than depositing available cash in the bank.

Given an initial inventory-cash state (x_n, y_n), if an order of size $q_n \geq 0$ is placed and the demand during the period is D_n, then we have the following operational and financial cash inflows:

• The cash flow from inventory operations (i.e., the cash received from inventory sales) at the end of the period is given by

$$\begin{aligned} R_n(D_n, q_n, x_n) &= p_n \cdot \min\{q_n + x_n, D_n\} - h_n \cdot [q_n + x_n - D_n]^+ \\ &= p_n \cdot [q_n + x_n - (q_n + x_n - D_n)^+] - h_n \cdot [q_n + x_n - D_n]^+ \\ &= p_n(q_n + x_n) - (p_n + h_n) \cdot [q_n + x_n - D_n]^+, \end{aligned} \tag{1}$$

where $[a]^+ = \max\{a, 0\}$ and the second equality holds by $\min\{a,b\} = a - [a-b]^+$.

• The cash flow from financial transactions at the end of the period is computed for each of the following investing and financing scenarios:

i) Investment transactions: If the order quantity satisfies $0 \leq q_n \leq y_n$, then the leftover amount $c_n \cdot (y_n - q_n)$ of cash is deposited in the bank and will yield a positive inflow of $c_n \cdot (y_n - q_n)(1+i_n)$ at the end of the period n.

ii) Financing transactions: If the order quantity satisfies $q_n > y_n$ (including the case $q_n = 0 > y_n$), then a loan amount of $c_n \cdot (q_n - y_n)$ will be borrowed during the period and will result in a cash outflow of $c_n \cdot (q_n - y_n)(1+\ell_n)$ at the end of the period.

In summary, the net cash flow from financial transaction (positive or negative) can be written in general as

$$K_n(q_n, y_n) = c_n \cdot (y_n - q_n) \left[(1+i_n)\mathbf{1}_{\{q_n \leq y_n\}} + (1+\ell_n)\mathbf{1}_{\{q_n > y_n\}}\right]. \tag{2}$$

Note that the cash inflow from inventory operations, $R_n(D_n, q_n, x_n)$, is dependent on D_n but independent of y_n. In a similar vein, cash flow from financial transactions, $K_n(q_n, y_n)$, is independent of the initial on-hand inventory, x_n, and the demand size, D_n, but is dependent on y_n. Note also that the ordering cost, $c_n \cdot q_n$, has been accounted for in Eq. (2) while the remaining cash, if any, is invested in the bank and its value at the end of the period is given by $K_n(q_n, y_n)$.

Since the order quantity $q_n = q_n(x_n, y_n)$ is decided at the beginning of period n as a function of (x_n, y_n), it is readily shown that the state process $\{(x_n, y_n)\}$ under study is a *Markov decision process* (MDP) with decision variable q_n; cf. Ross (1992). The state dynamics of the system, i.e., inventory flow and cash flow, are given as follows, for $n = 1, 2, ..., N-1$

$$x_{n+1} = [x_n + q_n - D_n]^+; \tag{3}$$

$$y_{n+1} = [R_n(D_n, q_n, x_n) + K_n(q_n, y_n)]/c_{n+1}, \tag{4}$$

where R_n and K_n are given by Eqs. (1)-(2), respectively.

Note that, at the beginning of period n, the interplay between finance and operations is typically one way. Namely, it is feasible to purchase products with available cash y_n (when $y_n > 0$), but inversion of on-hand inventory x_n into cash is restricted or imposibble. Conversely, the sales interplay between finance and operations is the opposite, namely, sales decrease the inventory and increase cash on hand. Thus, the requisite optimization can be formulated as the following:

$$V_n(x_n, y_n) = \sup_{q_n \geq 0} \mathbb{E}\Big[V_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n\Big], \qquad n = 1, 2, \cdots, N-1 \tag{5}$$

where the expectation is taken with respect to D_n, and x_{n+1}, y_{n+1} are given by Eqs. (3) and (4), respectively. For the final period N, we have

$$V_N(x_N, y_N) = \sup_{q_N \geq 0} \mathbb{E}\Big[R_N(D_N, q_N, x_N) + K_N(q_N, y_N)\Big]. \tag{6}$$

4. The Single Period Problem

The analysis of the last period, N, is a typical single-period problem and its objective function is given by Eq. (6). To simplify notation, we omit the subscript N from any related variable when denoting counterparts under the single period setting, without causing any confusion. Accordingly, for any given initial "inventory-cash" state (x, y) and order size, $q \geq 0$, the expected net worth at the end of the period is given by

$$G(q, x, y) = p\,(x+q) - (p-s)\int_0^{x+q}(x+q-t)f(t)dt + c\,(y-q)\Big[(1+i)\mathbf{1}_{\{q \leq y\}} + (1+\ell)\mathbf{1}_{\{q > y\}}\Big], \tag{7}$$

where the first term above is the initial inventory value, the second is the expected value of inventory sold over the period adjusted for salvage/disposal, and the third is the cash flow from financial transactions.

The following lemma summarizes the important properties of the function $G(q,x,y)$.

Lemma 1 *The function $G(q,x,y)$ is continuous in q, x and y, and satisfies the following*

i) It is concave in $q \in [0,\infty)$, for all x, y and all $s < p$.

ii) It is increasing and concave in x, for $s \geq 0$.

iii) It is increasing and concave in y, for all $s < p$.

Remarks.

1. It is important to point out that $G(q,x,y)$ might not be increasing in x if $s < 0$. To see this, we first write

$$\frac{\partial}{\partial x} G(q,x,y) = p\bar{F}(q+x) + sF(q+x). \tag{8}$$

In particular, if s represents a disposal cost, i.e., $s < 0$, then the right side of Eq. (8) might be negative, which implies that $G(q,x,y)$ is decreasing for sufficiently high values of x. Further, for the special case $s < 0$, it is of interest to locate the critical value, x' such that $G(q,x,y)$ is decreasing for $x > x'$. To this end, we set Eq. (8) to zero, which yields

$$(p-s) \cdot F(q+x) = p. \tag{9}$$

Therefore,

$$x' = F^{-1}\left(\frac{p}{p-s}\right) - q, \tag{10}$$

where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$. Eq. (10) shows that a higher disposal cost implies a lower threshold value for x' above.

2. Lemma 1 implies that higher values of initial assets, x and y, or of net worth, $\xi = x + y$, yield a higher expected revenue $G(q,x,y)$. Further, for any assets (x,y) there is a unique optimal order quantity q^* such that

$$q^*(x,y) = \arg\max_{q \geq 0} G(q,x,y).$$

We next define the critical values, α and β, as follows:

$$\alpha = F^{-1}(a); \tag{11}$$

$$\beta = F^{-1}(b), \tag{12}$$

where

$$a = \frac{p - c \cdot (1+\ell)}{p-s};$$

$$b = \frac{p - c \cdot (1+i)}{p-s}.$$

It is readily seen that $a \le b$, since $0 \le i \le \ell$ by assumption. This implies that $\alpha \le \beta$, since $F^{-1}(z)$ is increasing in z. The critical value β can be interpreted as the optimal order quantity for the classical newsvendor problem [cf. Zipkin (2000) and many others] corresponding to the case of sufficiently large Y of our model, in which case no loan is involved, but the unit "price" $c \cdot (1+i)$ has been inflated to reflect the opportunity cost of cash not invested in the bank at an interest rate i. Similarly, α can be interpreted as the optimal order quantity for the classical newsvendor problem corresponding to the case $Y = 0$ of our model, i.e., all the order is financed by a loan at an interest rate ℓ. Note also that in contrast to the classical newsvendor model, the critical values α and β above, are now functions of the corresponding interest rates and represent opportunity costs that take into account the value of time using the interest factors $1+i$ and $1+\ell$.

The following theorem identifies the optimal $(\alpha,\, \beta)$ policy.

Theorem 1 *For any given inventory-cash state (x, y), the optimal order quantity is*

$$q^*(x,y) = \begin{cases} (\beta - x)^+, & \beta \le x + y; \\ y^+, & \alpha \le x + y < \beta; \\ (\alpha - x)^+, & x + y < \alpha, \end{cases} \tag{13}$$

where α and β are given by Eq. (11) and (12), respectively.

Note that the optimal ordering quantity in the classical newsvendor model, can be obtained from Theorem 1 as the solution for the extreme case with $i = \ell = 0$, in which case the optimal order quantity is given by: $\alpha = \beta = F^{-1}\left(\frac{p-c}{p-s}\right)$.

Next we elucidate the structure of the (α, β) optimal policy below by discussing the utilization level of the initially available cash Y.

1. **Over-utilization**: $(x+y < \alpha)$. If $x \le \alpha$, then it is optimal to order $q^* = \alpha - x = y + (\alpha - x - y)$. In this case, $y = Y/c$ units are purchased using all available cash Y, if $Y > 0$, and the remaining $(\alpha - x - y)$ units are purchased with the proceeds of a loan in the amount $c \cdot (\alpha - x - y)$. If $x > \alpha$, then it is optimal not to order, i.e., $q^* = 0$.

2. **Full-utilization**: $(\alpha \le x + y < \beta)$. It is optimal to order $q^* = y = Y/c$ with all the available cash Y, if $Y > 0$; otherwise, no ordering. In this case, all available cash is fully utilized and no deposit and no loan are involved.

3. **Under-utilization**: $(x+y \ge \beta)$. In this case, it is optimal to order $q^* = (\beta - x)^+$. In particular, if $x < \beta$, it is optimal to order $\beta - x$ using $c \cdot (\beta - x)$ units of the available cash Y, and deposit the remaining cash to earn interest. However, if $x \ge \beta$, then $q^* = 0$, i.e., it is optimal not to order and deposit the entire amount Y if $Y > 0$, to earn interest.

The above interpretation is illustrated in Figure 1 for the case $x = 0$, which plots the optimal order quantity q^* as a function of y. Note that for $y \in (0, \alpha)$ there is over-utilization of y; for $y \in [\alpha, \beta)$ there is full-utilization of y and for $y \in [\beta, \infty)$ there is under-utilization of y.

We shall mention that the structure of this type of two threshold values is essentially caused by the two distinct (loan and deposit) interest rates. Actually, the optimal threshold policy with multiple critical values caused by a piecewise linear ordering cost has been studied in the literature; cf. Fu et al. (2012) and Bensoussan et al. (1983). In Section 7.1, we will derive the structural result of multiple threshold values for piecewise-type loan and deposit functions.

We are now in a position to derive an explicit form of the optimal expected net worth, $V(x, y) = \max_{q \geq 0} G(q, x, y)$.

Theorem 2 *For the single-period problem with initial state (x, y),*

i) V(x,y) is given by

$$V(x,y) = \begin{cases} p \cdot x - (p-s) \cdot T(x) + c \cdot y \cdot (1+i), & x > \beta; \\ p \cdot \beta - (p-s) \cdot T(\beta) + c \cdot (x+y-\beta)(1+i), & x \leq \beta,\ \beta \leq x+y; \\ p \cdot (x+y^+) - (p-s) \cdot T(x+y^+), & \alpha \leq x+y < \beta; \\ p \cdot \alpha - (p-s) \cdot T(\alpha) + c \cdot (x+y-\alpha)(1+l), & x+y < \alpha, \end{cases} \tag{14}$$

where $T(x) = \int_0^x (x-t) f(t) dt$;

ii) The function $V(x, y)$ is increasing in x and y, and jointly concave in (x, y), for $x, y \geq 0$.

From an investment perspective, it is of interest to consider the possibility of speculation [cf. Hull (2002)]. The following result shows that the (α, β) policy given in Theorem 1 yields a positive value with zero investment. Specifically, when the firm has zero initial inventory and cash, i.e., $x = 0$ and $y = 0$, the optimal policy yields a positive expected final asset value.

Corollary 1 *For $x = 0$ and $y = 0$, one has*

$$V(0,0) = (p-s) \int_0^\alpha t \cdot f(t) dt > 0.$$

Note that arbitrage usually means that it is possible to have a positive profit for any realized demand, i.e., of a net risk-free profit at zero cost; cf. Hull (2002). Thus the above speculation possibility does not in general imply an arbitrage, which only exists if the demand is deterministic.

5. Optimal Solution for the Multiple Period Problem

For ease of exposition, we shall denote $\xi_n = x_n + y_n$ and take $z_n = x_n + q_n$ as the decision variable, in lieu of q_n. Here, z_n represents on-hand inventory in product units after replenishment, and is

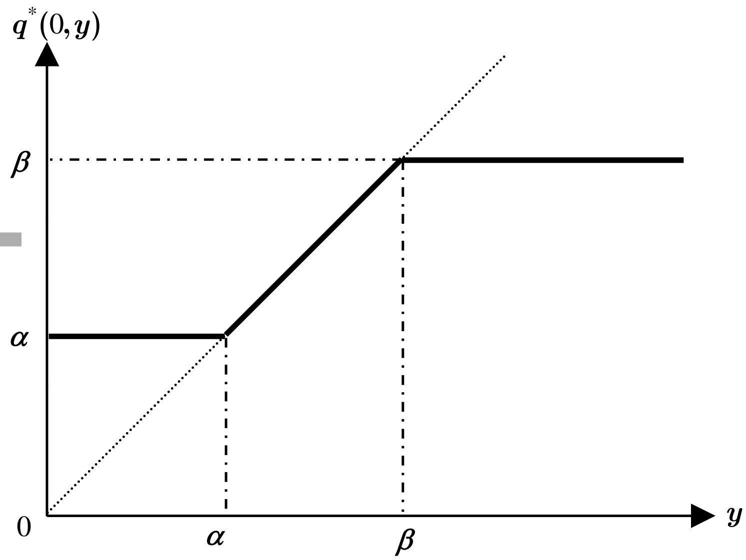


Figure 1 **The Optimal Order Quantity when** $x = 0$

constrained by $z_n \geq x_n \geq 0$ for each period n. The attendant MDP model defined by Eqs. (5)-(6) can be written as:

$$V_n(x_n, y_n) = \max_{z_n \geq x_n} G_n(z_n, x_n, y_n), \tag{15}$$

where $G_n(z_n, x_n, y_n) = \mathbb{E}\left[V_{n+1}(x_{n+1}, y_{n+1})\right]$. The inventory-cash states are periodically updated as follows:

$$x_{n+1} = [z_n - D_n]^+; \tag{16}$$

xi_n 应该是 x_n+y_n

$$y_{n+1} = p'_n \cdot z_n - (p'_n + h'_n)\,[z_n - D_n]^+ + c'_n \cdot (\xi_n - z_n)\left[(1+i_n)\mathbf{1}_{\{z_n \leq \xi_n\}} + (1+\ell_n)\mathbf{1}_{\{z_n > \xi_n\}}\right], \tag{17}$$

where $p'_n = p_n/c_{n+1}$, $h'_n = h_n/c_{n+1}$ and $c'_n = c_n/c_{n+1}$.

We first present the following result whose proof appears in Appendix A.

Lemma 2 *For* $n \in \mathcal{N}$,

(1) $G_n(z_n, x_n, y_n)$ *is increasing in* x_n *and* y_n, *and is concave in* z_n, x_n *and* y_n.

(2) $V_n(x_n, y_n)$ *is increasing and concave in* x_n *and* y_n.

We next present and prove the main result of this section.

Theorem 3 ***(The*** (α_n, β_n) ***optimal ordering policy)***.

For every period $n \in \mathcal{N}$ *with state* (x_n, y_n) *at the beginning of the period, there exist positive* *constants* $\alpha_n = \alpha_n(x_n, y_n)$ *and* $\beta_n = \beta_n(x_n, y_n)$ *such that* $\alpha_n \le \beta_n$*, which give rise to an optimal order quantity as follows:*

$$q_n^*(x_n, y_n) = \begin{cases} (\beta_n - x_n)^+, & \xi_n \ge \beta_n; \\ (y_n)^+, & \alpha_n \le \xi_n < \beta_n; \\ (\alpha_n - x_n)^+, & \xi_n < \alpha_n. \end{cases} \tag{18}$$

Further, α_n *is uniquely determined by*

$$\mathbb{E}\left[\left(\frac{\partial V_{n+1}}{\partial x_{n+1}} - (p_n' + h_n')\frac{\partial V_{n+1}}{\partial y_{n+1}}\right)\mathbf{1}_{\{\alpha_n > D_n\}}\right] = [c_n'(1+\ell_n) - p_n']\,\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\right], \tag{19}$$

and β_n *is uniquely determined by*

$$\mathbb{E}\left[\left(\frac{\partial V_{n+1}}{\partial x_{n+1}} - (p_n' + h_n')\frac{\partial V_{n+1}}{\partial y_{n+1}}\right)\mathbf{1}_{\{\beta_n > D_n\}}\right] = [c_n'(1+i_n) - p_n']\,\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\right]. \tag{20}$$

where the expectations are with respect to D_n*, conditioned on the initial state* (x_n, y_n)*.*

Theorem 3 establishes that the optimal ordering policy for period n is characterized by two threshold parameters, α_n and β_n. More importantly, these two threshold values can be computed recursively by solving the implicit equations, Eqs. (19) and (20), respectively.

Remark. The study in Chao et al. (2008) assumes that borrowing is not allowed, and thus the firm is strictly limited to order at most y_n units in period n. It was shown there that the optimal policy is determined in each period by one critical value. Our results in Theorem 3 subsume that study as a special case. To see that, set ℓ_n to be sufficiently large such that a loan is financially prohibitive. In this case, α_n becomes zero and β_n can be interpreted as the critical value developed in Chao et al. (2008).

Corollary 2 *For any period* $n < N$ *and the associated initial state* (x_n, y_n)*, the threshold parameters,* α_n *and* β_n*. are determined only by the total worth* $\xi_n = x_n + y_n$*, i.e., they are of the form:* $\alpha_n = \alpha_n(\xi_n)$ *and* $\beta_n = \beta_n(\xi_n)$*. However, for the last period* N*,* α_N *and* β_N *are independent of either* x_N *or* y_N*, or both.*

Not surprisingly, Corollary 2 implies that the operational decisions are affected by the firm's current financial status. Furthermore, one can first compute the α_n and β_n at the beginning of the period based on the total worth ξ_n. The decision on the order quantity q_n can then be made according to Eq. (18).

6. Myopic Policies and Bounds Analysis

As shown by Theorem 3, the computation of α_n and β_n is complex and costly. This fact motivates the study to follow of myopic ordering policies that are relatively simple to compute and implement. Such myopic policies optimize a given objective function with respect to any single period and ignore multi-period interactions and cumulative effects. To this end, we introduce two types of myopic policies. In the following two subsections, we will show that myopic policy (I) yields lower bounds, $\underline{\alpha}_n$ and $\underline{\beta}_n$ for the two threshold values, α_n and β_n, while myopic policy (II) yields upper bounds, $\bar{\alpha}_n$ and $\bar{\beta}_n$ for the same values.

Before presenting the myopic policies, we present the following lemma which will be used to derive the upper and lower bounds.

Lemma 3 *For real functions $f(x)$ and $g(x)$,*
(a) if $f(x)$ and $g(x)$ are both monotonically increasing or both decreasing, then

$$\mathbb{E}[f(X)\cdot g(X)] \geq \mathbb{E}[f(X)]\cdot \mathbb{E}[g(X)],$$

where the expectation is taken with respect to the random variable X.
(b) If $f(x)$ is increasing (decreasing), while $g(x)$ is decreasing (increasing), then

$$\mathbb{E}[f(X)\cdot g(X)] \leq \mathbb{E}[f(X)]\cdot \mathbb{E}[g(X)].$$

6.1. Myopic Policy (I) and Lower Bounds for α_n and β_n

Myopic policy (I) is the one-period optimal policy obtained by assuming that only the holding cost is assessed on any leftover inventory i.e., we assume the following modified "salvage value" cost structure:

$$\underline{s}_n = \begin{cases} -h_n, & n < N, \\ s, & n = N. \end{cases}$$

Let further,

$$\underline{a}_n = \frac{p_n - c_n\cdot(1+\ell_n)}{p_n - \underline{s}_n}; \tag{21}$$

$$\underline{b}_n = \frac{p_n - c_n\cdot(1+i_n)}{p_n - \underline{s}_n}. \tag{22}$$

and the corresponding critical values are given by

$$\underline{\alpha}_n = F_n^{-1}(\underline{a}_n); \tag{23}$$

$$\underline{\beta}_n = F_n^{-1}(\underline{b}_n). \tag{24}$$

For $n \in \mathscr{N}$, the order quantity below defines the *myopic policy* (I):

$$\underline{q}_n(x_n, y_n) = \begin{cases} (\underline{\beta}_n - x_n)^+, & x_n + y_n \geq \underline{\beta}_n; \\ (y_n)^+, & \underline{\alpha}_n \leq x_n + y_n < \underline{\beta}_n; \\ \underline{\alpha}_n - x_n, & x_n + y_n < \underline{\alpha}_n. \end{cases}$$

The next theorem establishes the lower bound properties of the myopic policy (I).

Theorem 4 *The following are true:*
i) For the last period N, $\alpha_N = \underline{\alpha}_N$ *and* $\beta_N = \underline{\beta}_N$.
ii) For any period $n = 1, 2, \ldots N-1$,

$$\alpha_n \geq \underline{\alpha}_n,$$
$$\beta_n \geq \underline{\beta}_n.$$

6.2. Myopic Policy (II) and Upper Bounds for α_n and β_n

Myopic policy (II) is the one-period optimal policy obtained by assuming that not only the holding cost is assessed but also the cost in the next period on any leftover inventory, i.e., we assume the following modified "salvage value" cost structure:

$$\bar{s}_n = \begin{cases} c_{n+1} - h_n, & n < N; \\ s, & n = N. \end{cases} \tag{25}$$

One can interpret the modified salvage values $\bar{s}_n$ of Eq. (25) as representing a fictitious income from *inventory liquidation* (or pre-salvage at full current cost) at the beginning of the next period, $n+1$, i.e., it corresponds to the situation that the firm can salvage inventory at the price c_{n+1} at the beginning of the period $n+1$. Note that the condition $c_n(1+\ell_n) + h_n \geq c_{n+1}$ is required if inventory liquidation is admitted. Otherwise, the firm will stock up at an infinite level and sell them off at the beginning of period $n+1$. Such speculation is precluded by the aforementioned condition.

Let further,

$$\bar{a}_n = \frac{p_n - c_n[1+\ell_n]}{p_n - \bar{s}_n}; \tag{26}$$

$$\bar{b}_n = \frac{p_n - c_n[1+i_n]}{p_n - \bar{s}_n}, \tag{27}$$

and let the corresponding critical values be given by

$$\bar{\alpha}_n = F_n^{-1}(\bar{a}_n), \tag{28}$$

$$\bar{\beta}_n = F_n^{-1}(\bar{b}_n). \tag{29}$$

For $n \in \mathcal{N}$, the order quantity below defines the *myopic policy* (II):

$$\bar{q}_n(x_n, y_n) = \begin{cases} (\bar{\beta}_n - x_n)^+, & x_n + y_n \geq \bar{\beta}_n; \\ (y_n)^+, & \bar{\alpha}_n \leq x_n + y_n < \bar{\beta}_n; \\ \bar{\alpha}_n - x_n, & x_n + y_n < \bar{\alpha}_n. \end{cases}$$

Let $V_n^L(x_n, y_n)$ denote the optimal expected future value when the inventory liquidation option is available only at the beginning of period $n+1$ (but not in the remaining periods $n+2, \ldots, N$), given the initial state (x_n, y_n) of period n. For notational simplicity, let $\xi_{n+1} = \xi_{n+1}(x_n, y_n, z_n, D_n)$ represent the total inventory and cash value in period $n+1$ when the firm orders $z_n \geq x_n$ in state (x_n, y_n) and the demand is D_n. Note that $\xi_{n+1} = x_{n+1} + y_{n+1}$, where x_{n+1} and y_{n+1} are given by Eqs. (16) and (17), respectively. Therefore, V_n^L can be written as

$$V_n^L(x_n, y_n) = \max_{z_n \geq x_n} \mathbb{E}\left[V_{n+1}(0, \xi_{n+1}) \,|\, x_n, y_n\right]. \tag{30}$$

Note that for any real functions $g(x)$ and $f(x)$, with $g(x)$ increasing and concave in x and $f(x)$ concave in x, the function $g\big(f(x)\big)$ is concave in $x \in \Re$. Thus, the concavity of $\mathbb{E}\left[V_{n+1}(0, \xi_{n+1}) \,|\, x_n, y_n\right]$ in z_n follows from the observations that $V_n(0, \xi)$ is increasing and concave in ξ (cf. part 2 of Lemma 2) and ξ_{n+1} is concave in in z_n, since both x_{n+1} and y_{n+1} are concave in z_n (cf. discussion in the proof of Lemma 2 regarding the second-order derivatives of x_{n+1} and y_{n+1} with respect to z_n). Consequently, V_n^L has an optimal policy characterized by a sequence of threshold value pairs, (α_n^L, β_n^L).

Before exhibiting the upper bounds of α_n and β_n, we present the following result.

Proposition 1 *For any period n with initial state (x_n, y_n),*
(i) $V_n(x_n - d, \, y_n + d)$ is increasing in d where $0 \leq d \leq x_n$.
(ii) The partial derivatives satisfy, $\partial V_n(x_n, y_n)/\partial y_n \geq \partial V_n(x_n, y_n)/\partial x_n$.

Part (i) of Proposition 1 states that, for the same total assets, incremental cash is preferable to incremental inventory. This is true because more cash provides more flexibility in tuning the inventory level. Part (ii) implies that the contribution margin of a unit of cash is higher than that of a unit of inventory. In view of Proposition 1, we have the following results.

Proposition 2 *For any state (x_n, y_n), and all $n = 1, \ldots N$,*
(i) $V_n^L(x_n, y_n) \geq V_n(x_n, y_n)$,
(ii) $\alpha_n^L \geq \alpha_n$ and $\beta_n^L \geq \beta_n$.

Note that inventory liquidation at the beginning of period $n+1$ provides the firm with additional flexibility, since the initial inventory x_{n+1} can be liquidated into cash, thereby allowing the firm to hold only $\xi_{n+1} = x_{n+1} + y_{n+1}$ in cash. Note further that the firm will chose to stock up to a higher level of inventory when liquidation is allowed. Indeed, the firm has no inventory risk in the sense that all leftover inventory in period n (after satisfying the period demand D_n) can be salvaged at full cost c_{n+1} at the beginning of the next period, $n+1$. In other words, the firm will take the advantage of inventory liquidation to stock product at a higher level than in the counterpart case which disallows such liquidation. The advantage of doing so is twofold: (1) more demand can be satisfied leading to more revenue, and (2) no extra cost accrues from liquidation of leftover inventory.

The next result establishes the upper bound properties of the myopic policy (II).

Proposition 3 *For any period $n = 1, 2, \ldots, N-1$, if $c_n(1+\ell_n) + h_n \geq c_{n+1}$, then the threshold parameters of the optimal policy given in Eqs. (19)-(20) and its myopic optimal policy given in Eqs. (28)-(29) satisfy*

$$\bar{\alpha}_n \geq \alpha_n;$$
$$\bar{\beta}_n \geq \beta_n.$$

For the last period N, $\alpha_N = \bar{\alpha}_N$ and $\beta_N = \bar{\beta}_N$.

Remark: It is not easy in general to compute solutions even for the simpler multi-period problems which involve base-stock levels. In the supply chain literature, computational methods are often based on heuristic algorithms or iterative search procedures. Determination of base-stock levels in each period or for each echelon and the minimization of the total supply chain cost are notoriously complex and computationally tedious, especially in our underlying two-dimensional state space with the attendant curse of dimensionality; cf. Lee and Billington (1993), Chen and Zheng (1994), Minner (1997), Shang and Song (2003) and Daniel and Rajendran (2006). Nevertheless, with the aid of the lower and the upper bounds developed above, one can devise an efficient search algorithm for the problem herein, e.g., a bisection or golden section search algorithm; cf. Press (2007).

6.3. Sell-Back to the Supplier: Upper Bound for the Value Function $V_n(x, y)$

The myopic policies above provide lower and upper bounds for α_n and β_n, respectively, and both of them provide lower bounds for the net worth value function $V_n(x, y)$. In this section we obtain upper bounds for the value function by considering a related problem in which the firm is given the option to sell unneeded inventory back to the supplier at full price. Thus, for some cases, the firm has more flexibility at the beginning of a period $n \in \mathscr{N}$ to readjust the inventory level as compared to just ordering from the supplier at cost c_n. For example, the firm is allowed to sell unneeded inventory back to the supplier or at a free-trading spot market at price c_n. We assume $c_{n+1} \le c_n + h_n$ to preclude situations where the firm can profit from selling product in inventory back to the supplier.

In practice this situation can occur either when there is an agreement between the supplier and the firm such as a *vendor-managed inventory* (VMI) contract, or there is a commodity spot market, e.g., energy sources and natural resources; cf. Secomandi (2010). We will refer to such a setting as a *sell-back* option. The sell-back option provides the firm with the flexibility to convert any excess inventory into cash prior to the regular selling season. Consequently, the state of the system is reduced from two variables, x_n and y_n, to a single variable ξ_n that represents the sum of the value of the on-hand inventory plus the value of cash on hand, expressed in product units, i.e., $\xi_n = x_n + y_n$.

Adding Eqs. (16) and (17) yields the dynamics equations for $\xi_{n+1} = x_{n+1} + y_{n+1}$, with the decision variables z_n,

$$\xi_{n+1} = p_n' \cdot z_n - (p_n' + h_n' - 1)\,[z_n - D_n]^+ + c_n' \cdot (\xi_n - z_n)\left[(1+i_n)\mathbf{1}_{\{z_n \le \xi_n\}} + (1+\ell_n)\mathbf{1}_{\{z_n > \xi_n\}}\right]. \quad (31)$$

The dynamic programming equations for the value function $V_n^S(\xi_n)$ are

$$V_n^S(\xi_n) = \max_{z_n \ge 0} \mathbb{E}[V_{n+1}^S(\xi_{n+1})], \quad (32)$$

where the terminal value $V_N^S(\xi_N) = V_N(0, \xi_N)$ is given by Eq. (6).

It can be readily shown that $V_n^S(\xi)$ is increasing and concave in ξ for any $n \in \mathscr{N}$. Consequently, we have the following result.

Theorem 5 ***(Optimal inventory-trading policy with sell-back)****.*
At the beginning of period $n \in \mathscr{N}$, if the firm has a sell-back option, then
(i) The optimal policy is the $(\alpha_n^S,\ \beta_n^S)$ policy, where $0 \le \alpha_n^S < \beta_n^S$. That is, the optimal target inventory is

$$z_n^*(\xi_n) = \begin{cases} \beta_n^S, & \xi_n \ge \beta_n^S; \\ \xi_n, & \alpha_n^S \le \xi_n < \beta_n^S; \\ \alpha_n^S, & \xi_n < \alpha_n^S. \end{cases}$$

(ii) For any x_n and y_n, the value functions satisfy

$$V_n^S(x_n + y_n) \ge V_n^L(x_n, y_n) \ge V_n(x_n, y_n).$$

Theorem 5 states that, with the sell-back option, the firm can freely retune its inventory via selling down or ordering up as follows. If $\xi_n \geq \beta_n^S$, then it is optimal to choose the target inventory z_n equal to β_n^S; if $\xi_n \leq \alpha_n^S$, then it is optimal to choose the target inventory z_n as α_n^S; and otherwise, if $\alpha_n^S < \xi_n < \beta_n^S$, then it is optimal to choose the target inventory z_n equal to ξ_n.

Note that V_n^S provides an upper bound of V_n which is easier to compute than that of $V_n^L(x,y)$, since the former is a function of only one variable.

7. Model Extensions

In this section, we study three extensions of the basic model. The first extension is a piecewise linear structure of the interest rate functions for both loans and deposits, the second is the constraint of a maximum loan limit and the third is backordering of unmet demand.

7.1. Piecewise-Type Loan and Deposit Functions

Hitherto, the loan function was assumed to be a linear function of the form $L(x) = (1+\ell)\cdot x$ with a constant loan rate ℓ. However, in practice, $L(x)$ can have a more complex form. In this section we investigate the common case of a piecewise-type linear $L(x)$, that is,

$$L(x) = (1+\ell^{(m)})\cdot x, \qquad x \in (x^{(m-1)}, x^{(m)}],$$

where $x^{(m-1)} < x^{(m)}$, $x^{(0)} = 0$ and $\ell^{(m)} < \ell^{(m+1)}$ for $m = 1,2,3,....$ In a similar vein, we assume that the deposit interest function is piecewise linear of the form,

$$M(y) = (1+i^{(k)})\cdot y, \qquad y \in (y^{(k-1)}, y^{(k)}],$$

where $y^{(k-1)} < y^{(k)}$, $y^{(0)} = 0$ and $i^{(k)} \leq i^{(k+1)}$ for $k = 1,2,3,....$ Without loss of generality, we assume that the loan interest rates are always greater than the deposit interest rates, that is $\bar{i} < \ell^{(1)}$ where $\bar{i} = \sup_k \{i^{(k)}\}$.

Before characterizing the optimal ordering policy, we introduce the threshold values of $\alpha^{(m)}$ and $\beta^{(k)}$ for $m, k = 1,2,3,...$ as follows

$$\alpha^{(m)} = F^{-1}(a^{(m)}); \tag{33}$$

$$\beta^{(k)} = F^{-1}(b^{(k)}), \tag{34}$$

where

$$a^{(m)} = \frac{p - c\cdot(1+\ell^{(m)})}{p-s};$$
$$b^{(k)} = \frac{p - c\cdot(1+i^{(k)})}{p-s}.$$

It is readily seen that

$$\beta^{(1)} \geq \cdots \beta^{(k)} \geq \beta^{(k+1)} \cdots \geq \bar{\beta} > \alpha^{(1)} \geq \cdots \alpha^{(m)} \geq \alpha^{(m+1)} \cdots \geq 0,$$

where $\bar{\beta} = F_n^-(\frac{p - c \cdot (1+\bar{i})}{p-s})$. The structure of the optimal ordering policy in exhibited in Theorem 6 below.

Theorem 6 *For any initial state* (x, y)*, the optimal order quantity is given by*

$$q^*(x,y) = \begin{cases} (\beta^{(k)} - x)^+, & \beta^{(k+1)} \leq x + y \leq \beta^{(k)}; \\ \cdots, & \cdots \\ (y)^+, & \alpha^{(1)} \leq x + y < \bar{\beta}; \\ \cdots, & \cdots \\ (\alpha^{(m)} - x)^+, & \alpha^{(m+1)} \leq x + y < \alpha^{(m)}, \end{cases}$$

where $\{\alpha^{(m)}\}$ *and* $\{\beta^{(k)}\}$ *are given by Eqs. (33) and (34), respectively.*

For the multi-period problem, the optimal order quantity for each period has a structure similar to that of its single-period counterpart, and the proof is analogous to that of Theorem 6. We note that for any period $n < N$, the threshold values, $\alpha_n^{(m)}$ and $\beta_n^{(k)}$, are determined by ξ_n. However, for the last period N, the threshold values, $\alpha_N^{(m)}$ and $\beta_N^{(k)}$, are independent of the initial state x_N and y_N, or ξ_N. We mention that if the lender charges a sufficiently high loan rate, such that $p \leq c \cdot (1 + \ell^{(m)})$ for loans exceeding $x^{(m-1)}$, then $\alpha^{(m)} = 0$ by virtue of Eq. (33), since $a^{(m)}$ is nonpositive. Theorem 6 also implies that for negative total asset $x + y < 0$, the firm will not finance orders via a loan.

7.2. Financing Subject to a Maximal Loan Constraint

In practice, the outstanding loan amount is often restricted not to exceed some prescribed maximum. Let $L_n > 0$ denote such maximum for period n; hence the maximal loan can finance $L'_n := L_n/c_n$ product units. In this case, we have the following structural results for the optimal ordering policy.

Theorem 7 ***(Optimal ordering policy under a maximal loan constraint).***
For period $n \in \mathcal{N}$ *with state* (x_n, y_n) *at the beginning of the period, and a loan maximum* L_n*, there exist positive constants* $\alpha_n^L = \alpha_n^L(\xi_n)$ *and* $\beta_n^L = \beta_n^L(\xi_n)$ *satisfying* $\alpha_n^L \leq \beta_n^L$*, which characterize the optimal order quantity as follows:*

$$q^*(x_n, y_n) = \begin{cases} (\beta_n^L - x_n)^+, & x_n + y_n \geq \beta_n^L; \\ (y_n)^+, & \alpha_n^L \leq x_n + y_n < \beta_n^L; \\ \alpha_n^L - x_n, & \alpha_n^L - L'_n \leq x_n + y_n < \alpha_n^L; \\ L'_n, & x_n + y_n < \alpha_n^L - L'_n. \end{cases}$$

Note that for period n, the inequality $\alpha_n^L < L'_n$ is admissible. In this case, the loan maximum is not binding, since the optimal order quantity satisfies $q_n^* \leq \alpha_n^L < L'_n$.

7.3. System with Backordering

Recall that our basic model assumes the lost-sale rule for unmet demand. This section considers the case in which backorders are allowed and shows that an ordering policy of a similar (α_n, β_n) structure is optimal.

Suppose that any unmet demand is backordered subject to penalty cost b_n per unit of unmet demand; cf. Shi et al. (2013), Shi et al. (2014) and Katehakis et al. (2015). Let $\tilde{V}_n(x,y)$ denote the optimal expected profit function of period n, given initial state $(x,y) \in \mathbb{R} \times \mathbb{R}$. Accordingly, we have the following dynamic programming formulation:

$$\tilde{V}_n(x_n, y_n) = \sup_{z_n \geq x_n} \mathbb{E}\Big[\tilde{V}_{n+1}(x_{n+1}, y_{n+1}) | x_n, y_n\Big], \qquad n = 1, 2, \cdots, N-1, \tag{35}$$

where the expectation is taken with respect to D_n, and the vector $(x_{n+1},\, y_{n+1})$ is given by

$$x_{n+1} = z_n - D_n; \tag{36}$$

$$y_{n+1} = [\tilde{R}_n(D_n, z_n) + \tilde{K}_n(z_n, \xi_n)]/c_{n+1}, \tag{37}$$

where

$$\tilde{R}_n(D_n, z_n) = (p_n + b_n) \cdot z_n - (p_n + h_n + b_n)\,[z_n - D_n]^+ - b_n \mathbb{E}[D_n]; \tag{38}$$

$$\tilde{K}_n(z_n, \xi_n) = c_n \cdot (\xi_n - z_n)\left[(1 + i_n)\mathbf{1}_{\{z_n \leq \xi_n\}} + (1 + \ell_n)\mathbf{1}_{\{z_n > \xi_n\}}\right]. \tag{39}$$

For the terminal period N, $\tilde{V}_N(x_N, y_N) = \sup_{q_N \geq 0} \mathbb{E}\Big[\tilde{R}_N(D_N, z_N) + \tilde{K}_N(z_N, \xi_N)\Big]$, where $h_N = -s$.

Theorem 8 *Under the backorder rule, for any period $n \in \mathcal{N}$ with initial state (x_n, y_n), there exist positive constants $\alpha_n = \alpha_n(\xi_n)$ and $\beta_n = \beta_n(\xi_n)$ satisfying $\alpha_n \leq \beta_n$, which characterize the optimal order quantity as follows:*

$$\tilde{q}^*(x_n, y_n) = \begin{cases} (\beta_n - x_n)^+, & x_n + y_n \geq \beta_n; \\ (y_n)^+, & \alpha_n \leq x_n + y_n < \beta_n; \\ \alpha_n - x_n, & x_n + y_n < \alpha_n. \end{cases} \tag{40}$$

Further, α_n is uniquely determined by

$$\mathbb{E}\left[\left(\frac{\partial \tilde{V}_{n+1}}{\partial x_{n+1}} - (p_n' + h_n')\frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}}\right)\mathbf{1}_{\{\alpha_n > D_n\}}\right] = [c_n'(1 + \ell_n) - p_n']\,\mathbb{E}\left[\frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}}\right], \tag{41}$$

and β_n is uniquely determined by

$$\mathbb{E}\left[\left(\frac{\partial \tilde{V}_{n+1}}{\partial x_{n+1}} - (p_n' + h_n')\frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}}\right)\mathbf{1}_{\{\beta_n > D_n\}}\right] = [c_n'(1 + i_n) - p_n']\,\mathbb{E}\left[\frac{\partial \tilde{V}_{n+1}}{\partial y_{n+1}}\right]. \tag{42}$$

where the expectations are taken with respect to D_n, conditioned on the initial state (x_n, y_n).

8. Numerical Studies

In this section, we present numerical studies designed to glean further insights into the interplay between cash flow and inventory flow. To this end, we fix the following parameters across periods: selling price $p = 2000$; ordering cost $c = 1000$; holding cost $h = 500$; deposit interest rate $r = 1\%$; loan interest rate $\ell = 15\%$; and salvage value $s = 600$. In the sequel, §8.1 considers the impacts of time horizon and initial states on the optimal solution; §8.2 studies the corresponding impact of various demand distributions, and §8.3 investigates the performance of the lower and upper bounds of the value function. An extensive test-bed study is provided in Appendix B.

8.1. Impact of Time Horizon

To examine the impact of the time horizon, N, on the value function, we assume that demand is uniformly distributed as $D \sim \mathcal{U}[0, 20]$. To emphasize the dependence of $V_1(x, y)$ on N we will write $V_{1:N}(x, y)$ and set N varying from 1 to 12. Figure 2 depicts the resultant optimal functions $V_{1:N}(x, y)$ for the three selected initial states $(x, y) = (0, 0)$, $(10, -50)$ and $(0, -50)$.

Observe that the total net worth $\xi = x + y$ exhibits a significant impact on the value function. In particular, $V_{1:N}(x, y)$ is increasing in N for sufficiently large net worth ξ, e.g., for $(x, y) = (0, 0)$ and even for $(10, -50)$; however, it is decreasing in N for sufficiently small ξ, e.g., $(x, y) = (0, -50)$. The former can be explained as follows: for moderately small values of ξ, the operational decisions play a more important role than the financial ones, yielding significantly larger magnitudes of $V_{1:N}(x, y)$ over the time horizon N. Conversely, for sufficiently small ξ, the optimal value is primarily determined by financial decisions as apposed to operational ones, e.g., there is a large cost stemming from loans when y is sufficiently negative, and accordingly, $V_{1:N}(x, y)$ is decreasing over time. The dominant role of operational decisions is visually demonstrated by the slopes of $V_{1:N}(x, y)$ in N as depicted in Figure 2.

8.2. Impact of Demand Distributions

In this section we study the impact of demand distributions on the value functions. We further provide some observations of the performances of myopic policies (I) and (II), in terms of uniform and ZIP distributed demands, with respect to different *coefficient of variation* (abbreviated as *c.v.*) levels.

In the computational examples below, we fix the time horizon at $N = 6$. We first consider several uniform demand distributions with the same mean but different variances. Figures 3 and 4 depict $V_1(x, 0)$ and $V_1(0, y)$, respectively, for $D \sim \mathcal{U}[0, 20]$, $D \sim \mathcal{U}[2, 18]$, $D \sim \mathcal{U}[4, 16]$ and $D \sim \mathcal{U}[6, 14]$. Both figures show concavity for each demand type. Given the common mean, it is seen that the demand variance has a significant impact on the value function. For example, there is roughly a

5,000 reduction from the case of $D \sim \mathcal{U}[6,14]$ to the case of $D \sim \mathcal{U}[4,16]$ for any x or y in each figure.

We next consider the impact of different types of demand distributions other than the uniform distribution. The *Zero-Inflated Poisson* ($\mathcal{ZIP}$) distribution is a useful generalization of the Poisson distribution; cf Lambert (1992). The ZIP distribution with parameters $\pi \in [0,1)$ and $\lambda \geq 0$ has the following probability mass function: $\mathbb{P}(X=0) = \pi + (1-\pi)e^{-\lambda}$ and $\mathbb{P}(X=k) = (1-\pi)e^{-\lambda}\lambda^k/k!$ for $k = 1,2,3\ldots$. Hence, it has mean $\mu_X = \lambda(1-\pi)$, standard deviation $\sigma_X = \sqrt{\lambda(1-\pi)(1+\lambda\pi)}$. Therefore, its *coefficient of variation* (defined as the ratio between the standard deviation and the mean) is $c.v. = \sqrt{\frac{1+\lambda\pi}{\lambda(1-\pi)}}$.

Figure 5 compares the optimal $V_1(x,y)$ with $D \sim \mathcal{ZIP}(0.18,10)$ and $D \sim \mathcal{U}[0,20]$, both having the same $c.v. = 0.58$. Here, it is seen that the uniform demand yields higher $V_1(x,y)$ than the $\mathcal{ZIP}$ demand for any x and y. This can be explained by the mean of the uniform demand ($\mu = 10$) being larger than that of the $\mathcal{ZIP}$ demand ($\mu = 8.2$), even though they have the same c_v.

Figure 2 **Function $V_{1:N}(x,y)$ with Selected Initial State (x, y)**

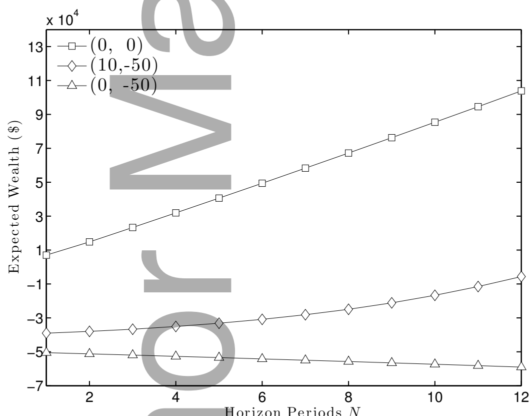


Figure 3 **Function $V_1(x,0)$ for $N = 6$ with Selected Uniform Demand Distributions**

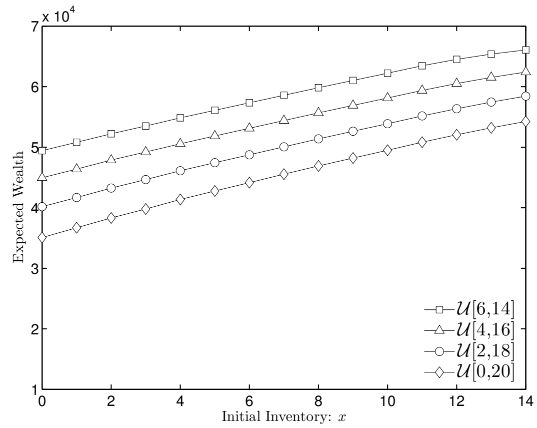


Table 1 displays the optimal objective function values and the performances of myopic policies (I) and (II), in terms of uniform and $\mathcal{ZIP}$ demands with respect to different $c.v.$ levels. In the table, for $x_1 = y_1 = 0$, $V_1 := V_1(0,0)$, $V_1^{(\mathscr{X})}$ is the expected net worth under *myopic policy* ($\mathscr{X}$), $\mathscr{X} \in \{I, II\}$, and the performance gap percentage $\Delta^{(\mathscr{X})}\% := (V_1 - V_1^{(\mathscr{X})})/V_1 \times 100\%$. Observe that, as $c.v.$ decreases, the value of $V_1(0,0)$ increases significantly for each type of demands. Observe further that under the uniform demand, myopic policy (II) with $\Delta^{(II)}\% < 0.2\%$ significantly outperforms myopic policy (I) with $\Delta^{(I)}\% > 3.5\%$; however the converse is observed under the $\mathcal{ZIP}$ demand. For the same $c.v.$ value, the $\mathcal{ZIP}$ demand has lower V_1 than Uniform demand; this can be explained by the smaller mean of $\mathcal{ZIP}$ demand, and the significantly higher probability of zero demand,

Figure 4 Function $V_1(0, y)$ for $N = 6$ with Selected Uniform Demand Distributions

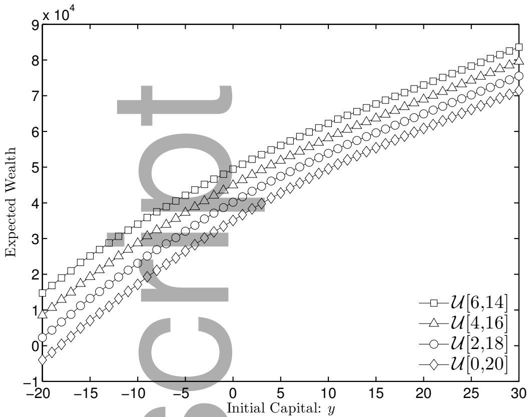
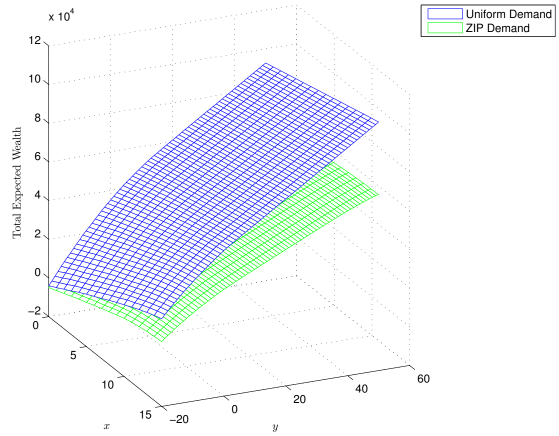


Figure 5 $V_1(x, y)$ for $N = 6$: Uniform v.s. $\mathcal{ZIP}$ Demand Distributions



e.g., when $c.v. = 0.46$, $\mathbb{P}(D = 0) = 9\%$ for the $\mathcal{ZIP}$ demand, but $\mathbb{P}(D = 0) = 0\%$ for the Uniform demand. Consequently, other characteristics of the demand distribution, such as the shape of its pdf function (in addition to the second order characteristic of D such as its $c.v.$), will influence relative myopic policy performance. Finally, note that for the last row in Table 1, $\mathcal{ZIP}(0, 10)$ is the standard Poisson distribution with mean $\lambda = 10$ — the same mean as $D \sim \mathcal{U}(6, 14)$. However, $\mathcal{ZIP}(0, 10)$ has a smaller V_1, because it has a larger standard deviation, and hence a higher $c.v.$ value.

Table 1 The Optimal v.s. Myopic Policies: Uniform and $\mathcal{ZIP}$ Demand with Various $c.v.$ ($x_1 = y_1 = 0$ and $N = 6$)

$D \sim \mathcal{U}[a,b]$							$D \sim \mathcal{ZIP}(\pi, \lambda)$						
		Optimal	Myopic I		Myopic II				Optimal	Myopic I		Myopic II	
$c.v.$	$[a,b]$	V_1	$V_1^{(I)}$	$\Delta^{(I)}\%$	$V_1^{(II)}$	$\Delta^{(II)}\%$	$c.v.$	(π, λ)	V_1	$V_1^{(I)}$	$\Delta^{(I)}\%$	$V_1^{(II)}$	$\Delta^{(II)}\%$
0.58	[0, 20]	35074	30271	13.69%	35016	0.16%	0.58	(0.18, 10)	23130	22800	1.43%	21757	5.94%
0.46	[2,18]	40174	36784	8.44%	40158	0.04%	0.45	(0.09, 10)	26920	26910	0.04%	26142	2.89%
0.35	[4,16]	44950	42329	5.83%	44886	0.14%	0.35	(0.02, 10)	29612	29484	0.43%	29115	1.68%
0.23	[6,14]	49428	47677	3.54%	49385	0.09%	0.31	(0, 10)	30355	30288	0.22%	29911	1.46%

8.3. Upper and Lower Bounds of the Value Function

As discussed in §6, each of the two myopic policies provides a lower bound for the value function, whence we can take the larger of the two as the better lower bound. Furthermore, V_1^S provides an upper bound. In what follows, we numerically examine their approximating performance using the same parameters as in the previous numerical studies.

Figure 6 compares $V_1^S(\xi)$ with $V_1(\xi, 0)$ and $V_1(0, \xi)$ for $N = 6$. Observe that, for any ξ, it is shown that $V_1^S(\xi) \geq V_1(0, \xi) \geq V_1(\xi, 0)$. Observe further that the difference between $V_1^S(\xi)$ and $V_1(0, \xi)$ is

Figure 6 $V_1^S(\xi)$ **v.s.** $V_1(0,y)$ **and** $V_1(x,0)$ **for** $N=6$ **with** $D\sim\mathcal{U}[0,20]$

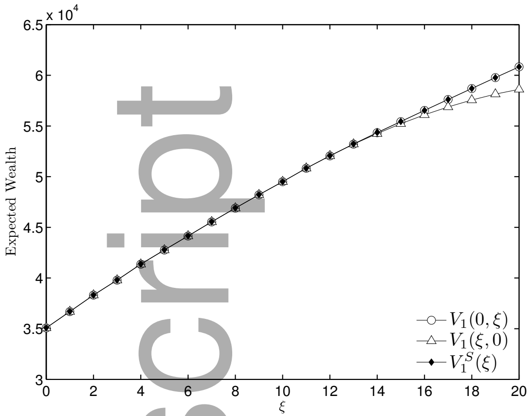
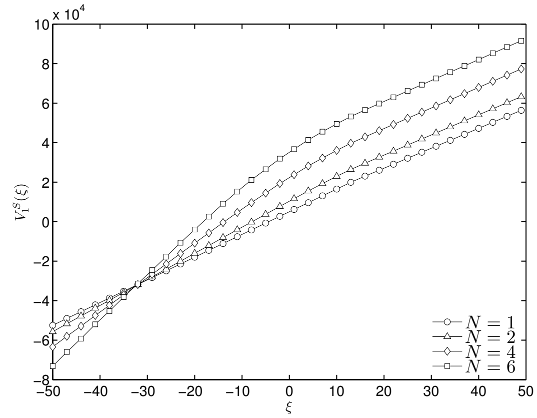


Figure 7 **Function** $V_1^S(\xi)$ **for** $N=1,2,4,6$ **with** $D\sim\mathcal{U}[0,20]$



almost imperceptible which implies that the upper bound is fairly tight for the state of high cash level y (or low product level x). Moreover, the bound gap between $V_1^S(\xi)$ and $V_1(\xi,0)$ gets larger as ξ increases.

Figure 7 depicts the effect of N on $V_{1:N}^S(\xi)$, for $N=1,2,4,6$. Observe that for any N, $V_{1:N}^S(\xi)$ is increasing and jointly concave in ξ. Observe further that in the top-right of Figure 7, $V_{1:N}^S(\xi)$ is increasing in N for large values of ξ, e.g., $\xi>10$; however, in the far bottom-left of the figure, $V_{1:N}^S(\xi)$ is decreasing in N for sufficiently small ξ, e.g., $\xi<-30$. Finally, we conclude that operational decisions play a more important role than financial ones for the middle range of ξ, e.g., $\xi\in(-30,10)$, since the slope at each N in the interval is significantly larger than elsewhere, and gets larger as N increases.

Table 2 displays the *optimal value function* (Opt.) and its lower and upper bounds for $D\sim\mathcal{U}[0,20]$ and $D\sim\mathcal{U}[6,14]$, $N=6,12$ and selected product level state $x=0,7,14$. In this case, the lower bound is obtained from myopic policy (II), since it performs better than myopic policy (I). Observe that each bound provides a fairly tight approximation of the optimal value with $\Delta^{(\mathscr{X})}\%$ less than 1%. It is important to point out that, compared with the myopic policies, the sell-back policy provides a better approximation for the value function when the initial net worth is not too high, e.g., for $x=0$ and 7. However, the opposite holds when the net worth gets larger, e.g., $x=14$. Observe further that, as the *c.v.* of the demand distribution decreases from $\mathcal{U}[0,20]$ to $\mathcal{U}[6,14]$, the optimal value function and its lower and upper bounds increase, due to the reduced volatility of the demand. Moreover, the lower bound gap $\Delta^{(II)}$ gets smaller, whereas this change is not evident for its upper bound gap Δ^S. Note that, for each N and each demand distribution, the bound gap Δ^S gets larger as x increases, whereas this behavior is not evident for $\Delta^{(II)}$. Finally, as the time horizon N gets longer, the optimal value function and its lower and upper bounds get larger, and the bound gap $\Delta^{\mathscr{X}}$ gets larger as well for both bounds $\mathscr{X} \in \{II, S\}$. However, such behavior of the corresponding gap percentage $\Delta^{(\mathscr{X})}\%$ is not evident.

Table 2 The Lower and Upper Bounds for the Value Function $V_1(x, 0)$

		$D \sim \mathcal{U}[0,20]$							$D \sim \mathcal{U}[6,14]$						
		Opt.	LowerBound			UpperBound			Opt.	LowerBound			UpperBound		
N	x	V_1	$V_1^{(II)}$	$\Delta^{(II)}$	$\Delta^{(II)}\%$	V_1^S	Δ^S	$\Delta^S\%$	V_1	$V_1^{(II)}$	$\Delta^{(II)}$	$\Delta^{(II)}\%$	V_1^S	Δ^S	$\Delta^S\%$
	0	35074	35016	58	0.16%	35080	-6	-0.02%	49428	49386	43	0.09%	49428	0	0.00%
6	7	45542	45435	107	0.24%	45550	-9	-0.02%	58575	58536	39	0.07%	58575	0	0.00%
	14	54248	54200	48	0.09%	54355	-107	-0.20%	66076	66036	40	0.06%	66634	-558	-0.84%
	0	75888	75693	194	0.26%	75923	-36	-0.05%	103872	103760	112	0.11%	103872	0	0.00%
12	7	87273	87057	216	0.25%	87290	-18	-0.02%	113564	113464	100	0.09%	113564	0	0.00%
	14	96528	96410	118	0.12%	96660	-132	-0.14%	121521	121419	102	0.08%	122114	-592	-0.49%

8.4. Impact of Maximal Loan Constraint

In this subsection, we numerically gauge the impact of the loan maximum L' which limits the firm borrowing to amounts to purchase at most L' product units. In this study, we assume $D \sim \mathcal{U}[0,20]$ and $N = 6$. Table 3 displays the optimal expected net worth and the performances of both myopic policies for $y = -10$ and varying x, with $L' = 5, 10, \infty$.

Table 3 Impact of Loan Miximum L'

	$L'=5$					$L'=10$					$L'=\infty$				
x	V	$V^{(I)}$	$\Delta^{(I)}\%$	$V^{(II)}$	$\Delta^{(II)}\%$	V	$V^{(I)}$	$\Delta^{(I)}\%$	$V^{(II)}$	$\Delta^{(II)}\%$	V	$V^{(I)}$	$\Delta^{(I)}\%$	$V^{(II)}$	$\Delta^{(II)}\%$
0	17026	7606	55.33%	9322	45.25%	17154	10761	37.27%	16745	2.38%	17163	10784	37.17%	16972	1.11%
2	21008	13472	35.87%	16285	22.49%	21106	14497	31.31%	20809	1.41%	21113	14511	31.27%	20854	1.23%
4	24511	18738	23.55%	22260	9.18%	24586	19255	21.68%	24488	0.40%	24591	19261	21.67%	24506	0.35%
6	28206	22063	21.78%	27447	2.69%	28257	22645	19.86%	28142	0.40%	28260	22649	19.86%	28152	0.38%
8	31841	27481	13.69%	31422	1.32%	31873	27595	13.42%	31785	0.27%	31875	27597	13.42%	31793	0.26%
10	35052	31644	9.72%	34820	0.66%	35073	31702	9.61%	35012	0.17%	35074	31704	9.61%	35015	0.17%
12	38156	34619	9.27%	38012	0.38%	38170	34668	9.17%	38121	0.13%	38171	34670	9.17%	38124	0.12%
14	40620	37155	8.53%	40495	0.31%	40634	37196	8.46%	40577	0.14%	40634	37198	8.46%	40579	0.14%
16	42607	39160	8.09%	42499	0.25%	42621	39201	8.02%	42570	0.12%	42622	39203	8.02%	42572	0.12%
18	44155	40750	7.71%	44060	0.22%	44170	40791	7.65%	44121	0.11%	44171	40794	7.65%	44123	0.11%
20	45214	41911	7.30%	45130	0.18%	45230	41953	7.25%	45187	0.10%	45232	41955	7.24%	45189	0.09%

We make the following observations. First, as L' increases, both myopic policies perform better. The impact of L' on the performance of each policy becomes significant when the total net worth ξ is relatively small and L' is low. For example, when $x = 0$ and $L' = 5$, $\Delta^{(I)}\% = 55\%$ and $\Delta^{(II)}\% = 45\%$, both of which are much higher than in the other cases. Furthermore, myopic policy (II) always outperforms myopic policy (I). This can be explained by the uniform distribution of the demand; cf. the discussion in §8.2. Finally, both myopic policies perform better as x increases (i.e., as the net worth ξ increases).

9. Concluding Remarks

In this paper, we studied the optimal operational-financial policy, for a single-item inventory system that admits both interest-bearing loans and interest-earning deposits, with goal of maximizing the net worth of the firm. We showed that the optimal ordering policy, called (α_n, β_n) policy, for each period is characterized by two threshold parameters. We presented two myopic policies which provide a lower and an upper bound, respectively, for the threshold parameters. The two myopic policies provide lower bounds of the value function. We further introduced an upper bound for the value function via the assumption that the firm can sell back unneeded inventory to the supplier.

The results of this paper suggest several directions for further research. These include (i) addition of a fixed ordering cost and/or a fixed financial transaction cost; (ii) addition of a fixed system operating cost proportional to the elapsed time; and (iii) non-risk neutral decision making. It would be of interest to analyze the risk-induced performance of the system, e.g., credit rollover risk and bankruptcy probabilities; cf. Babich et al. (2012), Gong et al. (2014), Shi (2016) and Li et al. (2013).

Acknowledgments

The authors are grateful to Dr. John Birge (the Departmental Editor), the Senior Editor and the reviewers for valuable comments. The authors are also indebted to the valuable comments and useful suggestions provided by Dr. Matthew Sobel at Case Western Reserve University, and Dr. Xiuli Chao at the University of Michigan. The first author gratefully acknowledges support for this project from the National Science Foundation (CMMI-14-50743). The third author thanks NJIT for financial support of this research.

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10. Appendix A: Proofs

Proof of Lemma 1. The continuity of $G(q,x,y)$ follows immediately from Eq. (7). We next prove its concavity via examining its first-order and second-order derivatives. To this end, differentiating Eq. (7) via Leibniz's integral rule yields

$$\frac{\partial}{\partial q}G(q,x,y) = \begin{cases} p - c\cdot(1+i) - (p-s)F(q+x) & \text{if } q<y, \\ p - c\cdot(1+\ell) - (p-s)F(q+x) & \text{if } q>y. \end{cases} \tag{43}$$

Therefore, for $q>y$ or $q<y$

$$\frac{\partial^2}{\partial q^2}G(q,x,y) = -(p-s)f(q+x). \tag{44}$$

The concavity in q now readily follows since $\frac{\partial^2}{\partial q^2}G(q,x,y)\le 0$ by Eq. (44).

Although $G(q,x,y)$ is not first-order and/or second-order differentiable at certain points (e.g., at $q=y$), we can still consider its derivatives to show its increasing or decreasing properties, which allows us to study the optimal solution. Such consideration will be used throughout the paper.

The increasing property of $G(q,x,y)$ in x and y can be shown by taking the first-order derivatives using Eq. (7):

$$\begin{aligned} \frac{\partial}{\partial x}G(q,x,y) &= p\bar{F}(q+x) + sF(q+x) > 0, \\ \frac{\partial}{\partial y}G(q,x,y) &= c\cdot\left[(1+i)\mathbf{1}_{\{q<y\}} + (1+\ell)\mathbf{1}_{\{q>y\}}\right] > 0. \end{aligned}$$

The joint concavity of $G(q,x,y)$ in x and y can be established by computing the second-order derivatives below using again Eq. (7),

$$\begin{aligned} \frac{\partial^2}{\partial x^2}G(q,x,y) &= -(p-s)f(q+x) < 0, \\ \frac{\partial^2}{\partial y^2}G(q,x,y) &= 0, \\ \frac{\partial^2}{\partial x\partial y}G(q,x,y) &= 0. \end{aligned}$$

Thus the Hessian matrix is negative semi-definite and the proof is complete. □

Proof of Theorem 1. For any initial state (x,y), Lemma 1 implies that there exists a unique optimal order quantity $q^*(x,y)$, such that the expected net worth value function $G(q,x,y)$ is maximized. To prove Eq. (13), we investigate the first-order derivative in Eq. (43). Figure 8 illustrates its functional structure for the three cases of $x+y$ in Eq. (13).

a) If $x+y<\alpha$, then $G(q,x,y)$ is strictly increasing in q as long as $q+x\le\alpha$, and decreasing thereafter, while $\partial G(q,x,y)/\partial q = 0$ for $q+x=\alpha$; cf. Figure 8 (a). It follows that in this case, $q^*=\alpha-x$ if $\alpha\ge x$, and $q^*=0$ otherwise.

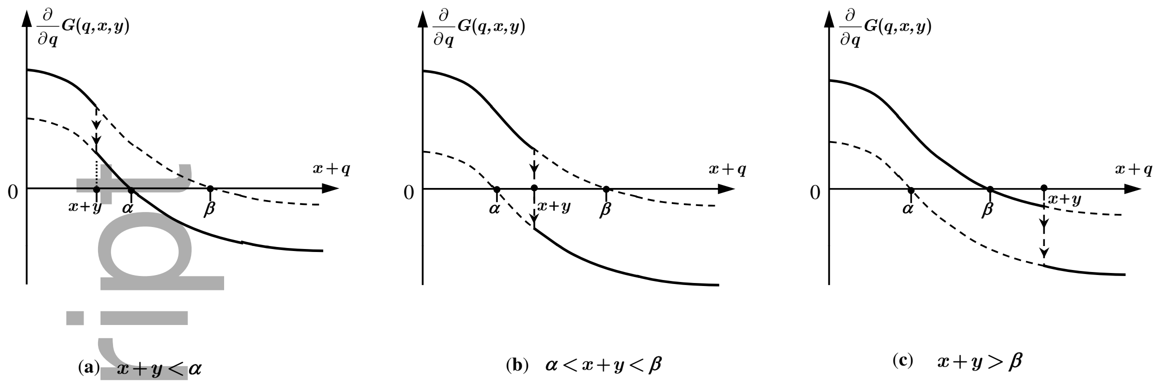

Figure 8 **Functional Structure for the Derivative of** $G(q,x,y)$ **with Respective to** q

b) If $\alpha \leq x+y<\beta$, then $G(q,x,y)$ is strictly increasing in q until $q=y$ in the real space $q\in\Re$, and decreasing thereafter; cf. Figure 8 (b). The optimal order quantity is then $q^*(x,y)=y$ if $y\geq 0$, and $q^*(x,y)=0$ otherwise.

c) If $x+y\geq\beta$, then $G(q,x,y)$ as function of $x+q$ is strictly increasing until β, and decreasing thereafter; cf. Figure 8 (c). The optimal quantity after ordering is the value of $q+x$ closest to β. Thus, the optimal order quantity is $(\beta-x)^+$.

This completes the proof. □

Proof of Theorem 2 . Part (i) follows by substituting q^* given by Eq. (13) into Eq. (7).

For part (ii), the increasing property of V can be readily justified. To show the concavity of V, note that by Lemma 1, $G(q,x,y)$ is concave in q, x and y. Maximizing G over q and using Proposition A.3.10 of Zipkin (2000), p. 436, we have that the concavity in x and y is preserved and the proof is complete. □

Proof of Corollary 1. The result follows readily by setting $x=y=0$ in Eq. (14). □

Proof of Lemma 2. We prove the result by backward induction. In particular, in each iteration, we will prove properties (1) and (2) by recursively repeating two steps: deducing the property of G_n from the property of V_{n+1} and obtaining the property of V_n from the property of G_n. Throughout the proof, we denote the transpose of a matrix or a vector w by w^T. The *Hessian Matrix* (if it exists) of a function $G=G(x,y)$ will be denoted by $\mathscr{H}^G(x,y)$. For example, the Hessian Matrix of $V_{n+1}(x_{n+1},y_{n+1})$ is denoted by

$$\mathscr{H}^{V_{n+1}}=\begin{bmatrix}\frac{\partial^2 V_{n+1}}{\partial x_{n+1}\partial x_{n+1}} & \frac{\partial^2 V_{n+1}}{\partial x_{n+1}\partial y_{n+1}}\\ \frac{\partial^2 V_{n+1}}{\partial y_{n+1}\partial x_{n+1}} & \frac{\partial^2 V_{n+1}}{\partial y_{n+1}\partial y_{n+1}}\end{bmatrix}. \tag{45}$$

1) For V_N, we have a one-period problem. In this case, the result for the function $G_N(z_N, x_N, y_N)$ is obtained by applying Lemma 1 with $z_n = x_n + q_n$, and the result for $V_N(x_N, y_N)$ is given by applying Theorem 2.

2) For $n = 1, 2, \cdots, N-1$, we prove the results recursively via the following two steps:

Step 1. We show that $G_n(z_n, x_n, y_n)$ is increasing in y_n and concave in z_n, x_n and y_n, if $V_{n+1}(x_{n+1}, y_{n+1})$ is increasing in y_{n+1} and concave in x_{n+1} and y_{n+1}.

We first compute the partial derivatives that will be used in the sequel. By Eq. (16), we have:

$$\frac{\partial x_{n+1}}{\partial z_n} = \frac{\partial x_{n+1}}{\partial x_n} = \mathbf{1}_{\{z_n > D_n\}}, \tag{46}$$

$$\frac{\partial x_{n+1}}{\partial y_n} = 0. \tag{47}$$

Similarly, from Eq. (17) we obtain:

$$\frac{\partial y_{n+1}}{\partial z_n} = p'_n \mathbf{1}_{\{z_n < D_n\}} - h'_n \mathbf{1}_{\{z_n > D_n\}} - c'_n \left[(1 + i_n)\mathbf{1}_{\{z_n < x_n + y_n\}} + (1 + \ell_n)\mathbf{1}_{\{z_n > x_n + y_n\}} \right]; \tag{48}$$

$$\frac{\partial y_{n+1}}{\partial x_n} = p'_n \mathbf{1}_{\{z_n < D_n\}} - h'_n \mathbf{1}_{\{z_n > D_n\}}; \tag{49}$$

$$\frac{\partial y_{n+1}}{\partial y_n} = c'_n \left[(1 + i_n)\mathbf{1}_{\{z_n < x_n + y_n\}} + (1 + \ell_n)\mathbf{1}_{\{z_n > x_n + y_n\}} \right]. \tag{50}$$

From Eqs. (46)-(50), it readily follows that the second-order derivatives of x_{n+1} and y_{n+1} with respect to z_n, x_n and y_n are all zero. In the sequel we interchange differentiation and integration in several places, which can be justified by the *Lebesgue's Dominated Convergence* Theorem; cf. Bartle (1995).

The increasing property of $G_n(z_n, x_n, y_n)$ in y_n can be established by taking the first-order derivative of Eq. (7) with respect to y_n, yielding

$$\begin{aligned}\frac{\partial}{\partial y_n} G_n(z_n, x_n, y_n) &= \mathbb{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial y_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n}\right] \\ &= \mathbb{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n}\right] \geq 0,\end{aligned}$$

where the second equality holds, since $\partial x_{n+1}/\partial y_n = 0$, by Eq. (47), and the inequality holds by Eq. (50) and the induction hypothesis that V_{n+1} is increasing in y_{n+1}.

To prove the concavity of $G_n(z_n, x_n, y_n)$ in z_n, x_n and y_n, we compute its Hessian matrix and show that it is negative semi-definite. To this end, we compute the first-order partial derivatives of V_{n+1} as

$$\frac{\partial G_n}{\partial z_n} = \mathbb{E}\left[\frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial z_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial z_n}\right]; \tag{51}$$

$$\frac{\partial G_n}{\partial x_n} = \mathbb{E}\left[\frac{\partial V_{n+1}(x_{n+1},y_{n+1})}{\partial x_{n+1}}\frac{\partial x_{n+1}}{\partial x_n} + \frac{\partial V_{n+1}(x_{n+1},y_{n+1})}{\partial y_{n+1}}\frac{\partial y_{n+1}}{\partial x_n}\right]; \tag{52}$$

$$\frac{\partial G_n}{\partial y_n} = \mathbb{E}\left[\frac{\partial V_{n+1}(x_{n+1},y_{n+1})}{\partial x_{n+1}}\frac{\partial x_{n+1}}{\partial y_n} + \frac{\partial V_{n+1}(x_{n+1},y_{n+1})}{\partial y_{n+1}}\frac{\partial y_{n+1}}{\partial y_n}\right], \tag{53}$$

and consequently, the second-order derivatives are

$$\frac{\partial^2 G_n}{\partial z_n \partial z_n} = \mathbb{E}\left[\left[\frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n}\right]\cdot \mathscr{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n}\right]^T\right];$$

$$\frac{\partial^2 G_n}{\partial z_n \partial x_n} = \mathbb{E}\left[\left[\frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n}\right]\cdot \mathscr{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right]^T\right];$$

$$\frac{\partial^2 G_n}{\partial z_n \partial y_n} = \mathbb{E}\left[\left[\frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n}\right]\cdot \mathscr{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right]^T\right];$$

$$\frac{\partial^2 G_n}{\partial x_n \partial x_n} = \mathbb{E}\left[\left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right]\cdot \mathscr{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right]^T\right];$$

$$\frac{\partial^2 G_n}{\partial x_n \partial y_n} = \mathbb{E}\left[\left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right]\cdot \mathscr{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right]^T\right];$$

$$\frac{\partial^2 G_n}{\partial y_n \partial y_n} = \mathbb{E}\left[\left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right]\cdot \mathscr{H}^{V_{n+1}} \cdot \left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right]^T\right],$$

where, by Eqs. (46) - (50), the terms involved in the second-order derivatives of x_{n+1} and y_{n+1} with respect to x_n and y_n vanish and the Hessian matrix $\mathscr{H}^{V_{n+1}}$ is given by Eq. (45). Thus, the Hessian matrix of G_n is:

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$$\mathscr{H}^{G_n}(z_n, x_n, y_n) = \begin{bmatrix} \frac{\partial^2 G_n}{\partial z_n \partial z_n} & \frac{\partial^2 G_n}{\partial z_n \partial x_n} & \frac{\partial^2 G_n}{\partial z_n \partial y_n} \\ \frac{\partial^2 G_n}{\partial x_n \partial z_n} & \frac{\partial^2 G_n}{\partial x_n \partial x_n} & \frac{\partial^2 G_n}{\partial x_n \partial y_n} \\ \frac{\partial^2 G_n}{\partial y_n \partial z_n} & \frac{\partial^2 G_n}{\partial y_n \partial x_n} & \frac{\partial^2 G_n}{\partial y_n \partial y_n} \end{bmatrix}. \tag{54}$$

To prove it is negative semi-definite, we consider the quadratic function below for any real w_1, w_2 and w_3,

$$[w_1, w_2, w_3]\cdot \mathscr{H}^{G_n} \cdot [w_1, w_2, w_3]^T = \mathbb{E}\left[\vec{v}\cdot \mathscr{H}^{V_{n+1}} \cdot \vec{v}^T\right], \tag{55}$$

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where $\vec{v} = w_1\left[\frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n}\right] + w_2\left[\frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n}\right] + w_3\left[\frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n}\right]$. Since by the induction hypothesis $\mathscr{H}^{V_{n+1}}$ is negative semi-definite, we have for any w_1, w_2 and w_3

$$\vec{v}\cdot \mathscr{H}^{V_{n+1}} \cdot \vec{v}^T \le 0,$$

and this implies that the left side of Eq. (55) is non-positive. This completes the proof for Step 1.

Step 2. We show that $V_n(x_n, y_n)$ is concave in (x_n, y_n), if $G_n(z_n, x_n, y_n)$ is concave.

Since $G_n(z_n, x_n, y_n)$ is concave in z_n and x_n and y_n, $V_n(x_n, y_n) = \max_{z_n \geq x_n} G_n(z_n, x_n, y_n)$ is concave in x_n, y_n by the fact that concavity is reserved under maximization; cf. Proposition A.3.10 in Zipkin (2000).

Finally, the induction proof is complete with steps 1 and 2. $\square$

Proof of Theorem 3. Given state (x_n, y_n) at the beginning of period $n \in \mathcal{N}$, we consider the following first-order equation:

$$\frac{\partial}{\partial z_n} G_n(z_n, x_n, y_n) = 0, \tag{56}$$

where $\partial G_n(z_n, x_n, y_n)/\partial z_n$ is given by Eq. (51). Substituting Eqs. (46) and (48) into Eq. (51) we consider the following cases:

(1) for $z_n \leq x_n + y_n$,

$$\frac{\partial G_n}{\partial z_n} = \mathbb{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}} \mathbf{1}_{\{z_n > D_n\}} + \frac{\partial V_{n+1}}{\partial y_{n+1}} \left(p'_n \mathbf{1}_{\{z_n < D_n\}} - h'_n \mathbf{1}_{\{z_n > D_n\}} - c'_n(1 + i_n)\right) \Big| x_n, y_n\right]; \tag{57}$$

beta

(2) for $z_n > x_n + y_n$,

$$\frac{\partial G_n}{\partial z_n} = \mathbb{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}} \mathbf{1}_{\{z_n > D_n\}} + \frac{\partial V_{n+1}}{\partial y_{n+1}} \left(p'_n \mathbf{1}_{\{z_n < D_n\}} - h'_n \mathbf{1}_{\{z_n > D_n\}} - c'_n(1 + \ell_n)\right) \Big| x_n, y_n\right], \tag{58}$$

alpha

where for each case above, the random variables x_{n+1} and y_{n+1} within the expectations are given by Eqs. (16) and (17), respectively.

The results follow readily by setting the right sides of Eqs. (57) and (58) equal to zero and simple manipulations. Note that $\partial G_n(z_n, x_n, y_n)/\partial z_n$ is monotonically decreasing in z_n due to its concavity, shown in part (1) of Lemma 2. Consequently, there are unique solutions, α_n and β_n, for Eqs. (58) and (57), respectively.

To show $\alpha_n \leq \beta_n$, we further compare Eqs. (57) and (58). For notational convenience, we simply denote Eq. (57) by $\frac{\partial G_n^+}{\partial z_n}$ and Eq. (58) by $\frac{\partial G_n^-}{\partial z_n}$, both of which are monotonically decreasing in z_n. It can be shown that $\frac{\partial G_n^-}{\partial z_n} = \frac{\partial G_n^+}{\partial z_n} - \mathbb{E}[\frac{\partial V_{n+1}}{\partial y_{n+1}} \cdot c'_n \cdot (\ell_n - i_n)]$, where the last term, $\mathbb{E}[\frac{\partial V_{n+1}}{\partial y_{n+1}} \cdot c'_n \cdot (\ell_n - i_n)]$, is nonnegative since $\frac{\partial V_{n+1}}{\partial y_{n+1}} \geq 0$ and $\ell_n > i_n$. Setting $z_n = \beta_n$ in the above equation, we have $\frac{\partial G_n^-}{\partial z_n}|_{z_n = \beta_n} = \frac{\partial G_n^+}{\partial z_n}|_{z_n = \beta_n} - \mathbb{E}[\frac{\partial V_{n+1}}{\partial y_{n+1}} \cdot c'_n \cdot (\ell_n - i_n)] < 0$. Because $\frac{\partial G_n^-}{\partial z_n}$ is decreasing in z_n, we must have $\alpha_n \leq \beta_n$; otherwise, $\frac{\partial G_n^-}{\partial z_n}|_{z_n = \alpha_n} < 0$ would contradict $\frac{\partial G_n^-}{\partial z_n}|_{z_n = \alpha_n} = 0$. This completes the proof. $\square$

Proof Corollary 2. For period N, the independence of x_N or y_N is obvious, since this is a single-period setting. For period $n < N$, revisit Eqs. (57) and (58), and note that x_{n+1} is independent of (x_n, y_n) by Eq. (16) while y_{n+1} is dependent of $x_n + y_n$ by Eq. (17). Consequently, α_n and β_n (which are implicitly given by Eqs. (57) and (58)) are dependent on $\xi_n = x_n + y_n$ only, which completes the proof. $\square$

Proof of Lemma 3. To prove (a), we only give the proof for the case of increasing $f(x)$ and $g(x)$, since the same argument can be applied in the case of decreasing $f(x)$ and $g(x)$.

Let X' be another random variable such that X' and X are i.i.d. Since $f(x)$ and $g(x)$ are increasing, we always have

$$[f(X)-f(X')][g(X)-g(X')]\geq 0.$$

因为独立同分布，所以波动性一样

Taking expectations with respect to X and X' yields

$$\begin{aligned}
&\mathbb{E}\left[[f(X)-f(X')][g(X)-g(X')]\right]\\
&=\mathbb{E}[f(X)g(X)+f(X')g(X')-f(X')g(X)-f(X)g(X')]\\
&=\mathbb{E}[f(X)g(X)]+\mathbb{E}[f(X')g(X')]-\mathbb{E}[f(X')]\mathbb{E}[g(X)]-\mathbb{E}[f(X)]\mathbb{E}[g(X')]\\
&=2\mathbb{E}[f(X)g(X)]-2\mathbb{E}[f(X)]\mathbb{E}[g(X)]\geq 0.
\end{aligned}$$

The result of part (a) readily follows from the above.

In a similar vein, we can prove part (b) via changing the direction of the inequality above. $\square$

Proof of Theorem 4. We only prove the result for α_n, since the same argument can be applied to prove the result for β_n.

In view of Eq. (57), α_n is the unique solution of the equation

$$\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\left(p_n'\mathbf{1}_{\{\alpha_n<D_n\}}-h_n'\mathbf{1}_{\{\alpha_n>D_n\}}-c_n'(1+\ell_n)\right)\right]=-\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}}\mathbf{1}_{\{\alpha_n>D_n\}}\right]. \tag{59}$$

Since $\frac{\partial V_{n+1}}{\partial x_{n+1}}\geq 0$ by Lemma 2 part (2), the equation above is negative, which implies

$$\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\left(p_n'\mathbf{1}_{\{\alpha_n<D_n\}}-h_n'\mathbf{1}_{\{\alpha_n>D_n\}}-c_n'(1+\ell_n)\right)\right]\leq 0. \tag{60}$$

Further, note that for any realization of demand $D_n=d>0$, the two terms on the left-hand side of Eq. (60),

$$\frac{\partial V_{n+1}(x_{n+1}(d),y_{n+1}(d))}{\partial y_{n+1}(d)},$$

and

$$p_n'\mathbf{1}_{\{\alpha_n<d\}}-h_n'\mathbf{1}_{\{\alpha_n>d\}}-c_n'(1+\ell_n),$$

are both increasing in d. Specifically, the first term is increasing by the concavity of V_{n+1} [cf. Lemma 2, part (2)] and Eq. (17). Thus, by Lemma 3 and Eq. (60), one has,

$$\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}}\right]\mathbb{E}\left[p_n'\mathbf{1}_{\{\alpha_n<D_n\}}-h_n'\mathbf{1}_{\{\alpha_n>D_n\}}-c_n'(1+\ell_n)\right]\leq 0. \tag{61}$$

Since $\frac{\partial V_{n+1}}{\partial y_{n+1}}\geq 0$ by Lemma 2 part (2), the above inequality implies

$$\mathbb{E}\left[p_n\mathbf{1}_{\{\alpha_n<D_n\}}-h_n\mathbf{1}_{\{\alpha_n>D_n\}}-c_n(1+\ell_n)\right]\leq 0,$$

which, after simple algebra, is equivalent to

$$p_n - c_n \cdot (1+\ell_n) - (p_n - s_n)F_n(\alpha_n) \le 0.$$

The above further simplifies to

$$F(\alpha_n) \ge \frac{p_n - c_n \cdot (1+\ell_n)}{p_n - s_n}.$$

By Eqs. (21) and (23), the right-hand side above is $F_n(\underline{\alpha}_n)$. Thus, we have $F_n(\alpha_n) \ge F_n(\underline{\alpha}_n)$, which completes the proof for $\alpha_n \ge \underline{\alpha}_n$ due to the increasing property of $F_n(\cdot)$. □

Proof of Proposition 1. To prove part (i), it suffices to prove that $V_n(x_n, y_n) \le V_n(x_n - d, y_n + d)$ for arbitrarily small $d > 0$. To this end, consider the initial state $(x_n - d, y_n + d)$. The firm can then always purchase d units without any additional cost to reset the initial state to be (x_n, y_n). This means that $V_n(x_n, y_n) \le V_n(x_n - d, y_n + d)$.

The proof for part (ii) follows from part (i). Considering the derivative of $V_n(x_n - d, y_n + d)$ with respect to d, we have

$$\lim_{d\to 0} \frac{V_n(x-d, y+d) - V_n(x,y)}{d} = -\partial V_n(x,y)/\partial x + \partial V_n(x,y)/\partial y \ge 0,$$

where the inequality follows from part (i). Finally, part (ii) readily follows by rearranging the above equation. □

Proof of Proposition 2. To prove Part (i), we note that:

$$V_n(x_n, y_n) = \max_{z_n \ge x_n} \mathbb{E}\left[V_{n+1}(x_{n+1}, y_{n+1})\right] \le \mathbb{E}\left[V_{n+1}(0, x_{n+1} + y_{n+1})\right] = V_n^L(x_n, y_n)$$

where the inequality above follows from Proposition 1(i) for period $n+1$ with $d = x_{n+1}$.

For part (ii), we only show that $\alpha_n^L \ge \alpha_n$, since the same argument can be applied to prove $\beta_n^L \ge \beta_n$.

To begin with, note that by Lemma 2, one can show that $\mathbb{E}[V_{n+1}(0, \xi_{n+1}(z_n))]$ is concave in z_n and its derivative is decreasing in z_n. Therefore, $z_n = \alpha_n^L$ is the unique solution of $\mathbb{E}[V_{n+1}(0, \xi_{n+1}(z_n))] = 0$. In addition, the derivative of $\mathbb{E}[V_{n+1}(0, \xi_{n+1}(z_n))]$ with respect to z_n is positive for $z_n < \alpha_n^L$ and negative for $z_n > \alpha_n^L$. Consequently, it suffices to show that at $z_n = \alpha_n$ the following is true

$$\frac{d}{dz_n}\mathbb{E}\Big[V_{n+1}\big(0,\, x_{n+1}(z_n) + y_{n+1}(z_n)\big)\Big]\Big|_{z_n=\alpha_n} \ge 0. \tag{62}$$

It is readily shown by backward induction that, $dV_{n+1}\big(0,\, x_{n+1}(z_n) + y_{n+1}(z_n)\big)/dz_n$ is bounded for every z_n in terms of a function of D_n (for the last period, $R_N(D_N, q_N, x_N) + K_N(q_N, y_N)$ is bounded for any D_N). The previous condition admits the interchange between differentiation and integration; cf. Widder (1989). Accordingly, we have the following:

$$\begin{aligned}\frac{d\mathbb{E}[V_{n+1}(0,\xi_{n+1})]}{dz_n} &= \mathbb{E}[\frac{dV_{n+1}(0,\xi_{n+1})}{dz_n}] \\ &= \mathbb{E}\left[\frac{\partial V_{n+1}}{\partial \xi_{n+1}}\mathbf{1}_{\{z_n>D_n\}} + \frac{\partial V_{n+1}}{\partial \xi_{n+1}}\left(p_n'\mathbf{1}_{\{z_n<D_n\}} - h_n'\mathbf{1}_{\{z_n>D_n\}} - c_n'(1+\ell_n)\right)\right], \\ &\geq \mathbb{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}}\mathbf{1}_{\{z_n>D_n\}} + \frac{\partial V_{n+1}}{\partial y_{n+1}}\left(p_n'\mathbf{1}_{\{z_n<D_n\}} - h_n'\mathbf{1}_{\{z_n>D_n\}} - c_n'(1+\ell_n)\right)\right] = 0,\end{aligned}$$

where for simplicity we write $\frac{\partial V_{n+1}}{\partial \xi_{n+1}} := \frac{\partial V_{n+1}(0,\xi_{n+1})}{\partial \xi_{n+1}}$, $\frac{\partial V_{n+1}}{\partial x_{n+1}} := \frac{\partial V_{n+1}(x_{n+1},y_{n+1})}{\partial x_{n+1}}$ and $\frac{\partial V_{n+1}}{\partial y_{n+1}} := \frac{\partial V_{n+1}(x_{n+1},y_{n+1})}{\partial y_{n+1}}$.

The inequality above is justified using backward induction as follows.

First, since the marginal contribution of profit of an additional unit of cash to $V_{n+1}(0,\xi_{n+1})$ is larger than that of $V_{n+1}(x_{n+1},y_{n+1})$, where $\xi_{n+1} = x_{n+1}+y_{n+1}$, one has

$$\partial V_{n+1}(x_{n+1},\, y_{n+1})/\partial y_{n+1} \leq \partial V_{n+1}(0,\, \xi_{n+1})/\partial \xi_{n+1};$$
$$\partial V_{n+1}(x_{n+1},\, y_{n+1})/\partial x_{n+1} \leq \partial V_{n+1}(0,\, \xi_{n+1})/\partial \xi_{n+1}.$$

Second, since $z_n = \alpha_n$ is the unique optimum of $\mathbb{E}[V_{n+1}(x_{n+1},y_{n+1})]$, its derivative at $z_n = \alpha_n$ is zero. This proves Eq. (62), thereby completing the proof. □

Proof of Proposition 3. For period N, the result readily follows from the optimal solution of the single-period model. For all other periods, we only prove the result for $\bar{\alpha}_n \geq \alpha_n$, since a similar argument can be used to prove $\bar{\beta}_n \geq \beta_n$ by replacing ℓ_n with i_n.

By Proposition 2, we have $\alpha_n \leq \alpha_n^L$ and α_n^L is determined by differentiating Eq. (30) and setting the derivative to zero, that is,

$$\mathbb{E}\left[\frac{\partial V_{n+1}(0,\xi_{n+1})}{\partial \xi_{n+1}}\left(\mathbf{1}_{\{\alpha_n^L>D_n\}} + p_n'\mathbf{1}_{\{\alpha_n^L<D_n\}} - h_n'\mathbf{1}_{\{\alpha_n^L>D_n\}} - c_n'(1+\ell_n)\right)\right] = 0. \tag{63}$$

For any realization of demand $D_n = d > 0$ the term

$$\frac{\partial V_{n+1}(0,\xi_{n+1}(d))}{\partial \xi_{n+1}(d)}$$

is decreasing in d by the concavity of V_{n+1} [cf. Lemma 2 part (2)] and the fact that ξ_{n+1} is increasing in d by Eqs. (16)-(17).

Furthermore, the term

$$\begin{aligned}&\mathbf{1}_{\{\alpha_n>d\}} + p_n'\mathbf{1}_{\{\alpha_n<d\}} - h_n'\mathbf{1}_{\{\alpha_n>d\}} - c_n'(1+\ell_n) \\ &= p_n' - (p_n'+h_n'-1)\mathbf{1}_{\{\alpha_n>d\}} - c_n'(1+\ell_n),\end{aligned}$$

is increasing in d.

By Eq. (63) and Lemma 3, one has

$$\mathbb{E}\left[\frac{\partial V_{n+1}(0,\xi_{n+1})}{\partial \xi_{n+1}}\right]\cdot \mathbb{E}\left[\mathbf{1}_{\{\alpha_n^L>D_n\}}+p_n'\mathbf{1}_{\{\alpha_n^L<D_n\}}-h_n'\mathbf{1}_{\{\alpha_n^L>D_n\}}-c_n'(1+\ell_n)\right]\ge 0.$$

Since $\partial V_{n+1}(0,\xi_{n+1})/\partial\xi_{n+1}\ge 0$ by Lemma 2 part (2), the above inequality implies

$$\mathbb{E}\left[\mathbf{1}_{\{\alpha_n^L>D_n\}}+p_n'\mathbf{1}_{\{\alpha_n^L<D_n\}}-h_n'\mathbf{1}_{\{\alpha_n^L>D_n\}}-c_n'(1+\ell_n)\right]\ge 0, \tag{64}$$

which, after simple algebra, is equivalent to

$$p_n-c_n\cdot(1+\ell_n)-(p_n+h_n-c_{n+1})F_n(\alpha_n^L)\ge 0.$$

The above further simplifies to

$$F(\alpha_n^L)\le\frac{p_n-c_n\cdot(1+\ell_n)}{p_n+h_n-c_{n+1}}.$$

Note that the right-hand side of the above is less than 1 since $c_n(1+\ell_n)+h_n\ge c_{n+1}$ by assumption. Next, by Eqs. (26) and (28), the right-hand side in the above inequality is $F_n(\bar{\alpha}_n)$. Thus, we have $F_n(\bar{\alpha}_n)\ge F_n(\alpha_n^L)$, which means $\bar{\alpha}_n\ge\alpha_n^L$, thereby completing the proof for $\bar{\alpha}_n\ge\alpha_n$, since $\alpha_n^L\ge\alpha_n$ by Proposition 2. □

Proof of Theorem 5. Since $V_n^S(\xi)$ is increasing and concave in ξ, we can prove part (i) by following the proof for Theorem 3 after simplifying the two-variable state (x_n,y_n) to ξ_n. The two threshold values, α_n^S and β_n^S, can be obtained via solving the first-order conditions, under loan with rate ℓ_n and deposit with rate r_n, respectively.

To prove part (ii), we only need to prove the first inequality since the second one has already been proved in Proposition 2. This can be done via backward induction. First, for the induction basis, note that for the last period, N, one has $V_n^S(x+y)=V_n^L(x,y)$. Next, for the induction step, assume that the inequality holds for periods $n+1,n+2,\dots,N$; namely, for any $\xi_{n+1}=x_{n+1}+y_{n+1}$, one has $V_{n+1}^S(\xi_{n+1})\ge V_{n+1}^L(x_{n+1},y_{n+1})$. Noting that ξ_{n+1} is a function of ξ_n, z_n and D_n and taking expectations w.r.t. D_n yields $\mathbb{E}[V_{n+1}^S(\xi_{n+1})]\ge\mathbb{E}[V_{n+1}^L(x_{n+1},y_{n+1})]$. Finally, taking the maximum over z_n, we obtain that the inequality $V_n^S(\xi_n)\ge V_n^L(x_n,y_n)$ holds, thereby completing the proof. □

Proof of Theorem 6. The proof readily follows via a straightforward modification of the proof of Theorem 1. □

Proof of Theorem 7. The proof follows the same steps as in Theorem 1 and Theorem 3, with some minor modifications. □

Proof of Theorem 8. The proof follows readily from the proof of Theorem 3 by replacing the inventory-cash flow equations by Eqs. (36)-(37). □

11. Appendix B: Test-Bed Numerical Study

Tables 4 and 5 display the results of an extensive numerical study of the performance of myopic policies with various time horizons and initial states, under Uniform and $\mathcal{ZIP}$ demand distributions, respectively. Based on the numerical study, we make the following observations: First, myopic policy (II) outperforms policy (I) under Uniform demand, while myopic policy (I) outperforms myopic policy (II) under $\mathcal{ZIP}$ demand. Second, ceteris paribus, each policy's performance improves as the planning horizon, N, grows longer. Third, for each fixed N, each policy's performance improves as the as the total initial net worth $\xi = x + y$ increases. Finally, as $\mathbb{E}[D]$ increases, both policies perform better.

Table 4 Performance of Myopic Policies with Uniform Demand Distributions

		$D \sim \mathcal{U}[2,10]$			$D \sim \mathcal{U}[6,14]$			$D \sim \mathcal{U}[0,20]$		
Horizon N	(x,y)	V_1	$\Delta^{(I)}\%$	$\Delta^{(II)}\%$	V_1	$\Delta^{(I)}\%$	$\Delta^{(II)}\%$	V_1	$\Delta^{(I)}\%$	$\Delta^{(II)}\%$
	$(0,-10)$	-1325	222.62%	115.29%	8860	0.05%	0.04%	1424	174.16%	2.77%
	$(0,0)$	13325	20.75%	10.76%	23333	0.04%	0.04%	16157	14.46%	0.02%
$N=3$	$(0,5)$	19622	7.83%	6.39%	29815	0.04%	0.04%	23102	7.95%	0.04%
	$(10,0)$	24412	4.72%	4.26%	35771	0.04%	0.04%	29456	2.97%	0.02%
	$(10,10)$	34712	3.32%	2.99%	46224	0.04%	0.04%	40379	3.43%	0.05%
	$(0,-10)$	8104	51.06%	2.23%	34045	7.22%	0.53%	17163	37.18%	1.11%
	$(0,0)$	25613	7.32%	0.18%	49428	3.54%	0.09%	35074	13.69%	0.16%
$N=6$	$(0,5)$	32337	2.43%	0.21%	56098	2.06%	0.08%	42766	10.85%	0.14%
	$(10,0)$	37475	1.61%	0.10%	62232	1.02%	0.06%	49503	7.87%	0.13%
	$(10,10)$	48075	1.25%	0.08%	72994	1.10%	0.07%	60834	7.44%	0.10%
	$(0,-10)$	21301	25.68%	1.01%	60460	4.97%	0.38%	35547	26.41%	1.62%
	$(0,0)$	39755	6.09%	0.21%	76266	3.01%	0.10%	55110	14.16%	0.23%
$N=9$	$(0,5)$	46665	2.72%	0.23%	83133	2.00%	0.09%	63216	12.39%	0.18%
	$(10,0)$	51958	2.10%	0.13%	89455	1.26%	0.08%	70202	10.12%	0.17%
	$(10,10)$	62860	1.74%	0.11%	100531	1.29%	0.08%	81877	9.50%	0.11%

Table 5 Performance of Myopic Policies with $\mathcal{ZIP}(\pi,\lambda)$ Demand Distributions

		$D \sim \mathcal{ZIP}(0,5)$			$D \sim \mathcal{ZIP}(0.18,10)$			$D \sim \mathcal{ZIP}(0,10)$		
Horizon N	(x,y)	V_1	$\Delta^{(I)}\%$	$\Delta^{(II)}\%$	V_1	$\Delta^{(I)}\%$	$\Delta^{(II)}\%$	V_1	$\Delta^{(I)}\%$	$\Delta^{(II)}\%$
	$(0,-10)$	-4668	0.00%	2.73%	1824	13.91%	32.51%	6262	0.17%	2.94%
	$(0,0)$	10076	0.00%	1.84%	14674	1.64%	4.05%	18559	0.00%	1.21%
$N=3$	$(0,5)$	16403	0.35%	1.35%	20553	0.71%	2.78%	23992	0.09%	0.96%
	$(10,0)$	20167	0.18%	0.57%	25944	0.27%	1.56%	28986	0.10%	0.62%
	$(10,10)$	30469	0.12%	0.37%	35093	0.19%	1.46%	37680	0.17%	0.68%
	$(0,-10)$	3681	0.19%	13.69%	10306	3.91%	14.60%	20042	0.12%	2.48%
	$(0,0)$	21474	0.74%	2.79%	23130	1.43%	5.94%	30356	0.22%	1.46%
$N=6$	$(0,5)$	28054	0.91%	2.51%	28364	0.92%	4.50%	34646	0.27%	1.40%
	$(10,0)$	31912	0.74%	1.83%	32916	0.70%	2.99%	38617	0.40%	1.00%
	$(10,10)$	42518	0.55%	1.38%	40635	0.59%	2.62%	45517	0.29%	1.08%
	$(0,-10)$	14360	0.87%	8.14%	16384	2.25%	10.89%	27948	0.29%	2.43%
	$(0,0)$	33236	1.08%	3.25%	28396	1.21%	7.10%	36304	0.37%	1.57%
$N=9$	$(0,5)$	40014	1.15%	3.03%	32891	1.01%	7.83%	39707	0.37%	1.56%
	$(10,0)$	43977	0.99%	2.22%	36708	0.85%	4.65%	42932	0.68%	1.22%
	$(10,10)$	54825	0.96%	1.87%	43227	0.74%	2.84%	48383	0.37%	1.25%