Omega 50 (2015) 126–140

Contents lists available at ScienceDirect

Omega

journal homepage: www.elsevier.com/locate/omega

Piecewise linear approximations for the static–dynamic uncertainty strategy in stochastic lot-sizing ☆

Roberto Rossi [a,*,1], Onur A. Kilic [b,2], S. Armagan Tarim [b,c,2,3]

[a] Business School, University of Edinburgh, 29 Buccleuch Place, EH8 9JS Edinburgh, UK
[b] Institute of Population Studies, Hacettepe University, Turkey
[c] Insight Centre for Data Analytics, University College Cork, Ireland

ARTICLE INFO

Article history:
Received 5 February 2014
Accepted 7 August 2014
This manuscript was processed by Associate Editor Kuhn
Available online 20 August 2014

Keywords:
Stochastic lot sizing
Static–dynamic uncertainty
First order loss function
Non-stockout probability
Fill rate
Penalty cost
Piecewise linearisation

ABSTRACT

In this paper, we develop a unified mixed integer linear modelling approach to compute near-optimal policy parameters for the non-stationary stochastic lot sizing problem under static–dynamic uncertainty strategy. The proposed approach applies to settings in which unmet demand is backordered or lost; and it can accommodate variants of the problem for which the quality of service is captured by means of backorder penalty costs, non-stockout probabilities, or fill rate constraints. This approach has a number of advantages with respect to existing methods in the literature: it enables seamless modelling of different variants of the stochastic lot sizing problem, some of which have been previously tackled via ad hoc solution methods and some others that have not yet been addressed in the literature; and it produces an accurate estimation of the expected total cost, expressed in terms of upper and lower bounds based on piecewise linearisation of the first order loss function. We illustrate the effectiveness and flexibility of the proposed approach by means of a computational study.
© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

We consider the non-stationary stochastic lot sizing problem — the stochastic extension of the well-known dynamic lot sizing problem [1]. This is a finite-horizon periodic review single-item single-stocking location inventory control problem in which demand is stochastic and non-stationary. Bookbinder and Tan [2] discuss three main control strategies that can be adopted in stochastic lot sizing problem: static, static–dynamic, and dynamic uncertainty. The static uncertainty strategy is rather conservative, since the decision maker determines both timing and size of orders at the very beginning of the planning horizon. A less conservative strategy is the static–dynamic uncertainty, in which inventory reviews are fixed at the beginning of the planning horizon, while associated order quantities are decided upon only when orders are issued. The dynamic uncertainty strategy allows the decision maker to decide dynamically at each time period whether or not to place an order and how much to order. Each of these strategies has different advantages and disadvantages. For instance, the dynamic uncertainty strategy is known to be cost-optimal [3]. The static uncertainty is appealing in material requirement planning systems, for which order synchronisation is a key concern [4]. The static–dynamic uncertainty strategy has advantages in organising joint replenishments and shipment consolidation [5–8].

In this study, we focus our attention on the static–dynamic uncertainty strategy, which offers a stable replenishment plan while effectively hedging against uncertainty [9,10]. An important question regarding the static–dynamic uncertainty strategy is how to determine order quantities at inventory review periods when a replenishment schedule is given. In this context, Özen et al. [11] showed that it is optimal to determine order quantities by means of an order-up-to policy. This result leads to the following characterisation of the static–dynamic uncertainty strategy: at each review period, the decision maker observes the actual inventory position (i.e. on-hand inventory, plus outstanding orders, minus backorder) and places an order so as to increase the inventory position up to a given order-up-to level. Key decisions for the static–dynamic uncertainty strategy include an inventory review schedule and an order-up-to level for each

☆This manuscript was processed by Associate Editor Kuhn.

* Corresponding author. Tel.: +44 131 6515239; fax: +44 131 650 8077.
E-mail addresses: roberto.rossi@ed.ac.uk (R. Rossi), onuralp@hacettepe.edu.tr (O.A. Kilic), armagan.tarim@hacettepe.edu.tr (S.A. Tarim).

[1] The author is supported by the University of Edinburgh CHSS Challenge Investment Fund.

[2] The authors were partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under Grant no. 110M500.

[3] This publication has emanated from research supported in part by a research grant from Science Foundation Ireland (SFI) under Grant number SFI/12/RC/2289.

http://dx.doi.org/10.1016/j.omega.2014.08.003
0305-0483/© 2014 Elsevier Ltd. All rights reserved.

review period — these decisions must be fixed at the beginning of the planning horizon.

We introduce a unified modelling approach that captures several variants of the problem and that is based on standard mixed-integer linear programming (MILP) models. Some of these variants have been previously addressed in the literature, whereas some other have not. More specifically, we consider different assumptions on the way unsatisfied demand is modelled: backorder and lost sales. We also consider different service quality measures commonly employed in the inventory control literature (see e.g. [5, pp. 244–246]): penalty cost per unit short per period, non-stockout probability[4] (α service level), cycle fill rate[5] (β^{cyc} service level), and fill rate[6] (β service level). Our models build on recently introduced piecewise linear upper and lower bounds for the first order loss function and its complementary function [12], which are based on distribution independent bounding techniques from stochastic programming: Jensen's and Edmundson–Madanski's inequalities [13, pp. 167–168]. In contrast to earlier works in the literature, we show that these bounds can be used to estimate inventory holding costs, backorder costs and/or service levels, and that they translate into readily available lower and upper bounds on the optimal expected total costs. Furthermore, for the special case in which demand is normally distributed, the model relies on standard linearisation parameters provided in [12].

Our contributions to the inventory control literature are the following:

- we develop a unified MILP modelling approach that enables seamless modelling of the non-stationary stochastic lot sizing problem under each of the four measures of service quality discussed;
- we discuss the first MILP formulation in the literature that captures the case in which service quality is modelled using a standard β service level in line with the definition found in many textbooks on inventory control, such as Hadley and Whitin [14], Silver et al. [5], and Axsäter [15].
- we discuss for the first time in the literature how to handle the case in which demand that occurs when the system is out of stock is lost, i.e. lost sales;
- in contrast to other approaches in the literature our MILP models bound from above and below the cost of an optimal plan by using a piecewise linear approximation of the loss function; by increasing the number of segments, precision can be improved ad libitum;
- we discuss how to build these MILP models for the case in which demand in each period follows a generic probability distribution; for the special case in which demand in each period is normally distributed, we demonstrate how the MILP formulations can be conveniently constructed via standard linearisation coefficients;
- we present an extensive computational study to show that (i) whenever other state-of-the-art approximations exist, our models feature comparable optimality gaps; (ii) the linearisation gap — the term linearisation gap is used here to denote the difference between the upper and lower bounds for the expected total cost obtained via Edmundson–Madanski and Jensen's bounds, respectively — shrinks exponentially fast as the number of segments in the piecewise linearisation increases; and (iii) the number of segments adopted only marginally affects computational efficiency.

2. Literature survey

Due to its practical relevance, a large body of literature has emerged on the static–dynamic uncertainty strategy over the last few decades. Here, we review some key studies which are of particular importance in the context of our work, and reflect upon our contribution. All the models discussed in the following sections operate under a static–dynamic uncertainty strategy. To keep our discussion focused, in what follows we only survey studies related to the static–dynamic uncertainty strategy and we disregard those addressing the static uncertainty (see e.g. [16–19]) or the dynamic uncertainty strategy (see e.g. [20,21]).

Early works on the stochastic lot sizing problem concentrated on easy-to-compute heuristics. Silver [22] and Askin [23] studied the problem under penalty costs, and proposed simple heuristics based on the least period cost method. These heuristics can be regarded as stochastic extensions of the well-known Silver–Meal heuristic [24].

Bookbinder and Tan [2] studied the problem under α service level constraints and introduced the terminology "static uncertainty," "dynamic uncertainty," and "static–dynamic uncertainty." They developed a method that sequentially determines the timing of replenishments and corresponding order-up-to levels for the static–dynamic uncertainty strategy. Following this seminal work, a variety of further studies — which significantly differ in terms of underlying service quality measures and modelling approaches — aimed to determine the optimal replenishment schedule and order-up-to levels simultaneously under Bookbinder and Tan's static–dynamic uncertainty strategy.

Tarim and Kingsman [25] discussed the first MILP formulation under α service level constraints. In contrast to [2], this formulation simultaneously determines the replenishment schedule and corresponding order-up-to levels. Efficient reformulations operating under the same assumptions were discussed in [26–28]. Rossi et al. [27] developed a state space augmentation approach; Tarim et al. [26] implemented a branch and bound algorithm; and Tunc et al. [28] developed an effective MILP reformulation. In addition, Constraint Programming reformulations based on a novel modelling tool, i.e. global chance constraints, were discussed in [29,30]. Finally, an exact, although computationally intensive, Constraint Programming approach was discussed in [31]. Extensions to the case of a stochastic delivery lead time were discussed in [32,33].

Tarim and Kingsman [34] developed the first MILP formulation for the case in which service quality is modelled using a penalty cost scheme. Rossi et al. [35] discussed an efficient Constraint Programming reformulation exploiting optimization oriented global stochastic constraints.

Özen et al. [11] discussed a dynamic programming solution algorithm and two ad hoc heuristics named "approximation" and "relaxation" heuristics, respectively; the authors analyse both the penalty cost and the α service level cases. The "approximation" heuristic operates under the assumption that scenarios in which the actual stock exceeds the order-up-to-level for a given review are negligible and can be safely ignored; while the "relaxation" heuristic operates by relaxing those constraints in Tarim and Kingsman's model that force order sizes in each period to be nonnegative.

Tempelmeier [36] introduced an MILP formulation for the case in which service quality is modelled via β^{cyc} service level constraints.

A key issue in all the aforementioned studies is the computation of the true values of expected on-hand inventories and stock-outs, and thereby associated costs and/or service levels. These values can only be derived from the (complementary) first-order loss function of the

[4] A lower bound on the non-stockout probability in any period over the planning horizon.

[5] A lower bound on the expected fraction of demand that is routinely satisfied from stock for each replenishment cycle, i.e. the time interval between two successive inventory reviews.

[6] A lower bound on the expected fraction of demand that is routinely satisfied from stock over the planning horizon.

demand (see [37, p. 338]) — a non-linear function that cannot be readily embedded into the proposed MILP models.

Tarim and Kingsman [25] — but also [26,27,29–33] — bypassed the issue by approximating the expected on-hand inventory by the expected inventory position. This approach could work well for inventory systems that operate under a very high non-stockout probability. However, it may result in sub-optimal solutions when the probability of observing a stock-out is not negligible.

Tarim and Kingsman [34] used a piecewise linear approximation of the standard loss function to approximate expected holding and penalty costs. The piecewise linear function is fitted to the nonlinear cost function by using an approach that minimises the maximum absolute approximation error. The power of this approach is that the piecewise linearisation is based on standard linearisation coefficients that can be computed once and then reused for any normally distributed demand. Unfortunately, the approximation proposed may either over or underestimate the original cost therefore it becomes hard to assess how far a given solution may be from the true optimal one, i.e. its optimality gap. Furthermore, this approximation is not easily extended to demands following a generic distribution or to models operating under service level measures rather than a penalty cost scheme.

Approximations in Özen et al. [11] require ad hoc algorithms and cannot be easily extended to handle β service level constraints or lost sales.

Tempelmeier [36] tabulates the complementary first order loss function and then uses binary variables to retrieve the holding cost associated with a given replenishment plan. A similar tabulation is employed to enforce the prescribed β^{cyc} service level. However, this tabulation is carried out by considering each possible replenishment cycle independently. This strategy disregards cost and service level dependencies that may exist among successive replenishment cycles. For this very same reason, it cannot be employed to model classic β service level constraints. To the best of our knowledge, no formulation exists in the literature for the case in which service quality is modelled via standard β service level constraints [14,5,15].

The issue of computing expected on-hand inventories and stock-outs is topical in inventory control, as witnessed by a number of recent works (see e.g. [38,39]). All approaches surveyed above address particular instances of the problem. The contribution of this paper is unique and novel in the sense that it introduces a unified modelling approach for static–dynamic uncertainty strategy based on linear approximations of the first order loss function. Furthermore, this unified modelling approach can be used to address the issue of computing static–dynamic uncertainty policy parameters under lost sales, which has not been addressed yet in the literature.

3. Piecewise linearisation of loss functions

For convenience, a list of all symbols used in the rest of the paper is provided in Appendix A.1. Consider a random variable ω with expected value $\tilde{\omega}$ and a scalar variable x. The first order loss function is defined as $\mathcal{L}(x,\omega)=E[\max(\omega-x,0)]$, where E denotes the expected value. The complementary first order loss function is defined as $\widehat{\mathcal{L}}(x,\omega)=E[\max(x-\omega,0)]$. It is known that there is a close relationship between these two functions, as stated in the following lemma.

Lemma 1 (*Snyder and Shen [37, p. 338, C.5]*).

$$\mathcal{L}(x,\omega)=\widehat{\mathcal{L}}(x,\omega)-(x-\tilde{\omega}). \tag{1}$$

The first order loss function and its complementary function play a key role in inventory models, since they are essential to compute expected holding and penalty costs, as well as a number of service measure such as the β^{cyc} and β service levels.

A common approach in computing near-optimal control parameters of the static–dynamic uncertainty strategy is to formulate the problem as a certainty equivalent MILP. Unfortunately, the loss function is non-linear and cannot be easily embedded in MILP models. To overcome this issue, we adopt a piecewise linearisation approach similar to the one recently discussed in Rossi et al. [12]. This approach is based on classical inequalities from stochastic programming: Jensen's and Edmundson–Madanski inequalities [13, p. 167–168] and can be applied to random variables following a generic probability distribution.

It is known that both the first order loss function $\mathcal{L}(x,\omega)$ and its complementary function $\widehat{\mathcal{L}}(x,\omega)$ are convex in x regardless of the distribution of ω. For this reason, both Jensen's lower bound and Edmundson–Madanski upper bound are applicable to the first order loss function and its complementary function. More formally, let $g_\omega(\cdot)$ denote the probability density function of ω and consider a partition of the support Ω of ω into W disjoint compact subregions $\Omega_1,\dots,\Omega_W$. We define, for all $i=1,\dots,W$

$$p_i=\Pr\{\omega\in\Omega_i\}=\int_{\Omega_i}g_\omega(t)\,dt \quad\text{and}\quad E[\omega|\Omega_i]=\frac{1}{p_i}\int_{\Omega_i}tg_\omega(t)\,dt \tag{2}$$

Lemma 2. *For the complementary first order loss function the lower bound $\widehat{\mathcal{L}}_{lb}(x,\omega)$, where*

$$\widehat{\mathcal{L}}(x,\omega)\ge\widehat{\mathcal{L}}_{lb}(x,\omega)=\sum_{i=1}^{W}p_i\max(x-E[\omega|\Omega_i],0)$$

is a piecewise linear function with $W+1$ segments. The i-th linear segment of $\widehat{\mathcal{L}}_{lb}(x,\omega)$ is

$$\widehat{\mathcal{L}}^i_{lb}(x,\omega)=x\sum_{k=1}^{i}p_k-\sum_{k=1}^{i}p_kE[\omega|\Omega_k]\quad E[\omega|\Omega_i]\le x\le E[\omega|\Omega_{i+1}], \tag{3}$$

where $i=1,\dots,N$; furthermore, the 0-th segment is $x=0$, $-\infty\le x\le E[\omega|\Omega_1]$.

This lower bound is a direct application of Jensen's inequality. Let then e_W denote the maximum approximation error for the lower bound in Lemma 2 associated with a given partition comprising W regions. A piecewise linear upper bound, i.e. Edmundson–Madanski's bound, can be obtained by shifting up the lower bound in Lemma 2 by a value e_W.

Lemma 3. *For the complementary first order loss function the upper bound $\widehat{\mathcal{L}}_{ub}(x,\omega)$, where*

$$\widehat{\mathcal{L}}(x,\omega)\le\widehat{\mathcal{L}}_{ub}(x,\omega)=\sum_{i=1}^{W}p_i\max(x-E[\omega|\Omega_i],0)+e_W$$

is a piecewise linear function with $W+1$ segments. The i-th linear segment of $\widehat{\mathcal{L}}_{ub}(x,\omega)$ is

$$\widehat{\mathcal{L}}^i_{ub}(x,\omega)=x\sum_{k=1}^{i}p_k-\sum_{k=1}^{i}p_kE[\omega|\Omega_k]+e_W\quad E[\omega|\Omega_i]\le x\le E[\omega|\Omega_{i+1}],$$

where $i=1,\dots,N$; furthermore, the 0-th segment is $x=e_W$, $-\infty\le x\le E[\omega|\Omega_1]$.

Having established these two results, we must then decide how to partition the support Ω in order to obtain good bounds. A number of works discussed how to obtain an optimal partitioning of the support under a framework that minimises the maximum approximation error [40,41]. In short, these works demonstrate that, in order to minimise the maximum approximation error, one must find parameters ensuring approximation errors at piecewise function breakpoints are all equal. This result unfortunately does not hold when optimal linearisation parameters must be found for a set of random variables.

Consider a set of random variables $\omega_1, \ldots, \omega_n, \ldots, \omega_N$ and associated complementary first order loss functions $\widehat{\mathcal{L}}(x, \omega_1), \ldots, \widehat{\mathcal{L}}(x, \omega_N)$. From (2) it is clear that, once all p_i have been fixed, all $E[\omega_n|\Omega_k]$ are uniquely determined. The particular structure of (3) makes it, in principle, possible to compute standard p_i coefficients for the whole set of random variables and then select the $E[\omega_n|\Omega_k]$ for a specific $\widehat{\mathcal{L}}_{lb}(x, \omega_n)$ via a binary selector variable y_n, that is

$$\widehat{\mathcal{L}}^i_{lb}(x, \omega) = x \sum_{k=1}^{i} p_k - \sum_{k=1}^{i} p_k E[\omega_n|\Omega_k] y_n \qquad \begin{array}{l} E[\omega_n|\Omega_i] \le x \le E[\omega_n|\Omega_{i+1}] \\ 1 \le n \le N \end{array} \tag{4}$$

where $\sum_{n=1}^{N} y_n = 1$. These expressions generalise those discussed in [12], which only hold for normally distributed random variables, and they will be used in Section 4.

Unfortunately, computing probability masses $p_1, \ldots, p_W$ that minimise the maximum approximation error over a set of random variables is a challenging task. As shown in [42] this is not a problem of convex optimisation as the one faced while computing optimal linearisation parameters for a single loss function. In this work, we adopt a simple and yet effective approximate strategy to compute probability masses $p_1, \ldots, p_W$: we split the support into W regions with uniform probability mass. Implementation details of this heuristic approach are discussed in Appendix A.2.

4. A unified MILP modelling approach

In this section we demonstrate how the results presented so far can be used to derive MILP formulations for the stochastic lot sizing problem under static–dynamic uncertainty strategy. First, we introduce the original stochastic programming formulation of the problem (Section 4.1). We then present MILP models for the problem under α service level constraints (Section 4.2), a penalty cost scheme. (Section 4.3), β^{cyc} (Section 4.4) and β(Section 4.5) service level constraints. Finally, we discuss how to extend these models to a lost sales setting. For a complete overview of the models presented the reader may refer to Appendix A.3.

4.1. Stochastic lot-sizing

The stochastic programming formulation of the non-stationary stochastic lot-sizing problem was originally presented in [2, pp. 1097–1098]. The formal problem definition is as follows. Customer demand d_t in each period $t=1,\ldots,N$ is a random variable with probability density function $g_t(\cdot)$ and cumulative distribution function $G_t(\cdot)$. There are fixed and variable replenishment costs: the fixed cost is a per order; the variable cost is v per unit ordered. Negative orders are not allowed. A holding cost of h is paid of each unit of inventory carried from one period to the next. I_0 denotes the initial inventory level. Without loss of generality, delivery lead-time is not incorporated in the model. When a stockout occurs, all demand is backordered and filled as soon as an adequate supply arrives. There is a service level constraint enforcing a non-stockout probability of at least α in each period — this is known in the inventory control literature as "α service level" constraint [5]. Finally, it is also assumed that the service level is set to a high value, i.e. $\alpha > 0.9$ in order to incorporate management's perception of the cost of backorders, so that shortage costs can be safely ignored. The objective is to minimise the expected total cost E[TC], which comprises fixed/variable ordering and holding costs. The resulting model is presented in Fig. 1. In this model I_t represents the inventory level at the end of a period; δ_t takes value 1 if an order is placed in period t; and Q_t represents the order quantity in period t. Constraints (6) are the inventory conservation constraints: inventory level at the end of period t must be equal to the initial inventory I_0, plus all orders received, minus all demand realised up to period t, since we assume — in line with the

$$\min \mathrm{E[TC]} = \int_{d_1}\int_{d_2}\ldots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + h\max(I_t,0) + vQ_t)\times \tag{5}$$
$$g_1(d_1)g_2(d_2)\ldots g_N(d_N)\,d(d_1)d(d_2)\ldots d(d_N)$$

subject to, for $t = 1, \ldots N$

$$I_t = I_0 + \sum_{i=1}^{t}(Q_i - d_i) \tag{6}$$
$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$
$$\Pr\{I_t \ge 0\} \ge \alpha \tag{8}$$
$$Q_t \ge 0,\ \delta_t \in \{0,1\} \tag{9}$$

Fig. 1. Stochastic programming formulation of the non-stationary stochastic lot-sizing problem under α service level constraints.

original model in the literature — that inventory cannot be disposed or returned to the supplier; constraints (7) set δ_t to one if an order is placed in period t; finally, (8) enforces the prescribed service level in each period.

The above model can be modified to accommodate a penalty cost scheme, in place of the original α service level constraints (see e.g. [34]). All one has to do is to drop (8) and to replace the original objective function with the following one:

$$\min E[\mathrm{TC}] = \int_{d_1}\int_{d_2}\ldots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + h\max(I_t,0) + b\max(-I_t,0) + vQ_t)$$
$$\times g_1(d_1)g_2(d_2)\ldots g_N(d_N)d(d_1)d(d_2)\ldots d(d_N) \tag{10}$$

where b denotes the penalty cost per unit short per period. β^{cyc} and β service level formulations are obtained by replacing (8) with an alternative service measure.

4.2. α service level constraints

We now consider the MILP formulation of Tarim and Kingsman [25] for computing near-optimal inventory control policy parameters under Bookbinder and Tan's static–dynamic uncertainty strategy. According to this strategy, inventory review times as well as their respective order-up-to-levels must be all fixed at the beginning of the planning horizon. However, actual order quantities are determined only after demand has been observed.

In what follows, M denotes a very large number and $\tilde{x}$ denotes the expected value of x. Tarim and Kingsman's model is presented in Fig. 2. This certainty equivalent model comprises three sets of decision variables: $\tilde{I}_t$, representing the expected closing inventory level at the end of period t; δ_t, a binary variable representing the inventory review decision at period t; and P_{jt}, a binary variable which is set to one if and only if the most recent inventory review before period t was carried out in period j. By observing that, for a period t in which an order is placed (i.e. $\delta_t = 1$) the order-up-to-level S_t is simply $S_t = \tilde{I}_t + \tilde{d}_t$, it follows that by solving the above model policy parameters are immediately obtained.

Constraints in the certainty equivalent model neatly reflect those in the original stochastic programming model. More specifically, (12) enforces the inventory conservation constraints; (13) is the reordering condition; and (14)–(16) enforce the prescribed service level α. In (14), $G^{-1}_{d_{jt}}(\alpha)$ denotes the α-quantile of the inverse cumulative distribution function of the random variable $d_j + \cdots + d_t$. Finally, the objective function is obtained by observing

$$E\left[v\sum_{t=1}^{N} Q_t\right] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + v\tilde{I}_N, \tag{19}$$

where $-vI_0 + v\sum_{t=1}^{N}\tilde{d}_t$ is a constant.

$$\min \mathrm{E[TC]} = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t) + v\tilde{I}_N \tag{11}$$

subject to, for $t = 1, \dots N$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \geq 0 \tag{12}$$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \leq \delta_t M \tag{13}$$

$$\tilde{I}_t \geq \sum_{j=1}^{t}\left(G_{d_{jt}}^{-1}(\alpha) - \sum_{k=j}^{t}\tilde{d}_k\right)P_{jt} \tag{14}$$

$$\sum_{j=1}^{t} P_{jt} = 1 \tag{15}$$

暗示 P_{11} = 1, 但是 delta_1 不一定为 1

$$P_{jt} \geq \delta_j - \sum_{k=j+1}^{t}\delta_k \qquad j = 1, \dots, t \tag{16}$$

$$P_{jt} \in \{0, 1\} \qquad j = 1, \dots, t \tag{17}$$

$$\delta_t \in \{0, 1\} \tag{18}$$

Fig. 2. Tarim and Kingsman's certainty equivalent MILP formulation of the model in Fig. 1 under the static–dynamic uncertainty strategy.

Following an assumption originally introduced by Bookbinder and Tan [2], Tarim and Kingsman [25] approximate the holding cost component in the original objective function, which we recall was $E[\max(I_t, 0)]$, via the expression $h\tilde{I}_t$. To overcome this limitation of the model, we introduce two new sets of decision variables: $\tilde{I}_t^{lb} \geq 0$ and $\tilde{I}_t^{ub} \geq 0$ for $t = 1, \dots, N$, which represent, respectively, a lower and an upper bound to the true value of $E[\max(I_t, 0)]$. The objective function then can be rewritten as

$$\min E[TC] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t^{lb}) + v\tilde{I}_N \tag{20}$$

if our aim is to compute a lower bound for the cost of an optimal plan, or as

$$\min E[TC] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t^{ub}) + v\tilde{I}_N \tag{21}$$

if our aim is to compute an upper bound for the cost of an optimal plan. We next discuss how to constrain $\tilde{I}_t^{lb}$ and $\tilde{I}_t^{ub}$. Our discussion applies to demand d_k, for $k = 1, \dots, N$ following a generic distribution. Consider the random variable d_{jt} representing the convolution $d_j + \dots + d_t$. We select a priori a number W of adjacent regions Ω_i into which the support of d_{jt} must be partitioned. This partitioning will produce a piecewise linear approximation comprising $W+1$ segments. We also fix a priori the probability mass $p_i = \Pr\{d_{jt} \in \Omega_i\}$ that must be associated with each region Ω_i. As discussed in Section 3, there are several possible strategies to assign probability masses p_i to regions. For instance, we may ensure uniform coverage, i.e. all region must have the same probability mass, or we may select regions — by using a heuristic or exact approach — in order to minimise the maximum approximation error over all possible convolutions d_{jt}, for $t = 1, \dots, N$ and $j = 1, \dots, t$. Regardless of the strategy we adopt, once all p_i are known, regions Ω_i are uniquely determined and the associated conditional expectation $E[d_{jt}|\Omega_i]$ can be immediately computed using off-the-shelf software. Finally, the maximum approximation error e_W^{jt} associated with the linearisation of $\hat{\mathcal{L}}(x, d_{jt})$ can be found by checking the linearisation error at the W possible breakpoints of the piecewise linear function obtained. Having pre-computed all these values for $t = 1, \dots, N$ and $j = 1, \dots, t$, we introduce the following constraints in the model:

分成几段就有几个约束条件

不能直接大于等于 piecewise 表达式

$$\tilde{I}_t^{lb} \geq \left(\tilde{I}_t + \sum_{j=1}^{t}\tilde{d}_{jt}P_{jt}\right)\sum_{k=1}^{i}p_k - \sum_{j=1}^{t}\left(\sum_{k=1}^{i}p_k E[d_{jt}|\Omega_i]\right)P_{jt},$$
$$t = 1, \dots, N; \quad i = 1, \dots, W \tag{22}$$

第一项可以通过正态分布与标准分布的转化性质削去 d_{jt}，成为包含 prob*sigma_{jt} 的项

This expression follows from Lemma 2 and closely resembles (4). Consider a replenishment in period j covering periods $j, \dots, t$ with associated order-up-to-level S. Our aim is to enforce $\tilde{I}_t^{lb} \geq \hat{\mathcal{L}}_{lb}^{i}(S, d_{jt})$ for $i = 1, \dots, W$, since $\tilde{I}_t^{lb}$ represents a lower bound for the expected positive inventory at the end of period t. By observing that $S = \tilde{I}_t + \tilde{d}_{jt}$, we obtain the above expression. We then obtain $\tilde{I}_t^{ub}$ from $\tilde{I}_t$, by noting that a piecewise linear upper bound can be derived by adding the maximum estimation error to Jensen's piecewise linear lower bound [12]:

$$\tilde{I}_t^{ub} \geq \left(\tilde{I}_t + \sum_{j=1}^{t}\tilde{d}_{jt}P_{jt}\right)\sum_{k=1}^{i}p_k + \sum_{j=1}^{t}\left(e_W^{jt} - \sum_{k=1}^{i}p_k E[d_{jt}|\Omega_i]\right)P_{jt},$$
$$t = 1, \dots, N,$$
$$i = 1, \dots, W; \tag{23}$$

下界刚好比上界上了 error

where $\tilde{I}_t^{ub} \geq \sum_{j=1}^{t} e_W^{jt} P_{jt}$ for $t = 1, \dots, N$. The special case in which demand in each period follows a normal distribution is discussed in Appendix A.4.

4.3. Penalty cost scheme

The model in Section 4.2 can be modified to accommodate a penalty cost b per unit short per period in place of the α service level constraints in Tarim and Kingsman [34]. As discussed in the previous section, our formulation is more accurate than Tarim and Kingsman [34], because the expected total cost of a plan can be now bounded from above and below. It is also more general, since the discussion in Tarim and Kingsman [34] is limited to normally distributed demand.

In the new model, we introduce two new sets of variables $\tilde{B}_t^{lb} \geq 0$ and $\tilde{B}_t^{ub} \geq 0$ for $t = 1, \dots, N$, which represent a lower and upper bound, respectively, for the true value of $E[-\min(I_t, 0)]$ and thus allow us to compute lower and upper bounds for the expected backorders in each period. The objective function then becomes

$$\min E[TC] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t^{lb} + b\tilde{B}_t^{lb}) + v\tilde{I}_N \tag{24}$$

if our aim is to compute a lower bound for the cost of an optimal plan, or

$$\min E[TC] = -vI_0 + v\sum_{t=1}^{N}\tilde{d}_t + \sum_{t=1}^{N}(a\delta_t + h\tilde{I}_t^{ub} + b\tilde{B}_t^{ub}) + v\tilde{I}_N \tag{25}$$

if our aim is to compute an upper bound for the cost of an optimal plan. Finally, we must remove constraints (14), since we are operating under a penalty cost scheme and not under a service level constraints.

Once more, we assume that demand in each period follows a generic distribution; we obtain $\tilde{B}_t^{lb}$ and $\tilde{B}_t^{ub}$ from $\tilde{I}_t$ by exploiting Lemma 1:

$$\tilde{B}_t^{lb} \geq -\tilde{I}_t + \left(\tilde{I}_t + \sum_{j=1}^{t}\tilde{d}_{jt}P_{jt}\right)\sum_{k=1}^{i}p_k - \sum_{j=1}^{t}\left(\sum_{k=1}^{i}p_k E[d_{jt}|\Omega_i]\right)P_{jt},$$
$$t = 1, \dots, N,$$
$$i = 1, \dots, W; \tag{26}$$

lb

where $\tilde{B}_t^{ub} \geq -\tilde{I}_t$ and

$$\tilde{B}_t^{ub} \geq -\tilde{I}_t + \left(\tilde{I}_t + \sum_{j=1}^{t}\tilde{d}_{jt}P_{jt}\right)\sum_{k=1}^{i}p_k + \sum_{j=1}^{t}\left(e_W^{jt} - \sum_{k=1}^{i}p_k E[d_{jt}|\Omega_i]\right)P_{jt},$$
$$t = 1, \dots, N,$$
$$i = 1, \dots, W; \tag{27}$$

where $\tilde{B}_t^{ub} \geq -\tilde{I}_t + \sum_{j=1}^{t} e_W^{jt} P_{jt}$. The case in which demand in each period follows a normal distribution is discussed in Appendix A.4.

有了新约束，老约束有时可以不要

后来又增加了一个新约束，新的形式计算效果更好 新约束 Pjt=1 => B_t == piecewise function

4.4. β^{cyc} service level constraints

The model discussed in Section 4.3 can be modified to accommodate β^{cyc} service level constraints, defined in [36] as

$$1-\max_{i=1,\dots,m}\left[E\left\{\frac{\text{Total backorders in replenishment cycle } i}{\text{Total demand in replenishment cycle } i}\right\}\right]. \tag{28}$$

β^{cyc} represents a lower bound on the expected fraction of demand that is routinely satisfied from stock for each replenishment cycle. The revised model resembles the one discussed in Tempelmeier [36]. However, we aim to stress that in this latter work the author enforces the prescribed service level by precomputing order-up-to-levels and cycle holding costs in a table, rather than using a piecewise linearisation of the loss function as we do. One of the advantages of our approach is that it is able to account for dependencies among opening stock levels and service levels of consecutive cycles. As we will show in Section 4.5, the ability to capture these dependencies is also important if we aim to generalise the model to the classical definition of "fill rate".

The discussion below is distribution independent; we modify the model discussed in Section 4.3 by introducing service level constraints

$$\tilde{B}_t^{lb} \le (1-\beta^{\mathrm{cyc}})\sum_{j=1}^{t} P_{jt}\tilde{d}_{jt}, \quad t=1,\dots,N, \tag{29}$$

if our aim is to compute a lower bound for the cost of an optimal plan; or

$$\tilde{B}_t^{ub} \le (1-\beta^{\mathrm{cyc}})\sum_{j=1}^{t} P_{jt}\tilde{d}_{jt}, \quad t=1,\dots,N, \tag{30}$$

if our aim is to compute an upper bound for the cost of an optimal plan. Constraints (29) and (30) directly follow from (28). Finally, the objective function is (20) if our aim is to compute a lower bound for the cost of an optimal plan, or (21) if our aim is to compute an upper bound.

4.5. β service level constraints

The model discussed in Tempelmeier [36] captures a definition of fill rate that is not conventional in the inventory literature. This issue has been discussed in Rossi et al. [43]. To the best of our knowledge, no modelling strategy exists for the conventional fill rate under a static–dynamic uncertainty control policy. In this section, we introduce an alternative MILP reformulation that captures a definition of β service level that is in line with the definition found in many textbooks on inventory control [14,5,15].

In the context of finite horizon inventory models the β service level is defined as "the expected fraction of demand satisfied immediately from stock on hand." In a number of works (e.g. [44,45]) this definition is formalized as

$$1-E\left\{\frac{\text{Total backorders within the planning horizon}}{\text{Total demand within the planning horizon}}\right\}. \tag{31}$$

The static–dynamic uncertainty strategy divides the finite planning horizon into a number, say m, of consecutive replenishment cycles. We can re-write (31) by taking these into account as

$$1-E\left\{\frac{\sum_{i=1}^{m}\text{Total backorders within the } i\text{th replenishment cycle}}{\sum_{i=1}^{m}\text{Total demand within the } i\text{th replenishment cycle}}\right\}. \tag{32}$$

However, in [36] the same β service level is imposed on each and every cycle within the planning horizon. Clearly (32) is different from (28): the original definition imposes a β service level throughout the planning horizon, whereas the definition in [36] imposes a β service level on each replenishment cycle within the planning horizon independently. This latter definition is thus more restrictive.

We modify as follows the model in Section 4.4 to implement the original definition outlined in (32). Also in this case the changes discussed below are distribution independent. We introduce two new sets of nonnegative variables $\tilde{C}_t^{lb}$ and $\tilde{C}_t^{ub}$ for $t=0,\dots,N$. These variables express a lower and an upper bound, respectively, to the expected total backorders within the replenishment cycle that ends at period t, if any exists. Hence, $\tilde{C}_t^{lb}$ (resp. $\tilde{C}_t^{ub}$) should be equal to $\tilde{B}_t^{lb}$ (resp. $\tilde{B}_t^{ub}$), if t is the last period of a replenishment cycle; otherwise $\tilde{C}_t^{lb}$ (resp. $\tilde{C}_t^{ub}$) should be equal to 0. We enforce this fact as follows:

$$\tilde{C}_t^{lb} \ge \tilde{B}_t^{lb}-(1-\delta_{t+1})\sum_{k=1}^{t}\tilde{d}_t, \quad t=0,\dots,N-1, \tag{33}$$

$$\tilde{C}_t^{ub} \ge \tilde{B}_t^{ub}-(1-\delta_{t+1})\sum_{k=1}^{t}\tilde{d}_t, \quad t=0,\dots,N-1. \tag{34}$$

where $\tilde{B}_0^{lb}=\tilde{B}_0^{ub}=\tilde{C}_0^{lb}=\tilde{C}_0^{ub}=I_0$. Finally, we must ensure that $\tilde{C}_N^{lb}=\tilde{B}_N^{lb}$ and $\tilde{C}_N^{ub}=\tilde{B}_N^{ub}$. We then use these new variables to build constraint

$$\sum_{t=1}^{N}\tilde{C}_t^{lb} \le (1-\beta)\sum_{t=1}^{N}\tilde{d}_t \tag{35}$$

which will replace (29), if our aim is to compute a lower bound for the cost of an optimal plan; and constraint

$$\sum_{t=1}^{N}\tilde{C}_t^{ub} \le (1-\beta)\sum_{t=1}^{N}\tilde{d}_t \tag{36}$$

which will replace (30), if our aim is to compute an upper bound. Constraints (35) and (36) directly follow from (32).

4.6. Lost sales

In this section we briefly sketch the extension of the models discussed in the previous section to the case in which demand that occurs when the system is out of stock is lost, i.e. lost sales setting. The discussion is purportedly short, since the derived models are quite similar to those already presented and there are only few adjustments that are necessary to adapt our models to this new setting. However, we wish to underscore once more that this is the first work in the literature discussing models for computing static–dynamic uncertainty policy parameters in a lost sales setting; and that it is the novel modelling approach we adopted, which is based on piecewise linearisation of loss functions, that enables such extensions.

Under lost sales we need to take into account the fact that if inventory drops to zero, demand that occurs until the next order arrives will not be met. Under a static–dynamic uncertainty control policy, this means that the actual order quantity will never exceed the order-up-to-level.

In this setting it is crucial to set up the model in such a way as to account for the opportunity cost associated with units of demand that are not met by a given control policy. For this reason, we must introduce a new parameter s that represents the selling price of a product; we then let $m=s-v$ be the margin — i.e. unit selling price minus unit ordering cost — for an item sold. The resulting stochastic programming model under lost sales is shown in Fig. 3. In this model $E[TP]$ represents the expected total profit, which we aim to maximise. δ_t is a binary decision variable that is set to one if we order items at period t, i.e. constraints (40). Q_t represents the order quantity in period t, which must be greater than or equal to zero. I_t is a random variable that represents the inventory level at the end of period t; despite lost sales, for convenience, we assume that I_t may take negative values. In the objective function, we multiply the margin m by the number of items Q_t ordered in period t; we then subtract ordering cost a, if an

$$\max \mathrm{E[TP]} = sI_0 + \int_{d_1}\int_{d_2}\dots\int_{d_N}\left[\sum_{t=1}^{N}(mQ_t - a\delta_t - h\max(I_t,0)) - s\max(I_N,0)\right]\times \quad (37)$$

$$g_1(d_1)g_2(d_2)\dots g_N(d_N)\,d(d_1)d(d_2)\dots d(d_N)$$

subject to, for $t=1,\dots N$

$$I_t + d_t - I_{t-1} \geq 0 \quad (38)$$

$$Q_t = I_t + d_t - \max(I_{t-1},0) \quad (39)$$

$$\delta_t = \begin{cases} 1 & \text{if } Q_t > 0, \\ 0 & \text{otherwise} \end{cases} \quad (40)$$

$$\Pr\{I_t \geq 0\} \geq \alpha \quad (41)$$

$$Q_t \geq 0,\ \delta_t \in \{0,1\} \quad (42)$$

Fig. 3. Stochastic programming formulation of the non-stationary stochastic lot-sizing problem under lost sales and α service level constraints.

可以递推出来

$$\max \mathrm{E[TP]} = sI_0 + \sum_{t=1}^{N}(m\tilde{Q}_t - a\delta_t - h\tilde{I}_t^{lb}) - s\tilde{I}_N^{lb} \quad (43)$$

目标函数简洁，适用于lostsale 的情况

subject to, for $t=1,\dots N$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \geq 0 \quad (44)$$

$$\tilde{Q}_t \geq \tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1}^{lb} + (1-\delta_t)M \quad (45)$$

大小顺序反了

$$\tilde{Q}_t \leq \tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1}^{lb} - (1-\delta_t)M \quad (46)$$

$$\tilde{I}_t + \tilde{d}_t - \tilde{I}_{t-1} \leq \tilde{Q}_t M \quad (47)$$

$$\tilde{Q}_t \leq \delta_t M \quad (48)$$

$$\tilde{I}_t \geq \sum_{j=1}^{t}\left(G_{d_{jt}}^{-1}(\alpha) - \sum_{k=j}^{t}\tilde{d}_k\right)P_{jt} \quad (49)$$

$$\sum_{j=1}^{t} P_{jt} = 1 \quad (50)$$

$$P_{jt} \geq \delta_j - \sum_{k=j+1}^{t}\delta_k \qquad j=1,\dots,t \quad (51)$$

constraints (22)

$$P_{jt} \in \{0,1\} \qquad j=1,\dots,t \quad (52)$$

$$\tilde{Q}_t \geq 0,\ \delta_t \in \{0,1\} \quad (53)$$

Fig. 4. MILP formulation of the model in Fig. 3 under the static–dynamic uncertainty strategy.

order is placed in period t (i.e. $\delta_t = 1$), and the holding cost h on items that remain in stock at the end of period t. There is a further term $-s\max(I_N,0)$ to reflect the fact that items in stock at the end of the planning horizon will not be sold and thus the associated selling price s should not be included in the total profit. Term $\max(I_{t-1},0)$ in constraints (39) makes sure that the order quantity does not include any lost sale from the previous period. Constraints (38) ensure that the inventory level at the end of period t is greater than or equal to the inventory level at the end of period $t-1$ plus the realised demand in period t; this makes sure that items in stock in a period and not sold are brought to the next period. Constraints (41) enforce an α service level and can be easily replaced by other service measures such as a β^{cyc} or a β service level. A formulation under a penalty cost scheme is also easily obtained by removing the service level constraints and by modifying the objective function as illustrated in previous sections.

An MILP formulation of the problem in Fig. 3 under the static–dynamic uncertainty strategy is shown in Fig. 4. In this model, by taking expectations, constraints (38) translate into (44). Constraints (39) translate into (45) and (46). Terms $\max(I_t,0)$ in the objective function and in constraints (39) can be handled by using an auxiliary variable $\tilde{I}_t^{lb}$ that represents a lower bound for $E[\max(I_t,0)]$ computed as before via a piecewise linearisation of the complementary first order loss function, i.e. constraints (22). Constraints (47) ensure that the expected inventory level at the end of period t is greater than the expected inventory level at the end of period $t-1$ minus the expected demand in period t if and only if an order has been placed in period t, i.e. $\tilde{Q}_t > 0$. Constraints (40) translate into (48). Finally, service level constraints (41) translate into (49)–(51) following a strategy similar to the one in [25], which we illustrated in Section 4.2. The model presented can be used to compute an upper bound for E[TP] — note that underestimating buffer stocks, i.e. $\tilde{I}_t^{lb}$ leads to lower holding costs and to an overestimation of the expected order quantity and associated margins $m\tilde{Q}_t$ in the objective function. If we aim to compute a lower bound instead, all occurrences of $\tilde{I}_t^{lb}$ should be replaced by $\tilde{I}_t^{ub}$ and constraints (22) should be replaced by constraints (23). Other MILP formulations under β^{cyc} and β service levels are obtained in a similar fashion, since only the service level constraints of the model are affected by this change. Finally, the reader should note that the penalty cost oriented MILP formulation presented in Section 4.3 charges penalty cost on a "per unit short per unit time" basis. This is not appropriate under lost sales, since it is common practice to charge penalty cost on a "per unit short" basis under this setting. If b is charged accordingly, then a penalty cost formulation under lost sales is easily obtained from the model in Fig. 4, by dropping the service level constraints and by modifying the objective function as follows:

$$E[\mathrm{TP}] = sI_0 + \max \sum_{t=1}^{N}(m\tilde{Q}_t - a\delta_t - h\tilde{I}_t^{lb} - b\tilde{C}_t^{lb}) - s\tilde{I}_N^{lb} \quad (54)$$

where $\tilde{C}_t^{lb}$, as defined in Section 4.5, bounds the expected units short at the end of each replenishment cycle.

In this section we presented a unified MILP modeling approach to compute near-optimal policy parameters for the non-stationary stochastic lot sizing problem under static–dynamic uncertainty strategy. We discussed settings in which unmet demand is back-ordered or lost and variants of the problem in which the quality of service is captured by means of backorder penalty costs, non-stockout probabilities, or fill rate constraints. A tabular overview of these models is given in Appendix A.3.

5. Computational experience

In this section we present an extensive computational analysis of the models previously discussed. The experiments below were conducted by using CPLEX 12.3 on a 2.13 GHz Intel Core 2 Duo with 4 GB of RAM. The test bed considered in our analysis is presented in Section 5.1. In our experiments, the expected total cost of near-optimal policies was estimated using Monte Carlo simulation, as a stopping criterion we imposed a maximum estimation error of 0.01% of the estimated cost at 95% confidence. The aim of our computational analysis is twofold. First, we investigate the performance of our models — both in terms of optimality gap with respect to the true optimal solution and in terms of computational efficiency — against other state-of-the-art models in the literature (Section 5.2); second, we investigate the behaviour of the linearisation gap and the impact on solution time for all models presented when the number of segments adopted for the piecewise linear approximation of the loss function increases (Section 5.3).

5.1. Test bed

We consider a test bed comprising 810 instances. More specifically, we carried out a full factorial analysis under the following factors. We considered ten different demand patterns illustrated in Fig. 5. The patterns include two life cycle patterns (LCY1 and

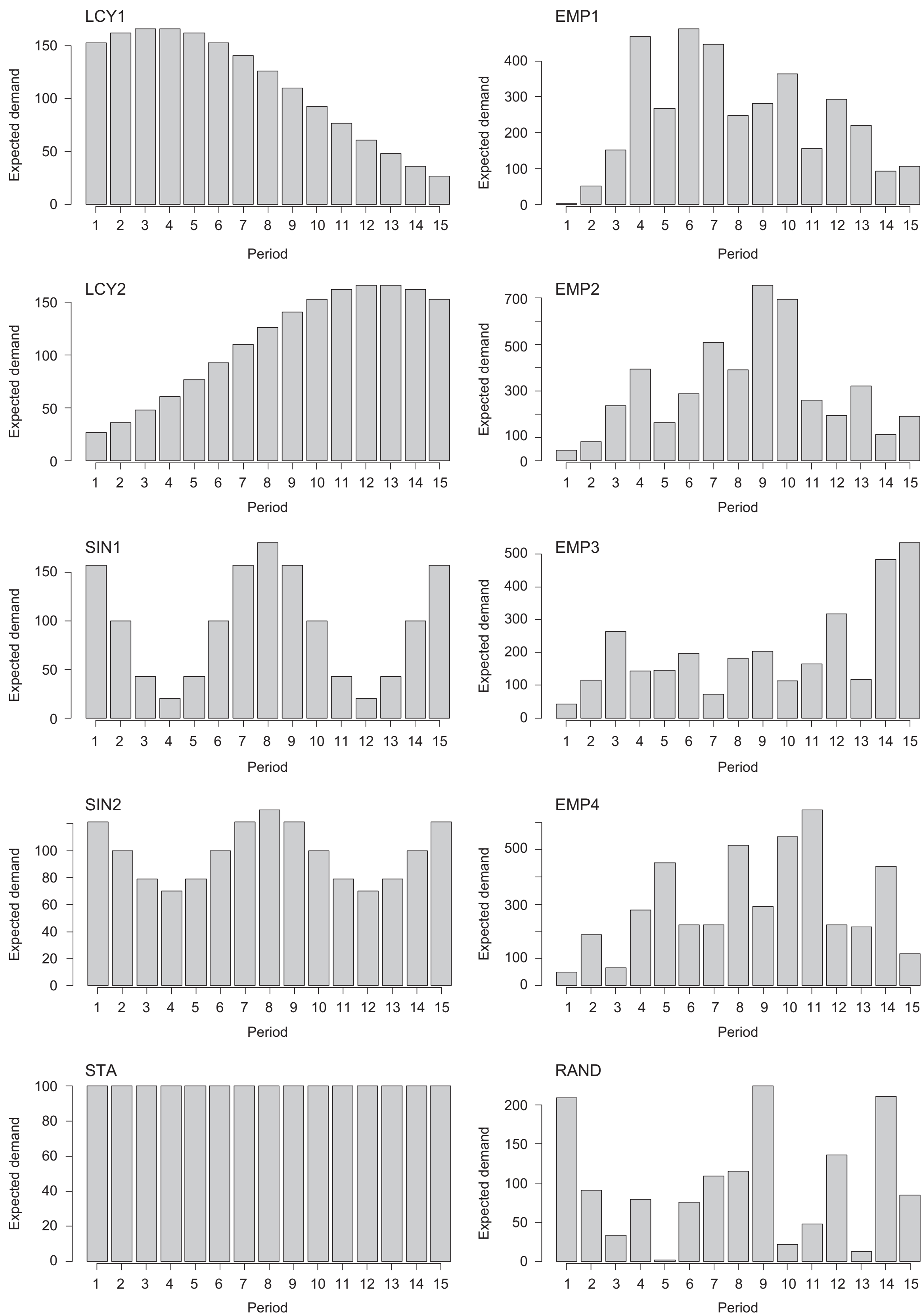

Fig. 5. Demand patterns in our computational analysis; the values presented denote the expected demand $\tilde{d}_t$ in each period t of the planning horizon.

LCY2), two sinusoidal patterns (SIN1 and SIN2), stationary (STA) and random (RAND) patterns, and four empirical patterns derived from demand data in [46] (EMP1,…,EMP4). Fixed ordering cost a takes values in $\{500, 1000, 2000\}$; inventory holding cost is $h=1$; while proportional unit cost v takes values in $\{2, 5, 10\}$. For models under service level constraints, we let the prescribed service level range in $\{0.8, 0.9, 0.95\}$; for model under a penalty cost scheme we let the penalty cost range in $\{2, 5, 10\}$.

5.2. Optimality gap

We assume that demand d_t in each period t is normally distributed with mean $\tilde{d}_t$ and standard deviation σ_{d_t}, and let the coefficient of variation $c = \tilde{d}_t/\sigma_{d_t}$, where $c \in \{0.10, 0.20, 0.30\}$. Expected values of the demand in each period are illustrated in Fig. 5 for each of the ten patterns considered. As discussed, when demand is normally distributed, general purpose linearisation parameters can be precomputed and immediately used in our models [12].

We consider three possible measures for the quality of service for which exact models to compute optimal policy parameters exist. More specifically, to compute optimal static–dynamic uncertainty policy parameters under α service level, we adopt the model in [31]. This model can be easily modified to accommodate a β^{cyc} service level and thus to compute optimal policy parameters under this service level measure. To compute optimal policy parameters under a backorder penalty cost scheme, we adopt the exact model discussed in [35]. Since the approach in [31] is computationally cumbersome, in our optimality gap analysis we considered shorter demand patterns: periods $5, 6, \ldots, 11$ for each pattern in Fig. 5.

We analysed the performance of the approach in Bookbinder and Tan [2] (BT88), of the approach in Tarim and Kingsman [25] (TK04), and of the lower and upper bound models discussed in Section 4.2 (MILP-11 LB & MILP-11 UB) against the optimal policy under α service level constraints. The piecewise linearisation we adopted features eleven segments. Optimality gaps are presented in Fig. 6. These models all provide excellent optimality gaps on average of 0.25% the optimal expected total cost.

Furthermore, we contrasted the performance of the model in [36] (T07), and of the lower and upper bound models we discussed in Section 4.4 (MILP-11 LB & MILP-11 UB) against the optimal policy under β^{cyc} service level constraints. The reader should note that in this case, the lower bound model in Section 4.4 produces a policy that does not strictly meet the prescribed service level, since the service level is determined from the piecewise linearisation of the loss function; this explains the negative optimality gaps observed for MILP-11 LB. Conversely, the upper bound model with ten segments MILP-11 UB always produces a feasible policy and features an average optimality gap (0.08%) that is slightly better than T07 (0.10%). To the best of our knowledge, this is the first analysis in the literature that investigates the performance of heuristics for the static–dynamic uncertainty policy under β^{cyc} service level constraints.

Finally, we contrasted the performance of the model in [34] (TK06), and of the lower and upper bound models we discussed in Section 4.3 (MILP-11 LB & MILP-11 UB) against the optimal policy under a backorder penalty cost scheme. Also in this case all models provide excellent optimality gaps on average of 0.15% the optimal expected total cost.

In Table 1 we report average optimality gaps (%) of methods considered in our study for different pivoting parameters: the coefficient of variation c, which reflects the impact of a smaller or larger demand forecasting error; the fixed setup cost a; the backorder penalty cost b; and the service level (α or β^{cyc}). An increase in c has a clear negative impact on all models except T07, which appears to be unaffected. An increase in a reduces the optimality gap under α service level constraints, while does not seem to have an effect on models operating under β^{cyc} service level constraints or backorder penalty cost. An increase of b tends to generate a slight increase of the optimality gap for both TK06 and MILP-11. Finally, an increase in the prescribed service level affects negatively the optimality gap for all models under α service level constraints; it does not affect the optimality gap of T07, and negatively affects MILP-11.

Two of the three exact models considered, i.e. the two based on the framework introduced in [31] addressing the α and β^{cyc} service level cases, were computationally cumbersome even for this reduced size test bed; with each instance taking several minutes to be solved. Instead, as the authors demonstrate, the exact model based on the framework in [35] is computationally efficient. All heuristic models considered — BT88, TK04, TK06, T07, and MILP-11 UB/LB — took on average a tenth of a second to solve an instance; this computational performance is comparable to that discussed in [35], also considering the fact that the machines used in these two studies do not significantly differ from each other.

5.3. Linearisation gap

We now investigate the behaviour of the linearisation gap for models in which demand that occurs when the system is out of stock is backordered (Section 5.3.1) and those in which demand is lost (Section 5.3.2).

5.3.1. Backorders

We concentrate on models presented in Section 4.1. Recall that in these models, when a stockout occurs, all demand is backordered and filled as soon as an adequate supply arrives. We investigate the behaviour of the linearisation gap and of the solution time for normally distributed demand. We then extend the analysis to the case in which demand in different periods follow different probability distributions.

Normal distribution: As in the computational study presented in Section 5.2 demand d_t in each period t is normally distributed and we let the coefficient of variation $c \in \{0.10, 0.20, 0.30\}$. Expected values of the demand in each period are illustrated in Fig. 5 for each of the ten

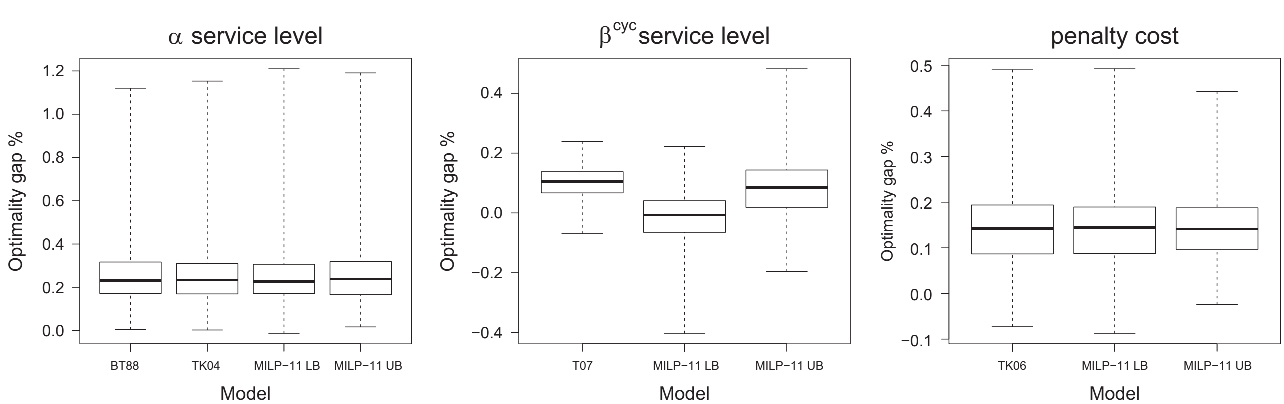

Fig. 6. Boxplots illustrating the optimality gap for different models considered in our study.

Table 1
Average optimality gaps (%) of methods for different pivoting parameters.

		α service level				β^{cyc} service level			penalty cost		
		BT88	TK04	MILP-11		T07	MILP-11		TK06	MILP-11	
				LB	UB		LB	UB		LB	UB
c	0.1	0.202	0.198	0.198	0.202	0.098	-0.002	0.033	0.118	0.107	0.119
	0.2	0.256	0.249	0.257	0.256	0.106	-0.023	0.067	0.160	0.162	0.153
	0.3	0.313	0.314	0.300	0.311	0.098	-0.024	0.165	0.160	0.169	0.166
a	500	0.351	0.346	0.341	0.345	0.099	-0.064	0.088	0.147	0.147	0.156
	1000	0.240	0.231	0.233	0.244	0.103	-0.007	0.080	0.155	0.155	0.148
	2000	0.180	0.184	0.180	0.180	0.100	0.022	0.097	0.137	0.136	0.134
b	2								0.133	0.131	0.121
	5								0.149	0.148	0.155
	10								0.157	0.159	0.162
service level	0.8	0.222	0.230	0.226	0.233	0.110	-0.008	0.066			
	0.9	0.258	0.242	0.245	0.240	0.097	-0.005	0.092			
	0.95	0.291	0.288	0.284	0.297	0.095	-0.036	0.107			

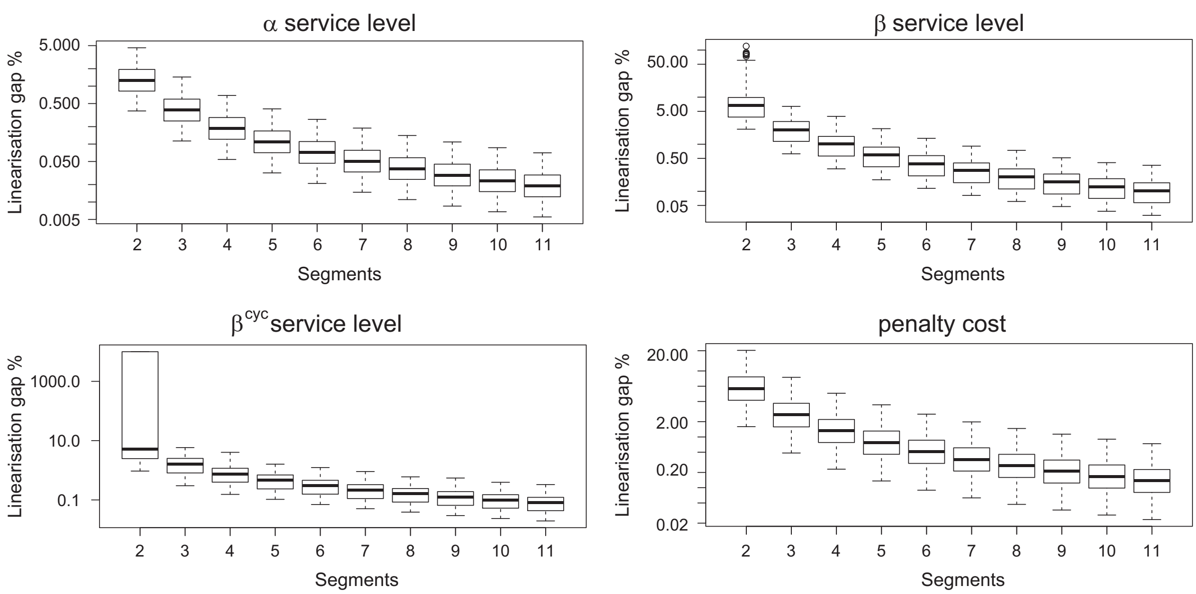

Fig. 7. Linearisation gap for different number of segments used in the piecewise linear approximation under normally distributed demand and backorders.

patterns considered. In contrast to our previous analysis for each pattern we now consider all periods.

In Fig. 7 we report, for each model discussed, boxplots illustrating the linearisation gap trend for different number of segments used in the piecewise linear approximation. It should be noted that the y-axis is displayed in logarithmic scale. This shows that the linearisation gap shrinks exponentially fast in the number of segments regardless of the model or parameter setting considered.

It is interesting to observe that a number of instances were found infeasible by the MILP model under β^{cyc} service level when two segments for the piecewise approximation were used — note the very large optimality gap. This is due to the fact that with only two segments the approximation error for the β^{cyc} service level was too large and no order-up-to-level could be found to enforce a service level as high as the prescribed one.

In Fig. 8 we report, for each model discussed, boxplots illustrating the computational time trend for a different number of segments used in the piecewise linear approximation. Computational times are only slightly affected by the number of segments in the approximation. Furthermore, all instances could be solved in just a few seconds.

Generic distribution: We now extend the analysis to the case in which demand in different periods follows different probability distributions. The test bed is the same that was previously analysed. However, demand in each period t now follows a gamma(a,b) distribution, where $b=10$ and $a=\tilde{d}_t/b$, so that the expected value reflects that of the demand in period t for the pattern considered. To the best of our knowledge, this is the first study in which a numerical analysis on demand that is not normally distributed is presented.

We limit our analysis to the model that operates under penalty cost scheme. We do this because the model that operates under penalty cost scheme embeds both the first order loss function — employed to compute expected shortages — and the complementary first order loss function — employed to compute holding costs; therefore we expect that results obtained for this model to be sufficiently representative of the overall optimality gap attained by our piecewise linear approximation of these two functions.

As discussed in Section 3, when demand follows a generic probability distribution we must compute dedicated linearisation parameters for the piecewise first order loss function. For a given number of segments better parameters will produce a tighter

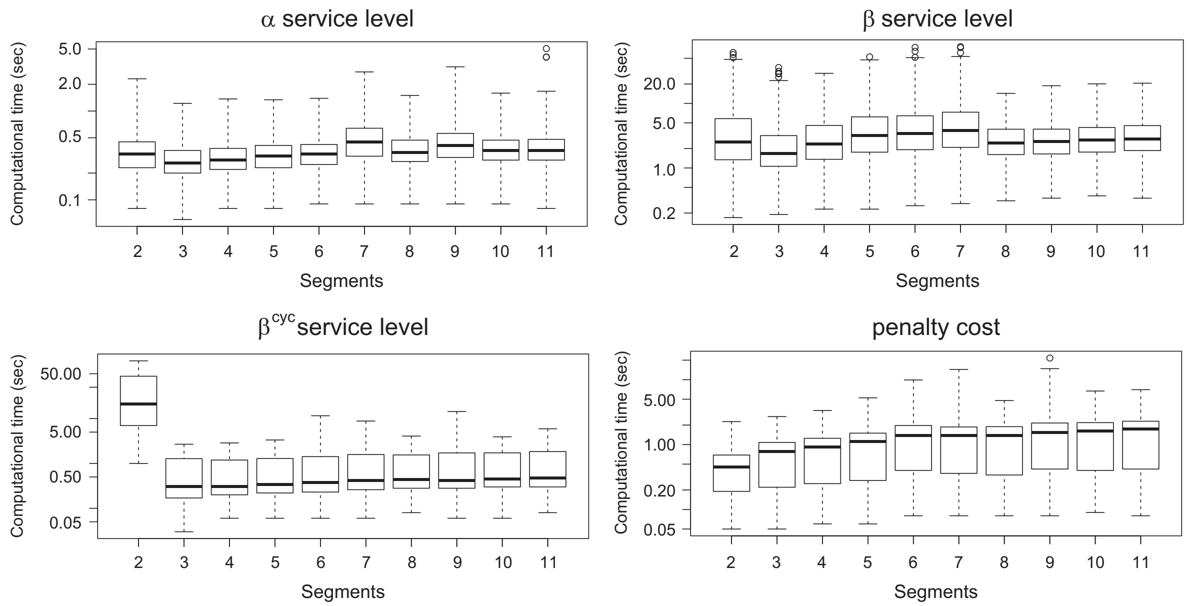

Fig. 8. Computational time for different number of segments used in the piecewise linear approximation under normally distributed demand and backorders.

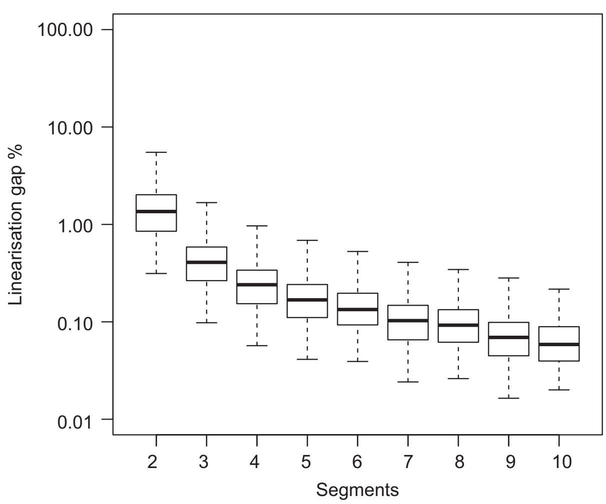

Fig. 9. Linearisation gap for different number of segments used in the piecewise linear approximation under gamma distributed demand and backorders.

optimality gap and therefore a better estimation of the cost of an optimal policy. Fig. 9 illustrates results for the case in which we split the support of the demand ω uniformly into W disjoint compact subregions $\Omega_1, \ldots, \Omega_W$ such that $p_i = \Pr\{\omega \in \Omega_i\} = 1/W$.

The results presented reveal that a uniform partitioning of the support of the demand provides excellent results — linearisation gap shrinks exponentially when the number of segments in the linearisation increases — note that axes are in logarithmic scale. However, as discussed in [42], the problem of computing optimal linearisation parameters is computationally challenging and future research should investigate global optimisation algorithms to address this open research question.

A final remark should be made on the computational efficiency of our approach under a generic demand distribution. Under a uniform partitioning of the support the computational efficiency of the model is essentially identical to the case in which demand is normally distributed. In other words, most of the instances can be solved in few seconds as shown in Fig. 8.

5.3.2. *Lost sales*

We finally extended our analysis to the case in which demand that occurs when the system is out of stock is lost. Models that operate under this setting were discussed in Section 4.6. We analyse the case in which demand d_t in each period t is normally distributed with coefficient of variation $c \in \{0.10, 0.20, 0.30\}$.

As shown in Fig. 10 also in this case the linearisation gap shrinks exponentially fast for all models considered when the number of segments in the linearisation increases. Furthermore, as shown in Fig. 11 all instances could be solved in few seconds. We do not discuss models that operate under lost sales and generic probability distributions since results obtained were comparable to those already discussed.

6. Conclusions

We developed a unified MILP modelling approach for the non-stationary stochastic lot sizing problem based on piecewise linearisation of the first order loss function and of its complementary function. This approach applies to settings in which unmet demand is back-ordered or lost; and it can accommodate variants of the problem for which the quality of service is captured by means of backorder penalty costs, non-stockout probabilities, or fill rate constraints. It has a number of advantages with respect to other existing approaches in the literature. It is versatile, as it enables seamless modelling of several variants of this problem. It is fully linear and, for the special case in which demand in each period is normally distributed, it does not require an offline evaluation of piecewise linearisation coefficients, as these can be derived from a standard table [12]. As shown in our computational experience, viable linearisation parameters for generic demand distributions can be derived by partitioning the support of the demand uniformly. Another advantage with respect to other existing approaches is that we bound from above and below the cost of an optimal plan; by increasing the number of segments in the piecewise linear approximation precision can be improved ad libitum. In our extensive computational experience we demonstrated that the optimality gap of our models is comparable to that featured by other state-of-the-art models in the literature. The linearisation gap shrinks exponentially fast in the number of segments used by the piecewise

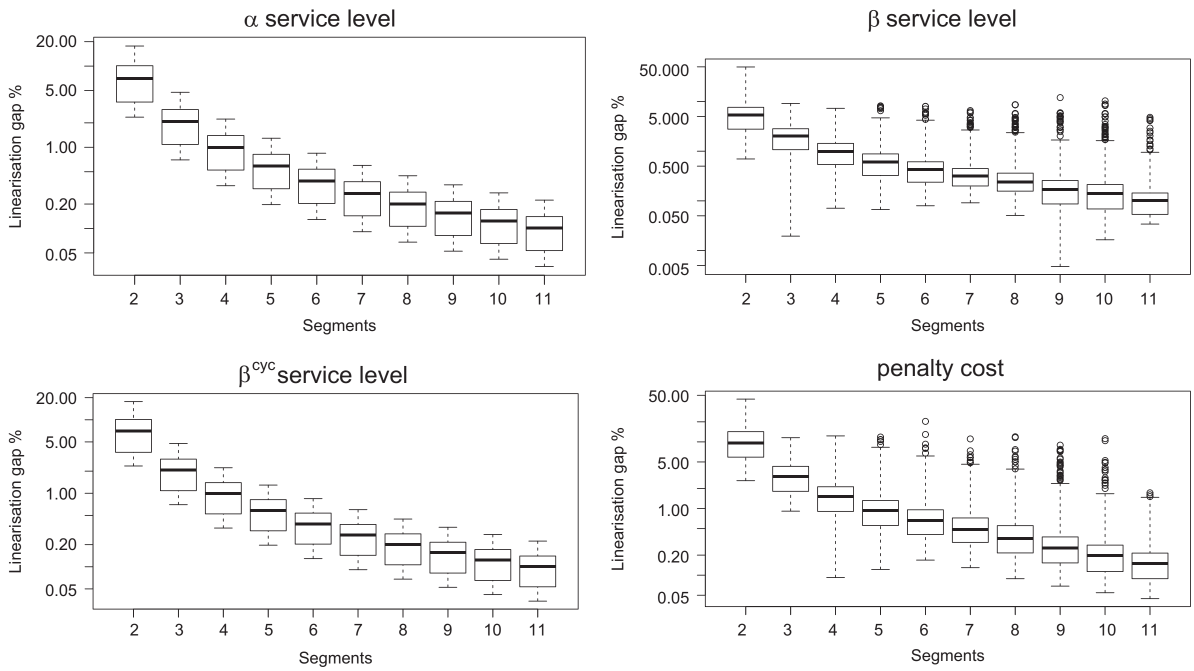

Fig. 10. Linearisation gap for different number of segments used in the piecewise linear approximation under normally distributed demand and lost sales.

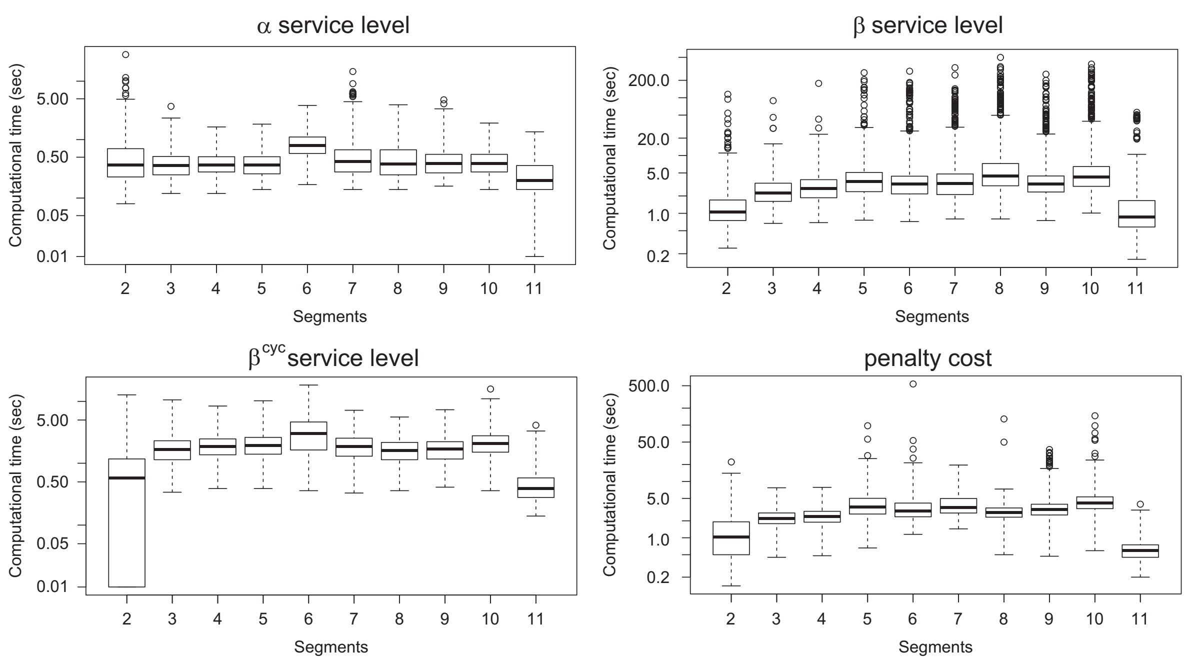

Fig. 11. Computational time for different number of segments used in the piecewise linear approximation under normally distributed demand and lost sales.

linearisation and all models developed can be generally solved in a few seconds when up to eleven segments are used in the linearisation.

Acknowledgements

R. Rossi would like to thank Peter Richtarik and Belen Martin–Barragan for the insightful discussions. The authors would like to thank the anonymous reviewers for their constructive suggestions.

Appendix A

In Appendix A.1, we present a list of all symbols used in the paper; in Appendix A.2, we briefly discuss the two approximate methods adopted in this work for computing good linearisation parameters; in Appendix A.3, we provide an overview of the MILP models discussed; in Appendix A.4 we discuss special cases that arise when demand is normally distributed.

A.1. List of symbols

x	a scalar value
ω	a random variable
Ω	support of ω
$g_\omega(x)$	probability density function of ω, where $x \in \Omega$
$\tilde{\omega}$	expected value of ω
σ_ω	standard deviation of ω

Z	a standard normal random variable
$\phi(x)$	standard normal probability density function
$\Phi(x)$	standard normal cumulative probability distribution function
W	number of regions in a partition of Ω
i	region index ranging in $1,\dots,W$
Ω_i	a compact region of Ω
p_i	$\Pr\{\omega \in \Omega_i\}$
$E[\omega\|\Omega_i]$	conditional expectation of ω in Ω_i
$\mathcal{L}(x,\omega)$	first order loss function
$\widehat{\mathcal{L}}(x,\omega)$	complementary first order loss function
$\widehat{\mathcal{L}}_{lb}(x,\omega)$	piecewise linear lower bound of $\widehat{\mathcal{L}}(x,\omega)$ with $W+1$ segments
$\widehat{\mathcal{L}}_{ub}(x,\omega)$	piecewise linear upper bound of $\widehat{\mathcal{L}}(x,\omega)$ with $W+1$ segments
e_W	maximum approximation error for $\widehat{\mathcal{L}}_{lb}(x,\omega)$ and $\widehat{\mathcal{L}}_{ub}(x,\omega)$
N	periods in the planning horizon
t	period index ranging in 1,…,N
d_t	demand in period t
c	demand coefficient of variation $\tilde{d}_t/\sigma_{d_t}$
$g_t(\cdot)$	probability density function of d_t
$G_t(\cdot)$	cumulative distribution function of d_t
a	fixed ordering cost, \$$a$ per order
v	proportional unit cost, \$$v$ per unit ordered
s	selling price, \$$s$ per item
m	margin, \$$s-v$ per item
h	holding cost, \$$h$ per unit of inventory carried to the next period
$E[TC]$	expected total cost
I_0	initial inventory, a scalar
I_t	inventory level at the end of period t
Q_t	order quantity issued (and received) at the beginning of period t
δ_t	binary variable set to one if and only if $Q_t>0$.
α	nonstockout probability
β	prescribed fill rate
β^{cyc}	prescribed cycle fill rate
b	penalty cost \$$b$ per unit short per unit time
M	a very large number
P_{jt}	a binary variable which is set to one if and only if the most recent inventory review before period t was carried out in period j
S_t	order-up-to-level in period t
d_{jt}	random variable representing the convolution $d_j+\dots+d_t$
$G^{-1}_{d_{jt}}(\alpha)$	α-quantile of the inverse cumulative distribution function of d_{jt}
e^{jt}_W	maximum approximation error associated with the piecewise linearisation of $\widehat{\mathcal{L}}(x,d_{jt})$
$\tilde{I}^{lb}_t$	a lower bound to the true value of $E[\max(I_t,0)]$
$\tilde{I}^{ub}_t$	an upper bound to the true value of $E[\max(I_t,0)]$
$\tilde{B}^{lb}_t$	a lower bound to the true value of $E[-\min(I_t,0)]$
$\tilde{B}^{ub}_t$	an upper bound to the true value of $E[-\min(I_t,0)]$
$\tilde{C}^{lb}_t$	a lower bound to the expected total backorders within the replenishment cycle that ends at period t, if any exists. $\tilde{C}^{lb}_t$ is equal to $\tilde{B}^{lb}_t$ if t is the last period of a replenishment cycle
$\tilde{C}^{ub}_t$	an upper bound to the expected total backorders within the replenishment cycle that ends at period t, if any exists. $\tilde{C}^{ub}_t$ is equal to $\tilde{B}^{ub}_t$ if t is the last period of a replenishment cycle

A.2. *Linearising the complementary first order loss function for a set of random variables*

We briefly discuss implementation details of the approximate method employed in this paper to determine linearisation parameters of the complementary first order loss function for a set of random variables.

Consider a set of generic random variables $\omega_1,\dots,\omega_N$ with complementary first order loss functions $\widehat{\mathcal{L}}(x,\omega_1),\dots,\widehat{\mathcal{L}}(x,\omega_N)$. From (2) it is clear that, once all p_i are fixed, all $E[\omega|\Omega_k]$ are uniquely determined for each loss function $\widehat{\mathcal{L}}(x,\omega_n)$; this computation can be carried out numerically by using off-the-shelf packages such as Mathematica[7] or libraries such as SSJ.[8] If our aim is to derive a piecewise linearisation with $W+1$ segments, then a possible strategy is to partition the support of each random variable ω_n into W disjoint compact subregions $\Omega_1,\dots,\Omega_W$, such that $\Pr\{\omega\in\Omega_i\}=1/W$. Once more, these regions can be determined numerically with one of the aforementioned off-the-shelf packages. By recalling that the complementary first order loss function is convex, it follows that the maximum approximation error will be attained at one of the breakpoints of its piecewise linear approximation. Since there are W breakpoints the maximum approximation error associated with the above partitioning can be easily determined for each $\widehat{\mathcal{L}}(x,\omega_1),\dots,\widehat{\mathcal{L}}(x,\omega_N)$.

[7] http://www.wolfram.com/mathematica/
[8] http://www.iro.umontreal.ca/~simardr/ssj/

A.3. List of MILP models

An overview of the MILP models discussed; numbers refer to the respective equations in the text.

	Lower bound		Upper bound	
	Objective	Subject to	Objective	Subject to
Backorders				
α Service level	(20)	(12) (13) (14) (15) (16) (17) (18) (22)	(21)	(12) (13) (14) (15) (16) (17) (18) (23)
Penalty cost	(24)	(12) (13) (15) (16) (17) (18) (22) (26)	(25)	(12) (13) (15) (16) (17) (18) (23) (27)
β^{cyc} Service level	(20)	(12) (13) (15) (16) (17) (18) (22) (26) (29)	(21)	(12) (13) (15) (16) (17) (18) (23) (27) (30)
β Service level	(20)	(12) (13) (15) (16) (17) (18) (22) (26) (33) (35)	(21)	(12) (13) (15) (16) (17) (18) (23) (27) (34) (36)
Lost sales				
α Service level	(43)	(22) (44) (45) (46) (47) (48) (49) (50) (51) (52) (53)		Replace $\tilde{I}_t^{ub}$, (22) with $\tilde{I}_t^{lb}$, (23)
Penalty cost	(54)	(22) (26) (44) (45) (46) (47) (48) (50) (51) (52) (53)		Replace $\tilde{I}_t^{ub}$, (22), (26), with $\tilde{I}_t^{lb}$, (23), (27)
β^{cyc} Service level	(43)	(22) (26) (29) (44) (45) (46) (47) (48) (50) (51) (52) (53)		Replace $\tilde{I}_t^{ub}$, (22), (26), (29) with $\tilde{I}_t^{lb}$, (23), (27), (30)
β Service level	(43)	(22) (26) (33) (35) (44) (45) (46) (47) (48) (50) (51) (52) (53)		Replace $\tilde{I}_t^{ub}$, (22), (26), (33), (35) with $\tilde{I}_t^{lb}$, (23), (27), (34), (36)

A.4. Normally distributed demand

In this appendix we discuss model variants for the case in which demand in each period is normally distributed. The assumption that demand is normally distributed plays a prominent role in inventory theory (see e.g. [47]) and most of the existing works on stochastic lot sizing focus on this distribution. An important property of the first order normal loss function is that it can be derived from its standard counterpart. Let ω be a normally distributed random variable with mean μ and standard deviation σ. Let $\phi(x)$ be the standard normal probability density function and $\Phi(x)$ the respective cumulative distribution function.

Lemma 4. *The complementary first order loss function of ω can be expressed in terms of the standard normal cumulative distribution function as*

$$\widehat{\mathcal{L}}(x,\omega)=\sigma\int_{-\infty}^{(x-\mu)/\sigma}\Phi(t)\,dt=\sigma\widehat{\mathcal{L}}\left(\frac{x-\mu}{\sigma},Z\right), \tag{55}$$

where Z is a standard normal random variable.

Rossi et al. [12] discussed how to obtain an optimal partitioning of the support under a framework that minimises the maximum approximation error. The same work reports standard linearisation parameters for the case in which ω is a standard normal random variable.

A.4.1. α Service level constraints

If demand in each period is normally distributed, we can exploit Lemma 4 to reduce the number of linearisation parameters that are needed in the model. Consider a partition of the support Ω of a standard normal random variable Z into W adjacent regions Ω_i. Recall that $p_i=\Pr\{Z\in\Omega_i\}$, by exploiting Jensen's piecewise linear lower bound, we introduce the following constraints in the model:

$$\tilde{I}_t^{lb}\ge\tilde{I}_t\sum_{k=1}^{i}p_k-\sum_{j=1}^{t}\left(\sum_{k=1}^{i}p_kE[Z|\Omega_i]\right)P_{jt}\sigma_{d_{jt}},\quad t=1,\dots,N;\ i=1,\dots,W$$

where $\sigma_{d_{jt}}$ denotes the standard deviation of $d_j+\dots+d_t$ and $\tilde{I}_t^{lb}\ge0$. This expression follows immediately from Lemmas 2 and 4: consider a replenishment cycle covering periods $j,\dots,t$ and associated order-up-to-level S. We aim to enforce $\tilde{I}_t^{lb}\ge\sigma\widehat{\mathcal{L}}_{lb}^{i}((S-\tilde{d}_{jt})/\sigma_{d_{jt}},Z)$ for all $i=1,\dots,W$. Observe that $S-\tilde{d}_{jt}=\tilde{I}_t$, the above expression follows immediately. We then derive $\tilde{I}_t^{ub}$ from $\tilde{I}_t$

$$\tilde{I}_t^{ub}\ge\tilde{I}_t\sum_{k=1}^{i}p_k+\sum_{j=1}^{t}\left(e_W-\sum_{k=1}^{i}p_kE[Z|\Omega_i]\right)P_{jt}\sigma_{d_{jt}},\qquad t=1,\dots,N,\ i=1,\dots,W;$$

where $\tilde{I}_t^{ub}\ge\sum_{j=1}^{t}e_WP_{jt}\sigma_{d_{jt}}$ for $t=1,\dots,N$; and e_W denotes the maximum approximation error associated with a partition comprising W regions; linearisation parameters that minimise the maximum approximation error e_W for a given number $W+1$ of segments, where $W=1,\dots,10$, can be found in [12].

A final remark that is worth making is the fact that it is possible to replace e_W with $e_W/2$ in the above equations to obtain an approximation of the first order loss function that minimises the maximum absolute error. However, this approximation does not allow one to establish if the cost produced by the model is an upper or a lower bound for the true cost of an optimal plan.

A.4.2. Penalty cost scheme

We discuss how to handle the case in which demand is normally distributed by exploiting Lemma 1. We obtain $\tilde{B}_t^{lb}$ and $\tilde{B}_t^{ub}$ from $\tilde{I}_t$ by exploiting the connection between Jensen's piecewise linear lower bound and the piecewise linear upper bound to the first order loss function (Lemma 1):

$$\tilde{B}_t^{lb}\ge-\tilde{I}_t+\tilde{I}_t\sum_{k=1}^{i}p_k-\sum_{j=1}^{t}\left(\sum_{k=1}^{i}p_kE[Z|\Omega_i]\right)P_{jt}\sigma_{d_{jt}},\qquad t=1,\dots,N,\ i=1,\dots,W; \tag{56}$$

where $\tilde{B}_t^{ub}\ge-\tilde{I}_t$ and

$$\tilde{B}_t^{ub}\ge-\tilde{I}_t+\tilde{I}_t\sum_{k=1}^{i}p_k+\sum_{j=1}^{t}\left(e_W-\sum_{k=1}^{i}p_kE[Z|\Omega_i]\right)P_{jt}\sigma_{d_{jt}},\qquad t=1,\dots,N,\ i=1,\dots,W; \tag{57}$$

where $\tilde{B}_t^{ub}\ge-\tilde{I}_t+\sum_{j=1}^{t}e_WP_{jt}\sigma_{d_{jt}}$.

References

[1] Wagner HM, Whitin TM. Dynamic version of the economic lot size model. Management Science 1958;5:89–96.
[2] Bookbinder JH, Tan JY. Strategies for the probabilistic lot-sizing problem with service-level constraints. Management Science 1988;34:1096–108.
[3] Scarf HE. Optimality of (s, S) policies in the dynamic inventory problem. In: Arrow KJ, Karlin S, Suppes P, editors. Mathematical methods in the social sciences. Stanford, CA: Stanford University Press; 1960. p. 196–202.
[4] Tempelmeier H. Stochastic lot sizing problems. In: Smith JM, Tan B, editors. Handbook of stochastic models and analysis of manufacting system operations, international series in operations research & management science, vol. 192. New York: Springer; 2013. p. 313–44.
[5] Silver EA, Pyke DF, Peterson R. Inventory management and production planning and scheduling. 3rd ed. New York: Wiley; 1998.
[6] Mutlu F, Çetinkaya S, Bookbinder JH. An analytical model for computing the optimal time-and-quantity-based policy for consolidated shipments. IIE Transactions 2010;42:367–77.
[7] Relvas S, Boschetto Magatão SN, Barbosa-Póvoa AP, Neves F. Integrated scheduling and inventory management of an oil products distribution system. Omega 2013;41:955–68.
[8] Pan F, Nagi R. Multi-echelon supply chain network design in agile manufacturing. Omega 2013;41:969–83.
[9] Kilic OA, Tarim SA. An investigation of setup instability in non-stationary stochastic inventory systems. International Journal of Production Economics 2011;133:286–92.
[10] Tunc H, Kilic OA, Tarim SA, Eksioglu B. A simple approach for assessing the cost of system nervousness. International Journal of Production Economics 2013;141:619–25.
[11] Özen U, Doğru MK, Armagan Tarim S. Static–dynamic uncertainty strategy for a single-item stochastic inventory control problem. Omega 2012;40:348–57.
[12] Rossi R, Tarim SA, Prestwich S, Hnich B. Piecewise linear lower and upper bounds for the standard normal first order loss function. Applied Mathematics and Computation 2014;231:489–502.
[13] Kall P, Wallace SW. Stochastic programming (Wiley Interscience Series in Systems and Optimization). Chichester: John Wiley & Sons; 1994.
[14] Hadley GH, Whitin TM. Analysis of inventory systems. Englewood Cliffs, New Jersey: Prentice Hall; 1963.
[15] Axsäter S. Inventory control (international series in operations research & management science). 2nd ed. New York: Springer; 2010.
[16] Tempelmeier H, Herpers S. ABC_{β}—a heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. International Journal of Production Research 2009;48:5181–93.
[17] Tempelmeier H, Herpers S. Dynamic uncapacitated lot sizing with random demand under a fillrate constraint. European Journal of Operational Research 2011;212:497–507.
[18] Tempelmeier H. A column generation heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. Omega 2011;39:627–33.
[19] Helber S, Sahling F, Schimmelpfeng K. Dynamic capacitated lot sizing with random demand and dynamic safety stocks. OR Spectrum 2013;35:75–105.
[20] Bollapragada S, Morton TE. A simple heuristic for computing non-stationary (s, S) policies. Operations Research 1999;47:576–84.
[21] Lagodimos AG, Christou IT, Skouri K. Computing globally optimal (s, S, T) inventory policies. Omega 2012;40:660–71.
[22] Silver EA. Inventory control under a probabilistic time-varying, demand pattern. AIIE Transactions 1978;10:371–9.
[23] Askin RG. A procedure for production lot sizing with probabilistic dynamic demand. AIIE Transactions 1981;13:132–7.
[24] Silver EA, Meal HC. A heuristic for selecting lot size quantities for the case of a deterministic time-varying demand rate and discrete opportunities for replenishment. Production and Inventory Management 1973;14:64–74.
[25] Tarim SA, Kingsman BG. The stochastic dynamic production/inventory lot-sizing problem with service-level constraints. International Journal of Production Economics 2004;88:105–19.
[26] Tarim SA, Dogÿru MK, Özen U, Rossi R. An efficient computational method for a stochastic dynamic lot-sizing problem under service-level constraints. European Journal of Operational Research 2011;215:563–71.
[27] Rossi R, Tarim SA, Hnich B, Prestwich S. A state space augmentation algorithm for the replenishment cycle inventory policy. International Journal of Production Economics 2011;133:377–84.
[28] Tunc H, Kilic OA, Tarim SA, Eksioglu B. A reformulation for the stochastic lot sizing problem with service-level constraints. Operations Research Letters 2014;42:161–5.
[29] Tarim SA, Hnich B, Rossi R, Prestwich SD. Cost-based filtering techniques for stochastic inventory control under service level constraints. Constraints 2009;14:137–76.
[30] Tarim SA, Smith B. Constraint programming for computing non-stationary (R,S) inventory policies. European Journal of Operational Research 2008;189:1004–21.
[31] Rossi R, Tarim SA, Hnich B, Prestwich S. A global chance-constraint for stochastic inventory systems under service level constraints. Constraints 2008;13:490–517.
[32] Rossi R, Tarim SA, Hnich B, Prestwich S. Computing the non-stationary replenishment cycle inventory policy under stochastic supplier lead-times. International Journal of Production Economics 2010;127:180–9.
[33] Rossi R, Tarim SA, Bollapragada R. Constraint-based local search for computing non-stationary replenishment cycle policy under stochastic lead-times. INFORMS Journal on Computing 2012;24(1):66–80. http://dx.doi.org/10.1287/ijoc.1100.0434.
[34] Tarim SA, Kingsman BG. Modelling and computing (Rn,Sn) policies for inventory systems with non-stationary stochastic demand. European Journal of Operational Research 2006;174:581–99.
[35] Rossi R, Tarim SA, Hnich B, Prestwich S. Constraint programming for stochastic inventory systems under shortage cost. Annals of Operations Research 2012;195:49–71.
[36] Tempelmeier H. On the stochastic uncapacitated dynamic single-item lotsizing problem with service level constraints. European Journal of Operational Research 2007;181:184–94.
[37] Snyder LV, Shen Z-JM. Fundamentals of supply chain theory. 1st ed.. New Jersey: Wiley; 2011.
[38] Silver EA, Bischak DP. The exact fill rate in a periodic review base stock system under normally distributed demand. Omega 2011;39:346–9.
[39] Axsäter S. A simple procedure for determining order quantities under a fill rate constraint and normally distributed lead-time demand. European Journal of Operational Research 2006;174:480–91.
[40] Gavrilović MM. Optimal approximation of convex curves by functions which are piecewise linear. Journal of Mathematical Analysis and Applications 1975;52:260–82.
[41] Imamoto A, Tang B. Optimal piecewise linear approximation of convex functions. In: Ao SI, Douglas C, Grundfest WS, Schruben L, Burgstone J, editors, Proceedings of the world congress on engineering and computer science 2008 WCECS 2008, October 22–24, 2008, San Francisco, USA: IAENG; 2008. p. 1191–4.
[42] Rossi R, Hendrix EMT. Piecewise linearisation of the first order loss function for families of arbitrarily distributed random variables. In: Hendrix EMT, editor, Proceedings of MAGO 2014, XII Global Optimization Workshop (GOW) September 1–4, 2014, Malaga, Spain, 2014, pp. 117–120.
[43] Rossi R, Tarim SA, Kilic OA. A note on Tempelmeier's β-service measure under non-stationary stochastic demand; 2011. arXiv:1103.1286.
[44] Chen J, Lin DKJ, Thomas DJ. On the single item fill rate for a finite horizon. Operations Research Letters 2003;31:119–23.
[45] Thomas DJ. Measuring item fill-rate performance in a finite horizon. Manufacturing & Service Operations Management 2005;7:74–80.
[46] Kurawarwala AA, Matsuo H. Forecasting and inventory management of short life-cycle products. Operations Research 1996;44:131–50.
[47] Axsäter S. When is it feasible to model low discrete demand by a normal distribution? OR Spectrum 2013;35:153–62.