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X-Y Band and (s,S) Policies

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This paper considers a single-item, periodic review inventory model with uncertain demands. We assume a finite production capacity in each period, and the production costs are composed of a fixed as well as a variable cost. With stationary data, a convex expected holding and shortage cost function, we show with counter examples that generally the modified (s,S) policy is not optimal to either finite or infinite time horizon inventory problems. Instead, the optimal policy exhibits a **X-Y band**: whenever the inventory level drops below **X**, order up to capacity; when the inventory level is above **Y**, do nothing; if the inventory level is between **X** and **Y**, however, the ordering pattern is different from problem to problem. We also provide one calculation for such **X** and **Y** boundaries which are tight in some cases.

1. INTRODUCTION. This paper deals with one of the most basic production and inventory models, in which the stock of a single item must be controlled under periodic review: There is a production cost if something is to be produced at the beginning of a period, and the expected holding and shortage cost charged at the end of a period is a convex function. Demand in each period is nonnegative and independently distributed from period to period. Stockouts are backordered. All data are stationary, and the planning horizon may be finite or infinite. Here we assume demands are discrete (integer-valued), although it does not make much difference if demands are assumed continuous. We also assume the expected-discounted-cost criterion.

In the past, many papers have tackled this problem. They mainly differ on the treatments of the following three aspects: a. Production cost; b. Production capacity; c. Planning horizon. The production cost may be linear in the amount produced or linear plus a fixed cost. The production capacity may be assumed unlimited or limited. The planning horizon may be finite or infinite.

If the production cost is linear and the production capacity is unlimited, then the optimal inventory policy can be described by a single critical number : When initial stock is below that number, enough should be produced to bring total stock up to the number; otherwise, nothing should be produced. This policy is sometimes called base-stock policy [1] [7] [8] [9].

If the production cost is linear and the production capacity is finite, the modified base-stock policy is then optimal: Follow a base-stock policy when possible; when the prescribed production quantity would exceed the capacity, produce to capacity [4]).

If the production cost is linear plus a fixed cost and the production capacity is unlimited, then the (s,S) policy is optimal: if the inventory level falls below a critical number s, produce enough to bring total stock up to S; otherwise nothing should be produced [7] [6] [8].

Although the critical number(s) of these optimal policies (base-stock-, modified base-stock-, or (s,S)- policy) may vary from period to period if the planning time horizon is finite, there exists stationary policies of corresponding types that are optimal to the infinite time horizon problems (e.g., [1] [2]).

Finally, if the production cost is linear plus a fixed cost and the production capacity is finite, the problem has not yet been solved. Wijngaard [10] gives an example of a finite horizon problem having a complex optimal policy. In this paper, we extend the work by Federgruen A. and Zipkin P [4] to the finite capacity with a fixed set up cost case. We show with a counter example that generally the modified (s,S)-policy is not optimal to this class of problems, where the modified (s,S)-policy, defined in [3], is as follows: When the inventory level falls below a critical number s, produce enough to bring total stock up to S or as close to it as possible, given the limited capacity; otherwise do not produce. Instead, Our paper proves that the optimal policy to the problem, finite or infinite time horizon, exhibits a **X-Y** band: When the inventory level drops below **X**, produce at full capacity; when the inventory level is above **Y**, produce nothing; if the inventory level is between **X** and **Y**, however, the ordering pattern is different from problem to problem. Thus, a modified (s,S)-policy is a special case of **X-Y** band policy if we set **X** = S - CP, and **Y** = s, where CP is the production capacity. The difference being that, with a **X-Y** band policy we know what to do only if the inventory level is outside the **X-Y** band, but it is undetermined within the **X-Y** band; with a modified (s,S)-policy, however, we not only know what to do outside, but also know within, as shown in the following figures:

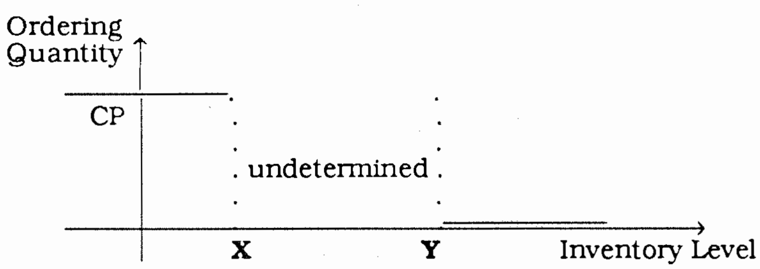

Figure 1. X-Y Band Policy

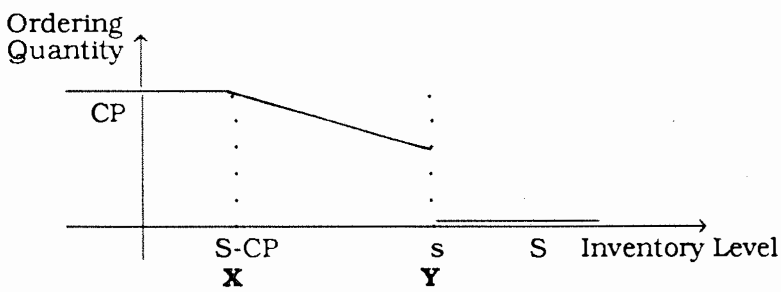

Figure 2. Modified (s,S)-policy

(Here we assumed that S - CP ≤ s, or, S - s ≤ CP, which will be justified later by proving that for any modified (s,S)-policy, we can always find S', such that the modified (s,S')-policy is equivalent to the modified (s,S)-policy, and S' - s ≤ CP.)

In this paper, we also give one calculation for **X** and **Y** boundaries which are tight in some cases.

Section 2 sets the notation and the basic assumptions, and presents the dynamic programming for solving the finite time horizon problem. Section 3 proves that $f_n(x)$ is a continuous function of x for any n, where $f_n(x)$ is the expected discounted cost over the last n periods, if the inventory level at the beginning of the last n periods is x and if the optimal inventory policy is followed over the n periods. Readers who prefer simplicity may skip this section by accepting the continuity of $f_n(x)$. Section 4 proves the existence of **X**. Section 5 proves the existence of **Y**, under the assumption that the maximum demand in one period MD, is less or equal to the capacity. Section 6 proves the existence of **Y** if MD exceeds the capacity, under an additional unrestrictive assumption. Section 7 proves the existence and continuity and some other properties of f(x), the limiting function of $f_n(x)$ when n approaches infinity. We show that the optimal policy for an infinite time horizon problem also exhibits a **X-Y** band. Section 8 presents some of the properties that f(x) should have given the modified (s,S)-policy is optimal to an infinite time horizon problem. Finally, we provide a counter example to the optimality of the modified (s,S)-policy to the infinite time horizon problems.

2. NOTATIONS, ASSUMPTIONS and the DYNAMIC PROGRAMMING FORMULATION.

D_t = demand in period t, t = 0,1,...

We assume that D_t are independently and identically distributed for all periods.

$p(j) = Pr(D_t = j)$, j = 0,1,... 需求是随机离散的

We assume that D_t is upper bounded, i.e., $D_t \le M$ for some positive integer M. The rationale for this will be given later. Practically this is of no restriction at all.

CP = production capacity, or limit on order size, a positive number.

K = set up cost.

c = per-unit order (production) cost.

c(q) = production cost (ordering cost) of producing (ordering) q units.

$$c(q) = \begin{cases} 0 & \text{for } q = 0 \\ K + cq & \text{for } q > 0 \end{cases}$$

T = lead time, a nonnegative integer.

α = discount rate, $0 \le \alpha < 1$.

y = inventory level just after an ordering decision has been made.

$Y(x) = \{ y : x \le y \le x + CP \}$, the feasible values of y given x, the beginning inventory level.

$O_n^*(x)$ = the optimal production (ordering) quantity, which is a function of x, the beginning inventory level, and n, the planning time horizon.

$y_n^*(x)$ = the optimal decision for y. $y_n^*(x) = x + O_n^*(x)$.

L(y) = one-period expected holding and shortage cost, which is calculated by the approach in [5] for any $T \ge 0$.

$f_n(x)$ = expected discounted cost for an n-period planning horizon problem, if the beginning inventory level is x, and if an optimal policy is followed over the n periods.

Assumption. (I) L(y) is a nonnegative and convex continuous function. 函数 L(y) 是一个假设

(II) $\lim_{|y| \to \infty} [\, cy + L(y) \,] = \infty$

Assumptions (I) and (II) are the two basic assumptions throughout the paper, and hence we may not mention them explicitly in the following sections.

By a similar approach in [5], the dynamic programming recursion appropriate to finding optimal policies and to calculating $f_n(x)$ is

$$f_t(x) = \underset{y \in Y(x)}{\text{minimum}} \left\{ c(y-x) + L(y) + \alpha \sum_{j=0}^{\infty} f_{t-1}(y-j)p(j) \right\} \quad \text{for } t = 1,2,\ldots,n \tag{2-1}$$

$f_0(x) \equiv 0.$

只需假设需求值不可能有无穷种情况，这样保证了积分的收敛性

In order to guarantee that $\sum_{j=0}^{\infty} f_{n-1}(y-j)p(j) < \infty$ (convergence), one has to make an assumption about the probability distribution p(j), or the cost function L(y), or both. To avoid this troublesome, we will simply assume that the demand cannot be infinite. Hence there is an upper boundary M such that $D_t \le M$, or $p(j) = 0$ for $j > M$. The latter expression allows us still to use (2-1). For convenience, we will use $\sum_{j \ge 0}$, or simply Σ, instead of $\sum_{j=0}^{\infty}$.

Define $g_n(y)$ as

$$g_n(y) = L(y) + cy + \alpha \sum_{j=0}^{\infty} f_{n-1}(y-j)p(j) \tag{2-2}$$

Hence,

$$f_n(x) = \underset{y \in [x, x+CP]}{\text{minimum}} \left\{ c(y-x) + L(y) + \alpha \sum_{j=0}^{\infty} f_{n-1}(y-j)p(j) \right\}$$

$$= \min. \begin{cases} L(x) + \alpha \sum_{j \ge 0} f_{n-1}(x-j)p(j) \\ \underset{y \in [x,x+CP]}{\text{minimum}} \left\{ L(y) + cy + \alpha \sum_{j \ge 0} f_{n-1}(y-j)p(j) \right\} - cx + K \end{cases}$$

$$= \min. \begin{cases} g_n(x) - cx \\ \underset{y \in [x,x+CP]}{\text{minimum}} \; g_n(y) - cx + K \end{cases} \tag{2-3}$$

Suppose $\underset{y \in [x,x+CP]}{\text{minimum}} \; g_n(y) = g_n(y^*)$, where $y^* \in [\, x, x+CP \,]$. (We will justify this by proving the continuity of $g_n(y)$ in Section 3.) Then the optimal decision is

$$y_n^*(x) = \begin{cases} y^* & \text{if } g_n(x) \ge g_n(y^*) + K \\ x & \text{if } g_n(x) < g_n(y^*) + K \end{cases} \tag{2-4}$$

In (2-4), if $g_n(x) = g_n(y^*) + K$, both $y_n^*(x) = y^*$ and $y_n^*(x) = x$ are optimal. Thus in the following sections we may select either of them as the optimal decision.

3. THE CONTINUITY OF $f_n(x)$.

f_n(x) 是连续函数

Lemma 1. $f_n(x)$ is a continuous function of x for all $n \ge 0$, i.e., $f_n(x) \in C$.

Proof. For $n = 0$, $f_0(x) \equiv 0$. So, $f_0(x) \in C$.

Since $g_1(y) = L(y) + cy$, by assumption (I), $g_1(y) \in C$. By writing $f_1(x)$ in terms of $g_1(x)$, and using the fact that $g_1(x)$ is a convex function, one can clearly show that $f_1(x) \in C$.

Suppose $f_{n-1}(x) \in C$. Now , we prove $f_n(x) \in C$.

Since

$$f_n(x) = \min \begin{cases} L(x) + \alpha \sum_{j\geq 0} f_{n-1}(x-j)p(j) \\ \underset{y\in[x,x+CP]}{\text{minimum}} \{ L(y) + cy + \alpha \sum_{j\geq 0} f_{n-1}(y-j)p(j) - cx + K \} \end{cases}$$

$$= \min \begin{cases} V_n(x) \\ \underset{y\in[x,x+CP]}{\text{minimum}} \ U_n(y,x) \end{cases}$$

where the definitions for $V_n(x)$ and $U_n(y, x)$ are obvious. Clearly, $V_n(x) \in C_X$, $U_n(y,x) \in C_Y$ for any given x. (Indeed, $U_n(y, x) \in C_{X\oplus Y}$) Thus, the first observation is that $f_n(x)$ is everywhere defined. Second, for any given $x = x_0$, and $\varepsilon > 0$, we can find $\delta > 0$ (we may assume $\delta < \varepsilon$), such that for all $x \in [x_0-\delta, x_0+\delta]$, all $y' \in [x_0-\delta, x_0+\delta]$, and all $y'' \in [x_0+CP-\delta, x_0+CP+\delta]$,

$$| V_n(x) - V_n(x_0) | < \varepsilon$$

连续函数的性质
因为 V(x) 与 U(y, x) 连续

$$| U_n(y', x_0) - U_n(x_0, x_0) | < \varepsilon$$

$$| U_n(y'', x_0) - U_n(x_0+CP, x_0) | < \varepsilon$$

Thus, for any $y_1', y_2' \in [x_0-\delta, x_0+\delta]$, $y_1'', y_2'' \in [x_0+CP-\delta, x_0+CP+\delta]$,

$$| U_n(y_1', x_0) - U_n(y_2', x_0) | < 2\varepsilon \qquad (3\text{-}1)$$

$$| U_n(y_1'', x_0) - U_n(y_2'', x_0) | < 2\varepsilon \qquad (3\text{-}2)$$

Now, let $y_\Delta^* \in [x_0-\delta, x_0+CP+\delta]$ be the point which satisfies

$$U_n(y_\Delta^*, x_0) = \underset{y\in[x_0-\delta,x_0+CP+\delta]}{\text{minimum}} U_n(y, x_0) \qquad (3\text{-}3)$$

(Since $U_n(y, x_0) \in C$, so $\exists\, y_\Delta^* \in [x_0-\delta, x_0+CP+\delta]$.)

y_\delta 全邻域的最优 y

Now, for any $x \in [x_0-\delta, x_0+\delta]$, define $y^*(x) \in [x, x+CP]$ (or simply, y^*) as the point which satisfies

y^\ast 较窄区间的最优 y

$$U_n(y^*, x) = \underset{y\in[x,x+CP]}{\text{minimum}} U_n(y, x) \qquad (3\text{-}4)$$

$$= \underset{y\in[x,x+CP]}{\text{minimum}} \{ L(y) + cy + \alpha \sum_{j\geq 0} f_{n-1}(y-j)p(j) - cx + K \}$$

f(x)

Note that,

$$U_n(y^*, x) - U_n(y^*, x_0) = - cx + cx_0 = c\cdot(x_0 - x) \qquad (3\text{-}5)$$

and $U_n(y^*, x) = K - cx + \underset{y\in[x,x+CP]}{\text{minimum}} \{ L(y) + cy + \alpha \sum_{j\geq 0} f_{n-1}(y-j)p(j) \}$

Hence, $U_n(y^*, x_0) = \underset{y\in[x,x+CP]}{\text{minimum}} U_n(y, x_0)$ f(x_0) (3-6)

Because $[x, x+CP] \subset [x_0-\delta, x_0+CP+\delta]$, by (3-3) and (3-6), we have

$$U_n(y^*, x_0) \geq U_n(y_\Delta^*, x_0) \qquad (3\text{-}7)$$

There are three cases possible:

a) $y_\Delta^* = y^*(x)$, if $y_\Delta^* \in [x, x+CP]$; (Remember $y^*(x) = y^*$)

b) $y_\Delta^* \in [x_0-\delta, x) \subset [x_0-\delta, x_0+\delta]$;

分了三种情况讨论

c) $y_\Delta^* \in (x+CP, x_0+CP+\delta] \subset [x_0+CP-\delta, x_0+CP+\delta]$.

If a), $| U_n(y^*, x_0) - U_n(y_\Delta^*, x_0) | = 0 < \varepsilon$.

If b), note that by (3-6)

$$U_n(y^*, x_0) \le U_n(x, x_0)$$

and use this together with (3-7) and (3-1), we have

x 在 x0 的邻域内

$$0 \le U_n(y^*, x_0) - U_n(y_\Delta^*, x_0) \le U_n(x, x_0) - U_n(y_\Delta^*, x_0) < 2\varepsilon$$

If c), note that by (3-6), we have

$$U_n(y^*, x_0) \le U_n(x+CP, x_0)$$

and combine this with (3-7) and (3-2), using the fact that $x+CP \in [x_0+CP-\delta, x_0+CP+\delta]$, we obtain

$$0 \le U_n(y^*, x_0) - U_n(y_\Delta^*, x_0) \le U_n(x+CP, x_0) - U_n(y_\Delta^*, x_0) < 2\varepsilon$$

Thus, for all the cases,

$$| U_n(y^*(x), x_0) - U_n(y_\Delta^*, x_0) | < 2\varepsilon \qquad (3\text{-}8)$$

for any $x \in [x_0-\delta, x_0+\delta]$. Especially, if $x = x_0$, we have

$$| U_n(y^*(x_0), x_0) - U_n(y_\Delta^*, x_0) | < 2\varepsilon \qquad (3\text{-}9)$$

Hence, for any $x \in [x_0-\delta, x_0+\delta]$

$$\begin{aligned} &| U_n(y^*(x), x_0) - U_n(y^*(x_0), x_0) | \\ &\le | U_n(y^*(x), x_0) - U_n(y_\Delta^*, x_0) | + | U_n(y_\Delta^*, x_0) - U_n(y^*(x_0), x_0) | \\ &< 2\varepsilon + 2\varepsilon = 4\varepsilon \end{aligned}$$

And using the fact of (3-5), we obtain

这一步很巧，也很关键

$$\begin{aligned} &| U_n(y^*(x), x) - U_n(y^*(x_0), x_0) | \\ &= | U_n(y^*(x), x) - U_n(y^*(x), x_0) + U_n(y^*(x), x_0) - U_n(y^*(x_0), x_0) | \\ &< | c\cdot(x - x_0) | + 4\varepsilon \le c\varepsilon + 4\varepsilon = (4 + c)\varepsilon \qquad (3\text{-}10) \end{aligned}$$

|x-x_0|<\epsilon

At the given point x_0, there are two possibilities:

A. Optimal to order something;

B. Optimal not to order.

If **A**, then $y_n^*(x_0) = y^*(x_0)$, $f_n(x_0) = U(y^*(x_0), x_0)$.

For any $x \in [x_0-\delta, x_0+\delta]$, if at x, it is optimal to order something, then $f_n(x) = U_n(y^*(x), x)$ and by (3-10), $| f_n(x) - f_n(x_0) | < (4 + c)\varepsilon$

Otherwise, it must be optimal not to order at x: $f_n(x) = V_n(x) < U_n(y^*(x), x)$

However, since $| V_n(x) - V_n(x_0) | < \varepsilon$, and note that

$$V_n(x_0) \ge U_n(y^*(x_0), x_0) = f_n(x_0) \quad \text{(because of } \mathbf{A}\text{)}$$

and combined with (3-10), we get

$$f_n(x_0) - \varepsilon \le V_n(x_0) - \varepsilon < V_n(x) = f_n(x) < U_n(y^*(x), x) < f_n(x_0) + (4+c)\varepsilon$$

which yields

$$| f_n(x) - f_n(x_0) | < (4+c)\varepsilon$$

If, at point x_0, **B** is true, then $y_n^*(x_0) = x_0$, $f_n(x_0) = V_n(x_0) \le U(y^*(x_0), x_0)$.

For any $x \in [x_0-\delta, x_0+\delta]$, if at x it is optimal not to order, then, $f_n(x) = V_n(x)$, and $| f_n(x) - f_n(x_0) | = | V_n(x) - V_n(x_0) | < \varepsilon$.

Otherwise, it is optimal to order at x, then $f_n(x) = U_n(y^*(x), x) \le V_n(x)$, and hence,

$$U_n(y^*(x), x) - V_n(x_0) \le V_n(x) - V_n(x_0) < \varepsilon \qquad (3\text{-}11)$$

Note that, by (3-10),

$$U_n(y^*(x), x) > U_n(y^*(x_0), x_0) - (4 + c)\varepsilon \ge V_n(x_0) - (4 + c)\varepsilon$$

Combined with (3-11), we have

$$| f_n(x) - f_n(x_0) | = | U_n(y^*(x), x) - V_n(x_0) | < (4 + c)\varepsilon$$

Thus, for both **A** and **B**, and $\forall\, x \in [x_0-\delta,\, x_0+\delta]$,

$$| f_n(x) - f_n(x_0) | < (4 + c)\,\varepsilon,$$ 证明了极限的存在以及等价性

which means that $f_n(x) \in C$. Lemma 1 is thus proven by induction. ∎

From the definition of $g_n(y)$, it is obvious $g_n(y) \in C$ for all $n \geq 1$.

4. THE EXISTENCE OF X.

Lemma 2. There exists $\mathbf{X}$, such that for all $x \leq \mathbf{X}$, and all $n \geq 1$, it is optimal to order up to full capacity, i.e.,

$y_n^*(x) = x + CP$, for $\forall\, x \leq \mathbf{X}$, and $\forall\, n \geq 1$.

Proof. Let x_m be the point at which the function $cy + L(y)$ reaches its minimum (since $cy + L(y)$ is a convex function). Hence $cy + L(y)$ is nonincreasing for $y \leq x_m$, and nondecreasing for $y \geq x_m$. For convenience, we will use decreasing to mean nonincreasing, and use increasing to mean nondecreasing.

Note that $g_1(y) = cy + L(y)$ by the definition of $g_n(y)$ (2-2) and of $f_0(x)(\equiv 0)$. Let x_s ($\leq x_m$) be the maximum point at which

$$c\cdot(x_s - CP) + L(x_s - CP) = g_1(x_s - CP) \geq cx_s + L(x_s) + K = g_1(x_s) + K \qquad (4\text{-}1)$$

That is, $(x_s - CP)$ is the highest inventory level at which full capacity ordering is required by the optimal policy for a single-period model ($n = 1$).

Now we will prove the following for all $n \geq 1$:

(a) $f_{n-1}(x)$ is a decreasing function for $x \leq x_s$.

(b) $g_n(y)$ is a decreasing function for $y \leq x_s$.

(c) $g_n(y) \geq g_n(y + CP) + K$ for $y \leq x_s - CP = \mathbf{X}$.

(d) $y_n^*(x) = x + CP$ for $x \leq x_s - CP = \mathbf{X}$.

(e) $f_n(x)$ is a decreasing function for $x \leq x_s$.

For $n = 1$, (a) is true by the definition of $f_0(x)$.

(b): $g_1(y) = cy + L(y)$, which is a decreasing function for $y \leq x_s \leq x_m$.

(c) is derived from the convexity of $g_1(y)$ and the definition of x_s.

(d) is derived from (c), (b) and (2-4).

(e) can be derived directly by expressing $f_1(x)$ in terms of $g_1(x)$, and using the convexity of $g_1(x)$ and the definition of x_s. Instead, we give the following proof, which is similar to the proof for any n.

By (d) and (2-3), for all $x \leq x_s - CP = \mathbf{X}$,

$$f_1(x) = g_1(x + CP) - cx + K$$

hence by (b), $f_1(x)$ is a decreasing function for $x \leq x_s - CP$. Thus we only have to prove that $f_1(x)$ is also a decreasing function for $x \in [x_s\text{-}CP,\, x_s]$.

Note first that by (b) for any $x \in [x_s\text{-}CP,\, x_s]$,

$$\underset{y\in[x,x_s]}{\text{minimum}}\; g_1(y) = g_1(x_s)$$

hence,

$$\underset{y\in[x,x+CP]}{\text{minimum}}\; g_1(y) = \min \begin{Bmatrix} g_1(x_s) \\ \underset{y\in[x_s,x+CP]}{\text{minimum}}\; g_1(y) \end{Bmatrix} \equiv F_1(x)$$

clearly, $F_1(x)$ is a decreasing function of x. And finally since

$$f_1(x) = \min. \begin{cases} g_1(x) - cx \\ F_1(x) - cx + K \end{cases}$$

and both ($g_1(x) - cx$) and ($F_1(x) - cx + K$) are decreasing functions, so is $f_1(x)$.

Now, suppose the result for n. We prove the result for n+1.

Part (a) for n+1 follows from (e), since (e) is true for n.

For part (b), since

$$g_{n+1}(y) = L(y) + cy + \alpha \sum_{j\geq 0} f_n(y-j)p(j)$$

since $y - j \leq x_S$ if $y \leq x_S$. Hence, (b) follows from (a) and the definition of x_S.

For part (c). If $y \leq x_S - CP$, then $y + CP - j \leq x_S$. Thus, by (a)

$$\alpha \sum_{j\geq 0} f_n(y-j)p(j) \geq \alpha \sum_{j\geq 0} f_n(y+CP-j)p(j), \qquad (4\text{-}2)$$

and by the definition of x_S and the convexity of $L(y) + cy$, for $y \leq x_S - CP$,

$$L(y) + cy \geq L(y + CP) + c\cdot(y + CP) + K, \qquad (4\text{-}3)$$

so, (c) follows from (4-2) and (4-3).

Part (d) follows immediately from (b), (c) and (2-4).

For part (e), the proof is very similar to the alternative proof for n = 1. Indeed, we need only to change the subscription 1 into n + 1.

Lemma 2, which is (d), is thus proved by induction. ∎

Note that if the function $g_1(y) = cy + L(y)$ is so "flat" that there is no point x_S satisfying (4-1), then it means, by (2-4), that it will never be optimal to order something for n = 1. (E.g., if the shortage cost is linear with per unit-period shortage cost π, and if $K > CP\cdot\pi$, then the reduction in shortage cost even with full capacity production cannot justify the set up cost.) If it were so (although practically rare), then by the development of $f_n(x)$ and $g_n(y)$, we can still find an integer $N > 1$, such that $g_N(y)$ has the properties of convexity and (4-1), and similar results can be shown for $n \geq N$. Otherwise it would be never optimal to order for any n (e.g., when $K \geq CP\cdot\pi/(1-\alpha)$), the case we wouldn't consider.

Define t_n as the maximum starting inventory level of a n-period system at which it is optimal to order something.

Lemma 3 $t_n \geq t_1$, for all $n \geq 2$.

Proof. Note first that by the convexity of $g_1(y)$, the modified (s,S) policy is optimal to a single-period model (i.e., for n = 1), and let (s_1,S_1) be that policy. By the definition of x_S and the continuity of $g_1(y)$, x_S, t_1 must satisfy $t_1 = s_1$, $x_S = S_1$, $x_S - CP \leq s_1 < x_S$, and

$$L(s_1) + cs_1 = L(x_S) + cx_S + K.$$

From (a) of Lemma 2, $f_{n-1}(x)$ is a decreasing function over $(-\infty, x_S]$, hence,

$$g_n(s_1) = L(s_1) + cs_1 + \alpha \sum_{j\geq 0} f_{n-1}(s_1-j)p(j)$$
$$\geq L(x_S) + cx_S + K + \alpha \sum_{j\geq 0} f_{n-1}(x_S-j)p(j) = g_n(x_S) + K$$

and thus by (2-4), $t_n \geq s_1 = t_1$. ∎

5. THE EXISTENCE OF Y, given MD $\leq$ CP.

Proposition 1. For any $x \in \Re$, and a, $0 \leq a \leq CP$, there exists a', $0 \leq a' \leq CP$, such that for any $n \geq 1$,

$$f_n(x) - f_n(x+a) \leq \max \begin{cases} K + ca \\ g_n(y') - g_n(y'+a') + ca \end{cases} \quad \text{where } y' = x + CP.$$

Proof. For any $x \in \Re$, and $a \in [0, CP]$, consider $g_n(y)$ and $f_n(x)$:
If at x+a, it is optimal not to order, then by (2-3), $f_n(x+a) = g_n(x+a) - c\cdot(x + a)$. And again by (2-3), $f_n(x) \leq g_n(x+a) - cx + K$. Hence

$$f_n(x) - f_n(x+a) \leq g_n(x+a) - cx + K - (g_n(x+a) - c\cdot(x + a)) = K + ca;$$

Otherwise, it's optimal at x+a to order, say, b units, $0 < b \leq CP$, then

$$f_n(x+a) = g_n(x+a+b) - c\cdot(x+a) + K,$$

if $x+a+b \leq x+CP$, then by (2-3), $f_n(x) \leq g_n(x+a+b) - cx + K$, and

$$f_n(x) - f_n(x+a) \leq ca;$$

otherwise, $x+CP < x+a+b \leq x+CP+CP$. Let $y' = x + CP$, $a' = a + b - CP$. Then,

$0 < a' \leq CP$, $x + a + b = y' + a'$, and by (2-3),

$$f_n(x) - f_n(x+a) \leq g_n(x+CP) - cx + K - (g_n(x+a+b) - c\cdot(x + a) + K)$$
$$= g_n(y') - g_n(y'+a') + ca.$$

Hence, Proposition 1, as a summary, is proven. ∎

Let x_L be the point at which L(y) is minimized.

Lemma 4. Suppose that the maximum demand in one period MD is less than or equal to the capacity CP, then for all $x \geq x_L = \mathbf{Y}$, all $n \geq 1$, $y_n^*(x) = x$, i.e., it is optimal not to order if the starting inventory level $x \geq x_L = \mathbf{Y}$. Specifically, the following are true:

(f) $f_{n-1}(x) \leq f_{n-1}(x+a) + ca + K$ for all $x \geq x_L - CP$, and all $0 \leq a \leq CP$;
(g) $g_n(y) < g_n(y+a) + K$ for all $y \geq x_L = \mathbf{Y}$, and for all $0 \leq a \leq CP$;
(h) $y_n^*(x) = x$ for all $x \geq x_L = \mathbf{Y}$;
(i) $f_n(x) \leq f_n(x+a) + ca + K$ for all $x \geq x_L - CP$, and all $0 \leq a \leq CP$.

Proof. For n = 1, $f_{n-1}(x) = f_0(x) \equiv 0$, so (f) is true. Since $x_L \geq x_m > s_1$, (It can be shown.) where x_m minimizes L(y) + cy , so (g) follows. (h) follows immediately from (g) and (2-4).
For part (i), for any a, $0 \leq a \leq CP$, first note that from (2-3), (2-4) and (g), for $x \geq x_L$, $f_1(x) = g_1(x) - cx$, and

$$f_1(x) - f_1(x+a) = g_1(x) - g_1(x+a) + ca \leq K + ca$$

Second, for any given $x \in [x_L\text{-}CP, x_L]$, $x + CP \geq x_L$. Hence, by Proposition 1 and (g),

$$f_1(x) - f_1(x+a) \leq K + ca.$$

Thus (i) follows. Indeed, (i) can be directly derived from Proposition 1 and (g).
Now, suppose the results for n, we prove for n + 1.

(f) is true from (i) and for n, (i) is true.

(g): If $y \geq x_L$, then, $y - j \geq x_L - j \geq x_L - CP$. Thus by the definition of x_L and from (f)

$$g_{n+1}(y) = L(y) + cy + \alpha \sum f_n(y\text{-}j)p(j) \leq L(y+a) + cy + \alpha \sum (f_n(y+a\text{-}j) + ca + K)p(j) < g_{n+1}(y+a) + K$$

(h) is derived from (g) and (2-4).

(i) follows from Proposition 1 and (g).

Thus, Lemma 4 is proven by induction. ∎

The following example is to illustrate that the **X** and **Y** boundaries given by Lemma 2 and Lemma 4 respectively are sometimes tight.

Example 1. Assume linear holding and shortage costs.

Let h be the holding cost per unit-period, π be the shortage cost per unit-period.

For our example, $h = 1.0$, $\pi = 10.0$, $K = 22.0$, $c = 1.0$, $T = 0$, $\alpha = 0.90$, $CP = 9$.

The demand probability distribution is: $p(6) = 0.95$, $p(7) = 0.05$.

One-period expected holding and shortage cost function is thus

$$L(y) = \begin{cases} 60.5 - 10\,y & \text{for } y \le 6 \\ -2.2 + 0.45\,y & \text{for } 6 \le y \le 7 \\ -6.05 + y & \text{for } y \ge 7 \end{cases} \qquad (\text{ note: } L(y) \in C\ )$$

It is obvious that for our example, $x_S = S_1 = x_m = x_L = 6$. Hence, by Lemma 2 and Lemma 4,

$$\mathbf{X} = x_S - CP = 6 - 9 = -3, \qquad \mathbf{Y} = x_L = 6,$$

and the optimal policy (optimal order quantity) is:

$$O_n^*(x) = \begin{cases} 9 & \text{for } x \le -3 \\ \text{undetermined} & \text{for } -2 \le x \le 5 \\ 0 & \text{for } x \ge 6 \end{cases}$$

The values of $O_n^*(x)$, the optimal order quantities, obtained through the dynamic program (2-1) up to 20 periods are given below:

n

		20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	-5	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
	-4	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
	-3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
	-2	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
	-1	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	9	7	7
	0	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	6	6
	1	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	5	5
x	2	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	9	4
	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	3
	4	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	0
	5	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	0	7	7	0
	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

As can be seen above, the **X** and **Y** boundaries are reached in this example.

The crucial assumption needed for Lemma 4 is that the demand never exceeds capacity. What happens if the condition is violated ? The following example is designed to say something about it.

Example 2. Assume linear holding and shortage costs, where $h = 0.2$, $\pi = 10.0$,

$K = 15.0$, $c = 1.0$, $T = 0$, $\alpha = 0.95$, $CP = 8$.

The demand probability distribution is: $p(0) = 0.3$, $p(10) = 0.7$.

After writing out the one-period expected holding and shortage cost function, one can clearly see that $x_S = S_1 = x_m = x_L = 10$. Hence, if Lemma 2 and Lemma 4 were still true, **X** and **Y** values would be: $\mathbf{X} = x_S - CP = 10 - 8 = 2$, $\mathbf{Y} = x_L = 10$. However, the solutions of $O_n^*(x)$, the optimal order quantities, from the dynamic program (2-1) up to 20 periods are as follows:

	n																			
	20	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	7
4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	6
5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	7	5
6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	8	6	4
7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	8	7	5	3
8	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	8	8	6	4	0
9	.	.	.	.	.	.	.	.	.	.	.	.	.	8	8	8	7	5	3	0
10	.	.	.	.	.	.	.	.	.	.	.	.	8	8	8	8	6	4	0	.
11	.	.	.	.	.	.	.	.	.	.	.	8	8	8	7	7	5	0	.	.
x 12	.	.	.	.	.	.	.	.	.	.	8	8	8	8	6	6	8	0	.	.
13	.	.	.	.	.	.	.	.	.	8	8	8	8	7	5	8	0	.	.	.
14	.	.	.	.	.	.	.	.	8	8	8	8	8	6	4	0	.	.	.	.
15	.	.	.	.	.	.	.	8	8	8	8	8	7	5	0	.	.	.	.	.
16	.	.	.	.	.	.	8	8	8	8	8	8	6	0	.	.	.	.	.	.
17	.	.	.	.	.	8	8	8	8	8	7	7	8	0	.	.	.	.	.	.
18	.	.	.	.	8	8	8	8	8	8	6	0	0	.	.	.	.	.	.	.
19	.	.	.	8	8	8	8	8	7	7	0	.	.	.	.	.	.	.	.	.
20	.	.	8	8	8	8	8	0	0	0	.	.	.	.	.	.	.	.	.	.
21	.	8	8	8	8	0	0	.	.	.	.	.	.	.	.	.	.	.	.	.
22	8	0	0	0	0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

It seems that the **Y** boundary is increasing as n increases. Note that for this example, the expected demand in a period (ED) is 7, less than the capacity CP. Thus, even the assumption $MD \leq CP$ in Lemma 4 is relaxed to $ED \leq CP$, it is hard to see the existence of **Y** boundary. Lemma 5 will deal with this problem.

6. THE EXISTENCE OF Y, if MD > CP.

In this section, we make an assumption about $L(y)$, in addition to **(I)** and **(II)** in section 2.

Assumption (III) $\quad L(y) - L(y + CP) \leq A + B\,|y|^{\tau}$ for some nonnegative integer τ, and for some positive constants A and B.

Alternatively, assumption **(III)** can be replaced by a stronger, but simpler, assumption, (assumption **(III')**), the one made in [4]:

Assumption (III') $\quad L(y) \leq A + B\,|y|^{\tau}$ where τ, A, B are constants, same as that in **(III)**.

Assumption (III) (or (III')) is virtually of no restriction. For instance, if the shortage cost is linear, and the holding cost is a increasing function of inventory on hand at the end of a period (normally the case), then the following is true:

$$L(y) - L(y + CP) \le \pi\, CP \qquad \text{for all } y$$

where, π is the shortage cost per unit-period. This is because that the 'slope' of $L(y)$ is always greater than or equal to $-\pi$.

If A is set big enough, say, $A \ge L(x_L\text{-}CP) - L(x_L)$, and suppose it has been fixed so, where x_L minimizes $L(y)$, then for all $0 \le a \le CP$,

$$L(y) - L(y+a) \le A + B\,|y|^{\tau} \qquad (6\text{-}1)$$

Lemma 5. Under assumptions (I), (II), and (III), there exists $\mathbf{Y} < \infty$, such that for all $\mathbf{x \ge Y}$, and all $n \ge 1$, it is optimal not to order, i.e., $y_n^*(x) = x$, for all $\mathbf{x \ge Y}$.

Proof. The proof for the case when $MD \le CP$ has been shown in Lemma 4. Now let's assume that $MD > CP$.

Let $Y_0 = x_L$, and $Y_{-m} = Y_{-m+1} - MD + CP = Y_{-m+1} - \Delta$, where $\Delta = MD - CP > 0$, $m = 1,2,\ldots$

Hence, $$Y_{-m} = Y_0 - m\,\Delta \qquad (6\text{-}2)$$

and $$Y_{-m} - CP = Y_{-m+1} - MD \qquad (6\text{-}3)$$

Since $g_1(y) = L(y) + cy$, by the convexity of $L(y)$ and (6-1), for $y \ge Y_{-m}$, $0 \le a \le CP$,

$$g_1(y) - g_1(y+a) = L(y) - L(y+a) - ca \le L(Y_{-m}) - L(Y_{-m}+a) - ca \le A + B\,|Y_{-m}|^{\tau} - ca.$$

Use (6-2) and the convexity of $|x|^{\tau}$, we have, for $y \ge Y_{-m}$, $0 \le a \le CP$,

$$|Y_{-m}|^{\tau} = |Y_0 - m\Delta|^{\tau} \le (1/2)|2Y_0|^{\tau} + (1/2)|2m\Delta|^{\tau},$$

and thus, $$g_1(y) - g_1(y+a) \le A' + B'\,|m\Delta|^{\tau} - ca \qquad (6\text{-}4)$$

where, $A' = A + B\,2^{\tau-1}|Y_0|^{\tau} = A + B\,2^{\tau-1}|x_L|^{\tau}$, $B' = B\,2^{\tau-1}$. For convenience, we will still use A and B to represent A' and B' respectively.

For **any** $\mathbf{x \ge Y_{-(m+1)} - CP}$, and a, $0 \le a \le CP$, let $x' = x + CP$, then $\mathbf{x'} \ge Y_{-(m+1)}$, by Proposition 1, there exists a', $0 \le a' \le CP$, such that

$$f_{n-1}(x) - f_{n-1}(x+a) \le \max \begin{cases} K + ca \\ g_{n-1}(x') - g_{n-1}(x'+a') + ca \\ \qquad\qquad x' \ge Y_{-(m+1)},\ 0 \le a' \le CP \end{cases}$$

$$\le ca + \max \begin{cases} K \\ \underset{\substack{y' \ge Y_{-(m+1)} \\ 0 \le a' \le CP}}{\text{maximum}} \left[\, g_{n-1}(y') - g_{n-1}(y'+a') \,\right] \end{cases}$$

Thus, for any $0 \le a \le CP$,

$$\underset{x \ge Y_{-(m+1)} - CP}{\text{maximum}} \left[f_{n-1}(x) - f_{n-1}(x+a) \right] \le ca + \max \begin{cases} K \\ \underset{\substack{y \ge Y_{-(m+1)} \\ 0 \le a' \le CP}}{\text{maximum}} \left[g_{n-1}(y) - g_{n-1}(y+a') \right] \end{cases}$$

Note that $g_n(y) - g_n(y+a) = g_1(y) - g_1(y+a) + \alpha \sum \left[f_{n-1}(y-j) - f_{n-1}(y+a-j) \right] p(j)$, and for $\mathbf{y \ge Y_{-m}}$, $y - j \ge Y_{-m} - MD = Y_{-(m+1)} - CP$. Hence, together with (6-4), for any $y \ge Y_{-m}$, $0 \le a \le CP$,

$$g_n(y) - g_n(y+a) \le A + B\,|m\Delta|^{\tau} + \alpha \max \begin{cases} K \\ \underset{\substack{y' \ge Y_{-(m+1)} \\ 0 \le a' \le CP}}{\text{maximum}} \left[\, g_{n-1}(y') - g_{n-1}(y'+a') \,\right] \end{cases}$$

Thus,

$$\underset{\substack{y \geq Y_{-m} \\ 0 \leq a \leq CP}}{\text{maximum}} [g_n(y) - g_n(y+a)] \leq A + B\,|m\Delta|^{\tau} + \alpha \max \begin{cases} K \\ \underset{\substack{y \geq Y_{-(m+1)} \\ 0 \leq a \leq CP}}{\text{maximum}} [g_{n-1}(y) - g_{n-1}(y+a)] \end{cases}$$

Hence, by induction and (6-4), and suppose we have fixed A (indeed A') such that $A \geq K$, we have

$$\underset{\substack{y \geq Y_{-m} \\ 0 \leq a \leq CP}}{\max} [\, g_n(y) - g_n(y+a)\,] \leq A + B\,|m\Delta|^{\tau} + \alpha\,(A + B\,|(m+1)\Delta|^{\tau}) + \cdots + \alpha^{n-1}(A + B\,|(m+n-1)\Delta|^{\tau})$$

$$n = 1,2,3,\ldots,\quad m = 1,2,3,\ldots$$

Note that, for $y \geq Y_0 = x_L$, $L(y) - L(y+a) \leq 0$, and hence $g_1(y) - g_1(y+a) \leq -ca$ for any $a \geq 0$. Thus by similar induction as above, we have

$$\underset{\substack{y \geq Y_0 \\ 0 \leq a \leq CP}}{\max} [\, g_n(y) - g_n(y+a)\,] \leq \alpha\,(A + B\,|\Delta|^{\tau}) + \alpha^2\,(A + B\,|2\Delta|^{\tau}) + \cdots + \alpha^{n-1}(A + B\,|(n-1)\Delta|^{\tau})$$

$$\leq \sum_{i=1}^{\infty} \alpha^i\,(A + B\,(i\,\Delta)^{\tau}) \equiv M \qquad (6\text{-}5)$$

Since $\dfrac{\alpha^{i+1}\,(i+1)^{\tau}}{\alpha^i\, i^{\tau}} \rightarrow \alpha < 1$, thus, $M < \infty$.

Let $Y_m = Y_{m-1} - CP + MD = Y_{m-1} + \Delta$, $m = 1,2,3,\ldots$. Hence,

$$Y_m = Y_0 + m\Delta \qquad (6\text{-}6)$$

$$\text{and} \quad Y_m - MD = Y_{m-1} - CP \qquad (6\text{-}7)$$

Since $L(y) - L(y+a) \leq 0$, for $y \geq Y_0$, $0 \leq a \leq CP$. Hence, for $y \geq Y_m \geq Y_0$, $0 \leq a \leq CP$, $n \geq 1$,

$$g_n(y) - g_n(y+a) \leq -ca + \alpha \sum [\, f_{n-1}(y-j) - f_{n-1}(y+a-j)\,]\, p(j). \qquad (6\text{-}8)$$

And note that when $y \geq Y_m$, $y - j \geq Y_m - MD = Y_{m-1} - CP$. So, by Proposition 1 and similar proof as above,

$$\underset{\substack{y \geq Y_m \\ 0 \leq a \leq CP}}{\text{maximum}} [g_n(y) - g_n(y+a)] \leq \alpha \max \begin{cases} K \\ \underset{\substack{y \geq Y_{m-1} \\ 0 \leq a \leq CP}}{\text{maximum}} [g_{n-1}(y) - g_{n-1}(y+a)] \end{cases} \qquad (6\text{-}9)$$

$$n = 2,3,\ldots,\ m = 1,2,\ldots.$$

Thus, the first observation, by induction and using the fact that $g_1(y) - g_1(y+a) < K$ for any $y \geq Y_0$ and $a \geq 0$, is that

$$g_n(y) - g_n(y+a) \leq K \quad \text{for all } y \geq Y_{n-1},\ 0 \leq a \leq CP,\ n = 1,2,\ldots.$$

$$\text{and hence} \quad y_n^*(x) = x \quad \text{for all } x \geq Y_{n-1},\ n = 1,2,\ldots. \qquad (6\text{-}10)$$

Second, let N be the smallest positive integer satisfying

$$\alpha^N M \leq K. \qquad (6\text{-}11)$$

Now, we show that Y_N is one of the **Y** boundaries.

Because $Y_N > Y_{n-1}$ for all $n \leq N$, thus, by (6-10), we only need to show that Y_N is a boundary for $n > N$.

For $n > N$, by (6-9) and induction, and using (6-5) and (6-11), we have

$$\underset{\substack{y \geq Y_N \\ 0 \leq a \leq CP}}{\text{maximum}} [g_n(y) - g_n(y+a)] \leq \max \begin{cases} \alpha K \\ \alpha^N \underset{\substack{y \geq Y_0 \\ 0 \leq a \leq CP}}{\text{maximum}} [g_{n-N}(y) - g_{n-N}(y+a)] \end{cases}$$

$$\leq \max\,[\alpha K,\ \alpha^N M] \leq K$$

Hence, for any $n \geq 1$, any y, $y \geq Y_N$, and any a, $0 \leq a \leq CP$,

$$g_n(y) - g_n(y+a) \leq K. \qquad (6\text{-}12)$$

So, by (2-3) and (2-4), $y_n^*(x) = x$ for all $x \geq Y_N$, and $n \geq 1$.
Lemma 5 is thus proven. ∎

7. INFINITE TIME HORIZON PROBLEMS.

Let r(x) be the discounted cost over the long run for an infinite time horizon problem, if the starting inventory level is x, and if an optimal policy is followed. Define q(y) as

$$q(y) = L(y) + cy + \alpha \sum_{j\geq 0} r(y-j)p(j)$$

Let $\delta(x)$ be defined as
$$\delta(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then, r(x) should satisfy the functional equation:

$$\begin{aligned} r(x) &= \underset{y\in[x,x+CP]}{\text{minimum}} \{ L(y) + c(y-x) + \alpha \sum_{j\geq 0} r(y-j)p(j) \}. \\ &= \underset{y\in[x,x+CP]}{\text{minimum}} \{ q(y) + K\,\delta(y-x) - cx \} \qquad (7\text{-}1) \end{aligned}$$

Now, the questions are: Is there any solution for r(x) in (7-1) ? Does the optimal policy show some systematic pattern ?

In this section, we will extend our earlier results to the infinite time horizon problems. Specifically, we show that $\lim_{n\to\infty} f_n(x)$ exists. Then we prove that the limiting function named f(x) is continuous, and satisfying the functional equation (7-1). And finally, we prove that the corresponding optimal policy to the infinite time horizon inventory problem also exhibits a **X-Y** band.

Proposition 2. $\{ f_n(x) \}$ is a nondecreasing series in n, for any given x.

Proof. first of all, $f_1(x) \geq 0 \equiv f_0(x)$, for all x.

Now, Assume that $f_n(x) \geq f_{n-1}(x)$, for all x. Then,

$$g_{n+1}(y) = L(y) + cy + \alpha \sum f_n(y-j)p(j) \geq L(y) + cy + \alpha \sum f_{n-1}(y-j)p(j) = g_n(y), \text{ for all } y.$$

Hence, by the definition of $y_n^*(x)$ and (2-3), for any given x,

$$\begin{aligned} f_{n+1}(x) &= g_{n+1}(y_{n+1}^*(x)) + K\,\delta(y_{n+1}^*(x)-x) - cx \geq g_n(y_{n+1}^*(x)) + K\,\delta(y_{n+1}^*(x)-x) - cx \\ &\geq \underset{y\in[x,x+CP]}{\text{minimum}} [\, g_n(y) + K\,\delta(y-x) - cx \,] = f_n(x) \end{aligned}$$

Thus, Proposition 2 is proven by induction. ∎

Lemma 6. Under either of the assumptions, MD ≤ CP or assumption (**III'**), $\lim_{n\to\infty} f_n(x)$ exists, and the convergence is uniform for all **x** in any finite interval. The limiting function, called f(x), is of course continuous.

Proof. Following similar notation in [1] [2], let

$$T(y,x,f) = L(y) + c\cdot(y - x) + K\,\delta(y-x) + \alpha \sum_{j\ge 0} f(y-j)\,p(j)$$

Thus, $f_n(x) = \underset{y\in[x,x+CP]}{\text{minimum}}\ T(y,x,f_{n-1}) = T(y_n^*(x),x,f_{n-1}) = T(y_n^*,x,f_{n-1})$.

For simplicity, we use y_n^* to represent $y_n^*(x)$.

Since $f_{n+1}(x) = T(y_{n+1}^*, x, f_n) \le T(y_n^*,x,f_n)$, hence, combined with Proposition 2,

$$0 \le f_{n+1}(x) - f_n(x) \le T(y_n^*,x,f_n) - T(y_n^*,x,f_{n-1}).$$

Expanding T(.,.,.) and cancelling terms yields

$$| f_{n+1}(x) - f_n(x) | \le \alpha \,| \textstyle\sum [\, f_n(y_n^*-j) - f_{n-1}(y_n^*-j)\,]\, p(j)\, |$$

$$\le \alpha \textstyle\sum |\, f_n(y_n^*-j) - f_{n-1}(y_n^*-j)\,|\, p(j) \qquad (7\text{-}2)$$

Now, choose two positive constants U and V, such that $-U \le \mathbf{X} - CP$, $V \ge \mathbf{Y} + CP$. Where **X**, **Y** are the boundaries of the **X-Y** band from Lemma 2, Lemma 4 and Lemma 5. First, consider the case where MD ≤ CP. For any $x \in [-U, V]$:

if $-U \le x \le \mathbf{X}$, then $y_n^*(x) = x + CP$, hence $y_n^*(x) - j = x + CP - j \ge x \ge -U$;

if $\mathbf{X} \le x$, then for $y_n^*(x) \ge x$, hence $y_n^*(x) - j \ge x - j \ge \mathbf{X} - CP \ge -U$;

if $x < \mathbf{Y}$, then $y_n^*(x) - j \le x + CP - j \le x + CP \le \mathbf{Y} + CP \le V$;

if $\mathbf{Y} \le x \le V$, then $y_n^*(x) = x$, and $y_n^*(x) - j = x - j \le x \le V$.

Thus, for all $x \in [-U, V]$, $(y_n^*(x) - j) \in [-U, V]$. Hence, by (7-2),

$$\underset{-U\le x\le V}{\text{maximum}} |f_{n+1}(x) - f_n(x)| \le \alpha \underset{-U\le x\le V}{\text{maximum}} \textstyle\sum |\, f_n(y_n^*-j) - f_{n-1}(y_n^*-j)\,|\, p(j)$$

$$\le \alpha \underset{-U\le x\le V}{\text{maximum}} |f_n(x-j) - f_{n-1}(x-j)|$$

$$\le \dots \le \alpha^n \underset{-U\le x\le V}{\text{maximum}}\ f_1(x)$$

$$n = 1,2,\dots$$

Second, if the assumption MD ≤ CP is violated, and instead, assumption (**III'**) is supposed, then by similar approach as above, it can be shown that, where $\Delta = MD - CP > 0$,

$$\underset{-U\le x\le V}{\text{maximum}} |f_{n+1}(x) - f_n(x)| \le \alpha \underset{-U\le x\le V}{\text{maximum}} \textstyle\sum |\, f_n(y_n^*-j) - f_{n-1}(y_n^*-j)\,|\, p(j)$$

$$\le \alpha \underset{-(U+\Delta)\le x\le V}{\text{maximum}} |f_n(x-j) - f_{n-1}(x-j)|$$

$$\le \dots \le \alpha^n \underset{-(U+n\Delta)\le x\le V}{\text{maximum}}\ f_1(x)$$

$$n = 1,2,\dots$$

Since $f_1(x) \le g_1(x) - cx = L(x)$, and by the convexity of L(y) and assumption (**III'**), we have

$$\underset{-(U+n\Delta)\le x\le V}{\text{maximum}}\ f_1(x) \le \max\{\, L(-(U+n\Delta)),\, L(V)\,\} \le A + B \max\{\, |U + n\Delta|^{\tau}, |V|^{\tau}\,\}.$$

But because,

$$\frac{\alpha^{n+1}\,(n+1)^{\tau}}{\alpha^n\, n^{\tau}} \to \alpha < 1, \text{ when } n \to \infty,$$

hence the series $\sum_{n=0}^{\infty} [\, f_{n+1}(x) - f_n(x)\,]$ converges absolutely and uniformly for all x in the interval $-U \le x \le V$ under either assumption. Since $\{ f_n(x) \}$ is a nondecreasing sequence in n for any given x, and U and V may be chosen arbitrarily large, we have shown that $f_n(x)$ converges monotonely and uniformly for all x in any finite interval. The functions $f_n(x)$ are continuous and the convergence uniform, thus the limiting function f(x), $f(x) = \lim_{n\to\infty} f_n(x)$, is also continuous. ∎

Lemma 7. The limiting function f(x) satisfies

$$\begin{aligned} f(x) &= \underset{y\in[x,x+CP]}{\text{minimum}} \{ L(y) + c(y-x) + \alpha \sum_{j\ge 0} f(y-j)p(j) \}. \\ &= \underset{y\in[x,x+CP]}{\text{minimum}} \{ g(y) + K\,\delta(y-x) - cx \} \end{aligned} \qquad (7\text{-}3)$$

the functional equation. Where $g(y) = L(y) + cy + \alpha \sum f(y-j)p(j)$.

Proof. First of all, since

$$g_{n+1}(y) - g(y) = \alpha \sum [\, f_n(y-j) - f(y-j) \,]\, p(j),$$

hence, $g_n(y)$ converges to g(y) monotonely (see the proof of Proposition 2) and uniformly for all y in any finite interval. It is obvious, $g(y) \in C$.

Second, for any given x, because

$$x \le y_n^*(x) \le x + CP,$$

thus, there exists a subseries $\{ y_{ni}^*(x) \} \subset \{ y_n^*(x) \}$, such that $\lim_{i\to\infty} y_{ni}^*(x)$ exists. Let

$$y^*(x) = \lim_{i\to\infty} y_{ni}^*(x).$$

Obviously, $y^*(x) \in [x,x+CP]$.

Now, we prove that

$$\begin{aligned} \lim_{i\to\infty} [\, g_{ni}(y_{ni}^*(x)) - cx + K\,\delta(y_{ni}^*(x)-x) \,] &= g(y^*(x)) - cx + K\,\delta(y^*(x)-x) \\ &= \underset{y\in[x,x+CP]}{\text{minimum}} \{ g(y) - cx + K\,\delta(y-x) \}, \end{aligned}$$

and $f(x) = g(y^*(x)) - cx + K\,\delta(y^*(x)-x)$.

Note first that, for any given x, since $g_n(y)$ converges uniformly to the continuous function g(y) in any finite interval, hence,

$$\lim_{i\to\infty} g_{ni}(y_{ni}^*(x)) = g(y^*(x)) \qquad (7\text{-}4)$$

Indeed, (7-4) can be clearly seen by considering the interval [x, x+CP] and the inequality

$$|\, g_{ni}(y_{ni}^*(x)) - g(y^*(x)) \,| \le |\, g_{ni}(y_{ni}^*(x)) - g(y_{ni}^*(x)) \,| + |\, g(y_{ni}^*(x)) - g(y^*(x)) \,|,$$

which can be as small as any given $\varepsilon > 0$ when i is big enough.

Now, If $\lim_{i\to\infty} y_{ni}^*(x) = y^*(x) > x$, then, there exists N, such that for all $i \ge N$,

$$y_{ni}^*(x) > x, \text{ and hence, } K\,\delta(y_{ni}^*(x)-x) = K.$$

Thus, by (7-4),

$$\begin{aligned} \lim_{i\to\infty} \{ g_{ni}(y_{ni}^*(x)) - cx + K\,\delta(y_{ni}^*(x)-x) \} &= g(y^*(x)) - cx + K \\ &= g(y^*(x)) - cx + K\,\delta(y^*(x)-x). \end{aligned}$$

If, on the other hand, $\lim_{i\to\infty} y_{ni}^*(x) = y^*(x) = x$, then there exists N, such that for all $i \ge N$,

$$y_{ni}^*(x) = x.$$

This is because

$$|\, g_{ni}(y_{ni}^*(x)) - g_{ni}(x) \,| \le |\, g_{ni}(y_{ni}^*(x)) - g(y^*(x){=}x) \,| + |\, g(x) - g_{ni}(x) \,| \to 0,$$

thus, there exists N, such that for all $i \ge N$,

$$g_{ni}(x) - g_{ni}(y_{ni}^*(x)) < K, \text{ i.e., } g_{ni}(x) < g_{ni}(y_{ni}^*(x)) + K,$$

and hence by (2-3) and (2-4), $y_{ni}^*(x) = x$.

Thus, $\lim_{i\to\infty} K\,\delta(y_{ni}^*(x)-x) = 0 = K\,\delta(y^*(x)-x)$, and

$$\lim_{i\to\infty} \{ g_{ni}(y_{ni}^*(x)) - cx + K\,\delta(y_{ni}^*(x)-x) \} = g(y^*(x)) - cx + K\,\delta(y^*(x)-x).$$

So, for both cases, we have shown that

$$\lim_{i\to\infty} \{ g_{ni}(y_{ni}^*(x)) - cx + K\,\delta(y_{ni}^*(x)-x) \} = g(y^*(x)) - cx + K\,\delta(y^*(x)-x). \qquad (7\text{-}5)$$

Note that for all $y \in [x, x+CP]$,

$$g_{ni}(y) - cx + K\,\delta(y-x) \geq g_{ni}(y_{ni}^*(x)) - cx + K\,\delta(y_{ni}^*(x)-x),$$

take limit both side, and use (7-5), we have

$$g(y) - cx + K\,\delta(y-x) \geq g(y^*(x)) - cx + K\,\delta(y^*(x)-x).$$

Which establishes the optimality of $y^*(x)$.

Lastly, since

$$f_{ni}(x) = g_{ni}(y_{ni}^*(x)) - cx + K\,\delta(y_{ni}^*(x)-x),$$

and take limit both side,

$$f(x) = g(y^*(x)) - cx + K\,\delta(y^*(x)-x).$$

Which establishes the functional equation of $f(x)$. ∎

Lemma 7 also indicates that there exists stationary optimal policies for the infinite time horizon problems, and the limit of any convergence subseries of $\{ y_n^*(x) \}$ can be used as the optimal solution of $y^*(x)$.

The following lemma, Lemma 8, is about the **X-Y** band of the optimal policy for an infinite time horizon problem.

Lemma 8.

(j) $f(x)$ is a decreasing function for $x \leq x_S$;

(k) $g(y)$ is a decreasing function for $y \leq x_S$;

(l) $y^*(x) = x + CP$ for all $x \leq x_S - CP = \mathbf{X}$;

(m) $y^*(x) = x$ for all $x \geq x_L = \mathbf{Y}$, if $MD \leq CP$.

(m') $y^*(x) = x$ for all $x \geq Y_N = \mathbf{Y}$, under assumption (III') if $MD > CP$.

Where x_S is defined by (4-1) (Lemma 2), and x_L minimizes $L(y)$ (Lemma 4), Y_N is defined by (6-6) and (6-11) (Lemma 5), and $y^*(x)$ is the optimal decision for y when the starting inventory level is x.

Proof. By (e) of Lemma 2, $f_n(x)$ are decreasing functions for $x \leq x_S$. Hence,

$$f_n(x) \geq f_n(x+b) \qquad \text{for any } x \leq x_S,\ x+b \leq x_S, \text{ where } b \geq 0.$$

Let $n \to \infty$ both side, we get $f(x) \geq f(x+b)$, which is (j).

Part (k) can be derived from (j), the definitions of x_S and $g(y)$.

For (l), note first that by (4-3), $L(y) + cy \geq L(y + CP) + c\cdot(y + CP) + K$ for all $y \leq x_S - CP = \mathbf{X}$, and combining this with (j), we have

$$g(y) \geq g(y+CP) + K \qquad \text{for all } y \leq \mathbf{X}.$$

Secondly, by (k), $g(y)$ is a decreasing function for $y \leq x_S$. Hence, by (2-4) or (7-3),

$$y^*(x) = x + CP \quad \text{for } x \leq \mathbf{X}.$$

For part (m) and (m'), the proof could be very similar to the proof for Lemma 4 and Lemma 5. Instead, we give the following simple proof.

Since $\quad y_n^*(x) = x \quad$ for all $x \geq \mathbf{Y}$, and all $n \geq 1$, hence,

$$\lim_{n\to\infty} y_n^*(x) = x \qquad \text{for all } x \geq \mathbf{Y}.$$

As pointed earlier, the limit of any convergence subseries of $\{ y_n^*(x) \}$ can be used as the decision for $y^*(x)$, thus, $y^*(x) = x$, for all $x \geq \mathbf{Y}$ ∎

Indeed, if we let $n \to \infty$ in (a) to (i) in earlier sections, we will get corresponding results in an infinite time version.

8. A COUNTER EXAMPLE.

The only pity of the **X-Y** band policy is that we are unable to describe the optimal ordering decisions for those inventory levels which are within the **X-Y** band. Example 1 and Example 2 (and many others we tried) clearly indicate that the optimal ordering pattern for inventory levels within the **X-Y** band is different from problem to problem, and of course the optimal policy is generally not the modified (s,S)-policy type, at least for the finite time horizon problems. But what is about the infinite-horizon case ? Will the optimal policy be a modified (s,S)-policy then ? The following simple example, example 3, is designed to give the answer: No!

Example 3. Again, assume linear holding and shortage costs. Let

$h = 1.0$, $\pi = 10.0$, $K = 27.$, $T = 1$, $\alpha = 0.88$, $CP = 11$.

The demand probability distribution is: $p(4) = 1.0$, i.e.,

the demand is a constant of 4 units per period, $d_t = d = 4$.

Thus, the demand D over 2 periods (lead time plus one review period) is 8, and hence

$$L(y) = \begin{cases} \pi \cdot (D - y) & \text{for } y < D \\ h \cdot (y - D) & \text{for } y \geq D \end{cases}$$

$$= \begin{cases} 10 (8 - y) & \text{for } y < 8 = D \\ y - 8 & \text{for } y \geq 8 = D \end{cases} \qquad (8\text{-}1)$$

In order to prove that the modified (s,S)-policy is not optimal to this example problem for any pair of s and S, we need to present further some analysis and results.

Lemma 9. For any modified (s,S)-policy, we can always find S', such that the modified (s,S')-policy is equivalent to the modified (s,S)-policy, and $S' - s \leq CP$.

Proof. The reason is very simple. First of all, we only need to consider the case when $S - s > CP$, because otherwise we can set $S' = S$. But if $S - s > CP$, then the order quantities O(x) will be

$$O(x) = \begin{cases} 0 & \text{for } x \geq s \\ CP & \text{for } x < s, \end{cases}$$

and if we set $S' = s + CP$, and simply by checking, it can be seen that the modified (s,S')-policy will yield exactly the same order quantities O(x) as above. Consequently, the two policies are equivalent, but $S' - s = CP \leq CP$. ∎

Lemma 9 tells us that we can restrict the search for the "optimal" s and S among those pairs of s and S that $S - s \le CP$. Thus, we can always (and will) assume that $S - s \le CP$.

Note that, although we have proven that $f(x)$, the limiting function of $f_n(x)$ from the dynamic programming, satisfies the functional equation (7-1), we are unable to prove the uniqueness of $r(x)$ in (7-1). However, designed specially for the counter example, we have the following lemma.

Lemma 10. Suppose that there is a modified (s,S)-policy that is optimal to an infinite time horizon problem with $MD \le CP$, and assume that the long run cost function resulted from the modified (s,S)-policy is $r(x)$. Let $f(x)$ be the limiting function of $f_n(x)$ from the dynamic programming to this problem. Then $r(x) \equiv f(x)$ for all x.

Proof. Let $y_f(x)$ and $y_r(x)$ be the minimizing value for y of the functional equation (7-1) for $f(x)$ and $r(x)$ respectively, as a function of x, i.e.,

$$f(x) = L(y_f) + c(y_f - x) + \alpha \sum f(y_f-j)p(j)$$
$$r(x) = L(y_r) + c(y_r - x) + \alpha \sum r(y_r-j)p(j)$$

where, $y_f = y_f(x)$, $y_r = y_r(x)$.

Note that both y_f and y_r are feasible decisions for y at x, hence by (7-1),

$$f(x) \le L(y_r) + c(y_r - x) + \alpha \sum f(y_r-j)p(j)$$
$$r(x) \le L(y_f) + c(y_f - x) + \alpha \sum r(y_f-j)p(j)$$

Thus, $\alpha \sum [f(y_f-j) - r(y_f-j)]\, p(j) \le f(x) - r(x) \le \alpha \sum [f(y_r-j) - r(y_r-j)]\, p(j)$

Hence,

$$| f(x) - r(x) | \le \alpha \max \{ \sum |f(y_f-j) - r(y_f-j)|\, p(j),\ \sum |f(y_r-j) - r(y_r-j)|\, p(j) \} \qquad (8\text{-}2)$$

Since the optimal policy corresponding to $f(x)$ is an **X-Y** band policy, and let X and Y be the **X** and **Y** boundaries respectively, and note that (s,S)-policy is a special case of **X-Y** band policy as pointed in section 1, if we let $\mathbf{X} = S - CP$, and $\mathbf{Y} = s$. Choose two positive constants U and V, such that

$$-U \le \min \{ X, S\text{-}CP \} - CP \quad \text{and} \quad V \ge \max \{ Y, s \} + CP,$$

then, following similar proof for Lemma 6, it can be shown that for all $x \in [-U, V]$, $(y_f - j) \in [-U, V]$, and $(y_r - j) \in [-U, V]$, for all demand j under the assumption $MD \le CP$. Thus, from (8-2), we obtain

$$\underset{-U \le x \le V}{\text{maximum}} | f(x) - r(x) | \le \alpha \underset{-U \le x \le V}{\text{maximum}} | f(x) - r(x) | \qquad (8\text{-}3)$$

But, since $\alpha < 1$, (8-3) is only possible when $\underset{-U \le x \le V}{\text{maximum}} | f(x) - r(x) | = 0$, that is $r(x) \equiv f(x)$ for all $x \in [-U, V]$. Since U and V can be chosen arbitrarily large, hence we have shown that $r(x) \equiv f(x)$ for all x. ∎

From Lemma 10, if a modified (s,S)-policy is optimal to an infinite horizon problem with $MD \le CP$, then the corresponding long run discounted cost function is nothing else but **f(x)**. As a consequence, all those results to f(x) we proved earlier apply to this cost function. Conversely, if we can (and will) show that for the problem presented at the beginning of this section, the optimal policy corresponding to f(x) cannot be modified (s,S)-policy type, then we can reject the optimality of modified (s,S)-policy in general.

Because of Lemma 10, we will use f(x) instead of r(x), defined in section 7, in the following, although Lemma 11 is true for any r(x) satisfying (7-1) without the assumption that $MD \le CP$ (except (3), which requires the continuity of r(x)).

Lemma 11. If the modified (s,S)-policy is optimal to an infinite horizon problem, then

(1) f(x) is a linear function of x with slope $-c$ over $[\, S\text{-}CP,\ s\,)$ if $S - CP < s$;

(2) g(y) is a decreasing function of y for $y < S$;

(3) $g(s) = g(S) + K,\quad f(s) = f(S) + c\cdot(S - s) + K$;

(4) $f(x-t) \ge f(x) + ct$ for any $x \le S$, and any $t \ge 0$, or equivalently,
$f(x_1) - f(x_2) \ge c\cdot(x_2 - x_1)$ for any $x_1 \le x_2 \le S$.

Proof. (1): Since at point S, it is optimal not to order, thus by (7-3),

$$f(S) = L(S) + \alpha \sum f(S\text{-}j)p(j).$$

For any x, $S\text{-}CP \le x < s$, it is optimal to order up to S, i.e., $y^*(x) = S$, thus,

$$f(x) = L(S) + c\cdot(S - x) + \alpha \sum f(S\text{-}j)p(j) + K = - cx + f(S) + cS + K, \tag{8-4}$$

which is (1).

For part (2), suppose (2) is not true, then there exists two points (call them counter-points) $x_1, x_2,\ x_1 < x_2 < S$, such that

$$g(x_1) < g(x_2) \tag{8-5}$$

First, we may assume that $x_2 - x_1 \le CP$, because otherwise, let $x_3 = (x_2 - x_1)/2$,

if a) $g(x_3) \le g(x_1)$, then replace x_1 with x_3, and by (8-5), we have

$$g(x_3) < g(x_2), \quad \text{and} \quad x_3 < x_2;$$

if b) $g(x_3) > g(x_1)$, then replace x_2 with x_3, and we obtain

$$g(x_1) < g(x_3), \quad \text{and} \quad x_1 < x_3.$$

So, whatever happens, we can always find two new counter-points which are 'nearer' each other. With finite steps, we can get two counter-points which have the property as assumed.

Now, consider x_2 - CP. Since $x_2 - CP < S - CP \le s$, so at x_2 - CP, it is optimal to order CP units, i.e.,

$$f(x_2\text{- } CP) = K + L(x_2) + c\cdot CP + \alpha \sum f(x_2\text{-}j)p(j) = K + g(x_2) - c\cdot(x_2 - CP). \tag{8-6}$$

However, note that since $x_2 - CP \le x_1 < x_2$, thus x_1 is a feasible decision for y at $(x_2 - CP)$. Hence,

$$f(x_2\text{-}CP) \le K + g(x_1) - c\cdot(x_2 - CP) \tag{8-7}$$

Combining (8-6) and (8-7) yields, $g(x_2) \le g(x_1)$, which contradicts to $g(x_1) < g(x_2)$. This contradiction proves (2).

For part (3), since f(x) is a continuous function, so let $x \to s^-$ in (8-4) for the case $S - CP < s$, we get

$$f(s) = - cs + f(S) + cS + K = f(S) + c\cdot(S\text{-}s) + K;$$

if $s = S - CP$, then since $f(S) = g(S) - cS$, and for $x < s = S - CP$,

$$f(x) = K + g(x+CP) - cx,$$

let $x \to s^-$, we get $f(s) = K + g(s+CP) - cs = K + g(S) - cs = f(S) + c\cdot(S - s) + K$.

Similarly, by the optimality of the modified (s,S)-policy,

$$g(y) - (\, g(S) + K\,) \leq 0 \qquad \text{for } s < y \leq S;$$

$$g(y) - (\, g(S) + K\,) \geq 0 \qquad \text{for } y < s \quad (\text{ if } y < S\text{-}CP, \text{ then use (2) })$$

Hence, by the continuity of $g(y)$, we get $g(s) = g(S) + K$. (Which also shows that at s, it is both optimal either ordering to S or doing nothing.)

For part (4), one simple way to prove it is by using (2).

First, for any x and t, such that $s \leq x - t \leq x \leq S$, it is optimal not to order from either x or x - t, hence

$$f(x\text{-}t) = g(x\text{-}t) - c\cdot(x - t) = g(x\text{-}t) - cx + ct \geq g(x) - cx + ct = f(x) + ct.$$

We used (2) for above inequality.

Second, for $S - CP \leq x - t \leq x \leq s$, (4) is derived by (1).

Third, for $x - t \leq x \leq S - CP$, since it is optimal to order CP units from both x and x - t, so,

$$f(x\text{-}t) = g(x\text{-}t\text{+}CP) - c\cdot(x\text{-}t) + K \geq g(x\text{+}CP) - cx + K + ct = f(x) + ct.$$

Thus, by the continuity of $g(y)$, (4) is true for all $x \leq S$, and all $t \geq 0$. E.g., suppose that x and t are as such that $s \leq x \leq S$, $x - t \leq S - CP$, since $x - t = S - CP - [\, S - CP - (\, x - t\,)\,]$, hence (4) can be derived by

$$f(x\text{-}t) \geq f(S\text{-}CP) + c\cdot[S - CP - (x - t)] \geq f(s) + c\cdot[s - (S - CP)] + c\cdot[S - CP - (x - t)]$$

$$= f(s) + c\cdot[s - (x - t)] \geq f(x) + c\cdot(x - s) + c\cdot[s - (x - t)] = f(x) + ct.$$

Indeed, it can be shown that $f(x\text{-}t) \geq f(x) + n\cdot K + ct$, where n is the largest integer satisfying $n \leq t/CP$, and $t \geq 0$. ∎

Now, we are ready to use our results to prove that the modified (s,S)-policy is not optimal to the example shown at the beginning of this section for any pair of s and S. To do this, we will first assume the optimality of the modified (s,S)-policy and then show that this assumption leads to contradiction.

Thus, suppose that there exists s and S such that the modified (s,S)-policy is optimal to the problem. First of all, we may assume $S - s \leq CP$ by Lemma 9.

Since for our example $MD = 4 < 11 = CP$, hence by Lemma 6, Lemma 7, and Lemma 10, the limiting function $f(x)$ of $f_n(x)$ from the dynamic programming to the example problem exists and is continuous. Moreover, the long run cost function resulted from the modified (s,S)-policy must be $f(x)$.

It is obvious from (8-1), $x_m = x_L = D = 8$, and (can be easily shown)

$$s_1 = D - K/(\pi - c) = 5, \qquad x_s = S_1 = x_L = 8.$$

Hence from Lemma 8 $\mathbf{X} = x_s - CP = -3$, $\mathbf{Y} = x_L = 8$, and thus, $s \leq \mathbf{Y} = 8$.

By Lemma 8 and Lemma 3 (or similar proof), we can obtain $s \geq s_1 = 5$, $S \geq x_s = D = 8$.

Thus, as a summary, we get

$$5 \leq s \leq 8, \quad S \geq 8, \qquad S - s \leq CP = 11 \;\; (\text{ or } S - 11 \leq s\,), \quad \text{and} \quad f(x) \in C.$$

If $S - 11 < s$, then by Lemma 11 (1), $f(x)$ is a linear function over $[\, S\text{-}11, s\,]$ with slope $- c$.

Let $t = \max\{\, S - 11 + d, D\,\} = \max\{\, S - 11 + 4, 8\,\}$, where d = demand per period.

Note that $s + d \geq 5 + 4 = 9 > 8 = D$, and since $S - 11 + d < s + d$, so $8 \leq t < s + d$, and hence, $[t, s + d)$ is nonempty. For any $y \in [t, s + d]$, we have

$$S - 11 \le t - d \le y - d \le s, \text{ i.e., } y - d \in [S-11, s],$$

since

$$g(y) = L(y) + cy + \alpha \sum f(y-j)p(j) = L(y) + cy + \alpha f(y-d)$$
$$= h(y - D) + cy + \alpha f(y-d),$$

and because that f(x) is a linear function over $[S - 11, s]$ with slope $- c$, hence **g(y) is also** a linear function of y over $[t, s+d]$ with slope

$$h + (1 - \alpha) c = 1.12 > 0.$$

Thus from Lemma 11 (2), we get

$$S \le t = \max\{ S - 11 + d, D \}.$$

So, either $S \le S - 11 + d$, or $S \le D$. But the former is not possible because it means $11 \le d = 4$. Combining the later with $S \ge D$ which we got earlier, we get $S = D = 8$.

Thus, S only has two possibilities:

1. $S = D = 8$;
2. $S - 11 = s$, or $S = s + 11 = s + CP$.

If $S = D = 8$, then since $S - d = 8 - 4 = 4 < s$ (for $s \ge 5$), hence

$$f(S-d) = K + cd + L(S) + \alpha f(S-d) = K + cd + f(S),$$

but,
$$g(S+d) - g(S) = L(S+d) + c\cdot(S + d) + \alpha f(S) - [L(S) + cS + \alpha f(S-d)]$$
$$= (h + c) d + \alpha [f(S) - f(S-d)] = (h + c) d - \alpha [K + cd]$$
$$= 8 - 0.88 \cdot (27 + 4) = - 19.28 < 0,$$

and note that $S + d - (S - d) = 8 < CP = 11$, hence it is better to order to $S + d$ from $S - d$ than order to S, which contradicts to the optimality of S.

Now, suppose $S = s + 11 = s + CP$. First, for $x \in [s - 11, s]$,

$$f(x) = K + c\cdot CP + L(x+CP) + \alpha f(x+CP-d),$$

but since $s \le x + CP \le s + CP = S$, so, $f(x+CP) = L(x+CP) + \alpha f(x+CP-d)$. Thus,

$$f(x) = K + c\cdot CP + f(x+CP). \qquad (8\text{-}8)$$

(That is, the function f(x) over $[s-11, s]$ is simply a shifting of f(x) over $[s, S]$).

Second, by Lemma 11 (3), $g(s) - g(S) = K$, i.e.,

$$L(s) - L(S) + c\cdot(s - S) + \alpha [f(s-d) - f(S-d)] = K,$$

while by (8-8), $f(s-d) = K + c\cdot CP + f(s-d+CP) = K + c\cdot CP + f(S-d)$, hence

$$L(s) - L(S) + c\cdot(s - S) + \alpha (K + c\cdot CP) = K,$$

solving for s by using (8-1) and the fact that $s \le D = 8$, $S \ge D$, **and** $S = s + CP$, **we get**

$$s = D - \frac{(1 - \alpha)(K + c\cdot CP) + h\cdot CP}{\pi + h} = 8 - \frac{15.56}{11} = 8 - 1.41\dot{4}\dot{5} = 6.58\dot{5}\dot{4}$$

Hence, $S = s + CP = 17.58\dot{5}\dot{4}$.

Let $\omega = D - s = 1.\dot{4}1\dot{4}\dot{5}$. Consider f(x) over the intervals: [s-d, D-d], [s, D], [s+d, D+d], [s+2d, D+2d] (note: $s + d > D$, $D + 2d = 16 < S$), we get by Lemma 11 (4):

$$f(s-d) - f(D-d) \ge c\cdot[(D - d) - (s - d)] = c\cdot(D - s) = c\omega,$$

$$f(s) - f(D) = L(s) - L(D) + \alpha\cdot[f(s-d) - f(D-d)] \ge \pi\cdot(D - s) + \alpha c\omega = (\pi + \alpha c)\,\omega,$$

$$f(s+d) - f(D+d) = L(s+d) - L(D+d) + \alpha [f(s) - f(D)] \ge h\cdot(s - D) + \alpha(\pi + \alpha c)\,\omega = (\alpha\pi + \alpha^2 c - h)\,\omega,$$

and similarly,

$$f(s+2d) - f(D+2d) \ge - h\omega + \alpha (\alpha\pi + \alpha^2 c - h)\,\omega = (\alpha^2\pi + \alpha^3 c - \alpha h - h)\,\omega.$$

And consider the intervals: [D, s+d], [D+d, s+2d] using Lemma 11 (4),

$f(D) - f(s+d) \geq c\cdot[(s + d) - D] = c\cdot(d - \omega)$,

$f(D+d) - f(s+2d) \geq c\cdot[(s + 2d) - (D + d)] = c\cdot(d - \omega)$.

Thus,

$$\begin{aligned} f(s) - f(S) &\geq [f(s) - f(D)] + [f(D) - f(s+d)] + [f(s+d) - f(D+d)] \\ &\quad + [f(D+d) - f(s+2d)] + [f(s+2d) - f(D+2d)] \\ &\geq (\pi + \alpha c + \alpha\pi + \alpha^2 c - h + \alpha^2\pi + \alpha^3 c - \alpha h - h)\ \omega + 2c(d - \omega) \\ &= 23.999872\ \omega + 8 > 23.999872 \cdot 1.4145 + 8 = 41.9478 > 38, \end{aligned}$$

which contradicts to Lemma 11 (3), since by Lemma 11 (3),

$f(s) - f(S) = K + c\cdot(S - s) \leq K + c\cdot CP = 27 + 11 = 38$.

Thus, all of the possibilities have been rejected. That is, we cannot find the values for s and S so that the modified (s,S)-policy is optimal to our problem. Indeed, the solutions of $On^*(x)$, the optimal order quantities, from the dynamic programming to this problem up to 20 periods are as follows:

x \ n	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
-4	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
-3	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
-2	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	10	11	10	10	10
-1	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	9	10	9	11	9
0	11	11	11	11	11	11	11	11	11	11	11	11	9	11	9	8	9	8	11	8
1	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	7
2	11	11	11	11	11	11	11	11	11	11	11	11	11	11	10	11	10	10	10	6
3	10	10	10	10	10	10	10	10	10	10	10	10	10	10	9	10	9	11	9	5
4	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	9	8	11	8	4
5	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	7	11	7	3
6	10	10	10	10	10	10	10	10	10	10	10	10	10	10	11	10	6	10	6	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

The results show that the optimal ordering patterns are the same for all $n \geq 9$, and they are not the modified (s,S)-policy type (except for n = 1). Suppose that we are going to follow the same policy as the optimal policy for n = 20 in the long run, although we are not so sure for the moment whether the optimal policy for n = 20 has already converged to the optimal one for the infinite time horizon without some further investigation which we will do next, then the interesting thing is that, simply by checking, whatever the starting point it is, after some periods running there will be only two states (state x = 4 and state x = 5) at which the system needs to order something: at x = 4, order 9 units; at x = 5, order 11 units (Notice here: higher inventory but ordering more) The system jumps periodically between the two states with 5 periods per cycle. Thus, if this policy could be proven optimal for the infinite time horizon directly, then it would be more visible to see the non-optimality of the modified (s,S)-policy than the proof we gave earlier. Fortunately, taking the advantage of the characteris-tics of the **X** and **Y** boundaries of the optimal policy, we can solve this problem by LP (linear programming) using the formulation presented in [5]. The results from LP show that the optimal policy to the example problem for the infinite time horizon is exactly the same as the one for n = 20 shown above.

We used constant demand in example 3 only for the purpose to simplifying the proof (still not easy), and even in the simplest case the optimality of the modified (s,S)-policy is rejected. In the case of random demand, we also tried some examples. The results are the same: the modified (s,S)-policy is optimal for some problems, and not for some others.

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